SUPPORT VARIETIES OF TILTING MODULES FOR GL_n

by

BOBBE J. COOPER

(Under the direction of Daniel K. Nakano)

ABSTRACT

Let G be a reductive algebraic group scheme defined over the finite field $\mathbb{F}_p$, with Frobenius kernel G_1. The tilting modules of G are defined as rational G-modules for which both the module itself and its dual have good filtrations. In 1997, J. E. Humphreys conjectured that the support varieties over the Frobenius kernel G_1 of tilting modules with regular highest weight should be given by the Lusztig bijection between cells of the affine Weyl group and nilpotent orbits of G, when $p \geq h$, where h is the Coxeter number. We present a conjecture for the support varieties of tilting modules when $G = GL_n$. Our conjecture is equivalent to Humphreys' conjecture for $p \geq h = n$ and regular weights, but our formulation allows us to consider small p or singular weights as well. We obtain results for several infinite classes of tilting modules, including the case $p = 2$, and tilting modules whose support variety corresponds to a hook partition. In the case $p = 2$, we prove the conjecture by S. Donkin for the support varieties of tilting modules.

INDEX WORDS: algebraic groups, support varieties, tilting modules

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Bobbe J. Cooper

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Bobbe J. Cooper

Approved:

Major Professor: Daniel K. Nakano

Committee: Brian D. Boe
Leonard Chastkofsky

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
May 2008

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Table of Contents

Chapter 1

Introduction[1]

1.1 Cohomology

Cohomology is a powerful tool which allows us to study the maps between algebraic objects and to determine how simple objects fit together to form more complicated structures. One of the most useful innovations in representation theory is the support variety, which is defined using cohomological operations. Support varieties were first defined for modules over finite groups by Carlson [3] in the early 1980's and have been a key to learning more about the representation theory of finite groups. The theory of support varieties was later extended to the Frobenius kernel of an algebraic group scheme G (denoted G_1) by Friedlander and Parshall [9]. For G_1, support varieties are intimately connected with the geometry of the restricted nullcone $\mathcal{N}_1(\mathfrak{g})$ which can be identified with the spectrum of the cohomology of the restricted Lie algebra $\mathfrak{g}$ of G.

The tilting modules of G are rational G-modules for which both the module itself and its dual have good filtrations. These modules are an integral part of the representation theory of G. There are connections between the theory of tilting modules and the theory of Young modules for symmetric groups [7, 8] when the underlying root system for G is of type A_n. Even more importantly, knowing the characters of the indecomposable tilting modules is equivalent to the determination of the characters of the simple modules for G [15, E.10]. The latter problem has a conjectural answer for large p through the Lusztig Conjecture, which

[1]Portions of the following chapters have been submitted to *Advances in Mathematics,* 10/24/2007, under the title *On the Support Varieties of Tilting Modules.*

connects the characters of the simple modules for G and the combinatorics of Kazhdan-Lusztig polynomials. Therefore, the character theory of tilting modules is connected to the Kazhdan-Lusztig combinatorics of the affine Weyl group of G.

1.2 Humphreys' Conjecture

In [13, §12], J.E. Humphreys stated a conjecture for the support variety of an indecomposable tilting module when the highest weight of the module is regular (which implies that $p \geq h$, where h is the Coxeter number), for arbitrary reductive G. The basic idea is that the support variety only depends on the Kazhdan-Lusztig two-sided cell region of the affine Weyl group W_p to which the weight belongs. Let O_w denote the nilpotent orbit corresponding to the two-sided cell of $w \in W_p$ via the Lusztig bijection, and let C be the set of integral weights in the bottom alcove.

Conjecture 1.2.1 (Humphreys)**.** *Let $\lambda \in X(T)_+$ be a regular weight in $w \cdot C$ for some $w \in W_p$; then the support variety of the indecomposable tilting module of highest weight λ is the closure of O_w.*

The aforementioned conjecture requires that the weight be regular, because it is not evident how to determine to which cell region the singular weights should belong. Humphreys' Conjecture has been shown to be true for quantum groups by Ostrik [20] (for type A_n) and Bezrukavnikov [1] (in general), by using the validity of the Lusztig Conjecture for quantum groups. Since we do not yet know that the Lusztig Conjecture is true for algebraic groups, Humphreys' Conjecture is still open in this setting. This means that any calculation of the support variety of a tilting module for G which agrees with Humphreys' Conjecture is more supporting evidence for the Lusztig Conjecture for algebraic groups.

A calculation of the support varieties of tilting modules is also closely related to the calculation of support varieties for $\text{ind}_B^G \lambda = H^0(\lambda)$ for $\lambda \in X(T)_+$, which was completed by Nakano, Parshall, and Vella in [19] for algebraic groups (cf. [20] and [2] for the quantum group case). This important result is essential to any tilting module calculation.

This dissertation centers on an attempt to compute support varieties of tilting modules for algebraic groups of type A_n. We have a new conjecture (see Conjecture 3.1.2) which predicts the support varieties of indecomposable tilting modules $T(\lambda)$ for *all* primes p and *all* dominant weights λ (regular or singular). This new conjecture agrees with Humphreys' Conjecture for type A_n in the cases where they overlap, and with Donkin's Conjecture for $p = 2$ [8, §6], with a small correction from the published version. The nontrivial proof of the agreement with Humphreys' Conjecture involves Greene's Theorem– an elegant result from the combinatorics of posets [11].

This conjecture comes along with two new ways of calculating the cell regions for type A_n. One of them involves root system structures and is very useful for calculating lower bounds on support varieties. The other is a new generalization of the Robinson-Schensted corresponence to the affine Weyl group. It involves a variation of Young tableaux. Other attempts at this generalization can be found in [21] and [24]. Our new method is an applicable algorithm which associates a finite tableau to each element $w \in W_p$; the shape of the tableau is the partition corresponding via the Lusztig bijection to the cell containing w. Compare [21] which associates an infinite tableau to each element w, and [24], whose generalization is not easily applicable to any elements in W_p not in the finite Weyl group.

The new conjecture and methods for calculating the cell regions allow us to determine the support varieties for several important classes of tilting modules. We prove that the conjecture is true for tilting modules whose highest weights lie in cell regions corresponding to hook partitions. We also calculate the support varieties of all indecomposable tilting modules when $p = 2$, thus proving Donkin's Conjecture. This is very strong evidence that our conjecture is correct for singular weights, since every weight is singular for $p = 2$ and $n > 1$. Furthermore, we calculate the support varieties for a large infinite class of tilting modules whose highest weights lie in a set of alcoves which intersects every cell region. In addition, we reduce the problem to the restricted region, thereby making the problem finite for given n.

Chapter 2

Background

2.1 Notation

The conventions and notation throughout this paper will generally follow those in [15] or [14]. Let k be an algebraically closed field of characteristic $p > 0$. Let $G = GL_{n+1}$ be considered as a reductive algebraic group scheme, defined and split over the finite field $\mathbb{F}_p$, with root system Φ of type A_n. We fix a base of simple roots $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_n\}$; the positive roots with respect to this basis are denoted Φ^+. In the ϵ-basis, we have

$$\Phi^+ = \{\epsilon_i - \epsilon_j : i < j\}.$$

Let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$.

For any $I \subseteq \Delta$, let L_I denote the standard Levi factor of the parabolic subgroup P_I of G corresponding to I, with root system Φ_I. Let Φ_I^+ be the set of positive roots of L_I. To each $I \subseteq \Delta$, we can assign a partition $\pi(I)$ of $n+1$ as follows: We have

$$\Phi_I \cong A_{i_1-1} \times A_{i_2-1} \times \cdots \times A_{i_t-1}$$

for some $\{i_1, i_2, ..., i_t\}, i_j \in \mathbb{Z}, i_j \geq i_{j+1} > 1$ for all j. Define $\pi(I) = (i_1, i_2, ..., i_t, 1, ..., 1) \vdash n+1$.

Let T be a maximal split torus, and let B be a Borel subgroup containing T corresponding to the negative roots. Let $X(T)$ be the set of integral weights of G. The inner product on $X(T)$ will be denoted by $\langle\ ,\ \rangle$. Let $\alpha^\vee = \alpha$ be the coroot corresponding to $\alpha \in \Phi$. The set of dominant integral weights is defined as

$$X(T)_+ = \{\lambda \in X(T) :\ 0 \leq \langle\lambda, \alpha^\vee\rangle \ \text{ for all } \alpha \in \Delta\}.$$

In the ϵ-basis, a dominant weight λ is a nonincreasing sequence of $n+1$ integers. We will often have occasion to use $\lambda+\rho$ in the ϵ-basis. When λ is a dominant weight and n is odd, this will be a strictly decreasing sequence of half-integers. For convenience, we prefer to work with integers. If n is odd, define ξ to be a sequence of $n+1$ numbers equal to $1/2$; otherwise, let $\xi = 0$. Then $\lambda+\rho-\xi$ is a strictly decreasing sequence of integers, with

$$\langle \lambda+\rho, \alpha^\vee\rangle \ = \langle \lambda+\rho-\xi, \alpha^\vee\rangle$$

for all $\alpha \in \Phi$.

Let W be the Weyl group corresponding to Φ and W_p be the affine Weyl group with translations scaled by p. We will consider the dot action of W_p on $E = X(T)\otimes\mathbb{R}$ defined by $w\cdot\lambda = w(\lambda+\rho)-\rho$ for $\lambda \in X(T)$. Write $N = \{1, ..., n+1\}$.

2.2 Orbit theory

Let $\mathcal{N}$ be the variety of nilpotent elements in $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{gl}_{n+1}(k)$. The group G acts on $\mathcal{N}$ by conjugation. The G-orbits are parameterized by partitions of $n+1$. We will use the exponent notation for partitions, where $(r_1^{d_1}, r_2^{d_2}, ...)$ denotes the partition with r_i appearing d_i times. Let $\pi = (r_1^{d_1}, r_2^{d_2}, ...)$ be a partition of $n+1$. We will use $\mathcal{O}_\pi$ to denote the G-orbit of $\mathcal{N}$ which contains the element in Jordan canonical form with d_i blocks of size r_i for all i.

The elements of W_p are partitioned into sets called *two-sided cells* or cells for short (cf. [18]). For $w \in W_p$, let $h(w)$ denote the partition of $n+1$ such that $\mathcal{O}_{h(w)} = \mathcal{O}_w$. Here, $\mathcal{O}_w$ is the nilpotent orbit corresponding to the two-sided cell containing w via the Lusztig bijection, as in Conjecture 1.2.1. We will use the combinatorial work of [24] as a reference for calculating cells. See Section 3.2.1 for the details.

