A contact structure $\xi$ on a three manifold $M$ is a completely non-integrable tangent two-plane distribution. A contact structure is called overtwisted if it contains an embedded disk $D$ such that $\xi$ is everywhere tangent to $D$ along $\partial D$. Eliashberg [6] showed that the topologically interesting case to study is tight contact structures. He did this by showing that the classes of overtwisted contact structures correspond to homotopy classes of two-plane distributions on $M$. The purpose of this work is to classify tight contact structures on $M = \Sigma_2 \times I$ with a specified boundary condition. This is done by applying cut and paste contact topological techniques developed by Honda, Kazez and Matić [15, 18].

**Index words:** Geometric Topology, Tight, Contact Structure, Convexity, Gluing
A Class of Tight Contact Structures on $\Sigma_2 \times I$

by

TANYA COFER

B.S., Augusta College, 1995

A Dissertation Submitted to the Graduate Faculty of The University of Georgia in Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

2003
A Class of Tight Contact Structures on $\Sigma_2 \times I$

by

Tanya Cofer

Approved:

Major Professor: Gordana Matić

Committee: Will Kazez
Clint McCrory
John Hollingsworth
Jason Cantarella

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
May 2003
ACKNOWLEDGMENTS

I would like to dedicate this work to my first and dearest math teacher, my father: John Cofer. Without his patience, support and love, I would have never learned to believe in myself. I would also like to thank my mother, Judith Cofer for her encouragement and for setting a wonderful example of a successful life. I wouldn’t have had the strength without the two of you.

I would also like to thank those who have been there for me with caring and kindness. Thank you to John Hollingsworth, a brilliant teacher whom I love as a member of my own family and who taught me the beauty of topology. Thank you also to the gifted teachers I have been blessed to know: Ted Shifrin, Clint McCrory, Jon Carlson, and Sybilla Beckmann-Kazez. Of course, my sincere gratitude to my major professor, Gordana Matić, for her time and dedication.

Many thanks to my fellow students who have, either directly or indirectly, helped me along my way, especially Daniele Arcara, Eric Pine, Blake Hindman, and Mike Beck. Thank you to Nancy Wrinkle, a great friend and confidant. You’ve kept me sane. Also, love to Beth Thompson, my oldest and dearest friend. You were there before the math.

Finally, I want to thank my partner and best friend, Dory Ruderfer for his enduring love and encouraging words, for putting up with me when I’m terrible, and for all of his indispensable computer help.
# Table of Contents

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>ACKNOWLEDGMENTS</strong></td>
<td>iv</td>
</tr>
<tr>
<td><strong>List of Figures</strong></td>
<td>vi</td>
</tr>
<tr>
<td><strong>Chapter</strong></td>
<td></td>
</tr>
<tr>
<td>1 Introduction, Background Results and Tools</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Convex Surfaces</td>
<td>3</td>
</tr>
<tr>
<td>1.2 Gluing</td>
<td>11</td>
</tr>
<tr>
<td>2 An Overview</td>
<td>17</td>
</tr>
<tr>
<td>2.1 Main Results</td>
<td>19</td>
</tr>
<tr>
<td>3 Characterizations of Potentially Tight Contact structures on $\Sigma_2 \times I$ with a specified configuration on the boundary</td>
<td>21</td>
</tr>
<tr>
<td>4 A Different Convex Decomposition for $(M,\Gamma_{\partial M})$ and a Proof of Tightness for the Unique Non-Product Contact Structure</td>
<td>47</td>
</tr>
<tr>
<td>4.1 A new perspective: a second convex decomposition for $M$</td>
<td>47</td>
</tr>
<tr>
<td>4.2 Existence of a unique non-product tight contact structure on $(M,\Gamma_{\partial M})$</td>
<td>58</td>
</tr>
<tr>
<td><strong>Bibliography</strong></td>
<td>70</td>
</tr>
</tbody>
</table>
List of Figures

1.1 Propeller Picture .................................................. 2
1.2 Overtwisted Contact Structure...................................... 3
1.3 The Dividing Set on a Cutting Surface .............................. 6
1.4 Edge Rounding .......................................................... 9
1.5 Bypass Disk ............................................................ 10
1.6 Abstract Bypass Moves ................................................. 10
1.7 Decreasing $tb(\gamma)$ .................................................. 15
1.8 Folding ................................................................. 15
2.1 $M = \Sigma_2 \times I$ ...................................................... 17
2.2 Classification of $\Gamma_A$ ............................................... 18
3.1 $M = M_1 = M\setminus A$ .............................................. 22
3.2 $\Gamma_A$ of type $T_{2k+1}^+$ ........................................... 23
3.3 $M_2 = M_1 (\delta \times I)$ ............................................... 24
3.4 $\Gamma_A$ of type $T_{2k+1}^+$ ........................................... 25
3.5 $\Gamma_A$ of type $T_{2k+1}^-$ ........................................... 26
3.6 $\Gamma_A$ of type $T_{2k+1}^+$ ........................................... 27
3.7 $\Gamma_A$ of type $T_{2k+1}^+$ ........................................... 27
3.8 $\Gamma_A$ of type $T_{2k+1}^+$ ........................................... 28
3.9 $\Gamma_A$ of type $T_{2k+1}^+$ ........................................... 28
3.10 $\Gamma_A$ of type $T_{2k+1}^+$ ......................................... 29
3.11 Convex decomposition of $M$ when $\Gamma_A = T_{12} (1)$ .......... 29
3.12 Convex decomposition of $M$ when $\Gamma_A = T_{12}$ (2) .................................. 30
3.13 Convex decomposition of $M$ when $\Gamma_A = T_{12}$ (3) .................................. 31
3.14 Convex decomposition of $M$ when $\Gamma_A = T_{12}$ (4) .................................. 31
3.15 Convex decomposition of $M$ when $\Gamma_A = T_{12}$ (5) .................................. 32
3.16 Convex decomposition of $M$ when $\Gamma_A = T_{20}$ ................................. 33
3.17 Convex decomposition of $M$ when $\Gamma_A = T_{22}$ ................................. 34
3.18 Statement of equivalence for non-product tight contact structures ........... 35
3.19 Convex decomposition #1 for $T_{11}$ and $T_{21}^+$ (1) ................................. 36
3.20 Convex decomposition #1 for $T_{11}$ and $T_{21}^+$ (2) ................................. 38
3.21 Convex decomposition #1 for $T_{21}^-$ and $T_{11}$ ................................. 39
3.22 Equivalence of $T_{11}$ and $T_{21}^+$ ................................................................. 41
3.23 Equivalence of $T_{11}$ and $T_{21}^-$ (1) .................................................. 43
3.24 Equivalence of $T_{11}$ and $T_{21}^-$ (2) .................................................. 45
3.25 Equivalence of $T_{21}^+$ and $T_{11}$ ................................................................. 46
4.1 $M = \Sigma_2 \times I$ ................................. 47
4.2 $M_1 = M \setminus A$ ................................................................. 48
4.3 Convex decomposition for $M$ with $\Gamma_A = T_{22+}^{2k+1}$ ......................... 49
4.4 Convex decomposition for $M$ with $\Gamma_A = T_{1-k}$ .................................. 50
4.5 Convex decomposition of $M$ with $\Gamma_A = T_{10}$ (1) ................................. 51
4.6 Convex decomposition of $M$ with $\Gamma_A = T_{10}$ (2) ................................. 52
4.7 $M_1 = M \setminus A$ ................................................................. 52
4.8 $M_1^+ = M_1 \setminus \epsilon$ with $\Gamma_{A_\epsilon} = T_{21}^+$ and possible bypasses ....... 53
4.9 $M_2 = M_1 \setminus \epsilon$ with possible $\Gamma_\epsilon$ ........................................ 54
4.10 Unique Non-Product $M_2 = M_1 \setminus \epsilon$ with $\epsilon = i$ and $ii$ ............. 55
4.11 Equivalence of $M_2 = M_1 \setminus \epsilon$ with $\epsilon = i$ and $ii$ ............. 56
4.12 Non-Equivalence of $M_2 = M_1 \setminus \epsilon$ with $\epsilon = ii$ and $iii$ .......... 57
4.13 A covering space for $T^2 - \nu(pt)$ .................................................. 60
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.14</td>
<td>Fundamental domain for the 3:1 cover of ((\Sigma_2 - \nu(pt)) \times I)</td>
<td>61</td>
</tr>
<tr>
<td>4.15</td>
<td>A covering space for (M_1)</td>
<td>62</td>
</tr>
<tr>
<td>4.16</td>
<td>The pullback of the contact structure</td>
<td>63</td>
</tr>
<tr>
<td>4.17</td>
<td>The pullback of the contact structure #2</td>
<td>64</td>
</tr>
<tr>
<td>4.18</td>
<td>The pullback of the contact structure #3</td>
<td>65</td>
</tr>
<tr>
<td>4.19</td>
<td>Decomposition of (\tilde{M}) into three types of balls</td>
<td>66</td>
</tr>
<tr>
<td>4.20</td>
<td>Some bypass possibilities</td>
<td>67</td>
</tr>
</tbody>
</table>
Chapter 1

INTRODUCTION, BACKGROUND RESULTS AND TOOLS

Let $M$ be a compact, oriented 3-manifold with boundary. A contact structure on $M$ is a completely non-integrable 2-plane distribution (smooth subbundle of the tangent bundle $TM$) $\xi$ given as the kernel of a non-degenerate 1-form $\alpha$ such that $\alpha \wedge d\alpha \neq 0$ at any point of $M$. By Darboux’s theorem, we know that all contact structures on three manifolds locally look like the standard one on $\mathbb{R}^3$ with cylindrical coordinates $(r, \theta, z)$ given as the kernel of $\alpha = dz + r^2d\theta$. This contact structure is invariant in the $z$-direction, and in the plane $z = 0$, starts out tangent at the origin, has radial symmetry, and becomes almost vertical as we move out along radial lines (see figure 1.1). This distribution is generally referred to as the propeller picture and defines what we call the standard contact structure on $\mathbb{R}^3$.

We say $\xi$ is tight if there is no embedded disk $D \subset M^3$ with the property that $\xi$ is everywhere tangent to $D$ along the $\partial D$. Such a $D$ is called an overtwisted disk and contact structures containing such disks are called overtwisted contact structures.

Consider the contact structure indicated in figure 1.2, which is given as the kernel of $\beta = \cos(\pi r^2)dz + \sin(\pi r^2)d\theta$. This contact structure is similar to the one given in the propeller example above, but the planes are allowed to twist through an angle of $2\pi$ for every $\sqrt{2}$ units we move out along radial lines. We can see that this contact structure is overtwisted because it is tangent to the boundary of the disk of radius $\sqrt{2}$ in the plane $z = 0$ centered at the origin. Bennequin [1] showed that the contact structure given by $\alpha = dz + r^2d\theta$ is tight.
In the 1970’s, Lutz and Martinet [22] showed that every closed, orientable three manifold admits a contact structure. Then, in the 1980’s and 1990’s, results of Bennequin [1] and Eliashberg [7] indicated that a qualitative difference exists between the classes of tight and overtwisted contact structures. By the early 1990’s, Eliashberg made it clear to us that the topologically interesting case to study really is tight contact structures [6]. He did this by showing that, up to isotopy, overtwisted contact structures are in one-to-one correspondence with homotopy classes of 2-plane fields on $M$. Soon after, he gave us the classifications of tight contact structures on $B^3$ (a foundational result for the classification of contact structures on three manifolds), $S^3$, $S^2 \times S^1$ and $\mathbb{R}^3$ [7]. Later studies include the classification of tight contact structures on the 3-torus [20], lens spaces [8, 11, 13], solid tori, $T^2 \times I$ [13, 21], torus bundles over circles [11, 14], circle bundles over closed surfaces [12, 14] and $\Sigma_g \times I$ with a particular boundary configuration [16]. Etnyre and Honda [9] exhibited a manifold that carries no tight contact structure, while Colin, Giroux and Honda, Kazez and
Matić proved that a closed 3-manifold supports finitely many isotopy classes of tight contact structures if and only if it is atoroidal [3, 4, 5, 18]. It is Giroux who gave us the theory of convex surfaces which allows us to translate contact topological questions into questions about certain curves on surfaces [10].

1.1 Convex Surfaces

We say that a curve $\gamma$ inside a contact manifold $(M, \xi)$ is Legendrian if it is everywhere tangent to $\xi$. Consider a properly embedded surface $S \subset (M, \xi)$. Generically, the intersection $\xi_p \cap T_p S$ at a point $p \in S$ is a vector $X(p)$. Integrating the vector field $X$ on $S$ gives us the a singular foliation called the characteristic foliation $\xi_S$. The leaves of the characteristic foliation are Legendrian by definition. A surface $S \subset M$ is called convex if there exists a vector field $v$ transverse to $S$ whose flow preserves the contact structure $\xi$. Such a vector field is called a contact vector field. Given a convex surface $S$, we define the dividing set $\Gamma_S = \{x \in S | v(x) \in \xi(x)\}$. Generically, this is
a collection of pairwise disjoint, smooth closed curves (dividing curves) for a closed
surface $S$ or a collection of curves and arcs (where the dividing arcs begin and end
on $\partial S$ if $S$ has boundary [10]). The curves and arcs of the dividing set are transverse
to the characteristic foliation and, up to isotopy, this collection is independent of the
choice of contact vector field $v$. Moreover, the union of dividing curves divides $S$
into positive and negative regions $R_{\pm}$. $R_{+}(R_{-}) \subset \partial M$ is the set of points where the
orientation of $\xi$ agrees (disagrees) with the orientation of $S$. If we take $v$ normal to
$S$, the dividing set is precisely where the contact planes are normal to the tangent
planes on $S$.

