THE DEVELOPMENT OF PRESERVICE MIDDLE SCHOOL TEACHERS' CONCEPTIONS OF PROOF DURING A GEOMETRY COURSE FOR TEACHERS

by

FILYET ASLI ERSOZ

(Under the Direction of Denise S. Mewborn)

ABSTRACT

Given that the importance and place of proof in mathematics education has been one of the emphases in the reform movement, it is crucial for teachers to be well equipped to teach mathematical reasoning and proof. The purpose of this study was to examine how pre-service middle school teachers' conceptions of proof evolved during a geometry content course for teachers. Six preservice teachers participated in this study. Multiple-case study method was used with each participant being the unit of analysis. Semi-structured interviews were conducted with the participants and the instructor of the course. Participants in this study recognized explaining why as an important role of proof and they developed an appreciation for the importance of proofs for themselves as teachers. The results also revealed that pre-service middle school teachers were unable to talk about the generality of proofs in the absence of a general argument if they had not developed a robust understanding of the fact that proofs prove for all cases. This study suggests that a nontraditional content class might help prospective teachers start valuing proofs and become aware of the explanatory power of proofs at least for themselves as teachers.

INDEX WORDS: Conceptions of proof, Preservice teachers, Middle school, Geometry, Content courses

THE DEVELOPMENT OF PRESERVICE MIDDLE SCHOOL TEACHERS' CONCEPTIONS OF PROOF DURING A GEOMETRY COURSE FOR TEACHERS

by

FILYET ASLI ERSOZ

B.S., Bogazici University, Turkey, 2002

M.S., University of Wisconsin-Madison, 2005

A Dissertation Submitted to the Graduate Faculty of The University of Georgia in Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSPHY

ATHENS, GEORGIA

2009

THE DEVELOPMENT OF PRESERVICE MIDDLE SCHOOL TEACHERS' CONCEPTIONS OF PROOF DURING A GEOMETRY COURSE FOR TEACHERS

by

FILYET ASLI ERSOZ

Major Professor: Denise S. Mewborn

Committee: Jeremy Kilpatrick
John Olive
AnnaMarie Conner

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
August 2009

DEDICATION

This work is dedicated to

My parents, Huseyin and Fatma Ersoz

My sisters, Didem and Ece

My husband Mehmet

With love and gratitude

ACKNOWLEDGEMENTS

First of all, I would like to express my deepest gratitude to my major professor, Denise Mewborn, for her guidance, encouragement, and support throughout my doctoral study. Her confidence in me, generous motivation, and valuable comments and suggestions helped me overcome the challenges of this doctoral study and complete my dissertation. Whenever I needed a person to talk to about academic or personal issues, I knew that I could turn to her and that she would point me in the right direction. I also thank her for giving me the opportunity to participate in her research and supporting me as a graduate assistant. She has contributed tremendously to who I am as a researcher and an educator. I am really grateful that I have had the chance to know her and work with her. I hope that we continue to work together for years to come.

I also would like to thank my committee members, Jeremy Kilpatrick and John Olive. Dr. Kilpatrick has been a great example for me of an analytical thinker and a questioning mind. I thank him for all the time and effort he devoted to reviewing my work and for all the great ideas he gave during my research. I also would like to thank Dr. Olive for the great questions that he asked and valuable feedback that he provided on my work. Thanks to his insightful comprehensive exam question and his encouragement to submit my work for publication, I presented at the International Commission on Mathematical Instruction Study 19: Proof and Proving in Mathematics Education in Taipei, Taiwan.

Special thanks goes to Anna Conner for agreeing to be a member of my committee and providing me with her invaluable experience in my research area. She spent hours with me to talk about my research and made insightful comments on my dissertation. I also would to thank her in particular for inviting me to participate in her research and supporting me during my doctoral program. I have also had the opportunity to teach with her and benefited tremendously from every step of the process from planning lessons to reflecting on our teaching.

I would like to thank the faculty in the UGA Mathematics Education Program both for the excellence of academic scholarship that they have modeled for us and for maintaining a warm and welcoming collegial environment for the doctoral students. I learned a lot from each of them throughout my doctoral study.

I am greatly indebted to many teachers in the past. I do not have room to mention them all, but I am thankful to each one of them for contributing to who I am today.

I owe gratitude to Sybilla Beckmann-Kazez for making this research possible. I am also grateful to the preservice teachers for agreeing to participate in my research, spending hours answering my questions, and letting me learn from them.

I would like to thank all friends here and at home, specifically Zelha and Hulya, who have always supported me. I specially want to thank my family in Madison, Wisconsin: Carol Ross, Jackie and Sherwood Seigel, Susan and Rick Herndon, and Carolyn and Jim Faren who have opened their homes and hearts to me, and made me feel at home so far away from home.

I extend special thanks to my family for their lifelong love and support. I cannot thank them enough for all they do and have done for me. I am grateful to my parents, Huseyin and Fatma Ersoz, for always believing in me and supporting my decisions. I also would like to thank my sisters, Didem and Ece, for always keeping me in their hearts and making me smile. I also thank my in-laws, Behzat and Nevin Iscimen, for their continuous encouragement and support.

This work would never have been possible – and I mean never – without the endless love and support of my husband, dearest friend, cheerleader, and voice of reason, Mehmet Iscimen.

Sharing with you makes everything more meaningful. As always, I would like to share this degree with *you*.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Mathematics is a discipline that has its unique ways of establishing truth known as "proof." Bell (1976) defined proof as "a directed tree of statements, connected by implications, whose end point is the conclusion and whose starting points are either in the data or are generally agreed facts or principles" (p. 26). As well as consisting of a "sequence of steps each of which has the form of justifying one claim by invocation of another, to which the first claim is logically reduced" (Ball & Bass, 2000, p. 203), proof is also a "sequence of ideas and insights" (Yackel & Hanna, 2003, p. 228).

Proof has an undeniably important role in both the activities of mathematicians and the development of mathematics. As Hanna (1983) explained, proof is an "indispensable tool of mathematics" (cited in Hanna, 1995, p. 42). Accordingly, the importance and place of proof in mathematics education has been one of the emphases in the reform movement. The National Council of Teachers of Mathematics' (2000) *Principles and Standards for School Mathematics* states that students in grades K-12 should be able to:

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs; and
- select and use various types of reasoning and method of proof. (p. 56)

Although several mathematicians and mathematics educators advocate that proof be central to mathematics education (Ball et al., 2002; Carpenter, Franke & Levi, 2003; Hanna, 1995; Knuth, 2002a; Ross, 1998; Schoenfeld, 1994), a large number of studies have revealed students' difficulties with proof (e.g., Chazan, 1993; Harel & Sowder, 1998; Healy & Hoyles, 2000). Despite the difficulties, there is also evidence that students can learn mathematical reasoning even in the elementary grades (Ball & Bass, 2003; Carpenter et al., 2003). Mathematical reasoning is something that students can learn to do and, hence, that teachers can teach (Ball & Bass, 2003). However, educators also recognize that

> the extent to which mathematical ideas such as proof and justification appear in classroom discourse will be influenced by both the teacher's choice of task and the questions and comments she makes during class, which are, in turn, influenced by the teacher's knowledge of proof. (Peressini, Borko, Romagnano, Knuth & Willis-Yorker, 2004, p. 81)

Background and Rationale

Given that proof has a central place in mathematics and mathematics education and that teachers' beliefs about the nature of a subject and subject matter knowledge influence what they teach and how they teach (Calderhead, 1996), it is crucial for teachers to be well equipped to teach mathematical reasoning and proof. Furthermore, Borko and Putnam (1996) emphasized that teachers should acquire syntactic structures of a discipline, which are defined as "rules of evidence and proof that guide inquiry in a discipline – the ways of establishing new knowledge and determining the validity of claims" (p. 676). Hence, teachers' conceptions of proof become an important area for researchers to investigate.

Despite the number of studies on students' conceptions of proof, there is limited research on teachers' conceptions of proof. Moreover, researchers have focused primarily on preservice elementary school teachers (e.g., Martin & Harel, 1989; Simon & Blume, 1996), inservice elementary school teachers (e.g., Ma, 1999), preservice secondary school teachers (e.g., Jones, 1997), and inservice secondary school teachers (e.g., Knuth, 2002a, 2002b), as well as undergraduate mathematics majors (e.g., Harel & Sowder, 1998). I chose to focus on preservice middle school teachers in this study.

Studying the middle school level is important and necessary because historically and traditionally, students' experiences with proof have been limited primarily to Euclidean geometry in high school (Ball & Bass, 2003; Chazan, 1993). However, given that proof is advocated as central to mathematics at all levels, middle school teachers have an important responsibility to incorporate proof into their teaching. Adding to the responsibilities of future middle school teachers is the fact that an analysis of randomly selected eighth-grade classrooms from the video study portion of the Third International Mathematics and Science Study (TIMSS) revealed that "there were no instances of mathematical reasoning in the United States lessons" (Manaster, 1998, p. 804). Hence, the purpose of the present study was to gain an understanding of how preservice middle school teachers' conceptions of proof develop, which, in turn, might aid teacher educators in preparing these teachers for their future responsibilities.

Harel and Sowder (2007) contended that the results of studies on mathematics teachers' conceptions of proof and their related practice in the classroom should draw the attention of "university mathematics departments, which have a primary responsibility for the preparation of mathematics teachers" (p. 49). The course that I investigated was being offered by a mathematics department and had the potential to provide future middle school teachers with a broader perspective with respect to proof and to help them become better equipped to teach proof.

In the study, I followed preservice middle school teachers throughout a course called Geometry for Middle Grades Teachers, a mathematics content course for prospective middle school teachers. I had observed this course in Spring 2007 (the year prior to my study) and chose it as my research site based on the goals of the course and my observations. This geometry course was structured and taught in a different way than a traditional geometry class in high school.

An important characteristic of the course was that it was problem and activity based. Once a problem was introduced, students were asked to work on it individually first and then discuss it with a neighbor. Then they were asked to share their solutions with the whole class. One of the objectives of the course, as listed on the course website, was “to strengthen the ability to communicate clearly about mathematics, both orally and in writing.” Students were expected to talk about their answers and explain them to others. Another course objective was “to show that many problems can be solved in a variety of ways.” To achieve this goal even when a problem was solved by a student correctly, the instructor would ask and encourage other students to share different approaches to the problem and different ways of solving it. One student who took the class the semester before the study described it as “a very safe environment to think out loud and to not be shut down in your thinking.”

Two of the objectives of the course that were particularly influential in my decision to choose it for the study were “to strengthen the understanding of and the ability to explain why various procedures and formulas in mathematics work and to promote the exploration and explanation of mathematical phenomena.” Although to construct proofs was not listed as a particular course objective, the instructor engaged the class in proving several formulas and theorems. She also noted in a private conversation that although she did not explicitly call the arguments proofs, a major goal for her was for the preservice teachers to be able to explain why things work the way they do in mathematics. Furthermore, several times during the class, she explicitly commented about why particular arguments were not proofs.

As Smith (2006) noted, the classroom environment portrayed above is “rare at the undergraduate level” (p. 73). She further stated that “there is little research focusing on how students *begin* the process of constructing a proof and how teaching strategies and learning experiences affect the development of students' understanding" (p. 76) and finally concluded that "there appears to be a need for more research investigating how students learn the process of proving in non-traditional classroom environments" (p. 88).

Furthermore, Harel and Sowder (2007), based on a review of the literature, concluded that teachers "do not seem to understand other important roles of proof, most noticeably its explanatory role" (p. 48). I expected that, based on its stated objectives and my preliminary interviews, this course would help preservice teachers develop an appreciation of this role of proof. The following quote by a student who took the course in Spring 2007 comparing her thoughts about proof at the beginning and at the end of the class supports my expectation:

> I used to think of proofs as just something to memorize, and it was just something that goes along with geometry that I didn't like to do, but now I see it. And I used to think of only those specific things that went along with proofs like that was outlined in the yellow box. I remember that in my textbook in high school that those were proofs, but … I guess coming out of this class I've got an understanding that you can look behind things to figure out why they are true, and you can prove it yourself rather than just memorizing something.

Research Questions

"Everybody knows what a proof is. Read, study, and you'll catch on. Unless you don't." says the Ideal Mathematician as portrayed by Davis and Hersh (1981) to a student who is questioning him about proof. I wanted to investigate what preservice middle school teachers know about proof after studying it in a mathematics content course. Hence, the following was my first research question:

1. How do preservice middle school teachers' conceptions of proof evolve as they participate in a mathematics content course titled Geometry for Middle Grades Teachers?

I investigated what middle school teachers thought a proof was prior to taking the course and as a result of studying proof in the way it was presented in the course. By *conceptions*, I mean both knowledge and beliefs, and I was interested in what proof meant for them, what constitutes a proof, what it means to prove something, the role of examples in a proof, how a proof could be general, and the purpose and importance of proof.

I also tried to determine what aspects of the course influenced the participants' conceptions of proof. When there were differences between students in how their proof conceptions evolved, I sought to uncover the factors that accounted for these differences. As I expected that their initial conceptions would be one of the factors influencing what they would learn, I posed a second research question:

2. How do preservice teachers' entering conceptions of proof influence what they learn in the course with regard to proof?

I expected that most, if not all, of the students' previous experiences with geometry and proof would be limited to their high school geometry classes. I wanted to know whether the image of proof they had constructed previously created a tension as they learned about proof in a different way in this course and how they negotiated or overcame the differences (if they realized them). Also, another important question to raise was whether experiencing proofs in a non-traditional way in this course would influence how they thought about the place of proofs in school mathematics, which led me to investigate the final research question:

3. What do preservice middle school teachers believe constitutes proof for middle school students?

There is research that shows that teachers value learning mathematics in a different way for themselves but do not see this way of learning as appropriate for their future students (Wilcox, Schram, Lappan & Lanier, 1991). Also, some secondary school teachers believe proof to be only for more able students (Knuth, 2002b). Thus, I investigated the beliefs of the participants in my study with regard to these matters.

CHAPTER 2

LITERATURE REVIEW

My research was informed by research on teachers' conceptions of proof. As mentioned in the previous chapter, existing research has investigated proof conceptions of preservice and inservice elementary school teachers, and preservice and inservice secondary school teachers. I organized the findings around four major themes: teachers' knowledge of proof, teachers' beliefs about the nature and role of proof in mathematics, teachers' beliefs about the role of proof in secondary school mathematics, and teachers' beliefs about themselves as mathematical thinkers in the context of proof.

Teachers' Knowledge of Proof

When researchers have investigated teachers' knowledge of proof, the majority have focused on teachers' acceptance of empirical versus deductive arguments as valid proofs. Some other common questions asked were the following: Is deductive argument convincing, or is there need for further evidence? Is one counterexample enough to reject an argument? Which characteristics of proof do teachers refer to while deciding whether an argument is a proof or not? Do teachers treat the argument that uses a particular example as the proof of a general case? What is the role of familiarity?

As a research strategy to reveal knowledge of proof, both Knuth (2002a), who investigated what constitutes proof for 16 inservice secondary school mathematics teachers, and Martin and Harel (1989), who assessed the notions of proof held by 101 preservice elementary school teachers, gave their participants statements accompanied by predetermined arguments and asked them to rate these in terms of their validity. Whereas Martin and Harel asked for written responses only, Knuth conducted in-depth interviews with his participants. In another study, Ma (1999) presented 23 U.S. and 72 Chinese elementary teachers with a scenario where a student comes to the teacher with a novel idea. The student's claim was supported only by empirical evidence. The teachers in the study were asked how they would react to the claim and how they would respond to the student. Simon and Blume (1996), in contrast to the other researchers, investigated the development of mathematical justifications by prospective elementary teachers in the context of a mathematics course that was designed as a constructivist teaching experiment.

Eleven of the teachers in Knuth's (2002a) study had undergraduate mathematics degrees and 13 held master's degrees, 2 of which were in mathematics. In the study by Martin and Harel (1989), the prospective teachers' experiences with proof were a high school course in geometry and a required undergraduate mathematics course with explicit attention to proof throughout the course. Despite the participants' varied mathematical backgrounds, both sets of researchers concluded that although most teachers correctly evaluate a valid argument, they wrongly accept invalid arguments as proofs. In Knuth's (2002a) study "a third of the ratings that the teachers gave to the nonproofs were ratings as proofs" (p. 391). Similarly, many preservice elementary teachers who "correctly accepted a general-proof verification did not reject a false proof verification" (Martin & Harel, 1989, p. 49).

The false proofs in Martin and Harel's study were in the form of a deductive argument and included variables and algebraic statements. Martin and Harel concluded that participants might have accepted these arguments as proofs based on their ritualistic aspects. Similarly, Knuth (2002a) found out that some teachers focused on the correctness of the algebraic manipulations or the form of the argument as opposed to the nature of the argument.

Similar to the results of studies on students' conceptions of proof, teachers tend to accept empirically based arguments as proofs. Several secondary school teachers in Knuth's (2002a) study rated an empirical argument showing that the interior angles of a triangle add up to 180 degrees as a proof. One teacher did not believe that it was a formal proof but still a proof, and two teachers found problems with it other than being empirical. Over 46 % of preservice elementary teachers simultaneously rated a general proof and at least one of the empirical arguments high (Martin & Harel, 1989). Furthermore, 9 % of U.S. teachers and 8 % of Chinese teachers immediately accepted the student's claim, which was based on empirical evidence, and 13 American teachers said that they would need more examples (Ma, 1999). Some of the prospective elementary teachers also accepted inductive reasoning as a means of justification (Simon & Blume, 1996).

According to Ma (1999), "These teachers ignored the fact that a mathematical statement concerning an infinite number of cases cannot be proved by finitely many examples – no matter how many. It should be proved by a mathematical argument" (pp. 86–87). This is a misconception "which would be likely to mislead a student" (p. 90). Martin and Harel (1989) claimed that "persons with limited experience in mathematics often hold the point of view that an inductive argument can also be a mathematical proof" (p. 42). This belief is grounded in people's everyday experiences, and it seems that extensive experience in mathematics might not be enough to help teachers to acquire correct ideas about what constitutes a proof (Knuth 2002a, 2002b) unless these concepts are explicitly addressed and discussed.

However, it is also important to note that even when the participating teachers accepted a general argument as a proof, several of them had reservations about whether it would be possible to invalidate the proof or whether the proof would be prone to counterexamples (Knuth, 2002a).

Although the teachers rated deductive arguments as valid proofs, they still did not find them convincing (Knuth, 2002a). In Knuth's (2002a) study, six of the teachers stated that it might be possible to find a counterexample or some other form of contradictory evidence. Inclusion of a concrete feature, specific examples, and visual reference were most convincing for these teachers. As Hanna (1990) said, "Understanding is much more than confirming that all the links in a chain of deduction are correct" (p. 12), and a proof might not be convincing enough when it is shown to be valid "by virtue of its form alone, without regard to its content" (p. 8).

Treating the proof of a particular case as the proof for the general case was also common among the teachers in these studies (Knuth, 2002a; Martin & Harel, 1989). Four of the secondary teachers rated the proof of a particular case as a proof of the general case (Knuth, 2002a). Similarly, the preservice elementary teachers also "showed high levels of acceptance of a particular proof" (Martin & Harel, 1989, p. 49). However, Knuth points out the possibility that the teachers mentally generalized the argument, using a particular example to generate a general argument.

Although familiarity with a statement (one they had previously seen or used) influenced the degree to which the secondary teachers were convinced of an argument (Knuth, 2002a), familiarity was not a factor in Martin and Harel's (1989) study. However, familiarity might be very subjective.

Employing a different method to assess knowledge of proof, Jones (2000) asked recent mathematics graduates who were enrolled in a one-year course to become secondary school teachers to construct concept maps reflecting their conceptions of mathematical proof. Analysis of the concept maps revealed that participants who had barely received passing grades in mathematics courses needed "considerable support in developing a secure knowledge base of mathematics" (p. 57). In addition, Jones reported that technical fluency in writing proofs did not necessarily imply richly connected knowledge of proof for these prospective secondary school teachers. This finding underlines the importance of distinguishing the ability to write proofs from a metaknowledge of proofs.

Teachers' Beliefs About the Nature and Role of Proof in Mathematics

One of the questions that Knuth (2002a) asked inservice secondary school teachers was "What purpose does proof serve in mathematics?" (p. 383). The themes that emerged corresponded to the proposed framework of the study. All 16 teachers mentioned establishing the truth of a statement as a role of proof in mathematics. Eleven of the teachers suggested that this establishment was done by means of a logical or deductive argument. However, when probed further, these teachers showed that they were not sure about the generality of the conclusion. The rest stated that proof was established by means of a convincing argument.

Three of the teachers mentioned the role of proof as explaining why something is true. However, this was a procedural focus rather than one of promoting understanding. Twelve teachers talked about the communicative role of proof in mathematics (social interaction, communicating and convincing others), and eight teachers mentioned the creation of knowledge and the systematization of results.

In a survey study with 30 preservice elementary school teachers and 21 students majoring in mathematics with an emphasis in secondary education, Mingus and Grassl (1999) asked the participants what constitutes a proof and asked about the role of proof in mathematics. In their definitions of proof, the secondary education majors emphasized the explanatory power, whereas the elementary education majors focused on the verification role of proofs. The majority of the participants also pointed out the importance of proofs in helping "students understand the mathematics they are doing" (p. 441). Furthermore, the secondary education majors "also considered the role of *proof* for maintaining and advancing the structure of mathematics" (p. 441).

Teachers' Beliefs About the Role of Proof in Secondary School Mathematics

The roles of proof in school mathematics that the teachers talked about (Knuth, 2002b) included all the roles they had mentioned for proof in mathematics (Knuth, 2002a) except for systematizing statements into an axiomatic system. At the same time, some new categories emerged. Thirteen of the teachers mentioned developing logical thinking skills, and four talked about displaying student thinking, which the teachers thought was beneficial for the student, the audience, and also helped them to assess student understanding. Contrary to Knuth's expectations, proof as a way of promoting understanding was not mentioned by these teachers.

For these teachers, proof in secondary school fell into three categories: formal proofs, which were identified according to their ritualistic aspects, format and language; less formal proofs, which were not necessarily mathematically rigorous but proved the general case and were mathematically sound; and informal proofs, which were explanations and empirically based arguments. The idea of a less formal proof is consistent with what Healy and Hoyles (2000) reported: "For many teachers it was more important that the argument was clear and uncomplicated than that it included any algebra" (p. 413).

Although the roles that the teachers attached to proof in secondary school mathematics seemed promising, their beliefs about the centrality of proof in school mathematics were very limited. Several teachers did not consider proof to be a central idea throughout secondary school. They believed that it was for advanced mathematics courses and for students who would study a mathematics-related area. On the other hand, all of the teachers considered informal proof to be a central idea throughout secondary school mathematics. They said that they would accept an empirically based argument as a valid argument from students in a lower level mathematics class. Two of them, however, explained that they would discuss the limitations of such "proofs."

The majority of the teachers in Knuth's study (2002b) viewed Euclidean geometry or upper level secondary mathematics courses as appropriate places to introduce proof to students. In a more recent study with 78 secondary mathematics teachers, Kotelawala (2009) found that "teachers vary greatly on when they feel proving is appropriate" (p. 254). In her study, 43 % of the teachers thought eighth grade or later to be appropriate for students to start focusing on proof-related activities, and 28 % believed that these activities did not have a place in the classroom until high school geometry, whereas 38 % of the participants indicated that all grades would be appropriate for proof-related activities. Another finding of this study was that "a majority of the teachers disagreed with the expectation of students being the ones proving" (p. 254).

It is likely that these beliefs were shaped by teachers' own experiences with proofs as high school Euclidean geometry is the "usual locus" (Sowder & Harel, 2003, p. 15) for introducing proof in U.S. curricula. Furthermore, "the only substantial treatment of proof in the secondary mathematics curriculum occurs" (Moore, 1994, p. 249) in this one-year geometry course. These findings also point out that teachers view proofs as a separate subject to be taught rather than a learning tool that could be integrated throughout mathematics.

On the other hand, the majority of the preservice teachers in the study by Mingus and Grassl (1999) advocated the introduction of proof before 10^{th} grade geometry classes. Furthermore, these participants who had taken collegiate level mathematics courses believed that proofs needed to be introduced earlier in the elementary grades as opposed to the participants whose only experiences with proofs were at the high school level. Mingus and Grassl argued that the former group “may have recognized that a lack of exposure to formal reasoning in their middle and high school backgrounds affected their ability to learn how to read and construct *proofs*” (p. 440).

Teachers’ Beliefs About Themselves as Mathematical Thinkers in the Context of Proof

A teacher’s attitudes toward using mathematical reasoning, her ability in constructing proofs, and her ability to deal with novel ideas are especially important because “ideas that surprise and challenge teachers are likely to emerge during instruction” (Fernandez, 2003, p. 267). In situations like those, teachers should be able to “reason, not just reach into their repertoire of strategies and answers” (Ball, 1999, p. 27). However, as Ma (1999) reported U.S. teachers are not mathematically confident to deal with a novel idea and investigate it. Similar to students, these teachers relied on some authority – a book or another teacher – to be confident about the truth of a statement. Although it was not the main purpose of their study, (Simon & Blume, 1996) also found evidence that showed that prospective elementary teachers appealed to authority. Ma claimed that “to empower students with mathematical thinking, teachers should be empowered first” (p. 105).

The study by Simon and Blume (1996) was different than others in the sense that Simon and Blume investigated preservice elementary school teachers’ conceptions of proof in the context of a mathematics course “which was run as a whole class constructivist teaching experiment” (p. 3). The preservice teachers’ previous mathematical experiences were limited to traditional classrooms where the authority resided in the teacher. The goal of the instructor in the study was to shift the “authority for verification and validation of mathematical ideas from teacher and textbook to the mathematical community (the class as a whole)” (p. 4). The authors argued that one of the reasons that this shift was significant was that it “can result in the students’ sense that they are capable of creating mathematics and determining its validity” (p. 4).

The findings are important as they illustrated how preservice teachers’ prior experiences with proofs or lack thereof and their views about mathematics influence how they initially respond to situations that need to be proved. In other words, at the beginning of the semester when the instructor asked for the justification of mathematical ideas, the preservice teachers referred to learning them in their previous mathematics courses or provided empirical reasons. Also, the preservice teachers did not necessarily make sense of others’ general explanations if they were not operating at the same level of reasoning. However, Simon and Blume claimed that “norms were established over the course of the semester, that ideas expressed by community members were expected to be justified and that those listening to the justification presented would be involved in evaluating them” (p. 29).

Although not explicitly focusing on teachers, a related study comes from Sowder and Harel’s (2003) proof understanding, production, and appreciation (PUPA) project, in which they studied how undergraduate mathematics majors’ ideas about proof evolved over the course of their undergraduate studies in mathematics. To achieve this goal, Sowder and Harel followed 36 randomly selected students during much of their undergraduate programs via interviews every semester. The students’ exposures to proof up to the point when the interviews started were minimal, and their “personal proof-writing” went back to their secondary school year-long course in Euclidean geometry. Sowder and Harel report three of these cases that illustrate the diversity among the students.

Ann’s PUPA was the weakest among the three students, and her proof skills remained the same throughout the program. This lack of change might point out the difficulty of helping students develop a better understanding of proof during their undergraduate mathematics courses if they have not already developed a basic level of understanding. Carla, who was in the middle, entered with weak PUPA but finished better. In response to a question about whether she had learned the necessity of a general argument in high school geometry, Carla said: “I didn’t really pay much attention to that back then. It’s like I was just trying to get the answer; I was not really concerned about how you got it, and why you got it” (Sowder & Harel, 2003, p. 15). The question remains as to what difference between Ann and Carla influenced the way that they benefited from their undergraduate experiences.

Ben, in contrast, entered with a strong PUPA and excelled even more. Ben remembered his secondary school geometry course as “a much fun course” and as “the first class I think I really excelled at” (Sowder & Harel, 2003, p. 13). Sowder and Harel attributed part of his success to entering the university “having had a very proof-favorable experience in secondary school geometry” and argued that “this preparation predisposed him to react positively to proof demands and opportunities” (p. 14). This is an encouraging conclusion because it suggests that having a positive experience with proof during secondary school is likely to be fruitful later. However, the data provided in this report and the level of analysis do not tell us much about how Ben might have invoked his knowledge of proving gained in secondary school in his undergraduate mathematics classes or the kinds of similarities he might have seen between the different contexts.

Theoretical Framework

One of the theoretical perspectives that shaped my study was the construct of “proof schemes” developed by Harel and Sowder (1998). By *proving*, Harel and Sowder meant “the process employed by an individual to remove or create doubts about the truth of an observation”

(p. 241). The process of proving includes two subprocesses: ascertaining and persuading. While ascertaining is “the process an individual employs to remove her or his own doubts about the truth of an observation,” persuading is “the process an individual employs to remove others’ doubts” (p. 241). In this framework, “a person’s proof scheme consists of what constitutes ascertaining and persuading for that person” (p. 244). As the product of their extensive work with their students, Harel and Sowder developed three categories of proof schemes, each with several subcategories. The three main categories are external conviction proof schemes, empirical proof schemes, and analytical (deductive) proof schemes. It is also possible for students to simultaneously hold more than one kind of scheme.

For students who hold external conviction proof schemes, what removes their doubts is either the form of an argument or the word of an authority. One of the subcategories of this scheme is the *ritual proof scheme* (Harel & Sowder, 1998, p. 246), where the appearance and form of an argument influence students’ judgments regarding the correctness of the argument. Another subcategory is the *authoritarian proof scheme*, and the main source of conviction for students who hold this scheme is a “statement appearing in a textbook or uttered by a teacher” (p. 247). In other words, something is accepted to be true because the teacher or the book says so, and there is no intrinsic need to provide a justification. Students at this level may believe that “proof is only a formal exercise for the teacher; there is no deep necessity for it” (Alibert, 1988, p. 31) as they do not view proof as a “functional tool” (p. 31). The manifestation of the third subcategory – *symbolic proof scheme* – occurs when students use the symbols without regard to the meanings that they might have in the context of the problem. For example, a student holding a symbolic proof scheme might start manipulating symbols immediately after reading a problem without trying to form an image of what the symbols stand for.

In an empirical proof scheme, students use “physical facts or sensory experiences” to validate conjectures. Students who possess an empirical proof scheme rely on either evidence from examples (sometimes just one example) of direct measurements of quantities, numerical computations, and substitutions of specific numbers in algebraic expressions, or perceptions. In contrast, all transformational proof schemes, a subcategory of deductive proof schemes, “share three essential characteristics: generality, operational thought and logical inference” (Harel & Sowder, 2007, p. 7). The generality characteristic means that an individual understands that the “goal is to justify a ‘for all’ argument” (p. 8). The idea of proof separates mathematics from the “empirical sciences as an indubitable method of testing knowledge which contrasts with natural induction from empirical pursuits” (Hoyles, 1997, p. 7). In other words, an important aspect of mathematical proof is the generality of the argument that is given. Knuth and Sutherland (2004) argued that understanding the generality aspect of proof is “critical to developing an understanding of the concept of proof” (p. 561).

The construct of proof scheme and its three main categories provided me with a useful tool in the process of establishing interview protocols and selecting participants. The construct of proof scheme was also useful in terms of answering the first and second research questions. I had hypothesized that some participants might hold more authoritarian proof schemes at the beginning of the semester but move toward seeing an intrinsic need to construct proofs even if they were not yet at the level of a deductive proof scheme. Based on the literature reviewed, I also expected some preservice teachers to start the course with empirical proof schemes. I included questions in all three interviews to reveal whether the participants held and would continue to hold empirical proof schemes. When I analyzed my data I looked for evidence that would help me to identify the proof schemes that the participants held at a certain point. On the other hand, a recent study by Felton (2007) helped me to focus on the importance of context when evaluating participants' conceptions of proof.

