MIDDLE SCHOOL STUDENTS’ SENSE-MAKING OF ALGEBRAIC SYMBOLS AND
CONSTRUCTION OF MATHEMATICAL CONCEPTS USING SYMBOLS*

by

JEONG-LIM CHAE

(Under the Direction of JOHN OLIVE)

ABSTRACT

The purpose of this study was to investigate how students constructed meaning for algebraic symbols and mathematical concepts with symbols in relation to narrative, tabular and graphical representations. This study focused on four seventh-grade students who were beginning to learn algebra with the mathematical context of representing changing situations with two variables in relation to each other.

Kaput’s (1991) referential relationship model guided the present study as the major theory in combination with the other complementary theories such as the Structure of Observed Learning Outcomes (SOLO) taxonomy (Biggs & Collis, 1982) and the procept model (Tall et al., 2001) to investigate students’ referential relationships.

This study was conducted within the activities of an ongoing project, Coordinating Students’ and Teacher’s Algebraic Reasoning (CoSTAR), funded by the National Science Foundation. The four participating students were selected from Ms. Moseley’s 7th grade class, who participated in one of the case studies of the CoSTAR project. Data were collected in the

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form of videotaped interviews with pairs of students based on classroom activities. The method for analysis of videotaped data was informed by iterative videotape analysis.

The results of the data analysis mainly explained the process and the nature of students’ referential relationships according to the three bi-directional ways of referential relationships centered around algebraic notations: algebraic ↔ narrative, algebraic ↔ tabular, and algebraic ↔ graphical. Going from algebraic to the other forms of representation appeared more difficult than the other way around. Also, students’ conceptions of variables and rates in the referential relationships played an explanatory role.

The conclusion of this study raised relevant issues for understanding students’ referential relationships. The issues included students’ appreciation of representing changing situations in various forms of representation in mathematics, their understanding of algebraic equations as an abstract form of representation, and their conceptions of variables and rates in the referential relationships. These issues also suggest instructional implications for teachers and mathematics educators to help enhance students’ understanding. Some implications for future research were also discussed.

INDEX WORDS: Algebraic symbolism, Algebraic reasoning, Multiple representations, Referential relationship, Middle school mathematics
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To my parents for their love and support
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CHAPTER I
INTRODUCTION

Background

Whenever we think of something and express or communicate about it, we need some tools. In doing algebra, symbols provide one such tool with which we can think of and communicate about our thoughts and ideas. Not only are symbols a tool for representation, they have also played a critical role in developing algebra. If we consider generality as what makes algebra most different from arithmetic, then the beginning of algebra is historically traced back to ancient Mesopotamia and Egypt. In spite of almost four thousand years of history of algebra, the history of modern algebraic symbols did not begin until the 16th century. It was François Viète who first used symbols purposefully and systematically after some mathematical symbols (e.g. +, −, =) were introduced with letters used for unknowns (Kline, 1972).

Before Viète, algebraic ideas were stated rhetorically, and special words, abbreviations, and number symbols were used as notations. For instance, Diophantus, as a syncopated algebraist, created numeral expressions with Greek letters of \(10 - x\) and \(10 + x\), and multiplied them to get \(100 - x^2\) in order to find two numbers such that their sum and product were 20 and 96, respectively (Sfard, 1995, p.19). Although he appeared to use the letter in a general way to solve similar problems, he denoted the letter \(x\) as the unknown solution with the given numbers in the problem. According to Sfard (1995), the mathematicians in the pre-Vietan era including Diophantus adopted symbols to explain “their computational method through concrete numerical examples rather than by universal prescriptions” (p. 20). Therefore, they were believed to
conceive expressions like $100 - x^2$ as only a procedural representation without a structural perspective, which would have enabled them to consider it as the product of the computational process.

While algebra had developed toward a science of generalized numerical computations in the pre-Vietan era, the way that Viète used letters made algebra develop toward a science of universal computations. By symbolizing numerical givens, Viète provided his contemporary mathematicians with a tool to seek general solutions for equations. Sfard (1995) quoted Viète’s differentiation between arithmetic and algebra stating, “whereas arithmetic is the science of concrete numbers (logica numerosa), his type of algebra is a science of species (logica speciosa) or of types of things rather than of the things themselves” (p. 24). Moreover, Viète’s idea was followed by more novel mathematical ideas of variables and functions, and the structural aspects of algebra became highlighted as opposed to the process.

Although algebra became developed more structurally thanks to Viète’s symbolism, it still had limitations in that algebra had dealt with the rules governing numerical operations, and it needed a logical basis to overcome the limitations. The efforts to get over the limitations, what Sfard (1995) called “the dearithmetization of algebra”, included the introduction of negative numbers and complex numbers, which made it possible to do subtraction with any numbers and take square roots of any numbers under the same algebraic rules, respectively (p. 29). That is, the invention of algebraic objects regardless of ontological origin became permitted within the laws of logic. Therefore, by loosening its connection with numbers and with numerical computations, modern algebra has developed as a science of abstract structures.

The aforementioned historical development of algebra suggested that algebra has been alienated from numerical ideas and required more structural thinking than procedural thinking.
Likewise, algebraic symbols are no longer simple substitutions for numbers and numerical operations but the objects to be studied. Moreover, it was algebraic symbolism that made it possible for algebra to develop into abstract structures. However, for young learners, symbolism is one of the major difficulties in learning algebra.

Hiebert et al. (1997) explained that the difficulties in dealing with symbols as a learning tool were attributed to the fact that “meaning is not inherent” in symbols (p. 55). The authors insisted that meaning was not attached to symbols automatically, and that without meaning symbols could not be used effectively. They suggested that students should construct meaning for and with symbols as they actively use them. Kieran (1992) also elaborated that symbolic language made algebra more powerful and applicable by eliminating “many of the distinctions that the vernacular preserves” and inducing the essences (p. 394). However, the powerful yet decontextualized language brought difficulties for young learners who were beginning to learn algebra:

Thus, the cognitive demands placed on algebra students included, on the one hand, treating symbolic representations, which have little or no semantic content, as mathematical objects and operating upon these objects with processes that usually do not yield numerical solutions, and, on the other hand, modifying their former interpretations of certain symbols and beginning to represent the relationships of word-problem situations with operations that are often the inverse of those that they used almost automatically for solving similar problems in arithmetic (Kieran, 1992, p. 394).

Whereas Hiebert et al. (1997) and Kieran (1992) have pointed out students’ difficulties in dealing with symbols, The National Council of Teachers of Mathematics’ (2000) Algebra Standard encouraged using symbols as a tool to represent and analyze mathematical situations and structures in all grade levels. In particular, students in Grades 6 – 8 are recommended to have

… extensive experience in interpreting relationships among quantities in a variety of problem contexts before they can work meaningfully with variables and symbolic
expressions. An understanding of the meanings and uses of variables develops gradually as students create and use symbolic expressions and relate them to verbal, tabular, and graphical representations. Relationships among quantities can often be expressed symbolically in more than one way, providing opportunities for students to examine the equivalence of various algebraic expressions (p. 225-226).

In this recommendation, NCTM put emphasis on using problem contexts to help students develop meaning for symbols and appreciate quantitative relationships. However, in order to inform mathematics teachers and educators and implement the recommendation successfully in classrooms, more studies on students’ experiences with symbolism are necessary to be added to the body of current literature of the area.

Research Questions

In line with the issues mentioned above, the present study intended to provide insight into students’ experiences with symbolism. In particular, the educational purpose of this study was to inform mathematics teachers and educators of how students construct meaning for algebraic symbols and learn mathematical concepts with symbols so that they will be able to enhance students’ learning of mathematics with symbols.

In fact, studies on students’ learning experience with algebraic symbolism appeared abundant relative to studies on the other areas of students’ learning experience. Some research focused on algebraic symbolism itself to study students’ difficulties in manipulating symbols as mathematical objects and modifying their interpretations of symbols (e.g. Stacey & MacGregor, 1997) and to investigate how meaning for symbols could be developed (e.g. Booth, 1988; Kieran 1981; Küchemann, 1978). These were examples of studies investigating how students drew meaning of symbols from inside of the symbol systems (Hiebert & Carpenter, 1992). The other way to develop meaning of symbols was by connecting them with other forms of representation. Hiebert and Carpenter (1992) explained that meaning development through connecting within
symbols was mostly accomplished by recognizing patterns within the symbol system. On the other hand, when symbols were connected to other forms of representation the meaning of the symbols came out of the meaning that the learners had already constructed for the representations. The authors warned us about acquiring meaning by considering concrete materials as other forms of representation:

In order for symbols to acquire meaning, learners must connect their mental representations of written symbols with their mental representations of concrete materials. The potential for these connections to create understanding is complicated by the fact that the concrete materials themselves are representations of mathematical relationships and quantities. Thus, the usefulness of concrete materials as referents for symbols depends both on their embodiments of mathematical relationships and on their connections to written symbols (p. 72).

They also insisted that symbols could serve two functions based on two types of connection. One was a public function in which symbols were used as a communication tool to convey mathematical ideas or actions already known or familiar based on the connection with other forms of representation. The other was a private function in which symbols were used “to organize and manipulate ideas” based on the connection within the symbol system (Hiebert & Carpenter, 1992, p. 73-74).

This analysis was informative for the design of the present study, which investigated students’ experience of symbolism. The present study specifically focused on middle school students who were beginning to learn algebra. Thus, they were already familiar with numeral symbols and operational symbols. However, they had not used algebraic symbols yet as tools for communication and representation. With the mathematical context of representing changing situations with two variables in relation to each other, the present study planned to study students’ meaning development of algebraic symbolism in relation to other forms of representation.
I presume that students’ prevalent experiences with algebraic symbolism occur in classroom learning situations where the learning experiences include listening to the teacher’s lectures, reading mathematics books, doing hands-on activities, observing how the teacher and other students use symbols, and discussing problems with other students. Thus, the present study is based on students’ mathematical activities in the classroom setting. In addition, it is most likely that the learning begins with the introduction of algebraic symbols that students try to make sense of throughout subsequent activities. Thus, when students are introduced to algebraic symbols for the first time, they do not yet have their own meanings for the symbols. In order to differentiate symbols before and after students develop their meanings for them, I will use the term ‘algebraic notations’ to denote ‘symbol’ used before their meaning development in the present study. The following questions guide this study:

1. How do students make sense of algebraic notations in relation to other forms of representation throughout mathematical activities?

2. How do students’ mathematical conceptions form and develop as they use the algebraic notations throughout mathematical activities?

The first research question is about how algebraic notations become symbols to students as students do mathematical activities in the classroom setting. Specifically, instead of studying students’ sense-making activities limited to algebraic notations, the present study investigates them through how students relate algebraic notations to other forms of representation. The main reason for this referential approach is that I believe studying students’ referential relationships between algebraic notations and other forms of representation provides rich explanations of how students make sense of algebraic notations in various contexts. Other forms of representation are narrative, tabular, and graphical representations in this study.
For the second research question, this study investigates the development of students’ mathematical conceptions in relation to students’ referential relationships between algebraic notations and other forms of representation. While investigating students’ referential relationships for the first research question, I expected to observe that students’ new mathematical conceptions would emerge or previous ones might change or develop. In particular, students’ emerging conceptions of variables and rates were studied in relation to the referential relationships.

Through the two research questions mentioned above, this study attempts to explain how students can develop meaning of algebraic notations and their emerging conceptions of variables and rates. The research questions will be elaborated with the theoretical orientations guiding the present study in the next chapter.
CHAPTER II
LITERATURE REVIEW AND THEORETICAL ORIENTATIONS

This chapter includes review of literature relevant to the present study and theoretical orientations through which the present study was guided. Reviewed literature will be discussed under three different approaches to symbolism: symbolism focusing on single symbols, symbolism representing mathematical situations within a symbol system, and symbolism representing mathematical situations outside of a symbol system. Theoretical orientations will include theories that provide a tool to analyze the data of the present study and elaboration of important terms in the study. Literature review and theoretical orientations will follow the discussion of what symbols and other terms mean in the present study.

Definition of Symbols and Other Terms

In order to study students’ experience with symbolism, it is necessary to define the term ‘symbols’ as opposed to algebraic notations and decide which symbols will be the focus of the present study. Cobb’s (2000) broad definition of symbols seemed to be a good place to start and his definitions was:

…to denote any situation in which a concrete entity such as a mark on paper, an icon on a computer screen, or an arrangement of physical materials is interpreted as standing for or signifying something else (p. 17).

His definition was very broad so that it could be applied not only for symbols in mathematics but also those in everyday life. As an example of symbols in everyday life, I could use two distinct erasers pretending they were cars in order to explain a traffic accident that I experienced. By moving the two erasers, I could explain how the accident happened. Here the erasers would be
symbols since they were concrete entities in the accident situation and stood for cars. As an example of mathematical symbols, the fractional numeral, \(\frac{3}{4}\), could stand for a fair share of sharing 3 apples among 4 children, a ratio of 3 out of 4, or an operator as in taking \(\frac{3}{4}\) of something in various contexts. Even when removing a specific context, \(\frac{3}{4}\) still could refer to the abstract concept of fraction. Both the erasers and \(\frac{3}{4}\) are legitimate symbols under the same definition, but they are very different. The erasers had quite clear meaning both to me and to the person listening to my traffic accident but lost the meaning as cars once leaving the accident context. Unlike the erasers, the fraction \(\frac{3}{4}\) has varying meanings in contexts and becomes an abstract entity even without the context. Considering the research questions, the second example is a definite interest of this study. So in order to define symbols in mathematics for this study I modified Cobb’s (2000) definition as to denote a mathematical situation in which a concrete entity written on paper, a board, or computer screen is interpreted as standing for or signifying something mathematical.

Since symbols should stand for or signify something, it has to be noted that a physical entity like \(\frac{3}{4}\) written on paper is not a symbol at all for a first grader who has just learned counting. However, the fraction \(\frac{3}{4}\) is a symbol for those who have meanings for it and it also has consensual meanings developed by the mathematics community. To differentiate these two, I will call such entities ‘notations’ for those who do not have their own meaning as yet.
Moreover, the term ‘algebraic notations’ is to be used for ‘notations’ under this study because they are used in the context of algebra.

With the definition above, the term, symbols, will be used in this study to encompass other terms such as symbolic representations and symbolic expressions. An equation, \( y = 65x \), has a string of symbols including the letters of \( x \) and \( y \), a numeral 65, and the equal sign that stand for something mathematical. However, at the same time, the whole equation may also signify a situation in which a distance (\( y \)) changes in relation to time (\( x \)) when driving a car at 65 miles per hour. Then, the equation, which may be considered as a symbolic representation or a symbolic expression for some other purposes, is a symbol to signify the changing situation. In fact, since the mathematical content of this study is the representation of changing situations with variables, it is frequent that equations should be considered as symbols. In addition to that, the differentiation between symbols and other similar terms seems unnecessary in order to highlight their relationships that students make between something to signify and something signified in this study.

I also need to clarify the definitions of some other terms used in this study. Some terms can be used in various ways, and they have to be defined to avoid possible confusions and also to limit the scope of this study to the investigation of students’ symbolism as defined above. The other terms to be discussed here include conception, variable, and rate. The term ‘conception’ is differentiated from the term ‘concept’ in this study. Leinhardt, Zaslavsky, and Stein (1990) clearly stated what conceptions meant in their review of research on teaching and learning functions. Their notion of ‘conceptions’ was borrowed for this study as opposed to ‘concepts’. They characterized the nature of conceptions as follows.

Conceptions are features of a student’s knowledge about a specific and usually instructed piece of mathematics. They are meaningful ideas that students develop that
can serve as powerful workhorses in the student’s continued efforts to reach deeper, more integrative levels of understanding. Conceptions are embedded in the domain in the sense that they consist of content that is specific to understanding in a particular domain. By nature, conceptions are in transition, or in the process of being fleshed out to their fullest realization or capacity. (That is why they can at times appear to be fragile, situation-based, or misapplied.) Nevertheless, they are well enough structured to do work for the student and explicit enough to be the object of communication with others. In their ideal form, conceptions are highly interactive with supportive intuitions (p. 6).

Students’ conceptions investigated in this study are located in their referential relationships among different forms of representation. Thus, their conceptions of variables and rates in this particular domain are not necessarily the same as those in different domains. In particular, it appeared that students in this study had their conceptions of variables and rates formed in previous learning experiences. Whereas their learning experiences about rates were extensive in mathematics, they seemed to have very little experience of variables in mathematics and only limited learning experiences about variables in science classes.

Regarding the meaning of variable, Schoenfeld and Arcavi (1988) developed their argument that the meanings of variables could be determined contextually. They also pointed out that the term ‘variable’ was used in multiple ways without deliberate distinction. Based on the context of this study, variable is used in a restricted manner. First of all, a variable in this study is a quantity that changes in relation to another in a certain situation. Thus, variable is differentiated from unknown and unspecified (generalized) quantities although all of them are represented by literal symbols. More discussion on the differentiation will be given in the section titled Conceptions of Variables in Chapter IV.

Also variable in this study has both static and dynamic aspects at the same time. According to Leinhardt, Zaslavsky, and Stein (1990), static aspects of variable are frequently shown in algebraic symbols by revealing generalization or patterns. On the other hand, since dynamic aspects of variables highlight “the variability and simultaneous changes” of two related
variables, they could be shown in other forms of representation than algebraic symbols (p. 22). Because students’ referential relationships between algebraic notations and other forms of representation are the focus of this study, both aspects of variable are discussed.

Just as there was for the term ‘variable’, there also appears to be different interpretations for the meaning of the term ‘rate’ in the literature. Thompson (1994) pointed out that ratio and rate had been used without definitions and distinction between the two. He cited several distinctions between ratio and rate from the literature as follows.

1. A ratio is a comparison between quantities of like nature (e.g., pounds vs. pounds) and a rate is a comparison of quantities of unlike nature (e.g., distance vs. time; Vergnaud, 1983, 1988).
2. A ratio is a numerical expression of how much there is of one quantity in relation to another quantity; a rate is a ratio between a quantity and a period of time (Ohlsson, 1988).
3. A ratio is a binary relation that involves ordered pairs of quantities. A rate is an intensive quantity – a relationship between one quantity and one unit of another quantity (Kaput, Luke, Poholsky, & Sayer, 1986; Lesh, Post, & Behr, 1988; Schwartz, 1988). (as cited in Thompson, 1994)

In defining rate for the present study, I modified the above notions. In this study, a rate is a comparison between two quantities, which are variables in changing situations. The two quantities are generally of unlike nature, and neither one has to be a period of time. However, the independent variable generates the unit quantity in the rate. In particular, a rate could generate series of ratios involving ordered pairs of quantities. However, the definition of rate as opposed to ratio was used for only analyzing data and therefore, students were not expected to appreciate the definition of rate and the distinction from ratio.

Literature Review

In this section, I will discuss selected literature on symbolism relevant to this study. I categorize the reviewed literature into three groups, according to how researchers approach algebraic symbolism. The first group of studies focused on single symbols in order to
understand how students developed their meaning of each symbol. The second group of studies focused on symbols as a tool to represent mathematical situations. The last group of studies also focused on symbols as a tool to represent mathematical situations, but they focused on meaning of symbols by connecting with other forms of representation. For the second and the third groups of studies, examples of algebraic symbolism included mostly equations and functions to represent situations with equivalent relations and functional relations.

Symbolism Focusing on Single Symbols

Studies in this category focused on single symbols to investigate how students understood and developed meaning for single symbols in general. Yet, they were very different in interpreting what symbols meant. Rubenstein and Thompson (2001) considered mathematical symbols as shorthand pointers. They classified symbols according to what these pointers indicated into six types: (1) naming a concept, (2) stating a relationship, (3) indicating an operation or a function with one input, (4) indicating an operation or function with two or more inputs, (5) abbreviating words, units, theorems and so on, and (6) indicating grouping. For instance, numerical symbols and literal symbols belong to the first type, and the equal sign and the congruent sign belong to the second type. With this classification, they argued that students had challenges in verbalizing, reading and writing these symbols for several reasons.

The authors, however, overlooked how challenging it was for students to understand concepts, operations, relationships and functions that these symbols indicated. They simply focused on challenges in using these symbols. For example, they showed that reading the expression $x - y$ was challenging because it could be read in multiple ways such as “$x$ minus $y$”, “$x$ take away $y$”, “$y$ less than $x$”, and so on (Rubenstein & Thompson, 2001, p. 267). For this challenge, their instructional suggestion was made to have students practice verbalizing. They
did not relate the challenges of reading the expression and the instructional suggestion to the concept of subtraction at all. Although some might consider this study informative about students’ challenges in using symbols, it was quite contrary to the present study in interpreting what symbols meant and understanding how students conceived meaning of symbols.

On the other hand, some studies investigated meaning of symbols in terms of relevant concepts. Among various symbols, the equal sign and literal symbols have been studied frequently. The equal sign, according to Gattegno (1974), represents various relationships of identity, equality, and equivalence between quantities. Whereas identity means the sameness, equivalence refers to an encompassing relationship, which allows replacing one quantity with the other. In fact, students are eventually expected to understand the concept of equivalence with the equal sign in doing algebra. However, Behr, Erlwanger, and Nichols (1980) found out that children considered the equal sign as an operator, not even a relational symbol. The authors provided children aged from six to twelve with various forms of number sentences with the equal sign in order to investigate children’s meaning of the equal sign. Children had a strong conception that the equal sign had to follow an operator such as addition and the result of the operation had to come to the right-hand side of the equal sign. In conclusion, the authors warned that students’ treating the equal sign as an operator tended to be persistent.

Kieran (1981) supported Behr and his colleagues’ (1980) warning by reviewing studies on the equal sign at various levels. According to Kieran (1981), the preschool children with basic counting skills concluded that two sets were the same if they had the same cardinality. As Behr and his colleagues (1980) showed, children at elementary school considered the equal sign as an operator, and they only accepted number sentences with the result of an operation following the equal sign. However, Kieran (1981) did not indicate whether or not children’s
symbolic operation (e.g., $3 + 5 = 8$) at this level was restricted to the cardinality of homogeneous sets. At high school and college levels, students overcame the restricted use of the equal sign as only allowing a result on the right-hand side, but they still considered the equal sign as an operator. For instance, what Kieran (1981) called “short-cut errors” suggested that students used the equal sign as an indicator of the results of operations (e.g., $3 + 2 = 5 - 1 = 4$). Therefore, Kieran’s (1981) results supported that students at various levels did not show evidence that they conceived the equal sign as a relational symbol as Behr, Erlwanger, and Nichols (1980) warned, and moreover, she implied that students’ understanding of the equal sign should be supported by the concept of equivalence.

As a way of developing meaning of the equal sign, Herscovics and Kieran (1980) recommended teachers to begin with arithmetic identities such as $4 \times 3 + 5 = 2 \times 9 - 1$. Next, students should pick one number in arithmetic identities and hide it with a finger to generate “arithmetic identity with a hidden number”. Gradually, they should replace the picked number with a box and later a letter (e.g., $4 \times x + 5 = 2 \times 9 - 1$). With this method, the authors suggested a way to overcome students’ limited understanding of the equal sign mentioned in the previous studies and to extend their understanding of arithmetic identities into equations with unknowns.

Along with the equal sign, letters were other symbols that have been studied frequently. Literal symbols can be used in many different ways such as representing unknowns or variables. Küchemann (1978) investigated how students aged from 13 to 15 interpreted literal symbols. He adopted the Collis’ (1975) classification of letters used in various ways to develop test items for students and analyze students’ responses. The six levels in the classification with short explanations are shown below.

Letter EVALUATED: The letter can be evaluated immediately.
Letter IGNORED: The letter is ignored or replaced without the concept of equivalence.
Letter as OBJECT: The letter is regarded as an object itself or as a name of an object.
Letter as SPECIFIC UNKNOWN: The letter is regarded as an unknown, but cannot be evaluated.
Letter as GENERALIZED NUMBER: The letter represents a set of numbers instead of one value.
Letter as VARIABLE\(^1\): A value for the letter is not specified, but a second order relationship needs to be found between two quantities expressed in terms of the letter.

As a result, Küchemann (1978) found out that most students interpreted letters as specific unknowns. Few students interpreted letters as generalized number or variable. In particular, the author pointed out that students resisted accepting the expressions with letters (e.g., \(2n\), \(8 + g\), etc.) as complete answers. It implied that students did not regard the expressions with letters as quantities that resulted from operating on some quantities.

Although Küchemann’s (1978) study showed students’ interpretations of letters used in various ways, it did not cover some other ways of using letters. In particular, it did not address the case when letters represented variables in the context of functions or represented changing situations. In terms of the definition of variable that I am using in this study, the case of when letters represent variables in a functional relation between two quantities, the level is higher than any of the levels in Küchemann’s (1978) hierarchy because students have to make a comparison between two varying quantities represented by different letters.

