On the Geometry of

Sets of Positive Reach

by

Jennifer Carol Ellis

(Under the direction of Joseph H. G. Fu)

Abstract

The reach of a set S in a metric space, denoted reach(S), is the supremal r such that any point within distance r of S has a unique nearest point in S. Sets of positive reach (PR sets) originate in the work of Federer in [9] with a theory of curvature measures on PR sets. More recently, Fu [10] defined the second fundamental form for PR sets, established Morse theory on PR sets, and revisited Federer's curvature measures.

We work exclusively with regular PR sets in Euclidean space. We further develop the theory of regular PR sets in Euclidean space by establishing regularity of geodesics and by determining a formula for reach using the second fundamental form. We prove that geodesics are $C^{1,1}$ in regular PR sets. Our formula for reach of regular compact PR sets is based on the technique in [3] for determining thickness of $C^{1,1}$ curves.

Index words: Differential geometry, Positive reach, Geodesic, Variation of arclength, Second fundamental form

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Chapter 1

Introduction: Sets of Positive Reach

1.1 The Basics

Sets of positive reach (PR sets) originate in the work of Federer in [9]. PR sets have many nice geometric properties that mimic those of smooth manifolds with boundary; we shall discuss some of these properties in this chapter. We shall consider only PR subsets of Euclidean space, although many of these notions make sense in more general settings. Let us begin our discussion of PR sets with some friendly examples.

Examples. *The following are examples of PR sets in Euclidean space:*

1. *Convex sets*
2. *Smooth compact submanifolds of Euclidean space*
3. *The pinched first quadrant of the plane (Figure 1.1)*
4. *A planar crescent moon shape*
5. *The region in $\mathbb{R}^3$ under a curved ridge (Figure 1.2)*

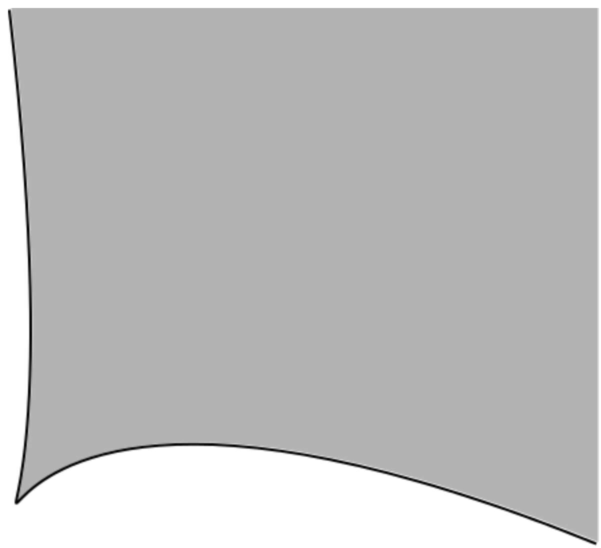

Figure 1.1: An unbounded planar PR set: the pinched first quadrant

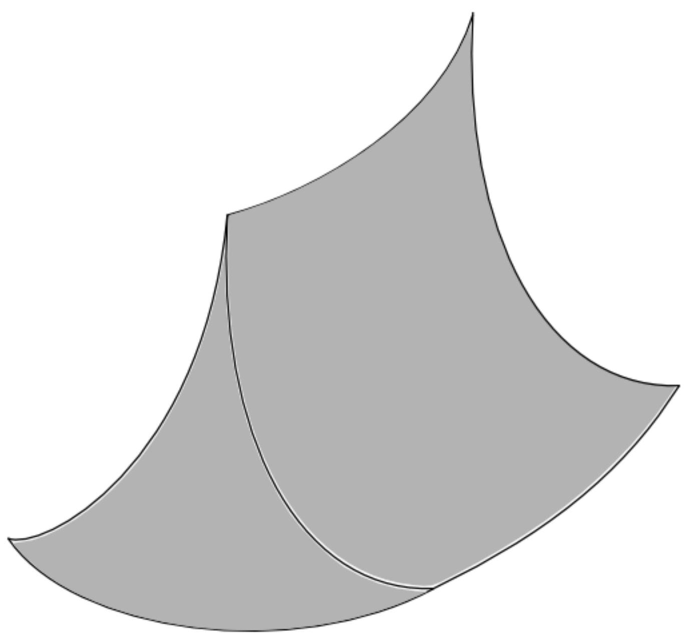

Figure 1.2: An unbounded PR set in $\mathbb{R}^3$: an infinite ridge set

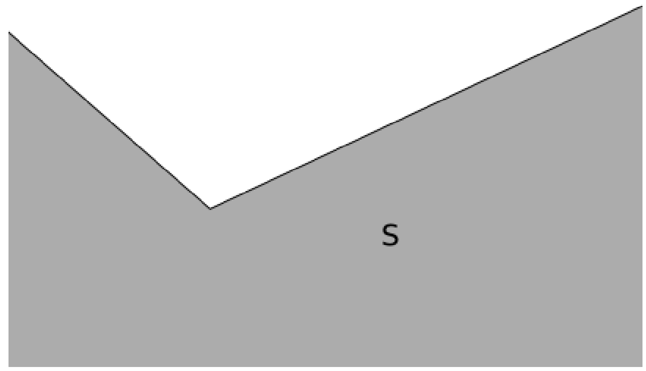

Figure 1.3: The shaded region S has $\mathrm{reach}(S) = 0$.

The key concept for reach is the *unique nearest point map*; the unique nearest point map is only defined for certain points outside of a given set, and the domain of the unique nearest point map determines the reach of the set. Let S be a set in a metric space, and let $d_S(x) := \inf\{d(x, a) : a \in S\}$, so that $d_S(x)$ is the distance from x to S.

Definition 1.1. *The set* $\mathrm{Unp}(S)$ *is the set of points having a unique nearest point in* S*. The* ***unique nearest point map*** $\pi_S : \mathrm{Unp}(S) \to S$ *is the map sending a point to its unique nearest point in* S*. For* $x \in \mathrm{Unp}(S)$*, we have that* $d_S(x) = d(x, \pi_S(x))$.

The reach of a set is a measurement of how far from the set one may go and still have a unique nearest point map:

Definition 1.2. *The* ***reach*** *of a set* S *in a metric space, denoted* $\mathrm{reach}(S)$*, is the supremal* r *such that any point within distance* r *of* S *has a unique nearest point in* S*. When* $\mathrm{reach}(S) > 0$*, we say that* S *is a PR set.*

Remark. A PR set must be closed: suppose S is a set and $x_0 \in \overline{S}$, where $\overline{S}$ is the closure of S. Then $d_S(x_0) = 0$, so either $\mathrm{reach}(S) = 0$ or $x_0 \in S$.

Example. *The set S in Figure 1.3 has* $\text{reach}(S) = 0$*, since points arbitrarily close to the corner point are in the complement of* $\text{Unp}(S)$*.*

Although PR sets are not in general smooth manifolds, the notions of tangency and normalcy carry through to this setting via the tangent cone and the normal cone. We will discuss these concepts for PR sets in Euclidean space. As in single-variable calculus, the tangent directions are limits of secant directions.

Definition 1.3. *Let S be a nonempty closed set. The **tangent cone** of S at a, denoted* $\text{Tan}(S, a)$*, is the cone over the set of all limits as y approaches a of $\dfrac{y-a}{|y-a|}$. The **normal cone** to S at a, denoted* $\text{Nor}(S, a)$*, is the set of all vectors v such that $v \cdot w \leq 0$ for all $w \in \text{Tan}(S, a)$. The **unit normal cone*** $\text{nor}(S, a)$ *is the set of unit vectors in the normal cone to S at a.*

Definition 1.4. *Let $C \subset \mathbb{R}^m$ be a cone. The **dual cone** to C, denoted* $\text{dual}(C)$*, is*

$$\text{dual}(C) := \{v \in \mathbb{R}^m : v \cdot w \leq 0 \textit{ for all } w \in C\}. \tag{1.1}$$

Remark. For a PR set S, the cones $\text{Tan}(S, a)$ and $\text{Nor}(S, a)$ are closed convex cones satisfying $\text{dual}(\text{Tan}(S, a)) = \text{Nor}(S, a)$ and $\text{dual}(\text{Nor}(S, a)) = \text{Tan}(S, a)$. (See Figure 1.4.)

The set $\text{Nor}(S, a)$ and maps π_S and d_S are related in the following way:

Lemma 1.5 ([9, Theorem 4.8 (2)])**.** *Let S be a nonempty closed set. For $a \in S$,*

$$\pi_S^{-1}(a) \subset \{a + v : d_S(a + v) = |v|\} \subset \{a + v : v \in \text{Nor}(S, a)\}, \tag{1.2}$$

Definition 1.6. *The **normal bundle*** $\text{Nor}(S)$ *is the set $\{(x, v) : x \in S$ and $v \in \text{Nor}(S, x)\}$. The **unit normal bundle*** $\text{nor}(S)$ *is the set $\{(x, v) : x \in S$ and $v \in \text{nor}(S, x)\}$.*

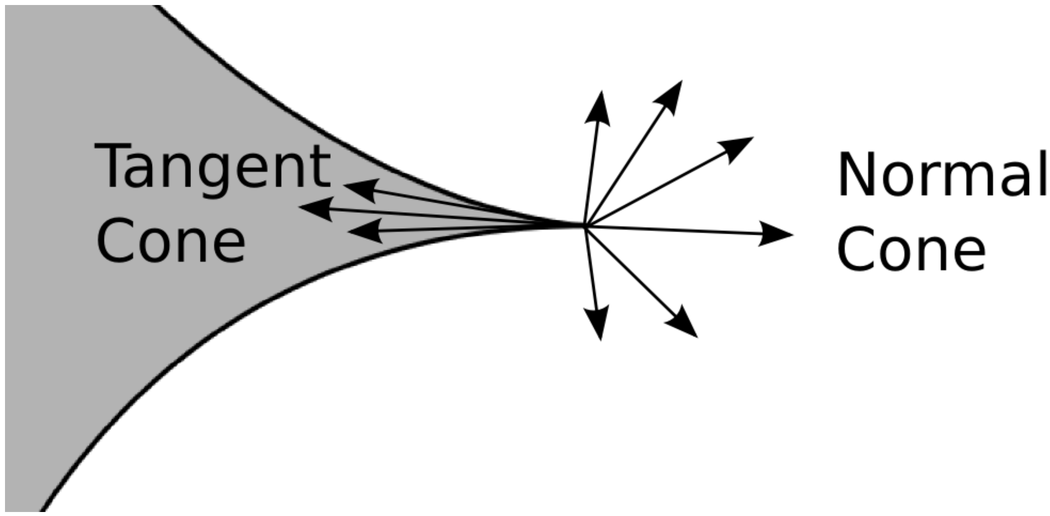

Figure 1.4: The tangent cone and normal cone

Remark. The normal bundle and unit normal bundle are not technically fiber bundles; we use this terminology because they can be thought of as generalized bundles.

The following is a slight modification of [9, Theorem 4.8 (7)].

Theorem 1.7 (Federer's Inequality)**.** *If* $v \in \operatorname{nor}(S, a)$ *and* $a, b \in S$*, then*

$$v \cdot (b - a) \leq \frac{|b - a|^2}{2\operatorname{reach}(S)}. \tag{1.3}$$

Federer's inequality implies that the boundary of a set of positive reach cannot be too outwardly curved. (See Figure 1.5.)

In [9], Federer shows that a set S has $\operatorname{reach}(S) \geq t$ if and only if $d_{\operatorname{Tan}(S,a)}(b - a) \leq |b - a|^2/(2t)$ for all $a, b \in S$. However, a more basic result is true; from the argument in the proof of [9, Lemma 4.17], it is possible to recover the reach of a set as the supremal t for which there is a Federer inequality:

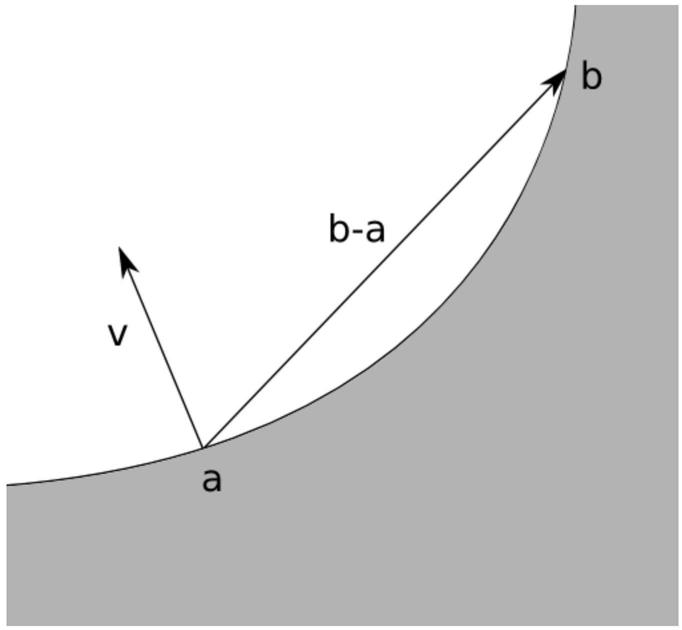

Figure 1.5: Federer's inequality states that the dot product of these two vectors cannot be too large.

Theorem 1.8. *A set S has* $\text{reach}(S) \geq t$ *if and only if* $v \cdot (b-a) \leq \frac{|b-a|^2}{2t}$ *for all $a, b \in S$ and all $v \in \text{nor}(S, a)$.*

Proof. ($\Longrightarrow$) If $\text{reach}(S) \geq t > 0$, then the result follows from Theorem 1.7.

($\Longleftarrow$) Suppose that $d_S(x) = |x-a| = |x-b|$ and $d_S(x) < t$. Assume that $a = 0$, so that $|x| = |b-x| < t$. By Lemma 1.5, we have that $x \in \text{Nor}(S, a)$. Thus

$$\frac{x}{|x|} \cdot b \leq \frac{|b|^2}{2t}. \tag{1.4}$$

Since

$$0 = |b-x|^2 - |x|^2 = |b|^2 - 2b \cdot x \geq |b|^2 - \frac{|x||b|^2}{t} = |b|^2(1 - \frac{|x|}{t}), \tag{1.5}$$

we must have that $a = b = 0$, so that $\text{reach}(S) \geq t$. $\square$

1.2 The Second Fundamental Form

Definition 1.9 ([10]). *Let $M \subset \mathbb{R}^m$ be an oriented C^1 hypersurface; let $n : M \to S^{m-1}$ be the Gauss map of M. Then we say that M is $C^{1,1}$ if n is Lipschitz.*

The second fundamental form for PR sets originated in the work of Fu in [10], where the author defined the second fundamental form first for $C^{1,1}$ hypersurfaces and then generalized to PR sets. The key idea is that, by Rademacher's theorem, a Lipschitz function is differentiable almost everywhere. By working with sets on which the Gauss map is Lipschitz, we have a second fundamental form almost everywhere. Thus we may generalize the second fundamental form to $C^{1,1}$ hypersurfaces.

The following result from [10] associates $C^{1,1}$ hypersurfaces to PR sets; the result follows immediately from [9, Theorem 4.8].

Lemma 1.10. *If $S \subset \mathbb{R}^m$ is a PR set and $0 < \epsilon < \mathrm{reach}(S)$, let $S_\epsilon := d_S^{-1}[0, \epsilon]$. Then ∂S_ϵ is a $C^{1,1}$ hypersurface.*

Definition 1.11. *Let $M \subset \mathbb{R}^m$ be a $C^{1,1}$ hypersurface. Define the set of **smooth points** of M, denoted $\mathrm{Sm}(M)$, to be the set of points at which the Gauss map is differentiable.*

Definition 1.12. *For $x \in \mathrm{Sm}(M)$, define the second fundamental form $\mathrm{II}(x)$ on $\mathrm{Tan}(M, x)$ by*

$$\mathrm{II}(x)(v, w) = -v \cdot dn_x(w). \tag{1.6}$$

Note that this definition differs from that in [10] by a sign. As in the smooth case, II is symmetric ([10, Proposition 3.5]). We shall use the second fundamental form on S_ϵ to define the second fundamental form on S. While in the smooth case (and in the $C^{1,1}$ case), the second fundamental form is defined on the tangent bundle, for sets of positive reach, the second fundamental form is defined over the tangent bundle of $\mathrm{nor}(S)$.

Let n_ϵ denote the Gauss map of S_ϵ. Define the maps ψ_ϵ and ϕ_ϵ by

$$\psi_\epsilon : \operatorname{nor}(S) \to \partial S_\epsilon, \quad \psi_\epsilon(p, v) := p + \epsilon v \tag{1.7}$$
$$\phi_\epsilon : \partial S_\epsilon \to \operatorname{nor}(S), \quad \phi_\epsilon(x) = (\pi_S(x), n_\epsilon(x)),$$

and note that $\phi_\epsilon = \psi_\epsilon^{-1}$.

Proposition 1.13 ([10, Proposition 4.2])**.** *Let $(p, v) \in \operatorname{nor}(S)$ and suppose that $0 < \epsilon < \operatorname{reach}(S)$. The following are equivalent:*

1. *$T := \operatorname{Tan}(\operatorname{nor}(S), (p, v))$ is an $(m-1)$-dimensional plane in $\mathbb{R}^m \times \operatorname{Tan}(S^{m-1}, v)$.*
2. *ϕ_ϵ is differentiable at $\psi_\epsilon(p, v)$*
3. *n_ϵ is differentiable at $\psi_\epsilon(p, v)$.*

Thus the Gauss map of ∂S_ϵ is differentiable precisely when the projection map from ∂S_ϵ onto $\operatorname{nor}(S)$ is differentiable.

Definition 1.14. *A point (p, v) satisfying the above is called a* ***smooth point*** *of $\operatorname{nor}(S)$, and we write $(p, v) \in \operatorname{Sm}(\operatorname{nor}(S))$.*

In addition, we have the following relationship between T and the Gauss map of ∂S_ϵ:

Proposition 1.15 ([10, Proposition 4.3])**.** *When the conditions of Proposition 1.13 are satisfied, if we put $x_0 := \psi_\epsilon(p, v)$, then the following hold:*

1. *Any $\xi \in \operatorname{Tan}(\partial S_\epsilon, x_0)$ has the form $\xi = \tau + \epsilon\sigma$ for some $(\tau, \sigma) \in T$.*
2. *If ξ has this form, then $d(n_\epsilon)_{x_0}(\xi) = \sigma$.*

We now have everything in place to define the second fundamental form on a set of positive reach. Let π_1 denote the projection onto the first component of T. Note that $T \subset \operatorname{Tan}(S, p) \times \operatorname{Tan}(S^{m-1}, v)$.

Definition 1.16 ([10, Definition 4.5]). *Suppose that the conditions in Proposition 1.13 are satisfied. Define the vector subspace T_1 by $T_1 := \pi_1(T)$. Then the* ***second fundamental form*** $\mathrm{II}(p, v)$ *on T_1 is given by*

$$\mathrm{II}(p, v)(\tau, \tau') = -\tau \cdot \sigma', \tag{1.8}$$

where $(\tau, \sigma), (\tau', \sigma') \in T$.

The second fundamental form is well-defined and symmetric due to the following lemma:

Proposition 1.17 ([10, Proposition 4.4]). *Suppose that $(p, v) \in \mathrm{nor}(S)$ and that the conditions in Proposition 1.13 are satisfied. If $(\tau, \sigma), (\tau', \sigma') \in T$, considered with the usual Euclidean inner product, then $\tau \cdot \sigma' = \tau' \cdot \sigma$.*

Remark. Note that for $\xi = \tau + \epsilon\sigma$ and $\xi' = \tau' + \epsilon\sigma'$, the second fundamental form II_ϵ on ∂S_ϵ is given by

$$\mathrm{II}_\epsilon(x_0)(\xi, \xi') = -\xi \cdot d(n_\epsilon)_{x_0}(\xi') = -(\tau + \epsilon\sigma) \cdot \sigma'. \tag{1.9}$$

Chapter 2

Semiconvex Function Theory

A function f is *semiconvex* if it can be written as the sum of a C^∞-smooth function and a convex function; any PR set is $f^{-1}(-\infty, 0]$ for a semiconvex function f such that 0 is a weakly regular value of f. For our work, we shall not use weak regularity; however, our main results assume the stronger condition that S is a *regular* PR set: S is the sublevel set $f^{-1}(-\infty, 0]$, where f is a semiconvex function and 0 is a *regular value* of f. Thus we can learn about PR sets by studying semiconvex functions; to this end, we shall begin by discussing properties of convex functions, all of which can be found in [11]. Following our discussion of convex function theory, we describe semiconvex function theory and regularity for semiconvex functions. We conclude the chapter by returning to regular PR sets S and describing $\text{nor}(S)$ in terms of *subgradients* of f.

2.1 Convex Function Theory

Recall that a function $k : \mathbb{R}^m \to \mathbb{R}$ is *convex* if $k(tx + (1-t)y) \leq tk(x) + (1-t)k(y)$ for all $x, y \in \mathbb{R}^m$ and all $t \in (0, 1)$. A convex function is locally Lipschitz and can be written as the pointwise supremum of a collection of affine functions (see [11]). In particular, for each

x there exists an affine function λ such that $\lambda(x) = k(x)$ with $\lambda \leq k$. We define **the set of subgradients to k at x** by $\partial k(x) := \{\text{lin}(\lambda) : \lambda(x) = k(x),\ \lambda \leq k\}$, where $\text{lin}(\lambda)$ is the linear part of λ.

Remark. There is a bijective correspondence between maps $x \mapsto w \cdot x$ and vectors w, and we shall use the vector w and the map $x \mapsto w \cdot x$ interchangeably. In particular, we shall think of the subgradient set as the collection of vectors $\{w\}$ instead of the collection of linear maps $\{x \mapsto w \cdot x\}$.

Lemma 2.1. $\partial k(x) = \{w : w \cdot (y - x) + k(x) \leq k(y)$ *for all* $y\ \}$.

Now we discuss some properties of the subgradient set $\partial k(x)$: it is convex, closed, and bounded.

Lemma 2.2. $\partial k(x)$ *is a convex set.*

Proof. Assume $v, w \in \partial k(x)$. Let λ_v be an affine function whose linear part is v, such that $\lambda_v(x) = k(x)$ and $\lambda_v \leq k$. Write $\lambda_v(y) = y \cdot v + c_v$, and define λ_w similarly. Then, for all $t \in (0, 1)$, the affine function $\lambda_{tv+(1-t)w} := t\lambda_v + (1 - t)\lambda_w$ with linear part $tv + (1 - t)w$ inherits the desired properties from λ_v and λ_w. $\square$

Lemma 2.3. $\partial k(x)$ *is closed.*

Proof. Suppose $w_i \to w$ and $w_i \in \partial k(x)$ for all i. Each $\lambda_i := \lambda_{w_i}$ has the form $\lambda_i(y) = w_i \cdot (y - x) + k(x)$, and each satisfies the inequality $w_i \cdot (y - x) + k(x) \leq k(y)$, so that, by continuity of the dot product, we have $w \cdot (y - x) \leq k(y) - k(x)$, from which it follows that $w \in \partial k(x)$. $\square$

Lemma 2.4. $\partial k(x)$ *is bounded.*

Proof. Assume $w \in \partial k(x)$. Then $w \cdot (y - x) \leq k(y) - k(x)$ for all y. Since k is locally Lipschitz, there exists $\alpha > 0$ such that $w \cdot (y - x) \leq |k(y) - k(x)| \leq \alpha|y - x|$ for all y in a neighborhood of x and all $w \in \partial k(x)$. Taking $y = x + tw$ for sufficiently small t, we have that $|w| \leq \alpha$. $\square$

While convex functions need not be differentiable, any convex function has one-sided derivatives at every point. Later, we shall see that these one-sided derivatives are related to the subgradient set above. We shall give the relationship by establishing a sequence of lemmas about the directional derivatives.

Definition 2.5. *We define the* ***one-sided directional derivative*** *of k at x with respect to v in the usual way:*

$$Dk_x(v) := \lim_{t \downarrow 0} \frac{k(x+tv) - k(x)}{t}. \tag{2.1}$$

Note that, in the case that k is differentiable, the one-sided directional derivative and the gradient are related in the usual way: $Dk_x(v) = \nabla k_x \cdot v$.

Lemma 2.6. *For any convex function k, $\dfrac{k(x+tv)-k(x)}{t}$ is a nondecreasing function of $t > 0$.*

Proof. For $t > s > 0$, note that

$$k(x+sv) = k(x + \frac{s}{t}tv) = k((1-\frac{s}{t})x + \frac{s}{t}(x+tv)) \leq (1-\frac{s}{t})k(x) + \frac{s}{t}k(x+tv). \tag{2.2}$$

Thus the difference quotient satisfies the inequality

$$\begin{aligned} &\frac{k(x+tv)-k(x)}{t} - \frac{k(x+sv)-k(x)}{s} \\ &= \frac{sk(x+tv) - sk(x) - tk(x+sv) + tk(x)}{ts} \\ &\geq \frac{sk(x+tv) - sk(x) - t[(1-\frac{s}{t})k(x) + \frac{s}{t}k(x+tv)] + tk(x)}{ts} \\ &= 0. \end{aligned} \tag{2.3}$$

$\square$

Lemma 2.7. *The one-sided directional derivative $Dk_x(v)$ is defined for all x and v.*

Proof. Since the difference quotient $\dfrac{k(x+tv)-k(x)}{t}$ decreases as $t \downarrow 0$, it suffices to show that the difference quotient is bounded below. Write $y = x + tv$. Then $k(x) = k(y - tv) = k(t(y-v) + (1-t)y) \leq tk(y-v) + (1-t)k(y) = tk(x+(t-1)v) + (1-t)k(x+tv)$, and

$$\begin{aligned} \frac{k(x+tv)-k(x)}{t} &\geq \frac{k(x+tv) - tk(x+(t-1)v) - (1-t)k(x+tv)}{t} \\ &= k(x+tv) - k(x+(t-1)v). \end{aligned} \tag{2.4}$$

Continuity of k and (2.4) together imply $\lim_{t\downarrow 0}(k(x+tv) - k(x+(t-1)v)) = k(x) - k(x-v)$, and the difference quotient is bounded below. □

We shall use the following result in the proof of Lemma 2.29.

Corollary 2.8. *Let k be a convex function. Then*

$$Dk_x(y-x) \leq k(y) - k(x). \tag{2.5}$$

Proof. Using convexity of k and Lemma 2.7, for $t < 1$ we have

$$\begin{aligned} Dk_x(y-x) &= \lim_{t\downarrow 0} \frac{k(x+t(y-x)) - k(x)}{t} \\ &\leq \lim_{t\downarrow 0} \frac{(1-t)k(x) + tk(y) - k(x)}{t} \\ &= k(y) - k(x). \end{aligned} \tag{2.6}$$

□

We now provide the following alternate characterization of the subgradient set; this characterization establishes the key relationship between the subgradients and directional derivatives.

Lemma 2.9. *The following are equivalent:*

- $w \in \partial k(x)$.
- $w \cdot v \leq \dfrac{k(x+tv)-k(x)}{t}$ *for all* $t>0$ *and all* v.
- $w \cdot v \leq Dk_x(v)$ *for all* v.

Proof. If w is a subgradient of k at x, then $w \cdot (y-x) + k(x) \leq k(y)$ for all y. Taking $y = x + tv$ yields $w \cdot v \leq \dfrac{k(x+tv)-k(x)}{t}$ for all t, and then $w \cdot v \leq Dk_x(v)$ follows from letting $t \downarrow 0$.

Now, assume that $w \cdot v \leq \dfrac{k(x+tv)-k(x)}{t}$ for all $t>0$ and all v. Then, taking $t=1$ and setting $v = y-x$, we have $w \cdot (y-x) + k(x) \leq k(y)$ for all y, so that $w \in \partial k(x)$. Finally, assume that $w \cdot v \leq Dk_x(v)$. Then $w \cdot v \leq Dk_x(v) \leq \dfrac{k(x+tv)-k(x)}{t}$ by Lemma 2.6, so that $w \in \partial k(x)$. $\square$

Now we shall establish key properties of the function $v \mapsto Dk_x(v)$. While these are obvious in the case that k is differentiable, they must be proved for the convex case.