Felton (2007), unlike other researchers studying teachers' conceptions of proof, focused on the importance of context on teachers' conceptions of proof. He drew researchers' attention to the difference between teachers' conceptions of proof in the abstract versus in the context of teaching. In his study, Felton compared preservice teachers' discussions of proof and its role in K-12 mathematics to their reflections on whether given student justifications were adequate proofs or not. From analyzing the former, he developed some codes, and using these codes and adding to them, he analyzed the latter. Then he compared the frequencies of each code for the two contexts and found differences between the two. He found that "the importance of establishing the validity of a claim and understanding new mathematics dropped when considering student work, while the finding of a solution took on greater importance" (p. 85).

Felton's distinction between proof in the abstract and in context gave me a useful tool to organize the findings of my data as I also saw shifts between the two for participants in this study. However, I used proof in context to mean validating arguments, whereas Felton used it to mean evaluating student work. The codes that emerged in Felton's study were the following: universal, explain why, establish validity, general argument, develop understanding, is an equation, and answer the question. I used these codes to do an initial analysis of my data. During this process, I eliminated codes that did not match my data, modified others, and added codes to fit my data.

In another recent study, Smith (2006) compared the perceptions of and approaches to mathematical proof of undergraduates enrolled in lecture-based and problem-based transition-to-proof courses. The problem-based course employed the "modified Moore method" (p. 74). In the traditional lecture-based course, the instructor presented theorems and proofs, whereas in the problem-based course, the instructor gave students a list of problems to work on and then asked them to present their solutions to the class, followed by a whole-class discussion led by the instructor. Smith expected the students in the two sections to differ in constructing and validating proofs. Smith found that the students in the lecture-based course “demonstrated conceptions of proof that reflect those reported in the research literature as insufficient and typical of undergraduates” (p. 73). For instance, appealing to authority or searching for a proof technique without knowing why it would be appropriate were among the approaches taken by these students. In contrast, the students in the problem-based course were “found to hold conceptions of and approach the construction of proofs in ways that demonstrated efforts to make sense of mathematical ideas” (p. 73).

The framework that emerged in Smith’s (2006) study provided me another lens to analyze my participants' strategies for constructing proofs or validating arguments. In Smith’s study four primary differences were revealed between the ways students from different sections of a course approached proof construction tasks. These differences were related to use of initial strategies, use of notation, use of prior knowledge and experiences, and use of concrete examples. For example, when students from the lecture-based section faced a proof construction task their initial response was to begin searching for proof techniques, whereas students from the problem-based section tried to make sense of the statement first. Although students from both groups tried to make use of prior knowledge, that use was based on surface features for the traditional group but on the concepts involved for the problem-based group.

The characteristics identified by Smith (2006) as indicators of learning how to prove in a lecture-based versus problem-based class provided me with an initial set of cues to look for in the data. For example, I paid attention to a student's initial reaction when faced with a proof construction task. I had expected the participants to try to remember strategies at the beginning of the semester as reminiscent of traditional learning but expected them to try to make sense of problems toward the end of the semester as a consequence of their experiences in this course. Although the problem-based transition-to-proof course that provided the context of Smith's study was not identical to the course that I investigated, these results were promising as they suggested "that such a problem-based course may provide opportunities for students to develop conceptions of proof that are more meaningful and robust than does a traditional lecture-based course" (p. 73).

Finally, I used Knuth's (2002a, 2002b) and De Villiers' (1999) frameworks as I looked into the roles of proof in mathematics and in mathematics education identified by my participants. Knuth (2002b) proposed the following framework outlining the wide range of roles proof plays in mathematics:

- to verify that a statement is true
- to explain why a statement is true
- to communicate mathematical knowledge
- to discover or create new mathematics
- to systematize statements into an axiomatic system. (p. 63)

De Villiers argued that the roles of proof in a school mathematics curriculum should reflect the roles of proof in the field of mathematics: verification, explanation, systematization, discovery, communication, and intellectual challenge. These frameworks provided an initial tool to analyze what the preservice teachers in my study said when they talked about how they envisioned the role of proof in their future classrooms.

CHAPTER 3

METHODOLOGY

Methods

The purpose of this study was to examine how preservice middle school teachers' conceptions of proof evolved during a geometry content course for teachers. Achieving this goal required rich descriptions of the participants' conceptions of and experiences with proof throughout the semester. Thus, a case study with each participant being a "unit of analysis" (Patton, 2002, p. 226) was an appropriate method for this study as "a qualitative case study seeks to describe that unit in depth and detail" (p. 55). In contrast, Ginsburg (1997) asserted that an "obvious weakness of individual case study … is its ambiguity with respect to generalization" (p. 189), and he indicated that "one approach taken by researchers is to aggregate them" (p. 190). Toward this end this study encompassed a collection of six cases; in other words, the multiple-case study method was used (Merriam, 1998). I also conducted a cross-case analysis to reveal the patterns in the data.

As Hays (2004) asserted, "Generalization is not a goal in case studies, for the most part, because discovering the uniqueness of each case is the main purpose" (p. 218). As such, each case investigated in this study had unique characteristics. Patton (2002) also suggested that "while one cannot generalize from single cases or very small samples, one can learn from them-and learn a great deal, often opening up new territory for further research" (p. 46). Ginsburg (1997), however, criticized students who wrote that their studies "cannot be generalized beyond the subjects employed" (p. 260). He believed that what was meant was that "they are not sure how widely the results can be generalized" (p. 260), and I agree. Even though the cross-case analysis revealed patterns in the data, one needs to be careful about generalizing the results to any population.

Setting

The course that provided the context of my study was offered in Spring 2008. The instructor of this course, Dr. Benson, was a professor of mathematics. She was also interested in the preparation of mathematics teachers. Toward this end, she had developed three content courses focusing on arithmetic, geometry, and algebra for preservice elementary and middle school teachers. She also wrote a textbook that accompanied these courses.

Dr. Benson had previously mentioned her interest in having mathematics education graduate students "doing studies that look into what students get out of these math courses" in an email message to all mathematics education graduate students. So, when I approached her and told her about my study, she readily agreed to participate. In Spring 2008, 18 students were enrolled in this course; 15 of them were undergraduate students, and 3 were graduate students. One of the undergraduate students was enrolled in the secondary education program. Most of these students had been enrolled in a content course titled Arithmetic for Middle Grades Teachers with the same instructor the previous semester. I had observed some sessions of that course both to become acquainted with the participants and also to familiarize them with me.

The topics covered in the course included "visualization, angles, geometric shapes and their properties, constructions with straightedge and compass, transformation geometry: reflections, translations, rotations, symmetry, congruence, similarity, measurement, especially length, area, and volume, converting measurements, principles underlying calculations of areas and volumes, why various area and volume formulas are valid, area versus perimeter and the behavior of area and volume under scaling" (MATH 5030 website).

Furthermore, this course was part of a Writing Intensive Program, which was "designed to help courses teach the writing process within various disciplines" (MATH 5030 website). This program supported the idea that "mathematical writing has its own special features." It was explained on the course Web site that "in mathematics, we seek coherent, *logical* explanations, in which the desired conclusion is deduced from starting assumptions" and students were encouraged to use writing to "deepen [their] understanding of the course concepts."

With respect to the place of proof and learning to prove in this class, Dr. Benson said the following in an interview:

> It's not, not really explicit proof per se, but, you know, it's never really clear, you know, where is the line between explaining why something is true and proof. I think people mean different things when they say that. I mean, when you say proof, sometimes that means you're working within sort of an axiomatic system, and it's all built up very carefully kind of from, from uh, some starting points, and you're rather systematic about it. Or not quite that, I don't think we get quite to that level of systematicness, … but if you take a proof as an explanation of why something is true based on things that you've already discussed or thought about, or things that you take to be true just, you know, from common sense notions in the world around us, then from that point of view, yeah, we are explaining why. We're, that's one of the key things that we do, you know. Why are the area formulas and volumes the way they are? And you know, why, why do various shapes have additional properties that, you know, you didn't construct them to be that way but those, those properties are there? Why is that so? So, you know, can you explain it in terms of some other things that you've already thought about? (Interview, Lines 23–41)

Some of her other goals for students were to realize that mathematics builds on itself, to connect ideas, and

> to have the sense of, you know, you don't just say things are true, you have to justify them. … You have to explain why …especially … in the context of teaching. You don't just tell the kids this is true. … 'Cause that's not math, you know. It's, it's not okay to just tell them a pile of facts and expect them to accept those facts just sort of on faith. (Interview, Lines 101–108)

She wanted them to be motivated to look for an explanation "if they notice that something appears to be true" (Interview, Lines 115–116).

In this class, rather than being a separate topic to be taught, the idea of a proof seemed to be integrated throughout the course as a learning tool. Dr. Benson described this integration of proofs in her class in the following way:

> We kind of weave in and out of proofs. … We did the sum of the angles of the triangle. And, and we're gonna do a little bit more with, um, kind of explaining how the various congruence criteria imply certain, that certain properties hold: … The base angles of an isosceles triangle are equal, and that the diagonals of a rhombus are perpendicular, and bisect the angle. So that, we'll talk a little bit about that. I'm not sure I'm gonna emphasize too much you know the careful proofs of that. … But then when we get into similarity, I'm kind of, you know it's not gonna be so much proof in the standard geometric sense, but it's gonna be like different ways of reasoning. (Interview, Lines 493–505)

Dr. Benson wanted the proofs that she and the students worked on in the class to be "accessible" to the students and also believed that proving a familiar statement, in other words, something that they knew to be true but did not think about why it was, would have an impact on them. She also said that in some cases they would "work something out in an example case, and then we just kinda say, 'Well it's really kind of the same thing if you do it in other cases' but that's not elaborated" (Interview, Lines 432–435). When I asked her the role of demonstrations in her class, she said:

> It's just so visually satisfying. … I see those like a first step. But then you know after that you could say, "Well, how do you know it's always gonna work for any triangle? We only had five here. We tried it for all five. That looks pretty convincing, but and how do you know it's not 179 degrees? We can't measure that accurately." (Interview, Lines 251–258)

I think she believed that those demonstrations would motivate the students to seek further explanations. With regard to the implicit nature of proofs in her classroom, she said, "You don't have to use the formal word *proof* or *prove*. But you could still be in that mind set of wanting to explain why it is" (Interview, Line 404–406).

Participants

I used purposeful sampling (Patton, 2002) "to adequately capture the heterogeneity in the population" (Maxwell, 2005, p. 89) with respect to Harel and Sowder's (1998) proof schemes. I selected the participants based on my observations of them in a course that they had taken the previous semester, recommendations from a previous instructor, and a proof survey. I wanted the participants to represent a broad spectrum in terms of the variety of knowledge and beliefs about proof they brought to this classroom. I approached seven students and asked them if they would participate. All of them agreed. However, because of scheduling conflicts, I was able to conduct the last interview with one of the participants only by phone, and she therefore could not complete the tasks that were part of the interview because they required looking at diagrams and producing diagrams. Because of the missing data, I analyzed the data for six participants only.

Two surveys (Appendix A) with open-ended questions that I developed were administered to all 18 students to gain background information with respect to conceptions of proof that they held at the beginning of the semester. The surveys asked the students a variety of questions ranging from their previous experiences with proofs to evaluating an argument. The survey data were used to select participants. Another reason for collecting some data from everyone was also to be able to describe the participants I selected in comparison to their peers. The same two surveys were administered at the end of the semester. Also, a Mathematical Proof Survey (Yoo, 2008) was administered both at the beginning and at the end of the semester. This survey consisted of questions asking about views of mathematics, the purpose of proofs, and learning of proofs (Appendix B).

After all the students filled in Survey 1 and Survey 2 (Appendix A), I tried to determine whether their responses demonstrated characteristics of one the following proof schemes: authoritarian scheme, empirical scheme, or deductive scheme (Harel & Sowder, 1998). Although it was difficult to strictly categorize some of the participants, their responses gave me cues as to which proof scheme they were likely to be holding when they filled in the surveys. For instance, one of the survey questions asked the students the sum of the measures of the interior angles of a triangle, and the next question asked how they knew that their answer was correct. If a participant responded to this question with reference to an authority – a book or a previous teacher – or simply said "by memorization," I categorized that participant as holding an authoritarian proof scheme. Another question in the survey asked the students how they knew that their answer to the sum of the interior angles question was correct for all triangles. The students were also asked to evaluate a given empirical argument. If they responded to the former question by saying that they would need to try several triangles and also did not find a problem with the given argument, I categorized them as having an empirical proof scheme. Finally, if the participants provided an argument – even if wrong – that demonstrated elements of being general and also rejected the given argument as a proof based on the reason of depending on examples, I categorized them as having a deductive proof scheme. There were also participants who demonstrated elements of being general in their own proof construction but rejected the given argument for reasons other than being empirical. I also grouped them under the deductive proof scheme category.

Among the 15 undergraduate students, 4 were in the authoritarian group, 3 were in the empirical group, and 8 were in the deductive group. Among the last 8, 3 students did not indicate that the given argument was empirical. Also, among the 8 students, 1 – the secondary education major – was able to construct a valid argument, and 1 was close to constructing a valid argument. From the 15 undergraduate students, I selected 2 from each category to participate in the interviews. All the undergraduate participants were in the middle school education program. From the 3 graduate students taking the course, I selected 1 to participate in the study mostly based on my observations of her the previous semester and because I believed that she would provide me with rich data. I also wanted to compare what she would learn about proofs in this class with what the undergraduate participants would learn.

Data Collection

My main methods of data collection were to administer surveys to all students at the beginning of the semester, conduct semi-structured interviews (Bernard, 2002, chapter 9) with the instructor and 6 focus students, videotape the class sessions, and take field notes, and collect students' quizzes and tests. For the purposes of the dissertation, I analyzed the interviews and the surveys.

As Hays (2004) indicated, "Interviews are one of the richest sources of data in a case study" (p. 229). All interviews in this study were semi-structured (Bernard, 2002, chapter 9) in the sense that I started with an interview protocol, but my follow-up questions depended on the participants' responses. The interviews were scheduled depending on the participants' available times, and they were all conducted one-on-one in a conference room on campus. Each interview lasted about an hour. I recorded all the interviews using two video cameras, one to capture the participants and one to capture what they wrote. I also kept all of the written work from the interview sessions. I transcribed the interviews in their entirety and added notes indicating what the students wrote and what hand gestures they used. I used the following conventions when preparing the transcripts:

{text} denotes guessed speech or questionable part of transcription.

(inaudible) denotes a time when I could not understand what a student said.

[ ] denotes a comment that I inserted while preparing transcripts.

Interviews with Students

I conducted 3 interviews with each participant during the semester. The goal of the first interview was to understand the participants' conceptions of proof at the beginning of the semester. These interviews were conducted during the first 2 weeks of the class. In the first part of this interview, I asked questions similar to the ones in Knuth's (2002a) study to elicit my participants' knowledge and beliefs about the nature of proof and the role of proof in mathematics and mathematics education. I started with these kinds of questions because I believed that their beliefs about the nature of proof could influence what they would accept as a proof and that trying to reveal these beliefs could give me insight as I tried to understand how they thought about proofs. The first interview protocol is in Appendix C.

In the second interview, some of the questions that I asked were similar to those in the first interview so that I would be able to trace the changes in participant's conceptions of proof. In this interview, I also asked participants questions about what they had learned about proofs up to this point in the semester and if there was anything about proofs that confused them. If a participant had raised a question about proofs in the first interview, I followed up on it in the second one. I also asked the participants about a recent proving experience (Conner, 2007) in an attempt to understand what they would describe as a proving situation. The second interview protocol is in Appendix D.

During the last interview I aimed to gain an understanding of what the students thought they had learned in this course about proofs and what aspects of the class were influential in their thinking. I also wanted them to compare this course to other courses where they might have learned about proofs. In addition, I wanted to learn about their beliefs about the importance of proof in general and also the place of proof in school mathematics at the end of the semester. Similar to the first two interviews, this interview also included questions that asked them to talk about proof in the abstract and also to evaluate arguments. The third interview protocol is in Appendix E.

All interviews included statements for the students to prove and arguments that went along with each statement to validate. Two of these statements were also presented in the classroom by the instructor and were proven by the students. The classroom activities involving these statements are discussed in detail in Chapter 4. These tasks gave me information about the participants' proving strategies as well as what they thought a proof was. Like Smith (2006), I assumed that "what students do when first presented with a statement to prove reveals a great deal about their understanding of proof" (p. 74). I also provided both valid and invalid arguments and asked them to evaluate these and decide whether they were proofs or not. I asked whether the arguments were convincing for them and for some arguments also asked about their appropriateness for middle school students.

Interview with the Instructor

I interviewed the instructor of the course once during the first month of the semester. Some of the goals of the interview with the instructor were to determine her goals for this class, her own conceptions of proof, and what she thought was important for middle school teachers to know about proof. The interview protocol for the instructor is in Appendix F.

Analysis

As Patton (2002) asserted, “The qualitative analysis process typically centers on presentation of specific cases and thematic analysis across cases” (p. 297). I started my analysis by going through transcripts and surveys for each case. To analyze the transcripts, I went through the entire transcript line by line and the corresponding written work and tried to make conjectures regarding what the participant was thinking. As I analyzed each subsequent transcript, I compared my conjectures with the previous ones and tried to construct an account of development for each case. During this process, I created a table for each participant summarizing that participant’s conceptions of proof for all three interviews along 5 themes: views about mathematics, experience with and attitude toward proof, proof in the abstract, proof in context and evaluation of arguments, and importance of proofs. (The layout of a table is in Appendix G.) These tables also helped me to see which aspects of participants’ conceptions of proof changed, stayed the same or were added. At the end of this process, I wrote a detailed record of each case. After writing each case, I used the same themes to run a cross-case analysis. During this analysis, I focused on not only similarities but also differences between the cases.

The themes that I used to organize my findings in the tables emerged as a result of interplay between the research questions, the literature, and the ongoing analysis of data. As I realized that some participants’ developing conceptions of proof might have been influenced by their views about mathematics, I made this one theme. I also believed that knowing about these beliefs would give general information about each participant. Based on the literature, I expected the participants’ prior experiences with proofs to influence what they were going to learn in this class. Because this was one of my research questions, it became another theme. The distinction that I made between proof in the abstract and proof in context, hence the two themes, was based on Felton's (2007) study and an initial analysis of my data. Finally, several researchers have argued that proof plays a wide range of roles both in mathematics and in mathematics education. Because I was looking into my participants' views on these topics, importance of proofs was my last theme.

A characteristic of a case study is that data from multiple sources need to be triangulated for reliable results (Patton, 2002). I triangulated my data using the different surveys administered both at the beginning and at the end of the semester and the three interviews. There were also a variety of questions in the interviews that were intended to reveal the participants' conceptions of proof in different ways. For instance the participants could talk about an aspect of a proof when they talked about things that they learned in this class, when they defined a proof, or when they talked about the importance of proofs. If a participant mentioned explaining why as an aspect of proofs in all these three different situations, that provided me with strong evidence that explaining why was an important element of that participant's conception of proof.

CHAPTER 4

FINDINGS

I start this chapter by describing two classroom activities as the two tasks that I used in the interviews (Question 17 in Appendix C and the first statement in Appendix D) were based on these activities. These activities also illustrate proving situations in the course. Next, I introduce each case under the following themes: views about mathematics, experience with and attitude toward proof, proof in the abstract, proof in context and evaluation of arguments, and importance of proofs. Finally, I present the cross-case analysis pointing out both the similarities and the differences between the cases.

Examples of Classroom Activities

The first activity was introduced on Jan 14. The task was to prove that the interior angles of a triangle add up to 180 degrees. The lesson started with a discussion of two lines in a plane and the angles formed by them. The students proved that vertically opposite angles are equal. Then they moved on to a similar discussion for three lines in a plane. They talked about the angles formed when two lines are parallel and the third one intersects the two. Dr. Benson introduced the phrase “parallel postulate” and explained that it actually could not be proven but was accepted as true by mathematicians. She also mentioned that there were other geometries where the parallel postulate would not hold true.

Next, they started talking about the case where each line intersects with the other two and a triangle is formed. When Dr. Benson asked what they knew about the triangle, several students said 180. Then she told them that they would do an experiment. She distributed papers and scissors. She asked them to draw triangles that would look different from the person next to them. Then she told them to cut the triangles, tear off the corners, and put the three corners together “point to point.” After the students realized that a straight angle was formed, she asked them if they had proven that the sum of the angles in a triangle is 180. One student volunteered to say that the argument did not necessarily prove it for every case but that it was a good argument. The next class period, January 16, they discussed what they liked about the argument and also why it was not a proof. One of the reasons was that it did not prove the given statement for all possible cases. The other reason was that when the corners were put together it may not have been perfectly 180 degrees. Dr. Benson concluded that it was a great demonstration but not a proof. Then students worked on proving the same statement using the parallel postulate. Finally they talked about whether the argument involving the parallel postulate would work for all triangles and concluded that it would.

The second activity was introduced the same day. The task was to prove that the exterior angles of a triangle add up to 360 degrees. A triangle was made with tape on the floor. Dr. Benson asked for two volunteers: a walker to walk around the triangle and a turner to face whatever direction the walker was facing. After the two volunteers demonstrated the activity, they talked about the directions that the turner faced as the walker was walking all the way around the triangle once. Based on this activity, they concluded that the turner had turned 360 degrees and this was the sum of the exterior angles of the triangle as the turner had covered the three exterior angles. After the physical demonstration, they also proved that the sum of the exterior angles is 360 using the parallel postulate.

Case 1: Kate

Kate was one of the undergraduate participants in this study. She had mathematics as her first specialization and science as her second. She was categorized in the deductive proof scheme group based on her response to the initial survey.

Views About Mathematics

Kate seemed to have a very static and formula-based understanding of mathematics. In the Mathematical Proof Survey, she ranked highest the option of mathematics being “a static body of absolute facts independent from human invention” both at the beginning and at the end of the semester. She believed that, although she admitted that she might be naïve, “we know everything we will know about math as far as formulas are concerned” (Interview 1, Lines 355–356). A characteristic that distinguished mathematics from other subjects for Kate was that “math builds off of previous knowledge. You have to start with the basics and work your way up. … Math requires previous understanding before learning something new” (Survey 1, Jan 08).

Kate liked the problem-solving aspect of mathematics: in other words, the fact that one can figure out what is missing and what is needed in a problem and solve it as opposed to memorizing facts as in a history class. She recognized that there were different ways to solve a problem but believed that “there is always a definite answer” (Interview 1, Line 48). This belief was reflected in her view of mathematics as being “black and white” (Interview 1, Line 37). In Dr. Benson’s class, Kate realized that she solved problems algebraically but that there were other methods as well. She also looked at mathematics “as something that you can apply to real life situations” (Interview 1, Lines 294–295).

When I asked Kate how we know that something is true in mathematics, her initial response was “To know that it’s true, I guess that goes back to the proof” (Interview 1, Line 73).

However, she then started talking about using different formulas to solve a problem and knowing that the formulas are true because they solve the problem. When further questioned about how we know that something is always true, she said, “To me it’s, it would be very hard to see if something would apply to everything. How do you test every situation, you know? So, I guess there I just trust [laughs] the mathematicians before us that have given us these formulas” (Interview 1, Lines 95–100). As I explain further in the next section, this responsewas also in line with her view in the first interview that only mathematicians work with proofs.

In contrast, Kate knew that “in science nothing’s proven. You can only disprove and that’s how they get the conclusions that they get. But in math we have actual proofs. We have formulas that work in everything” (Interview 1, Lines 223–225). However, she further said that

> as soon as a formula doesn’t work, as soon as you disprove it, then I guess it’s time to go back and see why that doesn’t work. And if it, if it didn’t work in this one case, is it just a particular case where it won’t work but 99 percent of the time it will? Um, kind of with the parallel postulate that we are talking about in class the other day that we are just gonna assume, um, the parallel postulate. And that it’s true in that case, but there is other cases of geometry that I am not even aware of that it doesn’t work. So, is it just a certain, certain type of geometry or a certain type of problem that only {applies to} a specific situation that it’s not gonna work. And then you have a separate formula for that. (Interview 1, Lines 226–236)

Although Kate recognized that establishing truth in mathematics was different than in science and that it depended on assumptions, she was not sure whether it would be possible to invalidate something proven in a geometry where the parallel postulate is accepted to be true. She said, “I don’t know. I would think not but I really, I don’t know the answer to that question” (Interview 1, Lines 245–246). It could be argued that Kate knew that a proof needed to be universal in mathematics. However, she also learned that there were situations that she thought were “above my head” (Interview 2, Line 226) where certain assumptions would not hold true.

Hence these two pieces of knowledge might have led her to believe that if a proof is invalidated, one needs to analyze the situation and see how it is different from other situations.

Experience With and Attitude Toward Proof

Kate started this course with very little experience with proofs. She did not remember doing proofs in high school, and, according to her response to a survey question, the word *proof* "reminded her of calculus (Survey 2, Jan 08). In the first interview, she said, “I really have very little experience with proofs, so, to me the term *proof* is kind of this word that I am not very fond of cause I don’t know anything about it” (Interview 1, Lines 256–259). Hence, she believed that she would not be able to supply me with the information that I was looking for. She associated the proofs that she remembered from her calculus class with writing a lot and found proving to be “an overwhelming task” (Interview 1, Line 211). Furthermore, she believed that proof is “not something that everyday people deal with” (Interview 1, Lines 317–318) but “something that the mathematicians are working with” (Interview 1, Lines 314–315).

Although Kate considered proof as something that mathematicians work on, when she was asked to prove something she interpreted this as showing how she found her answer. She wrote the following in response to the same survey question mentioned above: “I think proving your answer in all mathematics classes is something I have been asked to do since middle school, just in a less formal manner.” In the first interview about 2 weeks after this survey, she clarified that when she was responding to the survey questions about proof she “was thinking more along the lines of showing your work” (Interview 1, Lines 154–155). Furthermore, with this interpretation, “the idea of proving your answer rather than producing the answer w/ nothing to back it up” was something that she liked (Survey 2, Jan 08).

As the semester progressed, instead of being this overwhelming task that is only for mathematicians to deal with, Kate started viewing proofs as something that she could do and feel successful about. In the second interview, when she talked about a recent proving experience that she had had, she stated that it was like “satisfaction” (Interview 2, Line 76) to see that it all came together. In the third interview, she explicitly stated this change in her attitude toward proofs:

> I used to think about proof as something that like I couldn’t do. That it’s like for mathematicians and like really, really smart people can do that, but I mean that’s not something that I need to know. Um, but now I know that I can do a proof, and it’s not as intimidating or scary as I thought it was going into it. (Interview 3, Lines 164–168)

She also wrote the following in her last survey: “I really enjoy constructing proofs” (Survey 2, April 08).

Kate also started associating proofs with properties that she had attributed to mathematics in the first interview. One of the reasons that she liked proofs, as she expressed in the second interview, was that “if you know the facts and you know the properties then you can apply it, um, it’s kind of like black and white” (Interview 2, Lines 164–166). She explained what she meant by black and white as “it either fits or it doesn’t. Like you can’t just assume something, um, and that’s what I like about math in general” (Interview 2, Lines 176–177). It seems that the views about mathematics that she brought into this class provided her a lens through which she viewed proofs.

In the second and third interviews, Kate indicated that everything that she knew about proofs she learned in this class. She believed that in this class they “got to thinking in terms of proofs” (Interview 3, Line 53). As a result of the way she experienced proofs in this class, Kate developed not only a positive attitude toward proof but also a willingness to be able to reason for herself and figure out or explain something. Hence, her perspective changed from “tell me the formula, tell me how to do it and I’ll be okay with it” to “I want to be able to prove something, um, and come up with a reason" (Interview 3, Lines 54–57). She even thought that "to not, like, understand how to do it, it's frustrating because I want to figure it out. So I like the idea of knowing the proofs so that you can, like, it makes so much more sense" (Interview 3, Lines 858–860).

Furthermore, as Kate started questioning the teacher. In response to a question about the place of proof in this class, she said:

> I think at the beginning it was, we were learning even what a proof was. But then toward the end, we're like, we want, we want you to show us the proof for it. Even if she would tell us something, we're like, "Oh, you need to prove it, you know. Why is that so? Or would that apply to all, you know, figures like that?" So, yeah I think the proof was a very big part of the class. (Interview 3, Lines 37–42)

Proof in the Abstract

Although Kate viewed herself as knowing nothing about proofs, she came into this class with a basic but fundamental understanding of proof – something on which she could build the newly learned ideas – that helped her to grasp the ideas talked about in class and construct a more developed, connected, and rich understanding of proof. Kate's initial concept of proof encompassed both the idea that proof establishes truth and the idea that a proof must be universal. As she said in her first interview, to prove something is "to say that it's true, to prove it is to have enough evidence that, that it applies in every situation" (Interview 1, Lines 218–219). She also acknowledged that evidence is necessary to show that something is always true. However, she did not specify what kind of evidence when she talked about proof in the abstract at this point in the semester.

In the second interview, when Kate talked about what is involved in proving a mathematical statement, she went beyond what she had said in the first interview. She not only mentioned establishing truth and universality but also recognized that proving a mathematical statement entails “being able to put variables in there” (Interview 2, Line 127). So, in comparison to the first interview, as opposed to providing “enough evidence” (Interview 1, Line 219) to prove something, Kate realized that a proof needs to employ a general argument.

In addition, Kate also indicated that proving involves using previously established knowledge. She said, “You can’t just assume anything. Well, like, ‘Oh, those look like they’re congruent.’ … You have to prove it, um, through the facts you know of, like, triangles and, like, side-angle-side. Um, just all the properties have to come together and support each other” (Interview 2, Lines 130–135). She also acknowledged that the validity of a proof depends on the truth of the statements used in the proof: “If everything you used to prove it is accurate, I don’t think you can go back and disprove it, because the properties are always the same” (Interview 2, Lines 180–182). In this interview, Kate also said she viewed proving as putting pieces together like “a puzzle” (Interview 2, Line 26 & 71). This comment again resembles the way she talked about mathematics and problem solving in the first interview and may provide another piece of evidence that her developing beliefs about proof were filtered through her beliefs about mathematics.

The following quote from the last interview demonstrates how Kate’s conception of proof evolved throughout the semester. At the end of the semester, her conception of proof encompassed establishing truth, universality, using previous knowledge, and explain-why aspects. For Kate, to prove something meant the following:

> To use what I know, to use facts without assuming anything, um, but just the hard facts that we know about something and applying it, applying those facts to that to come up with a reason why, um, we have a certain formula. … I was thinking of the example of … the area of the parallelogram, so just proving to somebody that it would be true in all cases, yeah. (Interview 3, Lines 111–117)

When questioned about what it is about a proof that makes us sure that it is true for all cases, Kate said, “I guess working only with integers or, sorry, variables. And like not saying, ‘Oh, for example we’re gonna plug in these numbers and see if it works.’ Like, using variables to see if it would work in every situation” (Interview 3, Lines 437–441). So, in this interview she again acknowledged that a proof needed to employ a general argument. This reference to variables could also be interpreted as looking for a specific form of argument. However, she seemed to understand why using variables would make an argument general. She contrasted using a variable with using specific numbers and indicated that a variable would cover all possible situations.