Whereas Behr, Erlwanger, and Nichols (1980), Kieran (1981), and Küchemann’s (1978) mainly described students’ interpretation of symbols, Stacey and MacGregor (1997) focused more on what caused difficulties for students to understand algebraic symbolism. Stacey and MacGregor (1997) first presented students’ responses to represent David’s height when David was 10 \(cm\) taller than Con, whose height was \(h\) cm. In analyzing students’ various responses to

\(^1\) Letter as VARIABLE in this classification is different from the way of using letter as variable in the present study.
the questions, the authors found out that students’ previous experiences led them to misinterpret and misuse symbols. One example was using abbreviations such as $Dh$ to represent “David’s height” because of their linguistic experiences of using abbreviations in everyday life. A second was interpreting $a = 28 + b$ as “$a$ equals to 28, [then] to add $b$” because of their reading equations from left to right as when reading an English sentence (p. 112). The authors argued that teachers should recognize their students’ inappropriate interpretations and uses of symbolism, and they made instructional suggestions for teachers. Although students’ previous experiences would not explain all the causes of their difficulties in understanding symbols, Stacey and MacGregor’s (1997) study seemed meaningful in that they helped teachers understand some of their students’ ideas underneath their symbolism.

*Symbolism Representing Mathematical Situations within a Symbol System*

Studies presented in the previous group focused on single symbols to investigate how students understood the meaning of the symbols. Studies on the equal sign and literal symbols were mainly reviewed, and their focus was on the concept of equivalence and the concept of variables in a broad sense, respectively. In this group, studies focused on symbols as a tool to represent mathematical situations. For instance, whereas studies in the previous group focused on the equal sign as a symbol to study, studies in this group focused on the whole equation as a symbol to represent a mathematical situation. They were quite different approaches in that the former put an emphasis on one element (the equal sign) in the whole (the given equation), but the latter looked at the whole equation as a representation.

Generating equations or inequalities based on a given situation, and expressing functions with algebraic notations are examples of using algebraic symbols to represent mathematical situations. While students use algebraic symbols to write equations, inequalities, or functions,
the critical issue here includes whether they holistically conceive the algebraic expressions as symbols to represent mathematical situations. Usiskin (1988) argued that equations with the same operational structure could represent different mathematical situations and they had “a different feel” according to what they represented. He showed five equations as below and identified them as “(1) a formula, (2) an equation (or open sentence) to solve, (3) an identity, (4) a property and (5) an equation of a function of direct variation (not to be solved)” (p. 9).

1. \( A = LW \)
2. \( 40 = 5x \)
3. \( \sin x = \cos x \cdot \tan x \)
4. \( 1 = n \cdot 1/n \)
5. \( y = kx \)

Although some researchers referred to any literal symbol as a variable, Usiskin (1988) insisted that each equation gave us a different feel because the letters in each equation were used for different purposes. It implied that understanding what the letters stood for had to be accompanied with understanding what the equation in which the letters were embedded represented. For example, the equation (1) represented the area formula for a rectangle and it showed a quantitative relation between the area and the dimensions of the rectangle, while the equation (2) was an equation to be solved for the unknown \( x \). Figuring out what an equation represents is not easy for young students, but it has to be emphasized in order for them to understand algebraic symbolism.

It appears beneficial for young students to represent some situations with algebraic symbols in order to understand what equations represent. For instance, writing an equation for a given word problem would help students interpret equations with unknowns. In writing an equation for a word problem, what matters is whether students can appreciate that writing the equation is an algebraic way to represent the given word problem. However, studies on word
problems did not seem supportive of students’ appreciation of the equation to represent the word problem. Rather the focus seemed on how to find out the answer of the word problem.

When young students were given a word problem, they tried to find the answer directly from the situation rather than writing an equation to represent the situation (Carpenter & Moser, 1982). If they were asked to write an equation, they tended to write an equation after finding the answer first (Briars & Larkin, 1984). Kieran (1992) elaborated that even if students wrote an equation for a word problem, what they represented actually was mostly the operations that they performed to get the answer. In that case, the equation was not a symbol to represent the situation. Consider a typical word problem as below.

Alie gave 12 candies to Tesh and 9 candies to Jay, and she had 15 candies left. How many candies did she have at first?

Whereas writing an equation with a missing number represented by a box or a letter (e.g., \( \square - 12 - 9 = 15 \), or \( x - 12 - 9 = 15 \)) represented the given situation, writing an equation to find the answer (e.g., \( 15 + 9 + 12 = 36 \)) represented the operation (that is, addition) to perform.

Representing a mathematical situation with algebraic symbols as opposed to representing the operations to perform is important because representing a mathematical situation reveals the structure of the situation. Moreover, the structure highlights the quantitative relation among the quantities involved in the situation, and the relation leads to the operations. The operations in the structure, referred to as the “forward operations” in Kieran’s (1992) words, ought to be differentiated from the operations required to get the answer. Eventually, however, two types of operations are to be related by the quantitative relation. In the word problem example shown above, the equation \( x - 12 - 9 = 15 \) represented the situation of equality among the numbers of candies Alie had at the end, and it showed the subtraction as the forward operation. When solving the equation, students could use addition as the operation, but they could see how the
subtraction and the addition were related in the quantitative relation. Especially, when it comes to more complex mathematical situations, representing the situations seemed more beneficial since figuring out the operations to perform to get the answers sometimes seems unobtainable.

Some researchers studied how students represented mathematical situations in word problems. Chaiklin (1989) found that students represented mathematical situations either by a direct-translation or by a principle-driven approach. By a direct-translation, students syntactically translated each phrase into algebraic symbols, and it could be easily counterproductive since they did not necessarily have to look for the structure of the situation.

On the other hand, by the principle-driven approach, students adopted a mathematical principle to write an equation, and the various mathematical principles were described in schemata (Mayer, 1980). Instead of mathematical situations, Sherin (2001) investigated how students understood physics equations by the principle-driven approach. He hypothesized that successful students possessed symbolic forms, which associated “a simple conceptual schema with an arrangement of symbols in an equation” (p. 482). With symbolic forms, students could both understand and represent a physical situation with algebraic symbols.

According to Sherin (2001), symbolic forms had two components, a symbol template and a conceptual schema. The symbol template was a figurative structure for an equation, and the conceptual schema was represented in the equation by the symbol template. For example, when representing a force ($F$) that had two components ($f_i$ and $f_j$), two symbolic forms could be used; identity and parts-of-a-whole.

\[
\text{identity} \quad \Box = \Box \\
\text{parts-of-a-whole} \quad \Box + \Box + \Box \ldots
\]
Thus, the equation is $F = f_1 + f_2$, where $f_1$ and $f_2$ could be either a single symbol or a product of factors, respectively. The equation has the conceptual schemata that the right-hand side shows that two parts makes a whole and the whole is equivalent to the force ($F$).

In contrast to a direct-translation approach, a principle-drive approach encourages students’ conceptual understanding of situations to be represented. However, it has been also found that students had difficulties figuring out structures of given situations and realizing structural similarities underneath different cover stories (Reed, 1987).

*Symbolism Representing Mathematical Situations outside of a Symbol System*

Studies reviewed in this group considered symbols as a tool to represent mathematical situations as in the previous group, but the focus was on the relation between symbols and other forms of representation. In some studies, the connection between symbols and other forms of representation was investigated in the sense that other forms of representation could make symbols or symbolic manipulation meaningful. Filloy and Rojano (1989) conducted one such study. The authors observed that some linear equations could be solved without operating on the unknowns but others could not. They insisted that “arithmetical equations” such as $Ax ± B = C$, $A(Bx ± C) = D$, $x / A = B$, and $x / A = B / C$ could be solved by using the inverse operations (so-called “undo” method), but linear equations of the types of $Ax ± B = Cx$ and $Ax ± B = Cx ± D$ require students to operate on unknowns. To solve the non-arithmetical equations, the authors argued that “syntactical rules” had been traditionally taught but the approach lacked the connection with young students’ operations at a concrete level.

In order to help young students make sense of algebraic manipulations, instructions were provided with the geometric model and the balance model in the Filloy and Rojano’s (1989) study. At first, students had to translate a given equation into the models. In the geometric
model, the area of conjoined rectangles or a rectangle with unknown length represented the quantity for each side of the equation. By comparing the areas, students could find out the unknown length. In the balance model, the weights were compared to find out the unknown weight. With these models, the authors analyzed students’ abstraction of the operations on the unknowns.

The relation between algebraic symbols and two concrete models, however, was unidirectional because operating on both models was assumed spontaneous and meaningful for students and therefore, they could abstract the concrete operations into the syntactical rules. However, as the authors indicated in the findings, students still had difficulty operating on the concrete models. Thus, it seemed unreasonable to expect students’ successful abstraction from the concrete operations on the unknowns.

Rather than seeking the unidirectional relation, it might be better to consider bi-directional relations, with the understanding that both representations represented the mathematical relationship of equivalence in the case of Filloy and Rojano (1989). That is, students should be able to interpret the pictorial representations in both models and project them into algebraic equations. In contrast, some studies investigated multiple representations and the relations among them from the point of view that multiple representations enhance students’ understanding, not for the purpose of using other representations to provide semantics for algebraic symbols.

Specifically, Friedlander and Tabach (2001) quoted Ainsworth, Bibbly, and Wood (1998) to argue how multiple representations enhanced students’ understanding as follows.

... (a) it is highly probable that different representations express different aspects more clearly and that, hence, the information gained from combining representations will be greater than what can be gained from a single representation; (b) multiple representations constrain each other, so that the space of permissible operators becomes smaller; (c)
when required to relate multiple representations to each other, the learner has to engage in activities that promote understanding (pp. 174-175)

Then, Friedlander and Tabach (2001) suggested some instructional ways that encouraged using multiple representations. Their example task was the Savings Problem, which represented the savings of four people in various representations. The authors illustrated investigative questions and reflective questions that required students to work on each representation and relate multiple representations. In fact, the authors reported what they learned about how students used multiple representations of the saving problem. The task included seven questions about Danny’s and Moshon’s savings represented in a table and a graph together. The authors concluded that the initial presentation of a problem situation and the nature of the questions could encourage or discourage the use of multiple representations. They added that students chose one representation to answer a question according to their own interpretation of the problem situation, but they tended to convert into another representation when the former was disadvantageous to answer the question.

Also, Coulombe and Berenson (2001) supported the use of multiple representations to enhance students’ algebraic reasoning in the context of functions. They pointed out that a problem-based approach promoted various cognitive processes such as interpretation and construction of each representation, while the traditional approach promoted limited translation process such as computing values from algebraic equation to a table and plotting points from a table to a graph. The authors suggested three different problem situations of the Weight-Loss Problem, the Iced-Tea Problem, and the Allowance Problem, with tasks that required students to interpret a given representation for a situation and generate another form of representation. Each problem was initially represented in a graph, a table, or a narrative, respectively, and asked students to generate a table, verbal description, or a graph correspondingly. In particular, the
authors argued that generating another form of representation could deepen students’ understanding of important mathematical concepts, which were represented in the representations.

In fact, Coulombe and Berenson’s (2001) study was very similar to the present study in many ways. The mathematical topic to be studied and the types of representations were in common, but most importantly both studies share the reasons for supporting the use of multiple representations. As Coulombe and Berenson (2001) emphasized, the use of multiple representations would serve as tools to represent mathematical relations and help students’ algebraic reasoning. However, the present study focuses on students’ algebraic reasoning with symbols and therefore only the relations between symbols and other forms of representations are sought. The next section will explain how the present study is guided theoretically.

Theoretical Orientations

In this section, I will discuss some theories relevant to understanding symbolism and describe which theories guide the present study. In the definition of symbols aforementioned, a noteworthy point is what symbols stand for or signify. What is the nature of what symbols stand for? The literature on symbols seemed to agree on two complementary roles for symbols: symbols as signifying process and symbols as signifying concept. The frequent case that literature suggested was that a learner was introduced initially to symbols as a vehicle to signify process, got familiar with the process, and later conceived the symbols as an object carrying a concept. Thus, researchers have tried to explain the cognitive shift from symbol as process to symbol as concept. For example, Mason’s (1987) spiral model explained the shift of symbol uses as:

from confidently manipulable objects/symbols,
through their use to gain a ‘sense of’ some idea involving a full range of imagery
but at an inarticulate level,
through a symbolic record of that sense,
to a confidently manipulable use of the new symbols,
and so on in a continuing spiral (p. 74-75)

In particular, Mason (1987) not only explained the shift of symbol uses between symbol
as process and symbol as concept but also identified the continuous acquisition of new symbols
based on symbols that were used confidently. However, such studies seemed to divide up
symbol uses artificially and highlighted the linear or hierarchical development of symbol uses
from those signifying a process to signifying a concept and from one symbol to another. To me,
the demarcation between process and concept is not clear and they seem to grow together.

In contrast with the previous models, I believe that the notion of *procept* by Gray and
Tall (1991, cited in Tall et al., 2001, p5.) seemed to resolve the issues mentioned above. They
still used the notions of process and concept, but they focused on a powerful way of using
symbols to switch between process and concept from time to time rather than focusing on
symbol uses as progressing from one to the other. Tall et al. (2001) elaborated the notion of
procept as:

It [procept] is now seen mainly as a *cognitive* construct, in which the symbol can act as a
*pivot*, switching from a focus on process to compute or manipulate, to a concept that may
be *thought* about as a manipulable entity. We believe that procepts are at the root of
human ability to manipulate mathematical ideas in arithmetic, algebra and other theories
involving manipulable symbols. They allow the biological brain to switch effortlessly
from doing a process to thinking about a concept in a minimal way (p. 5).

Thus, procept enables a learner to conceive symbols not only as signifying a process (e.g.,
compute or manipulate) but also as a concept, and to switch between them flexibly. Thus,
“being able to think about the symbolism as an entity allows it to be manipulated itself, to think
about mathematics in a compressed and manipulable way, moving easily between process and
concept” (Tall et al., 2001, p. 8).
Before reaching the proceptual level, students can do a specific computation by knowing a specific procedure and then they become more flexible and efficient forming a process out of multiple procedures (see Figure 1). To distinguish procedure and process, Tall and his colleagues (2001) meant procedure as “a specific sequence of steps carried out a step at a time” and process as “in a more general sense any number of procedures which essentially have the same effect” (p. 7). As an example, solving a linear equation, $3x + 2 = 17$, is a process in general but solving this equation by subtracting 2 from both sides and then dividing both sides by 3 is a procedure. If a student can solve the equation with the procedure accurately and it is the only procedure available to him, he is at the procedural level in the spectrum. However, when he has other procedure(s) and he can relate procedures meaningfully, he is at the process level. So the
‘process’ here as a noun is more than the sum of multiple procedures and more advanced than processing the task to get a result.

In the procept spectrum (see Figure 1), I interpret ‘progress’ as being fluent at each level; for example, at the procedural level students become fluent at solving linear equations and they can solve them accurately with a procedure. As they move toward the proceptual level, students develop their mathematical sophistication. However, the procept model did not show explicitly how students make the development from a level to the next.

As a parallel model to the procept model, APOS theory (Dubinsky & McDonald, 2001) explained students’ learning of any mathematical concepts by mental constructions of actions, processes, objects, and schemas. Those with an action conception can perform an operation with external stimuli. Through repeating an action and reflecting on it, they can construct a process, which allows them to perform the action without external stimuli. From a process, the students construct an object when they consider the process “as a totality and realize that transformation can act on it” (p. 274). Finally, a schema for a mathematical concept is constructed as a collection of the mental constructions of actions, processes, and objects, and other schemas so that students can access and use the cognitive structure to solve related mathematical problem situations.

When APOS theory is applied to students’ learning of symbolism, it is quite similar to the procept model. Both suggest some mental constructions, which are hierarchical, although not necessarily linear. Also the mental constructions in both models correspond with each other. However, both seemed to explain students’ learning of symbolism within symbols rather than paying more explicit attention to the relationship with other forms of representation.
Since the present study will investigate students’ symbolism in relation to other forms of representation, adopting another model for explaining the relationship is necessary. In that sense, Kaput’s (1991) representation model will be helpful when considered together with the models previously mentioned.

Figure 2. Kaput’s referential relationship (1991, p. 60).

Kaput (1991) explained the referential relationships between ‘notation A’ \(^2\) as a representation and ‘referent B’. The model (see Figure 2) had two rectangles describing the referential relationships between A and B. The bottom rectangle represented what we considered as consensual (e.g. the symbolic expression \( y = 3x + 2 \) [A] represents the line [B] with slope 3 and \( y \)-intercept 2). Thus, through their instructions, teachers may expect their students to make the referential relationship that they believed was consensual. However, individual students may

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\(^2\) The term ‘notation’ was used as a collective term of representations.
not necessarily make the referential relationship that their teachers expected. Hence, the upper rectangle represented referential relationships that individual learners would make.

In the individuals’ referential relationships, Kaput (1991) insisted that the cognitive acts should be emphasized. Consequently, the upper rectangle described “acts of interpretation, mental operation, and projection to a physical display”, through which individuals could make A and B related referentially (p. 59). He also pointed that each individual was the agent of the cognitive acts and the acts happened in the individual’s subjective world. However, the result of the acts would be shown in a material form of referent B.

In the present study, Kaput’s (1991) model serves as the major theoretical framework to explain students’ sense-making of algebraic notations in relation to other representations, which is the first research question. Since the mathematical content of the present study focuses on representing changing situations including two variables, narrative, tabular, and graphical representations are frequently related to algebraic notations in algebra curricula including the Connected Mathematics Project (CMP; Lappan et al., 2004) materials. Thus, the present study investigates how students make referential relationships between algebraic notations and the three forms of representations. Among all possible combinations, three bi-directional ways of referential relationship are investigated in this study (see Figure 3). The other three ways are not investigated directly, but they may be inferred indirectly through symbolic representations. That is, students’ referential relationship between tabular and graphical representations may be implied by those between tabular and symbolic representations and between symbolic and graphical representations.
Regarding Kaput’s (1991) referential relationship model, algebraic notations is placed in ‘notation A’ in the diagram (see Figure 2) since symbolism is the main focus in this study. The other representations such as narrative, tabular, and graphical play the role of referents in order to explain how students make sense of algebraic notations. For instance, with the diagram in Figure 2 it can be explained how students interpret a narrative situation (referent B) where two variables change relative to one another, what mental operation they use and how they project the situation into algebraic notations (referent A). Since the referential acts are bi-directional, the other direction can also be explained.

Kaput’s (1991) model describes the processes involved in students’ referential relationships between algebraic notations and other forms of representation, but does not provide me with a way of describing the nature of that relationship. The Structure of Observed Learning Outcome (SOLO) taxonomy (Biggs & Collis, 1982) can provide me with a theoretical tool to describe the nature of students’ referential relationships. That is, the referential relationships that

Figure 3. Referential relationships in this study.

Narrative representation

Algebraic notations

Tabular representation

Graphical representation
each student makes can be different in nature. Also, the same student could make different kinds of referential relationships when going from one form of representation to another.

The SOLO taxonomy has five levels: prestructural, unistructural, multistructural, relational, and extended abstract. Olive (1991) used the taxonomy to analyze students’ performance of geometric tasks, and he explained that students who gave unistructural responses could “use one piece of information only in responding to the task” (p. 91). Regarding the referential relationship between algebraic notations and other forms of representation, students at this level would use and interpret one aspect of a given algebraic notation and barely relate the algebraic notation to other forms of representation.

Students at the multistructural level would be expected to use and interpret multiple aspects of a given algebraic notation but their interpretation and actions would not be necessarily conceptually related. Once students reach the relational level, they would be able to make their referential relationship conceptually and fluently (e.g., the referential relationship could be based on the student’s well-formed concept of variables and rates). Finally, the extended abstract level in the SOLO taxonomy was described by Olive (1991) as a state where students could “derive a general principle from the integrated data that can be applied to new situations” (p. 92). Students exhibiting this level of referential relationship would be able to see the similarity in the form of the algebraic notation for two very different situations (e.g., \( d = 55t \) as representing both a ‘distance-time’ situation and a ‘total cost-number of tickets’ situation).

In the present study, while Kaput’s (1991) model provides a tool to describe the processes of students’ referential relationships, the SOLO taxonomy provides a way to describe differences in nature and varying degrees of fluency among various referential relationships that students make. I had no intention in the present study to evaluate students’ referential
relationships adopting the SOLO taxonomy. Describing the nature of students’ referential relationships still seemed important and necessary in order to enhance my understanding of their sense-making of algebraic notations.
CHAPTER III

METHODOLOGY

The present study was conducted within the activities of an ongoing project, Coordinating Students’ and Teachers’ Algebraic Reasoning (CoSTAR), funded by the National Science Foundation (NSF). Since the spring of 2003, the project has investigated how teachers and students understand their shared classroom interactions and the ways that they work together on the teaching and learning of middle school algebra. With an emphasis on the linkage between teaching and learning in classrooms, the CoSTAR project has three main questions to answer. The first one is to what aspects of shared classroom interactions teachers and students attend and how they are the same or different. From the teachers’ perspective, the second question is how teachers use and build upon their existing subject-matter and pedagogical content knowledge when understanding and responding to mathematical problems of teaching in classroom interactions. The third one is how students use and build upon their existing mathematical understandings to make sense of classroom interactions and to solve new problems (Izsak et al., 2002, p. 2). While the CoSTAR project has investigated how both teachers’ and students’ algebraic reasoning are related and coordinated centering on the shared experiences, the present study was conducted from the students’ perspective only, so that some activities (e.g., classroom interactions) that would be approached through both the teachers’ and the students’ perspectives in the CoSTAR project, were interpreted only from the viewpoint of the students in this study.
As the research site of the CoSTAR project and of this study as well, Pierce Middle School\(^3\) was selected thanks to the established relationship between the school district and the Learning and Performing Support Laboratory (LPSL) at the University of Georgia. Pierce Middle School, a rural middle school in the northeast of Georgia, had 10 mathematics teachers and 685 students with racially and economically diverse backgrounds. Although the county where Pierce Middle School was located had steadily ranked in the middle on Georgia state assessments, it had made efforts to improve its mathematics education and adopted standards-based curricular since the 2001-2002 academic year.

The middle school adopted CMP materials (Lappan et al., 2004), developed through funding from the NSF. The materials were quite unconventional in terms of the way they were organized and the nature of the problems they contained. In each grade level, the CMP had 8 units, which corresponded to chapters in the traditional textbooks, and each unit had 5-6 investigations under one overarching theme that provided varying difficulty levels of problems in the section, Applications-Connections-Extensions. Most of the problems challenged students cognitively, and in addition they were open to multiple solution methods.

While the CoSTAR project focused on four units from grade 6 to grade 8 (Bits and Pieces II, Stretching and Shrinking, Variables and Patterns, and Moving Straight Ahead) throughout the project span, the present study focused on one unit in grade 7, Variables and Patterns. The CMP developed 8 units for grade 7, but Pierce Middle School selected 6 of the 8 units to implement. The unit, Variables and Patterns, was the second unit to be implemented in the seventh-grade and was aimed to provide the groundwork for algebra. It had a single overall context where some college students organized a bike tour and needed to figure out how much progress they could

\(^3\) All the names of the people in this study and the research site where data were collected are pseudonyms.
make each day during the bike tour, how much they would charge the customers for the tour, how they would compare the cost of renting bikes from two shops, how much they would make in profit from the bike tour, and so on.

Within this context, the unit was composed of 5 investigations: Variables and Coordinate Graphs, Graphing Change, Analyzing Graphs and Tables, Patterns and Rules, and Using a Graphing Calculator. This study was conducted on the first four investigations. The last investigation was excluded since it was mainly about how to draw a graph and make a table on a calculator, rather than on using a graphing calculator as a tool for investigating the mathematical topics. Throughout the first four investigations, the unit specifically asked students to (1) recognize problem situations in which two or more quantitative variables are related to each other, (2) find patterns that help predict the values of a dependent variable as values of other related independent variables change, (3) construct tables, graphs, and simple symbolic expressions that describe patterns of change in variables, and (4) solve problems and make decisions about variables using information given in tables, graphs, and symbolic expressions (Lappan et al., 2002, p.4).

Data Collection

The data collection activities of the CoSTAR project included videotaping classroom activities and interviews with teachers and students about the classroom activities. For each semester, the CoSTAR team had conducted two case studies, of which one case study included videotaping the activities in one classroom every day and interviewing the teacher and 3-4 pairs of students from that class each week. In particular, one teacher, Ms. Susan Moseley, participated in three case studies during the first three semesters of the CoSTAR project.
Ms. Moseley, whose initial certification was focused on social studies, had exclusively taught middle grade mathematics for the last 12 years. Since Pierce Middle School adopted the CMP curriculum, Ms. Moseley had taught the CMP materials at the sixth-grade level for two years. The next year when this study was conducted was her first year of teaching the CMP materials at the seventh-grade level. In general, Ms. Moseley was not fully confident of her mathematical knowledge, but she was enthusiastic about teaching mathematics and always willing to take all possible opportunities to improve her teaching of mathematics. In her class, Ms. Moseley was an informer and guide who tried to lead her students toward what she thought was the best for her students’ interest.