Lemma 2.10. *$Dk_x(v)$ is a positively homogeneous convex function in v; in particular $Dk_x(v)$ is continuous as a function of v.*

Proof. Use the definition of $Dk_x(tv+(1-t)w)$ and the fact that k is convex to get convexity (and therefore continuity) of Dk_x. To see the positive homogeneity, observe that for $\lambda > 0$,

$$Dk_x(\lambda v) = \lim_{t\downarrow 0} \frac{k(x+t\lambda v)-k(x)}{t} = \lambda \lim_{\lambda t\downarrow 0} \frac{k(x+t\lambda v)-k(x)}{\lambda t} = \lambda Dk_x(v). \tag{2.7}$$

$\square$

We are now ready to express the directional derivative in terms of the subgradient set.

Lemma 2.11. $Dk_x(v) = \max\{v \cdot w : w \in \partial k(x)\}$.

Proof. Dk_x is a convex function, so $Dk_x(v) = \sup\{\Lambda(v) : \Lambda \text{ affine}, \Lambda \leq Dk_x\}$. Assume w is a subgradient of k at x. Then $w \cdot y \leq Dk_x(y)$, so that $y \mapsto w \cdot y$ is in the family of affine functions above. Consider the affine function $\Lambda(y) = w \cdot y + c$, where w is a subgradient and $\Lambda \leq Dk_x$; evaluation at $y = 0$ implies that $c \leq 0$, so Λ is dominated by $y \mapsto w \cdot y$.

If w is not a subgradient of k at x, there exists z such that $w \cdot z > Dk_x(z)$ by Lemma 2.9. If $\Lambda(y) = w \cdot y + c$ and $\Lambda \leq Dk_x$, then $Dk_x(tz) \geq \Lambda(tz) = w \cdot tz + c$ for all $t > 0$. Using positive homogeneity of Dk_x, we have $t(Dk_x(z) - w \cdot z) \geq c$; since $Dk_x(z) - w \cdot z < 0$, letting $t \to \infty$ implies that c is less than every real number. Thus no such Λ exists, and $\{\Lambda(v) : \Lambda \text{ affine}, \Lambda \leq Dk_x\} = \{w \cdot v,\ w \in \partial k(x)\}$. Since $\partial k(x)$ is compact, the supremum is a maximum. ☐

Remark. This characterization of the subgradient set also follows from Clarke's Differentiation Theorem, which we shall discuss in Chapter 5.

Next, we show that the function $(x, v) \mapsto Dk_x(v)$ is upper semicontinous; we shall use this result when we develop semiconvex function theory in the next section. Recall that a function h is **upper semicontinuous** if whenever $\{x_i\}$ is a sequence of points converging to x_0 and $\lim_{i\to\infty} h(x_i)$ exists, we have $\lim_{i\to\infty} h(x_i) \leq h(x_0)$.

Lemma 2.12. *The function $(x, v) \mapsto Dk_x(v)$ is upper semicontinuous.*

Proof. Suppose that $(x_i, v_i) \to (x_0, v)$ and $\lim_{i\to\infty} Dk_{x_i}(v_i)$ exists. Further suppose that $w_i \in \partial k(x_i)$ satisfies $Dk_{x_i}(v_i) = w_i \cdot v_i$. Then $w_i \cdot (y - x_i) \leq k(y) - k(x_i)$ for all y. By taking $y = x_i + \dfrac{w_i}{|w_i|}$, we see that $|w_i| \leq k\left(x_i + \dfrac{w_i}{|w_i|}\right) - k(x_i)$. Since k is continuous, $|w_i|$ is at most twice the maximum of $|k|$ over a compact set containing x and all $x_i + \dfrac{w_i}{|w_i|}$. By passing to a subsequence the w_i converge to some w_0, and we may pass to the limit and obtain $w_0 \cdot (y - x_0) \leq k(y) - k(x_0)$ for all y. Thus $w_0 \in \partial k(x_0)$, so by Lemma 2.11 we have $Dk_{x_0}(v_0) \geq w_0 \cdot v_0 = \lim_{i\to\infty} w_i \cdot v_i = \lim_{i\to\infty} Dk_{x_i}(v_i)$.

☐

2.2 Semiconvex Function Theory

In this section, we generalize the theory of the previous section to semiconvex functions, define critical points and regular points for semiconvex functions, and discuss how subgradients determine the geometry of regular PR sets. Finally, we show that the "subgradient bundle" over a curve $\gamma : [0, L] \to \mathbb{R}^m$ is compact.

Definition 2.13. *Let f be a semiconvex function, and write $f = k + g$ for k convex and g smooth. Define the **subgradient set** of f at x, denoted $\partial f(x)$, by $\partial f(x) := \partial k(x) + \nabla g_x$.*

Lemma 2.14. *$w \in \partial f(x)$ if and only if $w \cdot v \leq Df_x(v)$ for all v.*

Proof. We know that $u \in \partial k(x)$ if and only if $u \cdot v \leq Dk_x(v)$ for all v, if and only if $(u + \nabla g_x) \cdot v \leq Dk_x(v) + \nabla g_x(v) = Df_x(v)$. □

Corollary 2.15. *$\partial f(x)$ is nonempty, convex, and compact.*

Proof. The set $\partial f(x)$ inherits these properties from $\partial k(x)$. □

Lemma 2.16. *$Df_x(v) = \max\{v \cdot w : w \in \partial f(x)\}$. The function $(x, v) \mapsto Df_x(v)$ is upper semicontinuous, and the function $v \mapsto Df_x(v)$ is convex and positively homogeneous.*

Proof. This is immediate, since $Df_x(v) = Dk_x(v) + \nabla g_x(v)$. □

We now define the subgradient bundle.

Definition 2.17. *Define the **subgradient bundle** $\mathbb{D}(f) \subset \mathbb{R}^{2m}$ by*

$$\mathbb{D}(f) = \{(x, v) : v \in \partial f(x)\}. \tag{2.8}$$

Remarks.

1. The subgradient bundle is not a fiber bundle, but it can be thought of as a bundle in a more generalized sense.

2. The subgradient bundle is the graph of the multifunction $x \mapsto \partial f(x)$, and a multifunction is upper semicontinuous if its graph is closed. Thus ∂f is upper semicontinuous as a multifunction if and only if $\mathbb{D}(f)$ is closed.

Lemma 2.18. $\mathbb{D}(f)$ *is closed.*

Proof. Suppose $\{(x_i, w_i)\}_{i=1}^{\infty} \subset \mathbb{D}(f)$ and $(x_i, w_i) \to (x, w)$. By Lemma 2.14, $w_i \cdot v \leq Df_{x_i}(v)$ for all v. By Lemma 2.16, we have that $\overline{\lim} Df_{x_i}(v) \leq Df_x(v)$ for all v. Thus $w \cdot v \leq Df_x(v)$ for all v, and $w \in \partial f(x)$ by Lemma 2.14. □

2.2.1 Subgradients, Regularity, and Sublevel Sets

Definition 2.19. *The set of* ***critical points*** *of f, denoted* $\mathrm{crit}(f)$*, is the set*

$$\mathrm{crit}(f) = \{x : 0 \in \partial f(x)\}. \tag{2.9}$$

The set of ***regular points*** *of f, denoted* $\mathrm{reg}(f)$*, is the complement of* $\mathrm{crit}(f)$*. A value of f is a* ***regular value*** *if all of its preimages are regular points. A value of f is a* ***critical value*** *if at least one of its preimages is a critical point.*

Lemma 2.20. *The point $p \in \mathrm{reg}(f)$ if and only if there exists v such that $Df_p(v) < 0$.*

Proof. ($\Longleftarrow$) Assume $p \notin \mathrm{reg}(f)$. By Lemma 2.16, we have $Df_p(v) = \sup\{v \cdot w : w \in \partial f(p)\}$, so $Df_p(v) \geq v \cdot 0 = 0$ for all v.
($\Longrightarrow$) Assume $p \in \mathrm{reg}(f)$. Write $f = k + g$ for k convex and g smooth. Then, since $0 \notin \partial f(p)$ and $\partial f(p) = \partial k(p) + \nabla g_p$, we have $-\nabla g_p \notin \partial k(p)$. By Lemma 2.9 there exists v such that $-\nabla g_p \cdot v > Dk_p(v)$, so that $Df_p(v) = Dk_p(v) + \nabla g_p(v) < 0$. □

We now return to our discussion of PR sets.

Definition 2.21. *Define* $\text{reach}(S,p) = \inf\{r : B(p,r) \subset \text{Unp}(S)\}$*, where* $B(p,r)$ *denotes the ball of radius* r *centered at* p*. We say that* S *has the* ***unique footpoint property*** *if* $\text{reach}(S,p) > 0$ *for all* $p \in S$*.*

Note that a PR set has the unique footpoint property. A compact set S is PR if and only if it has the unique footpoint property.

Definition 2.22. *We say that* p *is a* ***weakly regular point*** *of the semiconvex function* f *if* p *has the following property: if* $p_i \to p$ *with* $f(p_i) > f(p)$ *and* $w_i \in \partial f(p_i)$*, we have* $\lim_{i\to\infty} w_i \neq 0$*. A value* α *is a* ***weakly regular value*** *of* f *if each* $p \in f^{-1}(\alpha)$ *is a weakly regular point.*

Theorem 2.23 ([2])**.** *A set* $S \subset \mathbb{R}^m$ *has the unique footpoint property if and only if* $S = f^{-1}(-\infty, 0]$*, where* f *is a semiconvex function and* 0 *is a weakly regular value of* f*.*

We shall not use weak regularity; however, we shall frequently require that $S = f^{-1}(-\infty, 0]$, where 0 is a regular value of f. If 0 is a regular value of f, it is automatically a weakly regular value of f by Lemma 2.18.

Definition 2.24. *We say that a PR set* S *is* ***regular*** *if* $S = f^{-1}(-\infty, 0]$*, where* f *is a semiconvex function and* 0 *is a regular value of* f*.*

Remark. When f is a semiconvex function and 0 is a regular value of f, the sublevel set $A = f^{-1}(-\infty, 0]$ automatically satisfies $\text{reach}(A,p) > 0$ for all $p \in A$, but a set A with this property need not have $\text{reach}(A) > 0$. If A is compact, we can conclude that $\text{reach}(A) > 0$. However, it is possible to construct a noncompact set A with $\text{reach}(A,p) > 0$ for all p and $\text{reach}(A) = 0$: see Figure 2.1.

2.2.2 Subgradients and Normal Cones

Before we characterize the normal cone at a point of ∂S, we confirm that $\partial S = f^{-1}(0)$.

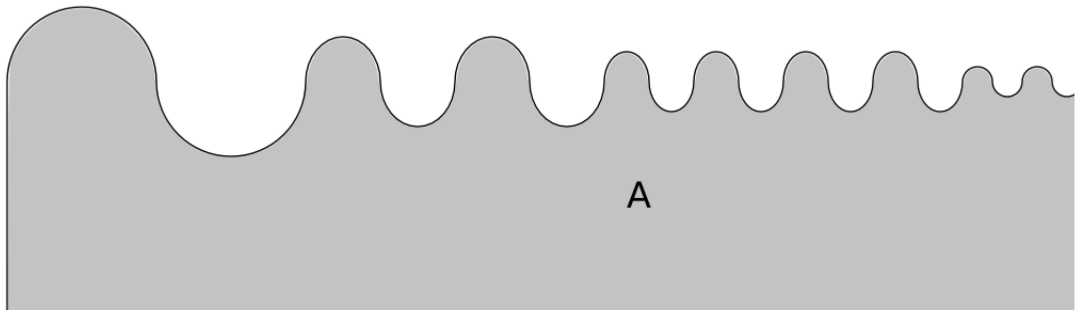

Figure 2.1: Assume the pattern continues. The set A has $\operatorname{reach}(A) = 0$ but $\operatorname{reach}(A, p) > 0$ for all $p \in A$.

Lemma 2.25. *Suppose $S = f^{-1}(-\infty, 0]$ is a regular PR set. Then $\partial S = \{x : f(x) = 0\}$.*

Proof. ($\supseteq$) Suppose $f(x) = 0$. By Lemma 2.20 there exists v with $Df_x(v) < 0$. Since $Df_x(v) = \lim_{t \downarrow 0} f(x + tv)/t$, for t sufficiently small we have $f(x + tv) < 0$ and $x + tv \in S$. Let $w \in \partial f(x)$. Then $w \neq 0$ since S is regular, and $Df_x(w) \geq |w|^2$ by Lemma 2.14. Thus $x + tw \notin S$ for all sufficiently small t, so that $x \in \partial S$.

($\subseteq$) Suppose $x \in \partial S$. Then there exists a sequence of points x_i such that $x_i \notin S$ and $x_i \to x$. Thus $f(x_i) > 0$ for all i, and $f(x) \geq 0$ by continuity of f. However, since S is closed, we have that $x \in S$ and $f(x) \leq 0$. □

Let $\operatorname{cone}(\partial f(x)) = \{\alpha w : w \in \partial f(x) \text{ and } \alpha \geq 0\}$. Note that $\operatorname{cone}(\partial f(x))$ is convex because $\partial f(x)$ is convex. The following theorem relates the normal cone to the subgradient set. The proof of the theorem follows from the lemmas below.

Theorem 2.26. *Let $S = f^{-1}(-\infty, 0]$ be a regular PR set. Then $\operatorname{cone}(\partial f(x)) = \operatorname{Nor}(S, x)$ for $x \in \partial S$.*

Proof. First, $\operatorname{Tan}(S, x) = \{z : Df_x(z) \leq 0\}$ follows from the lemmas below. Also, $Df_x(z) \leq 0$ iff $\max\{w \cdot z : w \in \partial f(x)\} \leq 0$, iff $w \cdot z \leq 0$ for all subgradients w at x, if and only if $z \in$

dual(cone($\partial f(x)$)). Thus $\mathrm{Tan}(S,x) = \mathrm{dual}(\mathrm{cone}(\partial f(x)))$. Taking the dual of each side yields the result. $\square$

Lemma 2.27. *Let $S = f^{-1}(-\infty, 0]$ be a regular PR set. Then*

$$\mathrm{cl}\{z : Df_x(z) < 0\} = \{z : Df_x(z) \leq 0\}, \tag{2.10}$$

where cl *denotes the closure of the set.*

Proof. ($\subseteq$) Suppose $z_i \to z$ and $Df_x(z_i) < 0$ for all i. Then $Df_x(z) \leq 0$ by Lemma 2.16. ($\supseteq$) Suppose $Df_x(v) = 0$. By Lemma 2.20, there exists w with $Df_x(w) < 0$. Let $\epsilon > 0$. By Lemma 2.16, we have $Df_x(v + \epsilon w) \leq Df_x(v) + \epsilon Df_x(w) = \epsilon Df_x(w)$. Since $v + \epsilon w \to v$ and $Df_x(v + \epsilon w) < 0$, we conclude that $v \in \mathrm{cl}\{z : Df_x(z) < 0\}$. $\square$

Lemma 2.28. *Let $S = f^{-1}(-\infty, 0]$ be a regular PR set. Suppose $x \in \partial S$, and let $w \in \partial f(x)$ satisfy $|w| = \min\{|v| : v \in \partial f(x)\}$. There is an open set U containing x such that, for all y in U, we have $Df_y(-w) < 0$ and $Df_y(w) > 0$.*

Proof. We have $w \neq 0$ because S is regular; also, we claim that $|w|^2 \leq v \cdot w$ for all $v \in \partial f(x)$. Let $F(\lambda) = |\lambda v + (1-\lambda)w|^2 - |w|^2$. Then $F(\lambda) \in \partial f(x)$ for all $\lambda \in [0, 1]$, so that F is nondecreasing in a half neighborhood of 0. Thus $F'(0) = -2|w|^2 + 2v \cdot w \geq 0$. By Lemma 2.16, $Df_x(-w) = \max\{-w \cdot z : z \in \partial f(x)\} = -\min\{w \cdot z : z \in \partial f(x)\} = -|w|^2 < 0$.

Since $Df_y(-w)$ is upper semicontinuous as a function of y, there exists a neighborhood U of x satisfying $Df_y(-w) \leq -|w|^2/2$ for all $y \in U$. Now, using Lemma 2.16, $0 = Df_y(w-w) \leq Df_y(w) + Df_y(-w) \leq Df_y(w) - |w|^2/2$. $\square$

Lemma 2.29. *Let $S = f^{-1}(-\infty, 0]$ be a regular PR set, and suppose $x \in \partial S$. Then*

$$\mathrm{Tan}(S,x) = \mathrm{cl}\{z : Df_x(z) < 0\} = \{z : Df_x(z) \leq 0\}. \tag{2.11}$$

Proof. We need only establish the equality $\mathrm{Tan}(S,x) = \mathrm{cl}\{z : Df_x(z) < 0\}$.

($\subseteq$) First, note that $v \in \mathrm{Tan}(S,x)$ is a limit of vectors of the form $\frac{|v|}{|y-x|}(y-x)$ for $y \in S$. By 2.20, there exists u such that $Df_y(u) < 0$. Thus, by perturbing each y slightly in a decreasing direction (at y) if necessary, we may assume that $f(y) < 0$ for all y. By Lemma 2.8,

$$Df_x(v) \leq |v| \lim_{y \to x} \frac{f(y) - f(x)}{|y-x|}. \tag{2.12}$$

Since $f(x) = 0$ by assumption, and each $f(y) < 0$, we have $Df_x(v) \leq 0$.

($\supseteq$) Suppose $Df_x(z) < 0$. Since $Df_x(z) = \lim_{t \downarrow 0} \dfrac{f(x+tz) - f(x)}{t}$, we have that $f(x+tz) < 0$ for all sufficiently small $t > 0$. Hence each $x + tz \in \mathrm{Interior}(S)$, and $x + tz \to x$ with $\dfrac{(x+tz)-x}{|(x+tz)-x|} - \dfrac{z}{|z|} = 0$, so $z \in \mathrm{Tan}(S,x)$. Since $\mathrm{Tan}(S,x)$ is closed, $\mathrm{cl}\{z : Dk_x(z) < 0\} \subseteq \mathrm{Tan}(S,x)$. $\square$

Thus we have shown that the subgradients determine the normal cones along the boundary. Now we prove some results about PR sets; we shall use these results in Chapter 6.

Lemma 2.30. *Let $S \subset \mathbb{R}^m$ be a regular PR set. Then for any $x \in \partial S$, there exists a neighborhood U of x such that $U \cap \partial S$ is the graph of a Lipschitz function. In other words, there is an open set $V \subset \mathbb{R}^{m-1}$ and a Lipschitz function ϕ on V such that $U \cap \partial S = \{(y, \phi(y)) : y \in V\}$.*

Proof. Let $x \in \partial S$, and write $S = f^{-1}(-\infty, 0]$. By Lemma 2.28 there exists a neighborhood U containing x and a direction v such that, for all $y \in U$, we have $Df_y(v) > 0$ and $Df_y(-v) < 0$. Let $z = \dfrac{v}{|v|}$. By applying a rigid motion, assume that $x = 0$ and $z = e_m = (0, \ldots, 0, 1)$. For $y \in (U \cap \{x_m = 0\})$, we may move up or down from y until either we reach a point $y' \in \partial S \cap U$ such that $f(y') = 0$ or we leave U. It is obvious that if such a $y' \in \partial S \cap U$ exists, it must be unique, since f is strictly monotone on each vertical segment in U.

We claim that there exists an open ball B in $\{x_m = 0\} \cong \mathbb{R}^{m-1}$ such that ∂S is a graph over B. Such a neighborhood B exists: if not, there exists a sequence of points $y_i \in P$

converging to 0 such that f is nonzero on the vertical segment in U through each y_i. By passing to a subsequence, we may assume $f(y_i) > 0$ for all i or $f(y_i) < 0$ for all i. Suppose that $f(y) > 0$ for all y on the vertical segment through y_i for all i. Then by continuity of f, $f(y) \geq 0$ for every y on the vertical line thru 0. This is impossible, since $f(0) = 0$ and $-z$ is a decreasing direction. A similar contradiction follows from assuming $f(y_i) < 0$ for all i.

For $y = (y_1, \ldots, y_{m-1}) \in B \subset \{x_m = 0\}$, suppose that $y' = (y'_1, \ldots, y'_m) \in \partial S$ and $y_i = y'_i$ for all $1 \leq i \leq m-1$. Let $\phi : B \to \mathbb{R}$ by $\phi(y) = y'_m = y' \cdot z$. Then ϕ is well-defined, and $(y, \phi(y)) = y' \in \partial S$. Further, ϕ is Lipschitz if and only if there exists M such that $|(z \cdot (p-q)| \leq M|(p-q) - ((p-q) \cdot z)z|$ for all p, q in the graph of ϕ. Thus ϕ is Lipschitz if and only if $|z \cdot (p-q)|^2 \leq M^2 |(p-q) - ((p-q) \cdot z)z|^2$, if and only if $|(p-q) \cdot z|^2 \leq \dfrac{M^2}{M^2+1}|p-q|^2$, if and only there exists $\alpha \in [0,1)$ such that $\dfrac{|(p-q) \cdot z|}{|p-q|} \leq \alpha$.

By replacing B with a smaller open set if necessary, ϕ is Lipschitz. If not, there exist sequences of points p_j and q_j in ∂S such that $p_j, q_j \to 0$ and $\dfrac{p_j - q_j}{|p_j - q_j|} \to y$ with $\left| \dfrac{(p_j - q_j) \cdot z}{|p_j - q_j|} \right| \to 1$. Then $|y \cdot z| = 1$ and the Cauchy-Schwarz inequality implies that $y = z$ or $y = -z$. Without loss of generality, assume that $y = z$.

Since $Df_x(-z) < 0$ and Df is upper semicontinuous, for sufficiently large j, we have that $q_j - p_j$ is a decreasing direction on a neighborhood of p_j. Since f is locally Lipschitz, we can recover $f(q_j) - f(p_j)$ by integrating along the segment from p_j to q_j:

$$0 = f(q_j) - f(p_j) = \int_{[0,1]} Df_{(1-t)p_j + tq_j}(q_j - p_j)dt. \tag{2.13}$$

Thus there exists y_j in between p_j and q_j satisfying $Df_{y_j}(q_j - p_j) > 0$. By upper semicontinuity of Df,

$$Df_x(-z) \geq \overline{\lim_{j\to\infty}} Df_{y_j}\left(\frac{q_j - p_j}{|q_j - p_j|}\right) \geq 0. \tag{2.14}$$

This is a contradiction. $\square$

Now we establish that the subgradient bundle restricted to a curve is a compact set. By Lemma 2.18, we have that $\mathbb{D}(f)$ is closed, or equivalently that the multifunction $x \mapsto \partial f(x)$ is upper semicontinuous. The graph of an upper semicontinuous multifunction is compact whenever its domain is compact and the (range of the) multifunction is locally bounded. Thus we only need to show that the multifunction ∂f is locally bounded.

Lemma 2.31. *The multifunction $x \mapsto \partial f(x)$ is locally bounded.*

Proof. For purposes of contradiction, assume ∂f is unbounded on every neighborhood of x_0. Then there exists a sequence (x_i, v_i) of points of $\mathbb{D}(f)$ satisfying $|(x_i, v_i)| \to \infty$ and $x_i \to x_0$. By Lemma 2.16, $Df_{x_i}(v_i) \geq |v_i|^2$ and $Df_{x_i}\left(\dfrac{v_i}{|v_i|}\right) \geq |v_i|$. By passing to a subsequence, we may assume that the $\dfrac{v_i}{|v_i|} \to z_0$; therefore by upper semicontinuity of Df, $Df_{x_0}(z_0) \geq \overline{\lim_{i\to\infty}} Df_{x_i}\left(\dfrac{v_i}{|v_i|}\right) = \infty$. This is a contradiction. $\square$

Corollary 2.32. *Let $\gamma : [0,1] \to \mathbb{R}^m$ be continuous, and let $Y = \mathbb{D}(f) \cap (\gamma \times \mathbb{R}^m)$. Then Y is compact.*

For $x \in \partial S$, write $\operatorname{nor}(x)$ for $\operatorname{nor}(S, x)$. It follows from Lemma 2.26 and Lemma 2.31 that the multifunction $x \mapsto \operatorname{nor}(x)$ is upper semicontinuous on ∂S.

Corollary 2.33. *If S is a regular PR set, then $x \mapsto \operatorname{nor}(x)$ is upper semicontinuous on ∂S.*

Proof. Suppose that $(x_i, v_i) \to (x, v)$ and $v_i \in \operatorname{nor}(x_i)$ for all i. Since ∂S is closed, we have $x \in \partial S$ as well. By Lemma 2.26 for each i there exists $\lambda_i > 0$ such that $\lambda_i v_i \in \partial f(x_i)$. By passing to a subsequence, we may assume that the $\lambda_i v_i$ converge to some nonzero $w \in \partial f(x)$ by Lemma 2.31, Lemma 2.18, and the assumption that S is regular. However, then $\dfrac{w}{|w|} \in \operatorname{nor}(x)$, and $v = \lim_{i\to\infty} v_i = \lim_{i\to\infty} \dfrac{\lambda_i v_i}{|\lambda_i v_i|} = \dfrac{w}{|w|}$. Thus $v \in \operatorname{nor}(x)$, and $x \mapsto \operatorname{nor}(x)$ is upper semicontinuous. $\square$

Chapter 3

Geodesics

In this chapter, we discuss geodesics in smooth Riemannian manifolds, in smooth Riemannian manifolds with boundary, and in PR sets.

3.1 Background

Definition 3.1. *Let (M, d) be a metric space. A* ***geodesic*** *of M is a curve $\gamma : [0, L] \to M$ such that there exists $\delta > 0$ for which $|t_0 - t_1| < \delta$ implies $d(\gamma(t_0), \gamma(t_1)) = |t_0 - t_1|$. When this is the case, we shall say that γ* ***locally minimizes length.***

Examples. *The following are examples of geodesics in subsets of Euclidean space.*

1. *In Euclidean space, a curve is a geodesic if and only if it is a straight line segment.*

2. *In the unit sphere, a curve is a geodesic if and only if it is a great circle arc.*

3. *In the open unit ball and other convex subsets of Euclidean space, a curve is a geodesic if and only if it is a straight line segment.*

Recall the well-known fact that any rectifiable curve has a parametrization by arclength.

Lemma 3.2. *Let (M, d) be a metric space. Any rectifiable curve in M has a parametrization by arclength.*

Proof. Assume that $\gamma : [a, b] \to M$ has length L, and define $l(t) := \text{length}\left(\gamma\,\big|_{[a,t]}\right)$, the length of the restriction of γ to $[a, t]$. Then l is a continuous and increasing function, and we may assume it is strictly increasing. For $t \in [a, b]$, let $s = l(t)$, and define the reparametrization $\tilde{\gamma}$ of γ by $\tilde{\gamma}(s) = \gamma(t)$. Then $\tilde{\gamma} : [0, L] \to M$ is well-defined. Write $s_1 = l(t_1)$ and $s_2 = l(t_2)$ for $t_1 < t_2$. We then have that $\text{length}\left(\tilde{\gamma}\,\big|_{[s_1,s_2]}\right) = \text{length}\left(\gamma\,\big|_{[t_1,t_2]}\right) = l(t_2) - l(t_1) = s_2 - s_1$, so that $\tilde{\gamma}$ is arclength parametrized. $\square$

In [7], geodesics in Riemannian manifolds are defined by the following, which is equivalent to our definition. (See [7, Proposition 3.6 and Corollary 3.9].)

Lemma 3.3. *Let ∇ be the Riemannian connection associated to the Riemannian manifold (M, g). An arclength parametrized curve $\gamma \subset M$ is a geodesic if and only if $\nabla_{\gamma'}(\gamma') \equiv 0$.*

When $M \subset \mathbb{R}^m$ has the Riemannian metric induced by the Euclidean metric on $\mathbb{R}^m$, geodesics are characterized by the following normality condition.

Theorem 3.4. *If $M \subset \mathbb{R}^m$ has the Riemannian metric induced by the Euclidean metric, then γ is a geodesic if and only if its Euclidean acceleration vector is normal to M.*

Proof. Write ∇ for the connection on M, and write v^T to denote the tangential part of $v \in \mathbb{R}^m$; let γ'' denote the (Euclidean) acceleration vector of γ. Using the fact that $\nabla_{\gamma'}\gamma' = (\gamma'')^T$ by [7, Chapter 2], we have that $\nabla_{\gamma'}(\gamma') \equiv 0$ exactly when the projection onto the tangent space is zero, i.e. when the acceleration vector of γ is normal to M. $\square$

In the setting of smooth Riemannian manifolds without boundary, it is well known that geodesics are smooth. We shall see that, even in the case of smooth Riemannian manifolds with boundary, geodesics are not in general smooth. For a proof of smoothness of geodesics in smooth Riemannian manifolds, see, for example [7, Chapter 3, Corollary 3.9].