A characteristic of proof that Kate included in her definition in the last interview in addition to all other aspects was that a proof explains why. This characteristic was evident not only when she defined a proof but also when she talked about the goals of this course and what knowing proofs would afford for teachers. First of all, a part of her definition of a proof was “taking an everyday formula or just idea, the interior angles of a triangle add up to 180 degrees, and saying why we know that” (Interview 3, Lines 96–98). Second, she emphasized the conceptual understanding that came with knowing proofs both in her last interview and in the last survey. Referring to the figures and formulas that the students learned in the class, she said that they learned “finding out why they have the properties that they do and literally proving, you know, how we came up with all those formulas. Yeah, just to give us a more conceptual understanding” (Interview 3, Lines 16–18). One of the purposes of proof that she listed in her survey was “to gain a conceptual understanding of the mathematics” (Survey 2, April 08). Finally, she also indicated that if a student says, “I don’t understand why that’s true” the teacher can prove it to them, in other words she or he “can explain it to them” because she or he has “the reasons behind it” (Interview 3, Lines 104–105).

Proof in Context and Evaluations of Arguments

When Kate talked about proof in context and evaluated arguments, characteristics that were not apparent in her definition became visible. For instance, when she evaluated an argument in the first interview, she paid attention to whether it employed a general argument. Toward this end, she examined whether the argument used numbers, letters, or variables. With this criterion, she correctly rejected empirical arguments and accepted general arguments as proofs. For example, according to Kate, the transforming-the-lines argument about the sum of the interior angles of a triangle (Argument 3 in Appendix C) “works in all cases because you’re using variables. … However much you move it from the outside here, you’re gonna gain that on the inside” (Interview 1, Lines 587–593).

She also employed other criteria as she examined arguments. She paid attention to whether the figure that was being worked on represented a special case and whether the argument utilized a previously known fact: “We are not looking at, um, an equilateral triangle. We are not looking at all of these being 60 degrees, and … we are using the fact we know it’s 360 degrees” (Interview 1, Lines 439–443). Furthermore, she did not accept the tearing-corners argument as a proof, because there was “no other reasoning why it’s 180 degrees. … There is no way of going through and trying every single possible, every single triangle possible, um, that you could make” (Interview 1, Lines 513–517). Similarly, the empirical argument showing the number of diagonals of a polygon in the second interview was not a proof for her, because it did not “explain why,” and it also did not “explain the properties that they used to get there” (Interview 2, Lines 473–474).

As Kate's understanding of proofs developed, characteristics of proof that she used in evaluating arguments became a part of her definition of proving. For example, employing a general argument was not an aspect of proof that she mentioned when she talked about proof in the abstract in the first interview but was a criterion that she used to evaluate arguments in that same interview. As mentioned in the previous section, this aspect became a part of her definition of proving in the next two interviews. Similarly, although she paid attention to whether some arguments explained or did not explain why in the first two interviews, it was not until the third interview that she explicitly mentioned that a proof explains why.

Although Kate was successful at distinguishing an empirical argument from a general one throughout the semester, one argument in the second interview was difficult for her to decide as to whether it was a proof or not: the walking-around-the-triangle argument to show that its exterior angles added to 360 degrees. Although she tried to understand whether the argument would work for all triangles and whether there was any measurement involved in the argument, at the end she was "still iffy about that one" (Interview 2, Line312). I believe that it was hard for her to realize that the triangle in the argument was a particular triangle and hence the argument was empirical as opposed to an argument that explicitly said that it tested several different triangles or an algebraic empirical argument.

It is also important to note that Kate exhibited a disposition to understand the arguments presented to her in the third interview, but when she could not, she referred back to other features of proof that she knew to be important. This approach led to an overreliance on looking for numbers or variables when she was faced with an unfamiliar argument and could not make sense of it. After examining the empirical argument for the claim about the sum of the first *n* positive integers, she said, "I don't necessarily understand this statement, but just simply looking at the fact that there is only three different examples to explain it would make me say that it's not enough to have proven the statement" (Interview 3, Lines 409–412). Similarly, although she could not follow the induction argument and was frustrated by that, she was inclined to accept it as a proof "because it's using *k* as opposed to a number" (Interview 3, Line 469).

It could also be argued that if Kate did not understand the logic of a proof by induction but accepted it based on variables, she was relying on form, which is an evidence of an authoritarion proof scheme. When Kate was given an argument to analyze, she first tried to make sense of the argument. However, given an unfamiliar situation, if she could not understand the content of the argument, form dictated her decision as to whether the argument was a proof or not when she was explicitly asked to make a decision. It is important to point out that although Kate accepted the argument as true without understanding it, she did not prefer variables just because they were symbols; they still had some meaning of being general.

When Kate attempted to prove a statement about triangles in the last interview, she tried to use properties that she knew about triangles, which is consistent with what she came to believe proving involves: using the properties that we already know about figures to come up with new ideas. She also mentioned in this interview that she was looking for mathematical reasons to be involved in an argument different from a mere demonstration, as she said the following:

> I think if somebody would have said, um, kind of with these hands-on activities that we did, if someone would have shown that to me then I'd be, like, "Oh yeah that's a proof, you know, without any math behind it." Um, but now I wouldn't. (Interview 3, Lines 250–253)

In this last interview, Kate also used a criterion that she had not mentioned before: the fact that measurement might not be exact. She thought that the given argument "worked just because she, like, she measured it, and so you'd have to measure every triangle. And your measurements, I mean, might not be accurate. But regardless of whether they are or not, you'd have to measure every example. So that wouldn't work" (Interview 3, Lines 798–801).

In line with her tendency to solve problems algebraically, as she had mentioned in the first interview, Kate expressed in the last interview her preference for algebraic arguments over visual or geometric ones. In her explanation of why she would prefer a particular argument, she said that one was "more algebraic, and that's typically how my mind works about solving things" (Interview 3, Lines 514–515).

Importance of Proofs

One of the things that Kate thought she learned about proofs in this class was "the importance of proof" (Interview 3, Line 47). In the first interview, she thought "of proof as important in math when it comes to bringing all the information together, and … it's making sure it's true in all cases or it's only true in this case. Um, so then it can be applicable in other situations" (Interview 1, Lines 306–309). However, she added that she thought of a proof as being "something that the mathematicians are working with, and then they come up with this formula. And then that formula is what we can use kind of on an everyday basis when we apply math to our lives. So, I don't think a proof is something that everyday people deal with" (Interview 1, Lines 313–318). Hence, an important aspect of her beliefs about proof at the beginning of the semester was that proof is something that mathematicians work with, and she looked up to them for correct formulas.

Throughout the semester, she came to consider proofs as being important for herself both as a learner and a teacher. This was reflected in the second interview when she said that she learned in this class "not to assume anything but to try to solve it for yourself" (Interview 2, Lines 719–720). It can be argued that proofs gave her the tools necessary to establish truth, as she felt “more confident in an answer” because she knew “better how to go about a proof” (Interview 3, Lines 304–305). However, she still wanted “the teacher to say, ‘Yeah, that’s right.’ Or, ‘No it’s not’” (Interview 3, Lines 306–307). It is clear that this might be a difficult belief to change.

She also viewed proofs as important for herself as a teacher because she thought that it is important for a mathematics teacher to explain why (Interview 2). She expanded this belief and explained further in the third interview why it is important for a teacher to know proofs. First, as in the second interview, she thought that proofs were important when teaching “to be able to explain to the students” (Interview 3, Lines 62–63). Furthermore, she believed that being able to prove things helps her to “know where something is derived from” (Interview 3, Lines 83–84) and to develop a “deeper conceptual understanding” (Interview 3, Line 63), which, in turn, help her as a teacher to be able to evaluate different student solutions and decide whether they are valid or not. In this way she would be able to “encourage all different kinds of thinking” (Interview3, Line 71) among students.

Although Kate started viewing proofs as being important for herself as a teacher, her beliefs about the role of proof in a middle school classroom seemed to stay stable during the semester. When I asked her in the first interview about the place of proof in her future classroom, she said that she thought of proofs as being important for students because it is important for students to understand why things work and to know where things come from so that “they’ll better know how to apply” (Interview 1, Line 678). She also remembered her experience as a student and that she never knew why things worked but that she had learned about those in Dr. Benson’s class the previous semester. However, she added that she would not go with middle school students as far as Dr. Benson had gone with them.

Kate expressed similar beliefs at the end of the semester as she thought that it would be "too hard" (Interview 3, Line 329) for middle school students to come up with proofs by themselves. She believed that "it's good for the students to know, to be able to not necessarily do the proofs, um, but to know the ideas behind it so that they can, um, generate formulas on their own and not just memorize stuff" (Interview 3, Lines 319–322). Toward this end, she would have them "try things out on their own, manipulate things, talk in their groups" (Interview 3, Line 324) but she would not expect them "to be able to do like proving interior angles of a triangle are 180 degrees or proving the Pythagorean Theorem" (Interview 3, Lines 327–328). On the other hand, she also acknowledged that she had not been in a middle school classroom and that she did not know what students' abilities were.

Case 2: Tammy

Tammy was one of the undergraduate participants in this study. She had mathematics as her first specialization and history as second. She was categorized in the authoritarian proof scheme group based on the initial survey results.

Views About Mathematics

Tammy liked mathematics because she was "obsessed with numbers" (Interview 1, Line 36) and there was a "definite answer" (Interview 1, Line 49). However, she made a distinction between mathematics at the lower levels and higher levels: "I guess with the higher up it gets more subjective, but like in the lower grades it's more toward like one answer and working with numbers and having multiple ways to get to a solution to a problem. And I just think that's really interesting" (Interview 1, Lines 59–62). She thought that "math is something that like most people struggle with" (Interview 1, Line 50), and she would like to help them understand. She also mentioned in the second interview that mathematics is universal (Interview 2, Line 233). By this, she seemed to mean that everybody in the world uses the same rules and formulas.

When I asked Tammy how we know that something is true in mathematics, she said, “Proofs and theorems” (Interview 1, Line 69). However, she added, “I don’t know why it’s that way. I know that it is because it is, and that’s what the math people say” (Interview 1, Lines 96–97). When asked if proofs ever become invalid, Tammy said,

> They do. Now that we talked about that postulate thing, I guess maybe it’s technology or time goes on. Maybe they find different things and make them question whether a proof is right or wrong, and then they have to go back to like I guess like the drawing board, and come up with the idea of, “Yes it’s right, no it’s not.” Go back to the whole multiple, trying multiple different dimensions and ideas and things, to like prove whether it was correct or incorrect. But like the parallel postulate we always, we did not know that there’s another geometry out there. … I guess over time things can be proven incorrect. (Interview 1, Lines 307–317)

It can be argued that Tammy viewed the parallel postulate as something that can be proven rather than an assumption and interpreted that it did not hold true in a different geometry as evidence that proofs can be invalidated.

Experience With and Attitude Toward Proof

Tammy’s experience with proofs prior to this class was limited to two-column proofs that she learned about in her 10th grade geometry class. Although she had taken several college level mathematics courses, she did not remember doing proofs in those courses. She also briefly touched on the fact that she learned about proofs in geometry and not in algebra.

Tammy’s experience with proofs in her high school geometry class was an experience of memorizing as opposed to understanding and sense making, which left her with an image of proof as something that needs to be written in a two-column format. She was actually good at proofs in her high school geometry class but only because “with the two-column proofs I've had note-cards like in high school, and I memorized index cards. And I did really well in geometry because I memorized those” (Interview 2, Lines 203–206). She did not think that she “understood them better than anyone else did” (Interview 1, Lines 192–193). Hence, what Tammy remembered about proofs when she started this course were aspects related to form rather than content: in other words, how a proof is written. “I remember you make a table. And use theorems and stuff to prove them,” she wrote on her survey at the beginning of the semester (Survey 2, Jan 08). She further said in the first interview that

> they were like a rectangle, triangle thing and … all this is lined up on the lef-like on the left side. And on the right side, you had the theorem or the like the proof behind it. And then I remember some of them had three or four reasons and some of them {were} like ten. And we did this for homework like every night for months, but that’s, that’s really all I remember about them. (Interview 1, Lines 183–190)

Hence Tammy thought that for something to be a proof it had to be in two-column format: “Since that was the only way we were taught in high school, that's the only thing we came knowing. Like, we thought everything had to be in two-column, like, we didn't know you could prove things without having a two-column proof” (Interview 2, Lines 162–166). Furthermore, although she was good at the two-column proofs she had studied in high school, she found them confusing because “there were so many different things. There were like the reasons behind it, and what the theorems were called, and all that” (Interview 2, Lines 140–142).

In all three interviews and in the surveys at the beginning and end of the semester, Tammy mentioned some aspect of two-column proofs, which indicated that form was an important part of her conception of proof. For that reason, she was paying attention to the formal aspects of the proofs that the students were learning in Dr. Benson’s class. Given her previous experience, Tammy thought that in Dr. Benson’s class, they “were gonna have to pull out our books and go through everything and find, like, do two-column proofs again” (Interview 2, Lines 168–169). She said in the last interview that if she had been told the previous semester that they were going to learn about geometric proofs, then she would have changed to “history first” (Interview 3, Line 143).

However, “it wasn't at all” (Interview 2, Lines 169–170) like that. In other words, they did not work on writing two-column proofs. In this course, students were not expected to write proofs or provide explanations using one specific form. The way that Tammy experienced proofs in the course helped to change her perception of how a proof could be written. She learned “that you can prove things without showing a two-column proof and all that stuff” (Interview 2, Lines 146–147). Tammy described the proofs that they worked on in this class as “paragraph proofs” (Interview 2, Line 194). She indicated:

> A lot of our stuff and our homeworks is, like, “Why is this like that?” And, like, that could be reasoning and, like, our proof behind it is like writing it out. Like, our proofs are paragraphs; they're not two-column proofs. So I think that kinda changed my perspective on it. ‘Cause I thought proofs were only, like, those were the only kind we ever learned about. (Interview 2, Lines 174–179)

She reiterated this change both in the last interview and end-of-the-semester survey. In the last interview, she mentioned it as one of the things that she learned in this class and said, “I wish I knew, like, proofs could have been just huge paragraphs explaining, you know, what I'm saying. ‘Cause it was a lot easier than remembering all the different postulates and stuff like that” (Interview 3, Lines 154–157). In response to an end-of-the-semester survey question asking about her experience with proof, she wrote “2-column proofs in hs geom” and “paragraph proofs in this class” (Survey 2, April 08).

Tammy not only learned that a proof could be written in a paragraph form, but she also liked this way of writing better than two-column format. Whereas she had always memorized two-column proofs, she thought that she actually could make sense of paragraph proofs. Rather than trying to remember the names of the theorems and postulates and using them in a proof, in this class she could write proofs using the properties of the figures that were relevant to the proof.

To explain what she meant by a paragraph proof, she said:

> We have to back up everything and draw pictures and label it. That way when we're like describing it, we can refer [to] specific angles or lengths or whatever. But we don't have, like, the two-column part of it, I guess we don't have, we don't use theorems. And like we used the parallel postulate when we were doing like angles and stuff, but we never, I don't remember using any theorems to prove anything. Like, we literally did it on the basis of the triangle, I guess, you know, like it's an isosceles because blah blah blah. (Interview 2, Lines 194–201)

She thought that the paragraph proofs were easier because as opposed to memorizing two-column proofs,

> if you're proving it in a paragraph you obviously know what you're talking about. But it's not necessarily like specific theorems that you're learning and stuff like that. Like, you're making sense out of what you already know about triangles and stuff. So I think the way we do it now is easier than the two-column stuff. (Interview 2, Lines 215–219)

She wrote on her end-of-the-semester survey that she liked proofs more because of that. As Tammy was not acknowledging using theorems in proofs, there is a chance that Tammy left the class thinking that she rarely used theorems or that she did not recognize properties as implications of theorems.

Although it was important for Tammy to learn that proofs could be written in a paragraph form, I think it is important to note that at the end of the semester she valued the content of a proof more than its form. She said, “A proof is a proof whether it's written out in a two-column proof or written out in a paragraph. But I guess just the idea behind it was that, you know, they’re both there to prove why something worked, or why something was like that” (Interview 3, Lines 244–247).

Given the kind of experience Tammy had with proofs, it was not unexpected to hear that she knew that algorithms and formulas worked but not necessarily why they were that way. In the first interview, referring to a survey question, Tammy said,

> That homework that we did with the whole triangles really threw me off. 'Cause it was like when it asked what the degrees inside the triangle was, I knew it was 180, but I couldn't tell why. … I guess like they almost take that part out of schools for children to learn. Like they just give them the information, like, "Here is the algorithm. This is how you do it. This is how you get the answer." (Interview 1, Lines 70–77)

When she talked about the place of proof in this class at the end of the semester, she said, "I would say more than half of the class was like proving the mathematics that we had already learned that we didn't really have experience of why it was like that" (Interview 3, Lines 63–66). She illustrated this with some examples from class:

> Just like all the formulas for, like, the area of a parallelogram, why it works and the area of the triangle, and why you can do, why you can break it up into all the different things. And it wasn't necessarily like a formal written proof. But we had to prove, like, why it works and why, how we can explain it to students. (Interview 3, Lines 54–59)

It is likely that this approach to proofs made it easier for Tammy to make sense of them. She described in the last interview how she thought about proofs in comparison to the beginning of the semester: "I used to think proofs were impossible [laughs], but now I think they are more like easier to grasp, I guess. But, I used to think proofs were difficult, but now I understand them" (Interview 3, Lines 271–273).

Proof in the Abstract

One of the connotations that to prove something had for Tammy was to show how you got your answer. Referring to one of their teachers, she said, "When [the students] come up with an answer, she says, "Prove it." And, like, [makes] them explain how they got there. So I guess kind of proof would be, I guess that in a way is a proof. It's just like not as advanced" (Interview

1, Lines 204–206). She believed that "when kids are growing up, they make it more advanced, the more you get through math, 'cause you understand it more" (Interview 1, Lines 207–208).

Tammy's conception of what constituted proof when she talked about it in the abstract stayed relatively constant throughout the semester. She viewed proving as establishing truth with certainty. She said, "When I think of proof, it's just like knowing that it's right without the shadow of the doubt" (Interview 1, Lines 209–210). However, she explained:

> It's that because it's been proven. They've done studies on it, they proved that it works, and they tried a whole bunch of situations. … They just didn't do it one time and say, "Oh, it works." And then never did it again. Like, almost like repetition, I guess, different cases, different numbers, different types of triangles, or something. (Interview 1, Lines 210–215)

In other words, for Tammy something was proven by trying different examples. Depending on the situation, these examples could be numbers or triangles. The important point seemed to be doing something more than once and getting the same answer every time. So, although Tammy did not consider one example as constituting a proof, she did not talk about a proof needing to employ a general argument. Her conception of proving was empirical. She believed that something would be proven: in other words, correctness and certainty would be ensured if different examples were tried and resulted in the same answer. Confirming this thought, later in the same interview she said that to prove something meant

> to do it multiple times with many different, you are obviously trying to get the same outcome to prove that that's the way that it is. So trying multiple numbers and dimensions and equations, whatever it takes to get back to that same like core answer each time, like, 'cause if you keep getting different answers then it's not proven. It's, you know, it's just another problem, I guess. (Interview 1, Lines 256–261)

In the second interview, Tammy repeated that to prove something meant "looking at the example or whatever and like proving that it's right in that circumstance. Like in that instance and then proving it for different ones" (Interview 3, Lines 222–224). She said, "If you can prove basically it's always gonna be true, then you proved it to [be] correct” (Interview 2, Lines 81–82). Although she still thought that proving would require trying different examples, there is an indication in this statement of an understanding that a proof applies to all situations: in other words, the universality aspect of proof. She also emphasized again the need for trying different examples:

> ‘Cause just because it's right for one doesn't mean it works for everything. So I think different examples that it works for as well. And then once you do all those and you see that they're right or they're consistent, then I think it's proving it. (Interview 2, Lines 224–227)

In this interview, it seemed as though Tammy was using the words *examples* and *cases* interchangeably and that having done proofs by cases reinforced her view that an argument with examples was a proof. She said that proving a mathematical statement involved “showing that there's multiple cases where it's true” (Interview 2, Lines 81). When asked how she would approach a proof, she said:

> I do it several times with different cases. Because even in math {with} triangles you have right triangles and non-right triangles. And we had to tear from worksheets in Dr. Benson's class where we had to prove it for a non-right triangles and for right triangles … I guess I just mentally, like, I have to see multiple cases for it to be like, okay, I believe it. (Interview 1, Lines 89–97)

The universality aspect of proof that Tammy mentioned in the second interview also came up in the third interview when she talked about the role of proof in ensuring that people would not make up things. Proofs guarantee that people do not make up claims:

> If you didn't have the proof and reasoning behind it, like, you can make up something and it can maybe work for one or two cases, but it wouldn't be true for everything. And it's obviously not a proof if it can't like work. You know what I'm saying, like, it can't just be like the one or two cases out there {but/that} it's always true. And then the other ones where it's never true. (Interview 3, Lines 412–418)

However, when she was asked how it could be made sure that something is always true, she said by “sincerely smart mathematicians” working on a new idea with multiple different examples.

Another slight shift occurred when Tammy tried to define a proof in the third interview. She talked about not only examples but also explanations that go along with examples as constituting a proof. She said,

> I guess like examples and explanations on why it works. Yeah, I don't know. I'm a big ex- I think I've said this and all of them. But I'm a big example person. I'm not always convinced by one explanation, but like the explanations behind the examples leads to the proof and why it works. (Interview 3, Lines 303–307)

She stated that "a picture and then an explanation of why it works will be a proof" (Interview 3, Lines 317–318).

Proof in Context and Evaluations of Arguments

The characteristics that Tammy attributed to proving as described above led her in the first interview not only to accept empirical arguments as proofs but also to repeat a general argument multiple times before accepting it as a proof.

Despite the fact that Tammy accepted empirical arguments as proofs, she had reservations about the tearing-off-the-corners argument for showing that the interior angles of a triangle sum to 180° (Argument 1 in Appendix C). She thought:

> It's very interesting. Kinda blew my mind away, but I don't think I'd be like, "Oh, I'm completely convinced that that's why it's that way." I would, I guess I would want more proof anyway, like, I would want to get out the protractor and measure a lot of angles and see it's the same every time. (Interview 1, Lines 458–462)

However, she would not necessarily measure randomly drawn triangles; she would choose triangles from a book to measure. Hence when presented with the measuring argument, she accepted it as a proof and explained, "The measuring, I guess, to me is more exact than the tearing off the angles. And I like numbers, so I wanna see it written down and see them all added together that they equal 180" (Interview 1, Lines 479–482).

When she examined the general argument using the parallel postulate, her explanations indicated that she followed the reasoning of the argument. Yet in response to the question of whether it was a proof or not, she said “If you did it multiple times and it came out this way every time, which it would, uh, yes, I would say it is” (Interview 1, Lines 637–638). When asked what she meant by “multiple times,” she said, “A different angle type. Maybe draw it sideways or upside down” (Interview 1, Lines 641–642). Although Tammy did not recognize this argument as being general, it is important to note that she identified it as being “logical” (Interview 1, Line 657).

Tammy also made a distinction between the number of examples she would do it for herself and the number of examples she would do it for her students, noting that students would need to see more examples than she would. She could have different views of a proof for different audiences. In the second interview, for example, she indicated that a mathematician would not need more than one example.

In the second interview, Tammy worked on arguments showing that the exterior angles of a triangle add to 360 degrees. Although her evaluations of the empirical arguments were very similar to those in the first interview, she seemed to have developed an understanding of the general argument as showing the truth of a statement for all cases.

Tammy accepted the measuring argument (Argument 1 in Appendix D) as a proof because “he drew several different triangles and because it kept adding up to 360, he knew it was right” (Interview 2, Lines 300–302). She initially also accepted the walking-and-turning argument (Argument 3 in Appendix D) as a proof. Her analysis of the argument suggested that she was seeing generality in it. She said:

> It doesn't matter what shape you put on the ground or like what shape you're actually turning, it's still gonna be, you'll still end up in the same direction as it was if you went

> around. You just may not have to turn as much on one angle as the other or whatnot, but it will still be the same. (Interview 2, Lines 370–374)

(Yet she still would do it with another one for the skeptics.) She eventually said she did not think that “it's like as concrete as the other ideas are. ‘Cause it just says, like, the same number [of] degrees as the exterior angles, which it's, but like they didn't measure or anything” (Interview 2, Lines 394–396) and that she “probably still would have been skeptical and try to add up angles every time” (Interview 2, Lines 402–403). This comparison actually resembles the comparison that she made between the tearing corners argument and measuring argument in the first interview.

After Tammy examined the general argument (Argument 4 in Appendix D), she said that it proved it for all triangles

> because with every triangle it doesn't matter what it is, like, you still have 3 linear pairs every single time. And we know that the interior angles are always the same. So regardless of what your different angles are, it's still gonna be 180 on the inside and 540 for all the linear pairs. So when you subtract {them all}, it is still gonna always be the same. (Interview 2, Lines 430–435)

So, in this second interview, as indicated in the previous section, Tammy did not mention employing a general argument but rather talked about doing something multiple times when she talked about proof in the abstract. However, as the above quote illustrates, she was able to explain why an argument was general when she talked about proof in context.

Tammy was convinced the most by the general argument because it used “actual numbers” and “prior knowledge” (Interview 2, Line 459), and when she compared it with the measuring argument, she said,

> This student measured the angles so obviously it's gonna be right because he did measure it but that is like specific to each individual triangle, this is more like generic and it will work on all triangles because all triangles add up to 180 on inside so they have to be 360 on the outside so even though it's not technically measuring it I think this is your more valid case to prove that every single triangle work like this. (Interview 2, Lines 467–473)

However, it is also important to mention that although Tammy found this argument to be more convincing and more generic than the other arguments, she did not think that there was a problem with the other arguments. It seemed as though one factor influencing this acceptance was the fact that the other arguments also had the same outcome, or in Tammy's words, "They're all the right answer" (Interview 2, Line 477). Nonetheless, she realized a limitation of the measurement argument after comparing it with this more general argument:

> He measured each individual triangle for each one and even though he did a right and an obtuse and acute, the angles for a right, an obtuse and acute can be like infinite amount of angles, you know, depending on the degree. (Interview 2, Lines 478–481)

I conjecture that comparing arguments with different validities might have helped Tammy to start realizing the differences between an empirical argument and a general argument. Yet she still believed that all of them except the walking-and-turning argument would receive good grades from the teacher as they all showed that the student understood.

In the third interview, after examining the claim about the sum of the first *n* positive integers Tammy said, "I'd probably wanna see like some examples or like proof why it works. Like, how did you do this? Does this work in every case? And how did you like come up with that?" (Interview 3, Lines 475–477). Questioning whether it works in every case or not might indicate that one of her expectations for a proof was to show that the statement works for all cases: in other words, the proof needs to be universal. It also seems like she expected the proof to explain why the statement works. However, the fact that she wanted to see examples might mean that she still thought that examples constitute a proof. "If I said I proved it, then it would be several examples and justifying why the statement is true" (Interview 3, Lines 498–499).

After an initial examination of the argument and talking about what she would expect to see, Tammy was presented with the empirical argument claiming that the statement is true because it worked for 2, 3, and 7 (Argument 1 in Appendix E). Tammy accepted this argument as a proof, “I think since there are several different ones that it's a proof” (Interview 3, Lines 532–533). When asked if it proved that the statement worked for all possible cases, she first said that she “would probably do a couple more” (Interview 3, Lines 535). She also said she would be more skeptical if the only examples were 2, 3, and 4, but the fact that the examples did not follow a sequential order seemed to convince her that the statement was true. When asked if there would be a case where the statement did not work, she said she did not think so. But she said that it would be more convincing if a class of students tried 30 examples instead of the 3 in the argument. To be even more certain that the argument worked for all cases, she “would go into the double digits, and probably the triple digits, and like bigger numbers” (Interview 3, Lines 559–560). So, it can be argued that the number and variety of examples tried increased her level of conviction. However, one could also argue that she thought that there was a problem with an empirical argument or realized the limitations of an empirical argument. She recognized that the argument given showed “that it works, but it doesn't explain why it works” (Interview 3, Lines 573–574).

Tammy’s first reaction when asked whether the induction argument was a proof was as follows: “I don't know [laughs]. I don't like it. It's too complicated” (Interview 3, Lines 598). However, after examining it and after being asked to decide whether it was a proof or not, she said:

> They're doing where like they don't pick an exact number for the *n*; they're doing it just like for any number *k*. … I would say yes, that it's a proof cause they're saying like that way you don't have to do fifty examples to prove it, they're doing just like an overall example, I guess, if that makes sense. (Interview 3, Lines 632–638)

She seemed to recognize that using a variable could assure the truth of the statement for all possible cases. It is important that she recognized that if you use a variable, it could be a general argument. She recognized that there is a difference between using variables and specific examples. This was particularly important as Tammy had been depending on examples to prove a statement or had been accepting empirical arguments as proofs. Although it is not possible to predict whether she would accept any argument that used a variable, even if it did not logically follow, it is important to recognize this instance as an evidence of a possible start of an understanding of generality of an argument. Although it could also be argued that Tammy was focusing on the ritual aspects of the argument, I believe that there is a difference between recognizing that using a variable will get you out of using a bunch of examples and depending on form.

Importance of Proofs

In the last interview, Tammy said that one of the things that she had learned about proof in this course was "just how important it is" (Interview 3, Line 141). In the first interview, in line with how she thought about proofs, she talked about the importance of proof from the perspective of viewing proving as showing how you got your answer:

> I like the idea of saying like why, like prove it, even for simple things like how kids got an answer cause that way if it's incorrect, if you tell them to prove it to you, you could at least see their thinking behind it instead of coming up and being like, no that's wrong, try again, you're not discouraging them but you also get to see why they're thinking that and what their thought processes are and you can kind of correct them from there I guess, and like, or after they explain it, you could help them get to the right answer. (Interview 1, Lines 578–585)

She also talked about the role of proof in establishing common knowledge and building on that:

> The role of them is to help all of us kinda come to the basic level, if everyone is studying the same subject and no one agrees about anything, you're not gonna get anywhere, … proof behind it is, like, this is what's been established, this is what we've shown as being correct and from this set of information, we move into other problems and other stuff like that. (Interview 1, Lines 280–286)

She said, “Just without proofs, I don’t think we would be able to have math, period” (Interview 1, Lines 290–291). She said that we would have to start from scratch every time we wanted to work with an idea.

In all three interviews, Tammy talked about proofs being important because they ensure that people do not make up things. This claim is consistent with the way that she thought about proofs as establishing truth. In the first interview, she said,

> We have proofs to prove that what we are saying is correct because anyone could walk in and say 2 plus 3 is 4 and if that’s all the kids learn they’re gonna go okay, that’s right when that’s not the correct answer. (Interview 1, Lines 263–265)

In the second interview, she said that proofs are important because students will not be “making up formulas” (Interview 2, Line 229). She also mentioned that “math is universal” and “if Georgia made up their own rules and areas and formula and stuff like that, like, our math would be completely different than everyone else's in the world” (Interview 2, Lines 234–236). So proving would be important to provide “uniformity” and “so that everyone's kinda like at a base level so everyone understands” (Interview 2, Lines 242–243). In the last interview, she reiterated this point by saying that “[proof] definitely has like a role because without it people would just be making stuff up” (Interview 2, Lines 242–243).