In the spring of 2003, one of Ms. Moseley’s 6th grade classes participated in a case study of the CoSTAR project with 3 pairs of interview students. In the following school year, her 7th grade class with the same interview students continued to participate in a second case study. Since I had worked on the case study of Ms. Moseley, observing and videotaping her classes and her student interviews from the beginning, I was quite familiar with her class and her students when the data collection of this study began. For the present study, four students were selected based upon Ms. Moseley’s recommendation and avoiding overlap with the interview students for the CoSTAR project. In selecting the participating students, I mainly considered students’ participation in classroom activities, their abilities for self-expression, and compatibility with a partner to include them in this study. Thus, two boys, Greg and Jeffery, and two girls, Angela and Peggy, were included.

Greg was a thinker who tried to make sense of mathematical ideas under discussion for himself. He also expressed his thoughts quite well and he was more outspoken than his partner, Peggy. Jeffery was a very quiet boy but he did not hesitate expressing his thoughts whenever he
thought he was confident. He also had a good understanding of numerical relationships and fluent computational skills. Angela was always cheerful and was never afraid of expressing her thoughts. Although she was not confident in her computational skills, she appeared willing to learn from her confusions and errors. Peggy seemed to think that she had to learn what to do rather than why and how to do in mathematics. So Peggy often seemed pleased with knowing a way that led to the right answer. In the continuum of understanding instrumentally to relationally (Skemp, 1987), Peggy seemed the closest to understanding instrumentally and Greg was the closest to understanding relationally among the participating students.

The data collection method in this study had two layers. The first layer was also a part of the CoSTAR data collection and it was to videotape classroom activities on a daily basis throughout the unit, Variables and Patterns. For each class, two project staff members video-recorded the classroom activities from two different angles. One with a stationary camera recorded the teacher’s perspective, which contained teacher’s written work on the front board and the whole class view, and the other with a mobile camera recorded the students’ perspective, which included students’ written work at their seats and the group working view.

Once class taping was done, the views from the two cameras were synchronized and digitized through a video mixer using “picture-in-picture” technology and they were saved as a movie file on a computer. The mixed video file presented the video from one camera as the background and inserted the video from the other camera inside a small rectangle superimposed on top of the background video. The two video sources could be switched, reversing background and inserted video, during the mixing process. The video-insert rectangle could also be moved around the screen during mixing. Following the video mixing, a lesson graph (see Appendix A) was written for each class, which showed how classroom activities went and what mathematical
issues emerged in the class along with the time stamps. Mixing the two views and writing the lesson graphs were done on the same day of taping and these responsibilities were shared among the CoSTAR staff.

As the second layer of the data collection of this study, I conducted a series of interviews with the four participating students. Based on observation of classroom activities and lesson graphs, interview tasks were carefully chosen in order to answer the research questions. In particular, watching the tapes of classroom activities and reading lesson graphs over and over helped me determine what mathematical issues around algebraic notations this study would investigate. Thus, each interview aimed to understand students’ sense-making of the targeted algebraic notations and their conceptions development with them. However, as the interviewer, I did not limit the purpose of the interviews to understanding students’ experiences of symbolism but sometimes asked probing questions relating to their classroom activities. By doing so, I hoped to extend their experiences by actively interacting with the participants so that they could possibly have learning experiences during the interviews as a benefit of involvement with this study and I could also gain dynamic and rich data, which could help me answer the research questions directly or support my assertions in this study.

Each interview was also video-recorded with two cameras. One stationary camera provided a whole picture of an interview and the other captured a focused view of students’ work or a computer screen. In order to ask some interview questions, a laptop computer was used to retrieve a video clip from the classroom activities whenever it was necessary. During the interviews, I interacted with a pair of students; Angela and Jeffrey made one pair and Greg and Peggy made the other. The main reason for pair interviewing was that I believed students could reveal their thoughts more clearly by communicating with each other rather than having one-to-
one interaction with the interviewer. They also could provoke each other’s thinking through interactions and by reflecting on their own thinking relative to that of the other. However, neither interactions between students nor their possible interactions with the interviewer were the main focus of the analysis of this study. Rather each individual student’s thinking as it related to the research questions was to be the main focus of the analysis.

Retrospectively, interviews had two phases according to the time line and the nature of the interview tasks. The first phase of the interviews was conducted in the fall of 2003 when students were taught the unit, Variables and Patterns. Each pair of students had five 50 to 60 minute interviews, one each week from early November to mid December in 2003. In each week, the pair of Greg and Peggy was interviewed on one day and the other pair was interviewed on another available day. Interviews were conducted at around 8:00 am before they began their classes. The ten interviews in total were all synchronized and digitized, and the interview graphs were completed during the interview period. Interview graphs (see Appendix A) had the same format as lesson graphs, showing how the interviews proceeded along with the interviewer’s notes.

After reviewing the ten interviews of the first phase, the second phase of interviews was conducted with the same pairs of students in February 2004 in order to pursue issues that were not answered in the fall of 2003 due to the time constraint. Thus, each pair of students had three interview sessions in two weeks. Whereas the interview schedules in the first phase were constrained by both the classroom activities and the CoSTAR data collection schedule, the second phase of interviews was conducted without such constraints. In addition, while the intent of the first phase of ten interviews was to investigate students’ thinking focusing on emerging mathematical issues following the classroom activities, the second phase of six interviews was to
understand how students relate algebraic notations to the other forms of representation in order to make sense of algebraic notations for changing situations. Thus, the second phase of the interviews was not only more structured but also able to target the research questions more directly as they were supported by the interviews in the first phase.

The purpose of the interviews in the first phase was to see how students understand different forms of representation to make sense of algebraic notations used in the classroom activities and how their related mathematical conceptions develop. The specific goals of the interviews were as follows:

- Students were to describe a situation with two variables, to identify variables in a situation, to identify which variable was independent and which was dependent, and to plot the graph of two variables.
- Students were to make a table and/or a graph after reading a narrative of a changing situation, to interpret data given in a table and a graph, and to compare tabular, graphical and narrative representations of the situation.
- Students were to search for patterns of change in a graph and a table, to describe a situation with verbal rules, and predict a change.
- Students were to show their understanding of the relationship between rate, time, and distance, to identify and represent rates in a table and a graph, and to express the patterns in symbols.

In order to pursue these goals, I developed the interview protocols based on my own analysis of the unit, Variables and Patterns, and issues that emerged from observing students’ classroom activities. The emerging issues could be categorized under the following eight themes: (1) meaning of variables, (2) identifying independent and dependent variables, (3) representing
changing situations, (4) connecting data points on a graph, (5) distance between two data points and steepness of the segment connecting two data points on a graph, (6) constant rate of change, (7) slope of a line segment connecting two data points on a graph, and (8) writing symbolic representations. Under these themes, the actual interview questions were asked according to the context and the progress of the class. The following tables show the sets of prototype questions according to the themes and then the daily interview tasks in the first phase. The tasks were provided to give a picture of how the interviews went, not to show the exact schedules. In fact, the two pairs of students were not necessarily asked the exact same questions on the same day due to the progress of the class. Also similar questions were asked on different days to better understand students’ thinking.

Table 1

*The Emerging Interview Themes and Questions*

<table>
<thead>
<tr>
<th>Themes</th>
<th>Interview questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Variables</td>
<td>▪ What does variable mean to you?</td>
</tr>
<tr>
<td></td>
<td>▪ Create examples of variables.</td>
</tr>
<tr>
<td>2. Independent vs. dependent variables</td>
<td>▪ Create situations where two variables change relative to each other.</td>
</tr>
<tr>
<td></td>
<td>▪ What do independent and dependent mean to you?</td>
</tr>
<tr>
<td></td>
<td>▪ How do you tell which variable is independent and which is dependent variable in the changing situation?</td>
</tr>
<tr>
<td>3. Representing changing situations</td>
<td>▪ How do you represent a changing situation?</td>
</tr>
<tr>
<td></td>
<td>▪ Can you generate a data set of your changing situation based on your invented story?</td>
</tr>
<tr>
<td></td>
<td>▪ Given a table, can you draw a graph?</td>
</tr>
<tr>
<td></td>
<td>▪ How do you determine if a given data set is cumulative or not?</td>
</tr>
<tr>
<td></td>
<td>▪ Given a table or a graph, can you make up a story?</td>
</tr>
<tr>
<td></td>
<td>▪ Can you discern a pattern in a table or a graph?</td>
</tr>
<tr>
<td></td>
<td>▪ What does the scale mean to you in a graph?</td>
</tr>
<tr>
<td></td>
<td>▪ How does the scale help you understand the graph?</td>
</tr>
</tbody>
</table>
4. Connecting data points on a graph
- Does connecting coordinate points always make sense on a graph?
- When do you decide to connect the data points on a graph?
- Can you think of various ways to connect data points? How are they different from each other?
- Given a certain way to connect data points, what does that way indicate to you about the data?

5. Distance vs. steepness of segment connecting two data points on a graph
- Where do you see the most increase/decrease in a given graph? How about the least increase/decrease?
- Does connecting data points help you see the increase/decrease? How?
- What do you mean by the ‘distance between dots’, ‘(diagonal) space between dots’, or ‘biggest jump’?
- What does the steepness mean to you? How is it different from the ‘distance between dots’?
- How do you compare steepness of two different line segments? Can you measure the steepness?
- Can you draw a line segment of the same steepness with a given line segment? How about steeper one, or less steep one?

6. Constant rate of change
- What does the constant rate of change mean to you?
- How can you use the constant rate of change to answer some questions?
- How do a graph and a table show the constant rate of change?

7. Slope of line segment
- What does the slope mean to you?
- How do you determine the slope of a line segment?

---

4 These were students’ verbatim statements from the in-class discussion.
8. Symbolic representation

- What does a general rule mean to you?
- How can you use symbols to represent the general rule?
- How does the formula tell you the story?
- Can you generate a table with a given formula?
- Can you draw a graph with a given formula?
- How does a formula show independent and dependent variables of the changing situation?
- How does a formula show a constant rate of change?
- How does a formula show a slope of a line segment?

Table 2

The Interview Tasks in the First Phase

<table>
<thead>
<tr>
<th>Interview</th>
<th>Greg and Peggy</th>
<th>Angela and Jeffery</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Students were asked to create a situation with two variables and identify the types of variables. Then with the situation they created, they generated their own data set in a table and graphed them. In the graph, students were asked to tell which interval had the most increase and how they could determine this. (Themes 1, 2, 3, 5, and 7)</td>
<td>Students were asked what variables meant and the types of variables. With the graph available to them, they were asked to tell whether they could always connect two adjacent data points and how they could. Also they were asked to tell which interval had the most increase and how they could determine this in the graph. (Themes 1, 2, 4, and 5)</td>
</tr>
<tr>
<td>2</td>
<td>Students were asked to tell what scales were in a graph. Then they were asked to tell the most increased interval in a graph and to compare the steepness of line segments. Students were also asked to compare a cumulative graph and a non-cumulative graph in the textbook and to read information from them. (Themes 3, 4, and 5)</td>
<td>Students were asked to draw a graph using a given table and were asked about variables in the graph and scales. Then students were asked to critique a graph that was drawn with an uneven scale and to correct it using a given table. (Themes 1, 2, 3, and 5)</td>
</tr>
</tbody>
</table>
Students were asked to choose a bike rental shop given that one shop showed the prices in a table and the other showed the prices in a graph at a constant rate. Then students were asked to make sense of a time-versus-speed graph in the textbook. (Themes 3 and 6)

Peggy without Greg was asked to find a pattern in a graph that showed the profit of the bike tour increased at a constant rate according to the number of customers. Using the constant rate, she also was asked to tell the exact profits for the numbers of customers. (Themes 3 and 6)

With a constant speed of 55 miles per hour, Angela without Jeffery was asked to find the distance for each hour and graph the relationship. Then she was asked to find the rule and write an equation for it. She also was asked to calculate the distances for 4 1/2 hours and 5 1/4 hours with the equation. (Themes 3, 6, and 8)

With a constant speed of 55 miles per hour, students were asked to find the distance for each hour and graph the relationship. Then they were asked to find the rule and write an equation for it. They also were asked to calculate the distances for 4 1/2 hours and 5 1/4 hours with the equation. (Themes 3, 6, and 8)

With the situation of organizing a bike tour, students were asked to write equations of income, bike rental, food and camp cost, van rental and the total cost according to the number of customers using a given table. (Themes 3 and 8)

While the first phase of interviews addressed all the eight themes following the classroom activities, the second phase of interviews focused on the relationship between symbolic representation and the other forms of representation of changing situations. That is, I pursued questions of how students used algebraic notations to represent changing situations and how they made sense of algebraic notations with respect to narrative, tabular, and graphical representations. Some interview questions were adopted from the CMP unit being studied. The interview questions in the second phase are shown below. The interview numbers begin at Interview 6 to be consecutive with the 5 interviews for each pair conducted during the first phase.
Interview 6

1. What do you think a rule means?

2. Suppose you are asked to baby-sit your younger sister or brother and your parents would pay you $3 per hour. Can you find a rule for the total payment? Can you use symbols to express the rule as an equation?

3. Suppose you join a book club and you have to pay $10 membership fee and $4 per book purchased. Can you find a rule for the total cost? Can you use symbols to express the rule as an equation?

4. This table shows the relationship between the number of people on a club picnic and the cost of lunches. Can you find a rule for the cost? Can you use symbols to express the rule as an equation? Using the equation, can you find the lunch cost for 25 people? How many people could eat lunch if they had paid $89.25?

<table>
<thead>
<tr>
<th>Number of people</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost in dollars</td>
<td>4.25</td>
<td>8.50</td>
<td>12.75</td>
<td>17.00</td>
<td>21.25</td>
<td>25.50</td>
<td>29.75</td>
<td>34.00</td>
<td>38.25</td>
</tr>
</tbody>
</table>

5. This graph shows the relationship between the number of concert tickets and the total cost. Can you find a rule for the total cost? Can you use symbols to express the rule as an equation?

Interview 7

1. Given the equation, $d = 8t$, can you make up a story? What do $d$, 8, and $t$, represent in your story, respectively?

2. Make a table that shows the distance traveled for every hour up to 5 hours.
3. Can you make a table that shows the distance traveled for every half hour up to 5 hours? How do the two tables differ? What do they have in common?

4. Draw a graph of the equation, \( d = 8t \).

5. Make up a story with an equation \( C = 2n + 5 \) and draw a graph of it.

**Interview 8**

Sidney started a table to help the partners determine their cost for the bike tour.

<table>
<thead>
<tr>
<th>Number of customers</th>
<th>Bike rental</th>
<th>Food and camp costs</th>
<th>Van rental</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$30</td>
<td>125</td>
<td>700</td>
</tr>
<tr>
<td>2</td>
<td>60</td>
<td>250</td>
<td>700</td>
</tr>
<tr>
<td>3</td>
<td>90</td>
<td>375</td>
<td>700</td>
</tr>
<tr>
<td>4</td>
<td>120</td>
<td>500</td>
<td>700</td>
</tr>
<tr>
<td>5</td>
<td>150</td>
<td>625</td>
<td>700</td>
</tr>
<tr>
<td>6</td>
<td>180</td>
<td>750</td>
<td>700</td>
</tr>
<tr>
<td>7</td>
<td>210</td>
<td>875</td>
<td>700</td>
</tr>
<tr>
<td>8</td>
<td>240</td>
<td>1000</td>
<td>700</td>
</tr>
<tr>
<td>9</td>
<td>270</td>
<td>1125</td>
<td>700</td>
</tr>
<tr>
<td>10</td>
<td>300</td>
<td>1250</td>
<td>700</td>
</tr>
</tbody>
</table>

1. Can you find a rule for the total cost for any number of customers? Can you write the rule with symbols?

2. Theo’s father has a van he will let the students use at no charge. Write a new rule for the total cost for any number of customers with symbols.

3. If the partners require customers to bring their own bikes, write a new equation for the total cost.

**Data Analysis**

Video-recorded interviews were the primary data source along with students’ written work during the interviews in the present study. The method to analyze the video data was informed by iterative videotape analyses (Lesh & Lehrer, 2000), which had several interpretation cycles. The first interpretation cycle included producing interview graphs (see Appendix A) right after each interview, which had the same format as the classroom lesson graphs. The
interview graphs were produced to describe how the interviews went on along with snapshots of students’ work. The time stamps were included in order to locate interview portions in movie files. In particular, the interview graphs showed chunks of the interviews and the chunks were separated according to the interview tasks, the eight themes that emerged from observing students’ classroom activities, and students’ responses. Dividing the interviews into chunks was also iterative. The interview graphs also included brief analytic notes, which served as initial assertions, and feedback from both the interviewer and an observer. In fact, unlike the first phase of interviews, the second phase of interviews had an observer, Olive, who observed the interviews at the site while operating the camera capturing students’ written work and provided feedback for the interviews.

The second interpretation cycle included writing transcripts of selected interviews. Since the second phase of interviews was more directly oriented toward the research questions, they were transcribed completely. However, the first phase of interviews was selectively transcribed. In order to select interview portions that were relevant to the research questions, I coded the chunks of the interviews in the interview graphs with key words and developed the tables of coding. With the tables, I was able to locate the interview portions that provided supports to answer the research questions.

The third interpretation cycle was analyzing all the interviews across students to answer the research questions. In this cycle, I physically cut out the chunks of interviews from the selected interview portions in the first phase and the interviews in the second phase, and rearranged them under the same types of referential relationships. With the help of Kaput’s (1991) referential relationships model, I identified three kinds of students’ acts: acts of interpretation, mental operation, and projection to a physical display (p. 59) in each type of
referential relationship. These analyses developed into the assertions to answer the first research questions. For the second research question, I located chunks related to students’ conceptions of variables and rates in the rearrangement and extracted themes for assertions.

All three-interpretation cycles were to be repeated to produce the assertions to answer the research questions consistently with the help of theoretical orientations of the present study. While repeating the cycles, supplementary data sets, such as students’ homework, notebook, and video clips showing their participation and discussion in classroom activities, were revisited. Also, the assertions as well as the interview graphs and the transcripts were revised repeatedly.

As an illustrative example of analysis, I reproduce the first two chunks of Interview 7 with Jeffery from the interview graph as follows. The first column in the following table contains the time stamps and the second contains the description of each chunk. The description shows how each chunk of the interview went on from my perspective, which could be considered as an initial analysis. The third and the fourth columns illustrate how my analytic notes and comments for each chunk of the interview were elaborated or revised while repeating the interpretive cycles.

For instance, regarding the first chunk of this interview, I attended to the fact that Jeffery was not able to make a story with the given equation and I wondered whether the reason for his inability to identify the operation was that he simply forgot the syntax in writing an equation. However, my revised comment, as shown in the fourth column, was that he might not appreciate the given equation as representing a changing situation. After repeating the interpretive cycles, I also located supporting evidence in the second chunk when he considered only one letter in the equation, and related the claim with the evidence.
In the second chunk of this interview, I highlighted key words for analysis with bold fonts. Especially, I thought that Jeffery could use the word “any number” in a sense of an independent variable. However, as repeating my interpretation, I found a possible relation of his word “any number” to an expression used in the CMP materials and in class, “Write an equation for the rule to calculate each of the following costs for any number, \( n \), of customers” (Lappan et al., 2004, p. 53). Once I hypothesized that Jeffery might consider a letter as any number literally, I revisited the whole interview and tested my hypothesis. Thus, I reached the claims in the fourth column.

Table 3

*Example of Analysis of Interview 7 with Jeffery*

<table>
<thead>
<tr>
<th>Time</th>
<th>Description</th>
<th>Comment 1</th>
<th>Comment 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>8:09:29</td>
<td>Interview begins with giving an equation ( d = 8t ) and asking Jeffery to read it. He reads, “( d ) equals eight ( t )”. I ask him to make up a story, but he says no. I then ask him what putting a number and a letter together means and he could not answer. Thus, I write, ( d = 8 \times t ) and ask him about the meaning of the dot. He answers it can be “plus”. Since we had a discussion of writing an equation with symbols in Interview 6, I give the other equation ( d = 8 \times t ) in order to remind him of it. Then, he seems to remember all that three equations are basically same but look different.</td>
<td>Jeffery could not initiate a story with the equation. He has a difficulty to identify an operation between 8 and ( t ) – related it to how he read the equation initially. Can it be a syntactical matter?</td>
<td>Does he appreciate the equation as a representation for a changing situation? – not supportive because next chunk shows that he does not consider two changing quantities (or things) for the equation.</td>
</tr>
</tbody>
</table>
I ask again Jeffery whether he can make up a story now. He tells that it could be dance equals 8 times t. He means that d could be a cost to get in dance and t would be the number multiplied by 8. So I ask what t could be, and he says time first and any number later. I ask him to pick numbers and show some story with them. He writes, “16 = 8 × 2” and identifies d is 16 and t is 2 in his equation. But he still cannot make a story with the equation. So I ask whether he can pick other numbers for d or t. Then he explains that he can pick any numbers for d or t but 8 will stay the same (and he seems to think that once a number for d is picked the number for t will be decided with the relation of d = 8t). So I ask what if I pick 40 for d and then he writes 40 equals 8 times 5.

He names d as dance initially: not assigning a quantity – changes it into a cost to dance.

The letter t does not have meaning for him.

He conceives 8 as constant. What about the meaning of 8?

He thinks a letter could be any number: as an independent variable?

Interestingly, he keeps the relation of two letters – due to his numerical understanding.

Both letters can be any number as long as they keep the operational relation.

Assign a quantity (or naming a thing) for one letter (d) and break it down into two factors (8 & t).

A letter is any number as a generalized number – supported by the chunk when he makes a story with $C = 2n + 5$, and the expression in “find a rule working for any number”

He seems to consider the operation as given.
CHAPTER IV
RESULTS OF DATA ANALYSIS

In this chapter, the results of data analysis will be presented. The assertions were made mainly from the analysis of the second phase of interviews and the analysis of the first phase of interviews provided supporting ideas of the assertions. The results will be presented according to the research questions. For the first research question, three ways of referential relationship shown in Figure 3 were analyzed: algebraic notations and narrative representation, algebraic notations and tabular representation, and algebraic notations and graphical representation. Under each referential relationship, I will describe students’ referential actions responding to the interview tasks and the analysis of students’ actions. For the second research question, students’ conceptions of variables and rates were analyzed in different types of representation.

Figure 3. Referential relationships in this study.
Three Ways of Referential Relationship Centered around Algebraic Notations

This section will answer the first research question: How do students make sense of algebraic notations in relation to other forms of representation throughout mathematical activities? This question focused on students’ referential relationships between algebraic notations and other forms of representation. For each referential relationship, I will first describe how the students responded to the related interview questions. The description will be organized by the interview questions; that is, I will state an interview question and describe each student’s response. Following the description, the analysis of the responses across all the students will be presented along with a scripted diagram of Kaput’s (1991) referential model. The other theories aforementioned also provided supporting explanations of students’ referential relationship within Kaput’s (1991) model. After describing all three ways of bi-directional referential relationships, I will summarize student’s referential relationships centered around algebraic notations.

Algebraic Notations and Narrative Representation

In order to investigate students’ bi-directional referential relationships between algebraic notations and narrative representation, students were asked to write a rule for a changing situation given in a narrative form in Interview 6 and to make up a story for a given rule in Interview 7. Before asking to do so in Interview 6, I played a class video clip where Ms. Moseley introduced algebraic notations in class to remind the interview students of how algebraic notations were introduced in class. Since there was a 2-month break between the first and the second phases of interviews, it was necessary to remind them of how they were introduced to algebraic notations.

In the video clip, Ms. Moseley talked about the advantage of rule over table or graph (i.e., being able to predict values not included in a table or a graph) and showed how to write a rule or
formula using algebraic notations with an example of representing how much income would be earned when each customer paid $350. She used the letters of \( I \) and \( N \) standing for the total income and the number of customers, respectively, and wrote the equations as shortening the multiplication sign: \( I = 350 \times N \), \( I = 350 \cdot N \), and \( I = 350N \). At the end, Ms. Moseley commented about the letters in the equations:

This (as she double-underlines the letter \( N \)) is nothing more than something we decided to use for a number of customers. Why they use \( N \)? Because of number. Number of customers. Why they use \( I \) (as she pointes to the letter \( I \) in the equations) over here? (students answer income all together) Income. So a lot of times, they make sense, sometime they don’t make sense. They’re just random.