3.2 Geodesics in Manifolds with Boundary

Let (M, g) be a smooth Riemannian manifold with boundary. In [1], the authors prove the following result about regularity of geodesics in Riemannian manifolds with boundary:

Theorem 3.5. *Let (M, g) be a smooth Riemannian manifold with boundary. An arclength parametrized geodesic $\gamma \subset M$ is twice differentiable except at countably many points; at points where the curve fails to be twice differentiable, the one-sided acceleration exists.*

Remark. In this setting, geodesics in general do not have vanishing acceleration. However, on interior segments, the acceleration exists and is zero, and on boundary segments, the acceleration exists and is outwardly normal to M. (For a discussion, see [1].)

Consider the following example:

Example. *Let M be the complement of the open unit ball centered at the origin in $\mathbb{R}^3$. Then the curve given by*

$$\gamma(t) = \begin{cases} (0, t-1, 1) & \text{if } 0 \leq t \leq 1 \\ (0, \sin(t-1), \cos(t-1)) & \text{if } 1 < t \leq 2 \end{cases} \tag{3.1}$$

is a geodesic in M.

This curve has continuous velocity vector, but its acceleration is undefined at $t = 1$. Note that $t = 1$ is where the geodesic switches from the interior of M to the boundary of M. Also, note that this geodesic is $C^{1,1}$, i.e. its velocity is Lipschitz continuous. It is remarkable, then, that geodesics in regular PR sets are in fact $C^{1,1}$.

3.3 Geodesics in PR Sets

Let S be a PR set in Euclidean space.

Lemma 3.6. *If $\gamma : [0, L] \to S$ is a locally shortest path, then γ has finite length.*

Proof. It suffices to show that γ has finite length on a neighborhood of each $t \in [a, b]$, since then the domain can be covered by finitely many such neighborhoods by compactness. Fix t. If $\gamma(t) \in \text{Interior}(S)$, there exists $\epsilon > 0$ such that $\gamma(t - \epsilon, t + \epsilon)$ is contained in the interior of S, so γ parametrizes a Euclidean line segment on $(t - \epsilon, t + \epsilon)$; otherwise γ would not be locally shortest.

If $\gamma(t) \in \partial S$, let U be a neighborhood of $\gamma(t)$ satisfying $\overline{U} \subset \text{Unp}(S)$. For any two points $p, q \in U$, the Euclidean line segment $\beta(s) = (1 - s)p + sq$ satisfies $\beta \subset U$. By [9, Thoerem 4.8 (8)], we have that π_S is Lipschitz, so that by the chain rule $\pi_S \circ \beta$ is a finite length path in S joining p and q. □

Lemma 3.7. *Assume that $\gamma \subset S$ is a rectifiable curve. Then γ has a reparametrization $\tilde{\gamma}$ by arclength, and $|\tilde{\gamma}'| = 1$ almost everywhere.*

Proof. By Lemma 3.2, the reparametrization $\tilde{\gamma} : [0, L] \to M$ by arclength exists, and it has Lipschitz constant 1. By Rademacher's theorem, we have that $\tilde{\gamma}'$ exists almost everywhere, and $|\tilde{\gamma}'(s)| \leq 1$. Since $\tilde{\gamma}$ is absolutely continuous, $\text{length}(\tilde{\gamma}) = \displaystyle\int |\tilde{\gamma}'| = L$. Therefore $|\tilde{\gamma}'| = 1$ almost everywhere. □

From now on, we shall assume that the geodesic γ is parametrized by arclength. We shall use the following properties of geodesics to prove the first theorem.

Lemma 3.8. *Suppose γ is an arclength-parametrized geodesic in S. Then γ is locally injective and Lipschitz continuous with Lipschitz constant 1. On any subsegment of the image of γ on which it is defined, γ^{-1} is Lipschitz with Lipschitz constant 1.*

Proof. The curve γ is Lipschitz continuous with Lipschitz constant 1 by Lemma 3.2. Since $|\gamma(t_1) - \gamma(t_2)| = |t_1 - t_2|$, the inverse function γ^{-1} is also Lipschitz with Lipschitz constant 1 on any subsegment of γ on which it is defined. It is therefore sufficient to show that γ is locally injective. Fix t, and let $\epsilon > 0$ be small enough that γ is a curve of minimum length joining pairs of points in $\gamma([t-\epsilon, t+\epsilon])$. Then γ is injective on $[t-\epsilon, t+\epsilon]$, since if we had $\gamma(t_0) = \gamma(t_1)$ for $t_1,\ t_2 \in [t-\epsilon, t+\epsilon]$, one could shorten $\gamma\big|_{[t-\epsilon,t+\epsilon]}$ by removing the portion of γ from $\gamma(t_0)$ to $\gamma(t_1)$. Thus $\gamma\big|_{(t-\epsilon,t+\epsilon)}$ is a bijection onto its image. $\square$

Remark. From now on, we shall consider only geodesics γ with $\gamma \cap \partial S \neq \emptyset$. Later we shall prove that such geodesics have Lipschitz continuous velocity. Any geodesic with $\gamma \cap \partial S = \emptyset$ is a geodesic in Euclidean space, so it is C^∞-smooth. Thus, by considering only the case $\gamma \cap \partial S \neq \emptyset$, we focus on the geodesics whose regularity is not yet known.

Chapter 4

Generalized Kuhn-Tucker Theorem

In this chapter we discuss the generalized Kuhn-Tucker Theorem of [4, Theorem 5.4]. It can be thought of as a generalization of the theory of Lagrange multipliers from calculus; in the generalized Kuhn-Tucker Theorem, however, we seek an integral over function values (which can be thought of as an infinite linear combination over some gradients) in place of a linear combination of gradients. We shall use the generalized Kuhn-Tucker Theorem in Chapter 6.

In this chapter and the next, we shall make frequent use of the Riesz Representation Theorem.

Definition 4.1. *Let Y be a locally compact space, and let Ω denote the smallest σ-algebra of Y containing the open subsets of Y. Let μ be a signed measure on Y, and let $\mu = \mu^+ - \mu^-$ denote the Hahn-Jordan decomposition of μ. Write $|\mu| = \mu^+ + \mu^-$. We say that μ is a* ***regular Borel measure*** *if*

1. *$|\mu|(K) < \infty$ for every compact $K \subset Y$,*

2. *for any $E \in \Omega$, we have $|\mu|(E) = \sup\{|\mu|(K) : K \subset E$ and K is compact$\}$, and*

3. *for any $E \in \Omega$, we have $|\mu|(E) = \inf\{|\mu(U)| : U \supset E$ and U is open$\}$.*

Theorem 4.2 (Riesz Representation Theorem [6, Theorem C.18])**.** *Let Y be a locally compact space, and let $M(Y)$ denote the set of real-valued regular Borel measures on Y. Define the norm mass(μ) by mass(μ) $= \mu^+(Y) + \mu^-(Y)$. Let $C(Y)$ denote the set of continuous functions on Y endowed with the sup norm $\|\cdot\|_\infty$. Let $C^*(Y)$ denote the space of continuous linear functionals on $C(Y)$, endowed with the operator norm. For $\mu \in M(Y)$, define $F_\mu : C(Y) \to \mathbb{R}$ by*

$$F_\mu(f) = \int f d\mu. \tag{4.1}$$

Then $F_\mu \in C^*(Y)$ and the map $\mu \mapsto F_\mu$ is an isometric isomorphism of $M(Y)$ onto $C^*(Y)$.

4.1 Statement of the Theorem and an Example

For a continuous function g, let g^+ and g^- denote the positive and negative parts of g, so that $g = g^+ - g^-$. The following is the generalized Kuhn-Tucker theorem of [4, Theorem 5.4].

Theorem 4.3. *Let X be any vector space and Y be a compact Hausdorff space. For any linear functional λ on X and any linear map $L : X \to C(Y)$, the following are equivalent:*

1. *There exists $\epsilon > 0$ such that $\|(L\xi)^-\|_\infty \geq \epsilon$ for all $\xi \in X$ with $\lambda(\xi) = -1$.*
2. *There exists a nonnegative regular Borel measure $\mu \in M(Y)$ such that $\lambda(\xi) = \mu(L\xi)$ for all $\xi \in X$.*

To gain some insight into the theorem and the proof, consider the special case of $X = \mathbb{R}$ and $Y = \{p\}$ = a single point $\subset \mathbb{R}$. Here we consider linear functionals λ on $\mathbb{R}$; any such λ has the form $\lambda(x) = c_\lambda x$. Further, in this case the linear map L has the form $L(x) = c_L x$.

Theorem 4.4. *For any linear functional λ on $\mathbb{R}$ and any linear map $L : \mathbb{R} \to C(Y) \cong \mathbb{R}$, the following are equivalent:*

1. *There exists $\epsilon > 0$ such that $(L\xi) \leq -\epsilon$ for all $\xi \in \mathbb{R}$ with $\lambda(\xi) = -1$.*

2. *There exists a nonnegative number μ such that $\lambda(\xi) = \mu \cdot (L\xi)$ for all $\xi \in \mathbb{R}$.*

Proof. If $\lambda = 0$, the theorem is trivially true. Assume $\lambda \neq 0$. The first condition says that if $c_\lambda \xi = -1$, then $c_L \xi \leq -\epsilon$. However, if $c_\lambda \xi = -1$, we know that $\xi = \dfrac{-1}{c_\lambda}$ and then $-c_L/c_\lambda \leq -\epsilon$, and c_L and c_λ have the same sign. In particular $c_L \neq 0$, so that $c_\lambda = (c_\lambda/c_L)c_L$ and $\lambda = (c_\lambda/c_L)L$.

Assuming the second condition, since $c_\lambda = \mu c_L$, we know that if $\lambda(x) = -1$, $c_\lambda x = -1$, so that $c_L x = (c_\lambda/\mu)x$, and then $c_L x = -1/\mu$, and in particular, $Lx \leq -1/\mu$. □

4.2 Proof of the Generalized Kuhn-Tucker Theorem

Our proof is a slight elaboration of the one in [4].

Proof. (2 $\implies$ 1) Suppose there exists a nonnegative regular Borel measure μ such that $\mathrm{mass}(\mu) < \infty$ and, for all $\xi \in X$, $\lambda(\xi) = \int_Y L\xi d\mu$. Decomposing $L\xi$ into positive and negative parts, $\lambda(\xi) = \int_Y ((L\xi)^+ - (L\xi)^-)\, d\mu$. Thus if $\lambda(\xi) = -1$,

$$-1 = \int_Y (L\xi)^+ d\mu - \int_Y (L\xi)^- d\mu. \tag{4.2}$$

Since μ is nonnegative, the first term on the right-hand side of (6.10) is nonnegative, so that

$$-\int_Y (L\xi)^- d\mu \leq -1. \tag{4.3}$$

Since $\mathrm{mass}(\mu) < \infty$ and $(L\xi)^-$ is nonnegative, we have the estimate

$$1 \leq \int_Y (L\xi)^- d\mu \leq \mu(Y) \|(L\xi)^-\|_\infty, \tag{4.4}$$

and $\|(L\xi)^-\|_\infty \geq \dfrac{1}{\mu(Y)}$.

($1 \implies 2$) Define a metric on $\mathbb{R} \times C(Y)$ by $\|(a,g)\| = \sqrt{a^2 + \|g\|_\infty^2}$. Let $I = \text{Image}(\lambda, L)$, and let $\mathcal{O} = (-\infty, -1] \times P$, where $P \subset C(Y)$ is the set of nonnegative continuous functions. By hypothesis, $I \cap \mathcal{O} = \emptyset$.

Claim. $\text{dist}(I, \mathcal{O}) > 0$

Take sequences of points (t_i, z_i) and $(\lambda\xi_i, L\xi_i)$ in $\mathcal{O}$ and I, respectively, such that the difference $v_i := (t_i - \lambda\xi_i, z_i - L\xi_i)$ satisfies $\|v_i\| \to \text{dist}(I, \mathcal{O})$. Without loss of generality we may assume that $v_i \in \mathbb{R}_{\leq 0} \times P$, by taking $t_i = \min\{\lambda\xi_i, -1\}$ and $z_i = (L\xi_i)^+$. If $\lambda\xi_i \leq -1$, then the nearest element of $(-\infty, -1]$ to $\lambda\xi_i$ is $\lambda\xi_i$. If $\lambda\xi_i > -1$, then -1 is the nearest element of $(-\infty, -1]$ to $\lambda\xi_i$. In either case, we have that $\lim |t_i - \lambda\xi_i| = \lim |\min\{\lambda\xi_i, -1\} - \lambda\xi_i| = \lim |\min\{0, -1 - \lambda\xi_i\}|$. Since $(L\xi_i)^+$ is the nearest element of P to $L\xi_i$, we have $\lim \|z_i - L\xi_i\|_\infty = \lim \|(L\xi_i)^+ - L\xi_i\|_\infty$, and we may take $z_i = (L\xi_i)^+$.

Thus, following these simplifications, we have that $v_i = (\min\{-1 - \lambda\xi_i, 0\}, (L\xi_i)^-)$. To finish the proof of the claim, we shall show that the first coordinate of v_i is uniformly negative:

Claim. *The $t_i - \lambda\xi_i$ are uniformly negative, i.e. there exists $\beta > 0$ such that $t_i - \lambda\xi_i \leq -\beta < 0$ for all sufficiently large i.*

When $t_i - \lambda\xi_i = 0$, we know that $\lambda\xi_i \leq -1$, and otherwise we have $t_i - \lambda\xi_i = -1 - \lambda\xi_i$ with $\lambda\xi_i > -1$. If the $t_i - \lambda\xi_i$ are not uniformly negative, then $t_i - \lambda\xi_i \to 0$, and then we must have $\lim \lambda\xi_i \leq -1$ and in particular $\lambda\xi_i < 0$ for all sufficiently large i. Then, by rescaling the ξ_i, there exists a sequence of ξ_i satisfying $\lambda\xi_i = -1$ for all i and $(-1, L\xi_i) \in I$. Note that by rescaling the ξ_i in this way we have not increased $\lim \|(L\xi_i)^-\|_\infty$, so that $\lim \|v_i\| = \text{dist}(I, \mathcal{O})$ still holds.

Define $d_i := \text{dist}((-1, L\xi_i), \mathcal{O}) = \text{dist}(L\xi_i, P) = \|(L\xi_i)^-\|_\infty$. By hypothesis, $\|(L\xi_i)^-\|_\infty \geq \epsilon$, so $d_i \geq \epsilon$. Assume without loss of generality that $\epsilon < 1$. Consider the distance from $(1 - \epsilon^2)(-1, L\xi_i) = (-1 + \epsilon^2, (1 - \epsilon^2)L\xi_i) \in I$ to $\mathcal{O}$: the distance from $-1 + \epsilon^2$ to $(-\infty, -1]$

is ϵ^2, and the distance from $(1-\epsilon^2)L\xi_i$ to P is $(1-\epsilon^2)\|(L\xi_i)^-\|_\infty$. Since $d_i \geq \epsilon$, the distance from $(1-\epsilon^2)(-1, L\xi_i)$ to $\mathcal{O}$ satisfies

$$\begin{aligned} \operatorname{dist}((1-\epsilon^2)(-1, L\xi_i), \mathcal{O}) &= \sqrt{\epsilon^4 + (1-\epsilon^2)^2\|(L\xi_i)^-\|_\infty^2} \\ &= \sqrt{\epsilon^4 + (1-\epsilon^2)^2 d_i^2} \\ &= d_i\sqrt{\epsilon^4/d_i^2 + (1-\epsilon^2)^2} \\ &\leq d_i\sqrt{\epsilon^2 + 1 - 2\epsilon^2 + \epsilon^4} \\ &= d_i\sqrt{1-\epsilon^2(1-\epsilon^2)}. \end{aligned} \tag{4.5}$$

Thus $\operatorname{dist}(I, \mathcal{O}) \leq \lim_{i\to\infty} \operatorname{dist}((1-\epsilon^2)(-1, L\xi_i), \mathcal{O}) < \lim_{i\to\infty} d_i = \operatorname{dist}(I, \mathcal{O})$. This is a contradiction, so the $t_i - \lambda\xi_i$ are uniformly negative.

Let $\beta > 0$ satisfy $t_i - \lambda\xi_i < -\beta < 0$ for all i. Then, in particular, we have $t_i - \lambda\xi_i = -1 - \lambda\xi_i$ and $t_i = -1$ for all i. By the Hahn-Banach Theorem [6, Theorem 6.8] there exists $(c_i, \nu_i) \in \mathbb{R} \times C^*(Y)$ such that

$$\begin{aligned} &(c_i, \nu_i)(v_i) = 1, \\ &(c_i, \nu_i)|_I = 0, \text{ and} \\ &\|(c_i, \nu_i)\| = \frac{1}{\operatorname{dist}(v_i + I, (0,0))} = \frac{1}{\operatorname{dist}(v_i, I)}. \end{aligned} \tag{4.6}$$

Since $(c_i, \nu_i)(t, f) = c_i \cdot t + \nu_i(f)$, we have

$$\begin{aligned} \frac{1}{\operatorname{dist}(v_i + I, (0,0))} &= \|(c_i, \nu_i)\| \\ &= \sup_{\|(t,f)\|=1} |c_i t + (\nu_i(f))| \\ &= \sqrt{c^2 + (\operatorname{mass}(\nu_i))^2} \\ &\geq \operatorname{mass}(\nu_i). \end{aligned} \tag{4.7}$$

Claim. *There exists $\beta' > 0$ such that $c_i \leq -\beta' < 0$ for all i.*

Since

$$\begin{aligned}
&1 = (c_i, \nu_i)(v_i) = c_i(-1 - \lambda(\xi_i)) + \nu_i((L\xi_i)^-), \\
&\beta < 1 + \lambda\xi_i, \text{ and} \\
&\|(L\xi_i)^-\|_\infty = \sqrt{\|v_i\|^2 - (1 + \lambda\xi_i)^2},
\end{aligned} \tag{4.8}$$

by (4.7) we have

$$\begin{aligned}
c_i &= \left(\frac{1}{1 + \lambda(\xi_i)}\right)\big(\nu_i((L\xi_i)^-) - 1\big) \\
&\leq \frac{1}{\beta}\big(\text{mass}(\nu_i)\|(L\xi_i)^-\|_\infty - 1\big) \\
&\leq \frac{1}{\beta}\left(\frac{\sqrt{\|v_i\|^2 - (1 + \lambda\xi_i)^2}}{\text{dist}(v_i + I, (0,0))} - 1\right).
\end{aligned} \tag{4.9}$$

Using the facts that $v_i = (-1 - \lambda\xi_i, (L\xi_i)^-)$, and $(-1, (L\xi_i)^+) \in \mathcal{O}$,

$$\begin{aligned}
\text{dist}(v_i + I, (0,0)) &= \inf\{\|v_i + u\| : u \in I\} \\
&= \inf\{\|(-1 - \lambda\xi_i, (L\xi_i)^-) + u\| : u \in I\} \\
&= \inf\{\|(-1 - \lambda\xi_i, (L\xi_i)^-) + (u + (\lambda\xi_i, L\xi_i))\| : u \in I\} \\
&= \inf\{\|(-1, (L\xi_i)^+) + u\| : u \in I\} \\
&\geq \text{dist}(I, \mathcal{O}).
\end{aligned} \tag{4.10}$$

Also, since $(0,0) \in I$,

$$\text{dist}(v_i + I, (0,0)) = \inf\{\|(-1 - \lambda\xi_i, (L\xi_i)^-) + u\| : u \in I\} \leq \|v_i\|. \tag{4.11}$$

Since $\|v_i\| \to \mathrm{dist}(I, \mathcal{O})$, from (4.9) and (4.10) we conclude that

$$\mathrm{dist}(v_i + I, (0,0)) \to \mathrm{dist}(I, \mathcal{O}). \tag{4.12}$$

Passing to the limit in (4.9) and using (4.12) and the fact that $1 + \lambda\xi_i \geq \beta$, we obtain

$$\varlimsup_{i\to\infty} c_i \leq \frac{1}{\beta}\left(\frac{\sqrt{\mathrm{dist}^2(I, \mathcal{O}) - \beta^2}}{\mathrm{dist}(I, \mathcal{O})} - 1\right) < 0. \tag{4.13}$$

Thus there exists $\beta' > 0$ such that $c_i \leq -\beta' < 0$. By (4.10), we have $\|(c_i, \nu_i)\| = \dfrac{1}{\mathrm{dist}(v_i + I, (0,0))} \leq \dfrac{1}{\mathrm{dist}(I, \mathcal{O})}$; by Alaoglu's Theorem [13, Theorem 3.17], by passing to a subsequence there exists (c, ν) such that

$$\begin{aligned} (c_i, \nu_i)(t, z) &\to (c, \nu)(t, z), \\ \nu_i^+(z) &\to \nu^+(z), \\ \nu_i^-(z) &\to \nu^-(z), \text{ and} \\ c_i &\to c \end{aligned} \tag{4.14}$$

for all $(t, z) \in \mathbb{R} \times C(Y)$. Note that (4.6) and (4.7) imply that $\mathrm{mass}(\nu) < \infty$, $c < -\beta' < 0$, and $(c, \nu)_I = 0$.

Define $\mu = \dfrac{1}{|c|}\nu$; then μ is a regular Borel measure by Theorem 4.2. For any $\xi \in X$,

$$0 = (-1, \mu)(\lambda\xi, L\xi) = -\lambda\xi + \mu(L\xi), \tag{4.15}$$

or equivalently $\lambda\xi = \mu(L\xi)$. Thus we only need to show that the measure ν is nonnegative. Decompose ν into positive and negative parts: $\nu = \nu^+ - \nu^-$. Since $(L\xi_i)^-$ is nonnegative, by applying the Cauchy-Schwarz inequality at (4.16) we obtain

$$
\begin{aligned}
1 &= (c_i, \nu_i)(v_i) \\
&= c_i(-1-\lambda\xi_i) + \int_Y (L\xi_i)^- d\nu_i^+ - \int_Y (L\xi_i)^- d\nu_i^- \\
&\leq c_i(-1-\lambda\xi_i) + \int_Y (L\xi_i)^- d\nu_i^+ \\
&\leq |c_i||(-1-\lambda\xi_i)| + \nu_i^+(Y)\|(L\xi_i)^-\|_\infty \\
&= (|-1-\lambda\xi_i|, \|(L\xi_i)^-\|_\infty) \cdot (|c_i|, \nu_i^+(Y)) \qquad (4.16) \\
&\leq \|v_i\|\sqrt{|c_i|^2 + (\nu_i^+(Y))^2} \\
&\leq \|v_i\|\sqrt{|c_i|^2 + (\mathrm{mass}(\nu_i))^2} \\
&= \|v_i\|\|(c_i, \nu_i)\|
\end{aligned}
$$

Since $\|v_i\|\|(c_i, \nu_i)\| \to 1$, $\mathrm{mass}(\nu_i) \to \mathrm{mass}(\nu)$, and $\nu_i^+(Y) \to \nu^+(Y)$, we conclude that $\lim_{i\to\infty} \|v_i\|\sqrt{|c_i|^2 + (\nu_i^+(Y))^2} = \lim_{i\to\infty} \|v_i\|\sqrt{|c_i|^2 + (\mathrm{mass}(\nu_i))^2} = 1$, and $\nu^+(Y) = \nu(Y)$. □

Chapter 5

Clarke Differentiation Theorem

5.1 Statement of the Theorem and an Example

The Clarke Differentiation Theorem of [5] allows one to compute the one-sided derivative of a function which is itself defined as a maximum (or minimum) over a family of functions. First we discuss a motivating example; then we state the theorem and discuss an application to the theory of PR sets, which we shall apply in the proof of the first theorem.

Example. *Consider the function* $f(x) = |x| = \max\{-x, x\}$.

The one-sided directional derivative of a function f at x with respect to v is given by $Df_x(v) = \lim_{t \downarrow 0} \frac{f(x+tv) - f(x)}{t}$. Because this function is defined on $\mathbb{R}$, denote by $f'_+(x)$ the right-hand derivative $Df_x(1)$. If we want to calculate the right-hand derivative at any value other than 0, we simply take the derivative of x when $x > 0$, and we take the derivative of $-x$ when $x < 0$. Thus we are taking the derivative of the function which attains the maximum. Suppose now that we want to calculate $f'_+(0)$, the right-hand derivative of f at 0. Using the definition of the one-sided directional derivative, $f'_+(0) = 1$, which is the derivative of x.

Definition 5.1. *Define the* ***Clarke generalized gradient*** $\partial_C f(x)$ *to be the convex hull of the set* $\left\{ \lim_{i \to \infty} \nabla f_{x+h_i} : \nabla f_{x+h_i} \text{ exists and } h_i \to 0 \text{ as } i \to \infty \right\}$. *In the special case that* $f =$

$g+k$, for k convex and g smooth, from [5, Proposition 1.2] we have the formula $\partial_C f(x) = \partial k(x) + \nabla g_x$, where $\partial k(x)$ is the usual subgradient set, so that $\partial_C f(x) = \partial f(x)$.

Definition 5.2. *Define the **generalized directional derivative** $f^\circ(x;v)$ by*

$$f^\circ(x;v) = \lim_{h\to 0} \sup_{\delta \downarrow 0} \frac{f(x+h+\delta v) - f(x+h)}{\delta}. \tag{5.1}$$

By [5, Propositions 1.2 and 1.4], in the case of a smooth function, $f^\circ(x;v) = \nabla f_x \cdot v$, and in the case of a convex function, $f^\circ(x;v) = \max\{\zeta \cdot v : v \in \partial_C f(x) = \partial f(x)\}$.

Suppose that $G : \mathbb{R}^m \times U \to \mathbb{R}$, and let $\partial_C G(x,u)$ denote the Clarke subgradient set of the function $x \mapsto G(x,u)$ at the point (x,u). Let $DG_{(x,u)}(v)$ denote the one-sided directional derivative of $x \mapsto G(x,u)$ at (x,u) in the v direction, and let $G^\circ((x,u),v)$ denote the generalized directional derivative in the v direction of the function $x \mapsto G(x,u)$.

Theorem 5.3. (Clarke Differentiation Theorem [5, Theorem 2.1]) *Let U be a sequentially compact space, and let $G : \mathbb{R}^m \times U \to \mathbb{R}$ have the following properties:*

(a) $G(x,u)$ is upper semicontinuous in (x,u).

(b) G is locally Lipschitz in x, uniformly for u in U; in other words, for every compact $K \subset \mathbb{R}^m$ there exists L such that for all $y_1, y_2 \in K$ and all $u \in U$, we have $|G(y_1,u) - G(y_2,u)| \leq L|y_1 - y_2|$.

(c) Fixing u, $DG_{(x,u)}(v) = G^\circ((x,u);v)$

(d) $\partial_C G(x,u)$ is upper semicontinuous in (x,u).

Then, if we let $f(x) = \max\{G(x,u) : u \in U\}$,

(1) $f'(x;v) = \max\{\zeta \cdot v : \zeta \in \partial_C G(x,u), u \in M(x)\}$, where $M(x) = \{u \in U : G(x,u) = f(x)\}$.

(2) $\partial_C f(x)$ is the convex hull of $\{\partial_C G(x,u) : u \in M(x)\}$.

In the example above, we can take $U = \{1, -1\}$, and let $G(x,u) = x \cdot u$. Then G is upper semicontinuous, Lipschitz in u and x, and

$$DG_{(x,u)}(v) = \lim_{h \downarrow 0} \frac{G(x+hv,u) - G(x,u)}{h} = v \cdot u \tag{5.2}$$

Further, since $G(x,u)$ is smooth, $\partial_C G(x,u)$ agrees with the usual subgradient, so that $\partial_C G(x,u) = \{u\}$, and $G^\circ((x,u);v) = v \cdot u$. Thus the conditions (a)-(d) are satisfied, and Theorem 5.3 states that $f'(x;v) = \max\{\zeta \cdot v : \zeta \in \partial_C G(x,u), u \in M(x)\}$. For $x < 0$, $M(x) = \{-1\}$, so that $f'(x;v) = -1 \cdot v$. Similarly, for $x > 0$, $M(x) = \{1\}$, so that $f'(x;v) = v$. We also have $M(0) = \{-1,1\}$, so that $f'(0;v) = \max\{v,-v\} = |v|$ and, in particular, $f'_+(0) = 1$.