In the first interview Tammy did not think that middle school students could come up with proofs by themselves. Although she recognized her lack of experience with middle school students, she still did not think that she would expect proofs from regular students:

> I haven’t worked with enough middle school students and haven’t done math yet but I don’t think they’re that advanced. Like if you had like a gifted math class maybe. But as for like a regular, just standard middle school student, I don’t think that, unless they really liked math. (Interview 1, Lines 609–613)

In line with the way that she thought about teaching, Tammy thought that students would need “to be guided to get the right answer” (Interview 1, Line 615). For instance, toward this end, she would give students a worksheet and ask them to measure angles of different triangles and guide them to the result of 180 degrees (and she thought that this was a proof). She believed that the purpose for having proofs in a mathematics class would be the following:

> Just deeper understanding. For them to understand why they're using an equation or why they're solving the problem the way they are that way. I think it's more interesting that way. You're not just making up numbers or doing whatever. You get to see it more and understand the why behind it. (Interview 1, Lines 618–622)

Although this is a promising goal to have for students, it is not clear how she would achieve this by using the empirical arguments that she believed to be proofs.

When Tammy talked about the importance of proofs in the last interview, she focused on conceptual understanding, knowing where things come from, and making students more interested in mathematics. She said that we prove things "just to make sure you have more of like a conceptual understanding" (Interview 3, Line 73). She added, "The proving part is to show you how we came up with the formula" (Interview 3, Lines 78–79). She also believed that if students know more about how things work in mathematics, it would make them more interested in the subject:

> I think if the kids had the background of like seeing how things are proven and why it was just not an old man created the formula and we still use it today, but like seeing how it works, I think that would really get them interested and like help them [in the classroom] more. (Interview 3, Lines 133–137)

Tammy also talked about the importance of knowing proofs for herself as a teacher. She believed that as "the curriculum now is moving people more toward like discovering it and seeing why it works" (Interview 3, Lines 97–98) the teachers have to know why.

> It's more of like the investigations, they don't necessarily kinda tell them the area formula, they have to kind of discover it and why it works and I think with that type of teaching we have to know why it works because if the students are discovering it and we don't know why it works, all we do is like, good job, here is the formula, it's not gonna really connect everything for them. (Interview 3, Lines 84–90)

She believed that “if we can't even prove it, like, understand it ourselves, there is no way we can explain it to thirty kids and five classes” (Interview 3, Lines 98–100). As in the second interview, she implied that rather than giving students a formula and asking them to explain why it works and “figure that out right away,” she would guide them to an understanding of where it comes from (Interview 3, Line 358).

Case 3: Casey

Casey, the graduate student participant, was in the career changer program. She had a BS in applied psychology. She was in the middle school master’s program and was going to be certified for the first time. She held deductive proof scheme based on the initial survey results.

Views About Mathematics

Casey defined mathematics as being “a very quantitative way of describing patterns in the world” (Interview 1, Lines 46–47). She believed that “it is basically like a form of reasoning and logic; it is a way of thinking through things that’s quantitative” (Interview 1, Lines 50–51). In comparison to science, she thought that “math is different because it, uh, it’s a little more basic. It explores just patterns and kind of states what patterns exist. And you can manipulate rules and theorems and postulates to find other patterns” (Interview 1, Lines 54–56).

When asked how we know that something is true in mathematics, Casey said, “That requires proof” (Interview 1, Line 58). She also acknowledged the importance of assumptions that we make and explained that “there is Euclidean geometry and hyperbolic geometry, and so that is an assumption that you state upfront. That in these conditions, this is true, and in these conditions, it is not true. But something else is true” (Interview 1, Lines 78–81). With respect to establishing truth, she believed that “math seems to be more certain” (Interview 1, Lines 73–74) than science. She indicated that “scientists would say that they don’t deal with truth. … They would say that whatever they find is always open to correction” (Interview 1, Lines 70–73). She also mentioned that scientists are investigating the world, and everything is open to discussion.

Casey expressed the same beliefs in the last interview when she said:

> I'm very much more skeptical of science than I'm of math. I'm much more ready to trust when Dr. Benson says, this formula is true, you know, (inaudible) shows you that it's true, but you just have to trust me for a formal proof. I'm less likely to trust a scientist saying because I can't see all of the thought that went into it, um, I can see with math, if I really wanted to know the proof, I could go, look it up and you know, it might take ten pages you know to figure it out or possibly even longer but I could actually go investigate it myself and see that that's true. (Interview 3, Lines 96–104)

Although she wanted to see the proof for a given statement, Casey also expressed trust of mathematicians (more than scientists): “I know there is a system of checks and balances in academia where, I mean, there is so many people studying this that if it's not true, probably somebody's already, you know, caught it” (Interview 3, Lines 48–50). She also believed that mathematics “is abstract, but you can record accurately what you're doing” whereas “in science experiments, there is this big human factor that's involved, And it just brings this uncertainty and possibly bias into it” (Interview 3, Lines 122–125).

Experience With and Attitude Toward Proof

Casey started this course with a well-developed understanding of proofs, as explained in the next section, and a positive disposition toward proving. Although she had hated mathematics in high school (that was actually why she wanted to be a mathematics teacher) and did not remember whether they did proofs then or not, she remembered that she had to prove things during her undergraduate studies. Still that was a long time ago, so she indicated that it had taken some effort for her to get her “brain working” on proofs in this course (Interview 1, Line 152).

Casey believed that in general, “The idea of a proof is that it lets *you* investigate yourself and see that something is true” (Interview 1, Lines 226–228). Mathematical proofs ensure this truth because they "are very standardized, um, not standard in the sense that there can only be one proof for something, but standard in the sense that it uses these logical procedures and statements and symbols to represent things" (Interview 1, Lines 230–233).

Rather than taking the word of an authority, Casey used proof – the tool of the discipline – to convince herself of the truth of a statement. A proof for Casey was something that convinced her about the truth of a statement, which actually might be counterintuitive. The following experience that Casey described both illustrates her comment and also demonstrates that she had actually adopted this way of thinking in practice. When Dr. Benson told Casey that .9999… is equal to 1, she did not believe her at first. Then Casey looked up the proof and found two different proofs. These proofs convinced her that the statement was true. She explained it as follows:

> I like math because you can look at a proof, and even though I was completely convinced that those two numbers couldn't be equal, and it still seems counterintuitive, um, I can look at the proof and agree that those were logical steps. And I agree there is nothing fishy going on here. And I don't have to trust the person who did it, or the methods that I didn't see that were used. You know, what kind of instruments they used when they were writing the proof. I can see, and I can repeat it myself and test myself that these things are true. (Interview 1, Lines 192–200)

According to Casey, one of the big ideas of this course was "learning what it takes to actually prove something" (Interview 3, Line 13). When I asked her in the second interview what she had learned about proofs in this course, she indicated that given her previous experience with proofs and "training in logical arguments," she "had a pretty solid base on the understanding of what a proof is" (Interview 2, Lines 215–216). But on the other hand, she also said, "I don't think I knew any of those proofs that we've done in her class before her class" (Interview 2, Lines 214–215):

> I don't know if I've learned anything new about the concept of proof, but I've learned new proofs. And I've also, um, I didn't really think about using geometric proofs like the,

> the whole square with the, you know, the inscribed square on the inside for the Pythagorean Theorem. I wouldn't have thought to use geometric representations as part of a proof. (Interview 2, Lines 219–224)

Casey repeated the same belief at the end of the semester: "I don't know if I learned more about the concept of proof, um, but I definitely learned some proofs that I didn't know before" (Interview 3, Lines 133–134). She added that she learned "specific strategies for proving" (Interview 3, Line 139). Some of the strategies that she thought she learned were "moving and additivity [sic] principle," "using geometric figures," "visualizing," and "seeing relationships between figures." She thought that

> relationships are really big in proving things, um, for example understanding that a straight line is 180 degrees and the triangle's interior angles are 180 degrees. So if you can show that those angles are also on the line then you've proven it, um, so I think relationships between shapes. (Interview 3, Lines 330–334)

One thing that she knew about proofs but became clearer for her during this class was the difference between a proof and a demonstration:

> I think I understood this before, but it just helped bring it to the front of my mind that you, um, need to be explicit about when you're actually proving something and when you're just demonstrating it. Um, for example, the triangle thing again, um, the proof is different than just showing that it's true for one triangle. Although it's nice to see that it's true with physical manipulatives, that's not the actual proof. (Interview 3, Lines 143–149)

Although Casey knew several things about a proof at the beginning of the semester, she still learned in this class what could be involved in writing a proof. First, she learned in a previous class with Dr. Benson that "when you're trying to solve a problem, drawing a picture is not cheating" (Interview 3, Lines 193–194). She further explained,

> I think we've been made to think it's like counting on your fingers, you know. It's a crutch, but really, um, it can be a really powerful tool because you can see similar shapes and visualizing something can really help you understand it better. So especially if you're a visual learner like I am. And so I think her proofs have made more use of geometric reasoning and, and visualization than I've had in the past. (Interview 3, Lines 190–200)

When Casey further compared her prior experiences with proof to her experiences in this class, she indicated that she did not “remember doing geometric proofs really before” and that she has “always looked at stuff algebraically” (Interview 3, Lines 186–187). She further said:

> I think in the past I've been able to just prove it with a line of, you know, maybe ten lines of algebra. And in her class you need to, you know, show the ten lines of algebra but explain why you did each step and maybe draw a picture to help connect it to the problem so I think you do more explaining, um, in her class. (Interview 3, Lines 203–208)

Although Casey did not necessarily define a proof using its explanatory role, she recognized that the proofs that the class did in Dr. Benson’s class involved explaining. I think she did not see this involvement as an essential component of a proof but an element that could be involved:

> I used to think about proof as strictly an algebraic, um, exercise, and now I think of it more as a, as a combination of, like, the algebra is just a way of communicating. So it's a, it's a combination of, um, different ways of communicating, showing the logical steps. And the algebra still comes into play but it's okay to use the diagrams and to write out, you, know, [a] paragraph explaining why it makes sense. (Interview 3, Lines 215–221)

Proof in the Abstract

Casey started the course with a rich conception of proof encompassing several aspects. Some words that she associated with proof were “without doubt” (Interview 1, Line 114), “certainty, evidence” (Interview 1, Line 116) and “convincing” (Interview 1, Line 118). She believed that a proof would establish truth for all possible cases; in other words, it would be universal. She also recognized that a proof would need to employ a general argument; in other words, it would use variables, for instance, instead of a lot of examples. In addition, she was also aware that there are assumptions on which a proof is based. According to Casey, a proof

> would cover all possible scenarios within a certain set of restrictions - meaning, well, everything is based on assumptions. So you start by stating your assumptions and the conditions, and then, um, if you were to prove something, you couldn’t, for example, just show a lot of examples. You’d have to show something probably with variables to show

> that it would be true no matter what the quantities are, that the pattern is always the same. (Interview 1, Lines 58–64)

Furthermore, an important characteristic of a proof for Casey was being logical. She said that a proof "is a set of statements, um, that step by step, logically shows that something is true" (Interview 1, Lines 129–131). She thought that proofs were like puzzles: "It's more, it's more like a puzzle trying to think of what step would be logically next to show that a relationship's going to be true in the end" (Interview 1, Lines 154–155).

In addition to all the aspects of proof that she mentioned in the first interview, Casey mentioned in the second interview the use of prior knowledge and previously established facts in proving a mathematical statement as she said that you would "use different facts that you already knew to be true or you assume to be true to come to some other conclusion" (Interview 2, Lines 104–106). Perhaps as she worked on proofs in this class, this aspect either became a new element of her conception of proof or came to the forefront.

She mentioned this aspect again when in the last interview she said that to prove something "means to show using, to state your assumptions and to show using logical steps um that for any scenario in the defined parameters what you said is actually true based on previously proven things and assumptions" (Interview 3, Lines 381–384). This was the first time that she also implied having a hypothesis.

It was interesting to note that explaining why did not become a part of Casey's definition of a proof. It could be because she viewed an explanation as something "more loose" than a proof (Interview 1, Line 247). When I asked her if by proving we do anything other than showing that a statement is true, she said that only things that she could think of was "connecting ideas together" and "using logical arguments" (Interview 2, Lines 230–231).

Proof in Context and Evaluations of Arguments

As mentioned before, Casey demonstrated confidence when asked to prove a statement or validate a proof. In the first interview, she thought that she could figure out a proof using the parallel postulate that would show that the interior angles of a triangle add up to 180 degrees. As she started, she acknowledged that "it might take me a minute to figure out" (Interview 1, Lines 288–289) and that she "might go on the wrong track for a little bit" at the beginning (Interview 1, Line 292), and actually she did figure it out. She did not expect to prove a statement in just a few minutes, and even if it took some time, that did not necessarily mean that she would not be able to prove the statement. In one of the interviews, when she realized that her initial path would not lead her to a proof, she tried three different strategies starting from scratch every time.

An important characteristic of Casey's conception of proof was the distinction that she made between conceptualizing a proof and writing a proof. An argument that passed her minimum criterion of establishing truth and being universal would be a proof, but because she believed that a proof needs to communicate an idea to a reader so that the reader sould be able to follow each step without "making a leap of faith," she expected to see assumptions and reasons listed as part of a proof. In other words, it was important that a proof be written in a way that a reader could follow all the steps and the reasoning that had gone into writing each step. She utilized the phrases "formal proof" and "written proof" to make the distinction.

In the first interview after proving a given statement, Casey said that with the addition of a few more details she would call it a formal proof (Interview 1, Lines 676–678). Similarly after she finished another argument about the interior angle of a triangle, she said "We see that this is true" (Interview 1, Line 311). Then I asked her whether what she had written was a proof, and she said, "I think there is a couple of assumptions that you probably have to list as part of a formal proof” (Interview 1, Lines 313–314) and listed the items that she would add including assumptions, prerequisite knowledge, and reasons and then said, “I would call that a proof” (Interview 1, Lines 330–331).

Casey said that before writing all those assumptions and other items, she would not call it a “written proof” (Interview 1, Line 334) and added,

> I think I had proved it in my head, but in order to communicate it to somebody who may be reading this, they may look at this and say, well, why is C up here when it is supposed to be down here, and I think writing it out step by step so that you don’t assume that your audience is going to be able to make a leap of faith to explicitly write out B is equal to D and I might even say because of this, um, and C is equal to E, because of this, and these are equal to 180 because it is a line and so then you can see if you substitute in, that this is equal, and that I would call a formal proof. (Interview 1, Lines 334–342)

However, she had difficulty in explicitly explaining how a formal proof differed from what she called a proof. When I asked her the difference between formal proof and proof, she said:

> Formal proof, I would say, goes through each step like I said, explicitly listing any assumptions that are made, um, explicitly listing reasons why each step can be made, um. And the logical reasoning and then clearly stating the conclusion. Um, proof, I guess, in my mind I think of a proof as a formal proof. (Interview 1, Lines 347–351)

Then she added, “This just seemed more formal to me because I actually wrote out every step” (Interview 1, Lines 353–354).

Casey also expected the same level of detail when she analyzed a given argument. After examining the argument involving the transformation of the lines in the first interview, she said that she would call it a proof but not a formal proof and explained:

> I would call it a proof because it does show that in any situation, this is true. Um, I would not call it a formal proof because this way of explaining it, saying however this amount lost, that this lost amount is gained with angle y since *BA’* and *DA* are parallel, and x and y are alternate interior angles, um. I would rather see a list, something written out, like x is equal to y. And you could say, “Because it is alternate interior angles.” Um, and, um, that let’s label this angle z, um, that x plus z equals 90. And we know that this is 90. Um, so then you could say y plus z equals 90, and we know that that’s 90, so 90 plus 90 equals 180. I think, I think this is a proof. But a more formal proof, if I were to read it in a textbook an explanation like this would be nice, but a list of statements that I can

> logically follow and see what thing was changed each time would be nicer for a formal proof. (Interview 1, Lines 491–503)

It is important to note that although she emphasized having a list of statements here, in one of the later interviews she said that a proof could be in a paragraph form.

In the second interview, Casey was better able to articulate what she meant by a formal proof, and her explanation also confirmed what I had conjectured that she had meant in the first interview:

> I guess I give the label of *formal* on something I feel counts as a um, very solid proof. That it counts for any situation and it is written out in a step-by-step form, so that another person could pick it up and read it. And um, you know, given that they had enough background knowledge in math could understand how you logically got from one step to another. (Interview 2, Lines 323–328)

Consistent with her conception of proof in the abstract, throughout the interviews Casey rejected empirical arguments as proofs. In the first interview, she said she did not believe that the tearing off the corners argument was a proof “because it doesn’t account for every single triangle that could exist” (Interview 1, Lines 438–446). She further said, “I think proofs to me also tend to be a little more simple and elegant in the sense that you can write it out, um, with a written explanation” (Interview 1, Lines 443–445). She also acknowledged that one “could keep trying it and keep seeing that it is true, but there is no way that you could say you’ve tested every single possible triangle that exists” (Interview 1, Lines 441–443).

Similarly, in the second interview, Casey said that the empirical argument showing the number of diagonals in a polygon was not a proof. She said:

> That is a good demonstration, but it is not convincing. Um, I mean, I would want to actually prove it to know that it was true. And I would want to understand why this relationship is true. It seems to be true from what you’ve shown me, but I would like to show why it is true, and that it is true for all polygons. (Interview 2, Lines 445–449)

It is important to note that this time, unlike the first interview, she indicated that she expected the proof to explain why the relationship was true. Although she mentioned this aspect in the context of an argument, it did not become a part of her definition of a proof, as mentioned in the previous section.

Casey also emphasized the importance of understanding a method of reasoning in the process of constructing a proof. After comparing two general arguments for the same statement, she said:

> The reason I liked this is because, um, I don’t feel like I have this as an algorithm in my head of how to do it. I feel like I can figure it out from scratch each time. Because I understand the concept of like that I could take all these things and add it together, that is like a method of reasoning and then I could subtract out pieces. So I feel like that’s given you more confidence and more ability to reason in new situations. (Interview 2, Lines 275–281)

Although Casey knew that trying a few examples would not be a proof, she would use examples to decide whether a statement was likely to be true or not: “This is enough to convince me that it's probably true … but to actually believe it, I would wanna a formal proof” (Interview 3, Lines 412–417). Casey was also able to distinguish between whether she already knew that a statement was true and whether a given argument proved the statement to be true. Even in a situation where she knew that the statement was true, she was able to identify an argument as not showing that the statement was true for all possible cases:

> My main reason for not calling it a proof is because it does not necessarily hold true. I mean we know that it does, but there would be no way to show for sure that this holds true for every possible situation. (Interview 1, Lines 460–463)

In line with the shift in her description of proof in the abstract from the first to the second interview, Casey emphasized in the second interview the fact that previously established knowledge could be used in a proof: “we know that the, for the previous knowledge which is another kind of thing to keep in mind, if you have already shown something to be true you can use it" (Interview 2, Lines 259–262).

Importance of Proofs

Casey viewed proofs as being important as a way of establishing a common base of knowledge and communicating in a convincing way. She believed that by way of proofs people "can agree about what things are true and what things aren't and have a way to communicate to other people in a convincing way that a certain relationship we found is true" (Interview 1, Lines 181–183). After this statement she gave the example of .9999… being equal to 1 to illustrate how a proof convinced her. She said

> I think that the role of proof within mathematics is for us to have some common grounds with each other to be able to dialog about certain ideas and patterns and not have to just assume that what you found is probably true, you can show me and I can say okay, that is true so then this must be true also. So, it is a way for us to communicate our findings and to communicate different patterns and relationships that exist so that we can continue to build our understanding off of those that have been established. (Interview 1, Lines 214–221)

In other words, proofs in mathematics would "show that the reasoning methods are valid and so that our people can build on previously done work" (Interview 1, Line 371–372).

When Casey talked about the role of proofs in mathematics education she stressed that she viewed them as being important more for the teacher. As a prospective teacher Casey thought that it was important to learn proofs as she said "I'm not going to teach anything that I'm not convinced is true" (Interview 1, Line 615–616). Also with the level of understanding that she developed as a result of learning proofs she believed that she would be "ready for any questions that students will have about this" (Interview 1, Line 627–628). She also believed that it was important for a teacher to distinguish a proof from an explanation:

> I think it's important for us as teachers to have a much more solid understanding which involves understanding proofs and understanding what a proof is versus, you know, a

> reasonable explanation that may or may not be true for every situation. (Interview 1, Line 630–633)

With respect to how she would use proofs in her classroom she said:

> I think we'll use a lot of explanations, um, I picture myself having my kids prove things a lot but not necessarily with formal proofs, um, I would ask them to do what I would call a more informal explanation that would be convincing. (Interview 1, Line 568–571)

She further said that

> if a student shows me that they found something um whether it was right or wrong I would say prove it to me and get them in the habit of showing their line of reasoning and um, learning how to convince others that what they're doing is true with the goal of them being able to later move onto more formal ways of proving things. (Interview 1, Line 582–587)

She also mentioned that proofs would help "to convince the students that what you're sharing with them is true" and "to teach them reasoning methods that they may not know on their own" as she believed that "learning to analyze and understand the thinking of other people is really valuable in developing your own critical thinking skills so I think that's a big emphasis, um, I think that's probably the main thing" (Interview 1, Line 372–378).

Although she wanted her students to provide explanations and to be able to reason through things, she would reserve more general questions and "actual" proofs for her more advanced students:

> I think if you come across some more advanced students you could certainly start asking harder questions like well, how do you know this would always be true, um, do you think there might be a way to show (inaudible) numbers that you can think of. (Interview 1, Lines 573–576)

Furthermore she said, "I think it would be nice to be able to pull out a proof if I needed to with more advanced student that's really having questions" (Interview 1, Lines 642–643). In this way she would "challenge my [her] students further" (Interview 1, Lines 649–650).

Case 4: Nora

Nora was one of the undergraduate participants in this study. She had science as her first specialization and mathematics as second. She was categorized in the empirical proof scheme group based on the initial survey results.

Views About Mathematics

A distinguishing characteristic of mathematics for Nora was that it builds on itself. She explained that as

> you get one, one thing down and later that's gonna be useful for the next, you know, to go further in your mathematical understanding and then it kinda builds, you know, you start off kinda small and kinda all builds on each other, um, and so the basic fundamentals are very important to, to know and to understand. (Interview 1, Lines 55–59)

Nora also thought that mathematics was different because "it makes sense" and "you can work through it" as opposed to the memorization required for other subjects, and you can visualize things (Interview 1, Lines 54–68). She believed that it does not make sense for students to say "I'm not good at math" because there are a lot of connections in mathematics; the ideas all connect to each other and build on previous knowledge so one just needs to think through it. For Nora, mathematics just makes sense to her:

> it all connects to each other there is a lot of connections in math I think, um, and so in that aspect I think it's different because you're just connecting to things that, you know, previous knowledge that you already have or previous concepts that you've already learned, um, they kinda build on each other and so I don't know it just makes sense to me. (Interview 1, Lines 76–81)

When I asked Nora how we know that something is true in mathematics she said

> you first look at something and you probably have to make conjectures about why, um, {the things like} why do you think this is true and kinda with that you have your different, um, like suggestions, well, it could be this way, it could be this way and then kinda you just have to like prove it like why this is working, you have to like verify it … to be true you have to, um, have evidence I guess, have like, um, have good reasoning behind why you think it's true and why things work. (Interview 1, Lines 114–124)

Then she started talking about something that confused her:

> It's kinda confusing 'cause sometimes when you think something is true and then someone tells you, well not really, you know, there is exceptions and it's like uh, so frustrating, … because some things can be true for one, for one, you know, one problem but maybe not true for another problem. (Interview 1, Lines 114–124)

She talked about the parallel postulate as an example of something that confused her. It seemed that Nora had a better understanding of how truth is established in science, and her confusion about the parallel postulate led her to interpret the situation using her science lens and eventually conclude that mathematics and science are similar in that respect:

> I still don't understand like why we can't say that this like this isn't, you know, um, a proof or whatever like, it's true in all the cases that we worked with but Ms. Benson says, you know, there is an exception like there is, I don't know up higher or I don't know where but that makes it not proof so I mean I guess when we think that things are true there's always can be an exception so it's kinda like, I guess it's kinda like in science too we believe things to be true but, you know, in science, well that's, in science it's like, um, nothing's ever proven. (Interview 1, Lines 132–140)

Experience With and Attitude Toward Proof

Nora said that thinking about proofs made her nervous because she had not worked much with them. She actually had some experience with proofs in her high school geometry class and in her mathematics education classes. Although she liked geometry, what she remembered about proofs from her high school geometry class was "hating them" (Interview 1, Line 155). She felt "like it was something that was just like pushed on" them; they were shown the proofs without any explanation of why things worked as they did. She remembered that there was a certain way – a format or a table – they had to write proofs in geometry but she did not recall much about it. Her experience in college level mathematics classes sounded similar. She thought that she was "taught so many procedural ways of doing things" (Interview 1, Line 193) and she did not have "a deep conceptual knowledge of why you do this" (Interview 1, Line 194).

Nora believed that she did not have "much experience with actual formal geometric proofs," (Interview 1, Line 170) but her experience was more like what is expected in "math ed classes. Like you say this is true, okay, like why? Show me some evidence so um, but not really a formal proof I guess more of just why do you think this is true and proof that way proof and reason that way" (Interview 1, Lines 166–169).

In the second interview she talked about how much sense it made to be able to explain why something works as opposed to just learning procedures for doing things:

> when I was in school like I was given, um, like this is how a lot of procedures like this is how you do things but it makes so much more sense when you, um, when you can explain like how and why that works every time and, and stuff so, I guess when you, when you figure out that one thing works for one thing I feel like you feel successful when you realize well this can work for all these other reasons and this is why. (Interview 2, Line 60–66)

Nora indicated that in this class she learned some proofs that she did not know before. As an example she mentioned the Pythagorean Theorem proofs. On the other hand, she believed that she did not learn what a proof is as they did not talk explicitly about it:

> I think we, we learned like a few different ways, a few different proofs in the class, I don't know like, I didn't really learn much about like what a proof is in the class at all … about actually proofs I don't think I learned anything from the class besides the few examples that we did. (Interview 3, Lines 51–56)

She thought that the instructor "never really talked about like a proof and like what's a proof" (Interview 3, Line 40–41). Nora did not relate the things that they were doing in class to proofs:

> Whenever we were doing things like I don't, I never related it to a proof you know. What I'm saying like I was just, I was just working it out, figuring out how this worked or whatever but, in my mind like I didn't really relate it to like a proof even though like we were, it was a proof like I don't know. (Interview 3, Line 73–78)

It was interesting that she was not treating this experience as being similar to her experience with proof in her mathematics education classes. She actually said that “if we didn’t meet I don’t think I would have thought about proofs as much” (Interview 3, Lines 120–121).

Proof in the Abstract

The defining characteristic of a proof for Nora was that a proof shows why something works, and this can be achieved through evidence, which seemed to imply examples for Nora. The first thing that proof reminded her of was “evidence,” and she said that “to prove something to be true, um, you need to back it up with, you know, you say something is true you need to have evidence and back it up why, why does this work” (Interview 1, Lines 222–225). Nora believed that

> to prove something you have to, um, you have to state why it works, you have to give examples of, you know, why that particular solution works for that method, um, provide examples and through those examples kinda draw a conclusion, you know, um, so I guess to have an understanding of why you do what you do I guess. (Interview 1, Lines 227–231)

Although Nora expected a proof to explain why, her conception of what would constitute a proof was empirical as she said that “in math I think a proof, um, is going to, um, just provide you with examples and, um, provide you with a reason of why, um, something works I guess” (Interview 1, Lines 262–264).

Nora believed that reasoning through a problem would help to build conceptual understanding and also make it easier to work with other problems. She emphasized the importance of conceptual understanding provided by a proof or in other words provided by showing why something works in the second interview as well. She would want to see the proof not necessarily to verify the truth of a statement given by the professor but to understand why the statement worked:

> I have enough trust in Dr. Benson that she wouldn't be lying to me or giving me wrong answer but I think that I'm gonna, I'm gonna understand it conceptually if I like see how, how you get that and how you know that that's right. (Interview 2, Lines 294–297)

Nora had an important question at the end of the first interview with respect to disproving a "proof." She knew that a theory in science could be disproven, but she did not how it worked in mathematics. The word proof implied certainty to her, but as mentioned above she "knew" that there could be exceptions as in the case of the parallel postulate. She could not conceive of the parallel postulate as a given but thought of it as something that is proven to be true in some cases but not others. Hence, she raised the following concern:

> I guess one question is that I know I have is kind of like, um, a proof, I guess the relationship that I was talking about earlier between a proof like, um, (inaudible) science you don't use that word I guess cause it's more like a theory because it can be disproven but like if it's a proof like I guess there is no way that it can be disproven is that what I'm, is that like make sense? … If it's, if in math if something is a proof like if this was the proof of why this works so that means that that it can't be disproven? (Interview 1, Lines 682–690)

When I asked her what she thought she said "if they're putting the word proof on it I guess that in my understanding that's what it would mean because in science you don't do that" (Interview 1, Lines 692–694).

Although the fact that the word proof is attached to an argument seemed to imply for Nora that that argument shows the truth of a statement without doubt, she had not necessarily developed an understanding of why a proof can achieve that status. Furthermore, she had an empirical understanding of a proof, yet she believed that things that they have worked on and turned out to be true for some situations might be false in other cases. In addition to all these, she knew how things worked in science. All of this created a tension for her.

In the second interview Nora demonstrated a better understanding of the fact that a proof needs to employ a general argument. She could not remember an unsuccessful proving experience but when asked to think about a hypothetical situation she said that if she figures out a procedure to do something and then tries it for another example but it does not work then she has to find another way that would work for both problems. She still referred to examples as being a part of a proof but mentioned the need for more than one situation. She also acknowledged that a proof is universal; in other words, works for all:

> In proving a mathematical statement, probably, um, has to be, um, reasons of why it works and examples of it working for more than just one situation, um, like for instance the pr-like proving the Pythagorean Theorem works, it works for all right triangles, so, um, for it to be a proof you have to make sure that how you got that proof is, can be applied to different examples, um, different size right triangles. (Interview 2, Lines 103–108)

So, she knew that a proof needs to work for all situations, but how a proof ensures that was the problem. Because she referred to “examples of it working for more than just one situation” I argue that she still had an empirical understanding of a proof. On the other hand, when she said “that proof is, can be applied to different examples, um, different size right triangles” she might have been thinking of a general argument as working in the same way for different situations. Another interpretation would be that Nora was referring to a generic example. In other words rather than the example itself, she was focusing on the underlying reason for why that example was working and realized that that reasoning could be applied to any other example; hence the argument would be genral. Generic example can be thought of as a bridge from empirical to general understanding. Although she was not at a point where she could articulate that a proof needed to employ a general argument, she realized that an empirical argument would not be a proof as she recognized in this interview a problem with just providing examples:

> Um, I think it has to work for every, every type of right triangle for it to be, um, to be true so I mean if you do ten examples there still might be another example out there that it

> won't work for so but to be a proof it has to work for all of them. (Interview 2, Lines 116–119)

She eventually explicitly talked about how she did not know how a proof works for every possible situation as she said,

> I feel like to be a proof it has to go through a lot of like testing and stuff and so, how do you know if you've tried every possible situation, and how do you know if your proof works for every possible situation? I guess that could be challenging, um, in proofs, and that's challenging to me in proofs. I feel like, um, like this way it works, but how am I sure that it works for every situation like how am I sure that this can't be disproven I guess?" (Interview 2, Lines 256–263)

Although she was more confident in this second interview about the fact that a mathematical proof is different than what she thinks of as proof in science, she was not at a point to really understand how the generality of a mathematical proof is attained: "Cause comparing it to science, yeah, I know you can in science but I don't think you can in math but I don't know" (Interview 3, Lines 390–391).