Giroux proved that a properly embedded closed surface in a contact manifold can
be $C^\infty$-perturbed into a convex surface [10]. We will refer to this as the perturbation
lemma. If we want to keep track of the contact structure in a neighborhood of a
convex surface, we could note the characteristic foliation. However, the characteristic
foliation is very sensitive to small perturbations of the surface. Giroux Flexibility [10]
highlights the usefulness of the dividing set $\Gamma_S$ by showing us that $\Gamma_S$ captures all of
the important contact topological information in a neighborhood of $S$. Therefore, we
can keep track of the dividing set instead of the exact characteristic foliation.

Given a singular foliation $\mathcal{F}$ on a convex surface $S$, a disjoint union of properly
embedded curves $\Gamma$ is said to divide $\mathcal{F}$ if there exists some $I$-invariant contact structure
$\xi$ on $S \times I$ such that $\mathcal{F} = \xi|_{S \times \{0\}}$ and $\Gamma$ is the dividing set for $S \times \{0\}$.

**Theorem 1.1 (Giroux Flexibility).** Let $S \subset (M^3, \xi)$ be a convex surface in a
contact manifold which is closed or compact with Legendrian boundary. Suppose $S$
has characteristic foliation $\xi|_{S}$, contact vector field $v$ and dividing set $\Gamma_S$. If $\mathcal{F}$ is
another singular foliation on $S$ which is divided by $\Gamma$, then there is an isotopy $\phi_s$, $s$
in $[0,1]$ of $S$ such that

1. $\phi_0(S) = S$
2. $\xi|_{\phi(s)} = \mathcal{F}$

3. $\phi$ fixes $\Gamma$

4. $\phi_s(S)$ is transverse to $v$ for all $s$

The *Legendrian realization principle* specifies the conditions under which a collection of curves on a convex surface $S$ may be realized as a collection of Legendrian curves by perturbing $S$ to change the characteristic foliation on $S$ while keeping $\Gamma_S$ fixed. In general this is not a limiting condition. The result is achieved by isotoping the convex surface $S$ through surfaces that are convex with respect to the contact vector field $v$ for $S$ so that the collection of curves on the isotoped surface are made Legendrian.

Let $C$ be a collection of closed curves and arcs on a convex surface $S$ with Legendrian boundary. We call $C$ *nonisolating* if:

1. $C$ is transverse to $\Gamma_S$.

2. Every arc of $C$ begins and ends on $\Gamma_S$.

3. The elements of $C$ are pairwise disjoint.

4. If we cut $S$ along $C$, each component intersects the dividing set $\Gamma_S$.

An isotopy $\phi_s$, $s \in [0,1]$ of a convex surface $S$ with contact vector field $v$ is called *admissible* if $\phi_s(S)$ is transverse to $v$ for all $s$.

**Theorem 1.2 (Legendrian Realization).** If $C$ is a nonisolating collection of disjoint, properly embedded closed curves and arcs on a convex surface $S$ with Legendrian boundary, there is an admissible isotopy $\phi_s$, $s \in [0,1]$ so that:

1. $\phi_0 = \text{id}$

2. Each surface $\phi_s(S)$ is convex
3. $\phi_1(\Gamma_S) = \Gamma_{\phi_1(S)}$

4. $\phi_1(C)$ is Legendrian

A useful corollary to Legendrian realization was formulated by Kanda [19]:

**Corollary 1.1.** Suppose a closed curve $C$ on a convex surface $S$

1. is transverse to $\Gamma_S$

2. nontrivially intersects $\Gamma_S$

Then $C$ can be realized as a Legendrian curve.

Suppose $\gamma \subset S$ is Legendrian and $S \subset M$ is a properly embedded convex surface. Recall that $\gamma$ intersects $\Gamma_S$ transversely. We define the *Thurston-Bennequin number* $tb(\gamma, Fr_S)$ of $\gamma$ relative to the framing, $Fr_S$, of $S$ to be the number of full twists $\xi$ makes relative to $S$ as we traverse $\gamma$, where left twists are defined to be negative. It turns out that $tb(\gamma, Fr_S) = -\frac{1}{2} \#(\Gamma_S \cap \gamma)$ (see figure 1.3). When $\gamma$ is not a closed curve, we will refer to the *twisting* $t(\gamma, Fr_S)$ of the arc $\gamma$ relative to $Fr_S$.

Given any Legendrian curve $\gamma = \partial S$ with non-positive Thurston-Bennequin number, the following relative version of Giroux’s perturbation lemma, proved by Honda [13], asserts that we can always arrange for the contact planes to twist monotonically in a left-handed manner as we traverse the curve. That is, an annular

![Figure 1.3: The Dividing Set on a Cutting Surface](image-url)
neighborhood $A$ of a Legendrian curve $\gamma$ with $tb(\gamma) = -n$ inside a contact manifold $(M, \xi)$ is locally isomorphic to \( \{ x^2 + y^2 \leq \epsilon \} \subset (\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}, (x, y, z)) \) with contact 1-form $\alpha = \sin(2\pi nz)dx + \cos(2\pi nz)dy$, $n \in \mathbb{Z}^+$ [13, 15]. This is called standard form. Once this is achieved, it is then possible to perturb the surface $S$ so as to make it convex.

**Theorem 1.3 (Relative Perturbation).** Let $S \subset M$ be a compact, oriented, properly embedded surface with Legendrian boundary such that $t(\gamma, Fr_S) \leq 0$ for all components $\gamma$ of $\partial S$. There exists a $C^0$-small perturbation near the boundary which fixes $\partial S$ and puts an annular neighborhood $A$ of $\partial S$ into standard form. Then, there is a further $C^\infty$-small perturbation (of the perturbed surface, fixing $A$) which makes $S$ convex. Moreover, if $v$ is a contact vector field on a neighborhood of $A$ and transverse to $A$, then $v$ can be extended to a vector field transverse to all of $S$.

1.1.1 THE METHOD: DECOMPOSING $(M, \xi)$

We will be analyzing $(M^3, \xi)$ by cutting it along surfaces into smaller pieces and then analyzing the pieces. A familiar way to do this is to cut down $M$ along a sequence of incompressible surfaces $\{S_i\}$ into a union of balls. This is known as a Haken decomposition:

$$M = M_0 \overset{S_1}{\sim} M_1 \overset{S_2}{\sim} \cdots \overset{S_n}{\sim} M_n = \prod B^3$$

To do this in the contact setting, we want the cutting surfaces to be convex with Legendrian boundary. We know, by Giroux flexibility [10], that the contact structure in the neighborhood of a convex surface is determined (up to isotopy) by the dividing set. So, if we keep track of the dividing sets $\Gamma_{S_i}$ on the cutting surfaces, we will also keep track of the contact structure in a neighborhood of the boundary of $M$ union the set of cutting surfaces. Hence, the contact structure is determined on the complement of a union of balls.
Eliashberg proved that there is a unique tight contact structure on $B^3$ with $\partial B^3$ convex and $\#\Gamma_{\partial B^3} = 1$. So, if $(M, \xi)$ is tight, it is uniquely determined by $\Gamma_{\partial M}$ together with the dividing sets on the cutting surfaces.

**Theorem 1.4 (Eliashberg’s Uniqueness Theorem).** If $\xi$ is a contact structure in a neighborhood of $\partial B^3$ that makes $\partial B^3$ convex and the dividing set on $\partial B^3$ consists of a single closed curve, then there is a unique extension of $\xi$ to a tight contact structure on $B^3$ (up to isotopy that fixes the boundary).

Also, Giroux proved that we can determine when neighborhoods of convex surfaces have tight neighborhoods by looking at their dividing sets [10].

**Theorem 1.5 (Giroux’s Criterion).** Suppose $S \neq S^2$ is a convex surface in a contact manifold $(M, \xi)$. There exists a tight neighborhood for $S$ if and only if $\Gamma_S$ contains no homotopically trivial closed curves. If $S = S^2$, then $S$ has a tight neighborhood if and only if $\#\Gamma_{S^2} = 1$.

By Giroux’s criterion, we know that if a homotopically trivial dividing curve appears on a convex surface inside a contact manifold, then our contact structure is overtwisted. If our surface happens to be $S^2$, then more than one dividing curve indicates an overtwisted contact structure.

Define a convex decomposition of $M$ to be:

$$M = M_0 \leadsto M_1 \leadsto \cdots \leadsto M_n = \biguplus B^3$$

such that the cutting surfaces are incompressible $S_1, S_2, \cdots, S_n$ and, at each step, we have a configuration of dividing sets on the cutting surface. We may assume that each $\partial S_i$ is Legendrian by the Legendrian realization principle [10] (see theorem 1.2 and corollary 1.1) and that each $S_i$ is convex [13] (see theorem 1.3). If $\xi$ is tight, Eliashberg’s uniqueness theorem for $B^3$ gives us that contact structure on all of $M$ is uniquely determined by $\Gamma_{\partial M}$ together with the dividing sets on the cutting surfaces.
At each stage of the decomposition, we want to smooth out our cut-open manifold and connect the dividing sets. We do this by making use of a result called edge rounding that tells us what happens to the contact structure when we smooth away corners. A proof of the edge rounding lemma can be found in [13]. A useful and brief discussion of the result can be found in [15]. The following is a summary.

If we consider two compact, convex surfaces $S_1$ and $S_2$ with Legendrian boundary that intersect along a common Legendrian boundary curve, then a neighborhood of the common boundary is locally isomorphic to \( \{ x^2 + y^2 \leq \epsilon \} \subset (\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}, (x, y, z)) \) with contact 1-form $\alpha = \sin(2\pi nz)dx + \cos(2\pi nz)dy$, $n \in \mathbb{Z}^+$ (standard form). We let $A_i \subset S_i$, $i \in \{1, 2\}$ be an annular collar of the common boundary curve. It is possible to choose this local model so that $A_1 = \{ x = 0, 0 \leq y \leq \epsilon \}$ and $A_2 = \{ y = 0, 0 \leq x \leq \epsilon \}$. Then the two surfaces are joined along $x = y = 0$ and rounding this common edge results in the joining of the dividing curve $z = \frac{k}{2n}$ on $S_1$ to $z = \frac{k}{2n} - \frac{1}{4n}$ on $S_2$ for $k \in \{0, \ldots, 2n - 1\}$. See figure 1.4.

Now, we would like to know how $\Gamma_{S_i}$ changes if we cut along a different, but isotopic surface $S'_i$ with the same boundary. Honda and Giroux [11, 13] both studied the changes in the characteristic foliation on a convex surface under isotopy. Honda singled out the minimal, non-trivial isotopy of a cutting surface which he calls a bypass.
A bypass (figure 1.5) for a convex surface $S \subset M$ (closed or compact with Legendrian boundary) is an oriented, embedded half-disk $D$ with Legendrian boundary such that $\partial D = \gamma_1 \cup \gamma_2$ where $\gamma_1$, $\gamma_2$ are two arcs that intersect at their endpoints. $D$ intersects $S$ transversely along $\gamma_1$ and $D$ (with either orientation) has positive elliptic tangencies along $\partial \gamma_1$, a single negative elliptic tangency along the interior of $\gamma_1$, and only positive tangencies along $\gamma_2$, alternating between elliptic or hyperbolic. Moreover, $\gamma_1$ intersects $\Gamma_S$ in exactly the three elliptic points for $\gamma_1$ [13].

![Figure 1.5: Bypass Disk](image)

Isotoping a cutting surface past a bypass disk (figure 1.5) produces a change in the dividing set as shown in figure 1.6. Figure 1.6(A) shows how the dividing set changes if we attach a bypass above the surface. Figure 1.6(B) shows the change in the dividing set if we dig out a bypass below the surface. Formally, we have:

![Figure 1.6: Abstract Bypass Moves](image)
Theorem 1.6 (Bypass Attachment). Suppose $D$ is a bypass for a convex $S \subset M$. There is a neighborhood of $S \cup D$ in $M$ which is diffeomorphic to $S \times [0,1]$ with $S_i = S \times \{i\}$, $i \in \{0,1\}$ convex, $S \times [0,\epsilon]$ is $I$-invariant, $S = S \times \{\epsilon\}$ and $\Gamma_{S_i}$ is obtained from $\Gamma_{S_0}$ as in figure 1.6 (A).

A useful result concerning bypass attachment is the bypass sliding lemma [14, 16]. It allows us some flexibility with the choice of the Legendrian arc of attachment.

Let $C$ be a curve on a convex surface $S$ and let $M = \min\{\#(C' \cap \Gamma_S)| C'$ is isotopic to $C$ on $S\}$. We say that $C$ is efficient with respect to $\Gamma_S$ if $M \neq 0$ and the geometric intersection number $\#(C \cap \Gamma_S) = M$, or, if $M = 0$, then $C$ is efficient with respect to $\Gamma_S$ if $\#(C \cap \Gamma_S) = 2$.

Theorem 1.7 (Bypass Sliding Lemma). Let $R$ be an embedded rectangle with consecutive sides $a, b, c$ and $d$ on a convex surface $S$ so that $a$ is the arc of attachment of a bypass and $b, d \subset \Gamma_S$. If $c$ is a Legendrian arc that is efficient (rel endpoints) with respect to $\Gamma_S$, then there is a bypass for which $c$ is the arc of attachment.