5.2 An Application

Suppose that S is a PR set, so that $S = f^{-1}(-\infty,0]$ for a (weakly regular) semiconvex function f. Let $\gamma : [0,L] \to \mathbb{R}^m$ be a geodesic in S, and let $\xi : [0,L] \to \mathbb{R}^m$ be a Lipschitz continuous vector field.

Definition 5.4. *We say that a vector v **points out of** S at $p \in S$ if $v \notin \mathrm{Tan}(S,p)$.*

Remark. Note that v can only point out of S at $p \in \partial S$ since the complement of $\mathrm{Tan}(S,p)$ is empty at interior points $p \in S$.

Let $\delta_\xi \max(f) := \left.\dfrac{d}{dt}\right|_{t=0} \left(\max_{s\in[0,L]} f(\gamma(s) + t\xi(s)) \right)$ denote the first variation of the maximum of f with respect to ξ along γ. We shall use Theorem 5.3 to show that $\delta_\xi \max(f)$ exists; moreover, for S a regular PR set, if ξ points out of S then $\delta_\xi \max(f) > 0$.

Write $f = k + g$, where k is convex and g is smooth. Let $\mathcal{A}$ denote the set of all affine functions A satisfying $A \leq k$ and $A(p_0) = k(p_0)$ for at least one point $p_0 \in \text{Image}(\gamma)$. We identify $A(p) = w \cdot p + r$ with the pair (w, r).

Lemma 5.5. *The pair* $(w, r) \in \mathcal{A}$ *if and only if there exists* $p_0 \in \text{Image}(\gamma)$ *such that* $w \in \partial k(p_0)$ *and* $r = k(p_0) - w \cdot p_0$.

Proof. Suppose first that $w \in \partial k(p_0)$ for $p_0 \in \text{Image}(\gamma)$. Then, by definition of $\partial k(p_0)$, $w \cdot p + (k(p_0) - w \cdot p_0) \leq k(p)$ for all p, so that the proposed $(w, r) \in \mathcal{A}$. Now, suppose that $(w, r) \in \mathcal{A}$. Then there exists p_0 such that $w \cdot p_0 + r = k(p_0)$, so that $r = k(p_0) - w \cdot p_0$. Thus , by definition of $\mathcal{A}$, $w \cdot p + (k(p_0) - w \cdot p_0) \leq k(p)$ for all p, so that $w \in \partial k(p_0)$. $\square$

By construction, $f(p) = k(p) + g(p) = \max\{A(p) + g(p) : A \in \mathcal{A}\}$. Let $U = \{(A, s) : A \in \mathcal{A},\ s \in [0, L]\}$. Define $G : \mathbb{R} \times U \to \mathbb{R}$ by $G(t, (A, s)) = A(\gamma(s) + t\xi(s)) + g(\gamma(s) + t\xi(s))$.

Lemma 5.6. *The function G above satisfies the hypotheses of Clarke's differentiation theorem.*

Proof. U is sequentially compact: the topology on $\mathcal{A}$ is given by the association of $A(p) = w \cdot p + r$ with the point $(w, r) \in \mathbb{R}^{m+1}$. Note that $A \in \mathcal{A}$ if and only if w is a subgradient of k at some $p_0 \in \text{Image}(\gamma)$ and $r = k(p_0) - w \cdot p_0$. Thus $\mathcal{A}$ is compact since $\mathbb{D}(k) \cap (\gamma \times \mathbb{R}^m)$ is compact, and $\mathcal{A}$ is the image of $\mathbb{D}(k) \cap (\gamma \times \mathbb{R}^m)$ under the continuous map $(p, w) \mapsto (w, k(p) - w \cdot p)$, so that $U = \mathcal{A} \times [0, L]$ is a compact subset of a metric space and hence is sequentially compact.

$G(t, (A, s))$ is upper semicontinuous in $(t, (A, s))$: if $A = (w, r)$, then by definition of G,

$$G(t, (A, s)) = w \cdot (\gamma(s) + t\xi(s)) + r + g(\gamma(s) + t\xi(s)), \tag{5.3}$$

so that, by continuity of the functions on the right-hand side in (5.3),

$$\lim_{(t,(A,s))\to(t_0,(A_0,s_0))} G(t,(A,s)) = w_0 \cdot (\gamma(s_0) + t_0\xi(s_0)) + r_0 + g(\gamma(s_0) + t_0\xi(s_0)). \tag{5.4}$$

Thus G is continuous as a function of $(t,(A,s))$.

$G(t,u)$ is locally Lipschitz in t:

$$\begin{aligned}
&|G(t,u) - G(t',u)| \\
&= |\,(w\cdot(\gamma(s)+t\xi(s)) + r + g(\gamma(s)+t\xi(s))) - (w\cdot(\gamma(s)+t'\xi(s)) + r + g(\gamma(s)+t'\xi(s)))\,| \\
&= |(t-t')(w\cdot\xi(s)) + (g(\gamma(s)+t\xi(s)) - g(\gamma(s)+t'\xi(s)))| \\
&\le |t-t'||w\cdot\xi(s)| + M_g|t-t'||\xi(s)| \qquad (5.5)
\end{aligned}$$

where M_g is a Lipschitz constant for g; since g is only locally Lipschitz on $\{\gamma(s)+t\xi(s) : s \in [0,L], t\in\mathbb{R}\}$, this inequality is understood to hold only locally.

$G(t,u)$ is locally uniformly Lipschitz for $u \in U$: suppose $K \subset \mathbb{R}$ is compact. Then g has Lipschitz constant M_g on $\{\gamma(s)+t\xi(s) : s\in[0,L], t\in K\}$, so the result follows from (5.5).

Also, $G^\circ((t,u);v) = DG_{(x,u)}(v)$: by definition, $DG_{(x,u)}(v)$ is the one-sided directional derivative, with respect to t, in the direction v, i.e.

$$\begin{aligned}
&DG_{(x,u)}(v) = \lim_{\delta\downarrow 0}\frac{G(t+\delta v,u) - G(t,u)}{\delta} \\
&= \lim_{\delta\downarrow 0}\frac{A(\gamma(s)+(t+\delta v)\xi(s)) + g(\gamma(s)+(t+\delta v)\xi(s)) - A(\gamma(s)+t\xi(s)) - g(\gamma(s)+t\xi(s))}{\delta} \\
&= \lim_{\delta\downarrow 0}\frac{w\cdot\delta v\xi(s)}{\delta} + \lim_{\delta\downarrow 0}\frac{g(\gamma(s)+(t+\delta v)\xi(s)) - g(\gamma(s)+t\xi(s))}{\delta} \qquad (5.6) \\
&= v(w\cdot\xi(s)) + \nabla g_{\gamma(s)+t\xi(s)}\cdot(v\xi(s)) \\
&= v\left((w+\nabla g_{\gamma(s)+t\xi(s)})\cdot\xi(s)\right)
\end{aligned}$$

By definition,

$$G^\circ((t,u);v) = \limsup_{h\to 0;\delta\downarrow 0} \frac{G(t+h+\delta v,u) - G(t+h,u)}{\delta}; \tag{5.7}$$

by [5, Proposition 1.2 and 1.4], since A is convex and g is smooth,

$$G^\circ((t,u);v) = \max\{\zeta\cdot v : \zeta \in \partial_C G(t,u)\} = v((w + \nabla g_{\gamma(s)+t\xi(s)})\cdot\xi(s)), \tag{5.8}$$

where $\partial_C G(t,u)$ is the usual subgradient set for the function $t \mapsto G(t,u)$, since G is smooth in t.

To apply Theorem 5.3, we must also show that $\partial_C G(t,u)$ is upper semicontinuous in (t,u): since G is smooth in t, $\partial_C G(t,u) = \{(w + \nabla g_{\gamma(s)+t\xi(s)})\cdot\xi(s)\}$. To see upper semicontinuity, suppose that $(t_i,(A_i,s_i)) \to (t_0,(A_0,s_0))$, which means also that $A_i =: (w_i,r_i) \to (w_0,r_0) := A_0$. We know that $w_i \in \partial k(p_i)$ for some $p_i \in \text{Image}(\gamma)$. By passing to a subsequence, we can assume that the p_i converge to p_0. Note that $w_0 \in \partial k(p_0)$ by Lemma 2.32. Also, $r_0 = -w_0\cdot p_0 + k(p_0)$ implies that $(w_0,r_0) \in \mathcal{A}$ and $((w_0,r_0),s_0) \in U$. Then by continuity of ∇g, γ, and ξ,

$$\lim_{i\to\infty} (w_i + \nabla g_{\gamma(s_i)+t_i\xi(s_i)})\cdot\xi(s_i) = (w_0 + \nabla g_{\gamma(s_0)+t_0\xi(s_0)})\cdot\xi(s_0). \tag{5.9}$$

Since $\lim\limits_{i\to\infty} \big((w_i + \nabla g_{\gamma(s_i)+t_i\xi(s_i)})\cdot\xi(s_i)\big) \in \partial_C G(t_0,u_0)$, the multifunction $\partial_C G(t,u)$ is upper semicontinous.

Thus we may apply Theorem 5.3 to G. $\square$

Lemma 5.7. *Let $S = f^{-1}(-\infty,0]$ be a PR set, and let ξ be a Lipschitz vector field on $[0,L]$. If $\gamma\cap\partial S \neq \emptyset$, then*

$$\delta_\xi \max(f) = \max\{Df_{\gamma(s)}(\xi(s)) : \gamma(s)\in\partial S\}. \tag{5.10}$$

Proof. Let $h(t) = \max_u\{G(t,u)\} = \max_{A,s}\{A(\gamma(s)+t\xi(s)) + g(\gamma(s)+t\xi(s))\} = \max_s\{f(\gamma(s)+t\xi(s))\}$. Then by Theorem 5.3, the one-sided derivative of h at 0 is

$$h'_+(0) = \max\{\zeta : \zeta \in \partial_C G(0,u),\ u \in M(0)\}. \tag{5.11}$$

Since $\gamma \cap \partial S \neq \emptyset$ implies $h(0) = 0$, the set $M(0)$ is the set of u for which $G(0,u) = 0$, i.e. the set of $u = (A,s)$ for which $A(\gamma(s)) + g(\gamma(s)) = f(\gamma(s)) = 0$. This happens precisely when $A(\gamma(s)) = k(\gamma(s))$, in which case $A \leq k$ implies that $w \in \partial k(\gamma(s))$, and $A(\gamma(s)) = w \cdot (p - \gamma(s)) + k(\gamma(s))$. Thus

$$\begin{aligned}
\delta_\xi \max(f) &= \max\{(w + \nabla g_{\gamma(s)}) \cdot \xi(s) : \gamma(s) \in \partial S, w \in \partial k(\gamma(s))\} \\
&= \max\{Dk_{\gamma(s)}(\xi(s)) + \nabla g_{\gamma(s)}(\xi(s)) : \gamma(s) \in \partial S\} \qquad (5.12) \\
&= \max\{Df_{\gamma(s)}(\xi(s)) : \gamma(s) \in \partial S\}.
\end{aligned}$$

□

Lemma 5.8. *Let $S = f^{-1}(-\infty, 0]$ be a regular PR set, and let ξ be a Lipschitz vector field on $[0,L]$. If ξ points out of S at some point in $\gamma \cap \partial S$, then $\delta_\xi \max(f) > 0$.*

Proof. If ξ points out of S at $\gamma(s) \in \partial S$, then $Df_{\gamma(s)}(\xi(s)) > 0$ by Lemma 2.29. Thus by Lemma 5.7, we also have that $\delta_\xi \max(f) > 0$. □

Remarks.

1. The result of Lemma 2.16 is an immediate consequence of Theorem 5.3: let $\mathcal{A}$ and g be as above, and let $G : \mathbb{R}^m \times \mathcal{A} \to \mathbb{R}^m$ by $G(p, A) = A(p) + g(p)$.

2. If regularity of S is not assumed, we still have the weaker result that $\delta_\xi \max(f) > 0$ whenever $\xi(s) \in \partial f(\gamma(s))$ for some s with $\xi(s) \neq 0$. This is a consequence of Lemma 2.16.

Chapter 6

Regularity of Geodesics in Regular PR Sets

In this chapter, we show that geodesics in regular PR sets are $C^{1,1}$; in other words, they have Lipschitz continuous first derivatives. The two main tools in our argument are the Clarke Differentiation Theorem and the generalized Kuhn-Tucker Theorem. In the last chapter we showed that $\delta_\xi \max(f) > 0$ whenever ξ points out of S at some point of $\gamma \cap \partial S$. We use this fact to show that the first condition of the generalized Kuhn-Tucker Theorem holds in the setting of Lipschitz vector fields along γ with linear functional $\delta_\xi \text{length}(\gamma)$ and a linear map related to the integrand in the first variation for arclength. Throughout, assume that S is regular, the arclength-parametrized geodesic $\gamma : [0, L] \to S$ is a bijection onto its image, and γ is a curve of minimum length joining any two points in its image.

6.1 Kuhn-Tucker and Geodesics

6.1.1 The Setup

Let $S = f^{-1}(-\infty, 0] \subset \mathbb{R}^m$ be a regular PR set, and let $\gamma : [0, L] \to S$ be a geodesic. Let $\delta_\xi \text{length}(\gamma) := \left.\frac{d}{dt}\right|_{t=0} \text{length}(\gamma + t\xi)$ denote the first variation of arclength of γ along ξ. We apply the generalized Kuhn-Tucker Theorem to the following:

$$\begin{aligned} &X = \{\text{Lipschitz vector fields on } [0, L] \text{ vanishing at the endpoints of } \gamma\} \\ &\lambda\xi = \delta_\xi \text{length}(\gamma) = \int \langle \gamma', \xi' \rangle \qquad (6.1) \\ &Y = \mathbb{D}(f) \cap (\gamma \times \mathbb{R}^m) \\ &L : X \to C(Y) \text{ by } L\xi(\gamma(s), v) = -\langle \xi(s), v \rangle = -\left\langle \xi(\gamma^{-1}(\gamma(s))), v \right\rangle. \end{aligned}$$

Remark. The map L satisfies $L : X \to C(Y)$: by definition of L, $L\xi(p, v) = -\langle \xi(\gamma^{-1}(p)), v \rangle$. Continuity of $L\xi$ follows from continuity of γ^{-1}, ξ, and the dot product.

In the next subsection, we show that the first variation of arclength of γ with respect to ξ is well-defined.

6.1.2 The First Variation of Arclength

Let $\gamma : [0, L] \to S$ be a geodesic. Let ξ be a Lipschitz vector field on $[0, L]$. Vary γ along ξ by taking $\gamma_t(s) = \gamma(s) + t\xi(s)$. Then the length of γ_t is given by

$$\text{length}(\gamma_t) = \int_{[0,L]} |\gamma_t'(s)| \, ds. \qquad (6.2)$$

Lemma 6.1. *The first variation of arclength of γ with respect to ξ is well-defined.*

Proof. To show that the first variation of arclength is well-defined, we only need to apply the dominated convergence theorem to pull the derivative inside of the length integral

$$\delta_\xi \text{length}(\gamma) := \frac{d}{dt}\bigg|_{t=0} \int_{[0,L]} |\gamma_t'(s)| ds = \int_{[0,L]} \langle \gamma'(s), \xi'(s) \rangle \, ds, \tag{6.3}$$

and justify the second equality.

Since γ and ξ are Lipschitz, they are differentiable at almost every s by Rademacher's Theorem. Thus for any $t \geq 0$ and almost every s,

$$\begin{aligned} \frac{d}{dt}|\gamma_t'(s)| &= \frac{d}{dt} \langle \gamma'(s) + t\xi'(s), \gamma'(s) + t\xi'(s) \rangle^{1/2} \\ &= \frac{1}{|\gamma'(s) + t\xi'(s)|} \langle \gamma'(s) + t\xi'(s), \xi'(s) \rangle . \end{aligned} \tag{6.4}$$

Therefore, using (6.4) and the fact that $|\gamma'(s)| = 1$, for almost every s we have

$$\frac{d}{dt}\bigg|_{t=0} |\gamma_t'(s)| = \lim_{h \downarrow 0} \frac{|\gamma_h'(s)| - |\gamma_0'(s)|}{h} = \langle \gamma'(s), \xi'(s) \rangle . \tag{6.5}$$

Further,

$$\left| \frac{|\gamma_h'(s)| - |\gamma_0'(s)|}{h} \right| \leq \frac{|\gamma_h'(s) - \gamma_0'(s)|}{|h|} = |\xi'(s)|. \tag{6.6}$$

Since ξ is Lipschitz, ξ' is measurable (as a pointwise almost everywhere limit of measurable functions) and bounded, so that $|\xi'(s)|$ is integrable as a function of s. Thus, by the dominated convergence theorem, we may differentiate the integrand, and the first variation of arclength is well-defined. $\square$

6.1.3 Clarke's Theorem Implies Condition (1) of Kuhn-Tucker

By Lemma 5.8, we know that $\delta_\xi \max(f) > 0$ whenever the Lipschitz continuous vector field ξ points out of S at some point in $\gamma \cap \partial S$. We now use this fact to show that the first condition of the Kuhn-Tucker theorem is satisfied. In our proof, we shall use the following lemma.

Lemma 6.2. *There exists a Lipschitz vector field η on $[0, L]$ satisfying $\delta_\eta \max(f) < 0$.*

Remark. The vector field η need not vanish at 0 and L. Specifically, in the case that $\gamma(s) \in \partial S$ for $s = 0$ or L, then $\delta_\eta \max(f) < 0$ implies $\eta(s) \neq 0$.

Proof. Suppose that $\gamma(t) \in \partial S$. Then by Lemma 2.28 there exists a direction w and an open set U containing $\gamma(t)$ such that $Df_y(w) < 0$ for all $y \in U$. There exists an open interval (a, b) containing t such that $\gamma\big|_{[a,b]} \subset U$. Take $\xi(s) = \phi(s)w$, where ϕ is a piecewise affine continuous function that is 0 outside of (a, b), 1 at $\dfrac{a+b}{2}$, and affine in between. Cover $\gamma^{-1}(\partial S)$ by finitely many such open intervals (a, b). Now, let η be the sum over all such ξ. Then η is Lipschitz continuous, and $\eta(s)$ is a decreasing direction at $\gamma(s) \in \partial S$ by Lemma 2.16. Thus $\delta_\eta \max(f) < 0$ by Lemma 5.7. $\square$

Remark. For all $c > 0$, we have $\sup |c\eta| = c \sup |\eta|$ and $\delta_{c\eta}\text{length}(\gamma) = c\delta_\eta\text{length}(\gamma)$, so by rescaling we can assume that $\sup |\eta| < \alpha$ and $\delta_\eta\text{length}(\gamma) < \alpha$ for any $\alpha > 0$.

Lemma 6.3. *There exists $\epsilon > 0$ with the following property: if $\xi \in X$ and $\delta_\xi\text{length}(\gamma) = -1$, then there exist $s \in (0, L)$ and $v \in \partial f(\gamma(s))$ such that $\gamma(s) \in \partial S$ and $\langle \xi(s), v\rangle > \epsilon$.*

Proof. We claim that it suffices to show that there exists $\epsilon > 0$ such that whenever $\delta_\xi\text{length}(\gamma) = -1$, we have $\delta_\xi \max(f) > \epsilon$. By Lemma 5.7, if $\delta_\xi \max(f) > \epsilon$, there exist $s \in (0, L)$ and $v \in \partial f(\gamma(s))$ such that $\langle v, \xi(s)\rangle > \epsilon$, as desired.

If no such ϵ exists, then there is a sequence $\{\xi_i\} \subset X$ such that $\delta_{\xi_i}\text{length}(\gamma) = -1$ but $\delta_{\xi_i}\max(f) \to 0$. By Lemma 6.2, there exists a Lipschitz vector field η on $[0, L]$ satisfying

$\delta_\eta \max(f) < 0$, $\sup|\eta| < 1/4$, and $\delta_\eta \text{length}(\gamma) < 1/4$. Since

$$\delta_{\xi_i+\eta}\text{length}(\gamma) = \delta_{\xi_i}\text{length}(\gamma) + \delta_\eta\text{length}(\gamma) = -1 + \delta_\eta\text{length}(\gamma) < -3/4, \tag{6.7}$$

if we take $\beta_i := \xi_i + \eta$, by Lemma 5.7 and sub-additivity of $Df_{\gamma(s)}(v)$ as a function of v,

$$\delta_{\beta_i}\max(f) \leq \delta_{\xi_i}\max(f) + \delta_\eta\max(f). \tag{6.8}$$

Since $\overline{\lim}(\delta_{\beta_i}\max(f)) \leq \delta_\eta\max(f) < 0$ by hypothesis, we must have $\delta_{\beta_i}\max(f) < 0$ for sufficiently large i. For i fixed and all sufficiently small $h > 0$, $\text{length}(\gamma + h\beta_i) - \text{length}(\gamma) < -(1/2)h$ by (6.7). Fix h. Consider the curve $\tilde{\gamma}$ formed by concatenating γ and the Euclidean segments $[\gamma(0), \gamma(0) + h\beta_i(0)]$ and $[\gamma(L) + h\beta_i(L), \gamma(L)]$. Provided h is sufficiently small, the curve $\tilde{\gamma}$ is contained in S. Using the fact that $\xi_i \in X$, the length of $\tilde{\gamma}$ is

$$\text{length}(\tilde{\gamma}) = h|\eta(0)| + \text{length}(\gamma + h\beta_i) + h|\eta(L)| < (1/2)h + \text{length}(\gamma) - (1/2)h. \tag{6.9}$$

This contradicts the assumption that γ is a curve of minimum possible length joining $\gamma(0)$ and $\gamma(L)$.

□

Corollary 6.4. *The given L, λ, X, and Y satisfy the first condition of the generalized Kuhn-Tucker theorem.*

Proof. By Lemma 6.3, there exists $\epsilon > 0$ such that whenever $\delta_\xi\text{length}(\gamma) = -1$, there exist $s \in [0, L]$ and $v \in \partial f(\gamma(s))$ such that $\gamma(s) \in \partial S$ and $\langle \xi(s), v\rangle > \epsilon$. In other words, $L\xi(\gamma(s), v) < -\epsilon$, so that $-L\xi(\gamma(s), v)^- < -\epsilon$, and $\|(L\xi)^-\|_\infty > \epsilon$. □

6.2 Consequences of Kuhn-Tucker

By Theorem 4.3, there exists a nonnegative regular Borel measure μ on Y such that $\text{mass}(\mu) < \infty$ and $\lambda(\xi) = -\int_Y \langle \xi \circ \gamma^{-1}, v \rangle \, d\mu(y)$. Using the definition of λ, then, for all $\xi \in X$ and $y = (p, v) \in Y$,

$$\int \langle \gamma'(s), \xi'(s) \rangle \, ds = -\int_Y \left\langle \xi \circ \gamma^{-1}(p), v \right\rangle d\mu(y), \tag{6.10}$$

and we have the estimate

$$\left| \int \langle \gamma'(s), \xi'(s) \rangle \, ds \right| \leq \mu(Y) \sup_{s \in [0,L]} |\xi| \sup_{(p,v) \in Y} |v|, \tag{6.11}$$

where the right-hand expression is finite because ξ is continuous on $[0, L]$, and Y is compact.

Let e_i denote the ith standard basis vector. Using (6.10), the second distributional derivative of $\gamma_i := \langle \gamma, e_i \rangle$ satisfies

$$\gamma_i''(h) := -\int \gamma_i'(s) h'(s) ds = -\int \langle \gamma'(s), (h \cdot e_i)'(s) \rangle ds = \int_Y \left\langle (h \cdot e_i) \circ \gamma_p^{-1}, v \right\rangle d\mu(p, v),$$

and thus is an element of $C^*([0, L])$. By Theorem 4.2 applied to γ_i'', there exists a finite-mass signed, regular Borel measure η_i on $[0, L]$ such that $\gamma_i''(f) = \int_{[0,L]} f d\eta_i$ for all continuous functions f on $[0, L]$. Define the vector-valued measure η by $\eta := (\eta_1, \dots, \eta_m)$.

Lemma 6.5. *For all Borel $\sigma \subset [0, L]$, we have*

$$\eta(\sigma) = \int_{Y \cap (\gamma(\sigma) \times \mathbb{R}^m)} v d\mu(p, v), \tag{6.12}$$

where 1_σ denotes the characteristic function of σ.

In the proof and in the next section, we shall use the concept of total variation.

Definition 6.6. *Define the* ***variation*** *of f over the partition $\mathcal{P} = \{t_0, \ldots, t_j\}$ by* $\mathrm{Var}(f, \mathcal{P}) = \sum_{i=1}^{j} |f(t_i) - f(t_{i-1})|$. *Define the* ***total variation*** *of f by* $\mathrm{Var}(f) := \sup\{\mathrm{Var}(f, \mathcal{P})\}$.

Proof. Suppose first that $\sigma \subset [0, L]$ is an open interval (a, b). Note that $Y \cap (\gamma(\sigma) \times \mathbb{R}^m)$ is μ-measurable because μ is a regular Borel measure. Let 1_σ denote the characteristic function of σ. Approximate 1_σ by nonnegative continuous piecewise affine functions ϕ_j such that $\phi_j\big|_\sigma = 1$ and $\mathrm{Var}(\phi_j) \leq 2$. Then, applying the dominated convergence theorem twice,

$$\begin{aligned}
\int 1_\sigma d\eta_i &= \int \lim_{j\to\infty} \phi_j d\eta_i \\
&= \lim_{j\to\infty} \int \phi_j d\eta_i \\
&= \lim_{j\to\infty} \int_Y \left\langle (\phi_j e_i) \circ \gamma^{-1}, v \right\rangle d\mu \\
&= \lim_{j\to\infty} \int_Y \phi_j \circ \gamma^{-1} \left\langle e_i \circ \gamma^{-1}, v \right\rangle d\mu \\
&= \int_Y (1_\sigma \circ \gamma^{-1}) \left\langle e_i, v \right\rangle d\mu \\
&= \int_{Y \cap (\gamma(\sigma)\times\mathbb{R}^m)} \left\langle e_i, v \right\rangle d\mu \\
&= \left\langle e_i, \int_{Y \cap (\gamma(\sigma)\times\mathbb{R}^m)} v d\mu \right\rangle .
\end{aligned} \tag{6.13}$$

Thus, $\langle e_i, \eta(\sigma)\rangle = \int 1_\sigma d\eta_i = \left\langle e_i, \int_{Y\cap(\gamma(\sigma)\times\mathbb{R}^m)} v d\mu \right\rangle$, as was to be shown. Thus, the statement is also true for open sets and then all Borel sets (using regularity of the measure η). $\square$

6.2.1 Continuity of γ'

Definition 6.7. *We say that a function f on $[0, L]$ is* ***essentially BV*** *($f \in \mathrm{eBV}$) if the variation $\mathrm{Var}(f, \mathcal{P})$ is uniformly bounded for all partitions $\mathcal{P}$ of $[0, L]$ by Lebesgue points. A function f on $[0, L]$ has* ***bounded variation*** *($f \in \mathrm{BV}$) if the variation $\mathrm{Var}(f, \mathcal{P})$ is uniformly bounded for all partitions $\mathcal{P}$ of $[0, L]$.*

The necessary facts about eBV and BV functions are in the Appendix.

Corollary 6.8. $\gamma' \in \mathrm{eBV}$.

Proof. Apply the criterion for real-valued eBV functions in the Appendix to the components of γ'. The result follows from applying (6.11) to test functions h:

$$\begin{aligned} \left| \int_{[0,L]} \gamma_i' h' \, ds \right| &= \left| \int_{[0,L]} \langle \gamma'(s), h' \cdot e_i \rangle \, ds \right| \\ &= \left| \int_Y \left\langle (h \cdot e_i) \circ \gamma_p^{-1}, v \right\rangle d\mu(p, v) \right| \\ &\leq \mu(Y) \|h\|_\infty \sup |v|, \end{aligned} \tag{6.14}$$

since $\mathrm{Var}(\gamma_i') = \sup\{\int \gamma_i' \phi' : \phi \in C_0^1[0, L],\ \|\phi\|_\infty \leq 1\}$ □

Theorem 6.9. *γ has continuous first derivative.*

Proof. We first show that γ' has bounded variation. Let E denote the set of Lebesgue points of γ'. Since γ' is eBV, γ' has left and right essential limits everywhere: the limit

$$\lim_{\substack{x \downarrow a, \\ x \in E}} \gamma'(x) = M \tag{6.15}$$

exists for every $a \in [0, L]$. Also, γ has left and right derivatives everywhere:

Let $\epsilon > 0$. Then there exists $\delta > 0$ such that $0 < t - a < \delta$ and $t \in E$ implies that $|\gamma'(t) - M| < \epsilon$. Since γ is absolutely continuous, for $0 < x - a < \delta$,

$$\gamma(x) - \gamma(a) = \int_a^x \gamma'(t) dt, \tag{6.16}$$

so that, since almost every point is a Lebesgue point,

$$\left| M - \frac{\gamma(x) - \gamma(a)}{x - a} \right| = \left| M - \frac{1}{x - a} \int_a^x \gamma'(t) dt \right| \leq \frac{1}{x - a} \int_a^x |M - \gamma'(t)| dt \leq \epsilon. \tag{6.17}$$

Thus γ has a right derivative at a, and that right derivative agrees with the essential right-hand limit at a. Similarly, γ has a left derivative at a agreeing with the essential left-hand limit at a. Denote by γ'_+ and γ'_- the right and left derivatives of γ.