Nora had a hard time defining a proof in the third interview. However, her definition seemed to indicate that she possibly did not think of examples as part of a proof. Her final definition incorporated the explanatory and universality aspects of a proof: "I think that a proof, um, is, my gosh, um, I don't even know, it's probably an explanation, um, of why something works for all situations, um, about a type of problem" (Interview 3, Lines 141–143).

Proof in Context and Evaluations of Arguments

When Nora tried to prove that the interior angles of a triangle add up to 180, she demonstrated both empirical and circular reasoning. She assigned values to the interior angles of the triangle – in fact she chose 30-60-90 as that was a familiar triangle. Then she subtracted each measure from 180 to find each exterior angle. As she knew that each line formed by one interior and one exterior angle would be 180 degrees, she multiplied that by three and subtracted the sum of the exterior angles to find the sum of the interior angles. After a line of questioning she said if she did not know that special triangle then she could have measured the angles.

When Nora analyzed a general argument in the second interview, she could explain why the argument was general and applied to all cases:

> because this reasoning is true for all, for all triangles, all, every angle and angle plus the exterior angle is gonna add up to 180 and so in all triangles you're gonna have a total of 540 and then all the interior angles is always 180 and so you subtract that and it's always gonna be 360 for every, it's always gonna be true for every triangle that he use. (Interview 2, Lines 448–453)

As explained in the previous section, in this second interview when Nora talked about proof in the abstract she could not articulate that a proof needed to apply a general argument and questioned how we could know that a proof works in all possible cases. However, as her analysis of this argument shows, given a general argument she could explain why it worked for all triangles. This provides evidence for my hypothesis that students who have not yet developed a robust understanding of proof are better at talking about the generality aspect in context than in the abstract.

In the last interview, consistent with her expectation that a proof would explain why a statement works, given a formula for finding the sum of the first *n* positive integers, Nora said she would like to know how they came up with the formula: "Like why, like how they came up with the, with that, with this formula, like how do they know that, how did they get that" (Interview 3, Lines 439–440). On the other hand, she wanted to plug in numbers to see if it would work: "I mean, I guess you could try plugging different numbers, that's how you could start off seeing if that works for, um, different numbers that you try" (Interview 3, Lines 454–456). However, she knew that just trying numbers would not prove it:

> I think that just plugging in numbers doesn't prove it but I think I would need to like see like how they came up with the formula to see if it wor-if like the explanation of it, like I

> can plug in numbers and they're probably all gonna work but that's not really proving it that that wor-that that's true like I feel like I would need to see like how that, they derived that. (Interview 3, Lines 465–470)

She was willing to accept an argument as a proof because she thought that it showed how the formula worked although she did not completely understand it and even though she had said before that a proof needed to be convincing:

> I think it's because it, it's like, it's showing like how they got the formula, it's not just giving you a formula, um, so, yeah, I think that this is a proof, I don't know if I really understand it though, that's the only thing. (Interview 3, Lines 482–485)

She actually eventually explained why the argument was general:

> I think cause you can for any case you can make a, you can do the dots, and for any n number you can make that n squared and then make that your square that you're working with and then take half of it and then subtract the diagonal. (Interview 3, Lines 482–485)

When Nora tried to decide whether an argument was a proof or not she used being explanatory or not as her criterion. Comparing two different arguments, she said that "this [Argument 4 in Appendix E] is just giving you the formula, but this [Argument 3 in Appendix E] is explaining to you how you got the formula so that's what I, that's what I think a proof is so that's why I would say this is a proof [Argument 3]" (Interview 3, Lines 639–642). In other words, she expected a proof to explain where a formula comes from or why it works.

In the last interview, when she evaluated another argument about the triangles, for the first time she questioned whether a statement used in the proof needed to have been proven first. Another criterion that she used in the last interview to decide whether an argument proved that the statement is true for all cases was whether it used numbers or variables as she said that an argument that she accepted as a proof was employing a general argument "because it's using, um, instead of using like numbers, it's just, it's using variables" (Interview 3, Lines 744–747).

She also acknowledged in this interview that previously she would accept an argument using measurement as a proof but she would not anymore because measurement could be wrong: “I probably used to think it was but I don’t think it’s … because like you’re measuring so I mean that could be, you can measure wrong and so it’s just not, again not really reliable” (Interview 3, Lines 239–242). However she attributed this change to comparing different arguments – something that she had been asked to do in the interviews: “I don’t know. Um, maybe comparing it to what, comparing it to like other explanations that maybe seem more reliable or like more I guess justified or I don’t know if that’s the right word but” (Interview 3, Lines 55–59).

Importance of Proofs

Nora believed that proofs were important because of the reasoning they provided behind a solution:

> I just think that without a proof it's kinda like well you don't really, like, there's not much about what your solution is whatever, you know, in science what your hypothesis or your theory if you don't have, you know, a good reasoning behind it. (Interview 1, Lines 255–259)

She also mentioned the importance of being able to build on previously proved statements:

> I think it's probably important also when you get into higher level math like um if you already have, if you've already learned that this is a proof, you know, you don't have to go back to and work out why, why is that a proof, you know it then you can just go ahead and build on it like we talked about in the beginning, it's just like kinda math is like building, building concepts so I guess in that aspect it's important in math, too. (Interview 1, Lines 269–275)

She also believed that proofs are important for teachers to know as they need to be able to explain to their students why things work as she said that

> I think that it's important to, for the teacher to have a good understanding, a good basis of the proofs cause the students are gonna ask you know why is that work or why is that work every time, you know, what about this time, so, I think that it's very important for teacher to have a good understanding and evidence that backs up the reasons and also

> with having a proof and having the, the concrete reason why this works every time, I think it's also good to have like reinforcements like we can also see that you know and then also like try different triangles out and they also work so kinda, kinda different examples I guess is important to show in a classroom. (Interview 1, Lines 515–524)

Because she did not think she knew enough about proofs, she was not sure how she would bring it up in the classroom; she needed to learn more first:

> I think I don't know like I feel like I probably don't I don't feel like I have a very good understanding of a proof now like after all these questions and stuff so like I don't know if I would be comfortable like, you know, I think I'd have to learn more about it and like what it is and what isn't to really understand like what, how I would bring it into my classroom so I don't think I know enough to answer that right now. (Interview 3, Lines 276–281)

Nevertheless, Nora believed that explaining why things work would benefit students when they learn about proofs:

> I don't have much knowledge about proofs, but I think that I mean just I think that just about that a proof involves like explaining, you know, and like why do you think this answer works, I think that's gonna be important for all students, um, to be able to like, verbally like talk about it and also like, you know, like writing down, explain like their answers and stuff so I think that like in, if they're used to doing that when it does come time to like proof they're gonna understand more like why this is a proof you know like why this work for all situations and stuff. (Interview 3, Lines 300–308)

Case 5: Brenda

Brenda was one of the undergraduate participants in this study. She was different from other participants in that this was a second career for her. She had five children, and she worked. During the time of the study she was a second semester junior in the cohort. It was her fourth year in college, but she indicated that her first two years took her three years to finish. Although Barbara was not certified as a teacher before, she was involved in working with children in different roles. She also said that she helped a lot of students for whom English was not their first language and struggling readers. She was categorized in the empirical proof scheme group based on the initial survey results.

Views About Mathematics

When Brenda was in school she liked mathematics for several reasons. First of all, she liked it “because there was a right answer” (Interview 1, Lines 69–70) and everybody would come up with that same answer. She found it cool that the answer to a mathematics problem might take three pages to complete. She thought that she was a logical kind of person and to her mathematics and science were subjects that had to do with logical reasoning, so those were always her “favorite subjects in school” (Interview 1, Lines 74–75). When I asked her what she meant by logical reasoning, she said

> I even do that just in life in general when I come up with anything or do anything or find out why something happened I always like to figure out well this happens so that happens so this happens so that happened and it’s kinda like uh reasoning. (Interview 1, Lines 104–108)

Brenda believed that the way that she was learning mathematics and how to teach mathematics in her college classes was very different than how she had learned mathematics herself as she said

> I had one class of accounting about 20 years ago so this is all pretty new, yeah and especially like the way that I learned math is not like what they’re doing now, so it’s all brand new to me. (Interview 1, Lines 41–43)

For her mathematics was always “two plus two is four; no questions asked” (Interview 1, Line 101), and she “never equated math with anything infinite” (Interview 1, Line 100).

Brenda contrasted mathematics with reading and literature where “you have to pull like from nowhere” (Interview 1, Line 76) and history which “is just kind of like facts” (Interview 1, Lines 77–78). She was not comfortable with reading and literature which to her were theoretical - “why would they do it if this would have happened kind of thing” (Interview 1, Lines 79–80). However, she was learning that mathematics shared some of the characteristics of these subjects.

As mentioned above, Brenda liked that given a problem in mathematics everybody would come up with the same answer; however, in this geometry class she said she was learning that

> it's not finite … like with the proofs there is some things that you just can't prove, can't come up with a proof like in our last class, so that kind of eliminates the everything being this is the answer kind of thing. (Interview 1, Lines 88–91)

So, she believed that in college she was learning something that she did not know before that "there is not always a specific finite answer in math" (Interview 1, Lines 93–94). On the other hand, this did not change how she thought middle school students would learn mathematics; she indicated that they "will pretty much have an answer to everything" (Interview 1, 92–93).

With respect to the relationship between mathematics and science Brenda said that

> there is a difference but I think there is a big relationship, too. Yeah, definitely a relationship. To me they are both logical. Um, with math you have to do the, um, uh, you know, question, come up with a theory, figure it out, decide whether or not it works and you do the same thing in science, … when I went to school they didn't teach you that, this is the answer and that was that but with this stuff I think is very interesting because it's totally like science. I mean it's like scientific investigation is the way that they're teaching us to teach math which I think is really cool. (Interview 1, Lines 114–125)

She believed that this was better than having the students "just recite facts, which doesn't help with reasoning" (Interview 1, Lines 127–128).

When I asked how we know that something is true in mathematics, Brenda's initial response was that it "depends on what level of math you are at" (Interview 1, Line 133) and that she did not really know higher mathematics. She also added that before (taking the classes with Dr. Benson.) she would have said "because someone else told me" (Interview 1, Line 135). Later in the interview she said, "How do you know that something is true in mathematics? I guess it would be this here by just doing it over and over again and having that come up with the same answer" (Interview 1, Lines 604–606). I believe that by "doing it over and over" she meant trying several different examples.

Experience With and Attitude Toward Proof

Brenda had almost no experience with proofs prior to this class. She vaguely remembered them from high school. With respect to the place of proof in Dr. Benson's class she said that "I think one of the goals of the class was to get us to understand what proofs are and where they came from and why they exist, to me that was the bigger picture of the class" (Interview 3, Lines 71–73).

Because of the way that she had learned mathematics previously, Brenda believed that she did not know anything about proofs before this class. She explained that "when I was in high school, it was just like here is the math, do it and that's all there is to it. So I didn't, had no idea what it was at all" (Interview 3, Lines 126–128). So, when asked what she learned about proofs in this class she said:

> I never knew what they were so I learned what they were. I didn't have any idea, to me, I just used it as the general vocabulary word of proof as in prove it kinda thing. So I mean I never knew what a proof was so now I know. I don't know them all, but I know a few and even then I still have to study and to remember them. (Interview 3, Lines 106–110)

Brenda believed that talking about proofs in the interviews, in other words reflecting on what she was learning, and also knowing science helped her to understand proofs better:

> Really to tell you the truth, once I started sharing with you it's kinda when that all came together. You know like when you start talking and then it kinda just kinda, kinda, yeah, all comes together … so that and then yeah that helped knowing science. Now if I wouldn't be a science major, yeah that wouldn't, I don't think that would have happened. (Interview 3, Lines 173–185)

Although Brenda was intrigued by this new type of mathematics that she was learning, several times throughout the interviews she said that she did not know if an argument would be considered a mathematical proof or not but that it would be enough for her as a middle school teacher. Although she said in the first interview that she hoped to get to the point where she would rely on herself to prove something (Interview 1, Lines 268–295), in the last interview she said “I mean, it would prove it to me but I don’t know if that’s enough … I’m not the abstract high dollar mathematical thinker; I’m just a middle school teacher” (Interview 3, Lines 1104–1105).

Proof in the Abstract

The first time that Brenda was expected to respond to a question about proof for the purposes of data collection was the survey given on the first day of class. In the first interview a few weeks later she admitted that she was not able to answer the proof question in the survey without looking it up in the internet

> because I didn’t even know, she had that big question, think about proofs, what’s a proof do you remember, I had to look it up in the internet. I didn’t even know what a proof was to even answer the questions.” (Interview 1, Lines 208–211)

Based on what she gathered from the first few days of class and from what she read in the book, she said that “a ‘proof proof’… is something that mathematic- that you can prove mathematically with certainty” (Interview 1, Lines 201–202). It became even more evident in the later interviews that certainty was an important part of her conception of proof, and the way that she thought that they learned about proofs in this class contradicted this initial conception. Hence, this created a tension for her throughout the semester. She eventually thought that she resolved this issue by constructing a link between the way the word “theory” is used in science and everyday language and the way the word “proof” is used in mathematics and in everyday language.

She vaguely remembered the distinction that Dr. Benson had made between a proof and a postulate as something that cannot be proven;

> I can’t think of them specifically but I know we just did the triangle and the straight lines, and one of them you could prove and the other one was you had to assume from previous proofs but I can’t remember specifically now cause I just learned it. (Interview 1, Lines 238–241)

However, she interpreted it in a way that the thing that cannot be proven is actually shown to be true using some other proofs but not "mathematically" (Interview 1, Line 236).

She also implied an understanding of the logical structure of a proof when she said that "the absolute can be proved using mathematical equations and knowledge from step one to step two to step three" (Interview 1, Lines 223–224). There was also implication in her definition of proof that each step needed to be (or could be) justified as she said that a proof is something that "you would be able to actually write it down on a piece of paper and have show som- have something equal something because of this" (Interview 1, Lines 253–255) and

> You can show on paper one step to the other and all the numbers match up; you can show it physically on the piece of paper like with writing and numbers this equals this so this equals this so this equals this. (Interview 1, Lines 248–251)

When I asked her if it was possible to invalidate a proof, she talked about things being subject to new findings and indicated that this was her science background talking. She also mentioned that across history mathematicians have been proven wrong. When I asked the question using a specific example – the sum of the interior angles of a triangle being 180 degrees – she mentioned that in class they discussed that it could be 179.999. It seemed like she interpreted what they had talked about in class related to measurement error with the tearing the corners argument, which was intended to show that this argument was not a proof, as evidence that one can never prove that the sum is 180 degrees. Thus, she appeared to be conflating the idea of something not being a valid proof with the idea that an established proof could be invalidated.

By the second interview Brenda had developed a conception of proof that she had not had at the beginning of the semester. First, when I asked her about a recent proving experience, she raised the following question "Is that a proof when you take the circumference and the diameter and get pi? I don't know if that's a proof or not; that's just a way to, that just shows you where pi came from I guess" (Interview 2, Lines 18–21). As she was trying to explain whether it was a proof or not, she started talking about what she thought a proof was:

> From what I've done so far, the, uh, the proof confirms something that you already believe, … You just think that this is, that this is right and then you wanna prove it … To me a proof is kinda like working backwards to prove something. (Interview 2, Lines 30–36)

With this understanding of a proof, she seemed to conclude that "if you were given the number, to prove it you would go back and measure circles, lots and lots of circles" (Interview 2, Lines 37–39).

When I asked for clarification she said:

> It's an assumption that you make, and you want to prove that your assumption is correct. So in order to prove that your assumption is correct you have to come up with some kind of either, um, you know, visual thing or algebraic thing to prove that what you think might be true is true. (Interview 2, Lines 44–48)

To illustrate her point she used the example of the triangle:

> The one thing we did to prove that the interior angles of the triangle equaled 180 we actually like tore them off, and when you tear them off and you lay them down you can see they make a straight line so that's kinda like you know a visual estimation that that might be true but then to prove that it's true you need to, you need, like the parallel postulate helps to prove that it's true because, you know, you know that this line is straight and then all the angles equal that, um, so that's a way to prove it using the parallel postulate. (Interview 2, Lines 50–57)

I hypothesize that Brenda came to think about proofs this way as a result of their classroom routines in Dr. Benson's class which usually involved a demonstration or a few examples followed by an explanation or a general argument. The demonstration or the examples allowed students to make a hypothesis and then they tried to prove that their hypothesis was true. Brenda seemed to be equating "hypothesis" with "assumption."

In the second interview Brenda brought up an issue that she was having with proofs that she illustrated with two different examples. She said that “growing up in school you’re just told the interior angles of a triangle equal 180. That’s just a fact” (Interview 2, Lines 97–98) and she accepted this as absolute truth. Similarly, she had previously learned that pi equaled 3.14. However, in this class she was learning that neither of these were facts; they were just “assumption[s] that can’t be proved beyond a reasonable doubt” (Interview 2, Line 99–100). She said that because “you can’t test infinite amount of circles so you can really only assume that pi is true for all circles because you’ve tested it on thousands of circles” (Interview 2, Lines 80–82). She was getting this idea about pi from a class activity where they measured circumferences and diameters of several circles and then found the ratios. In the case of the triangles, “you’d work through it and you’d make logical steps and, and, um, you’d use accurate, um, beliefs and, um, stuff like that but, um, it’s still just not absolute” (Interview 2, Lines 134–137).

This piece of knowledge contradicted what she believed before and bothered her. She said that she would use the result – the sum of the interior angles being 180 degrees – with reservations, every time she would remember that it was not absolute. Although this bothered her a lot, she was willing to accept the uncertainty as she viewed it as something that is learned in higher mathematics and is “enlightening” (Interview 2, Line 101). However, it was not something that is mentioned when teaching children. Based on this newly gained information Brenda concluded that “proofs in math are not an exact science” (Interview 2, Line 76).

Although Brenda believed that she was learning that proofs are not absolute truths in mathematics, she was still learning things that she did not know before. She noted that in school she was not given an opportunity to understand pi; rather she was taught what pi is but not why. Similarly,

> In school you're just taught that a triangle equals 180 degrees. You're not really ever taught why that's that; it just exists, but being able to prove it using the parallel postulate was really interesting and eye-opening. That you know it's not just, that someone just didn't make this assumption up of a triangle equal 180 degrees that mathematicians actually wanted or needed to prove that that was true. So even knowing that they existed was, um, new to me because I didn't really know that that's how things worked in math, so that was very interesting. Yeah, I liked it. (Interview 2, Lines 258–266)

Hence, 180 or pi were not just numbers made up; there was mathematical necessity and ways to prove them. Although this seemed to contradict what was explained in the previous paragraph, Brenda seemed to reconcile these two contradicting ideas by believing that 180 degrees or pi were assumptions but not any assumptions: "They actually use other math to go back and prove that the information that they're giving you is you know 99.9 percent true" (Interview 2, Lines 281–283). In other words there were mathematical reasons that would explain their truth. It could also be argued that Brenda had starting seeing why things are true in mathematics but had not yet developed an understanding of the fact they could also be shown to hold true for all possible cases by way of proofs, in other words how a proof could afford to be general.

These developing beliefs were reflected in Brenda's explanation of the notion of proof. She said that to prove something to her "means that you've used logical means and uh, information from other things to prove that something does exist so it's really a lot of just I think, logical assumptions especially if you can, especially if you prove something based on another proof" (Interview 2, Lines 158–161). I believe that by logical assumptions she meant that the steps that have been followed are logical, but because you have based your proof on other assumptions, what you have proved is only a logical assumption. So, she defined a proof as follows: "I would define it an assumption and or an assumption based on another assumption but assumption that you know that you've used as many means that are available to you um, mathematically, um, to prove whatever it is that you're interested in proving" (Interview 2, Lines 176–180).

Something that reinforced Brenda's confusion about the uncertainty of proof was the introduction of other geometries where the parallel postulate does not hold true:

> I don't know the difference between Euclidean geometry and the other one, but she did mention that some of the postulates and stuff that we have in Euclidean geometry don't apply to the other one. So that I mean right there shows you that the assumption that you have made isn't always true. (Interview 2, Lines 167–171)

In the second interview Brenda also indicated that she was confused about what she was allowed to use and not allowed to use when doing a proof. Part of what added to her confusion was the fact that Dr. Benson had told them to prove that the area formula worked without actually using the area formula that they already knew. Brenda was not sure why she could not use it when she already knew it to be true.

In the last interview Brenda defined a proof as "a way to justify, um, a mathematical idea. It's not absolute, but it's a one way to justify it. One proof is one way; another proof is another way to justify a mathematical idea" (Interview 3, Lines 188–190). She was highly influenced by the fact that the Pythagorean Theorem had more than one proof, and she incorporated that into her expectations from a proof. She believed that you have to have more than one proof to get closer to 100 percent sure that something is true. In her eyes, each proof seemed to provide evidence that the statement would be true rather than showing the statement to be true for all cases through the general argument employed. However, she did not know if it would be possible or necessary to have more than one proof for every mathematical statement. On the other hand she also talked about proof as being a mathematical process and emphasized using mathematics to get the answer.

Similar to her response in the second interview, she also believed that a proof confirms:

> It would be like you know four apples on the table take two away it equals two; put four oranges on the table take away two it still equals two. In simpler terms it would just be, okay four minus two does equal two because here is a way it equals two, here is a way it equals two and here is a way it equals two. It just confirms what you already think you know or know that four minus two is two so it's just kinda I don't know if you'd call it double checking but {no}, but it would be a way to double check. (Interview 3, Lines 230–238)

When asked how something would be known in the first place, she said guess or estimation, trial and error, or maybe sticking the corners together (referring to the tearing the corners argument).

In the third interview Brenda made an explicit distinction between the ways that proof is used in mathematics and outside of mathematics. When asked what she believed proof is outside of mathematics she said that "outside of math, again see the term is to me is just like in science is used loosely, it's just, I mean to me, proof in layman's terms is absolute where in math it's not" (Interview 3, Lines 262–269). An example of proof outside of mathematics would be weighing a person or "the apple rotten proof would be open it up and it's nasty {rotten on the} inside" (Interview 3, Lines 278–279). She said:

> I used to think proofs were absolute and now I don't, I just think they're confirmations, manmade, man-theorized, man-thought of confirmations of stuff that's hard to, intangible, stuff that's hard to touch, stuff that you can't really reach but getting as close as you can to what it does mean. (Interview 3, Lines 307–311)

How Brenda related theory in science to proof in mathematics helped her to resolve the conflict that she had been having throughout the semester:

> Like theories in everyday language for us just means like a guess, but like a theory in science is like as positive as they can be about anything. It's like the ultimate, and then proofs in math to us in everyday language a proof is an absolute, but in math a proof is just not that less important but it's not an absolute but it's as close as you can get to a theory in science, that's kinda like how I equated it. (Interview 3, Lines 20–26)

Proof in Context and Evaluations of Arguments

When I asked Brenda in the first interview if she could prove that the sum of the interior angles of a triangle was 180 degrees, she first said that they just talked about it being 179.999 in class. The only thing that she could think of right away was that she had liked tearing the corners argument.

When I asked her about the response that she had given to the survey question asking for a proof of the same idea, she said that “I think I originally wrote here because my teacher said so” (Interview 1, Line 468). Thinking that this would not be an appropriate response to a homework question, she had changed it to “add angles of many triangles” (Survey 1, Jan 08). When I asked her what she thought about this response during the interview she said that “one of the things we discussed in class is can you really draw every single triangle there is, and to me the answer would be no because once again there is an infinite number of triangles so that’s where 179.999 comes in” (Interview 1, Lines 443–446).

Even during the first interview Brenda was starting to be “torn” between what she knew before and how she interpreted what she was just learning. Before this class she knew that the interior angles of a triangle added up to 180 degrees because

> I knew it because my teacher told me but then you know, thinking a little beyond that I just knew that no matter what kind of triangle I draw from what I’ve been taught it’s gonna add up to 180 degrees. (Interview 1, Lines 476–479)

On the other hand in the class they talked about the tearing the corners argument and discussed that it was not a proof because it was not exact. They also talked about the fact that it was not possible to draw every single triangle. With her limited knowledge about proofs and without having developed an understanding of the generality aspect of a proof Brenda interpreted all this knowledge as it is never possible to be certain/exact about 180.

> As far as an elementary school student or middle school student goes, this would be sufficient for them, but thinking now in school here, um, it just challenges your mind to know that, um, math actually has things that can't be perfect. (Interview 1, Lines 483–487)

Hence, she concluded that "math is not as straight forward as I thought it was; that's what I'm learning [laughs]. It's intriguing" (Interview 1, Lines 534–535).

She indicated that at this point she did not know if there was a way to show that it is exactly and certainly 180. One thing that she could think of was using engineering as she said that "If you build it and it works, then you can assume that your assumption was correct, but whether or not that actually makes it, proves it, I don't really know" (Interview 1, Lines 557–559).

As they discussed in class Brenda realized limitations of using examples, but this seemed to make her believe even more that you could never be 100 percent sure that a statement is true. If she could draw a million triangles, Brenda "would be comfortable" but she "still wouldn't be absolutely positive" (Interview 1, Line 581). She would be absolutely positive if she could "draw an infinite number of triangles and they would all equal 180" (Interview 1, Lines 584–585).

Although she said that tearing the corners argument was not a proof "because it's just torn pieces of paper, I think it would be enough of a proof for a middle schooler to show them that it equals, that they equal 180 degrees" (Interview 1, Lines 674–676) she also said that she did not know if it was mathematically considered a proof or not.

Similar to other participants Brenda could remember that the exterior angles of a triangle added to 360 by thinking about the walking and turning argument. When I asked her if it was a proof she said that to her: "This is kinda like ripping the triangle apart and putting them altogether" (Interview 2, Lines 481–482). She said that "it's more like a start" (Interview 2, Lines 486–497), "it's just like a, I think this exists let's see if we can prove it kinda thing"

(Interview 2, Lines 504–505). It was not a proof to her “because there is no mathematics involved” (Interview 2, Line 508). She also recognized that in measuring there was always human error and that a measuring argument was not a 100 percent proof either.

Brenda said that “that’s what we’ve learned in class that if we can prove it on you know a thousand triangles, we can assume that it happens for all of them and that’s what we’ve been learning in class” (Interview 2, Lines 584–586). I believe that Brenda might have misinterpreted an instance(s) in class where they would say that the proof is beyond their level so they would accept a statement to be true in that class after doing an example or demonstration.

Similar to her response in the first interview Brenda knew that “you can’t measure every triangle in the whole wide world” (Interview 2, Line 618). She again believed that there are things that get you closer to your 100 percent but never get there.

Brenda believed that a given general argument was a proof because it used mathematics. In other words she differentiated the general argument, which used properties that they knew, from an empirical argument. However, she also believed that it was still as much of a proof as the empirical ones because it was depending on something else being true. In other words the argument that was claiming that the sum of the exterior angles of a triangle add up to 360 degrees used the piece of knowledge that the sum of the interior angles is 180 degrees which was not absolute truth. On the other hand, she believed that its inclusion in this argument made her “even more sure that … the sum of the [interior angles of] triangles is 180 degrees” (Interview 2, Lines 716–717).

Although Brenda differentiated the general argument from the empirical ones, she believed that it did not prove it for all triangles. She also added that before this class she would have said yes though because “from reading the textbook and listening to Dr. Benson, I have found out that math is not an exact science which I grew up believing that it was" (Interview 2, Lines 723–725). She also said that the argument "proves it for all the triangles we've ever seen, (inaudible) used, or worked with" (Interview 2, Lines 729–730).

When Brenda was given an unfamiliar statement about the number of diagonals in a polygon in the second interview, she actually tried to make sense of it. She questioned where the minus three came from, and with a little help about what a diagonal is, she could see why there was a division by two. She almost figured out the formula but did not see it as a proof. She said that she would need to do lots and lots more examples, but she seemed inclined to believe that the formula was correct. However, she said the empirical argument would be good enough for her middle school class. She did not think the formula could be proven to work for all polygons.

The last thing she said in this interview referring to the relation between calculus ideas and proofs was that

> they can't be proved beyond a reasonable doubt that it might always be that one and that's because in this you can't always prove you just can get closer and closer but you can't get to the exact, exact thing, so you can get closer and closer and closer to proving it but you can never be a hundred percent, 99.999 (laughs). (Interview 2, Lines 1144–1148)

When Brenda was given the statement about the sum of the first *n* integers, she said that she would try different numbers to see if it works. She would use "a 5 and then a 100 and then an odd number like 256421 instead of a rounded ten number" (Interview 3, Lines 629–630). Again, this confirms her reliance on examples to convince herself that something is true.

As before she did not necessarily accept that an empirical argument was a proof in the sense that it proved the statement for all cases, but each example would get her closer to believing that the statement was true: "I don't know if they each individually would be called a proof because they're all coming at the same way but for me personally each one would be a, would be a proof, each individual one, so the more you had the more true that you could say that it was” (Interview 3, Lines 634–638). She was accepting each empirical example as “one proof” as she was not expecting a proof to show that the statement is true for all cases. Something was a proof as long as the answer came out the same. Each individual case seemed to be a confirmation of the truth of the statement.

In the third interview Brenda indicated what she would do to believe in the truth of a statement would depend on whether she already knew whether it was true or not:

> I mean if my teacher told me this and we plugged in ten different things, then I would say okay yeah that’s right. But if my teacher didn’t tell me that that’s right, I would have to do it personally. I would have to do it way more than three times, just to prove to myself and feel comfortable that that’s true. I mean like I would try it for square roots and odd numbers like you know this is just what 6 and 3, you know I’d like to stick 2563 in there you know that kinda thing; that would make me feel better. (Interview 3, Lines 671–678)

For the first time during the semester Brenda indicated in the last interview a sign of an understanding that an argument could be general and could prove that a statement is true for all possible cases. This incident happened in the context of an algebraic argument, which could mean that the algebraic context is likely to lend itself to an understanding that it is possible to write a general proof:

> Listening to her teach that that if you can use *n*, which means you can use any number, that it would true hol- would true, it would hold true for all cases if you can use *n*, which means again you would be proving it by putting in just bunches of different numbers, but like when she showed us proofs and she used *n*, that’s what she said. *n* means it works for any number; that that would justify that it’s true because you can use it for any number. (Interview 3, Lines 643–649)

It is important to point out Brenda’s reference to what the instructor said with regard to the use of the variable. Nonetheless, this was an important incident as it was the first time that Brenda could conceive of an argument as general. Furthermore, later in the interview she articulated the difference between using a variable and using specific measurements:

> And, um, and then using those numbers which you measured which you know could or could not be accurate, um, I mean you could prove it as far as you know regular people

> are concerned, but I don't think that that would be considered a mathematical proof that we would learn in the classroom because you're not really using *n*. You are using specific, um, measurements that can be incorrect, not accurate. (Interview 3, Lines 1085–1090)

Importance of Proofs

Brenda believed that we have proofs because everybody needs to prove they are right, and in engineering people would rather use things that have been proven. She said that proof is for a man to "back-up his thoughts" (Interview 1, Line 340) and to "confirm his theory" (Interview 1, Line 353).

In the second interview Brenda talked about other reasons as to why proofs are important with one of them being building on what we know:

> A proof is important because it helps you build on math and go further into why things are the way they are. Um, I mean if you just stopped at your proof and you didn't go any further we probably wouldn't have spaceships in space, satellites and stuff like that … I would think that all the mathematics that we use for those kinds of things that we do now are based on the proofs that we have and go beyond them; that's a guess. (Interview 2, Lines 216–227)

She also developed some unconventional ideas about how proofs are used to confirm previous proofs, which in her mind were actually assumptions that were not 100 percent correct. She said that if you use a proof – I think that she meant the end product of a proof – which you know is an assumption to build something else and if that something else works, your proof gets closer to being 100 percent but never becomes 100 percent (Interview 2).