1.2 Gluing

When we do a convex decomposition of $(M^3, \xi)$ and end up with a union of balls, each with a single dividing curve, we know that our contact structure is tight on this cut-open manifold. If we glue our manifold back together along our cutting surfaces, the contact structure may not stay tight. It may be that an overtwisted disk $D \subset M$ intersected one or more of the cutting surfaces, and, hence does not appear at any stage of the decomposition. Honda [15] discovered that, if we take a convex decomposition of an overtwisted contact structure on $M$ and look at all possible non-trivial isotopies (bypasses) of the cutting surfaces $S_i$, we will eventually come upon a decomposition that does not cut through the overtwisted disk.

So, in order to partition the set of contact structures on $M$ into isotopy classes of tight and overtwisted structures, we appeal to a mathematical algorithm (known
as the gluing/classification theorem) due to Ko Honda. We describe Honda's gluing/classification algorithm in the case where \( M \) is \( \Sigma_g \) and \( \xi \) is a contact structure on \( M \) so that \( \partial M \) is convex, since it is this case that is relevant for our work here. One can find a statement of the general case in Honda [15].

Consider the splitting \( M = \Sigma_g \leadsto \Sigma_g \setminus (D_1 \cup D_2 \cup \cdots \cup D_g) = \Pi B^3 = M' \), and let \( C = (\Gamma_1, \Gamma_2, \cdots, \Gamma_g) \) represent a configuration of dividing sets on the cutting disks. We now need to be able to decide if a configuration \( C \) corresponds to a tight contact structure on the original manifold \( M \). We call a configuration \( C \) potentially allowable (i.e. not obviously overtwisted) [13] if \( (M, \Gamma_{\partial M}) \), cut along \( D_1 \cup D_2 \cup \cdots \cup D_g \) with configuration \( C \), gives the boundary of a tight contact structure on each \( B^3 \) (after edge rounding). That is, \( \Gamma_{S^2} \) consists of a single closed curve.

Honda introduced the idea of the state transition and defined an equivalence relation on the set of configurations \( \mathcal{C}_M \) on a manifold \( M \) [15]. This equivalence relation partitions \( \mathcal{C}_M \) into equivalence classes so that each equivalence class represents a distinct contact structure on \( M \), either tight or overtwisted.

We have that a state transition \( C \xrightarrow{st} C' \) exists from a configuration \( C \) to another configuration \( C' \) on \( M \) provided:

1. \( C \) is potentially allowable

2. There is a nontrivial abstract bypass dig (see figure 1.6) from one copy of some \( D_i \) in the cut-open \( M' \) and a corresponding add along the other copy of \( D_i \) that produces \( C' \) from \( C \) (a trivial bypass does not alter the dividing curve configuration).

3. Performing only the dig from item 2) (above) does not change the number of dividing curves in \( \Gamma_{\partial B^3} \) (i.e. the bypass disk exists inside of \( M \)).

Define an equivalence relation \( \sim \) on \( \mathcal{C}_M \) as follows: \( C \sim C' \) if either \( C \overset{st}{\sim} C' \) or \( C' \overset{st}{\sim} C \). Then, an equivalence class under \( \sim \) represents a tight contact
structure provided every configuration in that class is potentially allowable (i.e. no configuration in the equivalence class is obviously overtwisted).

We summarize the section by formally stating Ko Honda’s classification theorem [15]:

**Theorem 1.8 (Honda’s Gluing/Classification Theorem).** Let $M$ be a compact, oriented, irreducible 3-manifold and suppose we have a decomposition of $M$ along a convex, incompressible surface $S \subset M$. If $\partial M \neq \emptyset$, then prescribe $\Gamma_{\partial M}$. Also, if $\partial S \neq \emptyset$, let $\partial S$ be Legendrian with $t(\gamma) < 0$ for each component $\gamma \subset \partial S$. Then the number of tight contact structures on $M$ with the specified dividing set on the boundary is precisely the number of allowable equivalence classes of configurations on $S$.

Sometimes, the gluing/classification theorem does not suffice. For example, any case where there are infinitely many possible dividing sets for a given cutting surface (as is the case for an annulus), there are infinitely many configurations. Checking that every representative of an equivalence class is potentially allowable may very well be impossible. In such a case, we may appeal to another gluing theorem due to Colin [2], Honda, Kazez and Matić [18]. This theorem allows us to conclude tightness of the glued-up manifold when the dividing sets on the cutting surfaces are boundary-parallel. That is, the dividing set consists of arcs which cut off disjoint half-disks along the boundary. We will see, in chapter 4, that we must adapt this theorem in order to complete the classification in this case.

**Theorem 1.9 (Gluing).** Consider an irreducible contact manifold $(M, \xi)$ with nonempty convex boundary and $S \subset M$ a properly embedded, compact, convex surface with nonempty Legendrian boundary such that:

1. $S$ is incompressible in $M$
2. \( t(\delta, Fr_S) < 0 \) for each connected component \( \delta \) of \( \partial S \) (i.e., each \( \delta \) intersects \( \Gamma_{\partial M} \) nontrivially), and

3. \( \Gamma_S \) is boundary-parallel.

If we have a decomposition of \((M, \xi)\) along \( S \), and \( \xi \) is universally tight on \( M \setminus S \), then \( \xi \) is universally tight on \( M \).

Let \( C = (\Gamma_1, \Gamma_2, \ldots, \Gamma_n) \) represent a configuration of dividing sets on the cutting surfaces. We now need to be able to decide if a configuration \( C \) corresponds to a tight contact structure on the original manifold \( M \). One tool we use to help us make this decision is \textit{Giroux’s criterion}.

\textbf{1.2.1 Some Special Bypasses}

Establishing the existence of bypasses requires work. In general, the bypasses along a convex surface \( S \) inside a contact manifold \((M, \xi)\) may be “long” or “deep” (i.e. they exist outside of an \( I \)-invariant neighborhood), and establishing existence requires global information about the ambient manifold \( M \). This is further evidenced by examining that the two Legendrian arcs, \( \gamma_1 \) and \( \gamma_2 \) (where \( \gamma_1 \subset S \)) that comprise the boundary of the bypass half-disk \( B \) (see figure 1.5). When we isotope the convex surface \( S \) past \( B \) to produce a new convex surface \( S' \), we see that \( t(\gamma_1, Fr_S) = -1 \) whereas \( t(\gamma_2, Fr_{S'}) = 0 \). Since a small neighborhood of a point on a Legendrian curve is isomorphic to a neighborhood of the origin in \( \mathbb{R}^3 \) with the standard contact structure, the Thurston-Bennequin number can be increased by one of the two moves in figure 1.7, yet Bennequin’s inequality tells us that \( tb(\gamma, Fr_S) \) is bounded above by the Seifert genus of \( \gamma \). So, although it is easy to decrease twisting number, it not always possible to increase it.

There are two types of bypasses, however, that can be realized locally (in an \( I \)-invariant neighborhood of a convex surface): \textit{trivial} bypasses and \textit{folding} bypasses.
Trivial bypasses are those that do not change the dividing curve configuration. Honda argues existence and triviality of trivial bypasses in [15], lemmas 1.8 and 1.9. Folding bypasses are the result of certain isotopies of a convex cutting surface inside an $I$-invariant neighborhood of the surface.

In general, establishing existence of a bypass involves cutting down the manifold $M$ into a union of 3-balls (see the following section), and invoking Eliashberg’s uniqueness theorem. Since there is a unique tight contact structure on the 3-ball with $\#\Gamma_{\partial B^3} = 1$ (rel boundary), only trivial bypasses exist in this case. Thus, if we can show that a proposed bypass along a convex surface $S \subset M$ is trivial on $B^3$, we can conclude it’s existence.

We say that a closed curve is nonisolating if every component of $S \setminus \gamma$ intersects $\Gamma_s$. A Legendrian divide is a Legendrian curve such that all the points of $\gamma$ are tangencies. The ideas for the following exposition are borrowed from Honda et. al. [18].
To produce a folding bypass, we must start with a nonisolationg closed curve $\gamma$ on a convex surface $S$ that does not intersect $\Gamma_S$. A strong form of Legendrian realization allows us to make $\gamma$ into a Legendrian divide. Then, there is a local model $N = S^1 \times [-\epsilon, \epsilon] \times [-1, 1]$ with coordinates $(\theta, y, z)$ and contact form $\alpha = dz - yd\theta$ in which the convex surface $S$ intersects the model as $S^1 \times [-\epsilon, \epsilon] \times \{0\}$ and the Legendrian divide $\gamma$ is the $S^1$ direction (see figure 1.8).

To fold around the Legendrian divide $\gamma$, we isotope the surface $S$ into an "S"-shape (see the figure) inside of the model $N$. Outside of the model, $S$ and $S'$ are identical. The result is a pair of dividing curves on $S'$ parallel to $\gamma$. In many of the applications that follow, we will be establishing the existence of folding bypasses on solid tori. Note that, on the boundary of a solid torus with $2n$ dividing curves, there are $2n$ Legendrian divides spaced evenly between the dividing curves. Thus, there is always a Legendrian divide located near an existing dividing curve.
Chapter 2

An Overview

We investigate the number of tight contact structures on $M = \Sigma_2 \times I$ with a specific dividing set on the boundary $\partial(\Sigma_2 \times I) = \Sigma_0 \cup \Sigma_1$. The dividing set we specify on the boundary consists of a single separating curve $\gamma_i$ on each $\Sigma_i$, $i \in \{0, 1\}$. These curves are chosen so that $\chi((\Sigma_i)_+^+) = \chi((\Sigma_i)_-^-)$. Here, $(\Sigma_i)_{\pm} i \in \{0, 1\}$ represent the positive and negative regions of $\Sigma_i \setminus \Gamma_{\Sigma_i}$. The position of the $\gamma_i$ is indicated in figure 2.1.

Figure 2.1: $M = \Sigma_2 \times I$

This work extends work done by Honda, Kazez and Matić [16] and is intended as the basis of future calculations on $\Sigma_g \times I$ with a boundary condition specified so that $\chi((\Sigma_i)_+^+) = \chi((\Sigma_i)_-^-)$.

The technique is to provide a convex decomposition of $M$ and partition the set of contact structures into equivalence classes using Honda’s gluing/classification theorem [15]. One difficulty here is that the first cutting surface $(\delta \times I$ as indicated in figure
of the decomposition is an annulus $A$. The infinitely many possibilities for $\Gamma_A$ fortunately fall into four natural categories (see figure 2.2). However, it is not always true that tight contact structures survive regluing. In order to show the existence of a tight contact structure on $M$, we would like to use either the gluing/classification theorem [15] (theorem 1.8) or the gluing theorem [2] (theorem 4.1). However, to apply the former we would have to investigate all possible state transitions of some $\Gamma_A$, being confident at every step that no structure is obviously overtwisted. The later does not apply since our final gluing is not across a boundary-parallel surface. Therefore, we begin by establishing an upper bound for the number of tight contact structures on $(M, \Gamma_{\partial M})$. In chapter four, we will prove a variation of the gluing theorem to suit our situation and establish tightness of two contact structures on $M$. 

In each $\Gamma_A$ category, it is possible to reduce, via state transitions, to a single, possibly tight configuration. It is then possible to merge some categories and show that one entire category is overtwisted. In the end, the set of contact structures partition themselves into three equivalence classes: two possibly tight and one overtwisted.
2.1 Main Results

Chapter three is dedicated to establishing an upper bound for the number of tight contact structures on \((M, \Gamma_{\partial M})\). The first step towards establishing this bound is to prove some reduction lemmas for dividing curve configurations on the annulus. We would like to know that if there is a tight contact structure on \(M\) so that the first cut along the convex annulus \(A\) is of type \(\Gamma_A = T^\pm_{2k} T^\mp_{2k+1}\), \(T^k_1 k > 1\), or \(T^k_1 k < 1\) then there is also one of type \(T^\pm_{1} T^k_1\) and \(T^\pm_{1-1}\), respectively. We also show that all contact structures on \(M\) with \(\Gamma_A\) of type \(T^\pm_{2k}\) are overtwisted. Thus, we know that the number of tight contact structures on \((M, \Gamma_{\partial M})\) is finite. These four lemmas are as follows:

1. Lemma 3.1 \(T^\pm_{2m+1}, m \in \mathbb{Z}^+\) can be reduced to \(T^\pm_{1}\).

2. Lemma 3.2 \(T^k_1 k \in \mathbb{Z}^+\) can be reduced to \(T^1_1\).

3. Lemma 3.3 \(T^k_1 k \in \mathbb{Z}^-\) can be reduced to \(T^1_{1-1}\)

4. Lemma 3.4 All \(T^\pm_{2m}, m \in \mathbb{Z}^+\) are overtwisted.

After proving these lemmas, we are left only to consider the possibility of tight contact structures whose annular dividing sets fall into the categories \(T^\pm_{1}, T^\pm_{1-1}\), and \(T^1_{0}\). We show that, when considered on the cut-open manifold \(M_1 = M\setminus A\), there is a non-product tight contact structure in each of the categories \(T^\pm_{1}, T^\pm_{1-1}\), \(T^1_{1}\), and \(T^1_{1-1}\). Moreover, we can find isotopies transforming each of these structures into one-another:

Theorem 3.1 There exists exactly one potentially tight contact structure on \(M\), not equivalent to \(T^1_{1}\).

Chapter four takes a different convex decomposition of \((M, \Gamma_{\partial M})\), one which facilitates the existence proof of the two tight contact structures on \((M, \Gamma_{\partial M})\). Using
this decomposition and a certain class of tight covers of $M \setminus A$, we can reinterpret the gluing theorem [2] to serve our purposes:

Theorem 4.2 [Main Theorem] There is a unique, non-product, tight contact structure on $M$. 