Since $\gamma' \in \mathrm{eBV}$, there exists a function $g \in \mathrm{BV}$ such that $\gamma' = g$ on the set E of Lebesgue points of γ'. Now, by the above, for all a

$$g_+(a) = \lim_{x \downarrow a} g(x) = \lim_{\substack{x \downarrow a, \\ x \in E}} \gamma'(x) = \gamma'_+(a). \tag{6.18}$$

Thus $\gamma'_+(a)$ agrees with g_+ everywhere, and similarly for $\gamma'_-(a)$. In particular, at points where g is continuous, $\gamma'(a)$ exists and agrees with g. There are at most countably many points, therefore, where $\gamma'(a)$ fails to exist, and these points are jump discontinuities of g. Since we already had that the essential left- and right- limits of γ' exist everywhere, now we know that these must in fact be (honest) left- and right-hand limits. Thus $\gamma' \in \mathrm{BV}$, and γ' can only have jump discontinuities.

The left-hand and right-hand limits of γ' must agree: fix $a \in [0, L]$, and let ϕ_δ be a sequence of nonnegative piecewise affine functions approximating $1_{\{a\}}$ such that the ϕ_δ converge to $1_{\{a\}}$ pointwise, $\phi_\delta(a) = 1$, each ϕ_δ has compact support contained in $(a-\delta, a+\delta)$, and each ϕ_δ has total variation $\mathrm{Var}(\phi_\delta) = 2$. By definition of the distribution γ''_i,

$$\gamma''_i(\phi_\delta) = \int \phi_\delta \gamma''_i := -\int \gamma'_i \phi'_\delta. \tag{6.19}$$

Let $\epsilon > 0$. Then there exists $\delta > 0$ such that $|\gamma'(t) - \gamma'_-(a)| < \epsilon/2$ whenever $0 < a - t < \delta$ and $|\gamma'(t) - \gamma'_+(a)| < \epsilon/2$ whenever $0 < t - a < \delta$. Thus

$$\begin{aligned}\left| \left(\int_{(a-\delta,a)} \gamma_i' \phi_\delta' \right) - (\gamma_i)'_-(a) \right| &= \left| \int_{(a-\delta,a)} \gamma_i' \phi_\delta' - (\gamma_i)'_-(a) \int_{(a-\delta,a)} \phi_\delta' \right| \\ &\leq (\epsilon/2) \int_{(a-\delta,a)} |\phi_\delta'| \\ &\leq \epsilon.\end{aligned} \tag{6.20}$$

Using a similar argument for γ_+, we obtain

$$\lim_{\delta \downarrow 0} \int \phi_\delta(\gamma_i'') = -\lim_{\delta \downarrow 0} \int \gamma_i' \phi_\delta' = (\gamma_i)'_+(a) - (\gamma_i)'_-(a). \tag{6.21}$$

However, $\gamma_i''(\phi_\delta) = \int \phi_\delta d\eta_i$ as well, and using the bounded convergence theorem,

$$\begin{aligned}\lim_{\delta \downarrow 0} \int \phi_\delta d\eta_i &= \int \lim_{\delta \downarrow 0} \phi_\delta d\eta_i \\ &= \int 1_{\{a\}} d\eta_i \\ &= \eta_i(\{a\}) \\ &= \left\langle e_i, \int_{Y \cap (\gamma(a) \times \mathbb{R}^m)} v d\mu(p,v) \right\rangle \\ &= \int_{\gamma(a) \times \partial f(\gamma(a))} \langle e_i, v \rangle \, d\mu(p,v)\end{aligned} \tag{6.22}$$

Combining (6.21) and (6.22), we have that

$$(\gamma)'_+(a) - (\gamma)'_-(a) = \int_{\gamma(a) \times \partial f(\gamma(a))} v d\mu(p,v). \tag{6.23}$$

However, $(\gamma)'_+(a) - (\gamma)'_-(a) \in \mathrm{Tan}(S, \gamma(a))$ and $\int_{\gamma(a) \times \partial f(\gamma(a))} v d\mu(p,v) \in \mathrm{Nor}(S, \gamma(a))$. See the lemma below for a proof of $(\gamma)'_+(a) - (\gamma)'_-(a) \in \mathrm{Tan}(S, \gamma(a))$. The right-hand side is in the normal cone because $\text{cone}(\partial f(\gamma(a))) = \text{Nor}(S, \gamma(a))$ and $\int_{\gamma(a)\times\partial f(\gamma(a))} v d\mu(p, v)$ is an average of elements over the closed convex set Cone(Nor($\gamma(a)$)), and thus in the convex set. Thus each side must necessarily be the zero vector, and

$$(\gamma)'_{+}(a) = (\gamma)'_{-}(a), \tag{6.24}$$

so that γ is C^1. □

Lemma 6.10. $(\gamma)'_{+}(a) - (\gamma)'_{-}(a) \in \text{Tan}(S, \gamma(a))$.

Proof. Since the limit $\gamma'_{+}(a) = \lim_{\delta\downarrow 0} \frac{\gamma(a+\delta)-\gamma(a)}{\delta}$ exists, if $\gamma'_{+}(a)$ is nonzero,

$$\frac{\gamma'_{+}(a)}{|\gamma'_{+}(a)|} = \lim_{\delta\downarrow 0} \frac{\gamma(a+\delta)-\gamma(a)}{|\gamma(a+\delta)-\gamma(a)|} \in \text{Tan}(S, \gamma(a)). \tag{6.25}$$

Thus $\gamma'_{+}(a) \in \text{Tan}(S, \gamma(a))$ as well. Further, $-\gamma'_{-}(a) = \lim_{\delta\uparrow 0} \frac{\gamma(a+\delta)-\gamma(a)}{-\delta}$ exists. If $\gamma'_{-}(a)$ is nonzero, the limit

$$\frac{-\gamma'_{-}(a)}{|-\gamma'_{+}(a)|} = \lim_{\delta\uparrow 0} \frac{\gamma(a+\delta)-\gamma(a)}{|\gamma(a+\delta)-\gamma(a)|} \in \text{Tan}(S, \gamma(a)), \tag{6.26}$$

so $-\gamma'_{-}(a)$ is also in the tangent cone at $\gamma(a)$. Since S is a PR set, the tangent cone is a closed, convex cone, so that $(\gamma)'_{+}(a) - (\gamma)'_{-}(a) \in \text{Tan}(S, \gamma(a))$. □

6.3 The Measure κ

So far, we applied Theorem 4.3 to obtain a measure μ on Y, and then we used Theorem 4.2 to obtain the measure η satisfying

$$\eta(\sigma) = \int_{Y\cap(\gamma(\sigma)\times\mathbb{R}^m)} v d\mu. \tag{6.27}$$

Now we shall use the measure μ to obtain the curvature measure κ on $[0, L]$, which we will think of as the "curvature" of γ.

Write the Hahn decomposition of the measure η as $\eta = (\eta_1, \ldots, \eta_m) = (\eta_1^+ - \eta_1^-, \ldots, \eta_m^+ - \eta_m^-)$. Write $\alpha = \sum(\eta_i^+ + \eta_i^-)$. Then η_i is absolutely continuous with respect to α, since $\alpha(A) = 0$ implies $\eta_i(A) = 0$. By the Radon-Nikodym Theorem [8, Theorem 2 of Section 1.6],

$$\eta(\sigma) = \int_\sigma D_\alpha \eta d\alpha, \tag{6.28}$$

where

$$D_\alpha \eta(p) = \lim_{\epsilon \downarrow 0} \frac{\eta(p-\epsilon, p+\epsilon)}{\alpha(p-\epsilon, p+\epsilon)}. \tag{6.29}$$

This limit converges α-a.e. Define $V(p) = D_\alpha \eta(p)$ and $N(p) = \dfrac{V(p)}{|V(p)|}$ for $V(p) \neq 0$. When $V(p) = 0$, define $N(p) = 0$. Define κ by $d\kappa(p) = |V(p)| d\alpha(p)$, so that

$$\int_{Y \cap (\gamma(\sigma) \times \mathbb{R}^m)} v d\mu(p, v) = \eta(\sigma) = \int_\sigma V(p) d\alpha = \int_\sigma N(p) d\kappa. \tag{6.30}$$

Lemma 6.11. *The vector field N satisfies $N(p) \in \operatorname{nor}(S, \gamma(p))$ whenever $N(p) \neq 0$.*

Proof. By definition,

$$\frac{\eta(p-\epsilon, p+\epsilon)}{\alpha(p-\epsilon, p+\epsilon)} \in Y_\epsilon := \operatorname{conv}\left(\bigcup_{s \in (p-\epsilon, p+\epsilon)} \operatorname{Nor}(S, \gamma(s)) \right), \tag{6.31}$$

where conv denotes the convex hull. Since $Y_{\epsilon'} \subset Y_\epsilon$ whenever $\epsilon' < \epsilon$, we have that $N(p) \in \bigcap_{\epsilon > 0} Y_\epsilon$. Thus it suffices to show that $\bigcap_{\epsilon > 0} Y_\epsilon \subset \operatorname{Nor}(S, p)$.

Suppose that $v \in \cap_{\epsilon > 0} Y_\epsilon$ and $|v| = 1$. By Caratheodory's Theorem [11, Theorem 17.1], for each $j > 0$ and $1 \leq i \leq m$ there exist vectors $v_{i,j} \in \operatorname{Nor}(S, x_{i,j})$ such that $x_{i,j} \in$

$\gamma\left(p-\frac{1}{j},p+\frac{1}{j}\right)$ and

$$v=\sum_{i=1}^{m}v_{i,j}. \tag{6.32}$$

If there exists M satisfying $|v_{i,j}|\leq M$ for all i,j, then by passing to a subsequence we may assume that $v_{i,j}\to v_{i,0}$ as $j\to\infty$ for $i=1,\ldots,m$. Thus $v=\sum_{i=1}^{m}v_{i,0}$, and at least one $v_{i,0}$ is nonzero. For each nonzero $v_{i,0}$, we have $\frac{v_{i,0}}{|v_{i,0}|}\in \operatorname{nor}(S,\gamma(p))$ by Lemma 2.33. Thus we may write

$$v=\sum_{v_{i,0}\neq 0}\frac{v_{i,0}}{|v_{i,0}|}|v_{i,0}|, \tag{6.33}$$

so that $v\in \operatorname{Nor}(S,\gamma(p))$, and since $|v|=1$, $v\in\operatorname{nor}(S,\gamma(p))$.

If $|v_{i,j}|\to\infty$ as $j\to\infty$ for some i, relabel the indices i so that $|v_{1,j}|=\max_i|v_{i,j}|$ for all j. Define $w_{i,j}:=\frac{v_{i,j}}{|v_{1,j}|}$. By passing to a subsequence, we may assume that the limit $w_{i,0}=\lim_{j\to\infty}w_{i,j}$ exists. Then $\frac{v}{|v_{1,j}|}\to 0$, and $w_{1,j}:=\frac{v_{1,j}}{|v_{1,j}|}$ satisfies $w_{1,j}\to w_{1,0}$ as $j\to\infty$ and $|w_{1,j}|=1$ for all j. Thus we have

$$0=\sum_{\{i:w_{i,0}\neq 0\}}w_{i,0}. \tag{6.34}$$

By Lemma 2.33, $\frac{w_{i,0}}{|w_{i,0}|}\in\operatorname{nor}(S,p)$ for each i in the sum (6.34). By solving (6.34) for $-w_{1,0}$ we see that $-w_{1,0}\in\operatorname{Nor}(S,p)$. By Lemma 2.26, we have that $0\in\partial f(p)$. This contradicts our assumption that S is regular. $\square$

Since $\operatorname{nor}(S,p)=\emptyset$ whenever $p\notin\partial S$, the following is an immediate consequence of Lemma 6.11.

Corollary 6.12. *Whenever $N(p)\neq 0$, we have that $p\in\partial S$.*

Let $T(p)$ denote the unit tangent vector to the curve γ at $\gamma(p)$. Then η is analogous to the derivative of T in the following sense:

Lemma 6.13. $T(p) - T(q) = \int_{(q,p)} N d\kappa$.

Proof. We shall show that the equality holds on each component. Let ϕ_j be the continuous piecewise affine function that is zero outside of $(q-1/j, p+1/j)$, one on $(q+1/j, p-1/j)$, and affine on $(q-1/j, q+1/j)$ and $(p-1/j, p+1/j)$. Then, by applying the dominated convergence theorem, since $|\langle \phi_j e_i, v\rangle| \leq \sup_Y |v|$, by (6.30) we have

$$\lim_{j\to\infty} \int_Y \langle \phi_j e_i, v\rangle \, d\mu = \int_{Y\cap(\gamma(q,p)\times\mathbb{R}^m)} \langle e_i, v\rangle \, d\mu = \left\langle e_i, \int_{(q,p)} N d\kappa \right\rangle. \tag{6.35}$$

Also, using Theorem 4.3,

$$\int_Y \langle \phi_j e_i, v\rangle \, d\mu = -\int_{(q-1/j,p+1/j)} \left\langle \gamma'(s), \phi_j'(s) e_i \right\rangle ds = -\int_{(q-1/j,p+1/j)} \gamma_i'(s)\phi_j'(s) ds, \tag{6.36}$$

and since

$$\phi_j'(s) = \begin{cases} j/2 & \text{if } s \in (q-1/j, q+1/j) \\ -j/2 & \text{if } s \in (p-1/j, p+1/j) \\ 0 & \text{otherwise,} \end{cases} \tag{6.37}$$

we can simplify the integral above by evaluating ϕ_j':

$$\int_Y \langle \phi_j e_i, v\rangle \, d\mu = -\left(\frac{j}{2}\int_{(q-1/j,q+1/j)} \gamma_i'(s) ds\right) + \left(\frac{j}{2}\int_{(p-1/j,p+1/j)} \gamma_i'(s) ds\right) \tag{6.38}$$

Using continuity of γ', by the Lebesgue differentiation theorem, this limit converges to $\gamma_i'(p) - \gamma_i'(q)$, so that $\gamma_i'(p) - \gamma_i'(q) = \left\langle e_i, \int_{(q,p)} N d\kappa \right\rangle$. □

6.4 Geodesics are $C^{1,1}$

In the remainder of this chapter, we shall frequently use the concept of absolute continuity of Radon measures.

Definition 6.14. *We say that a measure μ on $\mathbb{R}^m$ is **Radon** if it is a regular Borel measure and $\mu(K) < \infty$ for all compact subsets $K \subset \mathbb{R}^m$.*

Definition 6.15. ([8, Section 1.6.2]) *Let μ and ν be Radon measures on $\mathbb{R}^m$. Then μ is **absolutely continuous with respect to** ν, written $\mu << \nu$, if $\mu(A) = 0$ implies $\nu(A) = 0$ for all $A \subset \mathbb{R}^m$. We say that μ and ν are **mutually singular**, written $\mu \perp \nu$, if there exists a Borel subset $B \subset \mathbb{R}^m$ such that $\mu(\mathbb{R}^m - B) = \nu(B) = 0$.*

We shall also use the Lebesgue decomposition theorem, which allows us to break up a Radon measure into parts which are absolutely continuous and mutually singular with respect to another given measure.

Lemma 6.16. ([8, Section 1.6.2, Theorem 3]) *Let μ, ν be Radon measures on $\mathbb{R}^m$. Then $\nu = \nu_{ac} + \nu_s$, where ν_{ac} and ν_s are Radon measures on $\mathbb{R}^m$ satisfying $\nu_{ac} << \mu$ and $\nu_s \perp \mu$.*

Also, the Lebesgue differentiation theorem holds for N integrated against the measure κ. See [8, Theorem 1 in Section 1.7.1] for a proof.

Lemma 6.17. *For κ-a.e. p,*

$$\lim_{r \downarrow 0} \frac{1}{\kappa(B(p,r))} \int_{B(p,r)} |N - N(p)| d\kappa = 0. \tag{6.39}$$

In particular, for κ-a.e. p,

$$\lim_{r \downarrow 0} \frac{1}{\kappa(B(p,r))} \int_{B(p,r)} N d\kappa = N(p). \tag{6.40}$$

The following lemma provides the necessary control over $Nd\kappa$ using geometric properties of S.

Lemma 6.18. *Suppose that $N(p) \neq 0$ and the limits $\lim_{\epsilon \to 0^{+/-}} \frac{T(p+\epsilon)\cdot N(p)}{\epsilon}$ exist in $\mathbb{R} \cup \{\pm\infty\}$. Then*

$$\lim_{\epsilon \to 0^{+/-}} \frac{T(p+\epsilon)\cdot N(p)}{\epsilon} \leq \frac{1}{\mathrm{reach}(S)}. \tag{6.41}$$

Proof. Assume that $\lim_{\epsilon \to 0^{+}} \frac{T(p+\epsilon)\cdot N(p)}{\epsilon} > C$. Then there exists $\delta(C) > 0$ such that $T(p+\epsilon)\cdot N(p) > C\epsilon$ whenever $0 < \epsilon < \delta(C)$. Suppose that $0 < \epsilon < \delta(C)$. Then $T(p+s)\cdot N(p) > sC$ whenever $0 < s < \epsilon$, and

$$\begin{aligned}
(\gamma(p+\epsilon) - \gamma(p)) \cdot N(p) &= \int_{(0,\epsilon)} T(p+s)\cdot N(p) ds \\
&> \int_{(0,\epsilon)} sC ds \qquad (6.42) \\
&= \frac{C}{2}\epsilon^2. \qquad (6.43)
\end{aligned}$$

By Lemma 1.7, we have

$$(\gamma(p+\epsilon) - \gamma(p)) \cdot N(p) \leq \frac{\epsilon^2}{2\ \mathrm{reach}(S)}, \tag{6.44}$$

so that $C < \frac{1}{\mathrm{reach}(S)}$. A similar argument shows that $\lim_{\epsilon \to 0^{-}} \frac{T(p+\epsilon)\cdot N(p)}{\epsilon} \leq \frac{1}{\mathrm{reach}(S)}$ $\square$

We now prove that κ is absolutely continuous with respect to m.

Theorem 6.19. *κ is absolutely continuous with respect to Lebesgue measure m.*

Proof. Assume that κ is not absolutely continuous with respect to m. Decompose κ as $\kappa = \kappa_{ac} + \kappa_s$, where κ_{ac} is absolutely continuous with respect to m and κ_s is singular.

Claim. *There exists a set E with $\kappa(E) > 0$ such that $D_{\kappa+m}(\kappa_{ac} + m)(p) = 0$ for $p \in E$.*

Proof. Otherwise, $D_{\kappa+m}(\kappa_{ac}+m)(p) > 0$ for κ-a.e. p. But by the Radon-Nikodym theorem [8, Section 1.6.2, Theorem 2], for all Borel sets U,

$$(\kappa_{ac}+m)(U) = \int_U D_{\kappa+m}(\kappa_{ac}+m)(p)d(\kappa+m), \tag{6.45}$$

but then if $(\kappa_{ac}+m)(U) = 0$, we clearly also have $(\kappa+m)(U) = 0$, which implies that κ is absolutely continuous with respect to κ_{ac}. But then by transitivity, since κ_{ac} is absolutely continuous with respect to m, κ is absolutely continuous with respect to m, which contradicts the assumption that κ is not absolutely continuous with respect to m. □

Further, for $p \in E$, by [8, Section 1.6.1], $D_{\kappa+m}(\kappa_{ac}+m)(p) = \lim_{\epsilon \downarrow 0} \frac{(\kappa_{ac}+m)(B(p,\epsilon))}{(\kappa+m)(B(p,\epsilon))} = 0$, so for $p \in E$,

$$\lim_{\epsilon \downarrow 0} \frac{(\kappa+m)(B(p,\epsilon))}{(\kappa_{ac}+m)(B(p,\epsilon))} \to \infty, \tag{6.46}$$

and we also have

$$\lim_{\epsilon \downarrow 0} \frac{\kappa(B(p,\epsilon))}{m(B(p,\epsilon))} \to \infty \tag{6.47}$$

and

$$(\kappa_{ac}+m)(E) = \int_E D_{\kappa+m}(\kappa_{ac}+m)(p)d(\kappa+m) = 0, \tag{6.48}$$

so we must have $\kappa_{ac}(E) = m(E) = 0$ and $\kappa_s(E) > 0$.

Assume that $p \in E$ satisfies (6.40). Then by Lemma 6.18,

$$\begin{aligned}
\frac{\kappa(B(p,\epsilon))}{m(B(p,\epsilon))} \cdot \frac{1}{\kappa(B(p,\epsilon))} \int_{B(p,\epsilon)} N(p) \cdot N d\kappa &= N(p) \cdot \frac{1}{2\epsilon} \int_{B(p,\epsilon)} N d\kappa \\
&= N(p) \cdot \frac{T(p+\epsilon) - T(p-\epsilon)}{2\epsilon} \\
&\leq \frac{2}{\operatorname{reach}(S)}
\end{aligned} \tag{6.49}$$

for all sufficiently small $\epsilon > 0$. By letting $\epsilon \to 0$, we conclude that

$$\overline{\lim_{\epsilon \to 0}} \frac{\kappa(B(p,\epsilon))}{m(B(p,\epsilon))} \leq \frac{2}{\text{reach}(S)}, \tag{6.50}$$

which is a contradiction. Thus κ is absolutely continuous with respect to m.

$\square$

Theorem 6.20. *γ is $C^{1,1}$.*

Proof. Let

$$U = \{p : D_m\kappa(p) < \infty\} \cap \left\{p : \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_{B(p,\epsilon)} ND_m\kappa \; dm = N(p)D_m\kappa(p)\right\}. \tag{6.51}$$

Recall that, by definition, $N(p) = \dfrac{D_\alpha \eta(p)}{|D_\alpha \eta(p)|}$ and by [8, Section 1.6.1, Theorem 1] N is α-measurable, and therefore κ-measurable. N is κ-integrable because it is also bounded; by the Radon-Nikodym theorem, $ND_m\kappa$ is m-integrable. By the Lebesgue differentiation theorem [8, Theorem 1 of Section 1.7.1], since $ND_m\kappa$ is Lebesgue integrable, we have $\lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_{B(p,\epsilon)} ND_m\kappa \; dm = N(p)D_m\kappa(p)$ for m-almost every p. Also, $D_m\kappa(p)$ is finite m-almost everywhere by [8, Section 1.6.1, Theorem 1]. Thus U has full m-measure in $[0, L]$.

By the Radon-Nikodym Theorem [8, Theorem 2 of Section 1.6.2], since $D_m\kappa \geq 0$,

$$\begin{aligned} |\gamma'(p) - \gamma'(q)| &= \left| \int_{(q,p)} N d\kappa \right| \\ &= \left| \int_{(q,p)} ND_m\kappa dm \right| \\ &\leq |p - q| \cdot \text{essential sup}\{D_m\kappa\}. \end{aligned} \tag{6.52}$$

It therefore suffices to show that $D_m\kappa$ is m-essentially bounded; we show that for $p \in U$,

$$D_m\kappa \leq \frac{2}{\text{reach}(S)}. \tag{6.53}$$

For p in U,

$$N(p) \cdot \frac{T(p+\epsilon) - T(p-\epsilon)}{2\epsilon} = N(p) \cdot \frac{1}{2\epsilon} \int_{B(p,\epsilon)} N d\kappa \to D_m\kappa(p). \tag{6.54}$$

However, by Lemma 6.18,

$$\begin{aligned} \lim_{\epsilon \downarrow 0} N(p) \cdot \frac{T(p+\epsilon) - T(p-\epsilon)}{2\epsilon} &= \lim_{\epsilon \downarrow 0} \left(N(p) \cdot \frac{T(p+\epsilon)}{2\epsilon} + N(p) \cdot \frac{T(p-\epsilon)}{-2\epsilon} \right) \\ &\leq \frac{2}{\text{reach}(S)}. \end{aligned} \tag{6.55}$$

□

Corollary 6.21. *We have $T(p) \cdot N(p) = 0$ for m-almost every p.*

Proof. The second derivative $\gamma'' = T'$ exists almost everywhere by Rademacher's Theorem; we claim that $T \cdot N = 0$ at points of U where T' exists. Using the fact that $p \in U$, since $\frac{T(p+\epsilon) - T(p-\epsilon)}{2\epsilon} = \frac{1}{2\epsilon} \int_{B(p,\epsilon)} N D_m\kappa dm$, we have that

$$\begin{aligned} D_m\kappa(p) \cdot \langle N(p), T(p) \rangle &= \left\langle \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{B(p,\epsilon)} N D_m\kappa dm, T(p) \right\rangle \\ &= \lim_{\epsilon \to 0} \left\langle \frac{T(p+\epsilon) - T(p-\epsilon)}{2\epsilon}, T(p) \right\rangle \\ &= \langle T'(p), T(p) \rangle \\ &= 0. \end{aligned} \tag{6.56}$$

□

Chapter 7

The Reach Formula for Compact Regular PR Sets

7.1 The Setup

The reach of a compact PR set $S \subset \mathbb{R}^m$ can be expressed in the following way as in [3]:

Lemma 7.1. *The reach of a compact PR set S is the infimal r such that there exist points x, p, and y such that $x \neq y \in S$, and $p - x \in \operatorname{Nor}(S, x)$ with $|p - x| = r = |p - y|$.*

Proof. As before, let π_S denote the unique nearest point map and let $\operatorname{Unp}(S)$ denote the set of points having a unique nearest point in S. We first show that $\operatorname{reach}(S) \leq \inf r$. For $x \in S$ having such p, y, and r, we may write $p = x + rv$ where v is a unit normal vector to S at x. By [9, Theorem 4.8(6)], $\operatorname{reach}(S) \leq \tau := \sup\{t \geq 0 : \pi_S(x + tv) = x\}$. Here we know that $\tau \leq r$ since x cannot be the unique nearest point of S to p, so that $\operatorname{reach}(S) \leq r$, and $\operatorname{reach}(S) \leq \inf r$.

Let $B(S, \epsilon)$ denote the set of points at distance less than ϵ from S. By definition, $\operatorname{reach}(S) = \sup\left\{\epsilon > 0 : \overline{B(S,\epsilon)} \subset \operatorname{Unp}(S)\right\}$. Thus, for any $R > \operatorname{reach}(S)$, there exist $p \notin \operatorname{Unp}(S)$ and distinct points x and y in S such that $d_S(p) = |x - p| = |y - p| \leq R$.

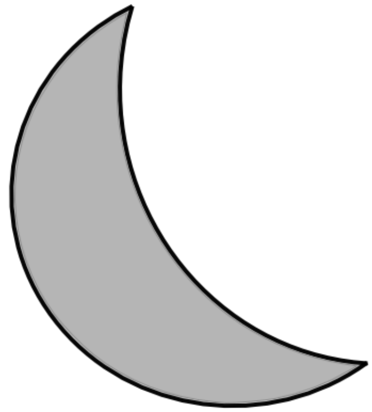

Figure 7.1: The outward curvature along the boundary determines the reach.

Then by Lemma 1.5, $p - x \in \mathrm{Nor}(S, x)$ (and also $p - y \in \mathrm{Nor}(S, y)$). Thus $R \geq \inf r$, so since $\mathrm{reach}(S) = \inf R$, $\mathrm{reach}(S) \geq \inf r$. $\square$

Now we shall obtain a formula for the reach of S that takes into consideration two different kinds of behavior: outward curvature at a point in ∂S and global obstructions to the projection π_S. (See Figures 7.1 and 7.2, respectively.) We first describe the reach as an infimum over a family of circle radii.