As Brenda believed that she was learning new ideas about mathematical proof in higher mathematics she indicated that "you just become uncomfortable with total {un}absolute certainty" (Interview 1, Line 619). She did not think that students were "capable of going into this theoretical stuff" (Interview 1, Line 615). Furthermore, she believed that "it's not what you wanna base your math on. I guess is not a good mathematical foundation; I think you'd have to start from solid, which is what we do with them" (Interview 1, Lines 615–618). Hence she indicated that "drawing 15 triangles … would be sufficient for them" (Interview 1, Lines 613–614).

In the second interview, Brenda reiterated the idea that "you have to have a beginning, you have to start somewhere" for kids and added that the measurement argument "would be enough for them" (Interview 2, Lines 652–653). In this interview, she also talked about how an argument that would not be accepted as a mathematical proof would be enough for middle school students as she said that

> from what I understood in class that's just like a conceptual thing like to show kids, yeah, look at this you know it does equal 180 degrees which for a middle schooler would be enough you know to prove it I think. But mathematically I don't think that's a proof, it's just like a, I think this exists let's see if we can prove it kinda thing. (Interview 2, Lines 501–505)

Although she did not totally dismiss the idea of proof in middle school she said that it would depend on the level of mathematics that she would teach. She would probably incorporate proof in a general mathematics class as an extra thing but would grade students on it in a higher level mathematics class. Brenda also made a distinction between students based on their mathematical abilities in terms of what they can learn about or do with proofs:

> I think this would apply to the, to the students that were a little higher up or that math came a little easier to and were into that reasoning kinda thing to prove to them, give them an actual proof from math I think would be good. I think the slower kids of the class wouldn't understand and it may confuse them and you may have to set it aside for them. But I think the kids that are more, um, developed mathematically or have better mathematical skills, I think that this would be good for them. (Interview 1, Lines 629–635)

Similar to the first interview, Brenda made a distinction between students with different ability levels with respect to what kind of a proof she would expect from each of them. She believed that the general argument proving that the exterior angles of a triangle add up to 360 degrees is something to be expected only from gifted students. She mentioned some of these ideas in the last interview, too; however, she also talked about the importance of starting early:

> I think some of them will, definitely not all of them. I really think just a few and then again it depends on what your– Some proofs are gonna be easier than others but, um, I don't think, I think that if they were taught to think like that from an earlier age, you would have a higher percentage that would be able to do it but, um– So that's dependent upon their past. And then you know some people just think, their minds just work like that, more easy than others do, so not for many but I think some of them could. (Interview 3, Lines 445–452)

Although she believed that not everyone could come up with a proof individually, she also recognized that it was more likely for students to come up with proofs in small groups. She emphasized the importance of group work in the last interview as well and said "When you put them altogether it works, so I'm definitely gonna do that" (Interview 3, Lines 453–454).

She also said that as a prospective teacher she was being taught,

> You would have to do the explanation and require the students to come up with justification for their final answer … Don't just give them the proof; force them to think through it so they can learn how to think for themselves. (Interview 1, Lines 395–399)

She emphasized this idea even more in the last interview:

> I don't know how many proofs there are that would be, uh, that a middle schooler could actually understand … but I would definitely instead of just saying, well, a triangle has 180 degrees or *a* squared plus *b* squared equals *c* squared like I was just given that information, I would definitely want them to explore it so that they would remember it and understand where it comes from because (that) not only make math better you know easier, not eas- well, yeah, easier to understand and yeah definitely think that belongs in there. (Interview 3, Lines 431–440)

In the last interview she also said that proofs are for mathematicians to confirm things that "they think they know" (Interview 3, Line 62). She explained that

> why we have them just might be like mathematicians confirming ideas … just going back to the triangles again or even like I was looking through, uh, Pythagoreous [sic] Theorem and all the proofs that there are for it, I was like, wow, there is a lot of proofs for this, but it's just mor– it feels like it's a mathematicians confirming to themselves that you know that they're right or that the mathematician before them was right. And um, and then our

> class was to show us you know where mathematicians come from, how they think and the things that they do that we never even thought about. (Interview 3, Lines 47–57)

She also said that “justification again, it’s kinda like a theory in science again, I had to keep going back to that but um, the more tests you do in science to not to prove but to not disprove your theory would kinda be the same thing with proof in math, it just, um, confirms, um, whatever theory it might be that you have in math or whatever I don’t know what the word to use for it is, whatever idea or whatever” (Interview 3, Lines 207–212).

Case 6: Andy

Andy was one of the undergraduate participants in this study. He had science as his first specialization and mathematics as his second. He was categorized in the authoritarian proof scheme group based on the initial survey results.

Views About Mathematics

Andy was a middle school education science major, and in the first interview he said that he picked mathematics as his second major because “science has a lot of applications that deal with math that you have to use, um. That’s one of the reasons I picked math” (Interview 1, Lines 56–58). He also found it more interesting to read a science book than “sit there and do math problems or read about math concepts” (Interview 1, Lines 64–65). In the last interview, he said, “I just pretty much picked [mathematics] because I thought it was easy” (Interview 3, Lines 201–204). He believed that mathematics was easy because of his previous learning experiences with it.

> [Mathematics] was easy because all you do was memorize the formulas. But I never really understood why anything like worked. So I just didn’t like it. So because all the teachers would teach you is, you know, the formula, the plug it in. Like, “Hey, just memorize this, you’ll be fine.” (Interview 3, Lines 213–217)

Andy also thought that you “have to keep your tools sharp, and you need to keep practicing or you’ll start forgetting stuff” (Interview 1, Lines 44–45). He indicated that mathematics was the one subject that he forgot the most. However, he had different feelings at the end of the semester: “I would say that after learning about proofs and stuff like that, I actually like [mathematics] more, ‘cause you are like, “Oh now I can figure this out, without knowing anything” (Interview 3, Lines 217–219).

Experience With and Attitude Toward Proof

Andy had limited experience with proofs before this course. He wrote, “High school geometry 9th grade” in response to the survey question asking about his prior experience with proof. In the first interview, he said that he did not think that he had done proofs since high school until this course. His last mathematics class was calculus, but it had been a long time ago, and he did not remember a lot about it. At the end of the semester, in response to the same survey question about his experience with proof, he wrote, “Geometry and this class.”

Andy demonstrated one indication of the authoritarian proof scheme in the initial proof survey when he said that he knew that his answer was correct because his teacher in the ninth grade had told him. He confirmed this when he said in the first interview that he disliked proofs in high school because “I guess I’ve always been that, that person, like, if you tell me, you know, if a teacher is telling me it’s pretty much true, and I believe it, you don’t have to prove it to me” (Interview 1, Lines 193–195). Although Andy’s appreciation for the importance of proofs increased throughout the semester as he valued knowing why, his responses to some of the interview questions indicated that this scheme could manifest itself in various forms.

One of the manifestations of this scheme was the difficulty that Andy had with respect to conceiving the need for proving. When I asked him in the second interview to tell me something that confused him about proofs, he talked about a difficulty that he had:

> When I was in younger schools or whatever, junior high or middle school, … we learned more formulas. … We never really learned why those formulas work. So I guess now when we're trying to prove them I've a little bit of trouble. 'Cause I'm, like, "Well, I know it's that, that's all I need." I know that formula. So, so I found it really hard at the beginning of actually showing why it works, started coming together a little bit. But that's still a little bit hard … to like getting that train of thought like. "Why is this working?" (Interview 2, Lines 188–200)

As this quote also suggests, it is hard for students to get into proving things if they already know that something is true by prior experience and have not had any need to go beyond knowing that it works.

One of the things that possibly contributed to Andy's recognition of the importance of proofs was the way he started thinking about them. He believed that by proving he learned why things worked, and by learning the why, he believed that he gained a better understanding of what he was learning. In each interview, Andy mentioned an aspect of the importance of knowing why. In the first interview, although he believed that he was "still not very good at explaining everything" and "still not very big into writing." He noted: "But I do agree now that it does probably give me a deeper comprehension of math" (Interview 1, Lines 581–582). In the second interview, he said, "It's kinda cool" (Interview 2, Line 202) to think about why and this way of learning "helps you like remember" (Interview 2, Line 207) better. He said, "I know that I can figure it out, so if I ever forget" (Interview 2, Lines 215–216). One advantage of knowing why that he emphasized more in the last interview was remembering better. As in the second interview, he said that he would be able to figure something out even if he forgot it.

One of the things that led Andy to thinking about why things work was the instructor's questioning. Andy talked several times about how he knew that things worked but not why before this course. He also mentioned in the first interview that since he started this program whenever he solved a mathematics problem he would again know that it worked but not why. However, unlike his previous learning experiences, Dr. Benson would ask the class why, and then they would have to prove. Hence, in this class he was learning why even basic mathematical things work, and eventually was getting a better understanding of mathematical concepts.

It is important to note that Andy started realizing the importance of explaining why even before this geometry class. This change had begun when he took another course with the same instructor and also in other courses in the program. When he talked about the difference between knowing that something works and knowing why something works he used the example of cross-multiplying fractions from the arithmetic class that he had taken from the same instructor the previous semester.

As in the first interview, he emphasized the importance of understanding why and also mentioned the satisfying feeling that came along with it. When he talked about proving in the context of a recent experience, which was proving the area formula, he said:

> When I was younger, all I did was count the, how many blocks are in there. And then just write down the answer. And now, like, I guess, we have a deeper understanding of why those pieces fit there, so I guess it makes a little bit more rewarding, I guess. (Interview 2, Lines 46–50)

Andy reiterated in this second interview how Dr. Benson always asked the class, "How do you know?" questions and said that these questions led the class to proving claims. Again, in the context of solving area problems with triangles, he said

> The area questions that she asked, she always asks like after like, um, we have to solve like the triangle as a whole, but in a lot of the cases we have to break down and fit pieces

> together she always asks us like how do you know that piece fits there so you have to prove. (Interview 2, Lines 19–22)

Another incident that could be interpreted as a manifestation of his authoritarian scheme happened during the second interview. When Andy explained how he would know that he was finished with a proof, it became apparent that his appeal to authority was not easy to change. He depended on either his group mates or his teacher to evaluate the correctness of an answer. When working in groups and taking turns to present their solutions he said that he would consider himself as "done with my proof when the last person is oh, oh, that makes sense, now I know what you're doing" (Interview 2, Lines 173–174). Similarly when he was writing a test he would

> try to do as much writing as possible so if, I don't wanna miss anything out so that so until I know that was like alright well this is Benson, she's got it, she knows what I'm talking about and then that's when I pretty much quit. (Interview 2, Lines 175–179)

When asked what he would do if a student came to him with an unfamiliar argument, Andy said

> I would look at it and if it made sense. Or if it wasn't the way we were doing it and I'd never seen that way before I guess I would try different scenarios, and then if it still worked for different scenarios I would give it to another colleague that was maybe a little bit better at math than I was and see if they could come up with any different scenarios to see if it didn't work. And then if not I'd just be like it works go for it; use it if that's what you are able to look at it that way better than go for it. (Interview 2, Lines 726–734)

In the last interview, when Andy talked about how much proof was part of this class, he said "A lot; I'd say it is pretty much everything" (Interview 3, Line 45). He further explained that

> Well, I guess like all the problems like we do, we always had to explain how we know it's true, why it's true, and does it like work every time, so I feel like that was pretty much proof, and why my answer was right, and then you are trying to explain like your way of thinking. (Interview 3, Lines 47–51)

Andy identified proving that the area formula for the triangle works as an instance that helped him understand proof better:

> I guess I just always thought that it was neat, the area of a triangle. I never realized where that came from until we proved it, or how we proved the, yeah, the area of the triangle, I thought that was pretty nifty after we proved it. (Interview 3, Lines 83–86)

When I further questioned him about what he learned about proofs as a result of this experience he said,

> Well, beforehand … the only thing I remembered was, you know, the formula, but then after proving it, I remember what like I learn like, she pretty much taught us like, you try to look for like the properties and stuff like that before and then you try to come up pretty much with your own formula. …, so I think I'll better remember like why it is that … like I could figure it out if I looked at it. (Interview 3, Lines 89–97)

So, he mentioned using prior knowledge and properties of figures to come up with new knowledge.

In the last interview Andy was able to think back and compare what he thought about proofs before this class and what he learned about them during this class. Prior to taking this class he did not see the purpose of proofs; he viewed them as a collection of justified steps but not like a meaningful and coherent whole. He said that in high school geometry they

> Learned about proofs, but it was more of like stating what you did, like step by step like, oh, I know that A equals this because this tells me that, and then, that kinda stuff. It never really was like, we never went into details like the area, or whatever the, what the interior angles equal. … We never really did stuff like she made us do. (Interview 3, Lines 116–124)

Although he thought that both high school proofs and proofs in this class were "both very long" and "that it always took a while," he believed that "the one that she made us do helped me more in the long run, so I would like it better the way she did it" (Interview 3, Lines 127–129). He further explained that

> I feel like the proofs that we did back then, we were just stating the obvious and then hers, you are pretty much have to like you are stating the obvious, but you are building off of that like you are building of that to explain something else kind of, and the other stuff was just like, you are just like listing stuff. (Interview 3, Lines 132–136)

Throughout the semester Andy went from believing that a proof is “just a waste of time” (Interview 3, Line 199) to believing that proofs are “useful in math” (Interview 3, Line 222). He said “I still don’t like doing them, but I really think that it’s helped me more, I actually like math a little bit better than I did when I came in” (Interview 3, Lines 201–203).

Proof in the Abstract

When asked how we know that something is true in mathematics, Andy said, “I guess if you could prove it true multiple, multiple times like, like what we are doing now with the triangle stuff.” By “multiple times” he might have meant multiple empirical examples or multiple different proof methods. Then he brought up the issue with other geometries and concluded that he did not “really know how to answer that question” (Interview 1, Lines 78–89). Although he had this idea that something needs to be proven to be true, his immature knowledge about other geometries seemed to confuse him and lead him to conclude that he did not know how truth is established in mathematics.

Andy thought about proof as an explanation – written or verbal. Writing was the first thing that came to his mind when I said proof. He explained that,

> When you say like prove, uh, like well, especially in that class I feel like an explanation pretty much so it’s either written out or I have to like stand up in the front visually show you. But like when you first say it, the first thing I think of as writing out {like} how, what’s happening, how it’s happening, and why does this happen so I think of writing. (Interview 1, Lines 217–222)

He also thought about proving as showing that something is either right or wrong: “I think more of it like you’re proving why, why it is what it is but also in another sense I feel like you’re trying to prove it wrong at the same time, does that make sense, but you’re proving it right too” (Interview 1, Lines 231–233). In other words when he was asked to prove something he interpreted it as not knowing whether the statement is true or not and hence ‘prove’ actually meant ‘prove or disprove’ for him.

Because Andy was trying to both prove and disprove that something was true, he approached proofs by trying examples to see if they would work and trying to find a counterexample. Although this is a legitimate and logical way to start a proof problem, what seemed to be problematic was that after seeing that several examples worked he would be inclined to believe that the statement was true:

> I feel like in a sense you are also looking to see, you’re looking well like I said prove it wrong. You’re trying different triangles so you’re trying to figure out-you’re trying to prove that one of them doesn’t work, but they-when they start working you’re like all right well, this is right so you proved it right. (Interview 1, Lines 244–248)

When I asked him if he was suggesting that if one cannot find a counterexample then the statement is proven, his first reaction was “I guess in my mind it does” (Interview 1, Line 252), but then he implied that if one kept working at it then it might be possible to find an example that did not work and then said that

> I guess you’d say like we proved it right in those cases, but I don’t know necessarily if we proved it right every time ‘cause we didn’t try every situation. But for the situations we did prove it always works, I feel it’s proved. (Interview 1, Lines 256–259)

As further questioning revealed, he could not talk about the generality of a proof in the abstract at this point in the semester. When I asked him if there was a way that we could prove it for all, he said he was not sure and mentioned the need to make assumptions. Eventually he said

> Well, like she said in class like some of the things you have to like assume. I guess I would after doing it like multiple times I would just assume that it works for all of them. So if I like try like extreme different triangles and it worked every time, I would just assume. I guess assumptions come in I guess so. (Interview 1, Lines 277–281)

As mentioned above, Andy’s newly learned knowledge about assumptions or accepting some things as givens was interfering with what he thought about proofs. In other words, at a point when he was not able to articulate that the generality aspect of a proof would ensure that a statement is true for all cases, he interpreted what he had heard about assumptions as giving him license to assume that a statement is true for all cases when it has been shown to be true for particular examples.

"A lot of work" (Interview 2, Line 110) was Andy's initial reaction when he was asked in the second interview what is involved in proving a mathematical statement. His further explanation showed that he was still focusing on the explanatory role of a proof and also emphasizing the questioning by the instructor that led to explaining why something worked. He also implied that by proving something one would acquire a deeper understanding than just giving the answer by itself:

> I'll say I guess when I think about, when you say proof in math like why is it that way or I'd say like when I get the answer Benson always goes well how did you come to that answer and then why does that work. So I would say that proof is your deeper comprehension of how that, why it works the way it does and why that's the answer so and you've proven why it does. So I guess that would be what I think of when I think of a proof. (Interview 2, Lines 112–118)

Similar to the first interview, he also mentioned that to prove something actually meant for him either prove or disprove.

In the second interview Andy also mentioned using previous knowledge and properties that they have learned to prove something new. When he was talking about the problems that they had recently worked on, he said "I had to pull out like the parallel postulate and stuff like that and then the principles that we learned to show that that piece does fit there" (Interview 2, Lines 23–25).

In the last interview Andy mentioned again both explaining why something is true and proving or disproving aspects of a proof. When I asked him to try to use a different word for proving, he said "I guess I would say a proof is like a process that you do to show why something works or how it works or, and then your process you can determine that it works, when you do it that way" (Interview 3, Lines 154–156). In this last interview, different than before, he said that proof is "visual stuff as well as writing" (Interview 3, Line 244) and added that he was a visual learner so he would always draw a picture off to the side even if it was given.

Proof in Context and Evaluations of Arguments

The way that Andy approached proof tasks was consistent with the way that he thought about proofs as explained in the previous section. Given a statement, he started by saying that he did not know whether the statement was right or wrong, tried a few examples in an effort to prove or disprove the statement, and when the examples started working he seemed to be convinced that the statement was true.

Also as a manifestation of his authoritarian scheme, as Andy was working on a proof task he usually asked for confirmation for his interpretation of the problem or for the correctness of a response that he gave. He was constantly worried about giving a wrong answer, did not seem to be confident about some of his responses, and believed that he made himself look "dumb." I also observed that he did not spend enough time on a problem, and he gave up even when he was very close to getting an answer.

Andy believed that the tearing the corners argument for showing that the sum of the interior angles of a triangle is 180 degrees was a "neat idea" but not a proof (Interview 1, Lines 104–108). He questioned whether the line formed by the three angles was straight line, and he believed that one "could eyeball it and say it's close to 180" (Interview 1, Lines 379–380). That technique was "a simple way to show it," but he thought "it wouldn't suffice or it wouldn't be accurate measuring it" (Interview 1, Lines 390–392). Nevertheless it was "convincing" (Interview 1, Line 394) and "more rememberable [sic]" (Interview 1, Line 404).

He compared this argument with the general argument using the parallel postulate in the following way:

> I liked her. I liked how in Dr. Benson's class we did the parallel lines to show that it's 180. I find that very more efficient way of proving it so … Uh, well like with the piece of paper like I'll never, well, I'm not very good with like cutting and stuff like that, but I know I'll never know like if it was perfectly 180 across. So it could be like 178 the way I have it placed so, I would say it's [the parallel postulate argument] kind of less flawed. (Interview 1, Lines 108–115)

Andy was better able to articulate why a proof would work for all possible cases when he talked about it in the context of an argument in contrast to when he talked about it in the abstract:

> If you had like the top point of the triangle no matter where like where it was like on the parallel line that you made with the other, the base (inaudible) we tried it like different spots on the parallel line where you just I guess you change the angles of the other ones but, uh, no matter where it was on the parallel line we always found that it was 180 so we showed that like we used different triangles and not just like the one corner thing. (Interview 1, Lines 122–128)

This explanation actually sounded like establishing the generality of the proof by making sure that the argument would work the same way no matter where the vertex was or no matter what the triangle looked like. However, it is possible that he did not see the situation as being continuous but each triangle as a discrete case. This might explain why he was able to talk about generality in a context but not in the abstract. I also believe that this way of thinking could explain why he said "prove it multiple times" when he was talking about establishing truth in mathematics rather than because of just a naïve empirical approach to proving.

Andy also liked the argument with the transformation of the lines proving that the interior angles of a triangle add up to 180 degrees (Argument 3 in Appendix C). He thought that "it's pretty interesting" and "a cool way to think about it" (Interview 1, Lines 643–644). As he tried to explain how the proof worked, he seemed to be focusing on understanding why the result was 180. He thought that it was a proof although he could not provide an articulate reason as to why it was a proof – nothing beyond being interesting and cool. However, in comparison to the other general argument using the parallel postulate, he said “I think they both go off pretty much the same way of thinking where you’re using like the alternate interior angles” (Interview 1, Lines 657–659). In other words, he seemed to pay attention to the fact that both arguments depended on the congruence of alternate interior angles. He also could explain why he thought that the argument would work for all triangles (Interview 1, Lines 661–681).

When Andy evaluated an argument that he constructed for a different problem, he used another criterion to claim that the argument would work for all triangles. He believed that it proved it for all triangles because “I would say that it works ‘cause I didn’t give, I didn’t assign any numbers to them so I would say that it works for all triangles” (Interview 1, Lines 708–785). Thus, he was employing the generality criterion to proofs.

In the second interview, Andy employed being able to explain why as a criterion for deciding whether an argument was a proof or whether a proving attempt was successful or not. He did not think that the walking and turning argument was a proof, but it was “just an easier way of thinking about” (Interview 2, Lines 257–258) the problem and “an easy way to remember it” (Interview 2, Line 432). This was consistent with the way that the thought about the tearing the corners argument. As an explanation of why these were not proofs he said “I guess you would need to … show that’s why it works the way it does” (Interview 2, Lines 264–266). Although he said that it did not show it for all triangles when I asked about it, he did not necessarily say that the argument was not a proof because of this reason.

When Andy analyzed the measuring argument he questioned whether a protractor was used and how the triangles were drawn. Then he said that it was not a proof because “there could have been like an error” and “I would like to show like why…they are what they are instead of oh, I measured them so it must be 360. So I guess he, he proved that it’s 360, but he didn’t really tell like why it’s 360 I guess” (Interview 2, Lines 343–349). It seemed that this argument made him believe that the statement was true but did not explain why: “I guess it would show that all of them but I guess it doesn’t really explain why all of them or sh- like I guess prove all, why they’re all like that but I guess, I guess I would believe it after a couple” (Interview 2, Lines 381–383). The next argument measured different types of triangles. It sounds like he had more “confidence” in this one compared to the first, but it still did not “teach” him “why” and at this point he did not know if it proved it for all triangles (Interview 2, Lines 402–425).

Andy liked the general argument given for the same statement. He thought that the argument would apply to every triangle and in his explanation of why he mentioned that the argument made use of previous knowledge about lines. It is important to note that although he could usually explain when asked why a general argument proved it for all cases, he did not use this as a criterion to decide whether an argument was a proof or not. Also, he was not consistent about what he would accept as general.

When asked which argument would get the best grade from his teacher he said “definitely” the general argument for Dr. Benson but he did not mention generality in his explanation. In his reasoning, he focused on the fact that this argument explained why the statement was true. He said that if he wrote the walking and turning argument in a test Dr. Benson would say “yeah it turns around in 360 but why does it turn around in a 360” (Interview 2, Line 556). Based on this explanation it can be argued that Andy had an understanding of the procedures that they had followed in class as demonstrating that something works and then explaining why it worked the way that it did.

I presented Andy with a formula for the number of diagonals in a polygon and several arguments for why the formula was correct. His initial reaction was to try the formula with a pentagon, speculating that the formula seemed like it would give a number that seems too big. Andy found the empirical argument convincing with respect to correctness, but the argument did not explain why the formula was correct.

When Andy first read the general argument for this statement, he said:

> It doesn't really prove to me, but it doesn't really show examples but I just don't know where they're getting all their numbers from. Like, like I don't know where they're getting it's possible to draw *n* minus 3 diagonals. Where does the 3 come in from? (Interview 2, Lines 751–755)

After I explained some of the details that were not written as part of the proof he said that he "would like that a lot better if that's written down" (Interview 2, Line 774). Once given the explanations he would consider the argument as a proof. However, he said that it would prove it for all cases if the empirical argument and the general argument were put together. He explained that "I like how it does the different examples in this one, but if you have all the explanations it would explain why the formula works instead of just throwing it all together" (Interview 2, Lines 793–795). There are several possible explanations for why he wanted to see the examples as part of the general proof, which is different from his work in other contexts. The problem was unfamiliar to him, the context was algebraic rather than geometric, and the general proof did not contain enough detail for him to make sense of it on his own. Any of these factors could have contributed to his desire to meld the two arguments instead of accepting the general argument.

As before, in the third interview when he was given the statement about the sum of first *n* positive integers, Andy started trying out some examples with his goal being to "prove it wrong"

(Interview 3, Lines 378). However, he was convinced that the formula was right after trying a few examples. He also believed that it would work for all cases.

After he analyzed the given arguments for the same problem he said that he liked the visual argument better than just using the examples, but in both cases the patterns would work for higher numbers, so he did not see them as being different in that sense. According to Andy neither argument explained why the formula worked; they just showed that it did (Interview 3).

I hypothesize that focusing on only the explanatory role of a proof might have given Andy a one-sided view of proof and might have prevented him from realizing that generality is also an important aspect of a proof. A question for future research might be "How does one think about generality if one's view of proof is only to explain something?"

Importance of Proofs

As mentioned above Andy thought about proving as explaining why something works, and he believed that explaining why is important for several reasons. First of all, he believed that showing the why gave "a better understanding [and] comprehension of what you're doing" (Interview 1, Lines 309–310). He also indicated that it was important to know why something worked because it helped him to "make connections with other stuff a lot easier" (Interview 1, Lines 207–209).

When I asked Andy about the role of proofs in mathematics, he mentioned being able to communicate something that one understands better. I think it is important to note the that he did not just talk about a proof communicating a mathematical idea but believed that proving something helped to communicate because one had a better understanding of why it worked as a result of proving: "Once you prove it you can visualize how it works and why it's working and then you’re able to better probably communicate with others to tell them how it works” (Interview 1, Lines 319–321).

Because Andy did not take geometry in middle school he was not sure what middle school students could do as far as proofs are concerned. However, he also added that if that was one of the standards that he would need to teach then he would like the students “to be able to do that and understand why” (Interview 1, Line 421). (His dependence on the standards could also be interpreted as an extension of his authoritarian scheme.) He also repeated this view in the last interview. He was not sure if middle school students were able to come up with proofs by themselves. Nonetheless, he recognized the importance of knowing why for his students as well. He believed that if students “are able to connect and use proofs to get a better comprehension, then it would make other stuff easier to comprehend and I think it’s more rememberable [sic]” (Interview 1, Lines 471–478).

In the first interview Andy talked about his practicum and his observation of a mathematics class using the *Connected Mathematics* curriculum materials and noted that students would need to “use previous information from other lessons to figure out the new information and then a lot of their questions were like … why does this work or how could you show that this works.” Thus he thought that proofs would have some place in his classroom because “after taking this class, I never would have thought writing or I guess connecting math would be very helpful, but then after taking this class and during observation I see like it’s got, it’s, I would say it’s a lot more useful” (Interview 1, Lines 424–435).

In the second interview he emphasized the importance of being able to explain for his students rather than just remember something: “I would rather my students be able to explain it than just, uh, just remember” (Interview 2, Lines 530–531). This expectation might have been influenced by his own experience of memorizing and remembering formulas but not necessarily knowing why. For instance among the exterior angle arguments he chose the general one to show to his students because he thought that that explained why the sum is 360 degrees.

In the last interview, Andy explained how he could use proofs in his future teaching as follows:

> I don’t know if I make them going into as much detail, but I would like, you know, I guess a couple like, you know like, maybe not so much as homework, but definitely on tests I would like them to explain why they did what they did. So, I don’t know, give me a little bit more insight than to what they are thinking and stuff like that. Maybe not always, you know, written proofs, but like when you are doing group work, make them like prove to you what they did was right or wrong. (Interview 3, Lines 283–290)

He also expected to see proofs playing out in his classrooms more than during his own experiences as a student:

> I guess I see it more in future classes than like how old school teachers teach, just from like lecturing. I guess now that they are gonna connect in math, I can see it stepping in a lot more than my experience in middle school so. (Interview 3, Lines 303–306)

Cross-case Analysis

Views About Mathematics

Although it was not the goal of this study to focus on prospective teachers’ conceptions of mathematics, some interview questions got at how they thought about mathematics or why they liked it. The analysis of this data and comparison with the participants’ conceptions of proof revealed similarities between some of the participants’ conceptions of mathematics and how they thought about or how they came to think about proof. For example, Kate, who viewed mathematics as “black and white” in the first interview, talked about proofs in the same way in the second interview. She also came to view proving as putting pieces together like “a puzzle” (Interview 2, Lines 26 & 71). This comment resembled the way she talked about mathematics and problem solving in the first interview. Another example is Tammy, who said that she was obsessed with numbers and always wanted to see numbers or examples as part of a proof. Finally, Casey seemed to highlight logical aspects of both mathematics and proofs.

When asked how we establish truth in mathematics, several of the participants did not know what to say. Even when they said proofs, they were not sure how proofs could establish truth. Some of them seemed to be more knowledgeable about how truth is established in science and mentioned how scientists do not talk about proving things but talk about disproving them. It seemed that what they knew about the epistemology of science was influencing what they thought about mathematics. In light of her science background Nora questioned whether there was a way to prove something for all cases in mathematics. Brenda said in the last interview that without knowing science it would have been difficult for her to comprehend some of the ideas that she was learning about mathematics. However, she did not realize that her view of science was confounding her view of mathematics.

Experience With and Attitude Toward Proof

All of the participants except Casey had limited experiences with proofs prior to this course. Kate and Casey indicated that they did not remember doing proofs in high school. Nora, Brenda, and Andy vaguely remembered doing them in their high school geometry classes. The participants rarely mentioned their college level mathematics classes when they talked about their previous experiences with proof. Kate said that proof reminded her of calculus; Tammy said that she did not remember doing proofs in her college level mathematics classes, and Andy said that he had calculus a long time ago so he did not remember it. Only Casey said that she had to prove things during her undergraduate studies. In contrast, Nora mentioned her other mathematics education courses as places where she worked on proving things but not necessarily in a formal way. Furthermore, several of the participants mentioned their previous course with the same instructor as providing an experience where they talked about explaining why things work the way they do.

Although I was expecting these prospective teachers to come to this class with a conception of proof that was overpowered by what they had learned in their high school geometry classes, and I expected these conceptions perhaps to be a hindrance for what they were going to learn in this class, was neither the case. The participants started this course with a wide range of conceptions of and dispositions toward proofs. Although the participants learned some common things about proofs in this course, there were differences in what they paid attention to or how they interpreted events given differences in their previous knowledge, beliefs, and experiences.

First, one student, Tammy, thought prior to this course that proofs could be written only in two-column format. That had been her experience in her high school geometry class. With this being an important element of her conception of proof, she paid attention to the format of how they wrote proofs in this class. She learned that a proof could be written in a paragraph form. At the end of the semester she had learned not only that a proof could be written in different ways but also that content was important regardless of the form.