For the reader's convenience, we quote the definition of dominance order on partitions. Let π, σ be partitions of $n+1$, and write $\pi = (\pi_1, ..., \pi_t)$, $\sigma = (\sigma_1, ..., \sigma_s)$. We have $\pi \geq \sigma$ if for all $1 \leq m \leq n+1$,

$$\sum_{j=1}^{m} \pi_j \geq \sum_{j=1}^{m} \sigma_j.$$

With this order we can calculate the (Zariski) closure of an orbit of G in $\mathcal{N}$ [6, 6.2]:

$$\overline{\mathcal{O}_\pi} = \bigcup_{\sigma \leq \pi} \mathcal{O}_\sigma.$$

We will need a least upper bound property for the dominance ordering on partitions, paraphrased from the French in [25, III, 3.3] below. Note that this property does not hold for arbitrary posets.

Lemma 2.2.1. *Let $P = \{\pi^1, \pi^2, ..., \pi^t\}$ be a set of partitions of $n+1$. There exists a unique minimal partition σ of $n+1$ such that $\sigma \geq \pi^i$ for all i.*

2.3 Tilting Modules

For $\lambda \in X(T)_+$, let $H^0(\lambda) = \text{ind}_B^G \lambda$ (respectively, $H^0_{L_I}(\lambda) = \text{ind}_{B(L_I)}^{L_I} \lambda$, where $B(L_I)$ is a Borel subgroup of L_I containing T). These will be referred to as the "induced modules". The simple G-module of highest weight λ will be denoted $L(\lambda)$. The Weyl modules are $V(\lambda) = H^0(-w_0\lambda)^*$ for $\lambda \in X(T)_+$. Here w_0 is the long element in W and $-^*$ denotes taking the dual module. A G-module M has a good (resp. Weyl) filtration if there exists a filtration of M such that all the subquotients are induced modules (resp. Weyl modules).

A (partial) tilting module is defined to be a module with both a good filtration and a Weyl filtration. See [15, II:E] for a discussion on tilting modules for G, and [22] for a discussion of the origins and application of the theory of tilting modules for hereditary algebras. The indecomposable tilting modules are indexed by their highest weights, which must be dominant. We will denote the indecomposable tilting module over G (resp. L_I) with highest weight λ by $T_G(\lambda) = T(\lambda)$ (resp. $T_{L_I}(\lambda)$). Tilting modules satisfy the following basic properties:

(i) If M, N are tilting modules, then $M \otimes N$ is a tilting module.

(ii) If M is a direct summand of a tilting module N, then M is a tilting module.

(iii) If N is a tilting G-module, then $N \downarrow_{L_I}$ is a tilting L_I-module for any Levi subgroup L_I.

2.4 Support Varieties

Let A be a finite k-group scheme and let

$$R = \begin{cases} \mathrm{H}^{2\bullet}(A, k) & \text{if char } k \neq 2 \\ \mathrm{H}^{\bullet}(A, k) & \text{if char } k = 2. \end{cases}$$

According to [10], the cohomology ring R is a commutative, finitely generated k-algebra. For finite dimensional $M \in A$-mod, define the *support variety* $V_A(M)$ as follows. Yoneda composition defines an action of R on $\mathrm{Ext}_A^{\bullet}(M, M)$. Let $J = J_A(M)$ be the annihilator ideal in R for this action. Set $V_A(M)$ equal to the maximum ideal spectrum of R/J.

We can consider G, L_I as algebraic group schemes (not finite) over k. For an algebraic group scheme H over k, let $F : H \to H^{(1)}$ be the Frobenius morphism. Set $H_1 = \ker F$. Now H_1 is a finite k-group scheme, so we can consider support varieties of H_1-modules. The Lie algebra $\mathfrak{h} = \mathrm{Lie}(H)$ is a restricted Lie algebra with p-mapping $x \to x^{[p]}$. In [23, (1.6), (5.11)] it is proven that $V_{H_1}(k)$ is homeomorphic to $\mathcal{N}_1(\mathfrak{h}) := \{x \in \mathfrak{h} : x^{[p]} = 0\}$. We will use this identification frequently without explicit mention. Note that if M is an H-module, then $V_{H_1}(M)$ is an H-stable subvariety of $\mathcal{N}_1(\mathfrak{h})$ under conjugation.

Support varieties behave nicely with respect to many module operations. We will use the following properties (cf. [19, 2.2]). Let M, N be H-modules, and K a closed subgroup of H, then

(i) $V_{H_1}(M) \cap V_{H_1}(N) = V_{H_1}(M \otimes N)$;

(ii) $V_{H_1}(M) \cup V_{H_1}(N) = V_{H_1}(M \oplus N)$;

(iii) $V_{K_1}(k) \cap V_{H_1}(M) = V_{K_1}(M \downarrow_K)$;

(iv) Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of H-modules. If Σ_3 is the symmetric group on three letters and $\sigma \in \Sigma_3$, then $V_{H_1}(M_{\sigma(1)}) \subseteq V_{H_1}(M_{\sigma(2)}) \cup V_{H_1}(M_{\sigma(3)})$.

In general $\mathcal{N}_1(\mathfrak{g})$ is a G-stable subvariety of $\mathcal{N}$ and when $p \geq h$, $\mathcal{N} = \mathcal{N}_1(\mathfrak{g})$. If $p < h$, then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}_{(p^d,r)}}$ where $n+1 = pd + r$, $0 \leq r < p$ [5, 3.1]. For $I \subseteq \Delta$, we have a Levi decomposition $\mathfrak{g} = \mathfrak{u}_I^+ \oplus \mathfrak{l}_I \oplus \mathfrak{u}_I$ where $\mathfrak{l}_I = \mathrm{Lie}(L_I)$. We have $G \cdot \mathcal{N}_1(\mathfrak{l}_I) = \overline{\mathcal{O}_{\pi(I)}} \cap \mathcal{N}_1(\mathfrak{g})$ and $G \cdot \mathfrak{u}_I = \overline{\mathcal{O}_{\pi(I)^t}}$ where $-^t$ denotes the dual partition [17, 2.2].

For each G-module M, $V_{G_1}(M)$ is a G-stable subvariety of $\mathcal{N}_1(\mathfrak{g})$, so $V_{G_1}(M) = \bigcup \overline{\mathcal{O}_{\pi^i}}$ for some set $\{\pi^i\}$ of partitions of $n+1$ which are pairwise incomparable. Note that each π^i must be dominated by (p^d, r).

We will frequently use [19, (6.2.1) Thm.] due to Nakano, Parshall, and Vella, which computes the support variety of an induced module. For $\lambda \in X(T)_+$, let $\Phi_\lambda = \{\alpha \in \Phi : \langle \lambda + \rho, \alpha^\vee \rangle \in p\mathbb{Z}\}$. Since Φ is of type A_n, there exists $w \in W$ such that $w(\Phi_\lambda) = \Phi_I$ for some $I \subseteq \Delta$.

Theorem 2.4.1 (Nakano-Parshall-Vella)**.** *For $\lambda \in X(T)_+$, choose $I \subseteq \Delta$ so that $w(\Phi_\lambda) = \Phi_I$ for some $w \in W$. Then $V_{G_1}(H^0(\lambda)) = G \cdot \mathfrak{u}_I = \overline{\mathcal{O}_{\pi(I)^t}}$.*

2.5 Notch Tableaux

We will consider a tableau to be a Young diagram with the boxes filled with integers, with one distinction: we do not require the rows of the diagram to be nonincreasing in length. We will call these tableaux *notch tableaux*. Note that re-arranging the order of the rows yields a traditional tableau. We obtain a partition as the shape of a notch tableau, where the parts of the partition are given by the lengths of the rows of the diagram. We re-arrange the parts to be in nonincreasing order if necessary. The partition whose parts are the column lengths is the transpose of the partition whose parts are the row lengths, just as for traditional tableaux. See Section 4.5 for examples of notch tableaux.

2.6 Visualizing Cells in the Affine Weyl Group of Type A_n

The affine Weyl group W_p of type A_n can be visualized by looking at its action on the n-dimensional Euclidean space $E = X(T) \otimes \mathbb{R}$, via reflections through hyperplanes corresponding to root vectors, and translations. Let $C_{\mathbb{R}}$ be the bottom alcove of E, defined as the set $\{\lambda \in E : 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p \text{ for all } \alpha \in \Phi^+\}$. Let C be the set of dominant integral weights in $C_{\mathbb{R}}$. Then for any $w \in W$, $w \cdot C_{\mathbb{R}}$ is an alcove. Since the closure of $C_{\mathbb{R}}$ is a fundamental region for W_p, we can label the alcoves by the elements of W_p. We will usually identify each element with its alcove. Throughout this thesis, we only consider alcoves that are contained in the dominant chamber.

The cells partition the elements of W_p; we will define a corresponding partition of E. In order to do so, we need a definition. Let $w \in W_p$ be given. As in [15, II 6.1], there exists a unique integer n_α for each $\alpha \in \Phi$ such that

$$n_\alpha p < \langle \lambda + \rho, \alpha^\vee \rangle < (n_\alpha + 1)p$$

for all $\lambda \in w \cdot C_{\mathbb{R}}$. We define the *lower closure* of the alcove $w \cdot C_{\mathbb{R}}$ to be the set

$$\{\lambda \in E : n_\alpha p \leq \langle \lambda + \rho, \alpha^\vee \rangle < (n_\alpha + 1)p \;\text{ for all } \alpha \in \Phi\}.$$

One should compare this to the definition of upper closure [15, II 6]. Note that the lower closure of an alcove can contain integral weights, even if the interior of the alcove does not (i.e. if $p < h$). Given any $\lambda \in E$, there exists a unique $w \in W_p$ such that λ is in the lower closure of $w \cdot C_{\mathbb{R}}$ (cf. [15, II 6.11]).

Now, we define a partition of E into *cell regions* via an equivalence relation: Given $\lambda, \mu \in E$, let $w, y \in W_p$ be such that λ is in the lower closure of $w \cdot C_{\mathbb{R}}$ and μ is in the lower closure of $y \cdot C_{\mathbb{R}}$. Then λ and μ are in the same cell region if and only if w and y belong to the same cell. So the weights which are in the lower closure of an alcove will belong to the same cell region as the alcove. In this thesis, we show that this is the correct region in which to place the singular weights.

It is easy to describe the cell regions in A_2 geometrically (cf. [24, p. 30]). We will describe their intersection with the dominant chamber. Consider $R_{(3)} = C_{\mathbb{R}}$, the bottom alcove. This alcove corresponds to the identity element of W_p, which is always the unique element in its cell region. This cell region is described geometrically as

$$R_{(3)} = \{\lambda \in E : 0 < \langle \lambda + \rho, (\alpha_1 + \alpha_2)^\vee \rangle < p\}.$$

If $p \geq 3$, then $R_{(3)}$ contains integral weights. Next we look at $R_{(2,1)}$. This cell region is described geometrically as

$$R_{(2,1)} = \{\lambda \in E : p \leq \langle \lambda + \rho, (\alpha_1 + \alpha_2)^\vee \rangle, 0 < \langle \lambda + \rho, \alpha_i^\vee \rangle < p \text{ for some } i \in \{1, 2\}\}.$$

For any p, $R_{(2,1)}$ contains integral weights. This region consists of the lower closure of all the alcoves inside and adjacent to the boundary of the dominant chamber, except the bottom alcove. Finally, consider $R_{(1,1,1)}$. This cell region is described geometrically as

$$R_{(1,1,1)} = \{\lambda \in E : p \leq \langle \lambda + \rho, \alpha_i^\vee \rangle \text{ for all } i \in \{1, 2\}\}.$$

For any p, $R_{(1,1,1)}$ contains integral weights. This region consists of the lower closure of alcoves in the dominant chamber translated by $p\rho$.