Remark. The term "plane" will always refer to a plane of dimension 2.

Definition 7.2. *Suppose S is a PR set. For distinct points $x, y \in \partial S$ and $v \in \mathrm{nor}(S, x)$, we say that y is **admissible** at $(x, v) \in \mathrm{nor}(S)$ if $\langle v, y - x \rangle > 0$. If y is admissible at (x, v), define the **radius function** $r(x, y; v)$ to be the radius of the planar circle through x and y with center on the ray $\{x + \lambda v : \lambda > 0\}$.*

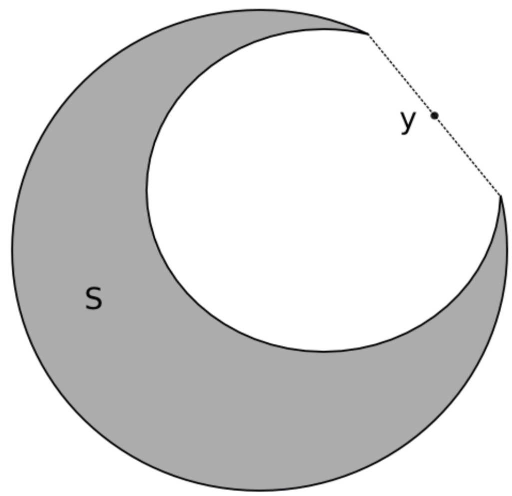

Figure 7.2: A global obstruction to π_S: the nearest point of S to y is not unique

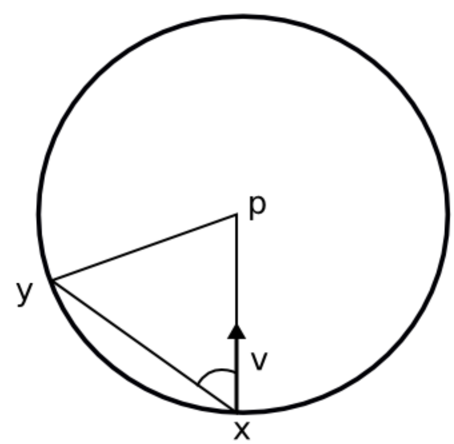

Figure 7.3: Determining the formula for $r(x, y; v)$.

Definition 7.3. *Suppose that y is admissible at $(x, v) \in \operatorname{nor}(S)$. Define $\psi(x, y; v) \in [0, \pi/2)$ to be the measure of the angle between $y - x$ and v.*

Lemma 7.4. *The radius function $r(x, y; v)$ is given by*

$$r(x, y; v) = \frac{|y - x|}{2 \cos \psi(x, y; v)} = \frac{|y - x|}{2 \left\langle v, \dfrac{y - x}{|y - x|} \right\rangle}. \tag{7.1}$$

Proof. Consider the planar circle through x and y with center $p = x + r(x, y; v)v$ (see Figure 7.3); we may compute the cosine of the base angle $\psi(x, y; v)$ by $\cos(\psi(x, y; v)) = \dfrac{|y - x|}{2r(x, y; v)}$. Further, since $\cos(\psi(x, y; v))$ is the length of the projection of v onto $y - x$, we have $\left\langle v, \dfrac{y - x}{|y - x|} \right\rangle = \dfrac{|y - x|}{2r(x, y; v)}$. $\square$

Remark. In the case that S is a $C^{1,1}$ m-dimensional submanifold with boundary, the radius $r(x, y; n(x))$ is also the radius of the sphere tangent to S at x and passing through y. In this case, we shall write $r(x, y)$ and $\psi(x, y)$ in place of $r(x, y; n(x))$ and $\psi(x, y; n(x))$. Similarly, we shall say that y *is admissible at* x because $\operatorname{nor}(S, x) = \{n(x)\}$.

Lemma 7.5. *Let S be a PR set with* $\operatorname{reach}(S) < \infty$. *Then*

$$\operatorname{reach}(S) = \inf_{\substack{x \neq y \\ y \text{ admissible at } (x,v) \in \operatorname{nor}(S)}} r(x, y; v). \tag{7.2}$$

Proof. Suppose that $r(x, y; v)$ is the radius of the planar circle through x and y with center $p = x + r(x, y; v)v$. The center p is a point satisfying $|p - x| = |p - y| = r(x, y; v)$ and $p - x \in \operatorname{Nor}(S, x)$. Moreover, given p and y satisfying $x \neq y \in \partial S$ and $p - x \in \operatorname{Nor}(S, x)$ with $|p - x| = |p - y|$, the planar circle through x and y which is normal to $v := \dfrac{p - x}{|p - x|}$ at x has radius $r(x, y; v) = |p - x|$. By Lemma 7.1, we have $\operatorname{reach}(S) = \inf_{\substack{x \neq y \\ y \text{ admissible at } (x,v) \in \operatorname{nor}(S)}} r(x, y; v)$. $\square$

Remark. In the case that $S \subset \mathbb{R}^m$ is a $C^{1,1}$ m-dimensional submanifold with boundary, the function r is a quotient of Lipschitz functions, so it is locally Lipschitz.

Definition 7.6. *Let $S \subset \mathbb{R}^m$ be a $C^{1,1}$ m-dimensional submanifold with boundary. Let $x \in \partial S$, and let P be a plane satisfying $P \not\subset \operatorname{Tan}(S, x)$. We say that $\vec{\kappa}$ is the **curvature vector of the curve** $P \cap \partial S$ **at** x if $\vec{\kappa}$ is the curvature vector at x of a parametrization of $P \cap \partial S$ by arclength near x.*

In the setting of $C^{1,1}$ submanifolds with boundary, the radii can be thought of as preliminary data for osculating circles:

Lemma 7.7. *Let $S \subset \mathbb{R}^m$ be a $C^{1,1}$ m-dimensional submanifold with boundary. Let P be a plane through $x \in \partial S$ such that $x + n(x) \in P$. Suppose that the curve $P \cap \partial S$ has curvature vector $\vec{\kappa}$ at x and that y is admissible at x for all $y \in P \cap \partial S$ sufficiently near x. Then*

$$\lim_{y \to x} r(x, y) = \frac{1}{|\vec{\kappa}|}. \tag{7.3}$$

Proof. See [3, Lemma 3.5]. □

Lemma 7.8. *Let $S \subset \mathbb{R}^m$ be a $C^{1,1}$ m-dimensional submanifold with boundary. Let P be a plane through $x \in \partial S$ such that $x + n(x) \in P$. Suppose that the curve $P \cap \partial S$ has unit tangent vector T and curvature vector $\vec{\kappa}$ at x. Then the second fundamental form $\mathrm{II}(x)(T, T)$ satisfies*

$$|\mathrm{II}(x)(T, T)| = |\vec{\kappa}|. \tag{7.4}$$

In particular, when $\langle \vec{\kappa}, n(x) \rangle > 0$, we have

$$\mathrm{II}(x)(T, T) = |\vec{\kappa}|. \tag{7.5}$$

Remark. We generalize this result in Lemma 7.11.

Proof. Let $\gamma = \gamma(t)$ be an arclength parametrization of $P \cap \partial S$ near x such that $\gamma(0) = x$. Then, since $\gamma'(0) = T$ and $\gamma''(0) = \vec{\kappa} = \pm|\vec{\kappa}| n(x)$, we may differentiate both sides of the equation $0 = \langle n \circ \gamma(t), \gamma'(t) \rangle$ at time zero and obtain

$$0 = \langle dn_x(T), T \rangle + \langle n(x), \vec{\kappa} \rangle = -\mathrm{II}(x)(T,T) \pm |\vec{\kappa}|. \tag{7.6}$$

In the case that $\langle \kappa, n(x) \rangle > 0$, we have $\vec{\kappa} = |\vec{\kappa}| n(x)$, so that $\mathrm{II}(x)(T,T) = |\vec{\kappa}|$. $\square$

Denote the the directional derivative of a function F with respect to the unit tangent vector $T = T(x)$ by $\dfrac{d}{dT}(F)(x) := dF_x(T)$.

Lemma 7.9 ([3, Lemma 3.6])**.** *Let S be a $C^{1,1}$ m-dimensional submanifold with boundary, and suppose $x \in \mathrm{Sm}(S)$. Let P be a plane through x containing $x + n(x)$, and suppose that the curve $P \cap \partial S$ has curvature vector $\vec{\kappa}$ at x. Further suppose that $y \in P \cap \partial S$ is admissible at x with $\dfrac{y-x}{|y-x|} \neq n(x)$. Orient $P \cap \partial S$ so that the unit tangent vector T at x satisfies $\langle T, y - x \rangle > 0$. Then the directional derivative $\dfrac{\partial r}{\partial T}(x,y)$ exists and is given by*

$$\frac{\partial r}{\partial T}(x,y) = (r(x,y)|\vec{\kappa}| - 1)\tan(\psi(x,y)). \tag{7.7}$$

Proof. Suppose that γ parametrizes $P \cap \partial S$ by arclength near x. Write $\cos\psi(x,y) = \left\langle n(x), \dfrac{y-x}{|y-x|} \right\rangle$ and take $\dfrac{d}{dT}$ of both sides; since $\dfrac{y-x}{|y-x|}$ is in the $T, n(x)$-plane and

$$\langle dn_x(T), n(x) \rangle = 0, \tag{7.8}$$

we have

$$
\begin{aligned}
&-\sin(\psi(x,y))\frac{d\psi}{dT} \\
&= \left\langle dn_x T, \frac{y-x}{|y-x|} \right\rangle + \left\langle n(x), \frac{1}{|y-x|}\left(\left\langle T, \frac{y-x}{|y-x|} \right\rangle \frac{y-x}{|y-x|} - T\right)\right\rangle \\
&= \left\langle dn_x T, \frac{y-x}{|y-x|} \right\rangle - \frac{1}{|y-x|}\langle n(x), T\rangle + \frac{1}{|y-x|}\left\langle T, \frac{y-x}{|y-x|} \right\rangle \left\langle n(x), \frac{y-x}{|y-x|} \right\rangle \\
&= \left\langle dn_x T, \frac{y-x}{|y-x|} \right\rangle + \frac{1}{|y-x|}\left\langle T, \frac{y-x}{|y-x|} \right\rangle \left\langle n(x), \frac{y-x}{|y-x|} \right\rangle \\
&= \langle dn_x T, n(x)\rangle \left\langle n(x), \frac{y-x}{|y-x|} \right\rangle + \langle dn_x T, T\rangle \left\langle T, \frac{y-x}{|y-x|} \right\rangle + \frac{\sin(\psi(x,y))\cos(\psi(x,y))}{|y-x|} \\
&= \langle dn_x T, T\rangle \left\langle T, \frac{y-x}{|y-x|} \right\rangle + \frac{\sin(\psi(x,y))\cos(\psi(x,y))}{|y-x|} \\
&= -|\vec{\kappa}|\sin(\psi(x,y)) + \frac{\sin(\psi(x,y))\cos(\psi(x,y))}{|y-x|},
\end{aligned}
\tag{7.9}
$$

so that

$$
\frac{d\psi}{dT} = |\vec{\kappa}| - \frac{1}{2r(x,y)}. \tag{7.10}
$$

Also, by taking the derivative of $2r\cos(\psi(x,y)) = \langle y-x, y-x\rangle^{1/2}$, we obtain

$$
-2r(x,y)\sin(\psi(x,y))\frac{d\psi}{dT} + 2\cos(\psi(x,y))\frac{dr}{dT} = -\left\langle \frac{y-x}{|y-x|}, T \right\rangle. \tag{7.11}
$$

Since $\left\langle \frac{y-x}{|y-x|}, T \right\rangle = \sin(\psi(x,y))$, by combining (7.11) and (7.10), we obtain the formula

$$
\frac{dr}{dT} = \tan(\psi(x,y))(r(x,y)|\vec{\kappa}| - 1). \tag{7.12}
$$

□

7.2 The Reach Formula in the $C^{1,1}$ Case

Throughout this section, assume that $S \subset \mathbb{R}^m$ is an m-dimensional $C^{1,1}$ submanifold with boundary.

7.2.1 Meusnier's Theorem

In this subsection, we shall further describe the relationship between the second fundamental form II and the curvatures of plane sections of ∂S. We shall use these results in the next section when we refine our formula for reach(S).

Lemma 7.10. *Let $x \in \partial S$, and let P be a plane containing x such that $P \not\subset \mathrm{Tan}(S, x)$. Then $\partial S \cap P$ is a C^1 curve near x.*

Proof. A standard codimension argument shows that $\partial S \cap P$ is a curve. It is continuously differentiable because the transverse intersection of two C^1 submanifolds is C^1. □

Lemma 7.11 (Meusnier's Theorem)**.** *Suppose that $x \in \mathrm{Sm}(S)$. If P is a plane through x satisfying $P \not\subset \mathrm{Tan}(S, x)$, then $P \cap \partial S$ has a curvature vector $\vec{\kappa}$ at x. If T is a unit tangent vector to $P \cap \partial S$ at x, then*

$$\mathrm{II}(x)(T, T) = \langle n(x), \vec{\kappa} \rangle. \tag{7.13}$$

Proof. Suppose that the arclength-parametrized curve γ parametrizes $P \cap \partial S$ on a neighborhood of x, and suppose that $\gamma(0) = x$. Project $n \circ \gamma$ onto P and then normalize. The resulting unit vector field $\tilde{n}$ along γ is normal to γ. This vector field is differentiable at $t = 0$ because $n \circ \gamma$ is differentiable at $t = 0$, and its projection onto P is non-vanishing in a neighborhood of x. Using $\mathbb{R}^2$ coordinates, write $\tilde{n}(t) = (\cos(\theta(t)), \sin(\theta(t)))$, so that $\tilde{n}'(t) = \theta'(t)(-\sin(\theta(t)), \cos(\theta(t)))$. Up to a sign, $\gamma'(t) = (-\sin(\theta(t)), \cos(\theta(t)))$; since $\tilde{n}'(0)$ exists, so does $\gamma''(0) = \theta'(0)(-\cos(\theta(0)), -\sin(\theta(0)))$. Now, using the fact that $0 = \langle n \circ \gamma, \gamma' \rangle$

and taking the derivative of both sides with respect to t at $t = 0$, we obtain

$$0 = \langle dn_x(T), T\rangle + \langle n(x), \vec{\kappa}\rangle = -\mathrm{II}(x)(T, T) + \langle n(x), \vec{\kappa}\rangle. \tag{7.14}$$

$\square$

Remark. It follows from Lemma 7.11 that we can compute $\mathrm{II}(x)(T, T)$ at $x \in \mathrm{Sm}(S)$ using any planar curve $P \cap \partial S$ containing x, so long as $P \not\subset \mathrm{Tan}(S, x)$.

7.2.2 Existence of Nearby Almost-smooth Curves

In this section we show that, on a sufficiently small open set in ∂S, any planar curve has a sufficiently close curve F_w such that $F_w(t) \in \mathrm{Sm}(S)$ for almost every t. We shall use the following lemma as the foundation for our approximations:

Lemma 7.12. *The oriented C^1 hypersurface $A \subset \mathbb{R}^m$ is a $C^{1,1}$ hypersurface if and only if, given any $p \in A$, there is a neighborhood $U \subset \mathbb{R}^m$ of p such that under some system of isometric coordinates on U the set $A \cap U$ appears as the graph of a C^1 function with Lipschitz gradient.*

Proof. First, A is locally a graph. Suppose that $\alpha : U \to A$ is a coordinate map on A near $p \in A$, and let $n : A \to S^{m-1}$ denote the Gauss map. Assume without loss of generality that $n(p) = e_m = (0, \ldots, 0, 1)$. Let $\Pi : A \to \{x_m = 0\}$ by $\Pi(x) = x - \langle x, e_m\rangle e_m$. We claim that Π is injective on a sufficiently small neighborhood of p. If not, there exist sequences of points y_i and x_i satisfying $x_i \to p$, $y_i \to p$, and $\dfrac{y_i - x_i}{|y_i - x_i|} = e_m$. Write $y_i' = \alpha^{-1}(y_i)$ and $x_i' = \alpha^{-1}(x_i)$. By passing to a subsequence, we may assume that $\dfrac{y_i' - x_i'}{|y_i' - x_i'|} \to v$. Then, since α is C^1, using the fundamental theorem of calculus and the dominated convergence theorem, we have

$$\begin{aligned}
d\alpha_p(v) &= \int_{[0,1]} \lim_{i\to\infty} d\alpha_{(1-t)x_i'+ty_i'}\left(\frac{y_i'-x_i'}{|y_i'-x_i'|}\right) dt \\
&= \lim_{i\to\infty} \int_{[0,1]} d\alpha_{(1-t)x_i'+ty_i'}\left(\frac{y_i'-x_i'}{|y_i'-x_i'|}\right) dt \\
&= \lim_{i\to\infty} \frac{\alpha(y_i')-\alpha(x_i')}{|y_i'-x_i'|} \\
&= \lim_{i\to\infty} \frac{|y_i-x_i|}{|y_i'-x_i'|}\frac{y_i-x_i}{|y_i-x_i|} \\
&= \lim_{i\to\infty} \frac{|y_i-x_i|}{|y_i'-x_i'|} e_m.
\end{aligned} \tag{7.15}$$

This is a contradiction. Since Π is locally injective, we may define a function ϕ by $(x,\phi(x)) = \Pi^{-1}(x)$. Up to a sign, $n(x) = \frac{1}{\sqrt{1+|\nabla\phi_x|^2}}(-\nabla\phi_x, 1)$. Thus $\nabla\phi$ is Lipschitz if and only if n is Lipschitz. $\square$

Lemma 7.13. *Let $p \in \partial S$. There exists an open set U containing p with the following property: for any pair of points q_1 and q_2 in $U \cap \partial S$, there exists a closed ball $\overline{B} \subset \mathbb{R}^{m-1}$ of positive radius and a $C^{1,1}$ map $F : [0,1] \times \overline{B} \to \partial S$ of ∂S such that*

1. *$F : (0,1) \times B \to \partial S$ parametrizes an open subset of ∂S.*
2. *The curve $F_w(t) := F(t,w)$ is a $C^{1,1}$ planar curve, and one of the curves F_w joins q_1 and q_2.*
3. $\mathrm{length}(F_w) \leq C'|q_1 - q_2|$, *where C' is independent of the vector w and points q_1 and q_2.*
4. *For almost every w, the curve F_w satisfies $F_w(t) \in \mathrm{Sm}(S)$ for almost every t, and the Lipschitz constant for the unit tangent to F_w is independent of the vector w and points q_1 and q_2.*
5. *Suppose $x = F_w(t) \in \mathrm{Sm}(S)$. Let T and $\vec{\kappa}$ denote the unit tangent vector and curvature vector of the curve F_w at x. Suppose that $\langle \vec{\kappa}, n(x)\rangle > 0$. Then there exists a constant*

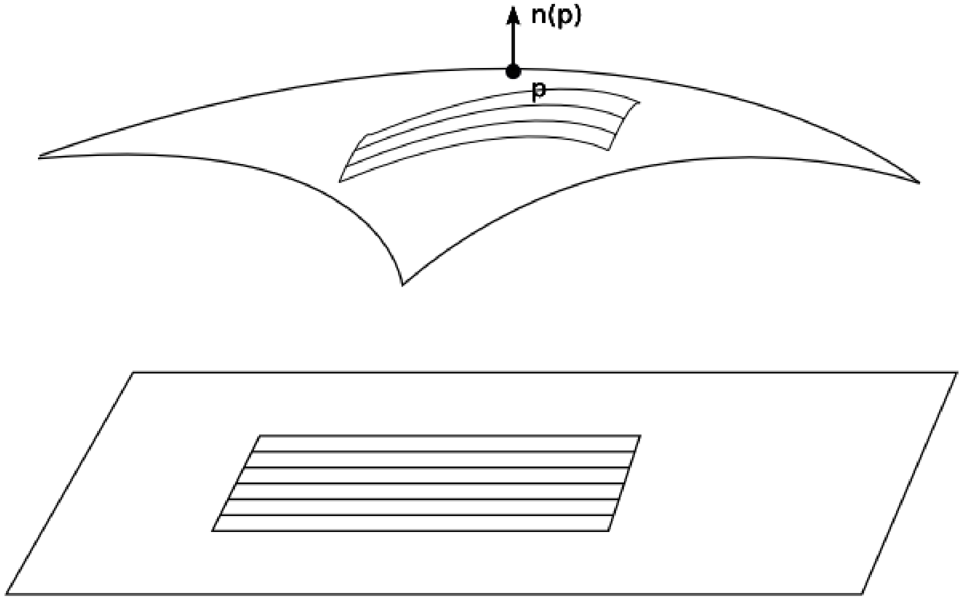

Figure 7.4: The curves F_w of Lemma 7.13.

C, independent of q_1, q_2, t, and w, such that

$$|\mathrm{II}(x)(T,T) - |\vec{\kappa}|| \leq C|p-x|. \tag{7.16}$$

Proof. Assume without loss of generality that $n(p) = e_m$, the m-th standard basis vector of $\mathbb{R}^m$ (see Figure 7.4). Using Lemma 7.12, let U be a neighborhood of p such that $U \cap \partial S = \{(x, \phi(x)) : x \in V \subset \{x_m = 0\}\}$.

Sub-lemma 7.14. *Suppose that $|u| = 1$ and that u and e_m are linearly independent. There exists a neighborhood $U' \subset \partial S$ of p on which the normalized projection of n onto the plane spanned by u and e_m is Lipschitz, and the Lipschitz constant is independent of u for all u with $|\langle u, n(p)\rangle| < \delta$.*

Proof. Let C be a Lipschitz constant for n. Write $n(x,u) := \langle u, n(x)\rangle u + \langle e_m, n(x)\rangle e_m$. The normalized projection has the form

$$\tilde{n}(x,u) := \frac{n(x,u)}{|n(x,u)|}. \tag{7.17}$$

Since $n(p,u) = 1$ for all $u \in \mathrm{Tan}(S,p)$, there exists $\delta > 0$ for which $|n(x,u)| > 1/2$ whenever $|\langle u, n(p)\rangle| < \delta$ and $|x-p| < \delta$. Thus $\tilde{n}$ is Lipschitz on $|x-p| < \delta$, and the Lipschitz constant depends only on C. $\square$

Now, assume without loss of generality that $U \cap \partial S \subset U'$ and V is convex. Let $\psi : U \cap \partial S \to V$ by $\psi(x_1, \ldots, x_m) = (x_1, \ldots, x_{m-1}, 0)$. For $q_1, q_2 \in U \cap \partial S$, assume without loss of generality that $\psi(q_1) = 0$ and $\psi(q_2) = ce_1$, so that $q_1 = (0, \phi(0))$ and $q_2 = (ce_1, \phi(ce_1))$. Note that $\psi(q_2) - \psi(q_1) = ce_1$. Let $\gamma : [0,1] \to \{x_m = 0\}$ be the straight-line path joining $\psi(q_1) = 0$ and $\psi(q_2) = ce_1$, parametrized by $\gamma(t) = cte_1$ for $t \in [0,1]$. Let $W = \{w : w \cdot e_m = 0 \text{ and } w \cdot e_1 = 0\}$, and let ϵ' be sufficiently small that $\gamma(t) + w \in V$ for all $t \in [0,1]$ and all $w \in W$ with $|w| \leq \epsilon'$. Let $\overline{B} = \{w \in W : |w| \leq \epsilon'\}$.

Consider the variation $F : [0,1] \times \overline{B} \to \partial S$ given by

$$F(t,w) = (\gamma(t) + w, \phi(\gamma(t) + w)) = (cte_1 + w, \phi(cte_1 + w)). \tag{7.18}$$

Let $F_w(t) = F(t,w)$, and note that F_0 joins q_1 and q_2.

Sub-lemma 7.15. *$F : (0,1) \times B \to \partial S$ parametrizes an open set in ∂S.*

Proof. This follows from smoothness of ψ and the fact that $\nabla\phi$ is Lipschitz along with the calculations

$$\frac{\partial}{\partial t}F(t,w) = (ce_1, \nabla\phi_{\gamma(t)+w}(ce_1)) \tag{7.19}$$
$$\frac{\partial}{\partial w}F(t,w) = (w, \nabla\phi_{\gamma(t)+w}(w)).$$

The partial derivatives span the tangent space at $F(t,w)$ since the vectors $\{e_1\}\cup\{w : w \in \overline{B}\}$ span $\mathbb{R}^{m-1}$. □

Sub-lemma 7.16. *Each curve F_w is a $C^{1,1}$ curve in the plane through $(w,0)$ spanned by e_1 and e_m, and the Lipschitz constant C is independent of w and q_1, q_2.*

Proof. It is clear that F_w satisfies the planarity condition. To see that the curve is $C^{1,1}$, note that the normal to the curve F_w at the point $F_w(t) = F(t,w)$ is given by $\tilde{n}\left(F(t,w), e_1\right)$, up to a sign. Thus, since $e_1 \cdot n(p) = 0$,

$$|\tilde{n}\left(F(t_1,w), e_1\right) - \tilde{n}\left(F(t_2,w), e_1\right)| \leq C|F(t_1,w) - F(t_2,w)|, \tag{7.20}$$

where C is a constant independent of w, q_1, and q_2 by Sub-lemma 7.14. Since $\tilde{n}$ is Lipschitz with constant C, so is the unit tangent vector field $T = T(F(t,w))$. □

Sub-lemma 7.17. length$(F_w) \leq C'|q_1 - q_2|$, *where C' is independent of the vector w and points q_1 and q_2*

Proof. Using the definition of F,

$$\begin{aligned}
\text{length}(F_w) &= \int_{[0,1]} |(ce_1, \nabla\phi_{\gamma(t)+w}(ce_1))|dt \\
&= c\int_{[0,1]} \sqrt{1 + |\nabla\phi_{\gamma(t)+w}(e_1)|^2}dt \\
&\leq c\sqrt{1 + \sup|\nabla\phi_x|^2} \\
&= C'|q_1 - q_2|,
\end{aligned} \tag{7.21}$$

where the supremum above is taken over all x in a compact set containing $\bar{U}$. □

Sub-lemma 7.18. *For almost every w, the curve F_w has the property that $F_w(t) \in \text{Sm}(S)$ for almost every t.*

Proof. Let $Y = \{(t, w) \in [0,1] \times \overline{B} : \text{II is undefined at } F(t, w)\}$. Then $F(Y) \subset S$ has measure zero by hypothesis, and

$$0 = \int_{F(Y)} d\text{vol}(\partial S) = \int_Y |\text{Jac}(F)| d\text{vol}(\{e_m = 0\}), \tag{7.22}$$

where $\text{Jac}(F)$ denotes the determinant of the map dF. Since the determinant $|\text{Jac}(F)|$ is non-vanishing by (7.19), we conclude that $Y \subset [0,1] \times \overline{B}$ has measure zero. Moreover, by Fubini's theorem,

$$\int_Y |\text{Jac}(F)| d\text{vol}(\{e_m = 0\}) = \int_{[0,1]} \int_{\overline{B}} \chi_Y |\text{Jac}(F)|, \tag{7.23}$$

where χ_Y denotes the characteristic function of Y. Thus we must have $(t, w) \notin Y$ for almost every w and t, so that almost every curve F_w satisfies $F_w(t) \in \text{Sm}(S)$ for almost every t. □

Sub-lemma 7.19. *Suppose $x = F_w(t) \in \text{Sm}(S)$. Let T and $\vec{\kappa}$ denote the unit tangent vector and curvature vector of the curve F_w at x. Suppose that $\langle\vec{\kappa}, n(x)\rangle > 0$. Then there exists a* *constant C, independent of q_1, q_2, t, and w, such that*

$$|\mathrm{II}(x)(T,T) - |\vec{\kappa}|| \leq C|p - x|. \tag{7.24}$$

Proof. From Lemma 7.11, we have

$$\begin{aligned}
\mathrm{II}(x)(T,T) &= \langle n(x), \vec{\kappa}\rangle \\
&= \langle n(x) - \frac{\vec{\kappa}}{|\vec{\kappa}|}, \vec{\kappa}\rangle + \langle \frac{\vec{\kappa}}{|\vec{\kappa}|}, \vec{\kappa}\rangle \\
&= \langle n(x) - \frac{\vec{\kappa}}{|\vec{\kappa}|}, \vec{\kappa}\rangle + |\vec{\kappa}|
\end{aligned} \tag{7.25}$$

so that, by the Cauchy-Schwarz inequality, we have

$$|\mathrm{II}(x)(T,T) - |\vec{\kappa}|| \leq |\vec{\kappa}| \left| n(x) - \frac{\vec{\kappa}}{|\vec{\kappa}|} \right|. \tag{7.26}$$

By Equation (7.20), we have $|\vec{\kappa}| \leq C$, where C is independent of q_1 and q_2 as well as t and w. Further, $\dfrac{\vec{\kappa}}{|\vec{\kappa}|}$ is the normalized projection of $n(x) = n(F(t,w))$ onto the plane spanned by e_1 and e_m, so that $\dfrac{\vec{\kappa}}{|\vec{\kappa}|} = \tilde{n}\left(F(t,w), e_1\right)$. This projection is Lipschitz continuous with constant C by Sub-lemma 7.14, so that, since $\tilde{n}\left(p, e_1\right) = n(p)$,

$$\begin{aligned}
\left| \frac{\vec{\kappa}}{|\vec{\kappa}|} - n(x) \right| &\leq \left| \frac{\vec{\kappa}}{|\vec{\kappa}|} - n(p) \right| + |n(p) - n(x)| \\
&= \left|\tilde{n}\left(F(t,w), e_1\right) - \tilde{n}\left(p, e_1\right)\right| + |n(p) - n(x)| \\
&\leq C|F(t,w) - p| + M_n|p - x| \\
&= (C + M_n)|x - p|,
\end{aligned} \tag{7.27}$$

where M_n is a Lipschitz constant for n. □

This concludes the proof of Lemma 7.13. □

7.2.3 The Penalized Distance and ρ

In this subsection, we adapt the techniques of [3] to the setting of $C^{1,1}$ submanifolds with boundary in Euclidean space in order to obtain a formula for the reach of such sets. We follow the same general techniques of proof as in [3], although we shall have to make some modifications.