After playing this video excerpt, I asked the interview students what they thought a rule meant. Peggy answered her meaning of a rule in the example of the class video clip saying, “What you did to get a certain number for the income.” Then, she restated What Ms. Moseley did in the video clip as “350 times how many of the customers there were led to get the income”. Angela could not complete what she thought a rule meant at first, but after Jeffery said that a rule was something that they used to solve a problem, she completed without full confidence that a rule was like a step of doing something.

*From Narrative Representation to Algebraic Notations*

*Description.* The interview students were asked to write a rule with algebraic notations for changing situations given in narrative form in Interview 6. Two scenarios of a changing situation included a linear relationship between two variables. The first scenario was that they were asked to baby-sit their younger sister or brother and their parents would pay them $3 per hour. In the interview with Greg and Peggy, I asked them how much the parents would pay for the baby-sitting in the scenario after students read the written scenario. Greg answered $3 per hour and I asked him what the phrase ‘$3 per hour’ meant. He answered, “That every hour you
get $3.” Then, I asked what rule they found in the situation, and Peggy responded that multiplying $3 by how many hours to work would give the total payment.

In the interview with the other pair of students, I asked students how much they would get. Jeffery answered that $3 times the number of hours was what the parents would pay. After Jeffery answered, Angela added that the longer they baby-sat the more money they would get. Then I asked her why they would get more money and she answered as follows:

Because it’s *per hour* and if you stay there, if you baby-sit for two hours you’ll get $6 since it’s like 3 times 2. And then, like you said, like if you stay there for 6 hours, which I don’t think you would, but… umm… So you’d like baby-sit 4 hours or something, you would times that by 4 and that would equal to how much money you’d get.

As Angela emphasized, all students seemed to pay attention to the phrase ‘$3 per hour’ and they figured out how to find out the payment using the context.

However, when writing a rule, each student wrote differently. In the first pair, Peggy wrote two equations, \( I = 3 \times H \) and \( I = 3 \cdot H \), and she explained that she put a dollar sign $ in front of 3 to represent 3 dollars. Greg wrote three equations, \( I = 3.00 \times N \), \( I = 3.00 \cdot N \) and \( I = 3.00N \) as Ms. Moseley showed in the class video clip, and he explained that \( I \) and \( N \) stood for income and the number of hours, respectively. For the equations that Peggy and Greg wrote, Peggy said that there was no difference among their equations and Greg said that they just used different letters to represent the same thing.

Of the other pair of students, Angela wrote, \( 3 \times \text{# of hour} = \text{pay} \), and Jeffery wrote, \( 3$ \times N = C \) and changed it into \( C = 3$ \times N \) shortly. After Jeffery explained that he wrote his equation as shown in the video clip and he used \( C \) for cost, Angela stated that Jeffery’s equation would be much easier to write because it had fewer words. Like Peggy, Angela said that she put the dollar sign $ in front of 3 to make sure this was about the money. However, she added that putting extra words in the equation was not necessary but it would make the situation of the
equation clear to anybody. Then, I asked her to rewrite hers in a simpler way and she wrote then, $C = 3N$.

The second question was also asked in Interview 6 and the question asked how much they had to pay if they joined a book club that charged $10 membership fee and $4 per book purchased. Peggy told that she could do ‘$4 per book purchased’ part but she had no idea about the membership fee. Greg claimed that the cost would be $10 membership fee for only join plus $4 times how many books purchased. When they were asked to write the rule, Peggy wrote $I = 4 \times N$, $I = 4 \cdot N$, and $I = 4N$ without the membership fee added. She explained that she used the letter $I$ for income, and so I asked whose income it was. Peggy answered it was the book club’s income. Greg wrote his equation effortlessly as $C = 10 + (4 \times N)$ and explained he put the parenthesis indicating that the multiplication had to be done first. Greg considered the payment as his cost to join the book club and used the letter $C$ for it.

In the interview with the other pair, Angela began to write $C = 10$ saying that it looked weird and completed her first equation $C = 10 + 4N$ first. Then, she explained her equation as saying, “the cost that it is to get in is $10 for membership fee and it’s $4 per book purchased”, and right after she changed it into $C = 104N$ saying that the second equation was better. For her second equation, Angela still explained that the cost was $10 and $4 times the number of books. Thus, I asked Jeffery to read Angela’s second equation, and he read, “The cost equals 10 plus 4 and then number.” When I told her that 104 in her equation looked like one number, Angela replied that she meant plus between 10 and 4. She seemed to think that she could shorten the equation by leaving out the addition sign like she could do with the multiplication sign. Since Jeffery had not written his equation for this situation, I asked Jeffery to write his equation and he wrote $M = 10$ meaning the membership was $10$. Unlike the other pair, Angela raised a
question how often they had to pay the membership fee, and I told them to assume a lifetime membership. Then, I asked Angela how much they had to pay in total for one book purchased with the membership. She answered $14 and she seemed to make sense of her first equation.

**Analysis.** This section will contain analysis of students’ reactions in terms of Kaput’s (1991) referential relationship model and the following figure will help understand the analysis including students’ interpretation of the narratives, cognitive operation in the referential relationship, and projection of the result of the cognitive operation. At first, all four students deliberately interpreted the given situations in the narratives to write equations for them as identifying what quantities changed. Identifying what quantities changed in the narratives did not seem challenging to the students, but it did not imply that they were able to identify the changing quantities as two variables successfully.

![Figure 4](image.png)

*Figure 4.* Referential relationship from narrative representation to algebraic notations.
In fact, all the students recognized that the payment for baby-sitting and the baby-sitting hour were changing in the first scenario as Angela said explicitly, “the longer they baby-sit the more money they would get”. In the second scenario, the interview students except Jeffery recognized successfully the total payment and the number of books as changing quantities. Thus, it was sufficient to say that students could identify quantities that changed in the situations but it was not enough to insist whether they were aware of those quantities as the dependent and the independent variables in each situation. In other words, something that changes, which was the definition of variable in Ms. Moseley’s class, was not necessarily registered as a variable that we as mathematically knowledgeable adults conceived. It seemed dependent upon each student’s conception of variables, which will be discussed later in answering the second research question.

Students also recognized key word(s) as interpreting the given situations. The key word(s) gave students a clue to figure out the operation(s) between two changing quantities. Both situations in the interview contained the word ‘per’ as in ‘$3 per hour’ and ‘$4 per book’. All students apparently interpreted ‘every hour you get $3’ and ‘every book you pay $4’ for the phrases and considered them as a constant rate. However, depending upon students’ conceptions of rates, some might have understood it additively or multiplicatively. For instance, students, who understand a rate ‘$3 per hour’ additively, keep adding $3 by increasing the number of hours and could simply switch the repeated addition into multiplication. On the other hand, students with multiplicative understanding of it could generalize the rate over the natural numbers and use it proportionally. Students’ conceptions of rates will be discussed more thoroughly in the section that will answer the second research question. Unfortunately, the first interview question did not uncover the differences in the students’ responses. That is, whether they understood the rate additively or multiplicatively, they could have written the rule with
multiplication since they knew that repeated addition could be written as multiplication as Angela explicitly mentioned. Thus, in either way they figured out that multiplication was the operation used for a rule and so they could verbalize the rule that the payment was the product of $3 and the number of hours in the baby-sitting situation.

On the other hand, in the second situation, students had to figure out the relation among the total payment, the book cost and the membership fee. The complicated relation caused some students difficulty figuring out operations. All students easily figured out that the book cost was the product of $4 and the number of books purchased as they did in the first situation by recognizing the key word ‘per’ in the phrase of ‘$4 per book purchased’. However, some seemed to have difficulty relating the key word ‘and’ in the phrase of ‘to pay $10 membership fee and $4 per book purchased’ to adding the membership fee and the book cost in order to get the total payment. Jeffery and Peggy could not write an equation for the situation and they only represented a partial payment as $M = 10$ and $I = 4N$, which might be due to their limited experience of writing rules with only a multiplicative relation. However, Greg and Angela seemed to have no difficulty figuring out the operations (e.g., addition and multiplication) for the total payment.

When they had to project the rule into algebraic notations after recognizing quantities that changed and figuring out the relation and the operation(s) between them, all students tended to pick the first letter of what they wanted to represent (e.g., $I$ for income, $C$ for cost, etc.) and

5 Relation and operation between quantities were differentiated in this study in that relation was used to describe how one quantity affected another (e.g., the payment of baby-sitting is proportional to the hour of baby-sitting) whereas operation specifically meant numerical relation between quantities such as addition and multiplication.
wrote the rule with the letters and operational symbols. In that process, several syntactical issues came in to play. At first, some extra words and signs were intentionally used in order that they wanted to make the rule clearer to others. For instance, Angela put extra words in $3 \times \# \text{ of hour} = \text{pay}$, and Jeffery and Peggy often put the dollar sign $\$ \text{ in front of money.}$ For the need of the dollar sign, Angela said:

Not really, but if you want to make sure that, like if you’ve never seen before, the question before, you don’t know what the question is. You might want to put the dollar sign to make sure that you’re talking about money. But you don’t really have to.

This implied that they were insecure in eliminating the context and abstracting the relation out of the context into an equation and, in addition, they might not be sufficiently informed of syntax in writing rules.

One of Angela’s equations, $C = 104N$, was also considered as a syntactical issue. Based on the fact that she wrote the equation $C = 10 + 4N$ first and changed it into $C = 104N$ by saying that the second one was better, she seemed to over-generalize the syntax that allowed omitting the multiplication sign into also omitting the addition sign. It implied that she simply focused on the succinctness of the syntax without realizing that it caused confusion. On the other hand, Greg did not omit both addition and multiplication signs in his equation, $C = 10 + (4 \times N)$, and used the parentheses to indicate that the multiplication had to be done before addition. Thus he seemed to clearly understand that his equation had to avoid any possible ambiguity.

The other syntactical issue was which side to put the independent and the dependent variables in the equation. In fact, this issue was not dealt with in Ms. Moseley’s class explicitly and students wrote the equations, as they wanted. This syntax might not be as important as the previous ones in that the equations written in either way still conveyed the same relation and
caused no confusion to understand it. However, by having this syntax discussed explicitly, students could have an opportunity to understand the relationship between the independent and dependent variables more sophisticatedly, and they could have a coherent way of writing equations to represent changing situations.

*From Algebraic Notations to Narrative Representation*

*Description.* Students’ referential relationship from algebraic notations to narrative representation was sought by asking them to make up a story with given equations in Interview 7. Two types of equations were provided: one was \( d = 8t \) and the other was \( C = 2n + 5 \). Since Angela was absent from the school on the day of Interview 7, Jeffery had to have a solo interview. The analyses below will include only three students: Greg, Peggy, and Jeffery.

For the equation \( d = 8t \), Peggy made up a story of traveling from Georgia to Washington, DC at 8 miles per hour. At first, she said, “Well, you can do \( d \) for distance, travel like 8 miles per time or something”. Then, when asked to be more specific, Peggy restated, “You’re traveling from Georgia to Washington, DC and you’re traveling 8 miles per hour. And you have different time intervals for when you stop.” She appeared to consider \( d \) as the distance and \( t \) as the time. Although I asked her what 8 meant in the equation, she simply repeated 8 miles per hour and so it was not clear how she related the speed to the distance and the time. Greg, interviewed with Peggy, did not make up his own story but commented that he thought just like what Peggy said.

When asked to make up a story with the equation, \( d = 8t \), Jeffery could not even begin. I asked him what putting a number and a letter together like \( 8t \) meant. Still he could not answer. Then, I wrote another equation \( d = 8 \cdot t \) and asked what the dot between 8 and \( t \) meant. Jeffery answered it could be “plus” meaning addition. I wrote another equation \( d = 8 \times t \) and reminded
him of the interview about writing an equation with algebraic notations in Interview 6. I again asked him to make up a story with the equation. The following excerpt from the interview shows the story Jeffery tried to make up with the equation.

Protocol 1: Interview 7 with Jeffery

Jeffery: It could be dance equal 8 times t. It could mean dance or something.
Interviewer: Dance? What do you mean by it?
J: Like the cost of, to get in a dance could be…
I: Oh, ok. So what does d stand for then?
J: Dance. It would be like what you were going to or something.
I: It’s more like a cost to go to the dance, right?
J: Uh-huh.
I: How about t then there?
J: It will be like the number that you are multiplying 8 by.
I: So what kind of number could that be?
J: Time or…
I: Time?
J: I think it could be any number.

Jeffery seemed to think of going to dance and paying some money for it, but he was not able to finish his story relating to the equation. Intending to help him, I asked him to pick some numbers for d and t and tell a story using the numbers. Jeffery was able to select a pair of numbers of d and t and completed the equation as $16 = 8 \times 2$, but he was still unable to tell a story with the equation. Since he told that t could be any number, I asked him whether d could be any number also. He answered;

You could use like for d, you could put another number, like whatever you want and then 8 would be the same and then the t would just be what this number that you would do it by 8.

As I picked 40 for d, Jeffery could tell t had to be 5. That is, Jeffery could pick numbers satisfying the given equation due to his understanding of multiplicative relations but he could not relate his numerical understanding into a changing situation with variables. Thus, what he meant
by his saying “I think it could be any number”, was that $d$ and $t$ could be any number as long as
the equation held true with the picked numbers.

For the second equation $C = 2n + 5$, Greg and Peggy worked together to make up a story
after Greg initiated that cost equals 2 times the number of people plus 5.

Protocol 2: Interview 7 with Greg and Peggy

I: Ok, can you make up a situation to be more specific?
G: There’s two people going on a trip. There’s two people going on a trip and umm, you
have to rent two water bottles, I guess, something like that. I don’t, … I don’t know.
I: Ok, why don’t you try, Peggy?
P: The cost equals $2 for the number of people going somewhere and you have to add $5
for like food or something.
I: Umm, does it make sense to you, Greg?
G: But don’t you have to multiply 2 times…
P: Yeah, multiply the number of people by the 2. Like if 15 people were going, like on a
getaway or something, they’ll pay like $2 or something. It’s like a cheap little thing.
You’ll pay $2 per that person and then you’re going to like add 5 for like food or
something.
I: Ok, so, ok. First of all I think you, both of you agreed on maybe $C$ is kind of
representing cost. Ok. And then what does $n$ represent for?
G & P: Number of people.
I: Ok, so what was 2 in your story, Peggy?
P: $2 to go somewhere.
I: $2 to go somewhere, like maybe admission fee or something like that?
P & G: Yeah
I: So $2 for people. How many people?
P: 15 or so.
I: I mean, how many people can go with $2?
G: 1
P: 1 or 2
I: One person? Ok, how about the 5? 5 is…
G: 5 per, umm, 5 is the same price for all the people that come, because you’re adding it
to whatever you get would be 2 times the number of people.
I: So 5 is just for the whole people, whatever, how many people you have.
G: It’s the same price.

As Greg told that two people were going on a trip and they rented water bottles, he seemed to
realize the story did not fit into the equation somehow and therefore he stopped. However,
Peggy continued the story by considering 2 as a cost to go somewhere and 5 as food cost, but she
was not clear whether she thought $2 for each person or all the people. When Greg told her that she had to multiply 2 by the number of people, Peggy seemed to make sense of the story appropriately by saying, “You’ll pay $2 per that person”. In order to make sure, I asked them how many people could go with $2. Whereas Greg confirmed that he interpreted the equation appropriately by answering 1, Peggy showed some confusion by answering 1 or 2. This implied that either Peggy meant that $2 was the constant cost for any number of people or she herself was not sure. She did seem to realize, however, that the terms, $2n$ and 5, in the equation should be of the same kind of quantities (dollars).

In making up a story with the equation $C = 2n + 5$, Jeffery suggested that cost equaled 2 times a number plus 5. When asked what cost he thought of, he finally answered that $C$ could be a candy bar cost, but he explained that $n$ was just any number. Again he was able to pick numbers for each letter to make an example as in making up a story with the equation $d = 8t$. So when Jeffery verbally completed $9 = 2 \times 2 + 5$ for the candy bar cost $9$, I asked him how many candy bars he would get. He answered one and he explained that the equation let him know the cost was $9$. So I asked him what 2 in the place of $n$ represented and he answered it represented $2$. Next, I asked him what if $n$ was $3$ and he completed $2 \times 3 + 5 = 11$. Again I asked him what 2 as a coefficient of $n$ represented and then he answered it meant 2 candy bars. Then, he told me that 5 would be a tax. So I asked how much one candy bar would be. He answered $1.50$, which he seemingly divided $3$ by 2 and he changed his answer into $6.50$ as saying, “It will be 11 if you had 2 candy bars, it would be $6.50$ for one.” I asked to him to find out the total cost if he had 2 candy bars, of which each one cost $1.50$ and paid $5$ tax as he previously explained. With the question, I intended to show his total cost, $11$, did not fit into his candy bar story. However, I had to stop the interview due to the time limit of the interview.
Overall, Jeffery seemed to think of the candy bar cost represented by $C$ in the equation first and he was able to find out any number for $n$, which corresponded to a certain $C$ in terms of the given equation thanks to his numerical understanding. Apparently Jeffery did not consider the cost as a variable that could vary depending upon something represented by $n$ in the equation. Rather he seemed to name the letter $C$ as the cost. Thus, the series of questions that I asked forced him to decompose a certain candy bar cost into the possible elements like the number of candy bars, dollars, or tax. Therefore, his answers did not make sense with the equation.

**Analysis.** In general, inventing a changing situation in narrative form based on a given algebraic notation was definitely not a simple reverse process of the opposite direction. It seemed even harder for the students to make up a story from an algebraic notation rather than writing a rule from a story, especially when their conceptions of variables were not fully developed.

Students did not seem to interpret a given algebraic notation so as to figure out what relation existed between two changing quantities and what situation would fit with the relation with understanding that the algebraic notation represented a changing situation with two quantities. Rather than interpreting a given algebraic notation as a whole, they seemed to focus on the letters in the algebraic notation and assign possible quantities to them. Mostly they assigned quantities that began with the given letter (i.e., $C$ for cost, $d$ for distance, etc.), which was implicitly encouraged through their experiences in class. Peggy simply made up a traveling story with $d = 8t$ by assigning distance and time for $d$ and $t$, respectively, which must have been familiar to her due to her experience in class. Sometimes, assigning things, not even quantities, to the letters was all they did to make up a story with an algebraic notation. Jeffery simply assigned a word, dance, to the letter $d$ without even explaining what he meant with dance in the
equation. When asked to be specific, he said that \( d \) would be the cost for dance without assigning anything to the letter \( t \). When asked what \( t \) could be, Jeffery assigned time to \( t \). So far, Jeffery did not seem to think of the relation between two quantities that they assigned to the letters.

When encouraged to complete their stories after assigning quantities to the letters, they seemed to pay attention to the operation in an algebraic notation. Peggy did not explicitly state the relation between the distance and the time traveled for \( d = 8t \). However, when she had a discussion over Greg’s story for \( C = 2n + 5 \), she showed clearly that she figured out operational relations between \( n \) and \( C \) when she said, “You’ll pay $2 per that person and then you’re going to like add 5 for like food or something”. It implied that Peggy and Greg began to think of the relation between two quantities.

In contrast, Jeffery showed very different interpretations of the operation in an algebraic notation. For the equation \( d = 8t \), he seemed to consider \( d \) as the only thing that changed and assigned the cost for a dance. Apparently he regarded \( t \) as any number that satisfied the equation \( d = 8t \) according to a value of \( d \) when asked. Since he had a good numerical understanding, he was able to generate several pairs of numbers \((d, t)\) for this equation. Nonetheless, he did not confirm that \( t \) could be a time that varied in relation to the value of \( d \), but I inferred that for Jeffery \( t \) could be “any number” as long as the product of the value of \( t \) and 8 made the value of \( d \). Also for the equation \( C = 2n + 5 \), he only named \( C \) as the candy bar cost without naming or assigning a quantity for \( n \). In fact, Jeffery seemed to initially mean \( C \) as the cost for one candy bar, but when picking up $9 for \( C \) and considering the corresponding value for \( n \) when asked, he broke down $9 into \( 2 \times 2 + 5 \) with constants 2 and 5 in the equation. It suggested that he did not have any meaning for 2 in place of \( n \) and the constants in advance, and he seemed to try to give
some meaning for each in an ad hoc manner. This lack of meaning for the constants and the letter $n$ seemed to be the reason why Jeffery contradicted himself when he considered $11$ for the candy bar cost.

In addition, the difference between Jeffery and the other students seemed to be whether they identified the letters in the equations as variables or not. Jeffery seemed to understand that the letters in the equations represented numerical values that could change and therefore he was able to generate different pairs of numbers for the letters. Nevertheless, Jeffery had shown no evidence that he interpreted the letters in the equations as variables that changed in relation to each other. There was also no evidence that he tried to figure out the relation between two variables. Peggy and Greg, however, first interpreted two letters as changing in relation to each other and thus, they figured out the relation between two quantities using the operations in the equation.

Once students figured out the relation between two quantities, they seemed able to understand what the situation would be with the equation although they did not completely state the story in terms of the independent and the dependent variables. Through discussion in the interview, Peggy and Greg were able to explain the changing situation of going on a trip with the equation $C = 2n + 5$, in which $C$ and $n$ stood for the cost and the number of people, respectively.
Thus, in this referential relationship, students, in fact, did not complete their projection of interpretation of algebraic notations and their mental operation in terms of Kaput’s model (1991). The lack of projective action is shown in the dotted box in Figure 5. Moreover, students would have been expected to state a story in terms of the changing quantities and the relation between them without worrying about structural syntax involved in the previous referential relationship from narrative to algebraic notation.

*Figure 5. Referential relationship from algebraic notations to narrative representation.*

For the referential relationship from tabular representation to algebraic notations, the participating students were asked in Interview 6 to write a rule for a given table showing the lunch cost for the numbers of people, and in Interview 8 to find rules for the total cost of a bike tour as changing situations given in a table. The questions in Interview 8 were adapted from the CMP textbook that had been used in a follow-up interview after the students had covered the
content in class at the end of the first phase of interviews. At that time, somehow the context underlying the questions caused confusion to students and therefore, in Interview 8, I streamlined the questions in order to see how the students formed the rules for the total costs based on the given table, wrote the equations for the rules, and operated with the equations. For the interview questions in this portion, the students would reveal not only how to write a rule with algebraic notations but also whether they were able to use the algebraic notations, think with the algebraic notations, and operate with them. For the other direction of the referential relationship, students were asked to generate a table for an equation \( d = 8t \) after making up a story with it in Interview 7.

From Tabular Representation to Algebraic Notations

**Description.** In Interview 6, the students were asked to find a rule for the lunch cost when the lunch costs for the numbers of people from 1 to 9 were given in a table.

<table>
<thead>
<tr>
<th>Number of people</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost in dollars</td>
<td>4.25</td>
<td>8.50</td>
<td>12.75</td>
<td>17.00</td>
<td>21.25</td>
<td>25.50</td>
<td>29.75</td>
<td>34.00</td>
<td>38.25</td>
</tr>
</tbody>
</table>

Peggy said that multiplying $4.25 by the number of people going on the picnic would lead to the total lunch cost. When asked why, Peggy explained that $4.25 times 2 was $8.50 for two people and it kept going on. To make it clear, I asked her how much the cost would be for three people and she explained that she obtained the answer by adding $4.25 to $8.50. Peggy seemed to pay attention to the additive pattern of adding $4.25 as increasing one more person. Greg agreed

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6 With the context of operating a bike tour business, the original tasks in the CMP textbooks asked students to write rules for the income, the costs, and the profits for any number of customers, and these business terms caused confusion because students were not familiar with the terms and the relationships among them.
with Peggy. Then, both wrote a rule $C = 4.25 \times N$ and Greg wrote two more equations with different multiplication signs.

In the interview with the other pair, Jeffery’s reasoning was the same as Peggy’s and he explained that the lunch cost for each person was $4.25 since when adding one more person, he could keep adding $4.25 to get the total lunch cost according to the table. Angela agreed with Jeffery. Then, Angela and Jeffery wrote the rules $C = 4.25 \cdot N$ and $C = 4.25 \times N$, respectively. In general, all the participating students recognized the additive pattern in the table, but they were able to write a rule with multiplication.