Chapter 3

Characterizations of Potentially Tight Contact structures on $\Sigma_2 \times I$ with a specified configuration on the boundary

In this chapter, we identify equivalence classes of tight contact structures $\xi$ on $M^3 = \Sigma_2 \times I$ with convex boundary and a specific dividing set $\Gamma_{\partial M}$. To do this, we investigate convex decompositions of $M$, making the first cut along an annulus. By cutting $M$ open along convex surfaces with Legendrian boundary and investigating the number of ways dividing curves can cross the cutting surface, we can establish an upper bound for the number of distinct tight contact structures supported by $(M, \Gamma_{\partial M})$.

First, we specify $\Gamma_{\partial M}$. Take $\Sigma_i$ to be $\Sigma_2 \times \{i\}$ for $i = 0, 1$. We want to define $\Gamma_{\Sigma_i} = \gamma$, where $\gamma$ is an oriented separating curve on $\Sigma$ ($\Sigma \setminus \gamma$ is disconnected) and $\chi((\Sigma_i)_+) = \chi((\Sigma_i)_-)$. Here, $(\Sigma_i)_\pm$, $i \in \{0, 1\}$ are the positive and negative regions of $\Sigma \setminus \Gamma_{\Sigma_i}$. Our choice of $\gamma$ is indicated in figure 2.1. Giroux flexibility [10] assures us that all essential contact topological information is encoded in the dividing set $\Gamma_{\Sigma}$, so choose a characteristic foliation $F$ on $\partial M$ which is adapted [13] to $\Gamma_{\Sigma_0} \cup \Gamma_{\Sigma_1}$ (i.e. choose $F$ so that its associated dividing set is isotopic to the one prescribed on the boundary).

Now, we define the first cut in our convex decomposition of $M$. We want $\delta$ to be an oriented closed curve chosen so that $\#(\delta \cap \gamma_i) = 1$ and $\langle \delta, \gamma_i \rangle = 1$ for $i \in \{0, 1\}$ (see figure 2.1). Let $A$ denote the annulus $\delta \times I$. Legendrian realization allows us to realize $\partial A$ as a Legendrian curve. Moreover, Giroux [10] proved that such an embedded surface $A$ may be transformed via a $C^\infty$-small perturbation so as to make it convex. Cut $M = \Sigma \times I$ along $A$, and assume $A$ has the orientation induced from $\delta$.
The resulting manifold $M_1 = M \setminus A$ contains two copies of the cutting surface $A$. The copy of $A$ in which the induced orientation on $A$ agrees with the boundary orientation of $M \setminus A$ will be called $A_+$. The copy of $A$ in which the induced orientation on $A$ disagrees with the boundary orientation of $M \setminus A$ will be called $A_-$. We will use this convention for all cutting surfaces throughout the chapter. $M_1$ is shown in figure 3.1.

![Figure 3.1: $M = M_1 = M \setminus A$](image)

Honda et. al. [16] also cut across an annulus that intersects $\Gamma_{\partial M}$ in two points on each component. Here, we keep the same conventions and a similar notation. The basic categories of dividing sets on $A$ will be denoted $T_{1k}$ and $T_{2n}^\pm$. Type $T_{1k}$ consists of two parallel, nonseparating dividing curves connecting $\delta \times \{0\}$ and $\delta \times \{1\}$. The value $k \in \mathbb{Z}$ is called the holonomy, and is defined as the spiraling of the dividing set $\Gamma_A$ around the boundary component of $A$ contained in $\Sigma_1$. Let $p_i$, $i \in \{0, 1\}$ be the intersection of $\Gamma_A$ with $\Sigma_0$. Then, type $T_{10}$ is defined to be $\Pi_{i \in \{0, 1\}}(p_i \times [0, 1])$, up to isotopy fixing the boundary. Type $T_{2n}^\pm$, $n \in \mathbb{Z}_{\geq 0}$ consists of two boundary-parallel dividing curves together with an even or odd number of homotopically essential closed curves. Type $T_{2n}^+$ ($T_{2n}^-$) consists of the class of dividing sets $T_{2n}$ such that the arc boundary-parallel to $\Sigma_0$ on $A_+$ becomes a positive (negative) region after edge rounding. Examples of both types are pictured in figure 2.2.
Note that there are an infinite number of possible dividing sets for the annulus \( A \) in each of the categories \( T_{1k} \) and \( T_{2n}^\pm \). The aim of the chapter will be to show that there are exactly two distinct tight contact structures on \( M_1 = M \setminus A \) with the specified boundary dividing curve configuration on \( M \): one of type \( T_{10} \) and a single non-product structure. Therefore, we must show that all of the other possibilities are associated to overtwisted structures or they reduce, via equivalence of tight contact structures, to one of the two aforementioned cases.

In lemma 3.1, we show that if there is a cutting of \( M \) along an annulus \( A \) of type \( T_{2k+1}^\pm, k \geq 1 \), then there is one of type \( T_{1}^\pm \). Similarly, in lemmas 3.2 and 3.3, we show that any cut of type \( T_{k}^\pm \) is equivalent to one of type \( T_{1}^\pm \). In lemma 3.4, we conclude that all cuts of type \( T_{2}^{\pm} \) are associated to overtwisted structures. Finally, in the main theorem of the chapter (theorem 3.1) and its associated lemmas, we prove the existence of a unique, non-product tight contact structure on \( M_1 = M \setminus A \).

**Lemma 3.1.** \( T_{2k+1}^\pm, k \in \mathbb{Z}^+ \) can be reduced to \( T_{1}^\pm \).

**proof.** We assume that there is a tight contact structure on \( M = \Sigma_2 \times I \) with \( \Gamma_A \) of type \( T_{2k+1}^\pm, k \geq 1 \). We want to show that \( A \) can be isotoped so that it is a cut of type \( T_{2k-1}^\pm \) by showing there is a bypass on \( A \) reducing \( T_{2k+1}^\pm \) to \( T_{2k-1}^\pm \). To do this,
we decompose $M_1 = M \setminus A$ by a sequence of cuts along convex disks. We see that we are forced to find the required bypass at some stage of this decomposition. The case $T^2_{2k+1}^-$ is virtually identical, though we rotate all figures by $\pi$.

Consider $M_1 = M \setminus A$. Now, $M_1$ has two copies of $A$ which we will call $A_+$ and $A_-$. Note that we could round edges and isotope the dividing curves on $\partial M_1$ so that they appear as in figure 3.2 (A). However, to see the bypass along $A$ more clearly, we leave them intersecting $A$ as in figure 3.2 (B). We round edges along $\partial A_\pm$ and choose a new cutting surface $\delta \times I$ where $\delta$ is indicated in figure 3.2 (B). Assume $\delta \times I$ is convex with efficient Legendrian boundary and notice that $\delta$ intersects $2k + 1$ curves on $A_+$ and $2k + 3$ curves on $A_-$. After cutting along $\delta$, we have a new manifold $M_2$ which is a punctured torus cross $I$ as pictured in figure 3.3. Two copies of the cutting surface $\delta \times I$ appear in $M_2$. Call them $\epsilon_+$ and $\epsilon_-$. There exist many choices of configurations for $\Gamma_{\epsilon_+}$. However, all but two of these configurations immediately induce bypass we seek.

For clarity of exposition, let us first consider the case $T^2_3^{+}$. In this case, we have three points of intersection of $\epsilon_+$ with $\Gamma_A$ on the left and five on the right. We would like to investigate the possible dividing sets for this cutting surface. Since we wish to deal with tight structures, Giroux’s criterion prohibits consideration of dividing sets on $\epsilon_+$ with homotopically trivial dividing curves. Therefore, the dividing set for
such a cutting disk will consist of four pairwise disjoint dividing curves connecting the boundary of the disk to itself. Moreover, the endpoints (or vertices) of the dividing set must be spaced evenly between the points of intersection of $\partial \epsilon_+$ with $\Gamma_{\partial M_1}$. It is easy to see that there are fourteen possible dividing curve configurations for $\epsilon_+$. The cutting surface $\epsilon_+$ with intersections $\partial \epsilon_+ \cap \Gamma_{\partial M_1}$ labelled 1 through 8 and vertices labelled a through h is given at the top of figure 3.4. The fourteen possible dividing sets are also given in this figure. All but one of the pictured configurations induce one of the bypasses A, B, C or D as pictured in figure 3.5. Isotoping $A$ past any of

Figure 3.4: $\Gamma_A$ of type $T2^+_{2k+1}$
these bypasses immediately reduces $T_{2_3^+}^+$ to $T_{1_1^+}^+$. We would like to isolate the only possibility that does not immediately produce a reduction. The possibilities when $a$ is connected to $b$ are given in $i$ to $v$. When $a$ is connected to $d$, we have possibilities $vi$ and $vii$. When $a$ is connected to $f$, we have possibilities $viii$ and $xiv$. Finally, if $a$ is connected to $h$, we have possibilities $x$ through $xiv$. We see that if vertex $a$ is connected to any vertex other than vertex $f$, we get a configuration straddling position 2, 5, 6 or 7 inducing bypass $A$, $B$, $C$ or $D$, respectively. We are then forced to connect vertex $c$ and $e$. Then, vertex $b$ and $g$ are connected to avoid a dividing curve straddling position 7. Finally, we connect vertex $d$ to vertex $h$. Therefore, the only possibility for $\Gamma_{\epsilon+}$ that does not immediately lead to a reduction is dividing curve configuration $xiv$.

\[ \text{Figure 3.5: } \Gamma_A \text{ of type } T_{2_{2k+1}^+} \]

In the general case $T_{2_{2k+1}}$, the only possibility for $\Gamma_{\epsilon+}$ that does not immediately induce a reducing bypass (by the same type of argument as above) is pictured in figure 3.6. This choice of $\Gamma_{\epsilon+}$ leads to the configuration $(M_2, \Gamma_{\partial M_2})$ as shown in figure 3.7. Continuing our search for a reducing bypass, we cut $M_2$ open along the cutting surface $\delta_1 \times I$ where $\delta_1$ is indicated in figure 3.7.

Assume $\delta_1 \times I$ is convex with efficient Legendrian boundary. After cutting $M_2$ along $\delta_1 \times I$, we obtain $M_3 \approx S^1 \times D^2$ with two copies of the cutting surface $\delta_1 \times I$.
appearing in $M_2$. Call them $\tau_+$ and $\tau_-$. There exist two choices for $\Gamma_{\tau_+}$ as pictured in figure 3.7 that do not have dividing curves straddling positions $2$ to $2k$ or $2k + 3$ to $4k + 1$. Such straddling dividing curves imply bypasses that may be slid so that they appear on $A_+$. Isotoping $M_1$ across these bypasses produces a new dividing set on the isotoped annulus, reducing $T_{2k+1}^+$ to $T_{2k-1}^+$.

Suppose $\Gamma_{\tau_+} = i$. Rounding edges along $\partial r_\pm$ yields $(M_3, \Gamma_{M_3})$ as in figure 3.8 with $\#(\Gamma_{M_3}) = 2k+4$. View $M_3$ as the solid torus in figure 3.9. Cutting open along the convex, meridional cutting surface $\delta_2 \times I$ with efficient Legendrian boundary as pictured, we obtain the ball $(M_4, \Gamma_{\partial M_3})$. Two copies of the cutting surface $\delta_2 \times I$ appear in $M_2$ that we will call $\eta_+$ and $\eta_-$. We have that $(\Gamma_{\partial M_3}) \cap \partial \eta_+ = 2k + 4$ and any dividing curve configuration on $\eta_+$ must contain a dividing curve straddling one of the positions $2$ through $2k + 2$. It is possible to slide the indicated bypass so
Figure 3.8: $\Gamma_A$ of type $T^+_{2k+1}$

that it realized along $A_+ \subset M_1$. Isotoping $M_1$ across these bypasses produces a new dividing set on the isotoped annulus equivalent to $T^+_{2k-1}$. So, in this case we may reduce $T^+_{2k+1}, (k \geq 1)$ to the case $T^+_1$.

Figure 3.9: $\Gamma_A$ of type $T^+_{2k+1}$

Suppose $\Gamma_{\tau_+} = ii$ in figure 3.7. Rounding edges along $\partial \tau_\pm$ yields $(M_3, \Gamma_{M_3})$ as in figure 3.10 with $\#(\Gamma_{M_3}) = 4$. Indicated in figure 3.10 is a possible bypass (also shown in figures 3.2 (B) and 3.7). We have that $M_4 \cong S^1 \times D^2$ and applying the proposed bypass increases $\#(\Gamma_{M_4})$ by two. Thus, the proposed bypass is a folding bypass on the ball. Such bypasses always exist [reference earlier discussion] [18]. Realizing the bypass on $A_- \subset M_1$ produces a new dividing set on the isotoped annulus equivalent to $T^+_{2k-1}$. Thus, in any case we may reduce $T^+_{2k+1}, (k \geq 1)$ to the case $T^+_1$. 
Lemma 3.2. $T_{1k}, k \in \mathbb{Z}^+$ can be reduced to $T_{11}$.

\textit{proof}. We assume that there is a tight contact structure on $M = \Sigma_2 \times I$ with $\Gamma_A$ of type $\Gamma_A = T_{1k}, k \geq 2$. We want to show the that there is also a cut of type $T_{1k-1}$ by showing there is a bypass on $A$ reducing $T_{1k}$ to $T_{1k-1}$. We provide a convex decomposition of $M_1 = M \setminus A$. We see that we encounter the required bypass at some stage of this decomposition.