Suppose $S \subset \mathbb{R}^m$ is an m-dimensional $C^{1,1}$ submanifold with boundary.

Lemma 7.20. *Suppose $y \in \mathrm{Sm}(S)$ and $v \in \mathrm{Tan}(S, y)$, and let P be the plane through y containing $y + n(y)$ and $y + v$. Suppose that γ parametrizes $P \cap \partial S$ by arclength on a neighborhood of $y = \gamma(0)$. Then $\mathrm{II}(y)(v, v) > 0$ only if $\gamma(t)$ is admissible at y whenever $|t|$ is sufficiently small.*

Proof. Suppose that $\mathrm{II}(y)(v, v) > 0$. By definition of II, we have $\langle n(y), \gamma''(0)\rangle > 0$. Thus, since $\langle n(y), \gamma'(0)\rangle = 0$, there exists $\delta > 0$ such that $\langle n(y), \dfrac{\gamma'(t)}{t}\rangle > 0$ whenever $|t| < \delta$. Since γ is $C^{1,1}$, by the Fundamental Theorem of Calculus, we have

$$\langle \gamma(\epsilon) - \gamma(0), n(y)\rangle = \int_{[0,\epsilon]} \langle \gamma'(t), n(y)\rangle dt > 0 \tag{7.28}$$

for all ϵ with $0 < \epsilon < \delta$. The result follows for $\gamma(-\epsilon)$ by an analogous argument. $\square$

Definition 7.21. *For $y \in \mathrm{Sm}(S)$, let $k(y) := \max\left\{0, \max_{|v|=1} \mathrm{II}(y)(v, v)\right\}$, and for x in S, let*

$$\rho(x) := \left(\limsup_{\substack{y \in \mathrm{Sm}(S) \\ y \to x}} k(y)\right)^{-1}. \tag{7.29}$$

Recall that a function g is *lower semicontinuous* if $g(x_0) \leq \lim_{i\to\infty} g(x_i)$ whenever $x_i \to x_0$ and $\lim_{i\to\infty} g(x_i)$ exists.

Lemma 7.22. *ρ is lower semicontinuous.*

Proof. If $x_i \to x_0$ and $\rho(x_i)$ converges, by definition $\rho(x_i) = \left(\limsup_{\substack{y\in \mathrm{Sm}(S)\\ y\to x_i}} k(y)\right)^{-1}$, so we can take $y_i \in \mathrm{Sm}(S)$ such that $|y_i - x_i| < 1/i$ and $\left|\rho(x_i) - \dfrac{1}{k(y_i)}\right| < 1/i$. Thus $y_i \to x_0$, and $\rho(x_0) = \liminf_{y\to x_0} \dfrac{1}{k(y)} \leq \lim_{i\to\infty} \dfrac{1}{k(y_i)} = \lim_{i\to\infty} \rho(x_i)$. □

Remark. A lower semicontinuous function $g : X \subset \mathbb{R}^m \to \mathbb{R}$ attains its infimum whenever X is closed.

Lemma 7.23. $\inf\limits_{\substack{y\in \mathrm{Sm}(S),\ |v|=1\\ \mathrm{II}(y)(v,v)>0}} \dfrac{1}{\mathrm{II}(y)(v,v)} = \inf\limits_{y\in \mathrm{Sm}(S)} \dfrac{1}{k(y)} = \min\limits_{x\in\partial S} \rho(x) \geq \mathrm{reach}(S)$

Proof. By Lemma 7.11, at a point $y \in \mathrm{Sm}(S)$, we have $|\vec{\kappa}| = \mathrm{II}(y)(v,v)$ for v a unit tangent vector to ∂S whenever $\vec{\kappa}$ points in the direction of $n(y)$. Thus $k(y) = \max\{0, \max|\vec{\kappa}|\}$ for $y \in \mathrm{Sm}(S)$, where the maximum is over all $\vec{\kappa}$ pointing in the direction of $n(y)$. Further, if $\vec{\kappa}(y)$ points out for some $P \cap \partial S$ containing y,

$$\frac{1}{k(y)} = \min \frac{1}{|\vec{\kappa}|} = \min_{\substack{|v|=1\\ \mathrm{II}(y)(v,v)>0}} \frac{1}{\mathrm{II}(y)(v,v)}, \tag{7.30}$$

where the minimum of $\dfrac{1}{|\vec{\kappa}|}$ is the minimum over all outward curvature vectors $\vec{\kappa}$ at y.

By definition of ρ, we have $\rho(x) = \liminf \dfrac{1}{k(y)}$, so that

$$\min_x \rho(x) = \inf_{\substack{y\in \mathrm{Sm}(S),\ |v|=1\\ \mathrm{II}(y)(v,v)>0}} \frac{1}{\mathrm{II}(y)(v,v)}. \tag{7.31}$$

Moreover, every outward curvature vector $\vec{\kappa}$ at $y \in \mathrm{Sm}(S)$ satisfies $\dfrac{1}{|\vec{\kappa}|} = \lim_{i\to\infty} r(y, z_i)$ for some sequence of (admissible) $z_i \to y$ by Lemma 7.7 and Lemma 7.20, so that $k(y)$ does as well, and

$$\min_{y\in \mathrm{Sm}(S)} \frac{1}{k(y)} \geq \inf r(x,z) = \mathrm{reach}(S). \tag{7.32}$$

$\square$

Definition 7.24. *We define the **penalized distance** between distinct points $x, y \in \partial S$ with y admissible at x by* $\operatorname{pd}(x,y) := \dfrac{|y-x|}{\cos^2(\psi(x,y))} = \dfrac{2r(x,y)}{\cos(\psi(x,y))}$. *We define* $\operatorname{pd}(x,y) = \infty$ *if $x = y$ or if y is not admissible at x.*

Definition 7.25. *We say that $(x,y) \in S \times S$ is a **critical pair** if x and y are distinct and $x - y \in \operatorname{Nor}(S,y)$ and $y - x \in \operatorname{Nor}(S,x)$.*

Lemma 7.26. *If* $\operatorname{reach}(S) < \infty$, $\operatorname{pd}(x,y) \geq 2\operatorname{reach}(S)$, *with equality only if (x,y) is a critical pair.*

Proof. It is clear that $\operatorname{pd}(x,y) \geq 2\operatorname{reach}(S)$, since $\operatorname{pd}(x,y) \geq 2r(x,y) \geq 2\operatorname{reach}(S)$. Also, if equality holds, we have $2r(x,y) \leq \operatorname{pd}(x,y) = 2\operatorname{reach}(S) \leq 2r(x,y)$, so that we must have $\psi(x,y) = 0$ and $y - x \in \operatorname{Nor}(S,x)$. If $x - y \notin \operatorname{Nor}(S,y)$, there is a tangential direction T in $\operatorname{Tan}(\partial S, y)$ such that $\langle x - y, T\rangle > 0$. Thus if we parametrize a planar section of ∂S at y by $\gamma(t)$, with $\gamma(0) = y$ and $\gamma'(0) = T$, we have that $\left.\dfrac{d}{dt}\right|_{t=0} |\gamma(t) - x| = \left\langle T, \dfrac{y-x}{|y-x|}\right\rangle < 0$, and

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{pd}(x,\gamma(t)) = \left\langle T, \frac{y-x}{|y-x|}\right\rangle < 0, \tag{7.33}$$

so that there exists y' with $\operatorname{pd}(x,y') < \operatorname{pd}(x,y) = 2\operatorname{reach}(S)$, which is a contradiction. $\square$

Summarizing our work above, we have shown that the reach of an m-dimensional $C^{1,1}$ submanifold with boundary satisfies the following inequality:

Corollary 7.27. *If $S \subset \mathbb{R}^m$ is an m-dimensional compact $C^{1,1}$ submanifold with boundary with* $\operatorname{reach}(S) < \infty$*, then*

$$\operatorname{reach}(S) \leq \min\left\{\min_{x,y\in S}\frac{\operatorname{pd}(x,y)}{2}, \min_{\partial S}\rho(x)\right\} \tag{7.34}$$

$$= \min\left\{\min_{x,y\in\partial S}\frac{\operatorname{pd}(x,y)}{2}, \inf_{\substack{x\in\operatorname{Sm}(S)\\ \mathrm{II}(x)(v,v)>0,\ |v|=1}}\frac{1}{\mathrm{II}(x)(v,v)}\right\}. \tag{7.35}$$

We shall later show that equality holds in the theorem above. We already know that the reach of S is the infimum over all $r(x,y)$. To express the reach in terms of $\operatorname{pd}(x,y)$ and $\rho(x)$, we shall use the following lemma:

Lemma 7.28. *If $r(x,y) = \operatorname{reach}(S)$ and $y - x \notin \operatorname{Nor}(S,x)$, then $\rho(x) = \operatorname{reach}(S)$.*

Proof. If not, since $\rho(x) \geq \operatorname{reach}(S)$, we have $\rho(x) > \operatorname{reach}(S)$. Also, recall that ρ is lower semicontinuous by Lemma 7.22. Keeping y fixed, by continuity of r where it is finite and lower semicontinuity of ρ, there is a neighborhood of x such that y is an admissible direction at x' and $r(x',y) < \rho(x')$ for all x'. But then, noting that $\operatorname{Sm}(S)$ has full measure, at every point x' of $\operatorname{Sm}(S)$ for which $r(x',y) < \rho(x')$, we have $r(x',y) < \dfrac{1}{|\vec{\kappa}|}$, where $\vec{\kappa}$ is the curvature vector at x' of the plane section through x' and y and containing $n(x')$. Thus by Lemma 7.9, we have that $\dfrac{dr}{dT}(x',y) < 0$. However, this contradicts the assumption that r is minimized (over the admissible set) at (x,y): by assumption S is $C^{1,1}$, so that $r(\cdot,y)$ is locally Lipschitz and we can recover r locally by integrating its derivative. □

Now, when determining the reach of a set, one encounters points where the boundary has large outward curvature, and one also encounters critical pairs; it turns out that these two behaviors completely determine the reach of a set. Our pd function identifies critical pairs, and the function ρ identifies the points with great curvature. To prove that our reach formula holds, we shall use the following inequality to relate curvature to $r(x,y)$.

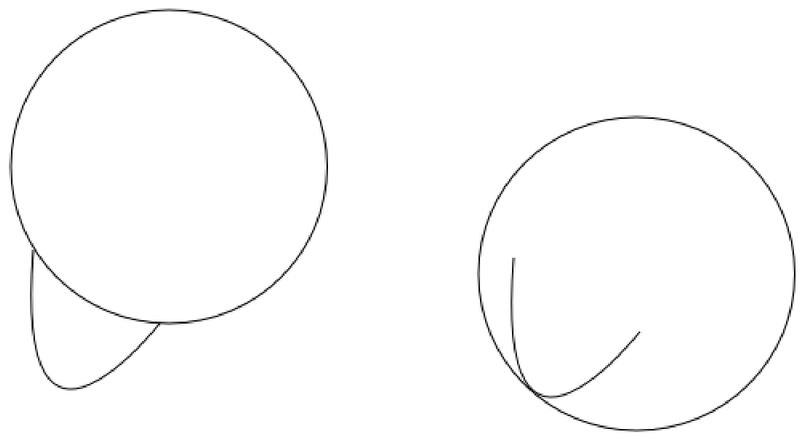

Figure 7.5: The circles C and $\tilde{C}$ in the proof of Lemma 7.29

Lemma 7.29. *Let $\gamma : [0, L] \to \partial S$ be an arclength parametrized curve such that $\gamma \subset P \cap \partial S$, where P is a plane. Suppose that $\gamma(0) = x$ and that $\gamma(L) = y$, and suppose that $C \subset P$ is a circle of radius R tangent to γ at x and passing through y. Let $N(s)$ denote the unit normal to $\gamma(s)$ at x, chosen so that $\langle N(0), n(\gamma(0)) \rangle > 0$. If $\mathrm{length}(\gamma) < R$, we have*

$$\sup_{\gamma} \langle N(s), \gamma''(s) \rangle \geq \frac{1}{R}. \tag{7.36}$$

Proof. Assume without loss of generality that γ is locally a graph over the x-axis and that C is centered at the origin. Either there exists a sub-curve $\overline{\gamma}$ from x to z which is disjoint from the interior of C, or there is a sequence $z_j \to 0$ with $\gamma(z_j)$ in the interior of C for all j.

In the first case, let $f(a, b) = b - \sqrt{R^2 - a^2}$, and consider the composition $f \circ \overline{\gamma}$. The level sets of f are circles of radius R, and $f^{-1}(0) = C$. The composition is well-defined because γ is a graph over the x-axis, and x and y lie on C. Moreover, the minimum of f on $\overline{\gamma}$ is attained at an interior point of $\overline{\gamma}$, since its endpoints lie on $f(a, b) = 0$ and the rest of $\overline{\gamma}$ lies outside $f^{-1}[0,\infty)$. Suppose that $f \circ \overline{\gamma}$ has a minimum at t_0. Then $\gamma(t_0)$ lies on a circle $\tilde{C}$ of radius R, and all sufficiently close points of γ lie inside the circle. (See Figure 7.5.)

Translate $\tilde{C}$ to the origin, and consider the function $g(t) = |\gamma(t)|^2$. Then g has a maximum at t_0, and we have

$$0 = g'(t_0) = 2\langle \gamma(t_0), \gamma'(t_0)\rangle.$$

Since g' is Lipschitz, by the Fundamental Theorem of Calculus, we have

$$g'(t) - g'(t_0) = \int_{[t_0,t]} g''(u)du = 2\int_{[t_0,t]} (\langle \gamma(u), \gamma''(u)\rangle + 1)du \tag{7.37}$$

and for all sufficiently small s near t_0, using the fact that $g'(t_0) = 0$, we have

$$\begin{aligned} 0 &> |\gamma(s)|^2 - R^2 \\ &= g(s) - g(t_0) \\ &= \int_{[t_0,s]} g'(v)dv \\ &= \int_{[t_0,s]} \int_{[t_0,v]} g''(u)dudv \\ &= 2\int_{[t_0,s]} \int_{[t_0,v]} (\langle \gamma(u), \gamma''(u)\rangle + 1)dudv. \end{aligned} \tag{7.38}$$

Thus there exists a sequence $u_j \to t_0$ such that $\gamma''(u_j)$ exists and $\langle \gamma(u_j), \gamma''(u_j)\rangle < -1$. Since $\gamma(t_0) = -RN(t_0)$, we have

$$\lim_{j\to\infty} \langle -RN(u_j), \gamma''(u_j)\rangle = \lim_{j\to\infty} \langle \gamma(u_j), \gamma''(u_j)\rangle \leq -1, . \tag{7.39}$$

so that

$$\sup_s \langle N(s), \gamma''(s)\rangle \geq 1/R. \tag{7.40}$$

In the second case, again consider the function $g(t) = |\gamma(t)|^2$. Since γ is tangent to C at $\gamma(0) = x$, we have $0 = g'(0) = 2\langle\gamma(0), \gamma'(0)\rangle$. Using (7.38), since $g(z_j) - g(0) < 0$, there exists $v_j \in [0, z_j]$ with $g'(v_j) < 0$. Thus, using (7.37) and the fact that $g'(0) = 0$, there exists $u_j \in [0, v_j]$ such that g is twice differentiable at u_j and $g''(u_j) < 0$. Since γ is parametrized by arclength, we have $0 > g''(u_j) = 2(\langle\gamma(u_j), \gamma''(u_j)\rangle + 1)$. Further, since $\gamma(0) = -RN(0)$, we have

$$\lim_{j\to\infty} \langle -RN(u_j), \gamma''(u_j)\rangle = \lim_{j\to\infty} \langle\gamma(u_j), \gamma''(u_j)\rangle \le -1. \tag{7.41}$$

Thus $\sup_s \langle N(s), \gamma''(s)\rangle \ge 1/R$. □

7.2.4 The Reach Formula in the $C^{1,1}$ Case

Now we are ready to establish the formula for reach of a compact $C^{1,1}$ m-dimensional submanifold with boundary of $\mathbb{R}^m$. The overall approach is the same as that of [3]. However, since we are working in the setting of manifolds with boundary, we have to be more careful in one of the cases in the proof. For that, we apply the results of Subsection 7.2.2

Theorem 7.30. *Let $S \subset \mathbb{R}^m$ be a compact m-dimensional $C^{1,1}$ submanifold with boundary satisfying* $\operatorname{reach}(S) < \infty$. *Then*

$$\begin{aligned} \operatorname{reach}(S) &= \min\left\{\frac{1}{2}\min_{x,y\in S} \operatorname{pd}(x,y), \min_S \rho\right\} \qquad (7.42) \\ &= \min\left\{\frac{1}{2}\min_{x,y\in S} \operatorname{pd}(x,y), \inf_{\substack{x\in \operatorname{Sm}(S) \\ \mathrm{II}(x)(v,v)>0,\ |v|=1}} \frac{1}{\mathrm{II}(x)(v,v)}\right\}. \end{aligned}$$

Proof. We have already shown that

$$\begin{aligned} \operatorname{reach}(S) &\leq \min\left\{\frac{1}{2}\min_{x,y\in S}\operatorname{pd}(x,y), \min_S \rho\right\} \\ &= \min\left\{\frac{1}{2}\min_{x,y\in S}\operatorname{pd}(x,y), \inf_{\substack{x\in \operatorname{Sm}(S)\\ \mathrm{II}(x)(v,v)>0,\ |v|=1}} \frac{1}{\mathrm{II}(x)(v,v)}\right\}. \end{aligned} \tag{7.43}$$

Also, we know that $\operatorname{reach}(S)$ is the infimum over all $r(x,y)$ for y admissible at x. Suppose that $r(x_i,y_i)\to\operatorname{reach}(S)$. By passing to a subsequence, we can assume that $x_i\to x_0$ and $y_i\to y_0$. Then one of the following is true:

- $x_0\neq y_0$ and $y_0\in\operatorname{Nor}(S,x_0)$
- $x_0\neq y_0$ and $y_0\notin\operatorname{Nor}(S,x_0)$
- $x_0=y_0$.

In the first case, by continuity of r where r is finite, we know that $r(x_0,y_0)=\operatorname{reach}(S)$; also, since $\psi(x_0,y_0)=0$, we have that $\operatorname{pd}(x_0,y_0)=2r(x_0,y_0)=2\operatorname{reach}(S)$. In the second case, we also have $\dfrac{y_i-x_i}{|y_i-x_i|}\to\dfrac{y_0-x_0}{|y_0-x_0|}$. Since $\operatorname{reach}(S)$ is finite by assumption, we also have $r(x_i,y_i)\to r(x_0,y_0)$, so that $r(x_0,y_0)=\operatorname{reach}(S)$. Lemma 7.28 implies that $\rho(x_0)=\operatorname{reach}(S)$.

In the third case, we have $x_i\to x_0$ and $y_i\to x_0$. Apply Lemma 7.13 at $p=x_0$. Using the fact that r is locally Lipschitz where it is finite, we may replace x_i and y_i with x_i' and y_i' such that

$$\begin{aligned} &|x_i-x_i'|\to 0, \\ &|y_i-y_i'|\to 0, \\ &|r(x_i,y_i)-r(x_i',y_i')|\to 0, \end{aligned} \tag{7.44}$$

and there exists an arclength parametrized $C^{1,1}$ planar curve $\gamma_i : [0, L] \to \partial S$ satisfying

- γ_i joins x_i' to y_i'
- II is defined almost everywhere along γ_i, and
- $\text{length}(\gamma_i) \to 0$.

Suppose that the plane P_i contains γ_i, and let C_i be the circle of radius R_i in P_i tangent to γ at x_i' and passing through y_i'. By construction of γ_i as in Lemma 7.13, the plane P_i contains $x_i' + n(x_0)$. By definition of $r(x_i', y_i')$, there is a ball B_i of radius $r(x_i', y_i')$ tangent to ∂S at x_i' and passing through y_i'. We have $C_i \subset \partial B_i$; since $n(x_i') \to n(x_0)$, we have $R_i - r(x_i', y_i') \to 0$.

Let $N_i(t)$ be the projection of $n(\gamma_i(t))$ onto P_i. By construction, we have $\langle N_i(0), n(x_i')\rangle > 0$. By Lemma 7.29, for all sufficiently large i, we have $\sup_s \langle N_i(s), \gamma_i''(s)\rangle \geq 1/R_i$. In particular, note that $\langle N_i(s), \gamma_i''(s)\rangle > 0$ implies that $\langle n(\gamma(s)), \gamma_i''(s)\rangle > 0$, since N_i is the projection of n onto P_i. Applying Lemma 7.13, we thus have

$$\lim_{i\to\infty} \left(\sup_{\gamma(s)\in \text{Sm}(S)} \text{II}(\gamma_i(s))(\gamma_i'(s), \gamma_i'(s)) \right) = \lim_{i\to\infty} \left(\sup_{\gamma_i} \langle N_i(s), \gamma_i''(s)\rangle \right) \geq \frac{1}{\text{reach}(S)}, \tag{7.45}$$

so that

$$\text{reach}(S) \geq \inf_{\substack{y\in\text{Sm}(S),\ |v|=1 \\ \text{II}(y)(v,v)>0}} \frac{1}{\text{II}(y)(v,v)}. \tag{7.46}$$

□

7.3 The Reach Formula for Regular Compact PR Sets

Let S be a regular compact PR set. In this section we establish a formula for reach(S) analogous to the one in Lemma 7.30. For this we shall use the second fundamental form for PR sets from Section 1.2.

Recall that we say that y is admissible at $(x, v) \in \mathrm{nor}(S)$ if $\left\langle v, \frac{y-x}{|y-x|} \right\rangle > 0$. We define the penalized distance for PR sets in the same way as for $C^{1,1}$ m-submanifolds with boundary:

Definition 7.31. *We define the **penalized distance** between distinct points $x, y \in \partial S$ with y admissible at (x, v) by $\mathrm{pd}(x, y; v) := \dfrac{|y-x|}{\left\langle v, \frac{y-x}{|y-x|} \right\rangle^2}$. Otherwise, set $\mathrm{pd}(x, y; v) = \infty$.*

It turns out that the reach formula for compact regular PR sets is nearly the same as the one for $C^{1,1}$ m-submanifolds with boundary:

Theorem 7.32. *The reach of a compact regular PR set $S \subset \mathbb{R}^m$ whose reach is finite satisfies*

$$\mathrm{reach}(S) = \min\left\{ \frac{1}{2} \min_{\substack{(x,y)\in S\times S \\ (x,v)\in \mathrm{nor}(S)}} \mathrm{pd}(x, y; v), \inf_{\substack{(x,v)\in \mathrm{Sm}(\mathrm{nor}(S)) \\ \mathrm{II}(x,v)(w,w)>0,\ |w|=1}} \frac{1}{\mathrm{II}(x, v)(w, w)} \right\}.$$

Proof. Recall that $S_\epsilon = \{x : d_S(x) \le \epsilon\}$, and note that $\mathrm{reach}(S_\epsilon) = \mathrm{reach}(S) - \epsilon$. Let pd_ϵ denote the penalized distance on S_ϵ, and let II_ϵ denote the second fundamental form on S_ϵ. We prove the theorem by showing that $\mathrm{reach}(S)$ is at most the right-hand quantity and at least the right-hand quantity.

Claim. *We have the inequality*

$$\mathrm{reach}(S) \le \min\left\{ \frac{1}{2} \min_{\substack{(x,y)\in S\times S \\ (x,v)\in \mathrm{nor}(S)}} \mathrm{pd}(x, y; v), \inf_{\substack{(x,v)\in \mathrm{Sm}(\mathrm{nor}(S)) \\ \mathrm{II}(x,v)(w,w)>0,\ |w|=1}} \frac{1}{\mathrm{II}(x, v)(w, w)} \right\}. \tag{7.47}$$

Sub-claim 1:

$$\mathrm{reach}(S) \le \inf_{\substack{(x,v)\in \mathrm{Sm}(\mathrm{nor}(S)) \\ \mathrm{II}(x,v)(w,w)>0,\ |w|=1}} \frac{1}{\mathrm{II}(x, v)(w, w)}. \tag{7.48}$$

There is a sequence of points $(\tau_j, \sigma_j) \in \mathrm{Tan}(\mathrm{nor}(S), (p, v))$ such that

$$\lim_{j\to\infty} \frac{1}{-\tau_j \cdot \sigma_j} = \inf_{\substack{(x,v)\in \mathrm{Sm}(\mathrm{nor}(S)) \\ \mathrm{II}(x,v)(w,w)>0,\ |w|=1}} \frac{1}{\mathrm{II}(x, v)(w, w)}. \tag{7.49}$$

By Lemma 1.15, for all j, we have that $\tau_j + \epsilon\sigma_j \in \mathrm{Tan}(S_\epsilon, x_0)$, where $x_0 = p + \epsilon v$. For fixed j, we have

$$\frac{1}{-(\tau_j + \epsilon\sigma_j)\cdot \sigma_j} \geq \inf_{\substack{y\in \mathrm{Sm}(S_\epsilon),\ |v|=1 \\ \mathrm{II}_\epsilon(y)(v,v)>0}} \frac{1}{\mathrm{II}_\epsilon(y)(v, v)} \geq \mathrm{reach}(S_\epsilon) \tag{7.50}$$

for all sufficiently small $\epsilon > 0$. Letting $\epsilon \to 0$, we have

$$\mathrm{reach}(S) \leq \frac{1}{-\tau_j \cdot \sigma_j}. \tag{7.51}$$

Letting $j \to \infty$, we have

$$\mathrm{reach}(S) \leq \inf_{\substack{(x,v)\in \mathrm{Sm}(\mathrm{nor}(S)) \\ \mathrm{II}(x,v)(w,w)>0,\ |w|=1}} \frac{1}{\mathrm{II}(x, v)(w, w)}. \tag{7.52}$$

Sub-claim 2:

$$\mathrm{reach}(S) \leq \min \frac{1}{2} \min_{\substack{(x,y)\in S\times S \\ (x,v)\in \mathrm{nor}(S)}} \mathrm{pd}(x, y; v). \tag{7.53}$$

Suppose that $\mathrm{reach}(S) > \frac{1}{2}\mathrm{pd}(x, y; v)$ for some triple $(x, y; v)$. Then there exists $\alpha > 0$ such that $\mathrm{reach}(S) - \alpha > \frac{1}{2}\mathrm{pd}(x, y; v)$. Consider $x(\epsilon) := x + \epsilon v,\ y(\epsilon) := y + \epsilon v' \in S_\epsilon$, where $v' \in \mathrm{nor}(S, y)$ is fixed and $\epsilon < \frac{\alpha}{2}$. Since $v \in \mathrm{nor}(S_\epsilon, x(\epsilon))$ (in fact $v = n(x(\epsilon))$),

$$\frac{1}{2}\mathrm{pd}_\epsilon(x(\epsilon), y(\epsilon)) = \frac{1}{2} \frac{|x(\epsilon) - y(\epsilon)|}{\left\langle v, \dfrac{y(\epsilon) - x(\epsilon)}{|y(\epsilon) - x(\epsilon)|} \right\rangle^2} \to \frac{1}{2}\mathrm{pd}(x, y; v) < \mathrm{reach}(S) - \alpha, \tag{7.54}$$

so that for sufficiently small ϵ, $\frac{1}{2}\mathrm{pd}_\epsilon(x(\epsilon), y(\epsilon)) < \mathrm{reach}(S) - \frac{\alpha}{2}$. Thus, for all sufficiently small ϵ, we have

$$\mathrm{reach}(S_\epsilon) = \mathrm{reach}(S) - \epsilon > \mathrm{reach}(S) - \frac{\alpha}{2} > \frac{1}{2}\mathrm{pd}_\epsilon(x(\epsilon), y(\epsilon)). \tag{7.55}$$

This contradicts the reach formula for S_ϵ.