Once they set up the equation for the total lunch cost, I asked them to calculate the lunch cost for 25 people using the equation. All of the students except Angela multiplied 4.25 by 25 and got the same answer $106.25$. Angela tried to multiply 25 by 4.25, which she knew was the same as multiplying 4.25 by 25, but she got an incorrect answer $1006.25$ because she put the third row multiplication (i.e., $25 \times 4$) to the left of the second row multiplication (i.e., $25 \times 0.2$) instead of underneath it (see Figure 6). When comparing Jeffery’s multiplication, Angela realized that her calculation did not make sense and said that she made a mistake. However, I focused on her operation itself not on her decimal multiplication since the decimal multiplication was not the focus of this study. All students apparently used the equation to calculate the total lunch cost correctly since they already figured out the relationship between the number of people and the lunch cost to set up the equation.
With the equation for the total lunch cost, the students were asked to find how many people could afford lunch with $89.25. Both Peggy and Greg attempted to divide 89.25 by 4.25, but neither did the decimal division directly. Instead they tried to multiply 4.25 by some numbers to find out how many times 4.25 would go into 89.25. Peggy began multiplying 4.25 by 9, 5, and 4 in order, which were unnecessary since these were already given in the table, and she explained, “I’m just trying to see how close I could get to 89”. Then Peggy added, “I went all the way to 9 and see if, what it would come to next, came out like 38.25 or something like that”. Without realizing that the product of 4.25 and 9 was much less than 89.25, she tried lower numbers of 5 and 4 instead of higher numbers. However, she explicitly stated a little later that she thought more than 9 people could eat. Thus, I asked her to estimate how many people she thought could eat and she answered 12 or 13 adding that she was guessing.

Greg started multiplying 4.25 by 20, which was quite reasonable estimate and then he tried to multiply 4.25 by 19 and 18. So I asked why he lowered the numbers to be multiplied and he said, “I messed up on the 20 and I kept going down, but I thought 20 was higher than 89. But it wasn’t”. In fact, Greg stopped his multiplication at 8500 in the second row, which he seemingly thought of his answer as 85.00 and his calculation was right. I inferred that what he
thought he messed up was that he tried 19 and 18 to be multiplied instead of numbers like 21 and 22 because he confusingly thought 85.00 was higher than 89. Thus I rephrased Greg’s words that he thought the product of 4.25 and 20 would be higher than 89 and therefore he tried to multiply 4.25 by numbers less than 20 to get the answer. Greg agreed with my words, and I asked Greg and Peggy to stop finding the answer for the division. Instead, I asked them whether they could write a rule for the number of people given the amount of lunch cost. Peggy stated that the cost equaled 4.25 times the number of people using the multiplicative relation, but Greg was able to tell that the number of people equaled the cost divide by 4.25. Then, he wrote an equation \( N = C \div 4.25 \) confidently.

Of the other pair of students, Jeffery answered quickly that 23 people could eat lunch, when I asked how many people could eat lunch with $98.25, which I mistakenly said instead of $89.25. He explained how he got the answer “because you know if for 25 it’s $106 so you would just take off $8.50 from it, which I think would give you $98”. Jeffery made an estimation based on his previous calculation that the lunch cost for 25 people was $106.25. Then I corrected myself and asked again how many people could eat lunch with $89.25 as intended. Jeffery answered as below.

I think you would do the same thing but you would have to take off more money. You would take off $4.25 for every time you go back a number of customers.

Then, I asked whether the equation \( C = 4.25 \times N \) could be helpful, and he said, “If you changed the times to a minus, minus.” He seemed to mean “minus” by the process of subtracting number of people from 25 in his method, but he was not able to relate his repeated subtraction to division. Angela, on the other hand, said that she would divide. Specifically, she explained that she would divide 4.25 by 89.25 but actually she wrote dividing 89.25 by 4.25 on the paper. However, when asked to estimate the division, Jeffery answered it would be about 19 or 18 as saying;
You would just estimate how many people you probably would need to take off, like how many, how much it would bring it down if you subtract it for [from] 25 for a customer.

Although Jeffery’s estimation did not lead to a correct answer, his reasoning was based on quantitative comparisons. Angela did not attempt to estimate and I did not ask them to answer the division.

In Interview 8, the students were to write a rule for the total cost of a bike tour with three different scenarios using the data given in a table. The table showed the bike rental cost, the food and camp cost, and the van rental cost for the numbers of customers from 1 to 10.

Noticeable was that the van rental cost stayed the same no matter how many customers came because the van would not be used for transporting customers, only supplies.

<table>
<thead>
<tr>
<th>Number of customers</th>
<th>Bike rental</th>
<th>Food and camp costs</th>
<th>Van rental</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$30</td>
<td>125</td>
<td>700</td>
</tr>
<tr>
<td>2</td>
<td>60</td>
<td>250</td>
<td>700</td>
</tr>
<tr>
<td>3</td>
<td>90</td>
<td>375</td>
<td>700</td>
</tr>
<tr>
<td>4</td>
<td>120</td>
<td>500</td>
<td>700</td>
</tr>
<tr>
<td>5</td>
<td>150</td>
<td>625</td>
<td>700</td>
</tr>
<tr>
<td>6</td>
<td>180</td>
<td>750</td>
<td>700</td>
</tr>
<tr>
<td>7</td>
<td>210</td>
<td>875</td>
<td>700</td>
</tr>
<tr>
<td>8</td>
<td>240</td>
<td>1000</td>
<td>700</td>
</tr>
<tr>
<td>9</td>
<td>270</td>
<td>1125</td>
<td>700</td>
</tr>
<tr>
<td>10</td>
<td>300</td>
<td>1250</td>
<td>700</td>
</tr>
</tbody>
</table>

For the first scenario when the total cost included all three sub-costs, Peggy wrote $T = N \times P$ with $T$, $N$, and $P$ standing for the total cost, the number of customers, and the price, respectively. She explained that her equation meant to put everything together and add them up. So I interpreted that she meant adding each row of the table to calculate the total cost for a certain number of customers. However, her equation did not represent what she explained, but it
seemed to me that she added the first row of the table, which was the total cost for one customer, and multiplied the sum by the number of customers.

On the other hand, Greg wrote $C = (N \times B) + (N \times F) + 700$ as saying, “The cost equaled the number of people times the bike rental, how much the bike rental was, plus the number of people times the food plus 700.” When asked how much the bike rental and the food and camp cost were, he changed his equation into $C = (N \times 30) + (N \times 125) + 700$. Then, I asked Peggy what Greg did to find the total cost and Peggy explained Greg’s equation correctly. At this point, Peggy did not know that her equation represented the different operations from Greg’s. When asked to find the total cost for 4 customers, Peggy added 500, 120 and 700 and got 1320. As I asked how she obtained each number to be added, Peggy figured out that 500 and 120 were the result of multiplying the number of customers by the unit costs of 125 and 30, respectively, and she, thus, seemed to make sense of Greg’s equation.

With the other pair, I asked Angela and Jeffery to calculate the total cost for one customer and 5 customers before asking them to write a rule. For these specific cases, both simply added three numbers read from the corresponding rows of the table. In answering the following questions, both analyzed how they obtained each three numbers. When asked to write a rule for the total cost, it was quite unsurprising for Angela to write a general equation without relating each sub-cost to the number of customers as she added three sub-costs for 5 customers since it was the same process she did for specific numbers of customers right before. Angela’s equation was $TC = Br + Fc + Vr$ meaning that the total cost equaled the sum of bike rental, food and camp cost and van rental.

Jeffery, at first, wrote the equation $NC \times Br + Fc + Vr = C$ and changed it into $Br + Fc + Vr \times NC = TC$. Since it was not clear whether he meant multiplying the number of
customers by the sum of three sub-costs or not, I asked him to use his equation to calculate the
total cost for 5 customers. Jeffery said, “I did 30 plus 125 plus 700 and got 855. Then times 5…I
don’t think it will be right.” As asked why he did not think he was right, he answered that he
got 1,475 for 5 customers when he calculated the total cost with the table, but he would get 4,275
with his equation. Jeffery, at this moment, did think his answer 1,475 using the table was correct
but did not seem to know what made his two answers so different. In order for him to figure out
what made the two answers different, I asked him questions as follows:

Protocol 3: Interview 8 with Angela and Jeffery

I: In this number (the total cost calculated from the table), when you got the total cost
$1,475, how many bikes are in there?
J: 5.
I: Ok, how many people can eat or stay, I mean, food and camp cost in?
J: 5.
I: 5 people. How many vans are there?
I: Ok, how about in the bottom calculation (the total cost, $4,275, calculated from
Jeffery’s equation)? How many bikes do you have?
J: 1.
A: Umm, 5.
I: Ok, you got different answers. Jeffery, tell me.
J: Umm, I think it’s right here for 30. It would be the cost for one customer.
I: Ok, how about, Angela? Why do you think it’s 5?
A: I think it’s 5 because, since you’re timing it by 5, it’s like adding the 5 into it. So it’s
like, since you already have like one, it’s sort of adding like the other ones with it, like
when you times it. And so you like, it’s the same number, so.
I: So you mean the 30 means one bike. But because he multiplied the 5…
A: It’s, at the end, it becomes like 5.
I: What do you think about what Angela said, Jeffery? Do you agree with that?
J: Yeah.
I: So how many people can afford for food and camp cost in this amount (the total cost,
$4,275, calculated from Jeffery’s equation)?
A & J: 5.
I: How many vans do we have?
I: Still one?
A: Yeah, because you don’t need a van for…
O: Wait, that’s what we need. How many are in this calculation?
J: 5.
I: $4,275.
J: That’s why I think it’s too much because you will only need one van but I’m doing 5 times 700.

Once Jeffery realized the error in his equation, he said that he could add 30 and 125 first, multiply the sum by 5, and add it to 700, and then he adjusted his equation into

$$((Br + Fc)) \times NC + Vr = C.$$  Angela did the same also. Since their equations were not specific enough, they were asked whether they could use the specific values in their equations. Both of them finally wrote $$(30 + 125) \times NC + 700 = TC$$ and Angela later simplified the equation as

$$155 \times NC + 700 = TC$$ by adding 30 and 125 together.

For the second scenario when the bike tour organizers did have a free van, I asked them what the total cost would be. Both Angela and Jeffery found out the new rule by taking off the term 700 in their previous equation $$155 \times NC + 700 = TC$$ and so they obtained $$155 \times NC = TC$$ very quickly. Angela explained her method.

Mine is 155 times number of customers equals total cost since he doesn’t need a van, you don’t need to do the plus 700 like we did. So you just cross out the 700 and you just don’t do that section of the problem so you just sort of take that out.

Jeffery’s explanation was basically the same as Angela’s that he just took out the 700 since they had a free van at this time. Both students at this point thought with the equation

$$155 \times NC + 700 = TC$$ and adjusted it into a new situation without actually calculating again.

With the other pair of students, Peggy wrote $$T = (30 \times N) + (125 \times N)$$ and Greg wrote $$C = (N \times 30) + (N \times 125).$$ Both seemed to get the equation by revisiting the table unlike Angela and Jeffery. Then, I asked them whether they could write the equation differently. Greg told that he could add 30 and 125 up first and multiply the sum by the number of customers because the sum of 30 and 125 was for one customer. He wrote $$C = (30 + 125) \times N$$ and simplified it into $$C = 155N.$$ After Greg, Peggy also simplified her equation into $$T = 155 \times N$$ but she did not
explain why she could do so clearly. Eventually, both pairs arrived at the same rule, but they seemed to take different paths. Angela and Jeffery clearly stated that they took off the van cost in the equation with the van cost, but it was not certain whether Peggy and Greg could do the same. However, Greg showed his confident understanding of distributive property by simplifying his equation into $C = 155N$.

At last, I asked them what the total cost would be in the third scenario when the tour organizers would rent a van but ask customers to bring their own bike. Peggy’s equation was $T = 825 \times N$ and she said, “Since they’re going to bring their own bike, they just pay food and van rental. So altogether that’s 825 and now you have to do is times it by how many people are going”. Then, she was asked to calculate the total cost for 3 customers and she answered 2,475 as multiplying 825 by 3. When I asked her to find the total cost for 3 customers in the table, she added 700 and 375 and got 1,075. As Peggy realized the difference of the total costs, she said that her equation $T = 825 \times N$ did not work. Then, she wrote a new equation $T = (125 \times N) + 700$ as explaining that she changed her mind “because 125 was for food and camp cost and you just do it times the number of customers and then add 700 since it stays the same”. Peggy’s new equation agreed with Greg’s equation $C = 125 \times N + 700$.

Like Peggy, Angela wrote a rule for a situation where customers brought their own bike as $825 \times NC = TC$. When asked to calculate the cost for 2 customers with her equation, Angela realized her answer 1,650 included 2 vans instead of 1 van and she corrected her equation into $125 \times NC + 700 = TC$, which Jeffery had already written.

**Analysis.** In order to write a rule based on a given table, students had to first identify quantities that changed. When a table had two rows (or columns), as given in Interview 6, identifying changing quantities seemed relatively easy as students could simply read the labels in
the table. Moreover, two changing quantities that students identified from the table were to be
the quantities shown in the equation for the rule later. That is, identifying changing quantities
helped students see between which quantities they had to look for a relation. In Interview 6,
students could see the lunch cost changed according to the number of customers, and
furthermore they could identify the lunch cost was the dependent variable and the number of
customers was the independent variable.

However, with the table given in Interview 8, the number of changing quantities was
more than two, and therefore students had to figure out among which quantities they had to look
for a relation to write a rule. In fact, students considered the number of customers and the total
cost as the two quantities for the rule because the interview tasks implied this by asking them to
write a rule for the total cost for any number of customers. So students were generally successful
in identifying two changing quantities for the rule. One of Angela’s equations,

\[ TC = Br + Fc + Vr \]

was the exception, which did not represent the total cost in terms of the
number of customers. Also students could determine the total cost as the dependent variable
depending on the number of customers after deciding two quantities for the rule.

Next, students had to figure out the pattern of how the dependent variable changed in
relation to the independent one and the operation(s) between them. They had three different
types of patterns to figure out throughout the interviews, including a directly proportional pattern
(e.g., \( y = ax \)), a linear pattern with a non-zero constant term (e.g., \( y = ax + b \)), and a constant
pattern (e.g., \( y = b \)). When they had a directly proportional pattern like the lunch cost question
in Interview 6, students seemed to recognize the pattern easily. All four students in common
stated the pattern additively first as adding a common difference while increasing the
independent variable by one and they converted the repeated addition into a multiplicative
relation. Since the interview questions dealt with independent variables whose domain were the
natural numbers, it was not surprising that students recognized an additive pattern first and
converted it into a multiplicative relation.

Students seemed to generalize the exact same method to the other cases. Thus, for a
linear pattern with a non-zero constant term, when writing a rule for the total cost composed of
the bike rental cost, the food and camp cost, and the van rental cost in Interview 8, students
attempted to sum up the sub-costs for one customer and multiply the sum by the number of
customers. In fact, for the first scenario, Peggy came up with her equation \( T = N \times P \) with \( T, N, \)
and \( P \) standing for the total cost, the number of customers, and the price, by which she meant the
sum of the sub-costs for one customer. Basically, although he omitted the parenthesis for the
additions, Jeffery’s equations, \( NC \times Br + Fc + Vr = C \) and \( Br + Fc + Vr \times NC = TC \), were the
same as Peggy’s in that both added the sub-costs for one costumer. Jeffery itemized the sum of
the sub-costs unlike Peggy. Since the van rental, represented as \( Vr \) in Jeffery’s equations, stayed
the same regardless of the number of customers, both Peggy’s and Jeffery’s equations did not
work. However, students’ overgeneralization continued for the third scenario, and Peggy and
Angela wrote \( T = 825 \times N \) and \( 825 \times NC = TC \), respectively. In order for them to figure out the
rules correctly, they had to revisit how to calculate the total cost for a specific number of
customers; they were then able to abstract their operations.

For a constant pattern, students tried to do some operation with two variables even
though they could verbalize that the dependent variable stayed the same regardless of the
independent variable. In Interview 8, I did not ask students to find a pattern for each sub-cost, so
students did not have a chance to write a rule for a constant pattern. However, in Interview 5 of
the first phase with Angela and Jeffery, I asked them to find a pattern for the van rental. Both
tried to incorporate the number of customers with the van rental cost $700 to write the pattern. Angela wrote $700 \times N = C$ and Jeffery wrote $700 \times N = 700$ with $N$ for the number of customers. I asked them to use their equations to calculate the van rental cost for 1, 2, and 3 customers. Both of them clearly understood that Angela’s equation did not produce the constant cost of $700 and Jeffery’s equation made no sense for plural number of customers. Both students seemed to apply the same operation that they used for a directly proportional pattern for this case.

Nonetheless, they still seemed to look for a way to include the number of customers in their equations. Next, Angela wrote $700 = 5$ and $5 = 700$ to represent that the van rental was 700 for 5 customers and she said, “For this problem, 5 would equal into 700 because you are not multiplying anything, adding anything, you actually do not do anything to it”. In fact, Angela’s comment, “you actually do not do anything to it”, was literally represented in her equation by placing 5 as the number of customers without any operation. Moreover, she did not consciously acknowledge that the equal sign between 700 and 5 indicated both quantities as the same. Her response implied that she might not conceive the equation as a symbol to represent the situation that the van rental was always $700, but she tried to represent what operation she had to perform to get the van rental. For her, ‘no operation on 5’ was represented in leaving 5 alone in the right hand side of her equation. In fact, representing operations in the other patterns successfully led Angela to the equation representing changing situations from the observers’ point of view.

Even with operational perspective on equation, Angela could have been successful to find out the pattern for the van rental if she had associated her operation of “you actually do not do anything to it” with multiplying zero by the number of customers. Then, she could have the equation $700 = 0 \times 5 + 700$ for 5 customers and generalize it into $C = 0 \times N + 700$ and $C = 700$ eventually.
Once they figured out the pattern and the operation(s), they had to think of how to write it in the equation form. Selecting letters to represent variables was not difficult for the students and they seemed to prefer using initials for the variables. Particularly, Jeffery and Angela used two-letter initials like Br, Fc, NC, etc. and I inferred that both students used them because they tried to include the context as much as they could. Some syntactical issues such as putting some extra words and signs in equations (e.g., “# of hour”, “pay”, or $) discussed in the referential relationship from narrative representation to algebraic notations, were not observed in this referential relationship because we talked about these issues in the previous interview sessions. However, the syntax of the place of variables in an equation was not addressed in the interview sessions and students still decided which variable was placed in which side of the equation as a personal choice.

![Figure 7. Referential relationship from tabular representation to algebraic notations.](image-url)
In Figure 7, the scripted diagram sums up students’ referential relationship from tabular representation to algebraic notation, and the following analysis will discuss how students used, thought with, and operated with the algebraic notations that they came up with. At first, all students could understand the operation embedded in an equation and use the equation to find out specific cases. For example, Peggy used her equation $T = 825 \times N$ to calculate the bike tour cost in the third scenario for a specific number of customers. That is, she understood that she had to multiply 825 by the number of customers to find out the bike tour cost. So she was able to show herself whether her equation told the same story with the given table by calculating the cost for 3 customers using her equation. This ability provided Peggy with feedback to correct her equation later.

Secondarily, students could infer the reverse operation embedded in an equation and use it for specific cases. All four students succeeded in figuring out how many people could have lunch with $89.25 using the equation $C = 4.25 \times N$. Although students struggled with division itself, they easily decided to divide 89.25 by 4.25 by thinking of the reverse operation of multiplication. Furthermore, Greg was able to generate the equation, $N = C \div 4.25$, with the reverse operation and it suggested that he was able to manipulate the algebraic notation confidently to represent the reverse operation.

Finally, students demonstrated some examples of operation with algebraic notations. One example was Greg’s operation with algebraic notations using the distributive property. He wrote an equation for the total cost of the bike tour in the second scenario with a free van, $C = (N \times 30) + (N \times 125)$. When I asked him whether he could write it differently, he explained that he could add 30 and 125 first since each price was for one customer. Then, he successfully simplified his equation into $C = 155N$. This implied that Greg not only understood the
distributive property in the context but also represented his understanding as the result of operating with the algebraic notations confidently. In fact, Angela and Jeffery wrote their equations, \((30 + 125) \times NC + 700 = TC\) for the total cost of the first scenario. However, with their equations, Angela and Jeffery showed supporting evidence of their understanding of the distributive property in the context, but not in operating with the algebraic notations. Another example of operating with algebraic notations occurred when Angela and Jeffery simply took off the constant term in their equation, \(155 \times NC + 700 = TC\), for the first scenario to get an equation for the scenario with a free van. At this point, both Angela and Jeffery conceived the previous equation as an object and operated on it to get another equation without revisiting the context.

In Chapter II, I considered three types of faculty as stepping-stones in understanding (Tall et al., 2001) of algebraic notations that students would pass through. There were three levels: procedural level, process level and proceptual level. At the proceptual level, students are able to consider an algebraic notation as signifying both a process and a concept so that they can think about mathematics symbolically. In terms of three levels of understanding of algebraic notations, the way Peggy used her equation \(T = 825 \times N\) mainly focused on process by using the equation in order to calculate the total cost and correct the equation after reflecting on her calculation. However, Greg’s equation \(N = C \div 4.25\) in the second case showed how he was able to project his understanding of division in relation to multiplication in the form of algebraic notations. The third cases were even closer to the proceptual level than the previous ones in that the students could think about the equations as manipulable entities and they could operate on them. Thus, these cases sufficiently implied that students began to increase their flexibility, switching focus between a process and a concept in algebraic notations.
Description. In Interview 7, after I asked the students to make up a story with the equation $d = 8t$, I asked them to generate a table with the equation. In the interview with Greg and Peggy, Peggy made up a story of traveling 8 miles per hour with $d$ for the traveled distance and $t$ for the time traveled. She said, “You’re traveling from Georgia to Washington, DC and you’re traveling 8 miles per hour.” Greg agreed with Peggy’s story without making up his own story.

When asked to make a table, Peggy made a table with two columns, and put time on the left and distance on the right. Peggy’s time column was for every hour and she explained that she kept adding 8 miles as increasing an hour because they were going at 8 miles per hour. When I pointed out to Peggy that she put 36 miles for 4.0 hours, she corrected it into 32 as adding 8 to 24. Greg made a table with two columns and put distance on the left and time on the right (see Figure 8 (a)). He wrote down the times first from 1 hour to 5 hours by a half hour interval but missing .5 hour and wrote 8 miles in the distance column for 1 hour. Then, he stopped writing distances and added .5 hour in the time column. Next he filled the distance column by adding 4 to each proceeding values, starting from 0. He described what he did as saying;

“I did my time and I went by 30 minutes intervals in all. And I went all the way to five and [for] each [half hour] I added 4 each time. And each hour it was 8 and you added 8 to each hour, like 4 then 8 and 12 and 16”.

From Algebraic Notations to Tabular Representation
When I asked Greg to explain differences between his table and Peggy’s, he considered both were basically the same. Below is the excerpt from the interview.

Protocol 4: Interview 7 with Greg and Peggy

I: Do you see the difference between these two (pointing to the two tables in Figure 8 above)?
G: Yes.
I: OK, would you tell me?
G: The intervals are different, the time and distance on each one.
I: The intervals are different. I mean the intervals in the time, right? How about the intervals in the distance?
G: Not really because all you’re doing is just breaking it down to where it’s half go with the…
I: OK, what do you mean by breaking down?
G: Like 4 is half of 8 so you just divide it by 2 and it’s just because it’s half hour that you do that.

The excerpt indicated Greg’s understanding of rate in changing situations over time. Greg not only considered ‘8 miles per hour’ and ‘4 miles for every half-hour’ as having the same rate but
also explained the proportional relation between them. So he suggested that two tables looked
different in appearance but they represented the same rates.

Jeffery, without Angela, had made up a dancing story with the equation $d = 8t$ in
Interview 7. He seemed to consider the letters, $d$ and $t$, as any numbers that satisfied the
equation without indicating that he considered them as variables in a changing situation. Thus, I
suggested thinking of $d$ as a distance since the class had dealt with the distance-time situation
and he could probably relate to it better. However, like in his dancing story and the candy bar
story, he was not able to identify what the letter $t$ could represent.

As Jeffery figured out that the letter $t$ could represent the time traveled with my help, he
made a table with distance and time without any difficulty. He put the distance on the left
column and the time on the right column in his table. It was interesting that he picked multiples
of 8 for the distances first and found out the corresponding times. About the table, Jeffery
explained that since he knew he would travel 8 miles for every hour, it would take 2 hours if he
went 16 miles. He continued to explain the other miles on the table that if he knew the number
of miles he could find how long it would take. Then, I asked him whether he could make a table
showing how far he could go for every half hour. Jeffery answered that he would cut 8 into a
half since he wanted to know for a half hour. As he was completing his table, he said that he
could add 4 more for every time like 4, 8, 12, etc. since a half of 8 was 4. He put the distance 12
miles for 2 and a half hour at first, but he corrected it into 20 when I asked him about it. He
seemed to make a simple mistake.
Analysis. The analysis of students’ referential relationship from a given equation to tabular representation continued from that of referential relationship from a given equation to narrative representation. Before they were asked to generate a table from the equation $d = 8t$, students were asked to make up a story with the same equation. By doing so, students already assigned quantities to the letters in the equation, although whether they conceived those quantities as variables depended upon their own conceptions of variables.