For the convex decomposition in this case, refer to figures 3.11 through 3.15. Let $M_1 = M \setminus A$ and round edges along $\partial A_{\pm}$. Then choose a new cutting surface
\( \delta \times I \) where \( \delta \) is indicated in figure 3.11 (A). Assume \( \delta \times I \) is convex with efficient Legendrian boundary. After cutting \( M_1 \) open along \( \delta \), we obtain a new manifold \( M_2 \cong (\mathbb{T}^2 - \nu(p)) \times I \) with two copies of the cutting surface that we will call \( \epsilon_+ \) and \( \epsilon_- \). There are many possible dividing curve configurations for \( \Gamma_{\epsilon_+} \). However, all but two of these configurations induce bypass half-disks straddling one of the positions \( 2k - 2 \) or \( 2k + 1 \) through \( 4k - 1 \). Any of these bypasses may be realized on \( A_+ \) or \( A_- \) contained in \( M_1 \) and isotoping \( M_1 \) across these bypasses produces a new dividing set on the isotoped annulus equivalent to \( T1_{k-1} \). The remaining possibility for \( \Gamma_{\epsilon_+} \) is shown in figure 3.11 (B).

Letting \( \Gamma_{\epsilon_+} \) be this choice and rounding edges along \( \partial \epsilon_\pm \), yields \((M_2, \Gamma_{\partial M_2})\) as in figure 3.12 (A). As in the previous case, we choose a new convex cutting surface \( \delta_1 \times I \) with efficient Legendrian boundary. Our choice of \( \delta_1 \times I \) is pictured in figure 3.12 (A). After cutting \( M_2 \) along \( \delta_1 \times I \), we obtain \( M_3 \cong S^1 \times D^2 \) with two copies of the cutting surface that we will call \( \tau_+ \) and \( \tau_- \). There exist many possibilities for \( \Gamma_{\tau_+} \). However all but one of these induce bypass half-disks straddling one of the positions \( 2 \) through \( 2k - 2 \) or \( 2k + 1 \) through \( 4k - 1 \) (see figure 3.12 (B)). Using the bypass sliding lemma, we may slide these bypasses so that they are realized on \( A_+ \subset M_1 \). Isotoping \( M_1 \) across these bypasses produces a new dividing set on the
isotoped annulus equivalent to $T_{1k-1}$. The remaining possibilities for $\Gamma_{\tau_+}$ are given in figure 3.12 (B)).

Suppose $\Gamma_{\tau_+} = i$ as in the figure. After rounding edges along $\partial\tau_\pm$, we obtain $M^3 \cong S^1 \times D^2$ with boundary dividing set consisting of four dividing curves as in figure 3.13. Here we note the existence of a folding bypass whose arc of attachment is indicated in the figure. It is possible to slide this bypass so that it realized along $A_+ \subset M_1$. Isotoping $M_1$ across these bypasses produces a new dividing set on the isotoped annulus equivalent to $T_{1k-1}$ (see figures 3.11 (A) 3.12 (A) and 3.13).

Suppose now that $\Gamma_{\tau_+} = ii$ as in figure 3.12 (B). After rounding edges along $\partial\tau_\pm$, we obtain $M^3$ with boundary dividing set consisting of $2k + 2$ dividing curves as in figure 3.14. If we view $M_3$ as the solid torus in figure 3.15 and cut it open along the convex, meridional cutting surface $\delta_2 \times I$ with efficient Legendrian boundary
as pictured, we obtain the ball \((M_4, \Gamma_{\partial M_4})\). Two copies of the cutting surface \(\delta_1 \times I\) appear in \(M_2\) that we will call \(\eta_+\) and \(\eta_-\). We have that \((\Gamma_{\partial M_2}) \cap \partial \eta_+ = 2k + 2, (k \geq 2)\) and any dividing curve configuration on \(\eta_+\) must contain a dividing curve straddling one of the positions \(2\) through \(2k\). It is possible to \textit{slide} the indicated bypass so that it realized along \(A_- \subset M_1\). Isotoping \(M_1\) across these bypasses produces a new dividing set on the isotoped annulus equivalent to \(T_{1k-1}\). Thus, in any case we may reduce \(T_{1k}, (k \geq 2)\) to the case \(T_{11}\).

\textbf{Lemma 3.3.} \(T_{1k}, k \in \mathbb{Z}^-\) \textit{can be reduced to} \(T_{1-1}\).

\textit{proof.} The proof here is virtually identical to the proof of lemma 3.2. Figures 3.11 through 3.15 may be used for this argument after a reflection through a vertical line passing through the center of each illustration has been performed.

\textbf{Lemma 3.4.} All \(T_{2k}^\pm, k \in \mathbb{Z}^+\) are overtwisted.

\textit{proof.} First, we suppose \(\Gamma_A = T_{20}^\circ\). The case \(T_{20}^\circ\) is virtually identical, though we rotate all figures by \(\pi\). The partial convex decomposition in this case is given in figure 3.16. Let \(M_1 = M \setminus A\), \textit{round edges} along \(\partial A_\pm\) and choose a new cutting
surface $\delta \times I$ that is convex with \textit{efficient} Legendrian boundary (see figure 3.16 (A)). Cutting open $M_1$ along $\delta$ yields a new manifold $M_2 \cong (T^2 - \nu(p)) \times I$ containing two copies of the cutting surface that we will call $\epsilon_+$ and $\epsilon_-$. Note that $tb(\partial \epsilon_\pm) = -1$. Hence, there is a unique possibility for the dividing set on $\epsilon_+$. After taking $\Gamma_\epsilon_+$ to be this possibility and \textit{rounding edges} along $\partial \epsilon_\pm$, we are left with an element of $\Gamma_{\partial M_2}$ bounding an embedded disk in $M_2$ as in figure 3.16 (B). By Giroux's criterion, we conclude the existence of an overtwisted disk. Thus, all $T^{2k}_0$ are overtwisted.

![Figure 3.16: Convex decomposition of $M$ when $\Gamma_A = T^{2k}_0$](image)

Now, suppose $\Gamma_A = T^{2k}_{2m}$, $k \geq 1$. The case $T^{2k}_{2m}$ is virtually identical, though we rotate all figures by $\pi$. The partial convex decomposition in this case is given in figure 3.17. Let $M_1 = M \setminus A$, \textit{round edges} along $\partial A_\pm$ and choose a new convex cutting surface $\delta \times I$ with \textit{efficient} Legendrian boundary as indicated in figure 3.17(A). After cutting along $\delta$, we have a new manifold $M_2$ which is a pair of pants cross $I$. Two copies of the cutting surface $\delta \times I$ appear in $M_2$ that we will call $\epsilon_+$ and $\epsilon_-$. There exist many possible dividing curve configurations for $\Gamma_{\epsilon_+}$. If any of these possibilities are obviously overtwisted, we are done. Suppose they are \textit{potentially allowable}. All but two of these induce bypass half-disks straddling one of the positions positions 2 through $2k-1$ or $2k+2$ through $4k-1$ (see figure 3.17 (B)). We may use these bypasses
Figure 3.17: Convex decomposition of $M$ when $\Gamma_A = T2^-_2$

to reduce to the case $T2^\pm_0$, which is obviously overtwisted. By the gluing/classification theorem (theorem 1.8, [15]), this means that these $T2^\pm_{2k}$ are overtwisted.

Now, if we take $\Gamma_{\epsilon_+} = i$ and round edges along $\partial \epsilon_\pm$, we are left with an element of $\Gamma_{\partial M_2}$ bounding an embedded disk in $M_2$ as in figure 3.17 (C). By Giroux's criterion, we conclude the existence of an overtwisted disk. Similarly, we can establish the existence of an overtwisted disk if we take $\Gamma_{\epsilon_+} = ii$ (see figure 3.17 (D)). We conclude that all $T2^\pm_{2k}$ are overtwisted.

Having shown that if there is a tight contact structure on $M$ and a cut of type $T2_{2k+1}^\pm$, $T1_k$, $k > 1$, or $T1_k$, $k < 1$ then there is also a cut of type $T2_{1}^\pm$, $T1_1$, $T1_{-1}$, respectively. To prove theorem 3.1, We find state transitions on $M_1 = M \setminus A$ and see that there are exactly two tight possibilities that are are potentially allowable. One is of type $T1_0$ and the other is the unique potentially tight non-product structure.
We will use two different convex decompositions (and one slight modification) of $M = \Sigma \times I$. With one of these decompositions, we show that there is a unique structure in each of the four categories. We use a slight modification of this decomposition to show that we can transition from $T_{1_1}$ to $T_{2_1^+}$, from $T_{2_1^-}$ to $T_{1_{-1}}$. With the second convex decomposition, we show that we can transition from $T_{1_1}$ to $T_{2_1^-}$ and from $T_{2_1^+}$ to $T_{1_{-1}}$.

The proof will be divided into five lemmas. In the first lemma, we show the existence of a unique tight contact structure on $M_1$ in each of the four categories. The subsequent lemmas will prove one of the four equivalences. This statement of equivalence is summarized in figure 3.18.

![Figure 3.18: Statement of equivalence for non-product tight contact structures](image)

**Theorem 3.1.** There exists exactly one potentially tight contact structure on $M_1$, not equivalent to $T_{1_0}$, in each of the categories $T_{1_{\pm 1}}$ and $T_{2_{\pm 1}}$.

**proof.** We show that there is exactly one non-product tight contact structure on $M_1$ with $\Gamma_A$ of type $T_{1_1}$. To do this, we provide a convex decomposition of $M = \Sigma \times I$ starting with the convex annulus $A$ together with $\Gamma_A = T_{1_1}$. The convex disks defining this decomposition are given in figures 3.19 and 3.20. The other cases are argued similarly.
Figure 3.19: Convex decomposition #1 for $T_{11}$ and $T_{21}^+$ (1)
Round edges along $\partial A_\pm \subset M_1$ and choose a new cutting surface $\delta \times I$ where $\delta$ is indicated in figure 3.19 (B). Assume $\delta \times I$ is convex with efficient Legendrian boundary. After cutting along $\delta$, we have a new manifold $M_2$ which is a punctured torus cross $I$ (see figure 3.19 (D)). Two copies of the cutting surface $\delta \times I$ appear in $M_2$ that we will call $\epsilon_+$ and $\epsilon_-$. Since $tb(\partial \epsilon_\pm) = -2$, there exist two possible dividing curve configurations for $\Gamma_{\epsilon_\pm}$. One of these configurations induces a bypass half-disk straddling position 3 as shown in figure 3.19 (C). Isotoping $A_+ \subset M_1$ across this bypass produces a new dividing set on the isotoped annulus $A'_\pm$ equivalent to $T_{1_0}$.

Let $\Gamma_{\epsilon_+}$ be the remaining choice. After rounding edges along $\partial \epsilon_\pm$, we obtain $(M_2, \Gamma_{\partial M_2})$ as in figure 3.19 (D). After choosing a new convex cutting surface $\delta_1 \times I$ with efficient Legendrian boundary as in figure 3.19 (D), we cut open $M_2$ and obtain $(M_3 \cong S^1 \times D^2, \Gamma_{\partial M_2})$ with two copies of the cutting surface $\delta_1 \times I$. Call these copies $\tau_+$ and $\tau_-$. Since $tb(\partial \tau_\pm) = -1$, there is exactly one possibility for $\Gamma_{\tau_\pm}$. Applying this configuration to $\tau_+$ and $\tau_-$ and rounding edges along $\partial \tau_\pm$ leads to the configuration $(M_3, \Gamma_{\partial M_3})$ as shown in figure 3.19 (E).

We view $M_3$ as a solid torus and choose a convex, meridional cutting surface $\delta_2 \times I$ with efficient Legendrian boundary as in figure 3.20 (A). We cut open $M_2$ along this surface to obtain $(M_4 \cong B^3, \Gamma_{\partial M_2})$ with two copies of the cutting surface $\delta_1 \times I$. Call these copies $\eta_+$ and $\eta_-$. Since $tb(\partial \eta_\pm) = -2$, there are two possibilities for $\Gamma_{\eta_\pm}$. One of these configurations induces a bypass half-disk straddling position 3 as shown in figure 3.20 (B). Isotoping $A_+ \subset M_1$ across this bypass produces a new dividing set on the isotoped annulus $A'_\pm$ equivalent to $T_{1_0}$.

Let $\Gamma_{\eta_+}$ be the remaining choice. After rounding edges along $\partial \eta_\pm$, we obtain $M_4 \cong B^3$ with $(\Gamma_{\partial M_4}) = 1$. Thus there is at most one non-product tight contact structure on $M_1$ with $\Gamma_A$ of type $T_{1_1}$.

Now, by Eliashberg’s uniqueness theorem, there is a unique universally tight contact structure on $M_4 = B^3$ which extends the one on the boundary. Also, the dividing
sets on the convex disks $\epsilon$, $\tau$ and $\eta$ corresponding to our unique non-product contact structure were all boundary parallel. Therefore, by the gluing theorem [2, 16], there is a unique, universally tight contact structure on $M_1$ with $\Gamma_A$ of type $T_{11}$. Similarly, we can establish the existence of a such a structure structure on $M_1$ with $\Gamma_A$ of types $T_{21}^+$ (given in figures 3.19 (F) through (I) and (C) and 3.20(C) and (D)), $T_{21}^-$ (figure 3.21 (A) through (D)) and $T_{1-1}$ (figure 3.21 (E) through (G) and (C)) . \[\square\]

**Lemma 3.5.** The unique, non-product potentially tight contact structures on $M_1$ of type $T_{1\pm 1}$ and $T_{2\pm 1}$ are all equivalent.

**proof.** We prove the equivalences $T_{11} \cong T_{21}^+$. The others are done similarly. We start by taking a convex decomposition of $M = \Sigma \times I$ starting with the convex annulus $A$ together with $\Gamma_A = T_{11}$. The convex disks defining this decomposition are given in figures 3.19 and 3.22. The purpose is to find a bypass half disk $B$ as indicated.
Figure 3.21: Convex decomposition # 1 for $T_{21}^{-}$ and $T_{11}^{-}$
in figure 3.19 (A) so that digging $B$ from one side of $A$ and adding it to the other transforms $\Gamma_A = T_1$ into $\Gamma_A = T_2^+$. The other cases are argued similarly.