Thus

$$\mathrm{reach}(S) \leq \min\left\{\frac{1}{2}\min_{\substack{(x,y)\in S\times S\\(x,v)\in\mathrm{nor}(S)}} pd(x,y;v), \inf_{\substack{(x,v)\in\mathrm{Sm}(\mathrm{nor}(S))\\ \mathrm{II}(x,v)(w,w)>0,\ |w|=1}} \frac{1}{\mathrm{II}(x,v)(w,w)}\right\}. \tag{7.56}$$

Using the formula for $\mathrm{reach}(S_\epsilon)$,

$$\mathrm{reach}(S) = \lim_{\epsilon\to 0}\left\{\min\left\{\frac{1}{2}\min_{x,y\in S}\mathrm{pd}_\epsilon(x,y), \inf_{\substack{x\in\mathrm{Sm}(S_\epsilon)\\ \mathrm{II}_\epsilon(x)(v,v)>0,\ |v|=1}} \frac{1}{\mathrm{II}_\epsilon(x)(v,v)}\right\}\right\}. \tag{7.57}$$

Let $\{\epsilon_i\}$ be a sequence with $\epsilon_i > 0$ for all i, and write $S_i := S_{\epsilon_i}$, $\mathrm{II}_i := \mathrm{II}_{\epsilon_i}$, and $\mathrm{pd}_i := \mathrm{pd}_{\epsilon_i}$. There exists a sequence $\{\epsilon_i\}$ such that $\epsilon_i \to 0$ and either $\mathrm{reach}(S_i) = \frac{1}{2}\mathrm{pd}_i(x_i, y_i)$ for all i, or $\mathrm{reach}(S_i) = \inf_{\substack{x\in\mathrm{Sm}(S_i)\\ \mathrm{II}_i(x)(v,v)>0,\ |v|=1}} \frac{1}{\mathrm{II}_i(x)(v,v)}$ for all i.

In the first case, since $\mathrm{reach}(S_i) = \frac{1}{2}\mathrm{pd}_i(x_i, y_i)$, the pair (x_i, y_i) is a critical pair for S_i, and $\frac{1}{2}\mathrm{pd}_i(x_i, y_i) = \dfrac{r(x_i,y_i)}{\left\langle n(x_i), \frac{y_i - x_i}{|y_i - x_i|}\right\rangle} = r(x_i, y_i) = |x_i - y_i|$. By passing to a subsequence, we can assume, using compactness of some fixed S_i, that $x_i \to x_0$ and $y_i \to y_0$, where $x_0, y_0 \in S$. Since S is a PR set and $|x_i - y_i| \to \mathrm{reach}(S)$, we have $|x_0 - y_0| = \mathrm{reach}(S) \neq 0$. Thus $\frac{y_i - x_i}{|y_i - x_i|} \to \frac{y_0 - x_0}{|y_0 - x_0|}$ as well. Now, $n(x_i) \in \mathrm{nor}(S, x_i - \epsilon_i n(x_i))$, and $x_i - \epsilon_i n(x_i) \to x_0$, so by upper semicontinuity of the multifunction nor, we may pass to a subsequence so that the $n(x_i)$ converge, and then $n(x_i) \to v \in \mathrm{nor}(S, x_0)$, and $\left\langle v, \frac{y_0 - x_0}{|y_0 - x_0|}\right\rangle = \lim_{i\to\infty}\left\langle n(x_i), \frac{y_i - x_i}{|y_i - x_i|}\right\rangle = 0$. Thus $\mathrm{pd}(x_0, y_0; v) = |x_0 - y_0| = \mathrm{reach}(S)$.

In the second case, $\mathrm{reach}(S_i) = \inf_{\substack{x \in \mathrm{Sm}(S_i) \\ \mathrm{II}_i(x)(v,v)>0,\ |v|=1}} \dfrac{1}{\mathrm{II}_i(x)(v,v)}$, so that by Lemma 1.15 there exist τ_i and σ_i satisfying $(\tau_i, \sigma_i) \in \mathrm{Tan}(\mathrm{nor}(S), (p_i, v_i))$ and

$$\mathrm{reach}(S_i) < \frac{1}{-(\tau_i + \epsilon_i \sigma_i) \cdot \sigma_i} < \mathrm{reach}(S_i) + \epsilon_i. \tag{7.58}$$

Since $\mathrm{reach}(S_i) > 0$, we have $-(\tau_i + \epsilon_i \sigma_i) \cdot \sigma_i > 0$ and

$$\begin{aligned} -\tau_i \cdot \sigma_i &\geq -\tau_i \cdot \sigma_i - \epsilon_i |\sigma_i|^2 \qquad (7.59) \\ &= -(\tau_i + \epsilon_i \sigma_i) \cdot \sigma_i > 0. \end{aligned}$$

Thus

$$\begin{aligned} \inf_{\substack{(x,v) \in \mathrm{Sm}(\mathrm{nor}(S)) \\ \mathrm{II}(x,v)(w,w)>0,\ |w|=1}} \frac{1}{\mathrm{II}(x,v)(w,w)} &\leq \frac{1}{-\tau_i \cdot \sigma_i} \qquad (7.60) \\ &< \frac{1}{-(\tau_i + \epsilon_i \sigma_i) \cdot \sigma_i} \qquad (7.61) \\ &< \mathrm{reach}(S_i) + \epsilon_i. \qquad (7.62) \end{aligned}$$

Letting $i \to \infty$, we have

$$\inf_{\substack{(x,v) \in \mathrm{Sm}(\mathrm{nor}(S)) \\ \mathrm{II}(x,v)(w,w)>0,\ |w|=1}} \frac{1}{\mathrm{II}(x,v)(w,w)} \leq \mathrm{reach}(S), \tag{7.63}$$

and since the reverse inequality also holds, we conclude that

$$\inf_{\substack{(x,v) \in \mathrm{Sm}(\mathrm{nor}(S)) \\ \mathrm{II}(x,v)(w,w)>0,\ |w|=1}} \frac{1}{\mathrm{II}(x,v)(w,w)} = \mathrm{reach}(S). \tag{7.64}$$

Thus

$$\operatorname{reach}(S) = \min\left\{ \frac{1}{2} \min_{\substack{(x,y)\in S\times S\\ (x,v)\in \operatorname{nor}(S)}} \operatorname{pd}(x,y;v)), \inf_{\substack{(x,v)\in \operatorname{Sm}(\operatorname{nor}(S))\\ \mathrm{II}(x,v)(w,w)>0,\ |w|=1}} \frac{1}{\mathrm{II}(x,v)(w,w)} \right\}. \tag{7.65}$$

□

Definition 7.33. *We say that $(x, y) \in S \times S$ is a* ***critical pair*** *if x and y are distinct and $x - y \in \operatorname{Nor}(S, y)$ and $y - x \in \operatorname{Nor}(S, x)$.*

As in the $C^{1,1}$ case, pd can only attain the reach at critical pairs.

Corollary 7.34. *If* $\operatorname{reach}(S) < \infty$*, then* $\operatorname{pd}(x, y; v) \geq 2\operatorname{reach}(S)$*, with equality only if (x, y) is a critical pair and* $v = \dfrac{y - x}{|y - x|}$*.*

Proof. The first assertion is clear from Theorem 7.32. If equality holds, then we have that $2r(x, y; v) \leq \operatorname{pd}(x, y; v) = 2\operatorname{reach}(S) \leq 2r(x, y; v)$. Then we must have equality throughout, so that $\left\langle v, \dfrac{y-x}{|y-x|} \right\rangle = 1$ and $v = \dfrac{y-x}{|y-x|} \in \operatorname{nor}(S, x)$. If $x - y \notin \operatorname{nor}(S, y)$, then there exists a tangential direction T to S at y such that $\langle x - y, T\rangle > 0$.

Using Lemma 2.30, let P be a plane through y such that $y + T \in P$ and $P \cap \partial S$ is a curve near y. There exists a sequence of points $y_j \in \partial S$ such that $y_j \to y$ and $\dfrac{y_j - y}{|y_j - y|} \to T$. Moreover,

$$\lim_{j\to\infty} \frac{|y_j - x| - |y - x|}{|y_j - y|} = \left\langle T, \frac{y - x}{|y - x|} \right\rangle \tag{7.66}$$

and, using the fact that $v = \dfrac{y-x}{|y-x|}$,

$$\lim_{j\to\infty} \frac{\operatorname{pd}(x, y_j; v) - \operatorname{pd}(x, y; v)}{|y_j - y|} = \left\langle T, \frac{y - x}{|y - x|} \right\rangle < 0. \tag{7.67}$$

Thus there is y' such that $\operatorname{pd}(x, y'; v) < \operatorname{pd}(x, y; v) = 2\operatorname{reach}(S)$, which is a contradiction. □

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Appendix A

Functions of Essentially Bounded Variation

In this section, we cover the foundations of the theory of functions of essentially bounded variation that we used in the proof Theorem 6.20. Throughout, assume that $f : [0, L] \to \mathbb{R}^m$.

Definition A.1. *Let $\mathcal{P} = \{t_0 \leq \ldots \leq t_j\}$ be a partition of $[0, L]$. We define the* ***variation of f on the partition*** *$\mathcal{P}$ by $\mathrm{Var}(f, \mathcal{P}) = \sum_{i=1}^{j} |f(t_i) - f(t_{i-1})|$. We define the* ***total variation of f on*** *$[0, L]$, $\mathrm{Var}(f)$, as the supremum, over all partitions $\mathcal{P}$, of $\mathrm{Var}(f, \mathcal{P})$. We say that a function f has* ***bounded variation*** *($f \in \mathrm{BV}$) if $\mathrm{Var}(f)$ is finite.*

Definition A.2. *An integrable function f is of* ***essentially bounded variation*** *($f \in \mathrm{eBV}$) if there exists a constant C such that $\mathrm{Var}(f, \mathcal{P}) < C$ whenever $\mathcal{P}$ is a partition of $[0, L]$ by Lebesgue points of f.*

In the case that $f \in \mathrm{eBV}$, we shall write

$$\mathrm{eVar}(f) := \sup\{\mathrm{Var}(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } [0, L] \text{ by Lebesgue points.}\} \tag{A.1}$$

The following is a well-known result about the total variation of smooth functions. We shall prove an analogous result for eBV functions.

Lemma A.3. *For a smooth function $f : [0, L] \to \mathbb{R}$,*

$$\mathrm{Var}(f) = \int |f'| = \sup \left\{ \int f\phi' : \phi \in C_0^1([0, L]),\ \|\phi\|_\infty \leq 1 \right\}. \tag{A.2}$$

Definition A.4. *Let m denote Lebesgue measure. A point x_0 is a* ***Lebesgue point*** *of the function f if*

$$f(x_0) = \lim_{\epsilon \downarrow 0} \frac{1}{m(B(x_0, \epsilon))} \int_{B(x_0,\epsilon)} f(x)dm. \tag{A.3}$$

When f is integrable, almost every point is a Lebesgue point. An integrable function $f \in \mathrm{eBV}$ is *almost* BV, in the following sense:

Lemma A.5. *If $f \in \mathrm{eBV}$ is integrable, then there exists a function $\tilde{f} \in \mathrm{BV}$ such that $f = \tilde{f}$ on the set E of Lebesgue points of f. Further, $f|_E$ has right and left limits everywhere (in E).*

Proof. First, consider the restriction of f to the set E of Lebesgue points. If $0 \notin E$, modify f at 0 as follows: set $f(0) = \liminf_{x \in E} f(x)$. Let $r_\delta = \inf_{0<|x|<\delta} f(x)$, so that $f(0) = \lim_{\delta \to 0} r_\delta$. Then for any $\epsilon > 0$ there exists $\delta_0 > 0$ such that $0 < \delta < \delta_0$ implies that $|r_\delta - f(0)| < \frac{\epsilon}{2}$. By definition of r_δ there exists x_δ such that $x_\delta \in E$ and $|x_\delta| < \delta$ with $f(x_\delta) < r_\delta + \frac{\epsilon}{2}$. Thus $|f(x_\delta) - f(0)| \leq \epsilon$.

With f modified as above, extend f to equal $f(0)$ on $[-1, 0)$, so that f is defined on $[-1, 0] \cup E$, and every point in $[-1, 0)$ is a Lebesgue point. Temporarily denote this extension by F. Let $\epsilon > 0$, and fix a partition $\mathcal{P} = t_0 < t_1 < \ldots < t_k$ by Lebesgue points. Suppose that $t_i < 0$ and $t_{i+1} > 0$, and suppose that $t_i \leq t_\delta < t_{i+1}$ with $|f(0) - f(t_\delta)| < \epsilon$. Then, since

t_δ is a Lebesgue point and $F(t_i) = f(0)$, we have

$$\begin{aligned}
\mathrm{eVar}(F, \mathcal{P}) &= \sum_{j=1}^{k} |F(t_j) - F(t_{j-1})| \\
&= \sum_{j=i+1}^{k} |f(t_j) - f(t_{j-1})| \\
&\leq \left(\sum_{j=i+2}^{k} |f(t_j) - f(t_{j-1})| \right) + |f(t_{i+1}) - f(t_\delta)| + |f(t_\delta) - f(t_i)| \qquad \text{(A.4)} \\
&= \left(\sum_{j=i+2}^{k} |f(t_j) - f(t_{j-1})| \right) + |f(t_{i+1}) - f(t_\delta)| + |f(t_\delta) - f(0)| \\
&\leq \mathrm{eVar}(f) + \epsilon
\end{aligned}$$

Thus $F \in \mathrm{eBV}$ on$[-1, L]$. From now on, replace E and f with $[-1, 0) \cup E$ and F.

The function $f\big|_E$ is bounded variation (on E) by construction of E. Given a set $t_0, \ldots, t_k$ of points of E with $t_i < t_{i+1}$ for all i, set

$$p = \sum_{i=1}^{k} [f(t_i) - f(t_{i-1})]^+ \tag{A.5}$$

$$n = \sum_{i=1}^{k} [f(t_i) - f(t_{i-1})]^- \tag{A.6}$$

$$t = \sum_{i=1}^{k} |f(t_i) - f(t_{i-1})|, \tag{A.7}$$

where $r^+ = \max\{r, 0\}$ and $r^- = \max\{-r, 0\}$. Then we have

$$p - n = f(t_k) - f(t_0). \tag{A.8}$$

Also, set $P_a^b := \sup p$, $N_a^b := \sup n$, and $T_a^b := \sup t = \mathrm{eVar}(f_{[a,b]})$, where each sup is understood to be over all partitions of $[a, b]$ by Lebesgue points E.

Claim. *If $f \in \mathrm{BV}$ on E, and a, b in E satisfy $a \leq b$, then*

$$f(b) - f(a) = P_a^b - N_a^b. \tag{A.9}$$

Proof. Note that if a partition does not include a and b, including a and b does not decrease p, n, or t over the partition, so we may as well define P_a^b, N_a^b, and T_a^b to be suprema over all partitions including a and b. For any collection $a = t_0 < \ldots < t_k = b$ of points of E, by (A.8) we have

$$p = n + f(b) - f(a) \leq N_a^b + f(b) - f(a) \tag{A.10}$$

so that

$$P_a^b \leq N_a^b + f(b) - f(a). \tag{A.11}$$

Similarly,

$$n = p + f(a) - f(b) \leq P_a^b + f(a) - f(b) \tag{A.12}$$

so that $N_a^b \leq P_a^b + f(a) - f(b)$, and combining these inequalities,

$$P_a^b - N_a^b = f(b) - f(a). \tag{A.13}$$

□

Fix a in $[-1, 0)$; for any $x \in E$ with $x > a$, we have

$$f(x) = P_a^x - (N_a^x - f(a)). \tag{A.14}$$

Claim. *For $x \in E$, let $g(x) := P_a^x$ and $h(x) := N_a^x$. Then g and h are increasing and bounded.*

Proof. By construction, $0 \leq P_a^x \leq T_a^x \leq \mathrm{eVar}(f)$ and $0 \leq N_a^x \leq T_a^x \leq \mathrm{eVar}(f)$ so that g and h are bounded.

Also, g is increasing: suppose $a \leq x_1 \leq x_2$ and $x_1, x_2 \in E$. Then

$$P_a^{x_1} + P_{x_1}^{x_2} = P_a^{x_2}. \tag{A.15}$$

Since $P_{x_1}^{x_2} \geq 0$, we conclude that $P_a^{x_1} \leq P_a^{x_2}$, so that $g(x_1) \leq g(x_2)$. An analogous argument shows that h is increasing on E. $\square$

Claim. *We may extend g and h to increasing functions on $[a, L]$.*

Proof of Claim. For $y \notin E$, define

$$\begin{gathered} g(y) := \sup_{x \in E,\ x \leq y} g(x) \\ \text{and} \\ h(y) := \sup_{x \in E,\ x \leq y} h(x). \end{gathered} \tag{A.16}$$

Then g is increasing: suppose $x_1 \leq x_2$.

- If $x_1, x_2 \in E$, then the assertion is clear.
- If $x_1 \in E$ and $x_2 \notin E$, then by definition of $g(x_2)$, we have $g(x_1) \leq g(x_2)$.
- If $x_1 \notin E$ and $x_2 \in E$, then $g(x_1) = \sup_{x \in E, x \leq x_1} g(x)$, and $g(x)$ is increasing on E, so $g(x_1) \leq g(x_2)$.

- If $x_1 \notin E$ and $x_2 \notin E$, we have

$$\sup\{g(x) : x \in E, x \leq x_1\} \leq \sup\{g(x) : x \in E, x \leq x_2\}, \tag{A.17}$$

  so that $g(x_1) \leq g(x_2)$.

The same argument shows that h is increasing. $\square$

Now, define $\tilde{f}$ on $[0, L]$ by

$$\tilde{f}(x) = g(x) - (h(x) - f(a)). \tag{A.18}$$

By (A.14), we have that $\tilde{f} = f$ on E. Since $\tilde{f}$ is a difference of bounded increasing functions, by [12, Section 2 of Chapter 5], we also have $\tilde{f} \in \mathrm{BV}$. Moreover, as a difference of bounded increasing functions, $\tilde{f}$ has left and right limits everywhere; thus f has left and right limits at points of E. $\square$

Definition A.6. *Let $C_0^1[0, L]$ denote the set of all continuously differentiable functions $\phi : [0, L] \to \mathbb{R}^m$ with $\mathrm{support}(f) \subset [a, b] \subset (0, L)$.*

Lemma A.7. *An integrable function $f \in \mathrm{eBV}$ if and only if there exists C such that for all $\phi \in C_0^1([0, L])$, $\int f\phi' \leq C\|\phi\|_\infty$. Moreover,*

$$\mathrm{eVar}(f) = \sup_{\phi \in C_0^1([0,L]), \|\phi\|_\infty \leq 1} \left\{ \int f\phi' \right\}. \tag{A.19}$$

Proof. ($\Longrightarrow$) Assume $f \in \mathrm{eBV}$. Suppose $\tilde{f} \in \mathrm{BV}$ satisfies $f = \tilde{f}$ at Lebesgue points of f. Let $\{\mathcal{P}_j\}$ be a sequence of partitions such that

1. $\mathcal{P}_j$ converges to a dense subset of $[0, L]$, and

2. $\mathrm{Var}(\tilde{f}) = \lim_{j \to \infty} \mathrm{Var}(\tilde{f}, \mathcal{P}_j)$.

Let $\tilde{f}_j$ be the piecewise affine function with $\tilde{f}_j\big|_{\mathcal{P}_j} = \tilde{f}$ such that $\tilde{f}_j$ is affine on each $[t_{i-1}, t_i]$. Then

$$\operatorname{Var}(\tilde{f}_j) = \sum_i \operatorname{Var}(\tilde{f}_j, [t_{i-1}, t_i]) = \sum_i \int_{[t_{i-1},t_i]} |\tilde{f}_j'| = \sum_i |\tilde{f}_j(t_i) - \tilde{f}_j(t_{i-1})| = \operatorname{Var}(\tilde{f}, \mathcal{P}_j).$$

Claim. *$\tilde{f}_j \to \tilde{f}$ pointwise except at jump discontinuities of $\tilde{f}$.*

Proof. Since $\tilde{f} \in \mathrm{BV}$, the function $\tilde{f}$ has left and right limits everywhere, and those limits agree except at the countably many jump discontinuities of $\tilde{f}$. Suppose x is a point at which the left and right limits of $\tilde{f}$ agree. Then there exist sequences $\{s_j\}$ and $\{t_j\}$ such that $s_j, t_j \in \mathcal{P}_j$, $s_j \leq x \leq t_j$ and $(s_j, t_j) \cap \mathcal{P}_j = \emptyset$.

By choice of x, we have $\tilde{f}_j(s_j) = \tilde{f}(s_j) \to \tilde{f}(x)$ and $\tilde{f}_j(t_j) = \tilde{f}(t_j) \to \tilde{f}(x)$. Thus we also have

$$\lim_{j\to\infty} \Big(\max\{\tilde{f}_j(s_j), \tilde{f}_j(t_j)\}\Big) = \tilde{f}(x) \quad \text{and} \quad \lim_{j\to\infty} \Big(\min\{\tilde{f}_j(s_j), \tilde{f}_j(t_j)\}\Big) = \tilde{f}(x). \tag{A.20}$$

Since $\min\{\tilde{f}_j(s_j), \tilde{f}_j(t_j)\} \leq \tilde{f}_j(x) \leq \max\{\tilde{f}_j(s_j), \tilde{f}_j(t_j)\}$, we have that $\tilde{f}_j(x) \to \tilde{f}(x)$. □

Claim. $\tilde{f}_j \to \tilde{f}$ in L^1.

Proof of Claim. Since $\tilde{f} \in \mathrm{BV}$, it is bounded. Also, $|\tilde{f}_j(x)| < \|\tilde{f}\|_\infty$, so by the dominated convergence theorem, $\tilde{f}_j \to \tilde{f}$ in L^1. □

Thus

$$\int \tilde{f}\phi' = \lim_{j\to\infty} \int \tilde{f}_j \phi' \tag{A.21}$$

for all $\phi \in C_0^1[0,L]$. Suppose $\phi \in C_0^1[0,L]$ with $\|\phi\|_\infty \leq 1$. Write $\mathcal{P}_j = \{t_i : t_0 < \ldots < t_k\}$. Then, using integration by parts and the definition of $\tilde{f}_j$, we have

$$\begin{aligned}
\int f_j \phi' &= \sum_i \int_{[t_{i-1},t_i]} \tilde{f}_j \phi' \\
&= \sum_i \left(-\int \tilde{f}_j' \phi + (\tilde{f}_j \phi) \Big|_{t_{i-1}}^{t_i} \right) \\
&= (\tilde{f}_j \phi) \Big|_{t_0}^{t_k} - \sum_i \int \tilde{f}_j' \phi \\
&= (\tilde{f}_j \phi) \Big|_{t_0}^{t_k} - \sum_i \int \left(\frac{\tilde{f}_j(t_i) - \tilde{f}_j(t_{i-1})}{t_i - t_{i-1}} \right) \phi \qquad \text{(A.22)} \\
&\leq (\tilde{f}_j \phi) \Big|_{t_0}^{t_k} + \sum_i \left| \frac{\tilde{f}_j(t_i) - \tilde{f}_j(t_{i-1})}{t_i - t_{i-1}} \right| \cdot |t_i - t_{i-1}| \|\phi\|_\infty \\
&= (\tilde{f}_j \phi) \Big|_{t_0}^{t_k} + \|\phi\|_\infty \mathrm{Var}(\tilde{f}_j, \mathcal{P}_j) \\
&= (\tilde{f}_j \phi) \Big|_{t_0}^{t_k} + \|\phi\|_\infty \mathrm{Var}(\tilde{f}, \mathcal{P}_j).
\end{aligned}$$

Letting $j \to \infty$, from (A.21) and (A.22), we have

$$\int \tilde{f} \phi' \leq \|\phi\|_\infty \mathrm{Var}(\tilde{f}). \qquad \text{(A.23)}$$

Since $f = \tilde{f}$ almost everywhere, we have

$$\int f \phi' = \int \tilde{f} \phi' \leq \|\phi\|_\infty \mathrm{Var}(\tilde{f}). \qquad \text{(A.24)}$$

($\Longleftarrow$) Suppose that there exists a constant C such that for any $\phi \in C_0^1([0,L])$, we have $\int f\phi' \leq C\|\phi\|_\infty$. Let ψ be a symmetric bump function with compact support contained in $[-1,1]$. Let $\psi_j(t) = j\psi(tj)$, and let $f_j = f * \psi_j = \int_{\mathbb{R}} f(t)\psi_j(x-t)dt$ denote the convolution of f and ψ_j. Let $\alpha > 0$, and fix a partition $\mathcal{P} = t_0 < \ldots < t_k$ of $[0,L]$ by Lebesgue points. Since $f_j \to f$ pointwise at Lebesgue points, there exists N such that $j > N$ implies that

$|f_j(t_i) - f(t_i)| \leq \dfrac{\alpha}{2k}$ for all i. Then

$$\begin{aligned}\mathrm{Var}(f, \mathcal{P}) &= \sum_i |f(t_i) - f(t_{i-1})| \\ &\leq \sum_i |f(t_i) - f_j(t_i)| + \sum_i |f_j(t_i) - f_j(t_{i-1})| + \sum_i |f_j(t_{i-1}) - f(t_{i-1})| \\ &\leq \alpha + \mathrm{Var}(f_j)\end{aligned} \tag{A.25}$$

Also, $\mathrm{Var}(f_j) = \sup\left\{\int f_j \phi' : \phi \in C_0^1([0, L]),\ \|\phi\|_\infty \leq 1\right\}$, and $(f * \psi_j)' = f * \psi_j'$. If $\phi \in C_0^1([0, L])$ with $\|\phi\|_\infty \leq 1$, by integration by parts and Fubini's theorem we have

$$\begin{aligned}\int f_j \phi' &= -\int (f * \psi_j)' \phi \\ &= \int_{[0,L]} \phi(x) \int_{\mathbb{R}} f(t) \psi_j'(x - t) dt dx \\ &= \int_{\mathbb{R}} f(t) \int_{[0,L]} \phi(x) \psi_j'(x - t) dx dt \\ &= \int_{[0,L]} f(t) \int_{[0,L]} \phi(x) \psi_j'(t - x) dx dt \\ &= \int_{[0,L]} f(t) (\phi * \psi_j)'(t) dt\end{aligned} \tag{A.26}$$

provided j is large enough that $(\phi * \psi_j)'(t)$ is compactly supported in $[0, L]$. By assumption,

$$\int_{[0,L]} f(t) (\phi * \psi_j)'(t) dt \leq C \|\phi * \psi_j\|_\infty \leq C \|\phi\|_\infty = C. \tag{A.27}$$

Thus

$$\mathrm{Var}(f, \mathcal{P}) \leq \alpha + C. \tag{A.28}$$

Since this is true for all partitions $\mathcal{P}$ of $[0, L]$ by Lebesgue points, we have $\mathrm{eVar}(f) \leq \alpha + C$. Since α is arbitrary, $f \in \mathrm{eBV}$ and $\mathrm{eVar}(f) \leq C$. Further, since $f \in \mathrm{eBV}$, there exists a function $\tilde{f}$ agreeing with f at Lebesgue points, so we can take $C = \mathrm{Var}(\tilde{f})$. In fact, the optimal C is $\mathrm{Var}(\tilde{f})$, since we can approximate $\tilde{f}$ by $\phi \in C_0^1([0, L])$. $\square$