As a next step, they interpreted the operation between changing quantities, which was also done when they made up a story with the equation. That is, they recognized that the multiplication was the operation between the distance and the time in the equation $d = 8t$. Then, they figured out the pattern to generate a table using the operation. Both Greg and Peggy interpreted 8 in the equation $d = 8t$ as 8 miles for every hour, and they thought of the additive pattern in the distances by setting up the time interval. So their coordination of specific times and distances was not by pairs. Instead, they thought of a series of times set by a certain interval and they generated a series of distances by repeated addition. That is, they added 8 miles
repeatedly as increasing the time by one hour instead of multiplying the time by 8 for each time. So they used the rate 8 miles per every hour additively at this point.

However, Greg was also able to use the rate to generate 4 miles for every half hour and he understood that his method was basically the same as Peggy’s. This showed that unlike Peggy, Greg understood the constant rate ‘8 miles per hour’ multiplicatively so that he could convert it into 4 miles per every half hour with his proportional understanding.

On the other hand, as mentioned previously, Jeffery conceived variables quite differently based on his stories invented with given equations, but he still could list out the pairs of numbers that satisfied the given equation. When he explained his table with one-hour time interval, he explained each pair of time and distance in terms of multiplication. However, he also generated the table with half-hour time interval by adding 4 miles repeatedly like Greg. It implied that he was able to relate the multiplication as the operation between the time and the distance to the additive pattern of distance according to time.

When projecting their interpretations and operations into a table, students seemed to have no syntactical difficulties in general. In fact, tables as a way of representing changing situations do not have as much of syntax as graphs. One syntactical issue, however, might be where to put the independent and the dependent variables in the table. This syntax was never discussed in Ms. Moseley’s class and I did not make it an issue during interviews.
Figure 10. Referential relationship from algebraic notations to tabular representation.

Algebraic Notations and Graphical Representation

For the third referential relationship, the students were asked to find a rule for the total ticket price given a graph in Interview 6. The given graph had the scale of 5 on the $x$-axis and the scale of 10 on the $y$-axis, and there was only one data point that the students could read exactly without estimation. In order to find a rule with the graph, the students had to realize that the price per ticket, which was the constant rate in the situation, was prerequisite. For the other direction in this referential relationship, Greg and Peggy were asked to generate graphs for equations $d = 8t$ and $C = 2n + 5$ after asking them to make up stories in Interview 7. Jeffery, without Angela, was asked to graph the equation $d = 8t$ in Interview 7, and Angela and Jeffery were asked to graph the equation $125 \times NC + 700 = TC$ that they generated for the total cost in Interview 8.
From Graphical Representation to Algebraic Notations

Description. The interview students were to find a rule for the total cost of concert tickets for any number of people given in a graph. The graph only showed the total cost for every 5 people so that the price per ticket had to be calculated in order to write a rule. In particular, the horizontal grid lines were only drawn for every 10 dollars and there was only one data point to read without estimation. Thus, students had to pick the data point (20, 70) showing that the total cost was $70 for 20 tickets in order to calculate the price for one ticket exactly.

When the graph was presented, Peggy claimed that the graph went in a constant rate and that is, it increased by the same amount. She explained she knew it because the data points had the same space in between each other. Greg also supported Peggy’s claim by saying, “the cost kept going up at the same, like you bought a certain number of tickets, and you bought that over again”. Then Greg explained what he meant by “over again”.

It just keeps going up by that same number. Like it was $18 and then you go add up another $18, another, another…

In fact, $18 was an estimate that Peggy and Greg agreed upon as the price for 5 tickets. However, both Peggy and Greg wrote a rule for the cost as \( C = 18 \times N \) without paying attention to the number of tickets on the x-axis at this point. When asked why they multiplied 18, Greg
said that he was not right. Peggy mentioned that they did not know what the cost was and then
Greg said that $18 was not the cost of one ticket. Peggy added that $18 was for 5 tickets and she
needed to know the cost of one ticket since otherwise she could not know the total cost for 7 or 8
people because it went by 5s. She also stated that she wanted to divide something by 5.

Interestingly, in the interview with the other pair of students, the exact same thing
happened. Angela read out all the data points from the graph saying, “Umm, 5 tickets would be
$18 about, umm, 10 would be about $35, 15 would be about $52, 20 would be around, umm,
about… Oh, not about, hold on… it would be $70.” Clearly, she demonstrated that she
estimated the prices for tickets except for 20 tickets. Nonetheless, Jeffery concluded, “I think
that maybe all tickets may go up by $18 every time since the first one starts at $18”. Then, both
wrote a rule \( C = 18 \times N \) as Peggy and Greg did. When asked to calculate the cost for 20 tickets
using the equation, Jeffery stated as below.

I don’t think it would be right then because 18 is just for 5 customers only. You could go
by 5s to get that, but you couldn’t find out for like 4 or 3 customers. You couldn’t find
out because 18 is only for 5.

In fact, I would have liked them to realize that the calculation of the cost for 20 tickets with their
equation would be different from what they read from the graph. However, with the same
reasoning as Peggy’s, Jeffery figured out that the equation would not work for any number of
customers, which meant that the equation could not be a rule. With both pairs of students, after
writing a rule \( C = 18 \times N \), they realized that they needed to know the price for one ticket because
the equation did not fit with the given graph.

Realizing the need for the price for one ticket, however, was one thing and finding out the
price was another for both pairs. Peggy attempted to divide 18 by 5 so as to get the price for one
ticket without noticing that the data point (5, 18) on the graph was an estimate. So I asked them
whether it was the only point read from the graph. Greg answered no and he picked the data point (20, 70) as saying, “Because that’s only [on] a grid line. That’s where it’s on a point with the two [vertical and horizontal] lines.” Then, I asked them what if they chose another point instead of (20, 70). Greg answered that the ticket price would be still the same and Peggy said that the reason was that the graph went by the same amount. They both said that they preferred (20, 70) “Because you can tell exactly how much it is for one ticket.” using that point. They then tried to divide 70 by 20 to get the price for one ticket, but they had a hard time performing the division. Greg got an answer 3.05 after his calculation (see Figure 11 (a)). When asked whether she agreed on Greg’s division, Peggy answered yes.

![Figure 11. Students’ division of 70 by 20.](image)

Thus, I changed my question and asked them how much 10 tickets would be when 20 tickets cost $70. Greg answered $30.50 by multiplying $3.05, which he got from his division of 70 by 20, by 10. He also said that two 30.50’s made 70 because 10 plus 10 made 20, and therefore he seemed to think his answer was supported by another method. So I asked him to add 30.50 and 30.50 on paper and he found his answer was not correct. When asked again what number made 70 added to itself, Greg found that 10 tickets cost $35. After that, Greg said that one ticket price was $3.50. Finally, both wrote a rule $C = 3.50 \times N$ for the total ticket price.
For Angela and Jeffery, after Jeffery figured out that they needed the price for one ticket, he tried to divide 18 by 5 like Peggy. So I reminded them that $18 was an estimate price for 5 tickets. I then asked him whether they had another data point that they could read exactly on the graph and use to calculate the price for one ticket. Angela read the point (20, 70) and said that she would divide 70 by 20. Angela calculated and answered $10 for one ticket, which I presumed she read in her remainder (see Figure 11(b)). Jeffery argued, “It’s couldn’t be $10 because two [10’s] would then equal 20 and that would be over already”. Then, Angela changed her mind and answered $3.01. Jeffery claimed that it probably would be $3.25 or $3.50.

Instead of asking about Jeffery’s two answers, I asked them how much 10 tickets would be when 20 tickets cost $70. Angela answered $35 without confidence but shortly she said, “We think it’s 35 because it’s exactly like in the middle” as a hand gesture of cutting. Then, Jeffery said that he would divide 35 by 2 by saying, “Because you’re like cutting it in half. And then once you find the price for 5 tickets, you divide it again.” Jeffery seemingly wanted to have the exact price for 5 tickets first and to calculate the price for one ticket using it. Angela attempted the calculation that Jeffery suggested, but she mistakenly divided 30 by 2 instead of 35 by 2. At this moment, Jeffery figured out that one ticket cost $3.50. Both finally wrote a rule

\[ C = 3.50 \times N \] with \( N \) for any number of tickets.

**Analysis.** For the referential relationship from graphical representation to algebraic notations, students identified quantities that changed as the first step. In order to do so, students easily read the labels of the axes in the given graph and considered them as two things that changed in the situation. Some students could correctly determine the types of variables according to which axis they were on. Which label would be the independent and the dependent
variables will be discussed in detail in the referential relationship from algebraic notations to graphical representation.

Next, students paid attention to the pattern of the data points, which lay on a straight line and concluded that the cost for tickets increased at a constant rate. All students seemed aware that a constant rate was $18 for every 5 tickets at some level, but they were not careful enough to think that $18 was their estimate and the price was for 5 tickets. Until they wrote the rule that the cost for tickets equaled $18 multiplied by the number of tickets, no one noticed that they had to have the price for one ticket. In addition, here $18 was not the cost for one ticket but an estimated cost for 5 tickets that all students coincidently read from the graph. Thus, they successfully recognized that the cost for tickets increased at a constant rate, but they did not yet succeed in finding out the constant rate.

Writing the equation $C = 18N$, however, provided the interview students with an opportunity to think of which constant rate they needed to write the correct rule. Then, students agreed that they had to find out the price for one ticket and they understood the reason for that as Peggy and Jeffery stated. Nonetheless, they still did not seem to understand why they had to pick a precise data point to calculate the price for one ticket. So they attempted to use the data point (5, 18) to calculate, which would not give the exact price for one ticket. After some discussion, they realized that the data point (20, 70) was the only data point leading to the exact unit price. Although students had a hard time calculating due to the lack of computational understanding, all of them knew that they could find out the price for one ticket by dividing 70 by 20.

After figuring out the price for one ticket, all students were able to write an equation without any difficulty. It implied that students could interpret ‘the price for one ticket’ as ‘the
price per ticket’ because the cost increased at a constant rate although I believed that they might not tell the subtle difference between them. Thus they could write the equation with the multiplication as they did in writing the rules for the previous problems (e.g., the baby-sitting problem and the book club problem). Even some syntactical issues raised previously (e.g., putting some extra words and signs in equations) were not shown at this time.

Figure 12. Referential relationship from graphical representation to algebraic notations.

From Algebraic Notations to Graphical Representation

Description. After asking students to make up a story and generate a table with the equation $d = 8t$, I asked them to draw a graph for the equation; they seemed to draw their graphs based on their tables. Peggy put a half hour interval on the $x$-axis and 4-mile interval on the $y$-axis, and plotted the data points for each hour first and then the data point for between hours later. I asked her what scale she used on the $y$-axis, and she read the marks on the $x$-axis as 30 minutes, an hour, an hour and 30 minutes, etc. So I asked her which was the $y$-axis and she pointed to the
She read the marks on the \( y \)-axis for the scale of the \( y \)-axis. I asked her what the scale meant. Peggy answered that it would be either from where to start to where to end or whatever it went by. Apparently she was not sure what the scale meant exactly. I also asked her why she plotted the data points for each hour first and for between hours later. Peggy simply said that it helped her find out what was half, like for the 30 minutes. However, she did not explain how it helped her.

![Figure 13. Peggy’s graph of \( d = 8t \).](image)

Greg put the time on the \( x \)-axis with the scale of .5 hour and the distance on the \( y \)-axis with the scale of 2 miles. Then, he plotted data points correctly. However, Greg wrote the numbers for his axes in the grid squares rather than on the grid lines as in Figure 14. When he explained about his graph, he knew which grid line he meant by a certain number. Also he mentioned that it was a little harder to read his, but it did not cause any trouble to him at this point. When asked why Peggy’s and Greg’s graphs looked different, Greg answered that his looked “more spread out” since his scale on the \( y \)-axis was 2.
Figure 14. Greg’s graph of \( d = 8t \) with the scale of 0.5.

Since I assumed that both Greg and Peggy drew the graphs from their tables, I asked them whether they could draw a graph from the equation without looking at their tables. Greg said yes right away and explained as saying (see Figure 15 (a)):

You can go by 8 each time for distance (pointing to the \( y \)-axis on his graph) and for each hour (pointing to the \( x \)-axis). You have to set, you would have to set how many hours, like it was 5. You would do 5 down here, then you could figure out, just plot it each time (pointing to the data points), like where they came together from what the line, you know, like that (drawing a vertical line from a point on the \( x \)-axis and turning to draw a horizontal line to the \( y \)-axis).
In fact, Greg was drawing a graph from the equation directly at this time, but in terms of Kaput’s (1991) referential model his interpretation and cognitive operation regarding the equation $d = 8t$ were the same as the referential relationship he made when going from equation to tabular representation (see Figure 10). The difference was how he projected his interpretation and cognitive operation into different representations. He seemed to set an hour time interval for his convenience so that he could plot the data points easily.

Once Greg finished plotting (see Figure 15 (a)), I noticed that he seemed to miss the data point when the time was one hour. So I asked him what happened after traveling for an hour, then he plotted where $(1, 16)$ was supposed to be and he erased the rest of his data points in order to attempt plotting them again. Thus, I asked where 8 was on the $y$-axis and he drew a hash mark where 16 was supposed to be. When I asked him where 0 was on the $y$-axis, he realized he had put the data point $(1, 8)$ at $(1, 16)$ and corrected his plotting (see Figure 15 (b)). Unlike the previous graph (see Figure 14), Greg actually confused himself by the way he labeled numbers in the squares and he admitted it caused confusion.

I asked Peggy whether she could plot data points for every half hour on Greg’s graph since she mentioned that plotting the data points for every hour helped her plot those for every half hour. Peggy tried to interpolate data points in between every hour, but she was misled by
Greg’s way of labeling and she plotted (.5, 4) and (1.5, 12) at (1, 4) and (2, 12), respectively (see two thin dots in Figure 16). Eventually, Peggy’s interpolation led to a discussion over Greg’s way of labeling between Greg and Peggy. In listening to what Peggy complained about his labeling, Greg reflected on his way of labeling numbers. So when asked what would be a better way to put the numbers on the axes so as to avoid confusion, Greg rewrote the numbers on the grid lines. Then, Peggy showed her interpolation correctly by beginning a vertical line from 30 minutes on the x-axis, drawing a line upward, and stopping the line at (.5, 4). She confirmed the coordinate of the point and she said that she could do the same process for every half hour.

![Figure 16. Peggy’s interpolation on Greg’s graph.](image)

Of all the students, Jeffery had the hardest time making up a story with the equation \( d = 8t \), so I suggested considering \( d \) as the distance traveled. He could then make a table with the story of traveling at 8 miles per hour. I asked Jeffery to draw a graph with the equation. He put the distances on the x-axis and the times on the y-axis. So I asked him whether there was a rule telling which one went to which axis. Jeffery answered that he thought there was but he did not remember it. I asked him to tell me the meaning of variable and the types of variable, but he could not answer. When I asked which was the independent variable and which was the dependent variable, Jeffery said, “I think this one (pointing to the x-axis with the distance) right here, it would be the dependent and this (pointing to the y-axis with the time) is the independent.” Nonetheless, it was not clear what he meant by ‘this’ and so I asked him. This
time he answered differently, “It’s like this right here (pointing to the x-axis) is depending on this one (pointing to the y-axis) like before you travel, before you have to know how many [miles] you travel before you know the time that you traveled.” Then, I rephrased what he said that he had to know the distance in order to find out how long it would take to travel the distance, and asked him what variable the distance was. He answered dependent first and changed it into independent right after. In fact, Jeffery seemed to know the fact that the dependent variable depended on the independent one, but he seemed confused about how to apply the dependency and identify the types of variables.

At this moment, Olive, the observer of the interview, presented two different situations with time and distance: how far could he go for 3 hours at 8 miles per hour and how long would it take to go 80 miles at 8 miles per hour? Jeffery answered both correctly as 24 miles for the first situation and 10 hours for the second one. Then, Olive asked Jeffery what depended on what in each situation. Jeffery told us that in the first situation the distance depended on the time and in the second the time depended on the distance. He also said that the second situation fitted his graph as the time depended on the distance. Therefore, his graph represented the equation $t = \frac{1}{8}d$, which in fact is the inverse function of $d = 8t$. However, Jeffery seemed to understand the dependency between two variables and he was able to tell which variable was the independent and which was the dependent variable in a given situation, I let him keep his graph as one for the equation $d = 8t$. 
Next, I asked him what the scale on the $x$-axis was, and Jeffery correctly answered that it went by fours. However, his way of writing the labeling numbers was quite confusing and it was hard to tell where he meant to mark a number on the $x$-axis, so I asked him to point out where 4 was on his $x$-axis. He pointed out the data point of $(4, 30)$. When asked again, he rewrote his labeling numbers in the grid squares rather than on the grid lines similar to Greg. Even so, he was able to answer where 4 was exactly on his $x$-axis when asked. I then asked what he noticed about his graph, and Jeffery told me that it went diagonal and it increased by 30 minutes. Jeffery kept explaining that it went by 30 minutes and the distance increased by 4, but he was not able to relate the two variables to describe the changing pattern. That is, Jeffery seemed to see two additive patterns in each variable separately instead of recognizing a pattern as relating to two variables (e.g., As the distance increases by 4, the time increases by 30 minutes).

Since Jeffery said that the table he made helped him draw the graph more than the equation $d = 8t$, I asked him to draw a graph from the equation. Jeffery drew two axes first and attempted to put marks for multiples of 8 at every other grid line on the $x$-axis and put marks for every hour on the $y$-axis (see Figure 18). Since Jeffery put 8 at the first grid line and then skipped grid lines to place 16, 24, 32, and 40 at the third, fifth, seventh, and ninth grid lines, respectively, the marks on the $x$-axis were not equally spaced out. After I asked him to tell me
about the distances between the marks, he realized where he had to put 8 and he put the correct marks with red pencil on the $x$-axis as shown in Figure 18. He also mentioned that the marks had to be equally spaced and he checked that the marks on the $y$-axis were equally spaced.

*Figure 18. Jeffery’s graph of $d = 8t$ from the equation.*

I asked Jeffery what differences there were between the graph that he drew with the help of the table and the graph that he drew from the equation. He answered that they differed in scales but they told the same story. He added that both graphs included the same data and so they went up the same way, but one looked more spaced out due to the scale of 8.

In Interview 7 with the other pair of students, I asked students to graph the equation $C = 2n + 5$ after making up a story. As an attempt to graph, Peggy used scales of 1 and 7 on the $x$-axis and the $y$-axis respectively, but she stopped working. Regarding her scales, Peggy explained that it increased by $7$ by adding 2 and 5 for each person. She added that she stopped graphing since she was not sure what to do.

Greg explained how he graphed the equation by calculating the cost for each number of people from 1 to 8. However, he used a scale of 1 on the $x$-axis but he started with 7 and went by 2’s on the $y$-axis. When I asked Peggy to explain how much Greg went by on the $y$-axis, she said that he went by 2’s and she pointed out that the first mark came from 0 to 7. At that
moment, Greg told us that he could not jump from 0 to 7 and he put the zigzagged mark between 0 and 7 (see Figure 19) to indicate there were “more numbers on it”.

![Figure 19. Greg’s graph of \( C = 2n + 5 \) with uneven scale on the y-axis.](image)

In order to encourage him to think about scale more carefully, I asked Greg what if he had to graph it again. The interview excerpt below shows how the discussion continued about the scale on the y-axis.

**Protocol 5: Interview 7 with Greg and Peggy**

I: If you have to do again, what do you want to do?
G: Not jump so much.
I: Not jump so much? So you might just begin zero and then went by twos evenly?
O: Could you just quickly draw, to the side, the scale for distance that you would use starting at 0? To the right, yes, there, like you’re starting a new graph.
G: (hesitating to draw) I don’t… You couldn’t go up by 2s because it wouldn’t come out to 7 because you have to add the 5 each time.
I: So you can’t go by 2s?
P: Can you do like 0 then go to 5 and then add 2 up, can’t you?
I: You mean, okay, would you show on your paper?
P: Like that, zero then now go to 5, then that’s 7, and 9, and 11, like that. You’re just adding 2. Can you do that?
I: Let’s think about that issue. She suggested starting from zero and the next grid line is 5 and then go by 2s. Is it okay?
G: I don’t know, I don’t think so.
P: I don’t really either, I don’t know. It’s just what I did.
I: Okay, if you do it and then what if I ask you, what is the scale on your y-axis?
G: It would be 2, but you started off with 5.
P: It would be 5 and then go up 2.
I: Okay, Greg, would you explain what you think you can’t go by 2s?
G: Because, umm, 2 wouldn’t equal to 7. You wouldn’t go 2, 4, 6, 7, like that.
I: You mean, if you go by 2s, you will go 0, 2, 4, 6…
G: You would probably have to start off at 1 and just go by 1s.

When he answered that he could not go by twos, Greg seemed to mean that he could not plot data points on grid lines if he went by twos. Later Greg said that he could go by ones and began to draw a graph (see Figure 20). Greg looked contented with his new graph because all the data points were on the grid lines.

![Figure 20. Greg’s graph of $C = 2n + 5$ with the scale of 1.](image)

Since Greg did not plot for the case of no person, I asked him what if nobody was going to a picnic, he answered it cost nothing and plotted (0, 0) on his graph at first. When Olive told him that in some cases he had to pay a reservation fee up front, Greg changed his point (0, 0) into (0, 5). I then asked what story the equation $C = 2n + 5$ told us about the case when nobody went,
and he answered it still cost $5. Also he figured out that with plotting (0, 5) the graph still went up by the equal amount.

In Interview 7, Jeffery did not have a chance to draw a graph for the equation \( C = 2n + 5 \) because of the time limit in this interview. Instead, Angela and Jeffery drew a graph for \( 125 \times NC + 700 = TC \) in Interview 8 with the situation where \( NC \) stood for the number of customers and \( TC \) stood for the total cost. When asked what went on the \( x \)-axis, Angela answered the number of customers but could not explain her reason. Then, when they were reminded of the types of variables, both Angela and Jeffery figured out that the number of customers was the independent variable, which had to go on the \( x \)-axis, and the total cost was the dependent variable based on the number of customers. Angela drew the \( x \)-axis and marked it with the scale of 1 up to 10.

When I asked Jeffery what scale would be used for the \( y \)-axis, he answered he would go by 125s since it would go up by 125 after glancing at the table and noticing the increment of 125, but Angela answered 100s. However, Angela said that it would not matter since different scales just made the graph either longer or shorter. Soon after Angela changed her mind by saying that using the scale 125 would lead to a more accurate graph because the data points would be right on the grid lines.

After Angela and Jeffery agreed on the scale of 125 on the \( y \)-axis, Jeffery put marks on the \( y \)-axis. They worked together to calculate the cost for each number of customers from 1 and plotted \((1, 825)\) first. Since the first data point was not on the grid line, I asked Angela whether it was what she expected as an accurate graph. Angela replied that she forgot about the van rental. When asked for 2 customers, Jeffery answered 950 quickly and explained that he took 700 and added it to 250 by referring to the marks on the \( y \)-axis. That is, because they used the
scale of 125 on the y-axis, they already knew the multiples of 125 and therefore they did not have to multiply 125 by the numbers of people. In fact, Jeffery figured out that they could add the multiples of 125, which were already marked on the y-axis, to 700 in order to find the total cost. Angela also understood how efficiently Jeffery found his answers.

Once Angela and Jeffery had plotted the first three data points, they were asked what they noticed about the graph. Angela said, “We’re not actually on the grid lines. I forgot about the van”. Before graphing, she seemed to expect all the data points would be on the grid line without thinking of the van rental $700, but later she realized the total cost would not be the multiples of 125 because of adding the van rental. In addition, Angela said that she could plot the points in the middle of the next grid square without calculation since the three points were in the middle. Jeffery added that he could keep going up diagonally. Angela put a ruler passing through three already plotted points and drew a line while saying that she remembered what Ms. Moseley told them in class. Then, she mentioned that this kind of graph kept gaining without dropping. So I asked her how it gained. Since Angela could not answer, I asked her how much it gained when having one more customer added. She could answer 125 without providing the reason, but Jeffery explained, “You know that your van rental stays the same for all of these, so that really, you’re just adding 125 every time” by referring to the equation. Angela then plotted the rest of the data points using the line she had drawn.
Next, I asked them, “What if we have no customers?” Angela answered the graph would go flat as drawing a horizontal line beyond the 10 customer point with her finger. She seemed to have interpreted my question to mean that there would be no more customers rather than zero customers and therefore the graph would become flat. Jeffery agreed with Angela. Then, Olive asked them where the zero customers were on their axis, and Angela eventually pointed to the origin after she changed her answers several times. Olive asked her to plot the data point for zero customers and she plotted at the point 1 unit to the right of the last data point (see the point not on the line in Figure 22). When asked, Jeffery could read the point as having the $x$-coordinate 12. Angela realized her error and changed the point into the origin. She explained that the total cost would be none if they had no customers. When I asked them whether the equation told them the same story, Jeffery argued that it would be still 700. Angela agreed with
Jeffery and plotted the point at (0, 700) correctly. Both students finally saw that the point would be on the extension of her line.