First, consider the partial convex decomposition $M_3 = M_1 \setminus (\epsilon \cup \tau)$ of the unique non-product $T_1$ as in the previous lemma and figures 3.19 (A) (B) (C) (D) and (E). We now proceed with a slightly different decomposition. View $M_3$ as a solid torus and choose a convex, meridional cutting surface $\delta_2 \times I$ with Legendrian boundary. In order to prove existence of the proposed bypass half-disk $B$, our choice of cutting surface shown in figure 3.22 (A) is not efficient.

We observe that the proposed bypass is a folding bypass, and such bypasses always exist [18]. However, it is easy to show existence explicitly in our case by proceeding with the decomposition.

After cutting $M_3$ open along $\delta_2 \times I$, we obtain the ball $(M_4, \Gamma_{\partial M_3})$. Two copies of the cutting surface $\delta_2 \times I$ appear in $M_4$. Call them $\eta_+$ and $\eta_-$. There exist five choices for $\Gamma_{\eta_+}$ as pictured in figure 3.22 (C). Applying choice $i$ or $ii$ of $\Gamma_{\eta_+}$ and rounding edges along $\partial \eta_\pm$ yields a dividing set on $\partial M_4 \approx S^2$ consisting of three dividing curves. By Giroux’s criterion, we conclude the existence of an overtwisted disk. Applying choice $iii$ of $\Gamma_{\eta_+}$ induces a bypass half-disk straddling position 5. Isotoping $A_- \subset M_1$ across this bypass produces a new dividing set on the isotoped annulus $A'_- \approx \partial B$ equivalent to $T_1$.

Both choices $iv$ and $v$ correspond to a dividing curve configuration on $S^2 \approx \partial B^3$ such that $\# \Gamma_{\partial B^3} = 1$ ($\Gamma_{\partial B^3}$ resulting choice $iv$ is illustrated in figure 3.22 (D)). We would like to find a state transition taking dividing curve configuration $iv$ to dividing curve configuration $v$. To do this, we must establish the existence of a bypass half-disk as indicated by it’s attaching arc in choice $iv$ of figure 3.22 (C). Applying choice $iv$ to $\partial M_4 \approx S^2$, we see that this bypass is the trivial one on the ball. Such a bypass is guaranteed to exist by the right to life principle. We conclude that both configurations
Figure 3.22: Equivalence of $T_{11}$ and $T_{21}^+$
represent the same tight contact structure on $M_3$. Thus, we have shown the existence
the proposed bypass $B$ taking $T_{11}$ to $T_{21^+}$

We proceed by performing the same convex decomposition on $M_1$ with $\Gamma_A$ of type $T_{21^+}$. We then argue that digging $B$ yields a state transition transforming the unique non-product $T_{11}$ into the unique non-product $T_{21^+}$. The convex disks defining this decomposition are given in figures 3.19 (F) (G) (C) (H) and (I) and 3.22 (E) (F) and (G).

We proceed exactly as in the previous case, noting that, in the third stage of the decomposition of $M_1$, the boundary of our cutting surface $\eta$ shown in figure 3.22 (E) is not efficient. Since $tb(\partial \eta_{\pm}) = -3$, we will have five possibilities for $\Gamma_{\eta_{\pm}}$. Cases $i$ and $ii$ in figure 3.22 (G) lead to $\#(\Gamma_{\partial B^3}) = 3$, and case $iii$ induces a bypass that can be realized on $A_- \subset M_1$ transforming $T_{21^+}$ into $T_{10}$. Finally, there exists a state transition, as indicated in figure 3.22 (G) $iv$ taking case $iv$ to case $v$.

We will now show that, at each stage of the convex decomposition, digging the bypass along a section of $A_+$ and adding it back along a section of $A_-$ transforms the unique non-product $T_{11}$ into the unique non-product $T_{21^+}$. First, consider $M_1 = M \setminus A$ with $\Gamma_A = T_{11}$ as pictured in figure 3.19 (A). If we dig the bypass along $A_+$ and add it back along $A_-$ as indicated, the result is exactly $M_1 = M \setminus A$ with $\Gamma_A = T_{21^+}$ as pictured in figure 3.19 (F). Second, consider $M_2 = M_1 \setminus (\delta \times I)$ as pictured in figure 3.19 (D). If we dig the bypass along $A_+$ and add it back along $A_-$ as indicated, the result is exactly $M_2 = M_1 \setminus (\delta \times I)$ with $\Gamma_A = T_{21^+}$ as pictured in figure 33.19 (H). Third, consider $M_3 = M_2 \setminus (\delta_1 \times I)$ as pictured in figure 3.19(E). If we dig the bypass along $A_+$ and add it back along $A_-$ as indicated, the result is exactly $M_3 = M_2 \setminus (\delta_1 \times I)$ with $\Gamma_A = T_{21^+}$ as pictured in figure 3.19 (I). Finally, consider $M_4 = M_3 \setminus (\delta_2 \times I)$ as pictured in figure 3.22 (D). If we dig the bypass and add it back as indicated, the result is exactly $M_4 = M_3 \setminus (\delta_2 \times I)$ with $\Gamma_A = T_{21^+}$ with $\#(\Gamma_{\partial M_4}) = 1$. Thus, we have that the unique non-product $T_{11}$ is equivalent to the unique non-product $T_{21^+}$.
Figure 3.23: Equivalence of $T_{1_1}$ and $T_{2_1}^{-}$ (1)
To obtain the equivalence $T_{2_1^-} \cong T_{1_{-1}}$, we use the same convex decomposition as in the previous cases. For the case $M_1$ with $\Gamma_A = T_{2_1^-}$ and $T_{1_{-1}}$, the convex decomposition is given partly in figure 3.21. The rest of the decomposition is as in figure 3.23.

For the equivalences $T_{1_1} \cong T_{2_1^-}$ and $T_{2_1^+} \cong T_{1_{-1}}$, we use a different convex decomposition to avoid cutting through the proposed bypasses. The convex decomposition for the first of these two transitions is given in figures 3.23 and 3.24. For the equivalence $T_{2_1^+} \cong T_{1_{-1}}$, the convex decomposition is given partly in figure 3.25. The rest of the decomposition is as in figure 3.24.

What we have established here is the existence of a unique, non-product tight contact structure on $M_1 = M\setminus A$ and $\Gamma_A = T_{2_1^{\pm 1}}$ or $T_{1_{\pm 1}}$. Unfortunately, there are an infinite number of possible state transitions on $A$, making the use of the gluing/classification theorem prohibitive (see [15] and theorem 1.8). Moreover, since none of the dividing sets ($\Gamma_A = T_{2_1^{\pm 1}}$ or $T_{1_{\pm 1}}$) are boundary-parallel, it is also impossible to apply the gluing theorem (see [2, 16] and theorem 4.1). Therefore, to establish tightness, we will provide an alternate convex decomposition and classification for $(M, \Gamma_{\partial M})$. Using this new decomposition, we will adapt the proof of the gluing theorem so that it applies to our manifold. We may then conclude tightness of the unique, non-product contact structure on $(M, \Gamma_{\partial M})$ (theorem 4.2).
Figure 3.24: Equivalence of $T_{11}$ and $T_{21}^-$ (2)
Figure 3.25: Equivalence of $T2^+_1$ and $T1_{-1}$
Chapter 4

A Different Convex Decomposition for \((M, \Gamma_{\partial M})\) and a Proof of
Tightness for the Unique Non-Product Contact Structure

Figure 4.1: \(M = \Sigma_2 \times I\)

It is the goal of this chapter to prove the tightness of the potentially tight non-
product contact structure of chapter 3. To do this, we provide a different convex
decomposition of \((M, \Gamma_{\partial M})\), one that facilitates the existence proof. Since the exis-
tence of at most two tight contact structures on \((M, \Gamma_{\partial M})\) was detailed in chapter 3,
we give here a brief description of the new decomposition and identify the potentially
tight contact structure from this new perspective. From there, we use the ideas of
the Gluing Theorem for tight contact structures [2, 18] to conclude tightness.

4.1 A NEW PERSPECTIVE: A SECOND CONVEX DECOMPOSITION FOR M

Here we choose our first cut to be \(\delta \times I\) where \(\delta\) is as in figure 4.1. We can assume
that \(\delta \times I\) is convex with Legendrian boundary since \(\delta\) intersects \(\Gamma_{\partial M}\) nontrivially.
$M_1 = M \setminus A = M_1^+ \cup M_1^-$ is given at the top of figure 4.2. Possibilities for $\Gamma_A$ fall into the same categories as before (see figure 2.2). However, it is immediately clear in this case that $\Gamma_A$ of type $T2_k^-$ are overtwisted. We show first that all tight $T2_{2k+1}^+, T1_k$, and $T1_{-k}$ for $k \geq 1$ can be reduced to $T2_1^+, T1_1$, and $T1_{-1}$, respectively.

**Lemma 4.1.** $T2_{2k+1}^+, k \in \mathbb{Z}^+$ can be reduced to $T2_1^+$.

**proof.** Suppose $\Gamma_A = T2_{2k+1}^+ + 1$ with $k \geq 1$, and cut the $A_+$ component of $M_1$ open along $\epsilon = \delta_1 \times I$ where $\delta_1$ is positioned as in figure 4.2. We see that all but two possibilities for $\Gamma_{\epsilon_+}$ contain a dividing curve straddling one of the positions 2 through $2k+2$ or $2k+5$ through $4k+5$ as shown in figure 4.3. Isotoping $A_+$ across any of these bypasses yields a dividing set on the isotoped annulus equivalent to $T2_{2k-1}^+$. The remaining possibilities are pictured. Both possibilities $i$ and $ii$ give us a dividing set on $M_2 \cong S^1 \times D^2 = M_1^+ \setminus \epsilon$ consisting of $2k+4$ longitudinal curves as shown in figure
4.3. If we choose a convex, meridional cutting surface $\eta$, we see, as in figure 3.9, that all possibilities for $\Gamma_{\eta+}$ contain a dividing curve straddling one of the positions 2 through $2k+2$ for $k \geq 1$. Isotoping $A_+$ across any of these bypasses yields a dividing set on the isotoped annulus equivalent to $T^+_{2k+1}$. Thus, $T^\pm_{2k+1}$, $k \in \mathbb{Z}^+$ can be reduced to $T^\pm_{1}$. 

Lemma 4.2. $T_{1k}$, $k \neq 0 \in \mathbb{Z}$ can be reduced to $T_{1\pm 1}$.

Proof. We show that $T_{1k}$, $k \leq -2$ can be reduced to $T_{1-1}$. The proof for positive $k$ is analogous. Consider the proposed bypass indicated in figure 4.4. After rounding edges, we see that this bypass is trivial. Although pictured for $T_{1-2}$, this is the case for all $k \leq -2$. Isotoping $A_+$ across this bypass yields a dividing set on the isotoped annulus equivalent to $T_{1k+1}$. 

Figure 4.3: Convex decomposition for $M$ with $\Gamma_A = T^+_{2k+1}$
Lemma 4.3. There is a unique potentially tight contact structure on $M$ of type $T_{1_0}$, $T_{1_1}$ and $T_{1_{-1}}$. These contact structures are all equivalent on $M$ via state transitions. Moreover, they are universally tight on $M_1 = M \setminus A$

proof. Consider $M_1 = M \setminus A$ with $\Gamma_A = T_{1_0}$ as pictured in figure 4.5. We show the decomposition for the component of $M_1$ containing $A_+$ ($M_1^+$). The argument for $M_1^-$ is virtually identical.

We cut open $M_1^+$ along the convex cutting surface $\epsilon = \delta_1 \times I$ with Legendrian boundary as indicated at the bottom of figure 4.5. Two copies of the cutting surface, $\epsilon_+$ and $\epsilon_-$ appear on the cut-open manifold $M_2 = M_1 \setminus \epsilon$. Since $tb(\partial \epsilon_\pm) = -1$, there is only one tight possibility for $\Gamma_\epsilon$. Assuming this choice for $\Gamma_\epsilon$ and rounding edges along $\partial \epsilon_\pm$, we get $(M_2 \cong S^1 \times D^2, \Gamma_{\partial M_2})$ as in figure 4.6.

We further decompose $M_2$ by cutting along a convex, meridional cutting surface $\eta$. Since $tb(\partial \eta_\pm) = -1$, there is only one tight possibility for $\Gamma_\eta$. Assuming this choice for $\Gamma_\eta$ and rounding edges along $\partial \eta_\pm$, we get $M_3 \cong B^3$ with $\# \Gamma_{\partial M_3} = 1$. By Eliashberg’s uniqueness theorem, there is a unique, universally tight contact structure on $M_3 \cong B^3$ which extends the one on the boundary. Moreover, since the dividing sets on our cutting surfaces $\epsilon$ and $\eta$ are boundary-parallel, we may apply the gluing
Theorem [2, 18] (Theorem 4.1) to conclude the existence of a unique, universally tight contact structure on $M_1$ with $\Gamma_A = T1_0$. If we consider $M_1$ with $\Gamma_{A_\pm} = T1_{\pm 1}$ and round edges along $\partial A_{\pm}$, we see that we obtain $M_1$ with a single homotopically essential closed dividing curve as in the $T1_0$ case (see figure 4.5). Proceeding with the convex decomposition of the $T1_0$ case shoes that there is a unique, universally tight contact structure on $M_1$ with $\Gamma_{A_+} = T1_1$ and another with $\Gamma_{A_+} = T1_{-1}$.