In this thesis, we discuss a similar geometric description for any dominant cell region in any dimension. We will have three different methods of calculating this geometric description. One method is a very useful way of associating a notch tableau to an alcove, where the shape of the tableau gives $h(w)$, the partition associated to the cell of the alcove. Note that we have labeled our cell regions in A_2 above with the partitions of $n + 1 = 3$. The other two methods arise from the fact that we can associate to any cell region certain hyperplanes which separate the cell region from $C_{\mathbb{R}}$ or certain hyperplanes which do not separate the cell region from $C_{\mathbb{R}}$. This distinction is surprisingly subtle.

Chapter 3

Two-sided Cells and a New Conjecture

3.1 A Conjecture for the Support Varieties of Tilting Modules

We start with a method for calculating the cell regions geometrically. Let $\lambda \in X(T)_+$ be given. Define

$$\Psi_\lambda = \{\alpha \in \Phi^+ : \langle \lambda + \rho, \alpha^\vee \rangle < p\}.$$

To each $\alpha \in \Phi^+$ there corresponds a hyperplane $H_\alpha = \{\mu \in E : \langle \mu + \rho, \alpha^\vee \rangle = p\}$. The elements in Ψ_λ are the roots for which H_α does not lie between $w \cdot C_\mathbb{R}$ and $C_\mathbb{R}$ where λ is in the lower closure of $w \cdot C_\mathbb{R}$. If λ and μ are in the lower closure of the same alcove $w \cdot C_\mathbb{R}$, then for all $\alpha \in \Phi^+$ such that $\langle \lambda + \rho, \alpha^\vee \rangle < p$ we have $\langle \mu + \rho, \alpha^\vee \rangle < p$. So in this case, $\Psi_\lambda = \Psi_\mu$.

Now, we define $c(\lambda)$ as the minimal partition such that $c(\lambda) \geq \pi(I)$ for all $I \subseteq \Delta$ satisfying $\Phi_I^+ \subseteq \Psi_\lambda$. Such a partition exists by Lemma 2.2.1.

Equivalently, $c(\lambda)_1$ is the maximal cardinality of a subset $X \subseteq N$ with $\langle \lambda + \rho, (\epsilon_i - \epsilon_j)^\vee \rangle < p$ for $i, j \in X, i < j$; in general, $\sum_{i=1}^t c(\lambda)_i$ is the maximal cardinality of a disjoint union of subsets $X_1, ..., X_t \subseteq N$ such that for each $1 \leq l \leq t$, we have $\langle \lambda + \rho, (\epsilon_i - \epsilon_j)^\vee \rangle < p$ for $i, j \in X_l, i < j$. Such a subset X_l corresponds to a set of simple roots $I_l = \{\epsilon_i - \epsilon_j : i < j, i, j \in X_l\}$ with $\Phi_{I_l}^+ \subseteq \Psi_\lambda$. This construction will create the parts of $c(\lambda)$ in nonincreasing order (cf. Section 3.3).

Suppose that $\Psi_\lambda = \Phi_I^+$ for some $I \subseteq \Delta$. For any $J \subseteq \Delta$ such that $\Phi_J^+ \subseteq \Psi_\lambda$, we must have $J \subseteq I$, which implies that $\pi(J) \leq \pi(I)$. Therefore, in this case $c(\lambda) = \pi(I)$.

Lemma 3.1.1. *For all p and all $\lambda \in X(T)_+$, $\overline{\mathcal{O}_{c(\lambda)}} \subseteq \mathcal{N}_1(\mathfrak{g})$.*

Proof. Recall that $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}_{(p^d,r)}}$ where $n+1 = pd+r$. Let $I \subseteq \Delta$ with $\Phi_I^+ \subseteq \Psi_\lambda$. If $\beta \in \Phi^+$ with the height of β greater than $p-1$, then $\langle \lambda + \rho, \beta^\vee \rangle \geq p$ because λ is dominant. So the roots in Ψ_λ must all have height less than p. Therefore, Φ_I cannot contain an irreducible subsystem with rank bigger than $p-1$, which implies that $\pi(I) \leq (p^d, r)$. The minimal partition dominating all such $\pi(I)$ must therefore be less than or equal to (p^d, r). □

Now, we formulate a conjecture for the support variety of any indecomposable tilting module for $G = GL_{n+1}$. Note that λ may be either regular or singular, in contrast to Humphreys' Conjecture 1.2.1, and thus this conjecture also covers $p < h$.

Conjecture 3.1.2. *Let $\lambda \in X(T)_+$. Then $V_{G_1}(T(\lambda)) = \overline{\mathcal{O}_{c(\lambda)}}$.*

Note that we have $V_{G_1}(T(\lambda)) \subseteq \mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}_{(p^d,r)}}$; compare Lemma 3.1.1.

3.2 Combinatorics and Cell Regions

3.2.1 Determining the cells using Γ_λ

A geometric method for calculating the cell regions in type A_n is given by Shi in [24] and quoted in [20]. We will paraphrase the relevant portion.

Let $\lambda \in X(T)_+$ be a regular weight, with $\lambda \in w \cdot C$ for some $w \in W_p$. Let

$$\Gamma_\lambda = \{\alpha \in \Phi^+ : \langle \lambda + \rho, \alpha^\vee \rangle \geq p\}$$

This corresponds to considering the set of roots α such that the hyperplanes H_α *do* lie between $w \cdot C_\mathbb{R}$ and $C_\mathbb{R}$ (the reader can contrast this with Ψ_λ). We say that $i, j \in \{1, 2, .., n+1\} = N$ are λ-*connected* if $\pm(\epsilon_i - \epsilon_j) \in \Gamma_\lambda$. A subset of N is λ-*connected* if its elements are pairwise λ-connected.

We can define a partition of $n+1$, $s(\lambda)$, by letting $s(\lambda)_1$ be the size of the largest λ-connected subset of N; $s(\lambda)_1 + s(\lambda)_2$ be the size of the largest subset of N which can be written as a disjoint union of exactly two λ-connected sets, and in general $\sum_{i=1}^m s(\lambda)_i$ is defined as the size of the largest subset of N which can be written as a disjoint union of exactly

m λ-connected sets. This construction will create the parts of $s(\lambda)$ in nonincreasing order (cf. Section 3.3). A λ-connected subset S of N corresponds to $R = \{\epsilon_i - \epsilon_j : i < j, i, j \in S\}$, and there exists $y \in W$ and $I \subseteq \Delta$ such that $y(\Phi_I^+) = R$ (cf. [16, 2.7]). This implies that $s(\lambda)$ is the least partition which dominates all $\pi(I)$, for all $I \subseteq \Delta$ satisfying $y(\Phi_I^+) \subseteq \Gamma_\lambda$ for some $y \in W$. Recall that $h(w)$ is defined in Section 2.2. It is shown in [24, 6.3] that

$$h(w) = s(\lambda)^t.$$

Note that this method, as given, does not require λ to be regular: Γ_λ is still well-defined if λ is singular. This is not explicitly mentioned in either [24] or [20], although [20] uses it implicitly. If λ is singular then $s(\lambda)^t = h(w)$ where $w \cdot C_{\mathbb{R}}$ is the alcove which contains λ in its lower closure. We will frequently use this extension of Shi's work in the sequel.

3.2.2 Determining the Cells via Tableaux

We now present another way of calculating the cell regions, which is a generalization of the Robinson-Schensted algorithm to the case of the affine Weyl group of type A_n. We have the following technical assumption on $S \subseteq \Phi^+$ which will be satisfied by all the sets to which we associate tableaux.

Assumption 3.2.1. *For $i < j < m$, if $\epsilon_j - \epsilon_m \notin S$, then $\epsilon_i - \epsilon_m \notin S$.*

If $\lambda \in X(T)_+$, then Ψ_λ satisfies Assumption 3.2.1. If $\langle \lambda + \rho, (\epsilon_j - \epsilon_m)^\vee \rangle \geq p$ for $j < m$, then $\langle \lambda + \rho, (\epsilon_i - \epsilon_m)^\vee \rangle \geq p$ for all $i < j$.

We can construct a notch tableau D_S with $n + 1$ boxes which satisfies the following condition:

Condition 3.2.2. *If i, j are numbers in the same column of D_S, then $\epsilon_i - \epsilon_j \notin S$.*

We construct D_S inductively. At each step, the numbers in D_S are strictly increasing across the rows and down the columns. For the first step, start with a box containing 1:

$$\boxed{1}$$

For the second step, we place a box containing 2 under the first box if $\epsilon_1 - \epsilon_2 = \alpha_1 \notin S$

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$$

Otherwise, we place a box containing 2 to the right of the first box

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$$

At the ith step, we will specify in which column and row to place a box containing i. We will place the box in the leftmost column such that Condition 3.2.2 holds. Note that Condition 3.2.2 is satisfied trivially if we start a new rightmost column with the box containing i. Let j be the index such that we are placing the box containing i in the jth column. If $j = 1$, we start a new row directly below the previous bottom row with the box containing i. Otherwise, let r be the number in the bottommost box in the $(j-1)$th column. We claim that the row containing r has exactly $j-1$ boxes. Assuming the claim, we will place the box containing i directly to the right of the box containing r. Note that there cannot be any boxes in the jth column below the box containing i, since there are no boxes in the $(j-1)$th column below the box containing r and each row is built from left to right.

To see the claim, we work by contradiction. If the row containing r has more than $j-1$ boxes, then there would be a box directly to the right of r, in the jth column, containing some number $q > r$. We have $\epsilon_q - \epsilon_i \notin S$ since we can place the box containing i in the jth column, which implies that $\epsilon_r - \epsilon_i \notin S$ by Assumption 3.2.1. Note also that the numbers above r in the $(j-1)$th column are less than r, so $\epsilon_s - \epsilon_i \notin S$ for any s in the $(j-1)$th column, by another application of Assumption 3.2.1. This implies that we can place the box containing i below r in the $(j-1)$th column, which is a contradiction. So the row containing r has exactly $j-1$ boxes. Finally, note that we have placed the box containing i at the end of a row and the bottom of a column, so the numbers in the boxes are still strictly increasing across the rows and down the columns.