As I said above, the participants' high school experiences were not as much of a hindrance as I had expected to be. Nonetheless they had not had very favorable experiences with proofs in high school or previous college courses. They either did not remember anything about proofs, except Casey whose experience was different, or they remembered that they hated proofs in high school. When Nora, Tammy and Kate talked about proofs at the beginning of the semester, they interpreted *prove* as "show your work" or "show that your answer to a given problem is true." Furthermore, almost all of them mentioned learning facts by memorizing and not knowing why things worked the way they did before this course. These prospective teachers certainly valued learning why, and I believe they would have benefited from an earlier experience emphasizing these points.

All of the participants except Nora believed that they learned something about proofs in this course. It is interesting to compare Nora with Kate with respect to what they thought they learned about proofs. Whereas Kate believed at the end of the semester that everything that she knew about proofs she had learned in this course, Nora believed that she learned nothing about proofs except for a few specific proofs. Although Nora did not see a connection between what the class did and proofs, because the instructor made no explicit emphasis, Kate suggested that this lack of explicit emphasis might have prevented other students from getting nervous about learning proofs:

> Dr. Benson never said we’re gonna learn about proofs, [or that] this semester is gonna be proving, I don’t remember that and being intimidated by it, so I feel like it was just, she would put it in front of us and let us explore with it. And then she would explain and then show us how the proof is done. But it was never saying, “Here’s a proof. You probably heard of proofs before, and we’re gonna learn about them.” It was throwing us into it, letting us explore it, and then at the end saying, “That’s a proof.” (laughs) You know. So it’s not as intimidating. (Interview 3, Lines 207–215)

Kate also commented on the power of the hands-on activities for motivating the need to prove something:

> I mean it’s the whole idea of inquiry learning, where, you know, she asks you to try to make something, and you make it and then you’re like, “Oh my gosh how does that work?” And then you try to come up with a formula. So I think that made it very successful. And then it gave us a bunch of ideas to use in our classroom, um, when we’re trying to get our kids [to] think that way. (Interview 3, Lines 221–228)

Casey talked about the way that Dr. Benson structured the class so that they would be curious about learning the proofs:

> She makes us wonder about it ourselves before she shows us the proof. So she'll pose the question. And that's something I would like to do with my students, too. She poses a

> question that makes you just curious enough to go, “Well, how could you tell this is true for every single triangle, you know?” ‘Cause you're kinda interested. Well it worked for all of these. How could we actually show it? And then you know once you've done that first proof in class, you get the idea that well you can probably prove most, most of these things. So it makes you start wondering, “Well okay this is true, why is it that way? Could I actually prove that it's that way?” (Interview 3, Lines 158–167).

In comparing this class to other classes where she had learned about proofs, Casey said the following:

> I think the difference is my other math professors have kind of come in and, and just kinda shown us, “Well, this is a proof, and this is it. And this is true.” And she makes us curious first. So we actually want to listen and learn the proof. (Interview 3, Lines 177–180)

These comments also suggest that the course was conducted very differently from Alibert’s (1988) observation of a traditional classroom where

> the students have no interest in proof as a functional tool. Proof is only a formal exercise to be done for the teacher. There is no deep necessity for it. It is true, of course, that the problems which the teacher’s proofs resolve have not usually been appropriated by the students. (p. 31)

Andy emphasized the way that Dr. Benson guided the class to the right answer without necessarily telling that they were right or wrong:

> She doesn’t really give us the answers right away. And even if we are wrong, she lets us keep going until we realize we are either wrong or we figure out what we were doing wrong and start doing it right. And then if not, she just helps us on the side, like, “Hey, this is what you are doing wrong, but what could you do to change that?” Or something like that. So, she doesn’t really give you the answer, but she guides you on the path to the right answer. (Interview 3, Lines 316–322)

Casey was unique in that she started this course with a developed understanding of proof emphasizing establishing truth, generality, and logical aspects. Hence, she believed that rather than learning about the concept of proof, she had learned new proofs and developed her repertoire of proof strategies. Throughout the semester, her definition of a proof seemed to stay rather stable and did not include the explanatory aspect that other participants came to value.

Even when asked in the last interview whether a given proof explained why the statement was true, she focused on whether the steps of the proof were explained rather than whether the proof was explanatory or not.

It appeared that different students interpreted the same class events in very different ways. In particular, they interpreted differently Dr. Benson's choice not to prove things that were beyond their mathematical capability. When asked about a class event that influenced the way she thought about proofs, Brenda described her confusion about an incident when the instructor chose not to prove a given statement. Brenda said:

> It was while we were doing proofs. And [Dr. Benson] said, "Well, you have to assume that it works for all triangles. Even though we can't do it an infinite amount of times, you know, we can pretty much assume that this applies for all triangles." So I guess that mention of infinite but we can't, I mean 'cause she would say, "Well so then we can prove or we can assume that it does work for all, you know, whatever, triangles or whatever." But she still always puts that "Of course we can't do it an infinite amount of times." But so that's where I guess I'm getting stuck. That has made more of an impact on me than, um, yeah, it's made more of an impact on me because once again I've always assumed math as an exact science. (Interview 2, Lines 1031–1041)

The instructor's reference to not being able to do an infinite number of examples led Brenda to conclude that it was impossible to prove a given statement for all cases; in other words, a proof could not be universal.

In contrast, Casey recognized that some things could be proven but the instructor was choosing not to. She gave the following example:

> There's some things that we didn't actually formally prove. We just sort of saw that it was probably true like the, I think the volume, I think the volume of the cylinder was like that, I can't remember. Maybe it was the volume of the sphere where we didn't actually formally prove it because it would probably take some calculus to prove it for every case. (Interview 3, Lines 33–38)

Casey was aware of the fact that the proof was omitted because it required higher-level mathematics, whereas Brenda came to believe that a universal proof was impossible.

Both Brenda and Adam started this course with limited knowledge about proofs and developed an understanding of a proof as confirming what is already known or believed to be true during the semester. They likely drew this conclusion from many class activities that involved experimenting with examples to convince oneself that something, such as a formula, was probably true and to figure out how the formula worked. Presumably the instructor intended for the students to use this experimentation and understanding of how the formula worked to build the proof, but these students interpreted proof simply as confirmation of what they already knew. This conclusion reflects their misunderstanding of the purpose of the class activities.

I also noticed a significant contrast between some of the participants with respect to their dispositions toward proving claims. For example, when she started constructing a proof, Casey would say that she did not know whether she was on the right track or not. She would try a strategy, and if it did not work, she either would check her work or would be comfortable with starting over with a new strategy. She did not expect to solve the problem in a short amount of time and demonstrated persistence. In contrast, Andy would try something, and if it did not seem to be working – despite the fact that he actually could be on the right track and might have made a calculation error – he would give up. I think it is important for teacher educators to recognize the differences between the two approaches and try to find ways to develop the kind of dispositions that Casey possessed as proving often involves false starts and missteps.

Proof in the Abstract

The cross-case analysis revealed three themes that were common to several students when they talked about proof in the abstract. The first theme involved a difference in their understanding of the generality of a proof when they talked about it in the abstract versus in context. The second theme brought together factors contributing to participants' confusions about what a proof is. The third theme revealed that an important aspect of participants' proof conceptions when they talked about proof in the abstract was explaining why. In other words, the common element of participants' definitions of a proof evolved to be explaining why something works.

As previous research suggests, some participants in this study started this course with the belief that an empirical argument is a proof or, in other words, showing that a statement is true for several examples proves it. As in previous research, these participants were sometimes aware of the limitations of using examples but would choose extreme examples to try. Three participants - Tammy, Nora, and Andy - my analysis indicates that they were able to articulate why an argument was general when presented with it and asked to analyze it, despite showing evidence of holding an empirical proof scheme. All three said that they would try examples to prove a statement, rated empirical arguments as proofs, and questioned how a proof could be general. This finding introduces a subtlety to the way we determine whether or not a student holds an empirical proof scheme (Harel & Sowder, 1998).

For example, in her second interview, when Tammy talked about proof in the abstract, she said that she would need to try different examples and see that they all came out to be the same to prove a statement. On the other hand, in that same interview, when she analyzed the general argument for the sum of the exterior angles of a triangle being 360, she could see and explain why that argument would work for all possible cases. Similarly, Nora also could explain why that same argument was general in her second interview even though she questioned how a proof could prove something for all possible cases when she talked about proof in the abstract. This finding contributes to research by extending Felton's (2007) findings. Felton found differences between preservice teachers' conceptions of proof when analyzed in the context of analyzing students' work versus in the abstract. The finding of the present study suggests that even though preservice teachers cannot talk about the generality aspect of a proof in the abstract, they may be able to explain why a given argument is general.

It was clear that some students who had an empirical understanding of proof at the beginning of the semester were not necessarily at a point where they could talk about the generality aspect in the abstract even by the end of the semester. They needed either more time or more explicit instruction as to how a proof could be general. One factor that may have contributed to the enhancement of an empirical scheme was the practice of proving by cases. It was difficult for some students to differentiate between a proof by cases and an argument by examples.

Furthermore, without having a robust understanding about proofs or perhaps the axiomatic system, taking the parallel postulate as a given and the mention of the geometries where this postulate would not hold true seemed to create tension for several students. Some students had difficulty understanding that the parallel postulate was a given, which caused them to question the soundness of any proof involving the postulate. They were concerned that because the parallel postulate had not been proven, any proof that relied on the parallel postulate could be invalid. These students were confused about the distinction between a postulate and a theorem and struggled to make sense of the boundary between what had to be proven and what could be accepted without proof. Because the students did not understand the nature of a postulate, they were confused by the idea that the parallel postulate would not hold in spherical geometry. This confusion led some to conclude that proving is not an exact science and there is no such thing as absolute truth.

Harel and Sowder (2007), based on a review of the literature, concluded that teachers "do not seem to understand other important roles of proof, most noticeably its explanatory role" (p. 48). I had hypothesized that, based on its given objectives, this course would help preservice teachers develop an appreciation of this role of proof. My analysis indicated that this indeed happened. In addition, it also seemed that explaining why not only was an important role of proof but also a defining characteristic of it as all participants except Casey mentioned that a proof explains why when they talked about proof in the abstract; in other words, when they defined a proof. The participants also talked about this aspect as an important reason for learning proofs. These results are also in line with findings of another research study where students in a problem-based classroom were compared with students in a traditional classroom. Yoo and Smith (2007) concluded that the problem-based class students had a more humanistic approach to proof, which they described as the tendency to try to make sense of arguments.

Proof in Context and Evaluations of Arguments

The students differed in their acceptance of different arguments as proofs or not depending on whether they held an empirical proof scheme. Despite this expected fact, all participants tried to make sense of given arguments before accepting them as proofs rather than accepting them on merely ritual or formal grounds. In other words, they employed "explain why" as a criterion to decide whether an argument was a proof or not. Whether it was a formula or the sum of angles for a figure, they wanted to know why it worked. They wanted to know where minus 3 or divided by 2 in a formula came from. They wanted to be able to explain why the sum of the exterior angles of a triangle was 360 by using other properties that they knew about triangles. One of the instructor's major goals for the course was for the students to understand why various mathematical ideas worked and for them to develop a proclivity to investigate why, and the students appeared to achieve these goals. This finding again confirms the finding of Yoo and Smith (2007) that in a problem-based mathematics course, students are likely to develop a sense-making approach to proofs.

Another commonality that I noticed was the reason that several participants provided when they believed that an argument was not a proof. Tammy, Kate and Brenda did not necessarily reject an argument because it was not general enough or because it involved measurement that could not be exact. Instead, they tended to reject it because no mathematics was involved. Although it was not clear what they meant by this, I believe that they were paying attention to whether the argument used previously established knowledge or the properties of a figure. When some participants rated an argument as a proof, they mentioned using previous knowledge or properties of a figure a characteristic of a proof. It is likely that some participants developed an understanding of a proof that focused on these as a result of the emphasis in the class on using the properties of a figure given to the students to figure out something new about that figure.

I was able to identify two things that helped some of the participants distinguish arguments with different levels of validity. Nora mentioned in the last interview that comparing different arguments might have helped her to realize what could be accepted as a proof and which arguments were more reliable than others. Brenda, who came to believe that it was not possible to prove a statement for all possible cases, revealed an understanding that an argument could be general when she talked about the use of a variable in an algebraic context. Perhaps an algebraic context, more than a geometric one, lends itself to a more fruitful discussion of whether an argument or proof could be general.

Importance of Proofs

Three participants - Andy, Kate and Tammy - said that they learned in this course how important proofs were. They gave several reasons that proofs were important: communicating, convincing, building on knowledge, establishing truth, explaining why, and creating a common base of knowledge. In all, they mentioned all the roles that proof plays in mathematics outlined by Knuth (2002b).

A common reason that almost all participants mentioned was that it was important to learn proofs because it was important to know why. Tammy, Kate, and Nora further believed that it was important for them as teachers to learn proofs and hence why things work because as teachers they would need to explain things to their students. Casey also emphasized the importance of being convinced of a statement's truth for her as a teacher so that she could teach that to her students. The way that the participants thought about proofs and the way that they viewed themselves as teachers made it important for them to learn proofs. Knowing proofs would provide them with the explanations that they would need to know to teach mathematics.

On the other hand, they had reservations about the place of proof in middle school. They definitely valued having students explore, investigate, and be able to reason for themselves. However, Andy, Kate, and Tammy mentioned that as they had not worked with middle school students yet, they did not know what to expect from them as far as proofs were concerned. Brenda and Tammy, although Tammy had admitted that she did not know enough about middle school students, indicated that they would expect only gifted students or students who were more interested in mathematics to be able to come up with proofs as opposed to "regular" students. Casey also reserved her presentation of proofs for more advanced students who would question things. This finding is similar to Knuth's (2002b) finding in his research with secondary school teachers. Several teachers in Knuth's study did not consider proof to be a central idea throughout secondary school. They further believed that it was for advanced mathematics classes and students who would study in a mathematics related area in college. One difference between the participants in the present study and Knuth's participants was the amount of experience they had had. Knuth's participants were experienced inservice teachers, and participants in this study were preservice teachers. That raises the question of whether experience alone would influence the way these preservice teachers would start thinking about the role and place of proofs in middle school classrooms or whether they would need more explicit guidance.

Furthermore Knuth (2002a) suggested that the meaningful experience that teachers have with proof might eventually influence the way that they think about proofs in the classroom:

> In short, teachers need, as students, to experience proof as a meaningful tool for studying and learning mathematics. Experiences of this nature may influence the conceptions of proofs that they develop as teachers, and these ideas, in turn, may influence the experiences with proof their students will encounter in secondary school mathematics classrooms. (p. 403)

However, my study suggests that Knuth's conclusion may not hold. The participants in the present study had considerable exposure to proof in the manner advocated by Knuth but did not see it as something they would do with middle school students. Furthermore, according the results of a study by Wilcox et al. (1991) investigating the influence of a teacher education program on prospective teachers' pedagogical content knowledge in mathematics, although the studied program "seemed to be a powerful influence on prospective teachers' thinking about mathematics for themselves, its impact did not seem to carry over to how they thought about mathematics for young children" (Borko & Putnam, 1996, p. 693). Hence, the participants in this study might have difficulty transferring their experiences to similar experiences for students.

CHAPTER 5

SUMMARY AND CONCLUSIONS

Summary

Proof has an undeniably important role both in the activities of mathematicians and the development of mathematics. Accordingly, the importance and place of proof in mathematics education has been one of the emphases in the reform movement. NCTM's *Principles and Standards for School Mathematics* (2000) states that students in grades K-12 should be able to:

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs; and
- select and use various types of reasoning and method of proof. (p. 56)

Although a large number of studies have revealed students' difficulties with proof (e.g., Chazan, 1993; Harel & Sowder, 1998; Healy & Hoyles, 2000), there is also evidence that students can learn mathematical reasoning even in the elementary grades (Ball & Bass, 2003; Carpenter et al., 2003). However, it is also recognized that "the extent to which mathematical ideas such as proof and justification appear in classroom discourse will be influenced by both the teacher's choice of task and the questions and comments she makes during class, which are, in turn, influenced by the teacher's knowledge of proof" (Peressini et al., 2004, p. 81). Hence, it is crucial for teachers to be well equipped to teach mathematical reasoning and proof.

Given the number of studies of preservice elementary school teachers (e.g., Martin & Harel, 1989; Simon & Blume, 1996), inservice elementary school teachers (e.g., Ma, 1999), preservice secondary school teachers (e.g., Jones, 1997), and inservice secondary school teachers (e.g., Knuth, 2002a, 2002b), as well as undergraduate mathematics majors (e.g., Harel &

Sowder, 1998). In this study I focused on preservice middle school teachers' conceptions of proof. I followed 6 preservice middle school teachers throughout a course called Geometry for Middle Grades Teachers, a mathematics content course.

The following research questions guided my study:

1. How do preservice middle school teachers' conceptions of proof evolve as they participate in a mathematics content course titled Geometry for Middle Grades Teachers?

2. How do preservice teachers' entering conceptions of proof influence what they learn in this course with regard to proof?

3. What do preservice middle school teachers believe constitutes proof for middle school students?

The main source of data for this study was three semi-structured interviews conducted with six participants at the beginning, middle, and end of the semester. These data were supplemented by surveys given to all students taking the class at both the beginning and the end of the semester. After examining each participant as a case and focusing on the development of their conceptions of proof throughout the semester, I went through each case and looked for both similarities and patterns and also contrasting differences.

Although the participants had some similarities, they also had unique beliefs and hence learned different things about proofs in the course. One student learned that proofs could be written in ways other than the two-column format she had learned in high school, and another student learned proofs of particular theorems and proof techniques that she did not know before. One student believed that she learned everything she knew about proofs in this course, and another believed that she did not learn anything about proofs other than the few specific proofs. The instructor deliberately emphasized reasoning and sense making but did not explicitly focus on the notion of proof. For some students, this subtle approach to proof avoided anxiety and allowed them to experience proof without realizing what they were doing. For other students, however, the lack of an explicit discussion of proof and proof techniques led to confusion about things such as the difference between a postulate and a theorem and whether it was possible to prove anything at all in mathematics.

A common theme was that the participants started to think about proofs as arguments that explain why a statement works and to realize the importance of proofs for themselves as teachers. They also seemed to understand the importance of using established knowledge and known properties of a figure to deduce new properties of that figure. Another common theme for the three participants who started with empirical proof schemes was the shift in their understanding of generality when they talked about proof in the abstract versus in the context of evaluating an argument. When these participants were asked to define a proof or to explain whether it was possible to prove a statement for all possible cases, they demonstrated empirical understanding. In other words, they talked about proofs as showing that a given statement is true for several examples and could not say whether a proof could employ a general argument. However, when they were given general arguments to evaluate, they were able to identify the argument as being general and explain why.

This study has important implications for both teacher education and also for future research. Although this content course helped the preservice teachers develop an appreciation for proofs for themselves as teachers, methods courses need to focus on students' conceptions of proof and the role of proofs in mathematics classrooms so that preservice teachers can develop an understanding of the importance of proof for students as well. The findings also point out the importance of explicit attention to how to handle the initial introduction of the axiomatic system, the parallel postulate, and treatment of proofs by cases because these topics were confusing for the study participants. The lack of explicit treatment of these topics led the students to form invalid conclusions. Because some of the participants in this study were preparing to be mathematics and science teachers at the middle school level, there was an interplay between their knowledge of how truth is established in science and how it is established in mathematics. Most of the students seemed to have a better understanding of the epistemology of science than of mathematics. Because the two epistemologies are so different, an explicit discussion about the differences between the two subjects might help preservice teachers better understand the characteristics of mathematics as a discipline.

Conclusions

Some of the literature regarding both students' and teachers' conceptions of proofs suggests that students' only exposure to proofs would be during their high school geometry classes in the context of two-column proofs emphasizing format rather than substance. Herbst (2002b) described this treatment of proof as the "reduction of mathematical reasoning to its logical, formal dimensions" (p. 285). Hence, I expected that the students would start this course with a formal understanding of proof and negative emotions toward proving. Although several of them expressed negative emotions towards proving, a lack of experience and knowledge was more dominant than a formal understanding of proof. Some of them vaguely remembered doing proofs in high school, whereas others – although rarely – had more recent experiences in college mathematics classes.

The participants started this course with very different prior experiences with proof, dispositions toward proving, and knowledge about proofs. An important conclusion of the study is that what the preservice teachers learned with respect to proofs in this course and which aspects of the course they paid attention to depended on their initial conceptions. Whereas one student who had a more formal and empirical understanding of proof focused on how proofs were written in this course, another student with a more developed understanding of proof at the beginning of the semester emphasized learning different proof techniques.

During the interviews, four participants mentioned that before this course they had not learned why things work the way they do in mathematics and where formulas or algorithms come from. They remembered their high school experiences and other mathematics classes as experiences of memorization rather than understanding and sense making. As they learned the reasons behind formulas and algorithms in this course, learning mathematics became a more meaningful experience for them. These preservice teachers were motivated by and valued learning why. I believe that they would have benefited from learning this earlier because the earlier they start making sense of mathematical concepts the more likely that they will expect to have similar experiences in their following mathematics classes.

Given the differences in their understanding of proofs, the participants tended to interpret some of the activities that they did in class in very different ways. In particular, they interpreted the parallel postulate, proofs by cases, and the omission of proofs that required higher level mathematics quite differently. A common problem for students who did not have a robust understanding of proofs and geometry as an axiomatic system was the introduction of the parallel postulate as a given and the fact that this postulate holds true only in Euclidean geometry. Some participants had difficulty distinguishing what is a given and what is a proof and understanding that the parallel postulate could not be proven. Furthermore, learning that the parallel postulate cannot be proven confounded their view of what proof is and whether a proof can be invalidated or not. This finding raises the question of when it is appropriate to introduce the axiomatic structure of geometry to students. The axiomatic system is complicated, and the way it is introduced needs more explicit attention and discussion. This finding also leads me to question whether geometry is an appropriate place to introduce proofs for the first time. Another conclusion that leads to questioning the appropriateness of geometry as the first place to introduce proofs was the difference between the students' abilities to distinguish between empirical and deductive arguments in a geometric context and an algebraic context. Sometimes it was easier for the students to distinguish an empirical argument from a general argument in the context of algebra and to notice that a general algebraic argument could prove a given statement for all possible cases because of the use of a variable.

The way that class activities were structured seemed to make those students who had limited experiences with proof believe that proofs confirm something that they already believe to be true. Although it is good to have an intuitive understanding of the truth of a statement to be proven, this approach caused problems for some students. Furthermore, the way that proofs were taught in this class – not explicitly mentioning the word *proof* or talking about what it means to prove but rather asking the students to explain why – seemed to have influenced the students in different ways. Although this way of teaching seemed to be beneficial with respect to not making students nervous about proofs, it deprived other students of the opportunity to confront some of their misconceptions about proof. In addition, one student said she did not learn much about proofs in this course. Based on the data I collected I can say confidently that she learned about proof, but the fact that she did not realize this consciously is problematic, given that she will be a teacher.

Another important conclusion of this study is that the students were unable to talk about the generality of proofs in the absence of a general argument if they had not developed a robust understanding of the fact that proofs prove for all cases. These students could not explain why a proof ensures generality, or they revealed an empirical understanding when asked to define a proof. Yet, when they were asked to evaluate a general argument, they realized that it was different than an empirical argument in some respects and could explain why the given argument would apply to all possible cases. This finding extends the finding of Felton (2007), who talked about the change in preservice teachers' conception of proof when it was analyzed in the abstract versus in the context of evaluating student work.

Another conclusion of the study was that the participants who had a science background or who had science as one of their majors seemed to have a better understanding of establishing truth in science. They knew more about scientific investigation and the importance of disproving in science. However, they tried to interpret mathematical proof with that lens and constructed unconventional links between how truth is established in science and mathematics, which sometimes confounded their views of mathematical proof. A better understanding of what a theorem meant in science could be the result of a more explicit attention to the epistemology and nature of science in the field of science education.

Finally, a common theme was that most of the participants questioned the value of proofs for middle school students. Some of them said that they did not know what to expect from middle school students as far as what they can prove or from middle school curriculum materials. Furthermore, they also indicated that they would expect proofs from gifted students or students who are more interested in mathematics, which is consistent with the findings of earlier studies (Knuth, 2002b).

Implications

Although the participants in this study developed an appreciation for the importance of proofs for themselves as teachers, they did not know what to expect from their future middle school students as far as proofs were concerned. They also indicated that they would use less general arguments in their classrooms and would expect proofs only from advanced, gifted students or students who are more interested in mathematics. An important implication is that as teacher educators we need to focus on the role and place of proof in school mathematics in methods classes. This focus could span a wide variety of topics ranging from the place of proof and reasoning in the NCTM standards, what research tells us about students' conceptions of proof, and what kinds of tasks and types of questions are likely to lead students to a better understanding of proofs. There is evidence that even young students can engage in proving activities (Ball & Bass, 2003; Carpenter et al., 2003), and thus it is important for prospective teachers to be equipped to reveal that potential.

The student who believed that she did not learn a lot about proofs in this class indicated that being asked to evaluate different arguments might have helped her to realize the difference between these arguments. Furthermore, several students who seemed to have an empirical understanding of proof in the abstract could explain why an argument that was given to them to evaluate was general..The implication is that this research technique might be a promising instructional strategy to help students develop an understanding of what is an acceptable proof. A similar suggestion was also made by Selden and Selden (2003). My analysis further suggests that talking about a proof in the context of analyzing, evaluating, and comparing arguments with various levels of validity is likely to help students understand the limitations of an empirical argument and understand the characteristics of a general one.

Given that there are students who have a better understanding of the nature of science and how truth is established in science and the fact that the two disciplines have very different epistemologies that have the potential to be confounding, an explicit discussion about the epistemologies of mathematics and science could benefit these students. This idea is also advocated by Conner and Kittleson (2009). In addition, Martin and Harel (1989) suggest that "attention to the similarities and differences of mathematics to everyday life may also be helpful" (p. 51).

A final implication is that geometry may not be appropriate as the first place to teach proof. Given that it was difficult for the preservice teachers in the study to conceive of the parallel postulate as a given and to comprehend the properties of an axiomatic system, algebra might be a better context to introduce proofs rather than geometry. Because of the use of variables in general arguments in algebraic contexts, preservice teachers might better understand the nature and use of a general argument if they were introduced to the idea in an algebraic context first. The discussion of which context is better to introduce proofs is not new. Wu (1996) argued that there was an "extramathematical reason" (p. 228) to prefer geometry as the first course to introduce proofs because "in learning to prove something for the first time, most people find it easier to look at a picture than to close their eyes and think abstractly" (p. 228). However, Harel (1999) observed in his research the problem with this approach as being "its robust influence on students in an advanced stage in their mathematical education" (p. 604). If "students are unable to be detached from a specific context" like the context of intuitive Euclidean space in geometry, they are called to have *contextual* conception (p. 603). A manifestation of the contextual proof scheme is that it makes it difficult for students to realize the nontriviality of a statement like $-1(x) = -x$. I believe that the findings of this study add another dimension to this discussion.

Future Research

In the first interview several of the participants recalled their experiences in their previous content course focusing on arithmetic with the same instructor. They seemed to have already started developing an understanding of proof as explaining why things work the way they do in mathematics and also an appreciation for being able to explain why. So, a study that looks at preservice middle school teachers' conceptions during the first class that is taught with a problem-solving orientation might allow for the documentation of the development of preservice teachers' ideas about proof from the beginning of their professional education. Such a study would also provide an opportunity to compare preservice teachers' knowledge about the role of proof with respect to formulas and algorithms with the findings of this study relative to geometry.

My analysis of the data revealed some differences between the algebraic context and geometric context with respect to realizing how an argument could afford to be general. A study exclusively investigating the differences between algebraic contexts and geometric contexts with respect to students' validations of proofs or initial realizations that a proof can be general could shed further light on the differences I hypothesized. This proposed study could be carried out with different populations and also with a large number of participants and aim to find out if the differences observed in this study apply to a large number of different populations. The proposed study could also look for other differences between these two contexts with respect to understanding other characteristics of proofs and even whether one of the contexts lends itself more easily to introducing proofs.

I believe that the finding that suggests a relationship between conceptions of mathematics and conceptions of proof also has some implications for a future study. A study focusing exclusively on investigating the relationship between beliefs about mathematics and proof could reveal further information about this relationship.

In a future study I would also like to investigate the differences between students who view proofs as being exclusively explanatory and students who view proofs as arguments that establish the truth of a given statement for all possible cases. My working hypothesis is that students who exclusively focus on proving as explaining why tend not to pay attention to other characteristics of proofs if they have not already developed a robust understanding of a proof. This study could be extended to investigate what it takes to develop an understanding of a proof that encompasses several roles that a proof could play both in mathematics and also in the mathematics classroom.

Another future study I would like to carry out would be to follow participants during their initial teaching experiences and investigate how their beliefs that were manifested in preservice courses would play out in their support of student reasoning in their own classrooms. Because the preservice teachers in this study had rather limited views about the role of proof in the middle school classroom, a future study could involve integrating a unit on students' conceptions of proof and the role and place of proof in the mathematics classroom into a methods course and investigating the influence on preservice teachers' beliefs about how proofs play out in school mathematics.

Concluding Comment

This study not only answered but also raised several questions with respect to the teaching and learning of proof for prospective middle school teachers and how to prepare them for their future teaching responsibilities. Although there are certainly aspects that these teachers lacked, I believe that the results of this study are promising in the sense that a nontraditional content class helped prospective teachers start valuing proofs and become aware of the explanatory power of proofs at least for themselves as teachers.

REFERENCES

Alibert, D. (1988). Towards new customs in the classroom. *For the Learning of Mathematics, 8*(2), 31-35, 43.

Ball, D. L. (1999). Crossing boundaries to examine the mathematics entailed in elementary teaching. In T. Lam (Ed.). *Contemporary Mathematics.* Providence: American Mathematical Society.

Ball, D., & Bass, H. (2000). Making believe: The collective construction of public mathematical knowledge in the elementary classroom. In D. Phillips (Ed.), *Constructivism in education,* (pp. 193-224). Chicago: University of Chicago Press.

Ball, D., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, J., W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 27-44). Reston, VA: National Council of Teachers of Mathematics.

Ball, D., Hoyles, C., Jahnke, H., & Movshovitz-Hadar, N. (August, 2002). *The teaching of proof*. Paper presented at the International Congress of Mathematicians, Beijing, China.

Bell, A. (1976). A study of pupils' proof-explanations in mathematical situations. *Educational Studies in Mathematics, 7,* 23-40.

Bernard, H. R. (2002). *Research methods in anthropology: Qualitative and quantitative approaches* (3rd ed.) Walnut Creek, CA: AltaMira Press.

Borko, H., & Putnam, R. (1996). Learning to teach. In R. Calfee & D. Berliner (Eds.), *Handbook of educational psychology* (pp. 673-708). New York: Macmillan.

Calderhead, J. (1996). Teachers: Beliefs and knowledge. In D. Berliner R. Calfee (Eds.), *Handbook of educational psychology* (pp. 709-725). New York: Macmillan.

Carpenter, T., Franke, M., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics*, *24*, 359-387.

Cobb, P., Wood, T. Yackel, E., & McNeal, B. (1992). Characteristics of classroom mathematics traditions: An interactional analysis. *American Educational Research Journal, 29*(3), 573-604.