Now we show the equivalence of the tight contact structures on $M_1$ with $\Gamma_{A_+} = T1_0, T1_1$ and $T1_1$. Let us consider $M_1$ with $\Gamma_{A_+} = T1_0$ as in figure 4.7. The proposed bypasses along $A_+$ and $A_-$ are trivial, and, hence, exist. They transform $T1_0$ into $T1_1$ along $A_+$ and into $T1_{-1}$ along $A_-$. These bypasses transform the unique, universally tight contact structure on $M_1$ with $\Gamma_A = T1_0$ into the unique, universally tight contact structure on $M_1$ with $\Gamma_A = T1_1$ and $T1_{-1}$, respectively.
Lemma 4.4. There is a unique, non-product potentially tight contact structure on $M$ of type $T2_1^+$. This contact structure is tight on $M_1 = M \setminus A$.

proof. Here we show the non-product potentially tight contact structure of chapter 3 from the perspective of the current convex decomposition. We will make the argument for the $A_+$ component of $M_1$ ($M_1^+$) since the argument for the other component is completely analogous.

Suppose $\Gamma_A = T2_1^+$. If bypass $a$ indicated in figure 4.8 exists, there is a decomposition of $M$ along an annulus $A'$, isotopic to $A$, with $\Gamma_{A'} = T1_0$. If bypass $b$ indicated in figure 4.8 exists, there is a decomposition of $M$ along an annulus $A''$, isotopic to $A$, with $\Gamma_{A''} = T1_{-1}$. Let $\epsilon = \delta_1 \times I$ be the convex cutting surface with Legendrian boundary indicated in the figure. Cutting $M_1$ open along this cutting surface yields
$M_2 = M_1 \setminus \epsilon$ with two copies $\epsilon_+$ and $\epsilon_-$ of the cutting surface. All but the three choices of $\epsilon_+$ given in figure 4.9 immediately lead to a homotopically trivial dividing curve, and, hence, by Giroux’s criterion, to the existence of an overtwisted disk. The boundary-parallel dividing curves of choice $iii$ may be realized along $A_+$ as bypass $a$ or $b$ transforming $T2^+_1$ into $T1_0$ or $T1_{-1}$. We want to show that there is a unique non-product structure on $M$ with $\Gamma_{\epsilon_+} = i$ and another with $\Gamma_{\epsilon_+} = ii$, and that the state transition indicated in choice $i$ of figure 4.9 takes one to the other. We will also show that the state transition indicated in choice $ii$ of figure 4.9 does not exist taking the non-product structure with $\Gamma_{\epsilon_+} = ii$ into any structure with $\Gamma_{\epsilon_+} = iii$.

By gluing/classification (theorem 1.8 [15]) we can then conclude there is a unique, non-product, potentially tight contact structure on $M$.

Suppose we have $M_2$ with $\epsilon_+ = i$ After rounding edges, we get $M_2 \cong S^1 \times D^2$ with four longitudinal dividing curves (see figure 4.9). By cutting $M_2$ open along a convex, meridional cutting surface $\eta$ as in figure 4.10 (A) yields $M_3 \cong B^3$ with two copies of the cutting surface, $\eta_+$ and $\eta_-$. Since $tb(\partial \eta_+) = -2$, there are two possibilities for $\Gamma_{\eta_+}$ (see figure 4.10 (B)). One choice contains a boundary-parallel dividing arc
Figure 4.9: $M_2 = M_1 \setminus \epsilon$ with possible $\Gamma_{\epsilon}$

straddling position 3. This indicates the existence of a bypass half-disk $B$ that can be realized along $A_\epsilon$. Isotoping $A_\epsilon$ across $B$ produces a dividing set on the isotoped annulus equivalent to $T_1$. Applying the remaining choice and rounding edges leads to $\# \Gamma_{\partial B^3} = 1$ as in figure 4.10 (C). By Eliashberg’s uniqueness theorem, there is a unique extension of this contact structure to the interior of $B^3$. Since the dividing set on $\eta$ is boundary-parallel, we may apply the gluing theorem (theorem 4.1 [2, 18]), we can conclude that this contact structure is tight on $M_2$ with $\epsilon = i$. If we similarly decompose $M_2$ with $\epsilon_+ = ii$ as in the figures 4.10 (D) through (F), we see that there is a unique, non-product, potentially tight contact structure on $M$ that is tight on $M_2$ with $\epsilon_+ = ii$.

Now, consider the proposed state transition from choice $i$ to choice $ii$ of $\epsilon_+$ on $M_2$. The bypass indicated on the left of figure 4.11 (A) is a folding bypass on $M_2 \cong S^1 \times D^2$, and such bypasses always exist [18]. We need to show that if we dig the bypass from $\epsilon_+$ and glue it back along $\epsilon_-$ on $M_2$ and on $M_3 \cong B^3$, we transform the unique non-
product $T_2^+$ with $\Gamma_{\epsilon^+} = i$ into the unique non-product $T_2^+$ with $\Gamma_{\epsilon^+} = ii$. To do this comparison, we need to take slightly different decompositions from the ones we used previously. Our new decomposition for $M_2$ with $\Gamma_{\epsilon^+} = i$ appears in figures 4.11 (A) through (E). Our new decomposition for $M_2$ with $\Gamma_{\epsilon^+} = ii$ appears in figures 4.11 (F) through (I). In order to avoid cutting through the bypass attaching arc, our first cutting surface (see figures 4.11 (B) and (G)) will be a convex cutting surface with Legendrian boundary that is not efficient.
Figure 4.11: Equivalence of $M_2 \approx S^1 \times D^2$ with $\epsilon = i$ and $ii$
Figure 4.12: Non-Equivalence of $M_2 = M_1 \setminus \epsilon$ with $\epsilon = i\bar{i}$ and $i\bar{i}\bar{i}$

We see that digging the bypass from $\epsilon_+ \in M_2$ and gluing it back along $\epsilon_-$ transforms our $M_2$ with $\Gamma_{\epsilon_+} = i$ into our $M_2$ with $\Gamma_{\epsilon_+} = i\bar{i}$ (see figures 4.11 (A) and (F)). By cutting $M_2$ inefficiently along $\eta = \delta_2 \times I$, we have five choices for $\Gamma_{i\eta+}$ as in figures 4.11 (B) and (G). Choices $i$ and $ii$ for both decompositions lead to a homotopically trivial dividing curve and hence, by Giroux’s criterion, an overtwisted disk. In each decomposition, choice $iii$ contains a boundary-parallel dividing curve straddling position 3. We conclude the existence of a bypass half-disk that can be realized on $A_+ \in M_2$ in each decomposition, transforming $\Gamma_{A_+}$ into $T1_0$, $T1_{-1}$ or $T1_1$. Choices $iv$ and $v$ are equivalent by the state transitions indicated in figures 4.11 (D) and (I) and represent the unique non-product $T2_1^+$ with $\Gamma_{\epsilon_+} = i$ and the unique non-product $T2_1^+$ with $\Gamma_{\epsilon_+} = i\bar{i}$, respectively. Digging the state transitioning bypass from a portion of $\epsilon_+ \in M_3 \cong B^3$ and gluing it back along a portion of $\epsilon_-$ as in figure 4.11 (C) transforms the unique non-product structure on $M$ with $\epsilon_+ = i$ and $\eta_+ = iv$ into the unique non-product structure on $M$ with $\epsilon_+ = i\bar{i}$ and $\eta_+ = v$. This establishes equivalence.

We know from the previous decomposition that the unique non-product contact structure on $M$ is tight on $M_2$. We need to see that the other possible state transition taking $\epsilon_+ = i\bar{i}$ to $\epsilon_+ = i\bar{i}\bar{i}$ does not exist (see figure 4.10 (F) and figure 4.12). From our choice of $\eta_+ = i\bar{i}$, we know that there is a bypass on the solid torus $M_2$.
straddling position 4 as in figure 4.10 (E). This is equivalent to adding a bypass straddling position 2 along the outside of the torus (see the Attach=Dig property, p.64-66 of [18]). Since both possibilities cannot exist inside a tight manifold, we can conclude that the state transition from $\epsilon_+ = i$ or $\epsilon_+ = ii$ is the only possible state transition. Thus, by Honda’s gluing/classification theorem [15] (theorem 4.1) that the non-product structure on $M_1$ with $\Gamma_{A_+} = T2^1_+$ is tight on $M_1$. This contact structure is potentially allowable on $M$.

4.2 Existence of a unique non-product tight contact structure on $(M, \Gamma_{\partial M})$

It is now necessary to establish the tightness of the unique potentially allowable contact structure of the previous section. The strategy here will be to use the ideas involved in the proof of the gluing theorem for contact manifolds with convex boundary and boundary-parallel dividing curves on all gluing surfaces. This proof was originally given by Colin [2] and subsequently formulated in terms of convex decompositions by Honda et. al. [18]. We say the dividing set $\Gamma_S$ on a convex surface is boundary-parallel if $\Gamma_S$ is a collection of arcs connecting $\partial S$ to $\partial S$ and this collection of arcs cuts off disjoint half-disks along the boundary of $S$. A contact structure $\xi$ on $M$ is universally tight if the pull-back $(\tilde{M}, \tilde{\xi})$ of the contact structure to any cover of $M$ is tight. The general statement is as follows:

**Theorem 4.1 (Gluing).** Consider an irreducible contact manifold $(M, \xi)$ with nonempty convex boundary and $S \subset M$ a properly embedded, compact, convex surface with nonempty Legendrian boundary such that:

1. $S$ is incompressible in $M$

2. $t(\delta, Fr_S) < 0$ for each connected component $\delta$ of $\partial S$ (i.e., each $\delta$ intersects $\Gamma_{\partial M}$ nontrivially), and
3. \( \Gamma_S \) is boundary-parallel.

If we have a decomposition of \((M, \xi)\) along \(S\), and \(\xi\) is universally tight on \(M\setminus S\), then \(\xi\) is universally tight on \(M\).

There are two main obstacles to applying the gluing theorem directly. First of all, the dividing set \(T2_1^+\) on \(A\) is not boundary-parallel. So, we cannot use the theorem directly to \(M_1\) glued along \(A\). In the gluing theorem, the boundary-parallel requirement is necessary in order to guarantee that any bypass on the gluing surface at most introduces a pair of parallel dividing curves. However, we know, in our case, precisely which bypasses exist along \(A\). They are trivial bypasses and folding bypasses along the central homotopically non-trivial closed dividing curve of \(T2_1^+\), which introduce a pair of dividing curves parallel to the original curve. Therefore, it is possible to establish the conditions necessary to apply the ideas of the gluing theorem and conclude tightness of \((M, \xi)\).

The second obstacle is that it is necessary in our case to use a state transition argument in establishing tightness of the contact structure on \(M_1 = M \setminus A\). This argument relies on Honda’s gluing/classification theorem which guarantees tightness but not universal tightness. Thus, we cannot pull our contact structure back to an arbitrary cover and expect that the structure remains tight. Instead, we will construct explicit covers \(\tilde{M}_i\) of \(M \setminus A\) and compute pull-back structures directly in order to establish tightness of \((\tilde{M}_i, \tilde{\xi})\).

The idea of the proof here, following the proof of the gluing theorem, will be as follows. First, we will construct finite covers of \(M_1\) in which all of the aforementioned folding bypasses are trivial and prove that these covers with the pull-back contact structures are tight. Then, we will assume the existence of an overtwisted disk \(D\) inside \(M\) and look at controlled pull-backs of the bypasses necessary to push \(A\) off of \(D\) to the specified tight covers. In this manner, we will construct a cover \((\tilde{M}, \tilde{\xi})\), a pull-back of \(\tilde{A}\) of \(A\) and a lift \(\tilde{D}\) of the overtwisted disk \(D\). In this cover, all the
bypasses needed to isotope \( \tilde{A} \) off of \( \tilde{D} \) will be trivial. Using tightness of \((\tilde{M}, \tilde{\xi})\), we can derive a contradiction to the existence of \( D \), thereby establishing tightness of \( M \).

4.2.1 Constructing tight covers of the unique, non-product tight contact structure on \( M \setminus A \)

Let us begin by constructing a 3 : 1 cover \( \tilde{S} \) of the punctured torus as in figure 4.13. Since this cover has a single boundary component, it must be a once punctured surface \( \Sigma_g \) for some \( g \in \mathbb{Z}^+ \). An Euler characteristic calculation tells us that this cover \( \tilde{S} \) must be \( \Sigma_2 - \nu(pt) \):

\[
\chi(\tilde{S}) = 1 - 2g = 3(1 - 2(1)) = -3
\]

Figure 4.13: A covering space for \( T^2 - \nu(pt) \)

Thickening each surface by crossing with an interval induces a cover of the punctured torus cross \( I \) by \((\Sigma_2 - \nu(pt)) \times I\). Thus, we have constructed a 3 : 1 cover of
each component \( M_1 = M \setminus A \). Now, we use this construction and the fundamental domain in figure 4.14 to construct a 3 : 1 cover of \((\Sigma_2 - \nu(pt)) \times I\). In this way, we construct a \(3^n : 1\) cover of \(M_1\) for each \(n \in \mathbb{Z}^+\).