*Figure 22.* Angela’s data point for zero customers.

*Analysis.* In order to graph a given equation, students had to assign quantities that changed to the letters in the equation as the first step from algebraic notation in Kaput’s (1991) model. Since the activities in this referential relationship followed the story-making with a given equation or generating an equation from a tabular representation, assigning changing quantities to the letters was accomplished easily. Moreover, students conceived the changing quantities as variables although their conceptions of variables varied from student to student. Students’ conceptions of variables will be discussed later in this chapter. Interpreting the operation(s) in the equation and finding a matching situation with the quantities and the operation(s) were also activities engaged in during the story-making or interpreting the given table.
The students also had to recognize the relations among the quantities in order to decide the types of variables. Although this issue concerning types of variables will be addressed in detail in the next section focused on the second research question of this study, some issues that emerged in this referential relationship will be discussed in this analysis. To identify the types of variables, students mostly applied the dependency criterion; that is, which one changed depending on the other. For the equations $C = 2n + 5$ and $125 \times NC + 700 = TC$, students decided unanimously that the cost relied on the number of people. However, for the equation $d = 8t$, Greg and Peggy’s pair considered that the distance depended on the time and Jeffery considered it the other way. In fact, both were legitimate and Jeffery had a chance to think of the two ways of possible dependencies. He came up with the dependency of time on distance that I did not expect because their learning experience in class generally considered time as the independent variable.

As the next step in Kaput’s (1991) model, students looked for the pattern between two variables to coordinate two variables. The way that students looked for patterns was similar to the way they did in the referential relationship from algebraic notations to tabular representation (compare Figure 10 and Figure 23). When students were asked to graph a directly proportional pattern, $d = 8t$, students tended to use the constant rate, 8 miles per hour. Thus they could picture that the distances increased by 8 miles as the time increased by one hour. When they had a linear pattern with non-zero constant term, $C = 2n + 5$ or $125 \times NC + 700 = TC$, students seemed to recognize that the dependent variables would increase by the number by which the independent variables were multiplied. That is, for the equation $C = 2n + 5$, Greg and Peggy appeared to know that the cost would increase by 2 when increasing the number of people by 1. Thus, Greg tried to use the scale of 2 on the $y$-axis although he jumped 7 from the origin. Angela
and Jeffery also noticed that the total cost of the bike tour would increase by 125 as the number of customers increased by 1, so they finally agreed on the scale of 125 on the y-axis for the equation \(125 \times NC + 700 = TC\).

With respect to the third step in Kaput’s (1991) model, the projection of the patterns they recognized to the graphical representation was well revealed by Greg’s action of how he graphed directly from the equation \(d = 8t\). Greg put his pencil at a value on the x-axis, went up by matching the appropriate y-value, turned his pencil to the left, and went horizontally to the y-axis, physically demonstrating where his pencil turned as the point to be plotted in the coordinate system. While actually plotting, he began at a value on the x-axis, went up by the matching y-value, and put a point without drawing the horizontal line. Apparently, he seemed to visually coordinate his position with the y-axis. I considered that this physical projection was the result of fundamentally the same cognitive operation, coordination of two variables, as he used for producing a tabular representation from algebraic notation.

When projecting students’ cognitive operation to the graphical representation, they had to deal with more syntactical issues in this referential relationship than any other relationships. At first, students had to decide which variable had to go to which axis. Since it was discussed in class, students seemed familiar with this syntax. What mattered was how they could decide the types of variables using the dependency relationship between the two variables.

The next issue was about scales on the axes. Peggy was not sure whether the scale referred to the numbers from the start to the end on the axes or the increase from one number to the next on the axes. A lot of cases were observed in both the classroom activities and the interviews in which students did not keep the scales even on the axes. During the interviews for this relationship, Jeffery intended to use a scale of 8 on the x-axis for the equation \(d = 8t\), but he
spaced one grid square between the origin and 8 and continued to space two grid squares between 8 and 16 (see Figure 18). For the equation $C = 2n + 5$, Greg jumped 7 units from the origin on the $y$-axis, although he seemed to understand that the scale had to be kept even and he set the scale of 2 units per grid line. That is, he put his first mark at 7, which was the value of the dependent variable when $n = 1$, on the $y$-axis and chose the scale of 2 based on the increment of the dependent variable.

These actions by the students implied that students had to understand why equal increment scales were important and what happened if the scales were not equally incremented. Also students seemed to presume that they had to put as many data points at the intersections of the grid lines of the scales as they could. Moreover, they seemingly presumed that that was the way to increase the accuracy of their graphs. The discussion concerning the scales providing the guidelines to help locate the data points at reasonably estimated positions seemed necessary so that the whole graph could deliver a fair story. However, all of the students were unaware of how the constant term in the linear equation would affect the value of the dependent variable when trying to figure out an appropriate scale for the $y$-axis in order to have the data points located on grid lines.

Another syntactical issue was how to label numbers on the axes. In particular, Greg consistently put numbers in the grid squares instead of on the grid lines to indicate that the numbers were assigned to the grid lines. After he confused himself and his partner Peggy by the way he placed his numbers, he chose to write the numbers on the grid lines to avoid confusion. While it was a syntactical issue for Greg, it seemed possibly related to Peggy’s conceptions of rates in graphical representations. Assuming that she understood that interpolation was possible because of the constant rate, 8 miles per hour, she could have plotted new data points in the
middle of existing data points even without looking at the marks on the \(x\)-axis. That is, mathematically competent adults could have interpolated correctly regardless of Greg’s labeling. Thus, her confusion during her interpolation might be either due to Greg’s way of labeling or due to her conception of constant rates. In fact, in the section of *Conceptions of Rates*, the discussion on Peggy’s conceptions of rates will continue regarding how she understood constant rates and ‘increased by a same amount’ in a graph.

**Figure 23.** Referential relationship from algebraic notations to graphical representation.

### Summary of Students’ Referential Relationships

In summarizing students’ making sense of algebraic notations, two directions of referential relationships will be discussed here in terms of students’ *interpretation of one representation, mental operation and projection to the other representation* as depicted in Kaput’s (1991) model. The first direction is students’ referential relationships from algebraic
notations to the other forms of representation. No matter what other representation they headed for, students had to contextualize a given algebraic notation. For instance, the equation \( d = 8t \) is an algebraic notation that represents a collection of changing situations abstractly, and the changing situations have in common two variables changing proportionally. Thus, ‘contextualizing’ means undoing the abstraction here. In order to contextualize a given algebraic notation, students interpreted the algebraic notation first.

Students’ interpretation included assigning changing quantities to the letters in the equation and recognizing the operation(s) between letters in the equation. Assigning different values to the letters, however, did not always guarantee that students fully understood the given equation represented a changing situation. Depending on students’ conceptions of variables, they could consider the letters as variables or not. For instance, Jeffery was able to assign various values for the letters in the equations of \( d = 8t \) and \( C = 2n + 5 \), but he did not conceive the letters as variables in a way that mathematically competent adults would do. Thus, I could hardly conclude that Jeffery considered the equations as representing changing situations. After assigning changing quantities to the letters, students could figure the pattern between two quantities by recognizing the operation(s) between the letters.

Students’ matching a situation to the given equation was their mental operation in the referential relationship. However, their action of matching a situation did not necessarily occur after their action of interpretation of the given equation, rather previously or simultaneously. That is, while adults tend to complete their interpretation of a given equation to figure out the relation between the variables and look for a situation matched with their interpretation, the interview students began to look for a situation even before completing their interpretation of the
equation. This suggests that three types of actions do not necessarily proceed in a linear manner as Kaput’s (1991) model described.

Next, students seemed to mentally coordinate specific quantities for the equation before actually projecting their interpretation and mental operation into the other forms of representation. For the equation \( d = 8t \), the coordination included, for example, \((t, d) = (1, 8)\) or \((2, 16)\) using their numerical understanding. Through the mental coordination, students could figure out how the pairs of quantities changed and anticipate a certain pattern in the other forms of representation.

At last, students actually projected the coordinated quantities into the other forms of representation. For the narrative representation, students did not state their story in a complete form in the interviews. However, students’ projections into tabular and graphical representations were complete and successful. Students seemed to project into tabular and graphical representation more confidently because they knew what to do rather than making up a story. The reason might be either because the narrative representation does not require a certain format or because students have less experience with the narrative.

The second direction of the referential relationship is students’ referential relationships from other forms of representations to algebraic notations. In contrast to the first direction, students had to abstract a changing situation into algebraic notations. However, all students seemed more fluent in this direction than making referential relationships from algebraic notations. At first, they identified changing quantities in a given form of representation. Next, they looked for a relation between two quantities by recognizing key word(s) in a narrative (e.g., “per hour”) or recognizing a pattern in a table or a graph. With the relation, students could decide the types of variables. In the section of Conceptions of Variables, it will be discussed
thoroughly how students decided the types of variables. Recognizing patterns mainly helped students identify operations among the quantities. Then, students projected the results of interpretation of a given representation and mental operation into algebraic notations. Their projective actions, including selecting letters for the changing quantities and writing an equation, occurred without difficulty.

So far, I have described referential relationships centering algebraic notations across students. The description focused on the nature of student actions as fundamental elements of the relationship. However, individual students showed varying degrees of fluency in making referential relationships. The different degrees of fluency could be explained by how well students related each action in their referential relationship with their understanding of representing changing situations. In particular, the SOLO taxonomy could be applied for explaining the different ways that students related each action in the referential relationship. According to how well the actions were related, the levels could be described as prestructural, unistructural, multistructural, relational and extended abstract. These levels were used as a tool to describe the nature of students’ referential relationships and explain the degrees of their fluency of making the relationships. The nature of the interview students’ referential relationships seemed to range from unistructural to relational.

An example of a student’s referential relationship at a unistructural level would be Jeffery’s relationship from algebraic notation to narrative representation. He barely began to engage himself in the task, but he only focused on the numerical aspect of the algebraic equation when responding to the interview questions. However, Jeffery’s referential relationship to algebraic notations seemed more advanced in general. These different levels for Jeffery implied that the degrees of fluency varied even within students.
In general, Greg showed more advanced referential relationships than the other students. Particularly, he achieved at least relational level in all types of referential relationships. For instance, when he wrote equations for the total costs in the bike tour scenarios, not only his actions in the referential relationship were well related, but also he demonstrated that his understanding of the changing situations supported his symbolic manipulations. In the summary section at the end of this chapter, different SOLO levels of individual students’ referential relationships will be discussed further.

Also the analysis of students’ referential relationships implied that students’ conceptions of variables and rates played a key role in tying together their actions in the relationships in this mathematical context. So the next section will address students’ conceptions of variables and rates in their referential relationship.

Students’ Conceptions of Variables and Rates in Referential Relationships

This section will address the second research question: How do students’ mathematical conceptions form and develop as they use the algebraic notations throughout mathematical activities? This question focused on how students’ conceptions of variables and rates incorporated the referential relationships. Based on the analysis of students’ referential relationships, students’ conceptions of variables and rates turned out to be crucial for making sense of algebraic notations. Therefore, here I will describe students’ conceptions of variables and rates and how they understood variables and rates in different forms of representation.

Conceptions of Variables

The meaning of variables was not discussed extensively either in the textbook or in class. The book explained simply, “A variable is a quantity that changes or varies” (Lappan et al., 2004,
In class Ms. Moseley referred to variables as things that changed. Thus, when the interview students were asked what variables meant, they unanimously answered something that changed. Understanding what each student thought of variables and how they understood and used them in different forms of representation of changing situations, however, was not as clear and simple as they had answered. In attempting to understand students’ conceptions of variables better, this section will discuss on what premise the definition of variables depended in the mathematical content of this study, how students understood and conceived variables, and how they identified the types of variables in different forms of representation.

First of all, in the context of the present study, students had to understand that the definition of variables, a quantity (of something)\(^7\) that changes or varies, was based on a changing situation where one quantity changed in relation to another. This premise differentiated variables\(^8\) from unknown quantities and unspecified quantities (or generalized quantities), which could be also represented with letters in equations.

Consider an equation \(3x + 2 = 14\) representing a situation that Cindy is 14 years old now and her age is 2 years older than three times of Betty’s. The equation has the letter \(x\) representing Betty’s age, and various values of \(x\) can be considered to find out Betty’s age. However, the various values of \(x\) are not based on a changing situation, and they are selected randomly, not by a relation to another quantity. Moreover, each value of \(x\) determines the truth

\(^7\) Quantity means the amount of something and quantity must be measurable in order to determine the amount of something. However, Ms. Moseley did not differentiate the terms, quantity and something, and the definition of variable, “something that changes” was accepted in her class.

\(^8\) The term, variable, was used and discussed in the context of representing changing situations and functions although some studies used the term broadly as referring to literal symbols.
or falseness of the given equation. Thus, the value of \( x \) that makes the equation true is simply an unknown quantity.

As another example, a linear equation \( y = mx + b \) represents a collection of situations where two quantities represented by the letters \( x \) and \( y \) have a linear relationship. The linear equation has letters \( m \) and \( b \) that can have various values. The values of \( m \) and \( b \) are not based on a changing situation involved with them, and they can vary independently not by a relation between them or to another quantity. Although they can be restricted to a certain set of numbers (e.g., integers, rational numbers, etc.), they are unspecified quantities (or generalized quantities) that represent the slope and the \( y \)-intercept of a linear equation. Thus, the quantities, represented by the letter \( x \) in the equation \( 3x + 2 = 14 \) and by the letters \( m \) and \( b \) in the equation \( y = mx + b \), can change or vary, but they are differentiated from variables in a changing situation.

The premise under the definition of variables helped me understand subtle differences among students’ conceptions of variables. When the changing situations were represented in the other forms of representation than algebraic notations, students literally could see what quantities changed and how they changed in relation to the other. That is, students could realize relatively easily that the variables were embedded in changing situations. However, with algebraic notations, students had to realize that the algebraic equations represented changing situations and the variables were in the changing situations before they attempted to identify the variables. In particular, when students were asked to make up a story with a given equation, their conceptions of variables were distinctively contrasted.

Peggy and Greg, in general, considered two changing quantities at the same time with the relation in the given equation to make up a story. For the equation \( d = 8t \), Peggy suggested a traveling situation with the time and the distance changing, and for the equation \( C = 2n + 5 \),
Greg and Peggy worked together on a changing situation where the cost and the number of people changed. It implied that they implicitly understood that variables were embedded in a changing situation where two quantities changed based on a relation.

However, Jeffery showed a very contrasting interpretation of variables. He seemed to conceive variables as any numbers without consideration of a changing situation. Thus, when he was asked to make up a story with an equation, neither could he think of a situation nor did he consider two quantities that were connected to the letters in the equation. Rather he named a letter in the equation when he was asked what the letter represented. That is, for the equation \( d = 8t \), he only named \( d \) as dance, which was not clear whether he considered \( d \) as a quantity, and he did not even mention what \( t \) could be. Even if Jeffery considered \( d \) as a quantity related to dance, he showed no evidence that he thought of a changing situation in which the quantity \( d \) changed in relation to the other quantity. Later when I suggested a distance-time situation for the equation, he could not successfully make a story but he was able to generate several pairs of numbers \((d, t)\) using his numerical understanding.

In making up a story with the equation \( C = 2n + 5 \), Jeffery’s conception of variables was further revealed. He named \( C \) as the candy bar cost without providing any changing situation involved with the candy bar cost. However, he mentioned that \( n \) could be any number and he could complete the number sentence \( 9 = 2 \times 2 + 5 \) with \( n = 2 \). Then, he said that he would get one candy bar with $9. This implied that Jeffery interpreted the definition of variables as, “something that changes”, literally so that he thought variables could be any number. Thus he did not relate a variable to a changing situation where the variable changed in relation to another variable. That is, Jeffery thought that \( C \), representing the candy bar cost in the equation \( C = 2n + 5 \), could be any number and \( n \), whatever it stood for, could be any number, too. It was
interesting that Jeffery considered $C$ as a random number but he seemed to think $n$ would be
determined once a specific number was chosen for $C$ and vice versa. Therefore, Jeffery
seemingly conceived that variables could be any numbers that changed and they could be related
to another number if he had another.

The way that students identified the types of variables in a changing situation also shed
light on their conceptions of variables. Students were introduced to the idea that the dependent
variable depended on the independent variable in Ms. Moseley’s class. The textbook also
suggested that students should think about how the two variables were related, and mentioned
that if one variable depended on the other, one was the dependent variable and the other was the
independent variable (Lappan et al., 2004, p. 8).

However, students had other ideas for categorizing the types of variables. Examples
included that the independent variables could not be manipulated and ‘time’ was always the
independent variable. All four students in Interview 1 expressed that they could not change the
independent variable because it changed on its own but they could change the dependent variable.
In reminding Greg and Peggy of the Jumping Jack experiment – collecting a data set of the total
number of jumping jacks after every 10 seconds up to a total time of 120 seconds – that they had
before as a classroom activity, I asked them to describe the types of variables in the experiment.
Peggy said that the time was the independent variable and the number of jumping jacks was the
dependent variable since she could not change the time but she could change the jumping jacks.
Then, I asked Greg whether he had a different way to determine the types. Together with Peggy,
Greg could explain that the number of jumping jacks depended on the time.

In the interviews with the other pair of students, Angela and Jeffery also confirmed that
whether or not they could control a variable determined the types of variables. Moreover, this
idea led them to think that time was always the independent variable in changing situations since
time could not be controlled. In Interview 1, I took an example data set from Ms. Moseley’s
class in order to ask them to identify the types of variables. The data set was given in a table
with Katrina’s height in every year from birth to 18 (see Appendix B; Lappan et al., 2004, p. 27)
and Angela and Jeffery graphed it as homework assignment. Jeffery said that the age was the
independent variable since they could not change age and therefore the height was the dependent
variable. Angela added that they could not control height but height did not go back. At this
point, she seemed to realize that ability to control a variable would not always clearly determined
the type of variable. So I suggested using the dependency relation, asking which one depended
on the other. Angela stated that the height depended on the age because they could not be six
foot high at age one.

In Interview 2, in order to challenge the idea that time was always the independent
variable, I asked Angela and Jeffery to identify the type of variables in a table showing the
winners and the winning times for the women’s Olympic 400-meters dash (see Appendix C;
Lappan et al., 2004, p. 32) in each year. Two variables in the data set were the years and the
winning times. Angela at first answered that she would put the years on the \(x\)-axis and the
winning time on the \(y\)-axis to graph it. Then, she tried to explain the reason by saying, “Because
it’s showing the time for the winning times, so that’s why it would be on the \(y\)-axis…”, but she
could not finish her sentence.

At that moment, Jeffery said, “I think the year depends on the time that the winning
people scored” because “You have to give it before you can decide the time.” In fact, Jeffery
tried to figure out the dependency between the years and the winning time, but his statement
sounded contradictory. If he meant that he needed a year to decide the winning time, he should
have answered that the time depended on the year. Angela insisted that both were “time” meaning that she could not decide which time would be the independent variable. So I asked Angela what she thought about Jeffery’s comment, she disagreed and explained that the independent variable was the years since time could not go on without year changing. Although Angela’s reasoning was a little ambiguous whether she meant ‘time’ by the winning time in her explanation, Jeffery changed his mind as saying that the winning time could not be without the year. As intended, all students eventually seemed to understand that considering which variable depended on the other would determine the types of variables better than considering which variable they could control or which variable was time. However, it did not seem easy for them to figure out the dependency between two variables.

Once students were equipped with the dependency criterion to determine the types of variables, they still had to deal with some issues with the relation between two variables. As aforementioned, variables are in a changing situation where one changes in relation to another and moreover some relations direct the dependency between two variables. For example, in the data set of Katrina’s height in every year (see Appendix B; Lappan et al., 2004, p. 27), two variables were Katrina’s age from birth to 18 and her height in each year. The relation between her age and height was that Katrina’s height increased as she got older, and it did not make sense that Katrina got older as her height increased. Such relations determine less controversially which variable is dependent on the other.

Some relations, however, allow both ways to be possible. Jeffery in Interview 7 was asked to graph the equation \( d = 8t \) and he put the distances on the \( x \)-axis and the times on the \( y \)-axis. Then, he was asked how he decided which variable went on which axis. He tried to use the dependency criterion to decide, but he seemed confused and changed his answer several times.
Consequently, Olive provided two possible situations with the two variables and asked Jeffery to answer which variable depended on the other. One situation was that the distance depended on the time by asking how far he could go for 3 hours at 8 miles per hour, and the other was that the time depended on the distance by asking how long it would take him to go 80 miles at 8 miles per hour. Jeffery could answer both questions correctly and had a chance to learn that two dependencies were possible. It implied that the dependency between two variables could be arbitrary in some relations. It also implied that understanding that they could set the types of variables under some relations would enable students to understand variables more flexibly and apply the concept of variable according to their purposes.

*Conceptions of Rates*

Whereas students had limited learning experiences about variables in their mathematics class when this study was conducted, I believed that they had somehow developed their understanding of rates through previous learning experiences. However, students still had some difficulties in understanding and using rates in changing situations. In fact, when it comes to represent changing situations, various rates could be considered and they might cause students’ difficulties understanding rates in changing situations. In a directly proportional pattern, say \( y = ax \), we can find out that both the rate of \( y \) to \( x \) and the rate of \( \Delta y \) to \( \Delta x \) are constant and equivalent as well (that is, \( \frac{y}{x} = \frac{\Delta y}{\Delta x} = a \) while \( \Delta x = x_2 - x_1 \) and \( \Delta y = y_2 - y_1 \)). Moreover, we can induce \( \frac{x_1}{x_2} = \frac{y_1}{y_2} \) from \( \frac{y}{x} = \frac{\Delta y}{\Delta x} = a \). It is interesting that all the rates mentioned just before are constant in a directly proportional pattern. Nonetheless, only the rate of \( \Delta y \) to \( \Delta x \) is constant in a linear pattern with a non-zero constant term (e.g., \( y = ax + b \)). Interview questions on rates in the present study were asked only in directly proportional patterns. Here I will
describe how students understood and used rates represented in various forms in order to compare two changing situations, to predict values and to generate rules.

In Interview 3 with Greg and Peggy, the interview task was to choose a bike shop over the other by comparing weekly rental fees, one set of fees shown in a table and the other in a graph (Lappan et al., 2004, p. 37). The task did not necessarily expect students to compare the rental rates of two shops, but one student, Jennie, in Ms. Moseley’s class did so. Here I will describe Jennie’s method that I considered evidence of her rich conception of rates and how Greg and Peggy made sense of her method. The following table shows the rental fees of Rocky’s Cycle Center and the following graph shows those of Adrian’s Bike Shop. Two days before the interview, students had a classroom activity related to the task and they had made a table of the rental fees of Adrian’s and graphed those of Rocky’s in the same coordinate system with Adrian’s.

Table 4

*Bike Rental Fees in Rocky’s Cycle Center (Lappan et al., 2004, p. 37)*

<table>
<thead>
<tr>
<th>Number of bikes</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rental fee</td>
<td>$400</td>
<td>535</td>
<td>655</td>
<td>770</td>
<td>875</td>
<td>975</td>
<td>1070</td>
<td>1140</td>
<td>1180</td>
<td>1200</td>
</tr>
</tbody>
</table>
Greg explained that he made a decision by comparing rental fees for every five bikes in two tables. In fact, he put the two tables side by side and calculated the differences for each row between two shops. He added that he chose cheaper fees by comparing each row, and concluded that Adrian’s would be cheaper up to 30 bikes and Rocky’s would be cheaper over 35 bikes. Peggy stated that she made a decision by comparing rental fees both in tables and graphs. She said that she compared the tables as Greg did. In the graph, she explained that Rocky’s was going to level off and Adrian’s shot up. Thus, she concluded that Adrian’s would be higher for more bikes. Peggy also mentioned that the two graphs intersected where they were at “around the same price”. Therefore, both students made a direct comparison of the prices using the patterns either in the tables or the graphs to make a decision.

In order to see whether Greg and Peggy could relate their methods to comparing rates, I showed a class video clip in which one of their classmates, Jennie, calculated the rental price per bike for every five bikes in both shops in order to make a decision. Jennie compared the rental price per bike for each case and mentioned that Adrian’s had always the same price per bike. She also supported her assertion by showing that the differences between adjacent data points were the same. Jennie’s method seemed quite different from other students’ and the other
members of Jennie’s group did not follow her method in the video clip. When I asked Greg and Peggy what Jennie did, Greg said that Jennie divided the price by the number of bikes. He explained that, for example, Jennie divided $600 by 20 for Adrian’s and obtained 30 meaning “30 dollars per piece”. So Greg seemed to understand that Jennie calculated the price per bike for each case.