The notch tableau D_S gives us a partition of $n+1$ whose parts are the lengths of the rows of D_S. Call this partition $\sigma(D_S)$. In order to apply this algorithm to a weight $\lambda \in X(T)_+$, we use $S = \Psi_\lambda$. We denote D_{Ψ_λ} by D_λ. It will be shown below that $\sigma(D_\lambda) = h(w)$ for λ in the lower closure of $w \cdot C_\mathbb{R}$.

3.3 Equivalence of these approaches

We now have three different ways to start with a weight λ (equivalently, an element $w \in W_p$ such that λ is contained in the lower closure of $w \cdot C_\mathbb{R}$) and obtain a partition: we have $\sigma(D_\lambda)$ using the algorithm to obtain a notch tableau, $s(\lambda)^t$ using Γ_λ, and $c(\lambda)$ using Ψ_λ.

In this section, we will show that, in fact, these methods always give the same answer. Thus we can use any of these methods to distinguish to which cell an element of w or its alcove belongs, using the labeling of the cells by partitions via the Lusztig bijection. Furthermore, we can use any of these methods in our conjecture to compute support varieties of tilting modules.

We will need the following combinatorial result from [**11**, Thms. 1.5, 1.6]. Given a poset P, a chain is a subset of P which is totally ordered; an antichain is a subset of P which contains no chains of length two.

Theorem 3.3.1 (Greene)**.** *Let P be a finite poset. For $i \geq 1$, let d_i be the size of the largest subset of P which can be partitioned into i chains; let $\widehat{d_i}$ be the size of the largest subset of P which can be partitioned into i antichains. Also, let $d_0 = \widehat{d_0} = 0$. Define partitions π, σ by $\pi_i = d_i - d_{i-1}$ and $\sigma_i = \widehat{d_i} - \widehat{d_{i-1}}$. Then $\pi_i \geq \pi_{i+1}$ and $\sigma_i \geq \sigma_{i+1}$ for all $i \geq 1$. In addition, we have $\pi = \sigma^t$.*

Note that in the course of this proof, we will show that $c(\lambda)$ and $s(\lambda)^t$ are in fact constructed from a partial order in the manner described in Theorem 3.3.1. Thus we know that their parts are constructed in nonincreasing order, as claimed in Sections 3.1 and 3.2.1.

Theorem 3.3.2. *Let $\lambda \in X(T)_+$. Let $w \in W_p$ be such that λ is in the lower closure of $w \cdot C_\mathbb{R}$. Then $\sigma(D_\lambda) = s(\lambda)^t = c(\lambda) = h(w)$, where we use the extension of the definition of $s(\lambda)$ to singular weights as in Section 3.2.1.*

Proof. First, we show that $\sigma(D_\lambda)^t = s(\lambda)$. By definition, $\sigma(D_\lambda)^t$ is the partition whose parts are the lengths of the columns of D_λ.

For $m \in \mathbb{N}$, define $S^m = \{i : i \text{ occurs in a box in the first } m \text{ columns of } D_\lambda\}$. We have

$$|S^m| = \sum_{j=1}^{m} \sigma(D_\lambda)^t_j.$$

By Condition 3.2.2, S^m is a subset of N which is a disjoint union of m λ-connected sets, for all m. Comparing the definition of $s(\lambda)$, we have $s(\lambda) \geq \sigma(D_\lambda)^t$.

Thus it suffices to show that $s(\lambda) \leq \sigma(D_\lambda)^t$. We proceed by contradiction. Suppose $s(\lambda) \not\leq \sigma(D_\lambda)^t$. This would mean there exists an index m such that $\sum_{j=1}^{m} \sigma(D_\lambda)^t_j < \sum_{j=1}^{m} s(\lambda)_j$. This in turn implies that there exists a set $R \subseteq N$ such that R is a disjoint union of m or fewer λ-connected sets, and $|R| > |S^m|$.

Write $S^m = \{s_1 = 1, s_2, ..., s_t\}$ with $s_i < s_{i+1}$; write $R = \{r_1, r_2, ..., r_{t+1}, ...\}$ with $r_i < r_{i+1}$. We claim that $r_{t+1} \notin S^m$, but $S^m \cup \{r_{t+1}\}$ is a disjoint union of m or fewer λ-connected subsets with cardinality $t + 1$. Assuming the claim, we have a contradiction: r_{t+1} must be λ-connected to some number in S^m because S^m is already a disjoint union of m (and no fewer) λ-connected subsets. This implies that the box containing r_{t+1} would be placed in one of the first m columns of D_λ, which is a contradiction.

It remains to show the claim. We use induction by showing that $\{s_1, ..., s_i, r_{i+1}, ..., r_{t+1}\}$ is a disjoint union of m or fewer λ-connected subsets, with $s_i < r_{i+1}$, for all $i \leq t$. Along the way, we will also show that $\{s_1, ..., s_i, r_{i+1}\}$ is a disjoint union of m or fewer λ-connected subsets for all $i \leq t$. The basis step may be argued thusly. We have $s_1 = 1 \leq r_1$ by the construction of D_λ. Suppose $r_j \in R$ is λ-connected to r_1: we want to show that r_j is λ-connected to $s_1 = 1$. We have $\epsilon_{r_1} - \epsilon_{r_j} \notin \Psi_\lambda$; that is, $\langle \lambda + \rho, (\epsilon_{r_1} - \epsilon_{r_j})^\vee \rangle \geq p$. Since λ is dominant, this implies that

$$\langle \lambda + \rho, (\epsilon_1 - \epsilon_{r_j})^\vee \rangle = \langle \lambda + \rho, (\epsilon_1 - \epsilon_{r_1})^\vee \rangle + \langle \lambda + \rho, (\epsilon_{r_1} - \epsilon_{r_j})^\vee \rangle \geq p.$$

So r_j is λ-connected to $1 = s_1$. This implies that $\{s_1, r_2, ..., r_{t+1}\}$ is a disjoint union of m or fewer λ-connected sets, and $s_1 = 1 < r_2$ is clear. Finally, to show that $\{s_1, r_2\}$ is a disjoint union of m or fewer λ-connected sets we consider the two cases $m = 1$ or $m > 1$. For $m > 1$, the condition is satisfied vacuously; for $m = 1$, we must have $\{s_1, r_2, ..., r_{t+1}\}$ to be λ-connected, so that $\{s_1, r_2\}$ is λ-connected.

For the inductive step, assume that $\{s_1, ..., s_i, r_{i+1}, r_{i+2}, ..., r_{t+1}\}$ is a disjoint union of m or fewer λ-connected subsets, and that the same property holds for $\{s_1, ..., s_i, r_{i+1}\}$. We claim that s_{i+1} may be characterized as the least integer such that $\{s_1, ..., s_i\} \cup \{s_{i+1}\}$ is a disjoint union of m or fewer λ-connected subsets with cardinality $i + 1$. It is such an integer, by the definition of S^m. It is the least such integer, because we have listed the integers in S^m in increasing order. Since r_{i+1} is such an integer by induction, we have $s_{i+1} \leq r_{i+1}$.

Now the same argument as in the basis step shows that for $r_j > r_{i+1}$, if r_j is λ-connected to r_{i+1}, then r_j is λ-connected to s_{i+1}. Thus, $\{s_1, ..., s_{i+1}, r_{i+2}, r_{i+3}, ..., r_{t+1}\}$ is a disjoint union of m or fewer λ-connected subsets with $r_{i+2} > r_{i+1} \geq s_{i+1}$, and $\{s_1, ..., s_{i+1}, r_{i+2}\}$ has the same property. So $\sigma(D_\lambda)^t = s(\lambda)$.

Next, we show that $c(\lambda) = s(\lambda)^t$ – that is, using Γ_λ or Ψ_λ yields the same the partition. For this portion of the proof, we will rely on notation and results from [24]. In the notation of [24, p. 97], $\sigma(w)^t = s(\lambda)^t$. We want to show that $c(\lambda) = \sigma(w)^t$.

Let $\overline{a}$ denote a modulo $n+1$. We will consider the representation of W_p as permutations acting on $\mathbb{Z}$ on the right: see [24, p. 67]. We define the action by letting the Coxeter generator s_i act by the transposition $(i, i+1)$ for $1 \leq i \leq n$ and letting the Coxeter generator s_{n+1} act by the product of transpositions $(1, 0)(n+1, n+2)$, and extending to all of $\mathbb{Z}$ by setting $(i+n+1)w = (i)w + n + 1$ for all $i \in \mathbb{Z}, w \in W_p$. For each $w \in W_p$, we can define a partial order $\lhd_w$ on $\mathbb{Z}$ by setting $i \lhd_w j$ if $\bar{i} = \bar{j}$ or for some $u_1, u_2 \in \mathbb{Z}$ with $(\overline{u_1}, \overline{u_2}) = (\bar{i}, \bar{j})$ we have $u_1 < u_2$ and $u_1 w^{-1} > u_2 w^{-1}$.

As in [24, 6.1], write k_{ij}^w for the unique integer which satisfies:

$$k_{ij}^w p \leq \langle \lambda + \rho, (\epsilon_i - \epsilon_j)^\vee \rangle < (k_{ij}^w + 1)p.$$

Thus we have $\epsilon_i - \epsilon_j \in \Psi_\lambda$ if and only if $k^w_{ij} = 0$. Hence $c(\lambda)_1$ is the maximal cardinal of a subset $X \subseteq N$ with $k^w_{ij} = 0$ for $i, j \in X, i < j$; in general, $\sum_{i=1}^t c(\lambda)_i$ is the maximal cardinal of a disjoint union of subsets $X_1, ..., X_t \subseteq N$ with $k^w_{ij} = 0$ for $i, j \in X_l, i < j, 1 \leq l \leq t$.

By [24, Lemma 6.3.2], we have $k^w_{ij} = 0$ if and only if for all $u_1, u_2 \in \mathbb{Z}$ with $\{\overline{u_1}, \overline{u_2}\} = \{\bar{i}, \bar{j}\}$, $u_1 < u_2 \implies u_1 w^{-1} < u_2 w^{-1}$. This lemma implies that each X_l above corresponds to an *antichain* in the partial order $\lhd_w$, so that $c(\lambda)$ is defined using antichains in $\lhd_w$, in the form of σ in Theorem 3.3.1.

Now, $\sigma(w)$ is defined using chains in the partial order $\lhd_w$, in form of π in Theorem 3.3.1. Thus we can apply Theorem 3.3.1 to obtain $c(\lambda) = \sigma(w)^t$.

Finally, we have $h(w) = s(\lambda)^t$ by [24, 6.3]. $\square$

Chapter 4

Calculations

4.1 Lower and Upper Bounds

4.1.1 Lower Bound

A major advantage of the approach using Ψ_λ is its amenability to the calculation of a lower bound for the support variety of a tilting module. In fact, we get almost exactly what we need by simply looking at the restriction of the tilting module to a well-chosen Levi subgroup.