![Figure 4.14: Fundamental domain for the 3:1 cover of \((\Sigma_2 - \nu(pt)) \times I\)](image)

We now have a \(m:1\)-fold covering space \((\tilde{M} = \Sigma_{m+1} \times I, p)\) of \(M = \Sigma_2 \times I\) such that \(m = 3^n\) for each \(n \in \mathbb{Z}^+\). The restriction of such a cover to \(M_1\) is the disjoint union of two copies of \((\Sigma_{\frac{m+1}{2}} - \nu(pt)) \times I\) (see figure 4.15). Note that the lift \(\tilde{A}\) of the annulus \(A\) is another annulus (enlarged from the original by a factor of \(m\)). Let \(M_1^+\) denote the \(A_+\) component of \(M \setminus A\). We will establish tightness of \((\tilde{M} \setminus \tilde{A}, \tilde{\xi})\) where \(\xi\) is the unique, non-product tight contact structure on \(M \setminus A\) by focusing on the covering space \((\tilde{M} \setminus \tilde{A}, p|_{p^{-1}(M_1^+)})\). The argument for the other component is completely analogous.

**Lemma 4.5.** \((\tilde{M} \setminus \tilde{A}, \tilde{\xi})\) is a tight contact manifold where \(\xi\) is the unique, non-product tight contact structure on \(M \setminus A\). Similarly, the pullback of the product structure is tight.

**proof.** It suffices to consider \((\tilde{M} \setminus \tilde{A}, p|_{p^{-1}(M_1^+)})\). Our aim is to prove tightness of \((\tilde{M} \setminus \tilde{A}, \tilde{\xi})\) by using Honda’s gluing/classification theorem on the convex decomposition of \((\tilde{M} \setminus \tilde{A}, \tilde{\xi})\) which is the pullback of the one on \(M \setminus A\) (see lemma 4.4 and figures 4.10 and 4.11). We will begin by lifting the cut on \(M \setminus A\) which is transverse to the double-arrow direction as indicated for the 3:1 cover in figure 4.16. Recall that the 3 : 1 cover of \(M_1^+\) is \(\Sigma_2 - \nu(pt)\).

Cutting open along the \(m\) pull-backs of the first cutting disk downstairs yields \(\frac{m-1}{2} + 1\) solid tori. Our covering projection \(p\) restricted to \(\frac{m-1}{2}\) of these tori is a 2:1
Figure 4.15: A covering space for $M_1$

cover, while $p$ restricted to the remaining torus is a 1:1 cover. These $\frac{m-1}{2} + 1$ solid tori fall into three categories according to the way the cutting surfaces appear on them. One of the 2:1 covers always contains the positive and negative copies of one cutting surface, the positive copy of a second cutting surface, and the negative copy of a third. The 1:1 cover always contains the positive copy of a cutting surface and the negative copy of another. The remaining $\frac{m-1}{2} - 1$ 2:1 covers contain copies of four different cutting surfaces, two positive and two negative. Examples of these three categories are given in figure 4.17 (note that no tori of the latter form appear in the decomposition of the 3:1 cover).

Now we begin pulling back the contact structure. At this stage, this means applying dividing curve configuration $i$ to the cutting surfaces (see lemma 4.4 and figure 4.9). The resulting boundary configurations (parallel sets of longitudinal dividing curves on the tori) are given for the 3:1 cover at the top of figure 4.18 along with three new convex cutting disks which are the pull-backs of the second
cut downstairs (transverse to the single arrow direction). Pulling back the contact structure at this level means applying the dividing curve configuration of figure 4.10 (B) \( \tilde{\tau} \) to these disks. The result is \( m \) 3-balls with a single dividing curve each.

This state is not obviously overtwisted (i.e. it is \textit{potentially allowable}). Recall that the state transition \( i \) to \( ii \) along \( \epsilon_+ \) exists downstairs while the state transition \( ii \) to \( iii \) does not. The state transition \( i \) to \( ii \) downstairs corresponds to doing all such transitions along all lifts of \( \epsilon_+ \) upstairs. It is necessary to check that doing any combination of these bypasses upstairs transitions us to a potentially allowable state. These state transitions can be checked explicitly and exist as trivial or folding bypasses on each torus. The results are dividing curve configurations on the tori consisting of either two, four or six parallel, longitudinal dividing curves. The non-existence of the state transition \( ii \) to \( iii \) downstairs does not imply the non-existence
of any such state transition upstairs. However, it can be checked that this possibility never exists. The finiteness of the check is a result of the fact that, for any cover, the pull-back decomposition results in the union of $m = 3^n$ balls (each containing two copies of the pullback of each of the two cutting disks downstairs). These balls fall into three categories. Two of the $m$ balls contain two different types of self-gluing, while the remaining $m - 2$ balls admit no self-gluing (see figure 4.19). Thus, we may conclude, by the gluing/classification theorem, that $(\tilde{M} \setminus \tilde{A}, \tilde{\xi})$ is tight.

Suppose the pullback $\tilde{\xi}$ is the pullback of the product structure. The cutting disks remain the same, but $\#\Gamma_{\tilde{A}} = 1$. Thus, there is only one possibility for the dividing set of the pullbacks of the first cutting surface as in figure 4.17. This possibility leaves us with $\#\Gamma_{\tilde{T}_2} = 2$ (two longitudinal dividing curves) on each of the $\frac{m-1}{2} + 1$ solid tori upstairs. We further decompose $(\tilde{M} \setminus \tilde{A}, \tilde{\xi})$ by pulling back the second cutting surface downstairs as in figure 4.18. There is only one possible dividing set on these cutting surfaces. The result is $m$ 3-balls with a single dividing curve each. Since all cutting
Figure 4.18: The pullback of the contact structure #3

surfaces had boundary-parallel dividing curves, the gluing theorem [2, 16] applies. Thus, the glued-up manifold \((\tilde{M} \setminus \tilde{A}, \tilde{\xi})\) is tight.

\(\square\)

4.2.2 Interpreting the gluing theorem to prove tightness of the unique non-product contact structure on \(M\)

The proof of the standard gluing theorem (theorem 4.1 [18]) depends on two facts that we don’t have:

1. The dividing sets on cutting surfaces are boundary-parallel.

2. The contact structure \(\xi\) is universally tight on \(M \setminus S\) where \(S\) is our cutting surface.

However, we have classified our contact structures on \(M \setminus A\) and know which bypasses are possible. By reviewing the proof of the gluing theorem, we see that the requirement that \(\Gamma_S\) be boundary-parallel is there so that the types of bypasses
possible along $S$ are strictly limited to those which are trivial or “long”. In our case the types of bypasses possible along $A$ with $\Gamma_A = T2_1^+$ inside $(M, \xi)$ are also limited (where $\xi$ is the unique non-product structure). We can conclude the following lemma.

**Lemma 4.6.** Let $A$ be a convex annulus whose boundary is as indicated in figure 4.1 with $\Gamma_A = T2_1^+$ or $T1_0$ inside the unique non-product $(M, \xi)$. Then, any convex annulus $A'$ that is obtained from $A$ by a sequence of bypass moves will have a dividing set $\Gamma_{A'}$ differing from $\Gamma_A$ by possibly adding an even number of parallel, closed dividing curves encircling the inner boundary component $\delta \times 1$ of $A$.

**proof.** We argue the case $\Gamma_A = T2_1^+$. The case $\Gamma_A = T1_0$ is argued identically. Recall that, for the non-product structure on $M \setminus A$ with $\Gamma_A = T2_1^+$, the bypasses indicated along $A_+$ at the left of figure 4.8 and their counterparts along $A_-$ do not exist. These bypasses also do not exist in any of the covers constructed in the previous section for similar reasons (see figure 4.18 and lemma 4.4). By examining all remaining possibilities, we see that the only bypasses that exist along $A$ in this case are trivial bypasses and folding bypasses. Further bypasses may be trivial (i.e. they produce no change in the dividing set), may add pairs of parallel dividing curves, or may delete pairs of dividing curves. Examples of these possibilities are given in figure 4.20. So,
the dividing set on any annulus $A'$ obtained from $A$ by a sequence of bypass moves has dividing set $\Gamma_{A'}$ that differs from $A$ by adding, at most, pairs of homotopically non-trivial closed dividing curves parallel to the original one.

Figure 4.20: Some bypass possibilities

To address our second problem, let us consider the strategy of the proof of the gluing theorem. The proof is by contradiction. The existence of an overtwisted disk $D \subset M$ is supposed, and we consider the sequence of bypass moves required to isotope $A$ off of $D$. The requirement that $\xi$ is universally tight on $M \setminus A$ is so that, when analyzing a single bypass move along $A$ (which is trivial or increases $\# \Gamma_A$), we can lift to a large enough cover so as to make this bypass trivial. We continue to lift to covers until we arrive at the cover $\tilde{M}$ of $M$ with lifts $\tilde{A}$ of $A$ and a lift $\tilde{D}$ of a proposed overtwisted disk $D$ in which all bypasses needed to isotope $\tilde{A}$ off of $\tilde{D}$ (producing the isotoped annulus $\tilde{A}'$) are trivial. But $\Gamma_{\tilde{A}'} = \Gamma_{\tilde{A}}$ where $\tilde{A}'$ is obtained from $\tilde{A}$ by a sequence of trivial bypass attachments. Since $\xi$ is universally tight on $M \setminus A$, this gives us a contradiction.

The objective of the previous section was to construct specific covers of $\tilde{M}$ so that $\tilde{M} \setminus \tilde{A}$ is tight. These covers suffice to satisfy the requirements of the gluing theorem. Thus, we are able to establish tightness of $(M, \xi)$.

We repeat here two lemmas from the proof of the gluing theorem [18]. The first one concerns isotoping $A$ off of an overtwisted disk $D$. By a result of Honda [15], we may perturb the characteristic foliation on the cutting surface and look at the local model near points on the boundary of the disk. We can arrange for $D$ to be transverse
to $A$ and for $\partial D$ to be contained in $\Gamma_A$. Moreover, after possibly modifying $D$, we can assume that the hypotheses of the Legendrian realization principle are satisfied so that $D \cap A$ consists of Legendrian arcs and curves. This is called the “controlled intersection” of an overtwisted disk with a convex cutting surface $A$.

Now, to push $A$ away from $D$ so that we eliminate a closed curve of intersection $\delta$. Let $D_\delta$ denote the subdisk of $D$ with $\partial D_\delta = \delta$. Since we assume $A$ has a tight neighborhood, we must have that $t(\delta, Fr_A) < 0$. We may perturb $D_\delta$ (rel boundary) so as to make it convex with Legendrian boundary.

We have that $\hat{D}_\delta$ is contained in $M \setminus A$, which is tight. So we may assume that the dividing set of $D_\delta$ consists only of embedded arcs with endpoints on $\partial D$. We may push $A$ to engulf the bypass which corresponds to one of these boundary-parallel dividing curves on $D_\delta$. We continue until all bypasses are consumed.

For an arc of intersection $\gamma$ in $D \cap A$, we proceed similarly, but we choose a disk $D_\gamma$ with $\gamma \subset \partial D_\gamma$. This concludes the sketch of the following lemma:

**Lemma 4.7.** It is possible to isotope $A$ off of $D$ in a finite number of steps, each of which is a bypass along $A$.

The second lemma concerns the effect of isotoping cutting surfaces across trivial bypasses. A trivial bypass along a convex surface $A$ can be realized inside an $I$-invariant (contact “product”) neighborhood of the surface.

**Lemma 4.8.** If $A$ is a convex surface with Legendrian boundary inside a contact manifold $(M, \xi)$ and $A'$ is a convex surface obtained from $A$ by a trivial bypass, then $A$ and $A'$ are contact isotopic and, hence, $(M \setminus A, \xi)$ is tight if and only if $(M \setminus A', \xi)$ is tight.

We are now ready to prove an adaption of the gluing theorem. Much of this proof is identical to the general proof.
Theorem 4.2. The product structure is tight on $M$. Moreover, there is a unique, non-product, tight contact structure on $M$.

Proof. We prove the existence of a unique, non-product tight contact structure on $M$. The proof for the product structure is identical.

Assume that $M$ is not tight. Then, there exists an overtwisted disk $D \subset M$. We can perform a contact isotopy so that $D$ and $A$ intersect transversely along Legendrian curves and arcs and so that $\partial D \cap A \subset \Gamma_A$ [15]. We note that closed curves in $D \cap A$ are homotopically trivial in $A$. We want to eliminate the innermost closed curves on $D$ by pushing $A$ across $D$. Consider a two-sphere $S$ formed by a disk on $D$ and one on $A$ whose common boundary is an innermost curve of intersection $\delta \subset D$. Then, $S$ bounds a ball across which we can isotope $A$.

By lemma 4.7, we can push $A$ across $D$ to decrease the number of intersections of $A$ with $D$ in a finite number of bypass steps that possibly change the dividing curve configuration on the isotoped annulus. If we consider a single bypass along $A$, we know it is either trivial or increases $\Gamma_A$ (see the proof of lemma 4.6). Recall that the covering spaces constructed in the last section enlarge $A$ by a factor of $m$. So we can lift to a large enough cover of this type so that any folding bypass attachment becomes trivial. We can continue lifting through covers of this type until we arrive at the cover $\tilde{M}$ of $M$ with lifts $\tilde{A}$ of $A$ and $\tilde{D}$ of a proposed overtwisted disk $D$ in which all bypasses needed to isotope $\tilde{A}$ off of $\tilde{D}$ are trivial. But $\Gamma_{\tilde{A}'} = \Gamma_{\tilde{A}}$ where $\tilde{A}'$ has $\tilde{A}' \cap \tilde{D} = \emptyset$ and is obtained from $\tilde{A}$ by a sequence of trivial bypass attachments. Since $\tilde{\xi}$ is tight on $M \setminus \tilde{A}$, then, by lemma 4.8, $\tilde{\xi}$ is tight on $M \setminus \tilde{A}'$. This contradicts the existence of an overtwisted disk in $M \setminus \tilde{A}'$. Thus, $(M, \xi)$ is tight. □
Bibliography