Conner, A. (2007). Student teachers' conceptions of proof and facilitation of argumentation in secondary mathematics classrooms. (Doctoral dissertation, The Pennsylvania State University, 2007). *Dissertations Abstracts International, 68*/05, Nov. 2007 (UMI No. AAT 3266090).

Conner, A., & Kittleson, J. M. (2009). Epistemic understandings in mathematics and science: Implications for learning. In F. Lin, F. Hsieh, G. Hanna, & M. de Villiers (Eds.), *Proceedings of the ICMI Study 19 Conference: Proof and proving in mathematics education*. Vol. 1, pp. 106-111. Taipei, Taiwan. National Taiwan Normal University: Department of Mathematics.

Davis, P. J., & Hersh, R. (1997). *The mathematical experience.* New York: Houghton Mifflin.

Felton, M. (2007). Context and preservice teachers' conceptions of proof. In T. Lamberg, & L. R. Wiest *(Eds.), Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.*

Fernandez, C. (2005). Lesson study: A means for elementary teachers to develop the knowledge of mathematics needed for reform minded teaching? *Mathematical Thinking and Learning, 7*(4), 265-289.

Ginsburg, H. (1997). *Entering the child's mind: The clinical interview in psychological research and practice.* New York: Cambridge University Press.

Hanna, G. (1995). Challenges to the importance of proof. *For the Learning of Mathematics, 15*(3), 42-49.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education III* (pp. 234-283). Washington, DC: Mathematical Association of America.

Harel, G., & Sowder, L. (2007). Toward a comprehensive perspective on proof, In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning,* National Council of Teachers of Mathematics.

Hays, P. A. (2004). Case study research. In K. deMarrais & S. D. Lapan (Eds.), *Foundations for research: Methods of inquiry in education and the social sciences* (pp. 217-234). Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education, 31*(4), 396-428.

Jones, K. (2000). The student experience of mathematical proof at university level. *International Journal of Mathematical Education, 31*(1), 53-60.

Knuth, E. J. (2002a). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education, 33*(5), 379-405.

Knuth, E. (2002b). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education, 5*(1), 61-88.

Knuth, E. & Sutherland, J. (October, 2004). Student understanding of generality. *Proceedings of the Twenty-sixth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, 561-567.

Kotelawala, U. (2009). A survey of teacher beliefs on proving. In F. Lin, F. Hsieh, G. Hanna, & M. de Villiers (Eds.), *Proceedings of the ICMI Study 19 Conference: Proof and proving in mathematics education*. Vol. 1, pp. 250-255. Taipei, Taiwan. National Taiwan Normal University: Department of Mathematics.

Ma, L. (1999). *Knowing and teaching elementary mathematics.* Mahwah, NJ: Erlbaum.

Manaster, A. B. (1998). Some characteristics of eighth grade mathematics classes in the TIMSS videotape study. *American Mathematical Monthly, 105*(9), 793-805.

Martin, W. G., & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education, 20*(1), 41-51.

Maxwell, J. A. (2005). *Qualitative research design: An interactive approach.* Thousand Oaks, CA: Sage.

Merriam, S. B. (1998). *Qualitative research and case study applications in education*. San Francisco: Jossey-Bass.

Mingus T. T. Y., & Grassl, R. M. (1999). Preservice teacher beliefs about proofs. *School Science and Mathematics, 99*(8), 438-444.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics, 27,* 249-266.National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Patton, M. Q. (2002). *Qualitative research and evaluation methods* (3rd ed.). Thousand Oaks, CA: Sage.

Peressini, D., Borko, H., Romagnano, L., Knuth, E., & Willis-Yorker, C. (2004). A conceptual framework for learning to teach secondary mathematics: A situative perspective. *Educational Studies in Mathematics, 56*(1), 67-96.

Ross, K. (1998). Doing and proving: The place of algorithms and proof in school mathematics. *American Mathematical Monthly, 3*, 252-255.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* Orlando, FL: Academic Press.

Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education*, *34*(1), 4-36.

Simon, M., & Blume, G. (1996). Justification in the mathematics classroom: A study of prospective elementary teachers. *Journal of Mathematical Behavior*, *15*, 3–31.

Smith, J. C. (2006). A sense-making approach to proof: Strategies of students in traditional and problem-based number theory courses. *Journal of Mathematical Behavior, 25*(1), 73-90.

Sowder, L., & Harel, G., (2003). Case studies of mathematics majors' proof understanding, production, and appreciation. *Canadian Journal of Science, Mathematics and Technology Education, 3*, 251-267.

Wilcox, S., Schram, P., Lappan, G., & Lanier, P. (1991). *The role of a learning community in changing preservice teachers' knowledge and beliefs about mathematics education* (Research Report 91-1). East Lansing: Michigan State University, National Center for Research on Teacher Education.

Yackel, E., & Hanna, G. (2003). Reasoning and proof. In J. Kilpatrick, J., W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics*, (pp. 227-236). Reston, VA: National Council of Teachers of Mathematics.

Yoo, S. (2008). *Effects of traditional and problem-based instruction on conceptions of proof and pedagogy in undergraduates and prospective mathematics teachers*. Doctoral dissertation, The University of Texas, Austin.

Yoo, S., & Smith, J. C. (2007). Differences between mathematics majors' views of mathematical proof after lecture-based and problem-based instruction. *Lamberg, T., & Wiest, L. R. (Eds.). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.*

APPENDIX A

SURVEYS

SURVEY 1

Name: Date:

Email:

1. What makes mathematics different from other subjects?

2. What is the sum of the measures of the interior angles of a triangle?

3. How do you know that your answer to the second question is correct?

4. How do you know that your answer to the second question is correct for all triangles?

5. How do we know that something is true in mathematics?

SURVEY 2

Name: Date:

Email:

1. What is your prior experience with proof?

2. What is the purpose of proof in mathematics?

3. How do you feel about mathematical proof–like/dislike, find it hard/intriguing, etc.?

4. How do you feel about yourself as a constructer of proofs?

5. What would be your reaction to the following statement: “I know that the sum of the measure of the interior angles of a triangle is 180 degrees because I drew several different triangles and measured the angles. Every time I measured, the angles added up to 180 degrees. Therefore, the sum of the measures of the interior angles of any triangle must be 180 degrees.”

APPENDIX B

MATHEMATICAL PROOF SURVEY

Name: ___________________________________ Instructor Name: ____________________

Email: ____

The following questions address your views along a continuum. The following example clarifies how to respond to each of the questions.

--*Example*---

My favorite types of mathematics problems are those that (a) require a complex computation, or (b) are theoretical and abstract in nature.

Mostly a	*Toward a* ←		*Both a and b equally*	→ *Toward b*		Mostly *b*	*Neither a nor b*
1	2	3	4	5	6	7	8

There are eight answer options as follows:
Options 1 or 7 to select alternative exclusively (a) or (b)
Options 2, 3, 4, 5, or 6 to select a weighted combination of the two alternatives
Option 8 to select neither (a) nor (b)

Please circle the response that best summarizes your opinion. Your answers to the questions that follow will help us to understand your current thoughts on these topics.

1. Do you believe that mathematics is (a) a dynamic field in which human create and construct knowledge or (b) a static body of absolute facts independent from human invention?

Mostly a			*Both a and b equally*			*Mostly b*	*Neither*
1	2	3	4	5	6	7	8

2. The main purpose of proof in mathematics is (a) to show the truth of a mathematical proposition or (b) to explain why the statement is true.

Mostly a			*Both a and b equally*			*Mostly b*	*Neither*
1	2	3	4	5	6	7	8

3. Do you consider proof to be (a) a step-by-step procedure given by an expert to follow or repeat or (b) something you have to construct based on your own understanding and knowledge of the topic?

Mostly a		*Both a and b equally*				*Mostly b*	*Neither*
1	2	3	4	5	6	7	8

4. To prove the truth of a mathematical statement, (a) there is one accepted best way to do it or (b) there could be several different ways as long as the proof is valid and convincing to your intended audience.

Mostly a		*Both a and b equally*				*Mostly b*	*Neither*
1	2	3	4	5	6	7	8

5. When you read and evaluate a mathematical proof, the most important thing you consider is (a) the content of the proof (for example, reasoning or logical structure of the proof) or (b) the form of the proof (for example, the surface features or use of precise notation).

Mostly a		*Both a and b equally*				*Mostly b*	*Neither*
1	2	3	4	5	6	7	8

6. Proof is (a) a tool for doing and understanding mathematics or (b) a tool for demonstrating the correctness of mathematical statements.

Mostly a		*Both a and b equally*				*Mostly b*	*Neither*
1	2	3	4	5	6	7	8

7. Ideally in mathematics class, the goal of the instruction is (a) to transmit established mathematical facts and procedures to students or (b) to guide students to construct mathematical knowledge and understanding on their own.

Mostly a		*Both a and b equally*				*Mostly b*	*Neither*
1	2	3	4	5	6	7	8

8. Based on your opinion and experience in a mathematics classroom, it is the role of a student to (a) actively participate in the learning activity to figure out and discuss solutions with others or (b) to absorb mathematical concepts and practice routine problems for accurate performance.

Mostly a		*Both a and b equally*				*Mostly b*	*Neither*
1	2	3	4	5	6	7	8

9. The primary purpose of proof in a mathematics classroom is (a) to confirm the truth of mathematical results already known to be true or (b) to promote students' understanding of why the mathematical results are true.

Mostly a		*Both a and b equally*				*Mostly b*	*Neither*
1	2	3	4	5	6	7	8

10. Seeing several different proofs of a theorem (a) helps students to understand better why mathematical statements are true or (b) rarely increases their understanding and may even be confusing.

Mostly a		*Both a and b equally*				*Mostly b*	*Neither*
1	2	3	4	5	6	7	8

11. In your experience, students are expected to consider proof as (a) a separate topic or prescribed algorithm to learn and master or (b) an essential tool for studying and doing mathematics.

Mostly a		*Both a and b equally*				*Mostly b*	*Neither*
1	2	3	4	5	6	7	8

12. Ideally in a proof-based mathematics course, students should (a) make conjectures, investigate them, and construct their own proofs or (b) practice proving statements or conjectures that they know have been proven to be true before by an expert.

Mostly a		*Both a and b equally*				*Mostly b*	*Neither*
1	2	3	4	5	6	7	8

13. In your opinion and experience, students can best learn how to prove (a) if they are shown the correct proof from a teacher or a textbook or (b) if they are engaged in activity to develop and evaluate mathematical arguments by themselves.

Mostly a		*Both a and b equally*				*Mostly b*	*Neither*
1	2	3	4	5	6	7	8

14. In order to prove a mathematical statement you need to (a) have seen the proof of a similar statement before or (b) know how to apply various types of reasoning and methods of proof.

Mostly a		*Both a and b equally*				*Mostly b*	*Neither*
1	2	3	4	5	6	7	8

15. Doing well in a proof-based mathematics course depend more on (a) how much effort students put into producing their own proof and reflecting on it or (b) how well they practice and recall a proof in the way it was presented in the classroom.

Mostly a		*Both a and b equally*				*Mostly b*	*Neither*
1	2	3	4	5	6	7	8

APPENDIX C

STUDENT INTERVIEW PROTOCOL 1

1. Reminder about the consent form
2. Ask about videotaping
3. Talk about interest on thinking
4. Background questions

   Program, year, previous career, previous courses with Dr. Benson
5. How do we know that something is true in mathematics?
6. What is your prior experience with proofs?
7. Tell me about something that first comes to your mind when I say proof.
8. What does the notion proof mean to you?
9. What does it mean to prove something?
10. Why do we have proofs?

    If not mentioned by the participant: What is the role of proof in mathematics?
11. Do proofs ever become invalid?
12. I am going to tell you three words. I want you to think how similar or how different they are: Proof, explanation and justification.
13. What is the role of a teacher in a mathematics classroom?
14. How do you envision the role of proof in your classroom as a future teacher?
15. What is the place of proof in math education/classrooms?

16. Do you think that middle school students are able to come up with proofs by themselves?
17. What is the sum of the interior angles of a triangle?
18. How do you know?
19. Given an argument:

    Is this a proof? Why or why not?

    What makes it a proof?

    Does it prove it for all cases?

    If participant says not proof: What is missing? How would you complete it/change it to be a proof?

    Is it convincing?
20. Is there anything else that you want to add?

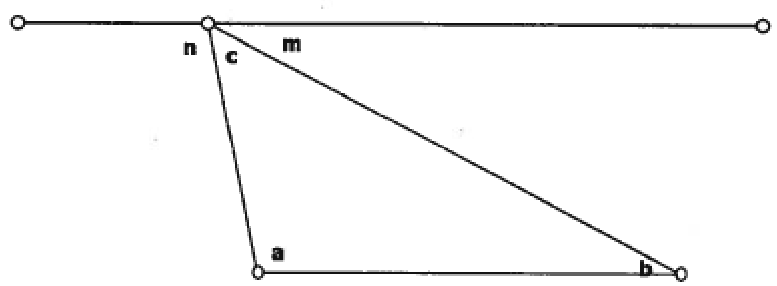
Argument 1:

I tore up the angles of the obtuse triangle and put them together (as shown below).

The angles came together as a straight line, which is 180 degrees. I also tried it for an acute triangle as well as a right triangle and the same thing happened. Therefore, the sum of the measures of the interior angles of a triangle is equal to 180 degrees. (Knuth, 2002b)

Argument 2:

I drew a line parallel to the base of the triangle.

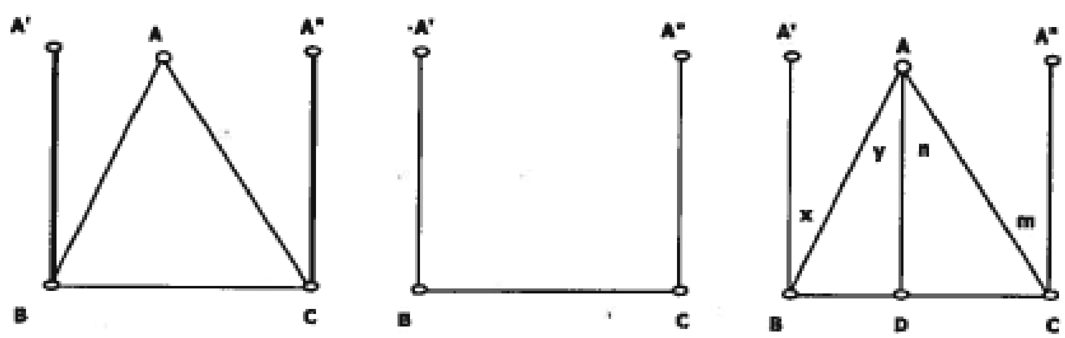


I know $n = a$ because alternate interior angles between two parallel lines are congruent. For the same reason, I also know that $m = b$. Since the angle measure of a straight line is 180 degrees, I know $n + c + m = 180$.

Substituting a for n and b for m gives $a + b + c = 180$. Thus, the sum of the measures of the interior angles of a triangle is equal to 180 degrees. (Knuth 2002b)

Argument 3:

Using the diagram below, imagine moving *BA* and *CA* to the perpendicular positions *BA'* and *CA"*, thus forming the second figure. In reversing this procedure (i.e. moving *BA'* to *BA*), the amount of the right angle, *A'BC*, that is lost is *x*. However, this lost amount is gained with angle *y* (since *BA'* and *DA* are parallel, and *x* and *y* are alternate interior angles). A similar argument can be made for the other case. Thus, the sum of the measures of the interior angles of any interior angle is equal to 180 degrees (Harel & Sowder, 1998).



APPENDIX D

STUDENT INTERVIEW PROTOCOL 2

1. Can you tell me about a recent proving experience that you’ve had?
    a. What is important about that experience to you?
2. Can you tell me about a recent proving experience that you’ve had and in which you felt successful?
    a. What do you think makes you feel successful in a proving experience?
3. Can you tell me about a recent proving experience that you’ve had and in which you haven’t felt that successful?
    a. What do you think makes you feel successful in a proving experience?
4. What is involved in proving a mathematical statement? / What does it mean to prove something?
5. How would you define a proof?
6. What did you learn about proofs so far that you did not know before?
7. Is there anything that is confusing to you about proofs that you can think of?
8. Once you prove something is it possible to disprove it again?
9. Given an argument:

    Is this a proof? Why or why not?

    What makes it a proof?

    Does it prove it for all cases?

If participant says not proof: What is missing? How would you complete it/change it to be a proof?

Is it convincing?

10. Which one is more convincing for you?
11. Which one would you use to convince a friend?
12. Which one would you use to convince a student?
13. Which one would you use to convince a mathematician?
14. Which argument would get a better grade from your teacher?

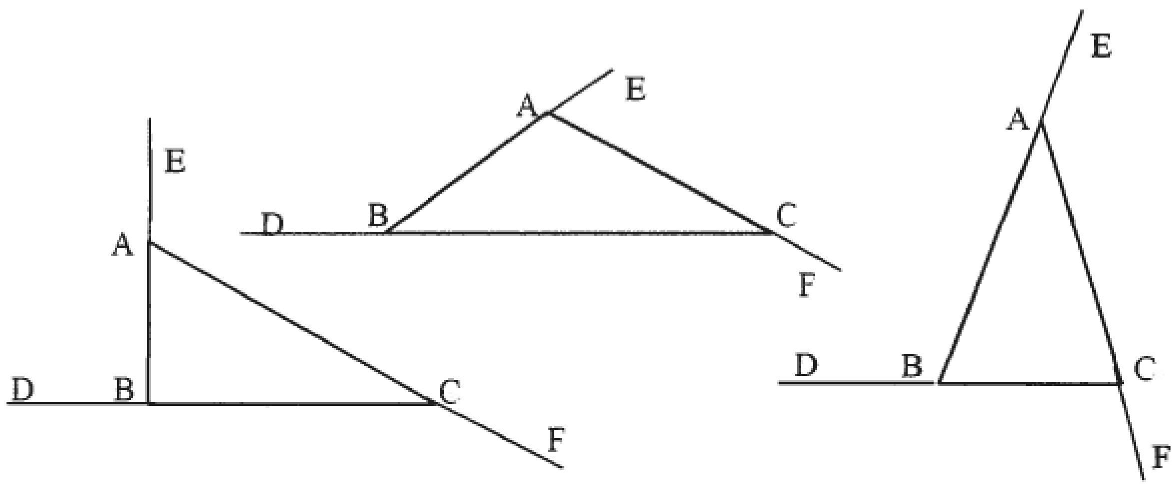
Statement: In a triangle, the sum of the exterior angles is 360 degrees.

Argument 1:

I drew several different triangles. I measured the exterior angles in each of these triangles and in each case the sum of the exterior angles was 360 degrees. Since it worked for these triangles, I can be sure that the statement is always true.

Argument 2:

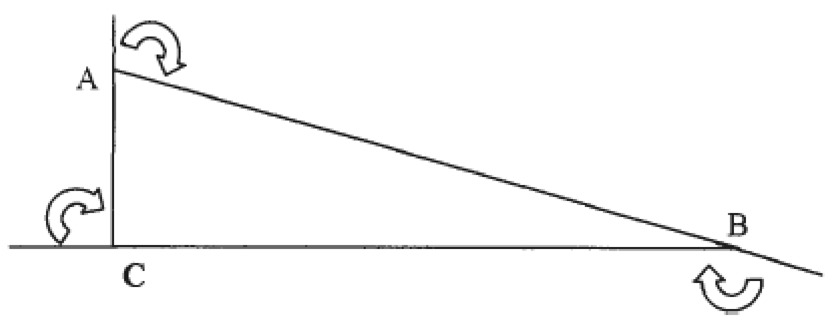
I drew three different triangles. I labeled each triangle ABC. Each triangle also has three exterior angles labeled $\angle ABD, \angle CAE,$ and $\angle BCF$. I measured the exterior angles in each of the three different triangles and in each case the sum of the exterior angles was 360 degrees. Since I checked all three kinds of triangles, namely, right, obtuse, and acute, I can be sure that the statement is always true.



(Retrieved from http://www.cas.ilstu.edu/proofproject/webpage/index/methodsinstruments.htm)

Argument 3:

If you start at vertex *A* facing north (up), then turn clockwise and walk along the sides of the triangle until you end up at vertex *A* again, you will end up facing the way you began. Because each of the three turns you make on your trip is the same number of degrees as each of the three exterior angles on the triangle, the number of degrees your body turns is the same as the exterior angle sum. As a result, the full 360 degrees turn (in which you end up facing the same direction you started) you completed shows that the sum of the exterior angles also must be 360 degrees.



(Retrieved from http://www.cas.ilstu.edu/proofproject/webpage/index/methodsinstruments.htm)

Argument 4:

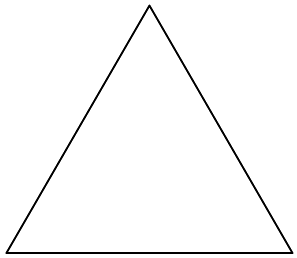
In a triangle, each exterior angle forms a linear pair with its associated interior angle. You can see that the sum of the angle measures in each linear pair is 180 degrees because both angles together form a straight line. Because there are three of these linear pairs, the sum of all of them is 540 degrees (3 x 180 = 540).

Because we know that the sum of the interior angles in any triangle is 180 degrees, we can subtract 180 from 540 (sum of all interior and exterior angles) to be left with the sum of the triangle's exterior angles only. We know that 540 – 180 = 360, so the statement is true for all triangles. (Retrieved from http://www.cas.ilstu.edu/proofproject/webpage/index/methodsinstruments.htm)

Statement: The number of diagonals in a polygon is $\frac{n(n-3)}{2}$, where n is the number of sides of the polygon.

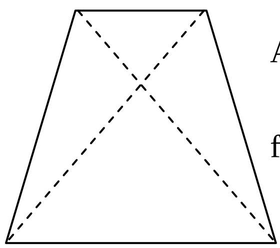
Argument 1:

I tried the formula for different polygons. I first drew a triangle.



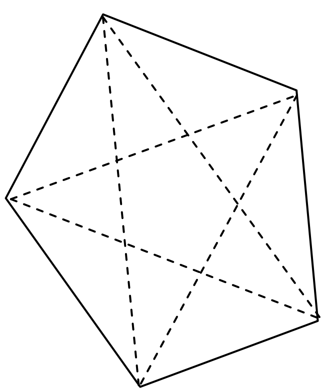
A triangle has three sides and no diagonals. If I plug in 3 in the formula I get 0. $\frac{n(n-3)}{2}=\frac{3(3-3)}{2}=\frac{3\cdot 0}{2}=0$

Then I tried it for a quadrilateral:



A quadrilateral has four sides and two diagonals. If I plug in 4 in the formula, I get 2. $\frac{n(n-3)}{2}=\frac{4(4-3)}{2}=\frac{4\cdot 1}{2}=2$

Finally, I tried it for a pentagon:



A pentagon has five sides and five diagonals. If I plug in 5 in the formula, I get 5. $\frac{n(n-3)}{2}=\frac{5(5-3)}{2}=\frac{5\cdot 2}{2}=5$

Since the formula works for a triangle, quadrilateral and pentagon, it works for all polygons.

Argument 2:

An n-sided polygon has n vertices. From one vertex, it is possible to draw $n - 3$ diagonals. So, the total number of diagonals that can be drawn is $n(n-3)$. However, this would mean that each diagonal is drawn twice. Hence, the expression must be divided by 2.

APPENDIX E

STUDENT INTERVIEW PROTOCOL 3

1. What do you think the main goal/big ideas of this class were?
2. What did you learn about proof in this class?
3. Did you learn anything about proof in this class that you didn't know before? Can you tell me about those?
4. Tell me about an instance in class which helped you understand proof better.
5. Tell me about an instance in class which influenced the way you think about proof.
6. How was this class similar to/different than your other classes where you learned about proof?
7. How do you compare the proofs that you worked on in Dr. Benson's class to proofs that you might have seen elsewhere?
8. Think back to the beginning of the class: I used to think … (about proof this way), now I think … (about proof this way).
9. I am going to tell you three words. I want you to think how similar or how different they are: Proof, explanation and justification.
10. Are there things that you would consider as proof now that you wouldn't consider before? Or vice versa?
11. How do you envision the role of proof in your classroom as a future teacher?
12. How do you see proof playing out in middle school curriculum?

13. Do you think that middle school students are able to come up with proofs by themselves?

14. What is your opinion about the importance of proof?

15. Given an argument:

    Is this a proof? Why or why not?

    What makes it a proof?

    Does it prove it for all cases?

    If participant says not proof: What is missing? How would you complete it/change it to be a proof?

    Is it convincing?

Statement: The sum of the first n positive integers is $\frac{n(n+1)}{2}$.

Argument 1:

I tried it for different numbers.

For n = 2, 1 + 2 = 3

$$\frac{n(n+1)}{2} = \frac{2(2+1)}{2} = \frac{2 \cdot 3}{2} = 3$$

3=3

So, it's true for n = 2.

For n = 3, 1 + 2 + 3 = 6

$$\frac{n(n+1)}{2} = \frac{3(3+1)}{2} = \frac{3 \cdot 4}{2} = 6$$

6=6

So, it's true for n = 3

For n = 7, 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28

$$\frac{n(n+1)}{2} = \frac{7(7+1)}{2} = \frac{7 \cdot 8}{2} = 28$$

28=28

So, it's true for n = 7.

Since, I've showed that the statement is true for three different numbers, I've proven the statement.

Argument 2:

For $n = 1$ it is true, since 1 = 1(1+1)/2.

Assume it is true for some arbitrary *k*, that is, $S(k) = k(k+1)/2$.

Then consider:

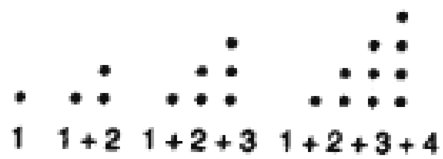
$$S(k+1) = S(k) + (k+1) = k(k+1)/2 + k + 1 = (k+1)(k+2)/2.$$

Therefore the statement is true for *k* + 1 if it is true for *k*.

By induction, the statement is true for all *n*. (Hanna, 1990)

Argument 3:

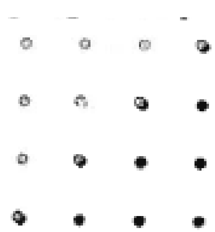
We can represent the sum of the first *n* positive integers as triangular numbers.



The dots form isosceles right triangles with the *n*th triangle containing:

$S(n) = 1 + 2 + 3 + 4 + \ldots + n$ dots.

Overlaying a second isosceles right triangle of the same size so that the diagonals coincide produces a square containing n^2 dots plus *n* extra dots due to the overlapping diagonals. To illustrate, the figure below represents the fourth isosceles right triangle and another of the same size overlaid so that the diagonals coincide. In this case, a square containing 4^2 dots plus 4 extra dots due to the overlapping diagonals is produced:



Therefore, in the general case (using the nth triangle, the number of dots produced by the two overlapping triangles is $2S(n) = n^2 + n$, so $S(n) = (n^2 + n)/2$. (Hanna, 1990)

Argument 4:

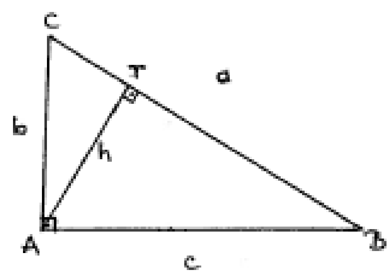
$$S(n) = 1 + 2 \qquad + 3 \qquad + ... + n$$
$$S(n) = n + (n-1) + (n-2) + ... + 1$$

Taking the sum of these two rows:

$$\begin{aligned} 2S(n) &= (1+n) + \left[2 + (n-1)\right] + \left[3 + (n-2)\right] + ... + (n+1) \\ &= (n+1) + (n+1) + (n+1) + ... + (n+1) \\ &= n(n+1) \end{aligned}$$

Therefore, $S(n) = n(n+1)/2$. (Hanna, 1990)

Statement: ABC is a right triangle. AT is perpendicular to BC. Prove that $\frac{1}{h^2} = \frac{1}{b^2} + \frac{1}{c^2}$.



Argument 1:

I drew several different triangles and measured b, c, and h for each triangle. Then using these numbers I showed that $\frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{h^2}$. So, I proved that the statement is true.

Argument 2:

The area of a triangle is $\frac{base \times height}{2}$. Any one of the three bases can be used to find the area.

$$\left.\begin{array}{l} Area(ABC)=\frac{b\cdot c}{2} \\ \\ \\ Area(ABC)=\frac{a\cdot h}{2} \end{array}\right\} \quad \frac{b\cdot c}{2}=\frac{a\cdot h}{2} \quad \Rightarrow \quad b\cdot c=a\cdot h \qquad\qquad 1$$

Also using the Pythagorean Theorem $a^2 = b^2 + c^2$ 2

Using 1 and 2, $a = \frac{b\cdot c}{h}$

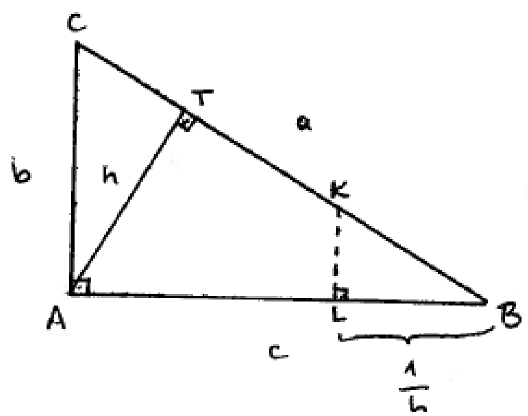
$$a^2 = \frac{b^2\cdot c^2}{h^2} = b^2 + c^2$$

$$\frac{1}{h^2} = \frac{b^2 + c^2}{b^2\cdot c^2}$$

$$\frac{1}{h^2} = \frac{b^2}{b^2\cdot c^2} + \frac{c^2}{b^2\cdot c^2}$$

$$\frac{1}{h^2} = \frac{1}{b^2} + \frac{1}{c^2}$$

Argument 3:



Draw KL perpendicular to AB such that $|LB| = \frac{1}{b}$. Since triangle ACB and triangle LBK are similar $|KL| = \frac{1}{c}$. Also $m(\angle LKB) = m(\angle TAB)$

So, $ATB \approx KLB$ (AAA)

$$\frac{|AT|}{|KL|} = \frac{|AB|}{|KB|}$$

$$\frac{h}{\frac{1}{c}} = \frac{c}{|KB|} \quad \Longrightarrow \quad |KB| = \frac{1}{h}$$

Now, look at triangle KLB.

$$|KL| = \frac{1}{c},\ |LB| = \frac{1}{b},\ |KB| = \frac{1}{h}$$

Using the Pythagorean Theorem,

$$\left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2 = \left(\frac{1}{h}\right)^2$$

$$\frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{h^2}$$

APPENDIX F

INSTRUCTOR INTERVIEW PROTOCOL

1. What are your objectives for this class?
2. Do you have any objectives for this class with respect to proof and learning how to prove?
3. What is important for pre-service teachers to know with respect to proof?
4. What are your goals for them/what would you like them to be able to do by the end of the class?
5. What is the role/importance of proof in math/in this class/in middle school math?
6. Do you have a particular approach to teaching proof or how to prove things?
7. Do you have any sense of, um, how the students come into this class in terms of what they know about proof?
8. How do you decide about what to prove in this class?
9. In your view, what does it mean to prove something?
10. Do you see any difference between proving, justifying and explaining?

APPENDIX G

ANALYSIS TABLE

PARTICIPANT NAME	INTERVIEW 1	INTERVIEW 2	INTERVIEW 3
Views about math			
Experience with, attitude toward proof			
Proof in the abstract			
Proof in context			
Evaluation of arguments			
Importance of proofs			