Lemma 4.1.1. *Let $I \subseteq \Delta$ and $\lambda \in X(T)_+$.*

(a) If $\langle \lambda + \rho, \alpha^\vee \rangle < p$ for all $\alpha \in I$, then $V_{(L_I)_1}(T_{L_I}(\lambda)) = \mathcal{N}_1(\mathfrak{l}_I)$.

(b) $T_{L_I}(\lambda)$ is a direct summand of $T(\lambda) \downarrow_{L_I}$ for all $I \subseteq \Delta$.

Proof. For part (a), if $\langle \lambda + \rho, \alpha^\vee \rangle < p$ for all $\alpha \in I$, then λ and 0 belong to the same facet for L_I (cf. [15, II 6]). This implies that $T_{L_I}(\lambda) = H^0_{L_I}(\lambda)$ [15, E.1]. Since λ and 0 belong to the same facet for L_I, we have

$$V_{(L_I)_1}(H^0_{L_I}(\lambda)) = V_{(L_I)_1}(H^0_{L_I}(0)) = V_{(L_I)_1}(k) = \mathcal{N}_1(\mathfrak{l}_I).$$

For part (b), since λ is a weight of $T(\lambda)$, there exists a vector $v_\lambda \in T(\lambda) \downarrow_{L_I}$ of weight λ. Because $T(\lambda) \downarrow_{L_I}$ is a tilting module for L_I, it is the direct sum of indecomposable tilting modules for L_I. Each component of v_λ in this direct sum will be a weight vector of weight λ. Thus we may assume that $v_\lambda \in T_{L_I}(\mu)$ for some $\mu \in X(T)_+$, where $T_{L_I}(\mu)$ is an indecomposable summand of $T(\lambda) \downarrow_{L_I}$. This implies that $\lambda \leq \mu$. Now, similarly, there must be a vector $v_\mu \in T_{L_I}(\mu)$ of weight μ; and since $T_{L_I}(\mu)$ is a direct summand of $T(\lambda) \downarrow_{L_I}$, this means that μ is a weight of $T(\lambda)$. This implies that $\mu \leq \lambda$; so $\mu = \lambda$. □

Theorem 4.1.2. *Let $I \subseteq \Delta$ and $\lambda \in X(T)_+$ with $\Phi_I^+ \subseteq \Psi_\lambda$. Then $G \cdot \mathcal{N}_1(l_I) \subseteq V_{G_1}(T(\lambda))$. Thus, for all I such that $\Phi_I^+ \subset \Psi_\lambda$, $\overline{\mathcal{O}_{\pi(I)}} \subseteq V_{G_1}(T(\lambda))$.*

Proof. First, note that I and λ satisfy the hypothesis of Lemma 4.1.1(a). Using Lemma 4.1.1, we calculate as follows:

$$\mathcal{N}_1(l_I) = V_{(L_I)_1}(T_{L_I}(\lambda)) \subseteq V_{(L_I)_1}(T(\lambda)\downarrow_{L_I}) = V_{G_1}(T(\lambda)) \cap V_{(L_I)_1}(k) \subseteq V_{G_1}(T(\lambda))$$

Since $V_{G_1}(T(\lambda))$ is closed under conjugation by G, the result is proved. □

In particular, if there exists an I such that $\Phi_I^+ \subseteq \Psi_\lambda$ and $c(\lambda) = \pi(I)$, then this theorem shows that $\overline{\mathcal{O}_{c(\lambda)}} \subseteq V_{G_1}(T(\lambda))$. Thus one inclusion of the conjecture is proven in this case. Note that this case obtains for all λ when $n \leq 4$, and appears from initial computations to be a common case for larger n. In general, we have $\bigcup_{i=1}^t \overline{\mathcal{O}_{\pi^i}} \subseteq V_{G_1}(T(\lambda))$ where $\{\pi^i\}$ is a set of pairwise incomparable partitions less than $c(\lambda)$. Each $\pi^i = \pi(I^i)$ for some I^i such that $\Phi_{I^i}^+ \subseteq \Psi_\lambda$. This reduces the problem of finding a lower bound of the support variety to showing that the variety is irreducible. For another reduction of the problem, see Theorem 4.4.1.

4.1.2 Upper Bound

The following proposition is the basis for all the upper bound calculations in this paper.

Proposition 4.1.3. *Let $\lambda, \mu \in X(T)_+$ satisfy $\lambda - \mu \in X(T)_+$. Then $V_{G_1}(T(\lambda)) \subseteq V_{G_1}(T(\mu))$.*

Proof. Consider $T(\mu) \otimes T(\lambda - \mu)$, which is a tilting module over G. If $T(\lambda)$ is a direct summand of this module, then $V_{G_1}(T(\lambda)) \subseteq V_{G_1}(T(\mu) \otimes T(\lambda - \mu))$; this implies the lemma. So it suffices to show that $T(\lambda)$ is a direct summand. There exists a vector in this tensor product of weight $\lambda = \mu + \lambda - \mu$; in particular, there exists $v_\lambda \in T(\gamma)$ of weight λ for some $T(\gamma)$ which is a direct summand of $T(\mu) \otimes T(\lambda - \mu)$. We have $\lambda \leq \gamma$. Since γ is a weight in the tensor product, we can write $\gamma = \gamma_1 + \gamma_2$ where γ_1 is a weight of $T(\mu)$ and γ_2 is a weight of $T(\lambda - \mu)$. So $\gamma_1 \leq \mu$ and $\gamma_2 \leq \lambda - \mu$; this implies that $\gamma \leq \lambda$. So $\lambda = \gamma$. □

4.2 Computations

4.2.1 When $\Psi_\lambda = \Phi_I^+$ for some $I \subseteq \Delta$

Given any partition π of $n+1$, there exists at least one set of simple roots $I \subseteq \Delta$ such that $\pi(I) = \pi$; and given any such set of simple roots I with $\pi(I) \leq (p^d, r)$, there exists a weight λ in $X(T)_+$ such that $\Psi_\lambda = \Phi_I^+$. In fact, if $I \neq \Delta$, then there are infinitely many such λ. So we are considering an infinite class of tilting modules whose highest weights lie in a set of alcoves which intersects every cell region corresponding to partitions $\pi \leq (p^d, r)$. We can prove our conjecture for this class of tilting modules.

Throughout this subsection, let $\lambda \in X(T)_+$, with $\Psi_\lambda = \Phi_I^+$ for some $I \subseteq \Delta$. Thus Ψ_λ is a complete set of positive roots for some Levi subgroup of G.

Theorem 4.2.1. *If $\lambda \in X(T)_+$ and $\Psi_\lambda = \Phi_I^+$, then $V_{G_1}(T(\lambda)) = \overline{\mathcal{O}_{c(\lambda)}}$.*

Proof. We start by showing that $V_{G_1}(T(\lambda)) \subseteq \overline{\mathcal{O}_{c(\lambda)}}$. We will construct a weight $\mu \in X(T)$ satisfying the following properties:

(i) $\lambda - \mu \in X(T)_+$;

(ii) $\Phi_\mu \cong A_{n_1} \times A_{n_2} \times ... \times A_{n_s}$ where $(n_1+1, n_2+1, ..., n_s+1, 1^l) = c(\lambda)^t$;

(iii) $\mu \in X(T)_+$.

For the moment, suppose such a weight exists. Let $S \subseteq \Delta$ be such that $w(\Phi_\mu) = \Phi_S$ for some $w \in W$. Using Theorem 2.4.1, we have $V_{G_1}(H^0(\mu)) = G \cdot \mathfrak{u}_S = \overline{\mathcal{O}_{\pi(S)^t}} = \overline{\mathcal{O}_{c(\lambda)}}$.

Since λ and μ satisfy the hypotheses of Proposition 4.1.3, $V_{G_1}(T(\lambda)) \subseteq V_{G_1}(T(\mu))$. By definition, we have a filtration of $T(\mu)$ with successive subquotients $H^0(\mu_1)$, ..., $H^0(\mu_i)$ where the μ_j are each strongly linked to μ (cf. [15, II 6]). Thus we have $\Phi_{\mu_j} = w_j(\Phi_\mu)$ for $w \in W$ and for all j, so $V_{G_1}(H^0(\mu_j)) = V_{G_1}(H^0(\mu))$ by Theorem 2.4.1. Recall that if $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of H-modules, then for $\sigma \in \Sigma_3$ (the group of permutations of $\{1, 2, 3\}$), we have $V_{H_1}(M_{\sigma(1)}) \subseteq V_{H_1}(M_{\sigma(2)}) \cup V_{H_1}(M_{\sigma(3)})$. By an induction argument, this result applies to finite filtrations, so we have

$$V_{G_1}(T(\mu)) \subseteq V_{G_1}(H^0(\mu)) = \overline{\mathcal{O}_{c(\lambda)}}.$$

Thus it will suffice to show that μ exists.

Consider the notch tableau D_λ. The shape of this tableau gives $c(\lambda)$ reading across the rows, and $c(\lambda)^t$ reading down columns. We build a new tableau E_λ, using the shape of D_λ, but with different numbers inside. Fill the top left box of E_λ with an arbitrary integer x. Now, fill the first row with the consecutive decreasing numbers $x-1, x-2, \ldots$. For the second row, start with $x-p$ in the leftmost box. Note that $x-p$ must be strictly less than the least number on the first row, because no row can have more than p boxes. Fill the second row with the consecutive decreasing numbers $x-p-1, x-p-2, \ldots$. Continue in this fashion: for the ith row, start with $x-(i-1)p$ in the leftmost box, which will be strictly less than the least number in the row above. Fill the row with consecutive decreasing integers, in order.

Now, we have constructed $\mu+\rho-\xi$: this is defined as the vector whose coefficients in the ϵ-basis of $X(T)$ are given by reading the integers across the rows of E_λ in order. Because these numbers are strictly decreasing, it follows that μ is dominant. We have $\Phi_\mu \cong A_{n_1} \times A_{n_2} \times \ldots \times A_{n_s}$, by construction. One can see this by reading down the columns, and remembering that $\langle \mu+\rho, \alpha^\vee \rangle = \langle \mu+\rho-\xi, \alpha^\vee \rangle$ for all $\alpha \in \Phi$.

It remains to show that $\lambda - \mu \in X(T)_+$. This is equivalent to $\lambda+\rho-\xi-(\mu+\rho-\xi) \in X(T)_+$; we can check this condition by looking at inner products with simple roots. We want to show that $\langle \lambda+\rho-\xi-(\mu+\rho-\xi), \alpha_i^\vee \rangle \geq 0$ for all $\alpha_i \in \Delta$, which is equivalent to $\langle \lambda+\rho, \alpha_i^\vee \rangle \geq \langle \mu+\rho-\xi, \alpha_i^\vee \rangle$ for all $\alpha_i \in \Delta$. If $\alpha_i \notin \Psi_\lambda$, then we have $\langle \lambda+\rho, \alpha_i^\vee \rangle \geq p$ by definition of Ψ_λ, and $\langle \mu+\rho-\xi, \alpha_i^\vee \rangle \leq p$ by construction. If $\alpha_i \in \Psi_\lambda$, then we have $\langle \lambda+\rho, \alpha_i^\vee \rangle \geq 1$ because λ is dominant. We claim that $\langle \mu+\rho, \alpha_i^\vee \rangle = 1$, so the condition holds for such α_i. In order to verify the claim, we will prove that if $\alpha_i \in \Psi_\lambda$, then the boxes containing i and $i+1$ are next to each other in the same row of D_λ. This follows from the fact that if $\alpha_i \in \Psi_\lambda$, then the box containing $i+1$ does not start a new row in D_λ, and is placed at the right end of the current bottommost row in D_λ during the construction. We prove this by induction. For $i = 1$, this can be verified from the construction. Now, suppose by induction that this is true for all $i \leq m$ for some m, and suppose that $\alpha_m \in \Psi_\lambda$. Consider the diagram D_λ, with only the boxes containing numbers less than or equal to m. We will show that the box containing $m+1$ is placed at the end of the current bottommost row in D_λ. Let l be the number in the bottom row, leftmost column. By induction, m was placed in the bottom row (possibly $m = l$ started this row), so either $\epsilon_l - \epsilon_m \in \Psi_\lambda$ or $l = m$. Either way, we have $\epsilon_i - \epsilon_{m+1} \in \Psi_\lambda$ for all i in the bottom row, since Ψ_λ is a root system. This means that we cannot place $m+1$ below any boxes on the current bottom row, so it cannot start a new row. Note that by induction, we must have $\epsilon_{l-1} - \epsilon_l \notin \Psi_\lambda$ (otherwise l would have been placed on the previous bottommost row), so $\epsilon_s - \epsilon_{m+1} \notin \Psi_\lambda$ for all $s < l$. This means we can place $m+1$ in the bottom row directly to the right of m. This proves the claim, and allows us to conclude that $\langle \mu + \rho, \alpha_i^\vee \rangle = 1$.

For the lower bound, note that we have $c(\lambda) = \pi(I)$ by the remark before Lemma 3.1.1. Thus, the results of Section 4.1.1 imply that $\overline{\mathcal{O}_{c(\lambda)}} \subseteq V_{G_1}(T(\lambda))$.

□

4.2.2 Hook Partitions

A partition is called a hook partition if it has at most one part bigger than 1. If λ is a weight for which $c(\lambda)$ is a hook partition, we show how the variety of $T(\lambda)$ can be calculated using the results of Section 4.1 and the notch tableau D_λ. The following theorem provides an upper and lower bound for the support variety of *any* indecomposable tilting module. When $c(\lambda)$ is a hook partition, the lower bound and upper bound agree and we have equality.

Theorem 4.2.2. *Let $\lambda \in X(T)$ be such that $c(\lambda) = (c_1, c_2, ..., c_t)$ with $c_t > 0$. Let $\sigma = (n+1-(t-1), 1^{(t-1)})$; let $\tau = (c_1, 1^{(n+1-c_1)})$. Then σ is the least hook partition dominating $c(\lambda)$ and τ is the greatest hook partition dominated by $c(\lambda)$.*

(a) For any $\lambda \in X(T)$, we have $\overline{\mathcal{O}_\tau} \subseteq V_{G_1}(T(\lambda)) \subseteq \overline{\mathcal{O}_\sigma}$.

(b) If $c(\lambda)$ is a hook partition, then $V_{G_1}(T(\lambda)) = \overline{\mathcal{O}_{c(\lambda)}}$.

Proof. Since $c_i \geq 1$ for all $i > 1$, we have $c(\lambda) \geq \tau$, and any hook partition greater than τ must have first part larger than c_1, so τ is the greatest hook partition dominated by $c(\lambda)$. To show that $\sigma \geq c(\lambda)$, we can calculate using the fact that $c_i \geq 1$. We have

$$\sum_{i=1}^{m} c_i = n + 1 - \sum_{i=m+1}^{t} c_i \leq n + 1 - (t - m) = \sum_{i=1}^{m} \sigma_i.$$

Any hook partition which dominates $c(\lambda)$ must not have more parts than $c(\lambda)$, so it must have less than or equal to t parts. The hook partitions are ordered linearly by the dominance ordering, and their order is determined by how many parts they have. Since σ has t parts, it is the least hook partition dominating $c(\lambda)$.

(a) As in the proof of Theorem 4.2.1, for the upper bound it suffices to construct a weight μ which satisfies three conditions:

(i) $\lambda - \mu \in X(T)_+$;

(ii) $\Phi_\mu \cong A_{n_1} \times A_{n_2} \times ... \times A_{n_s}$ where $(n_1 + 1, n_2 + 1, ..., n_s + 1, 1^l) \geq \sigma^t$ (that is, $n_1 + 1 \geq t$);

(iii) $\mu \in X(T)_+$.

We build a new tableau E_λ, using the shape of D_λ as in Theorem 4.2.1, but with a slightly different technique. First, fill down the first column of E_λ with $x, x - p, x - 2p, ...$ To fill in the other boxes, we proceed inductively. Suppose all boxes which correspond in D_λ to numbers less than s have been filled in E_λ, and the box corresponding to s has not yet been filled. Suppose the box corresponding to $s - 1$ is filled with the integer m. Then fill the box corresponding to s with $m - \langle \lambda + \rho, (\epsilon_{s-1} - \epsilon_s)^\vee \rangle$.

Working in this way, we can fill all the boxes. Now, we have constructed $\mu + \rho - \xi$: this is defined as the vector whose coefficients in the ϵ-basis of $X(T)$ are given by reading the integers across the rows of E_λ, in the order given by the tableau D_λ.

We need to verify the three conditions above. The second condition is verified from the construction, as in the proof of Theorem 4.2.1. Note that we only need to look at the first column of E_λ for this. To show the first condition holds, it suffices to show that $\langle \lambda+\rho-\xi-(\mu+\rho-\xi), \alpha_i^\vee\rangle \geq 0$ for all $\alpha_i \in \Delta$. For all $\alpha_i \in \Delta$, we want to show that $\langle \lambda+\rho, \alpha_i^\vee\rangle = \langle \lambda+\rho-\xi, \alpha_i^\vee\rangle \geq \langle \mu+\rho-\xi, \alpha_i^\vee\rangle$. In fact, equality holds unless $i+1$ is in the first column of D_λ. In this case, $i+1 > 1$ so there exists j in the first column of D_λ directly above $i+1$. Observe that $\langle \mu+\rho-\xi, (\epsilon_j-\epsilon_{i+1})^\vee\rangle = p \leq \langle \lambda+\rho, (\epsilon_j-\epsilon_{i+1})^\vee\rangle$. Also, since none of the numbers strictly between j and $i+1$ are in the first column of D_λ, we have $\langle \mu+\rho-\xi, (\epsilon_j-\epsilon_i)^\vee\rangle = \langle \lambda+\rho, (\epsilon_j-\epsilon_i)^\vee\rangle$. We have

$$\begin{aligned}\langle \mu+\rho-\xi, \alpha_i^\vee\rangle &= \langle \mu+\rho-\xi, (\epsilon_j-\epsilon_{i+1})^\vee\rangle - \langle \mu+\rho-\xi, (\epsilon_j-\epsilon_i)^\vee\rangle\\ &= p - \langle \mu+\rho-\xi, (\epsilon_j-\epsilon_i)^\vee\rangle\\ &= p - \langle \lambda+\rho, (\epsilon_j-\epsilon_i)^\vee\rangle\\ &\leq \langle \lambda+\rho, (\epsilon_j-\epsilon_{i+1})^\vee\rangle - \langle \lambda+\rho, (\epsilon_j-\epsilon_i)^\vee\rangle\\ &= \langle \lambda+\rho, \alpha_i^\vee\rangle.\end{aligned}$$

Finally, we need to show that $\mu \in X(T)_+$. We have $\langle \mu+\rho-\xi, \alpha_i^\vee\rangle = \langle \lambda+\rho, \alpha_i^\vee\rangle > 0$ for any i such that $i+1$ is not in the first column of D_λ. If $i+1 > 1$ is in the first column of D_λ, then there exists j in the first column of D_λ directly above $i+1$. We will argue by contradiction to show that in this case $\langle \mu+\rho-\xi, \alpha_i^\vee\rangle > 0$. As above, $\langle \mu+\rho-\xi, (\epsilon_j-\epsilon_{i+1})^\vee\rangle = p$. If $\langle \mu+\rho-\xi, (\epsilon_i-\epsilon_{i+1})^\vee\rangle \leq 0$, then $\langle \mu+\rho-\xi, (\epsilon_j-\epsilon_i)^\vee\rangle \geq p$. Since $\langle \lambda+\rho, (\epsilon_j-\epsilon_i)^\vee\rangle \geq \langle \mu+\rho-\xi, (\epsilon_j-\epsilon_i)^\vee\rangle$, this would mean that $\langle \lambda+\rho, (\epsilon_j-\epsilon_i)^\vee\rangle \geq p$. However, in this case some number smaller than $i+1$ would have been placed directly below j in the construction of D_λ, a contradiction. So μ is dominant.

For the lower bound, we want to apply the results of Section 4.1.1 to imply that $V_{G_1}(T(\lambda)) \supseteq \overline{\mathcal{O}_\tau}$. In order to do this, we need to find a set $I \subseteq \Delta$ such that $\Phi_I^+ \subseteq \Psi_\lambda$, with $\pi(I) = \tau$. Note that if $\tau = (1, 1, ..., 1)$, then we can take $I = \emptyset$. Otherwise, the shape of D_λ has a maximal row (not necessarily unique) with length equal to $c_1 > 1$.

Let j be the number in the rightmost box of this row in D_λ. When this box was placed on the diagram, at the jth step, we compared j to at least $c_1 - 1$ numbers less than j before placing it. Thus, $\langle \lambda + \rho, (\epsilon_{(j-c_1+1)} - \epsilon_j)^\vee \rangle < p$. We can therefore take $I = \{\alpha_{(j-c_1+1)}, ..., \alpha_{j-1}\}$.

(b) In the case that $c(\lambda)$ is a hook partition, we have $c(\lambda) = (c_1, 1, ..., 1) = \tau = \sigma$; so the result of (a) reduces to equality. □

4.2.3 The case $p = 2$

In the introduction, we noted that if $p < h$ then the interior of the alcoves contain no integral weights. As a result of this, certain cell regions will no longer contain integral weights. For example, $C_{\mathbb{R}}$ is always a cell region by itself and will not contain integral weights whenever $p < h$. For smaller p, there will be fewer and fewer cell regions containing integral weights. In fact, the cell regions containing integral weights will be exactly the cell regions corresponding to partitions dominated by (p^d, r). Compare Lemma 3.1.1, and note that these partitions correspond exactly to the orbit closures which can appear as the support varieties of tilting modules.

This behavior limits the possible support varieties of tilting modules for small p, which actually simplifies the problem. In fact, the lower and upper bounds proven in this dissertation are powerful enough to allow us to prove the conjecture when $p = 2$.

Theorem 4.2.3. *When $p = 2$, we have $V_{G_1}(T(\lambda)) = \overline{\mathcal{O}_{c(\lambda)}}$ for all $\lambda \in X(T)_+$.*

Proof. We know that $V_{G_1}(T(\lambda)) \subseteq \mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}_{(2^d,1^r)}}$ where $n + 1 = 2d + r$. All partitions of $n + 1$ which are dominated by $(2^d, 1^r)$ are of the form $(2^m, 1^{(n+1-2m)})$. These partitions are linearly ordered by the dominance ordering. The linear ordering of these partitions implies that $V_{G_1}(T(\lambda))$ must be an irreducible variety. Therefore, $\overline{\mathcal{O}_{c(\lambda)}} \subseteq V_{G_1}(T(\lambda))$ by Theorem 4.1.2 and the remarks afterward.

Write $V_{G_1}(T(\lambda)) = \overline{\mathcal{O}_\tau}$ where τ is a partition of $n + 1$. We have $\tau \geq c(\lambda)$. Write $c(\lambda) = (2^m, 1^{(n+1-2m)})$. By Theorem 4.2.2(a), we have $V_{G_1}(T(\lambda)) \subseteq \overline{\mathcal{O}_\sigma}$ where $\sigma = (m + 1, 1^{(n+1-(m+1))})$. So $V_{G_1}(T(\lambda)) \subseteq \overline{\mathcal{O}_\sigma} \cap \overline{\mathcal{O}_{(2^d,1^r)}}$. Thus, $\tau \leq \sigma$. This means that $\sum_{j=1}^{s} \tau_j \leq$

$\sum_{j=1}^{s} \sigma_j = m + s$ for all s. So when $m < s$, we must have $\tau_s \leq 1$; that is, τ has at least $n + 1 - (2m)$ parts equal to 1. So $\tau \leq c(\lambda)$, and the result is proved. □

4.3 Comparing Conjecture 3.1.2 to Donkin's Conjecture

In [8, §6], S. Donkin states a conjecture for the support varieties of tilting modules for $p = 2$ where the highest weight of the tilting module is a polynomial weight. A weight $\lambda \in X(T)_+$ is a polynomial weight if all coefficients of the weight in the ϵ-basis are nonnegative. We paraphrase the relevant portion of [8, §6]. For $\lambda \in X(T)_+$ where λ is a polynomial weight, let $\bar{\lambda}$ be a partition of $n + 1$ defined as follows: let a_i be the number of coefficients of λ in the ϵ-basis which are equal to i. Then $\bar{\lambda}$ is the partition formed by reordering $(a_0, a_1, ...)$ in nondecreasing order. We say that a partition μ of $n + 1$ is a *refinement* of a partition $\pi = (\pi_1, \pi_2, ..., \pi_t)$ if there exist $\mu^1 \vdash \pi_1, \mu^2 \vdash \pi_2, ..., \mu^t \vdash \pi_t$ such that μ is equal to the concatenation of the μ^i, reordered if necessary.

Conjecture 4.3.1 (Donkin). *Let $p = 2$, and let $\lambda \in X(T)_+$ be a polynomial weight, and μ a partition of $n + 1$. Then $\mathcal{O}_\mu \subseteq V_{G_1}(T(\lambda))$ if and only if μ is a refinement of $\bar{\lambda}$.*

As stated this is is not correct. One can take $G = GL_3$ and $\lambda = (1, 1, 1)$ in the ϵ-basis (i.e. the determinant representation). Then $T(1, 1, 1) = L(1, 1, 1)$ is one-dimensional so $V_{G_1}(T(1, 1, 1)) = \mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}_{(2,1)}}$. But according to Donkin's Conjecture as stated, $\bar{\lambda} = (3)$ and $\mathcal{O}_{(3)} \subseteq V_{G_1}(T(1, 1, 1))$. After an e-mail communication with Donkin, we have the following correction:

Conjecture 4.3.2 (Donkin). *Let $p = 2$, and let $\lambda \in X(T)_+$ be a polynomial weight, and μ a partition of $n+1$. Then $\mathcal{O}_\mu \subseteq V_{G_1}(T(\lambda))$ if and only if μ is a refinement of $\bar{\lambda}$ and all parts of μ are less than or equal to 2.*

We will now show that this corrected conjecture is equivalent to Conjecture 3.1.2 for $p = 2$. This means that Theorem 4.2.3 proves Donkin's Conjecture.

For $x \in \mathbb{Q}$, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Write $\bar{\lambda} = (\pi_1, \pi_2, ..., \pi_t)$; and let

$$m = \sum_{i=1}^{t} \left\lfloor \frac{\pi_i}{2} \right\rfloor .$$

Now define $d(\lambda) = (2^m, 1^r) \vdash n+1$. Note that $d(\lambda)$ is a refinement of $\bar{\lambda}$.

Lemma 4.3.3. *The conclusion of Conjecture 4.3.2 is equivalent to* $V_{G_1}(T(\lambda)) = \overline{\mathcal{O}_{d(\lambda)}}$.

Proof. We want to show that $\mathcal{O}_\mu \subseteq \overline{\mathcal{O}_{d(\lambda)}}$ if and only if μ is a refinement of $\bar{\lambda}$ with all parts less than or equal to 2. We have $\mathcal{O}_\mu \subseteq \overline{\mathcal{O}_{d(\lambda)}}$ if and only if $\mu \leq d(\lambda)$ in the dominance order. First, let $\mu \leq d(\lambda)$. Since $d(\lambda) = (2^m, 1^r)$, we have $\mu = (2^s, 1^q)$ with $s \leq m$. This implies that μ is a refinement of $d(\lambda)$, which implies that μ is a refinement of $\bar{\lambda}$. So μ is a refinement of $\bar{\lambda}$ with all parts less than or equal to 2. Conversely, note that if μ is a refinement of $\bar{\lambda}$ with all parts less than or equal to 2, then $\mu = (2^s, 1^q)$ for some s. Each part π_i of $\bar{\lambda}$ can contribute to μ at most $\lfloor \frac{\pi_i}{2} \rfloor$ parts equal to 2, so $s \leq m$. Therefore $\mu \leq d(\lambda)$. □

Theorem 4.3.4. *Let $p = 2$ and let $\lambda \in X(T)_+$ be a polynomial weight. Then $d(\lambda) = c(\lambda)$. Thus, Donkin's Conjecture is equivalent to Conjecture 3.1.2 for $p = 2$, and Donkin's Conjecture holds by Theorem 4.2.3.*

Proof. We work by induction on n. If $n = 1$, we split into two cases. If $\alpha_1 \in \Psi_\lambda$, then $\langle \lambda + \rho, \alpha_1^\vee \rangle = 1$ since $p = 2$, so $\lambda = s\epsilon_1 + s\epsilon_2$ for some nonnegative integer s. So $d(\lambda) = (2) = c(\lambda)$ in this case. Otherwise, if $\alpha_1 \notin \Psi_\lambda$, then $\lambda = s\epsilon_1 + t\epsilon_2$ for some $s \neq t$. So $d(\lambda) = (1,1) = c(\lambda)$ in this case.

Now, suppose that the conclusion holds for all ranks less than n, and we will show it holds for n. Note that $\Psi_\lambda \subseteq \Delta$ since $p = 2$. We split into two cases again. If $\Psi_\lambda = \Delta$, then $\langle \lambda + \rho, \alpha_i^\vee \rangle = 1$ for all $\alpha_i \in \Delta$. This means that $\lambda = s\sum_{i=1}^{n+1} \epsilon_i$ for some nonnegative integer s. So $\bar{\lambda} = (n+1)$ and $d(\lambda) = (2^d, 1^r) = c(\lambda)$ where $n+1 = 2d + r$, $0 \leq r < 2$.

Otherwise, fix an i such that $\alpha_i \notin \Psi_\lambda$. Set $I = \{\alpha_j : j < i\}$ and $J = \{\alpha_j : j > i\}$. Write $\lambda = (\lambda_1, ..., \lambda_{n+1})$ in the ϵ-basis; let $\lambda_I = (\lambda_1, ..., \lambda_i)$ and let $\lambda_J = (\lambda_i, ..., \lambda_{n+1})$. Note that since $\alpha_i \notin \Psi_\lambda$, we have $\alpha_i \notin K$ for any $K \subseteq \Delta$ such that $\Phi_K^+ \subseteq \Psi_\lambda$. So $c(\lambda)$ is the partition formed by concatenating $c(\lambda_I)$ and $c(\lambda_J)$ and reordering if necessary. Since $\lambda_i \neq \lambda_{i+1}$, $\bar{\lambda}$ is the partition formed by concatenating $\overline{\lambda_I}$ and $\overline{\lambda_J}$ and reordering if necessary; thus $d(\lambda)$ is formed similarly from $d(\lambda_I)$ and $d(\lambda_J)$. By induction, $d(\lambda_I) = c(\lambda_I)$ and $d(\lambda_J) = c(\lambda_J)$. Thus $d(\lambda) = c(\lambda)$. □

4.4 Reduction to the Restricted Region

In this section, we prove that it suffices to look at weights in the restricted region $X(T)_1 = \{\lambda \in X(T) : \langle \lambda, \alpha^\vee \rangle < p \text{ for all } \alpha \in \Delta\}$ in order to calculate the support varieties of all indecomposable tilting modules. This reduces the problem to a finite number of weights for any given n.

Theorem 4.4.1. *If $V_{G_1}(T(\mu)) = \overline{\mathcal{O}_{c(\mu)}}$ for all μ in the restricted region, then $V_{G_1}(T(\lambda)) = \overline{\mathcal{O}_{c(\lambda)}}$ for all $\lambda \in X(T)_+$.*

Proof. Suppose the condition holds for all weights in the restricted region. For the upper bound, let $\lambda \in X(T)_+$ be given, and let μ be a weight in $X(T)_1$ satisfying:

$$\langle \mu + \rho, \alpha_i^\vee \rangle = \begin{cases} \langle \lambda + \rho, \alpha_i^\vee \rangle & \text{if } \alpha_i \in \Psi_\lambda \\ p & \text{if } \alpha_i \notin \Psi_\lambda \end{cases}$$

Note that $\Psi_\mu = \Psi_\lambda$, so $c(\mu) = c(\lambda)$. We have $\lambda - \mu \in X(T)_+$, so by Proposition 4.1.3, $V_{G_1}(T(\lambda)) \subseteq V_{G_1}(T(\mu)) \subseteq \overline{\mathcal{O}_{c(\mu)}} = \overline{\mathcal{O}_{c(\lambda)}}$. Thus, the upper bound holds.

For the lower bound, we will argue by induction on n. For $n = 1$, if $\lambda \notin X(T)_1$ then $c(\lambda) = (1, 1)$ and $\overline{\mathcal{O}_{(1,1)}} = \{0\} \subseteq V_{G_1}(T(\lambda))$. Now, suppose the hypothesis is true for all $s \leq n$, and the conclusion holds for all $s < n$. We will show the conclusion holds for n.

First, note that if $\lambda \in X(T)_+$ but λ is not in the restricted region, then there exists $\alpha_i \in \Delta$ such that $\alpha_i \notin \Psi_\lambda$. Fix such an i. Set $I = \{\alpha_j : j < i\}$ and $J = \{\alpha_j : j > i\}$. Write $\lambda = (\lambda_1, ..., \lambda_n)$ in the fundamental weight basis; let $\lambda_I = (\lambda_1, ..., \lambda_{i-1})$ and let $\lambda_J = (\lambda_{i+1}, ..., \lambda_n)$. Note that since $\alpha_i \notin \Psi_\lambda$, we have $\alpha_i \notin K$ for any $K \subseteq \Delta$ such that $\Phi_K^+ \subseteq \Psi_\lambda$. So $c(\lambda)$ is the partition formed by concatenating $c(\lambda_I)$ and $c(\lambda_J)$ and reordering if necessary.

By our induction hypothesis, $V_{(L_I)_1}(T_{L_I}(\lambda_I))$ contains a nilpotent matrix with Jordan block sizes given by $c(\lambda_I)$. Similarly, $V_{(L_J)_1}(T_{L_J}(\lambda_I))$ contains a nilpotent matrix with Jordan block sizes given by $c(\lambda_J)$. Let $K = I \cup J$; these two results imply that $V_{(L_K)_1}(T_{L_K}(\lambda))$ contains a nilpotent matrix with Jordan block sizes given by the concatenation of $c(\lambda_I)$ and $c(\lambda_J)$. Let A be an $(n+1) \times (n+1)$ nilpotent matrix in Jordan form with block sizes given by the concatenation of $c(\lambda_I)$ and $c(\lambda_J)$.

By Lemma 4.1.1(b), $T_{L_K}(\lambda)$ is a direct summand of $T(\lambda) \downarrow_K$. As in the proof of Theorem 4.1.2, we have

$$V_{G_1}(T(\lambda)) \supseteq G \cdot A = \overline{\mathcal{O}_{c(\lambda)}}.$$

□

4.5 Examples

Throughout this section, we will use the following five examples to demonstrate the various theorems and explain subtleties which arise in proving upper and lower bounds in general. The first three examples are representatives of large classes of weights λ such that $T(\lambda)$ can be computed. In the last two examples, we cannot calculate the support variety of $T(\lambda)$ using the methods in this dissertation.

A word about the notation: we write, for example, $\lambda + \rho = (b_1, b_2, ..., b_n)$; this should be interpreted as "λ is a weight such that $\langle \lambda + \rho, \alpha_i^\vee \rangle = b_i$ for all $1 \leq i \leq n$". This notation allows us to ignore the ξ term.

(4.5.1) The complete subroot system case. Take $p = 5$ and $n = 7$. Let $\lambda_1 + \rho = (4, 5, 2, 2, 7, 3, 5)$. The tilting module $T(\lambda_1)$ is a representative of the largest class of tilting modules for which the variety can be calculated using the methods in this paper.

(4.5.2) The hook partition case. Take $p = 7$ and $n = 7$. Let $\lambda_2 + \rho = (14, 13, 2, 2, 1, 2, 87)$.

(4.5.3) The $p = 2$ case. Take $p = 2$ and $n = 9$. Let $\lambda_3 + \rho = (1, 1, 4, 1, 1, 1, 3, 1, 2)$. This example will be used to demonstrate the general proof for $p = 2$.

(4.5.4.1) A subtle upper bound case. Take $p = 5$ and $n = 4$. Let $\lambda_4 + \rho = (1, 4, 4, 1)$.

(4.5.4.2) A subtle lower bound case. Take $p = 5$ and $n = 5$. Let $\lambda_5 + \rho = (3, 1, 1, 1, 3)$.

4.5.1 Using Tableaux and Complete Root System

Let $S = \Psi_{\lambda_1}$. We have $\Psi_{\lambda_1} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_6, \alpha_3 + \alpha_4\}$. We can find the partition $c(\lambda_1)$ by looking at the structure of Ψ_{λ_1}. If we let $I = \{\alpha_1, \alpha_3, \alpha_4, \alpha_6\}$, then we have $\Psi_{\lambda_1} = \Phi_I^+$. So $c(\lambda_1) = \pi(I) = (3, 2^2, 1)$. We can also get this result using the notch tableau construction given in Section 3.2.2.

Since $\alpha_1 \in S$, the tableau after two steps is

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$$

Since $\epsilon_1 - \epsilon_3 = \alpha_1 + \alpha_2 \notin \Psi_{\lambda_1}$, Condition 3.2.2 will be satisfied if the box containing 3 is put at the bottom of the first column, so this is what we do:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Since $\epsilon_3 - \epsilon_4 = \alpha_3 \in \Psi_{\lambda_1}$, Condition 3.2.2 will not be satisfied if we place the box containing 4 in the first column; however, we can place the box containing 4 in the second column because $\epsilon_2 - \epsilon_4 = \alpha_2 + \alpha_3 \notin \Psi_{\lambda_1}$.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$$

Condition 3.2.2 will not be satisfied if the box containing 5 is placed in either the first or second column. It must be placed starting a new third column and according to the stated construction, we must attach it to the right of the bottommost box in the second column. This gives us:

$$\begin{array}{|c|c|c|}\cline{1-2} 1 & 2 \\ \hline 3 & 4 & 5 \\ \hline\end{array}$$

Continuing in this manner, we can construct D_S; when finished, it looks like this:

$$\begin{array}{|c|c|c|}\cline{1-2} 1 & 2 \\ \hline 3 & 4 & 5 \\ \hline 6 & 7 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array}$$

We can read off the partition from this notch tableau: $\sigma(D_\lambda) = (3, 2^2, 1)$.

Now we can use λ_1 to demonstrate the proof of Theorem 4.2.1. Let $x = 15$, and recall that $p = 5$. Then E_λ is the following notch tableau:

15	14	
10	9	8
5	4	
0		

Thus, $\mu + \rho - \xi = (15, 14, 10, 9, 8, 5, 4, 0)$ in the ϵ-basis, or $\mu + \rho = (1, 4, 1, 1, 3, 1, 4)$ in the notation for this section. The interested reader can check that $\Phi_\mu \cong A_3 \times A_2$; thus $V_{G_1}(T(\lambda_1)) \subseteq \overline{\mathcal{O}_{(4,3,1)^t}} = \overline{\mathcal{O}_{c(\lambda)}}$. For the lower bound, we can apply Theorem 4.1.2 with $I = \Psi_\lambda$.

4.5.2 Hook Partition

Consider $\lambda = \lambda_2$. To calculate the support variety, we will demonstrate the proof of Theorem 4.2.2. Constructing D_λ yields

1			
2			
3	4	5	6
7			
8			

Here $c(\lambda) = (4, 1^4)$. Recall that $p = 7$, and let $x = 35$. Following the proof of Theorem 4.2.2, we have E_λ as

35			
28			
21	19	17	16
14			
7			

Thus, $\mu + \rho - \xi = (35, 28, 21, 19, 17, 16, 14, 7)$ in the ϵ-basis, or $\mu + \rho = (7, 7, 2, 2, 1, 2, 7)$ in the notation for this section. The interested reader can check that $\Phi_\mu \cong A_4$; thus $V_{G_1}(T(\lambda_1)) \subseteq \overline{\mathcal{O}_{(5,1^3)^t}} = \overline{\mathcal{O}_{c(\lambda)}}$. For the lower bound, we can apply Theorem 4.1.2 with $I = \{\alpha_3, \alpha_4, \alpha_5\}$.

4.5.3 The case $p = 2$

Now, consider the case of λ_3. Constructing D_λ yields

1	2
3	
4	5
6	7
8	9
10	

Here $c(\lambda_3) = (2^4, 1^2)$. We have $t = 6$ in Theorem 4.2.2, which gives $V_{G_1}(T(\lambda_3)) \subseteq \overline{\mathcal{O}_{(5,1^5)}}$. In fact, using the fact that $p = 2$, we have

$$\begin{aligned} V_{G_1}(T(\lambda_3)) &\subseteq \overline{\mathcal{O}_{(5,1^5)}} \cap \mathcal{N}_1(\mathfrak{g}) \\ &\subseteq \overline{\mathcal{O}_{(5,1^5)}} \cap \overline{\mathcal{O}_{(2^5)}} \\ &\subseteq \overline{\mathcal{O}_{(2^4,1^2)}} \\ &\subseteq \overline{\mathcal{O}_{c(\lambda_3)}} \end{aligned}$$

by a dominance order calculation. For the lower bound, we can apply Theorem 4.1.2 with $I = \{\alpha_1, \alpha_4, \alpha_6, \alpha_8\}$.

4.5.4

Finally, we demonstrate some of the subtleties involved in extending the results in this dissertation to all indecomposable tilting modules.

Upper Bound

Unfortunately, Proposition 4.1.3 is not strong enough to provide the upper bound we want for all indecomposable tilting modules. Consider the case of λ_4. Constructing D_λ yields

1	2
3	
4	5

Here $c(\lambda_4) = (2^2, 1)$, and $c(\lambda)^t = (3, 2)$. In order to use Proposition 4.1.3 to obtain a tight upper bound on the variety, we would need to find a dominant weight μ such that $\lambda_4 - \mu$ is dominant, with $\Phi_\mu \cong A_2 \times A_1$. A brute force check shows that such a weight does not exist.

Lower Bound

Consider an example where the lower bound given by Theorem 4.1.2 is not quite big enough. Constructing D_λ for $\lambda = \lambda_5$ yields

1	2	3	5
4	6		

We have $c(\lambda_5) = (4, 2)$. Let $I_1 = \{\alpha_2, \alpha_3, \alpha_4\}$ and $I_2 = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5\}$: these satisfy $\Phi^+_{I_i} \subset \Psi_\lambda$ for $i = 1, 2$. Thus Theorem 4.1.2 gives

$$\overline{\mathcal{O}_{\pi(I_1)}} \cup \overline{\mathcal{O}_{\pi(I_2)}} = \overline{\mathcal{O}_{(4,1^2)}} \cup \overline{\mathcal{O}_{(3,3)}} \subseteq V_{G_1}(T(\lambda_5))$$

There is no I such that $\pi(I) = (4, 2)$ and $\Phi^+_I \subseteq \Psi(\lambda_5)$; therefore, we cannot get a tight lower bound from Theorem 4.1.2. Other methods will be needed to calculate the support variety of this tilting module.

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