However, when I asked Greg whether Jennie would make the same decision as they did, he kept saying yes and no several times and he finally said that he did not know. Peggy also said that she could not answer. Then, I asked them to calculate the price per bike for Rocky’s and they calculated up to 20 bikes. After I reminded them how they compared the total price for each case, Greg said that Jennie compared the price per bike for two shops while they compared the total price. He added that Jennie would get the same decision as his since Adrian’s set the price $30 for each bike but Rocky’s had more than $30 before renting 35 bikes and less than $30 after renting 35 bikes. Eventually Greg seemed to understand Jennie’s method but Peggy appeared still unsure. In fact, although Greg seemingly understood Jennie’s method, it was still not certain how he conceived her method and why it worked. Greg certainly knew that Jennie’s method resulted in the same decision that he made, and probably he was able to relate the total rental fee and the rental rate during this interview.

Secondly, I will describe how students used constant rates to calculate and predict values in changing situations. In particular, this description will focus on students’ understanding of constant rates additively and multiplicatively. Two tasks were provided for each pair of students. The first was to calculate and predict the profits for some numbers of customers (in the bicycle tour company) when the provided graph showed the profits for every 5 customers. In fact, the graph included only two data points that could be read without estimation. The second task was
to calculate and predict the distances traveled at a constant speed of 55 miles per hour. While the domain of the independent variable in the first task was the set of whole numbers, the second task extended the domain to rational numbers. Thus, the description here will not only include students’ understanding of rates additively and multiplicatively but also how they could extend their additive understanding into multiplicative understanding.

In Interview 3 with Angela and Jeffery, I provided the graph of estimated tour profits for the different numbers of customers (see Figure 25; Lappan et al., 2004, p. 39) and asked them to describe a pattern that they recognized in the graph. Angela answered that the graph went up at a constant rate since the data points were going up like a diagonal straight line. Jeffery agreed with her. Angela added another reason that there were the equal spaces between two consecutive data points.

![Figure 25. Estimated tour profits (Lappan et al., 2004, p. 39).](image)

I asked them how much profit would be made with 10 customers and Angela estimated $150 by reading the graph. I then asked them whether they had some data points that they could read exactly. Angela read $600 for 40 customers and Jeffery read $300 for 20 customers. Right then Angela recognized the relationship between two data points and said that 20 was a half of 40 as 300 was a half of 600. She added that she could predict the profit for 60 customers. In fact,
Jeffery followed Angela’s reasoning and they predicted the profits for 60, 80 and 100 together. Both explained that they could add $300 repeatedly as they increased by 20 customers. They used the data ‘$300 for 20 customers’ additively and predicted the profits for every 20 customers.

Next, I asked them to calculate the profits for every 5 customers using a similar reasoning. Angela answered that the profit for 5 customers was $90 because adding 90 repeatedly reached 300 and estimation from the graph agreed on 90, too. When I asked her whether she was sure, Angela did the repeated addition and got 90, 180, and 270, and Jeffery multiplied 90 by 2, 3, and 4. Thus, both found that Angela was not right. Here Angela seemed to guess the profit for 5 customers by reading an estimate in the graph since she mentioned that the profit was about $90. Then, she checked her guess by adding 90 repeatedly in order to see whether she could reach $300.

Since the previous strategy did not work, Angela suggested dividing 300 by 3. She explained, “since there are three in between them”, as pointing to the three rows for 5, 10 and 15 customers in their table containing the profit for every five customers. She apparently thought of dividing 300 by 3 since she noticed three rows of 5, 10, and 15 customers before 20 customers in the table. That is, she thought how many times she had to add the profit for 5 customers to get $300 for 20 customers. It implied that Angela’s reasoning with the rate was additive basically but she could relate the repeated addition to multiplication. So she did the division as the reverse operation of the multiplication. This method would have worked if she had been careful enough to think that she had to add the profit for 5 customers three times to itself, which led to multiplying the profit for 5 customers by 4. If she could use the rate multiplicatively, I believe that she would rather think how many fives were in 20 and divide 300 by the number to get the profit for 5 customers. Interestingly Jeffery agreed on Angela’s method and they divided 300 by
3. However, Jeffery finally said that it was not right since their estimate of the profit for 5 customers was less than $100.

   In order to encourage Angela and Jeffery to reason multiplicatively, I asked Angela what she said about the relation between the profits for 20 and 40 customers. She said that she could write down the relation into a fraction and she wrote \( \frac{20}{40} \) and reduced it to \( \frac{1}{2} \) as shown below. Jeffery then wrote \( \frac{1}{2} \) in the profit column. So they were able to not only recognize the relation between the profits for 20 and 40 customers but also write the relation as a fraction.

   ![Figure 26. Angela and Jeffery’s representation of the double relation as a fraction.](image)

   Then, Jeffery said that he could observe the exact same relation between the profits for 15 and 30 customers. When I asked whether he knew the exact profit for either number of customers, he instantly picked the case of 20 and 10 customers. He then figured out that the profit for 10 customers was a half of 300, $150. At this moment, Angela checked the graph to see whether $150 agreed with the data point. Shortly Angela said that the profit for 5 customers was $100 and she would add 50 more to find out the profit for 15 customers. Jeffery disagreed
with Angela by saying that $100 would be too many for 5 customers by the graph. He also convinced Angela that they had to make a half of 150 for 5 customers using the relation between 5 and 10. Angela agreed with Jeffery and they calculated $75 profit for 5 customers. For the profit for 15 customers, Angela said, “We just add 75 two times”, but she meant $75 + 75 + 75 since she confirmed that it was the same as 75 times 3. Jeffery agreed with her and they decided to fill out the table by adding 75 more as moving down the column. So they figured out some of the profits multiplicatively (or proportionally) but they filled out the rest of the profits additively.

After they had filled out the table, I asked them whether they could find some relations between two different numbers of customers. Since they looked at only the relation of half, I asked them to see the relation between 10 and 30 customers. Angela recognized the relation that 10 was $\frac{1}{3}$ of 30 and 150 was also $\frac{1}{3}$ of 450. Then, I asked them to find out the profit for one customer and Jeffery instantly answered $15 by dividing 75 by 5. After Jeffery explained how he got the answer to Angela, she said that she could find out the profit for any number of customers. However, when asked what the profit for 32 customers would be, Angela answered “75 divide by 32” with hesitation. Her response did not support her claim, but Jeffery answered confidently, multiplying 15 by 32.

Based on what they did so far, both Angela and Jeffery could figure out the profits for the multiples of $n$ customers by doing repeated addition when the profit for $n$ customers was given. This supported their additive understanding of rates. However, their additive understanding of rates did not succeed in doing the reverse way. It might be because their additive understanding was not fully developed, or because Angela made a simple mistake and Jeffery did not notice it. Nonetheless, it turned out to be an opportunity for them to think of the multiplicative aspect of rates. Although Angela seemed less confident than Jeffery, both of them could understand and
use rates multiplicatively to some extent. During this activity, writing the relation as a fraction seemed definitely helpful for them to understand the proportional relations. Thus, they began to develop their multiplicative understanding of rates using their conceptions of ratios and comparing the profits and numbers of customers proportionately. Angela, however, was still unsure that the rate had to be used to generate the profit for 32 customers.

In Interview 4 with the other pair of students, the same task was given to Peggy alone since Greg was absent. The task for Peggy was to figure out the exact profits for the number of customers by using the profits for 20 and 40 customers that she could read exactly from the given graph. However, she did not seem to understand even after I explained what she was asked to do several times. She seemed to understand the given graph showed that the profits increased at a constant rate by saying, “It increased by the same amount.” Instead of using a constant rate she could figure out in the graph, she seemed dependent upon her memory of some equivalent ratios of the constant rate including ‘$15 profit per customer’ and ‘$75 profit for 5 customers’ that she learned from the classroom activity about this task. Therefore, her responses during the interview were mathematically unrelated and challenging for me.

However, I could still infer some aspects of Peggy’s conception of rates. Generally, Peggy understood rates additively and she did not seem to have difficulty in figuring out the profits using repeated addition. However, she had difficulties in using rates multiplicatively including doubling. When I asked her to calculate the exact profits for every 5 customers, Peggy made a table with the number of customers and the profits and filled out two rows of 20 and 40 customers read from the graph. Then, I asked her to figure out the other profits and she put $75 for 5 customers without any evidence of using the given information. So I asked her how she
knew, and she responded that $15 was the profit for one customer. She said that she knew it but she forgot how to figure it out, which I inferred that she knew it by her memory.

In order to encourage Peggy to reason quantitatively rather than depending on her memory, I asked her whether she could find any relation between the profits for 20 and 40 customers. She then answered, “it doubles” and she said that she could use ‘doubling’. She took some time to think and finally said, “I was thinking that 75 times by 10 because 10 is the number of customers”. Because her table already had the profits for 5, 20, and 40 customers, she seemingly attempted to use the profit for 5 customers in order to find the profit for 10 customers. Here Peggy assumed that the profit $75 for 5 customers was not something that she had to explain but something already given. Even if I accepted her assumption, her reasoning would not be aligned with her comment that she could use the double relation. Possibly, Peggy seemed to consider $75 for 5 customers as the rate ‘$75 per customer’ and suggested multiplying 75 by 10. She seemed to know that 10 was double of 5 but she did not realize that the product of 75 and 10 would not be double of 75. Moreover, she related $75 for 5 customers to $15 per customer by saying that 15 times 5 made 75, but she could not explain correctly. When I asked how she knew $15 per customer, she replied: “Divide, umm, 75 by… Some divide by 2.” Her response implied that she knew the multiplication $15 \times 5 = 75$, but she could not relate the multiplication to the division to get $15$ per customer. All her responses here made me wonder what she meant by double and how she used doubling.

In another attempt to understand Peggy’s reasoning, I asked her to find out the profits after reminding her what she said about the given graph, “It increases by the same amount”. Then, she wrote $150$ for 10 customers and explained that she obtained it by multiplying 15 by
10. Without commenting on her response, I asked her to calculate the profit for 15 customers.

She wrote $225 but changed it into $235\textsuperscript{9} a little later. She explained her reason;

Since it doubles, if you double 150, it’s gonna go to 300. So half of 300… I think half of 300 would probably be 15, something like that since it doubles.

After her comment, I asked her what the half of 300 was and she answered 250 at this time. At the moment, she pondered a while and then she said, “It doesn’t go evenly”. Thus, I asked her what part she wanted to make even. She answered that she wanted to make 75, 150, 235, and 300 even as pointing these figures in her table. Then she added that 150 was even and she continued that she got 150 by doubling 75 and she doubled 150 to get 300. Here she seemed to expect 300 to be in the current place of 235 and therefore she looked a little confused. However, her confusion seemed to make sense to me as explained in the next paragraph.

Based on her reactions, Peggy seemed to equate ‘increase at a constant rate’ with ‘doubling relation’. When she commented about the pattern of the given graph, she said that it increased by the same amount. She also recognized the double relation between the profits of 20 and 40 customers. Here the major issue was whether she was fully aware of both patterns in terms of two variables. That is, she could have stated that the profits increased by the same amount without noticing that the number of customers also increased by 5. Likewise for the doubling relation, she would have paid attention to the doubling of the profits only, without noticing the doubling of the number of customers. Thus, provided that she used the fact ‘$75 for 5 customers’ by recalling it from the class activity, she found a conflict in the profits of 75, 150, 235, and 300 that she generated using the ‘increased by the same amount’ method but these results conflicted with her doubling method due to the existence of 235.

\textsuperscript{9} Peggy maintained this error until she found and fixed it later.
Next, I asked Peggy how many additional customers she had in 10 customers from 5 customers and she answered 5 customers. I then asked her to calculate the profits for 10 customers using that of 5 customers. She could answer that she added 75 to 75. Using repeated addition, she could find the profits for every five customers and she saw the profits were evenly distributed. When she said that it increased by 75, I asked her whether the number of customers increased by a same number. She answered it increased by 5 and she wrote the increment. Then, I asked her to make a statement about the increment and she said, “It doubles, and it increases by the same amount”. I asked her to point out which part doubled and she answered the profit. She pointed out that 150 was doubled from 75. I asked her whether $225^{10}$ was doubled. She said “no” and she was asked whether she could find double relations in the table. She pointed to 150 and 300. I then asked her to see the relation in the numbers of customers for 150 and 300. Peggy finally looked at the double relation in the numbers of customers. At this point, she seemed to make sense of ‘increase at a constant rate’ and ‘doubling relation’. However, her confusion appeared to remain.

With her table for the profits for every five customers from 5 to 50, Peggy was asked to find the profit for 100 customers. She answered 100 times 75. I asked her what the increment was in the table and then she seemed to know she could get the answer by adding 75 repeatedly. So I asked her to use another method by using the relation between 50 and 100 customers. Once she was asked to write 100 right underneath of 50, she answered, “I guess, 750 plus 75… Uh-umm, times by 10, 750 times 100…” . These responses supported that she used neither ‘increase at a constant rate’ nor ‘doubling relation’. These responses also convinced me that she paid little attention to the number of customers. Rather she seemed to engage in a random search pattern

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$^{10}$ The error was fixed while she was doing the repeated addition.
using any of the numbers in the situation to generate an answer. Such behavior suggested that in terms of the procept model (Tall et al., 2001) model, Peggy had not developed a process for this type of mathematical situation\textsuperscript{11}. In fact, she had trouble generating an appropriate procedure.

Then, I reminded Peggy of the double relation in the profits for 20 and 40 customers. She said that she could keep adding 20 for the number of customers and 300 for the profit and she got the profit 1500 for 100 customers. To make sure, I asked her whether she meant double by keep adding the same number and she said yes. She said that double meant also multiplying a number by 2. It again convinced me that Peggy equated ‘increase by the same amount’ with ‘doubling relation’ as I hypothesized. In particular, if she paid attention to the profits of 5 and 10 customers only in the table, it convinced her that increasing by $75 was equivalent to doubling the profits. Likewise, whenever she had only two initial terms \(( a_1, a_1 + a_1 )\) and \(( a_1, 2a_1 )\) in arithmetic and geometric sequences respectively, she could make a conclusion that ‘increase by the same amount’ and ‘doubling relation’ were the same without envisioning the continued patterns. Considering that she might have had no chance to generate two patterns before, the conclusion had worked for her till this interview. During the interview, I presumed that she experienced a number of conflicts of ‘increase by the same amount’ and ‘double relation’. However, she seemed unable to resolve the conflicts.

I then asked Peggy whether she could generate the table of the profits as she did in the interview if I gave one piece of information such as $600 profit for 40 customers. She said no and she needed more information such as the profit for 35 customers. Then later, she changed

\textsuperscript{11}Peggy’s inability of generating an appropriate procedure can also be explained by her lack of “composite units coordinating schemes”, which would have enabled her to coordinate two levels and types of units (Olive, 1999).
her mind and said that she needed the profit $75 for 5 customers. This response implied for me that she did not conceive all of the data pairs as having the equivalent rates. Thus Peggy seemed to have a limited additive understanding of rates but she showed no evidence of multiplicative understanding of rates.

So far, I have discussed the cases in which students were asked to use and reason with rates when the independent variables belonged to the domain of natural numbers. Consequently, once students understood rates additively, they could extend the additive operation into the multiplicative operation. That is, students extended additive understanding of rates into multiplicative understanding by proportional reasoning as in Interview 3 with Angela and Jeffery. In order to understand how students use rates multiplicatively, the cases where the independent variables belong to the domain of rational numbers should be addressed.

In Interview 4 with Angela, the context was provided that the bicycle riders made the return journey by van at an average constant speed of 55 miles per hour after the bike tour (Lappan et al., 2004, p. 50). Angela, with Jeffery absent, completed the table of distance for every hour from 0 hour to 8 hours. Her method was multiplying 55 by the number of hours. She commented that adding 55 repeatedly would be easier but takes longer. When she found her error in the multiplication for the distance traveled for 3 hours, she began the repeated addition.

Once Angela completed the table of distance, I played the class video clip showing that her group worked on the same task on the previous day. In the clip, students in Angela’s group tried to figure out the distance traveled for \( \frac{5}{4} \) hours, and Angela did interpolation of the graph to read the distance. After watching the video clip, she explained that she marked \( \frac{5}{4} \) hours on the x-axis as partitioning in between 5 and 6 into 4 equal parts, drew a vertical line from the first
mark next to 5, and read 300 miles for $5\frac{1}{4}$ hours. She also said that the distance was not exact since her graph was not accurate without a ruler. Angela seemed to understand how to interpolate the graph to read the distance. Then, I continued playing the class video clip showing that Angela answered Ms. Moseley’s question regarding whether they could use the table to find the distance. In the video clip, Angela said that she would divide some number by 4. I asked her to explain more and she said that she was thinking to divide 55 by 4. However, in the class she actually did divide 275 by 4 because 275 was the distance traveled for 5 hours, but she said that she did not mean to do so. Angela seemed to understand that she had to find out 1/4 of 55 miles, which was the distance for one hour as relating to her interpolation of the graph. This response implied for me that she could understand rates multiplicatively even though she did not complete her calculation.

In fact, Angela calculated the distance traveled for $4\frac{1}{2}$ hours with the same reasoning. While calculating, she had some computational difficulty in dividing 55 by 2, but she eventually got the answer 27.5 after she figured out 15 divided by 2 was 7.5 using 7 times 2 was 14 and 8 times 2 was 16. Then, when she attempted to add 27.5 to 220, which was the distance traveled for 4 hours, she wrote 27 1/5 as reading it out “twenty seven and a half” and answered 490 1/5 miles. So I asked her whether her addition was correct, she realized that she did not line up two numbers by the place values and she corrected her previous answer into 247 1/5. However, she read her answer as “two hundred forty seven and a half”. Since the time was running out, I could not ask her whether she meant “a half” by 1/5. A possible explanation for this discrepancy between what she wrote and what she said could be the transformation from her decimal representation of a half (.5) to her fractional representation (1/5) in which she maintained the
digit "5". In spite of her difficulties in calculation, she was able to extend the additive understanding into multiplicative understanding using proportional reasoning based on her additive understanding of rates.

In Interview 5 with Greg and Peggy, the same task was provided but they had a different approach. At first, I asked them to complete the table of the distance for each hour from 0 to 8 hours. Greg explained how he calculated the distances by saying, “You can add 55 each time to the number before it because you keep in a constant rate of speed.” Later I asked them to find out a rule for the distance for any number of hours using the table. Greg came up with the rule and wrote, “\( \text{speed} \times \text{time} = \text{distance} \)”. When I reminded him of using algebraic expression in the rule, he wrote \( S \cdot T = D \). I asked them to find out the distances for some whole numbers of hours with the equation and they could calculate them easily using the equation.

Next, I asked Peggy to find out the distance traveled for \( 4 \frac{1}{2} \) hours. Peggy instantly converted \( 4 \frac{1}{2} \) hours into 4.5 hours and seemed to multiply 55 by 4.5 without writing the multiplication on the paper. When she whispered the answer 220 miles without confidence, Greg disagreed with her and said that because 220 miles was the distance traveled for 4 hours the answer would be more than 220. Instead of doing multiplication, Peggy looked at the graph and did interpolation to estimate 250 miles this time. When I asked them whether they could find out the exact distance, Greg did the multiplication correctly to get 247.5 miles.

Then, I asked them whether they could use the fraction without converting, and Greg said that he forgot how to do the calculation. So I said that they had to find out the distance traveled for 4 and half hours. Greg explained that he multiplied 55 by 4 to get 220 miles and he wrote, “\( \frac{1}{2} \text{ of } 55 = 27.5 \)”. Then, for some reason, he changed his answer 27.5 into 22.5 and added it to
220 to get 240.5 miles. However, by referring to 247.5 that he got from decimal multiplication, he said that his answer 240.5 was wrong. Shortly, he realized that his addition was wrong and corrected it into 242.5, which was still wrong. Thus, I asked him to check the calculation, 

$$\frac{1}{2} \times 55 = 22.5.$$ 

Peggy suggested dividing 55 by 2 and Greg realized that his first answer was right. Greg then found out that he got the same answer by both decimal and fractional calculations.

I asked Peggy to calculate the distance traveled for \(5 \frac{1}{4}\) hours. She converted \(5 \frac{1}{4}\) into \(5.25\) right away and multiplied 55.0 by 5.25 vertically. She then answered 2887.50, but Greg disagreed. Peggy changed her error and answered 288.75 this time. When I asked her why she multiplied 55.0 instead of 55, she said that she used zero for the extra spot. She seemed to think that she had to make two numbers with the same number of digits. Then, I asked her whether she could do the calculation without conversion. Peggy said that she did not know how to do.

Greg said that he could do and he began the multiplication. He wrote, “\(55 \times 5 = 275\)” and “\(\frac{1}{4} \text{of} \ 55 = \)” . While he was thinking of \(\frac{1}{4} \text{of} \ 55\), he said, “You can divide \(\frac{1}{4}\), 55, equals…” umm”. When I asked him what he wanted to divide, Greg answered, “55, but you divide \(\frac{1}{4}\), would go how many times \(\frac{1}{4}\) on 55.” Then, Peggy said that she would divide 55 by 4 and Greg said that it would be the same. Peggy tried the division of 55 by 4, but she got 13.3 by putting the remainder as the decimal. Greg disagreed her division and corrected it into 13.75 finally.

Thus although Greg said that he would divide 55 by \(\frac{1}{4}\) to get \(\frac{1}{4} \text{of} \ 55\), he seemed to mean dividing 55 by 4 based on his action.
Here Greg and Peggy set up the equation for the distance traveled at 55 miles per hour and the equation seemed helpful for them to extend their additive understanding of rates into multiplicative understanding of rates. However, when they used the equation to calculate distances, they had difficulties to calculate with fractions. Rather than considering the difficulties as computational, I considered them as difficulties that they had to go through to develop their understanding of rates multiplicatively.

Thirdly, generating rules with constant rates also illuminated students’ conceptions of rates. In particular, students had to realize that the unit rates\textsuperscript{12} were necessary to be calculated. For example, in Interview 6, students were asked to find the rule for the cost when provided with a graph showing the cost for every 5 tickets. Once students read that the cost increased at a constant rate, they had to figure out the constant rate as acknowledging that the constant rate could be represented in various equivalent ratios. That is, ‘$70 per 20 tickets’, which was the only data point that students could read from the graph without estimation, shared the same constant rate with ‘$35 per 10 tickets’ and ‘$7 per 2 tickets’ in this situation. However, what was necessary for the rule was the rate ‘$3.50 per ticket’. Eventually, students figured out the unit rate in the example as described in the referential relationship from graphical representation to algebraic notations. In particular, Greg and Jeffery understood the necessity of the unit rate quite well based on the fact that the rules had to work for any number of tickets by reflecting on their

\textsuperscript{12} The term, the unit rate, was carefully used in order to indicate that the rate represented a relationship between one quantity and one \textit{unit} of another quantity. In particular, the \textit{unit} had to be a singleton as in ‘3 dollars per ticket’ and ‘55 miles per hour’, as opposed to a complex unit as in ‘20 dollars per 5 tickets’ and 30 miles per half hour’. 
So far, I have described how students understood and used constant rates in changing situations. Particularly, three ways of understanding were described that included comparing two changing situations, predicting values, and generating rules. Under these contexts, constant rates were to be understood and used as relationships between two variables. In most cases, students used the idea that the rate of $y$ to $x$ was constant. Only when Angela and Jeffery calculated the profits in a table (see Figure 26), they used the ratio of $\frac{x_1}{x_2} = \frac{y_1}{y_2}$ based on $\frac{y_1}{x_1} = \frac{y_2}{x_2} = a$ and therefore they could connect their additive understanding and their multiplicative understanding of rates. No students used the rate of $\Delta y$ to $\Delta x$ in this study, which would be the constant rate in a linear pattern with a non-zero constant. Furthermore, because the rate of $\Delta y$ to $\Delta x$ is the slope of a linear equation, the notion of the rate would be critical to understand rates in various forms of representation and make a referential relationship in representing a changing situation.

In addition, students’ understanding of constant rates had to be bi-directional: one was to construct rates based on two quantities in a changing situation, and the other was to decompose constructed rates into two quantities as Lobato and Thanheiser (2002) suggested. When a changing situation was given in one form of representation, students needed to construct a rate according to the pattern between two variables. The tasks that included comparing two changing situations and generating rules asked students to construct rates. For the opposite direction, tasks that included predicting values asked students to decompose a given constant rate into two specific values of variables. Understanding constant rates in both ways was complicated because
students had to associate a rate as one quantity with two changing quantities. However, coupled with their conceptions of variables, students’ understanding of rates definitely supported or constrained their referential relationships around algebraic notations.

Summary of Individual Students’ Referential Relationships with Their Conceptions

In this section, I will summarize how the participating students made their referential relationships between algebraic notations and other forms of representation and how their conceptions of variables and rates affected the process of making their referential relationships. Particularly, this summary will suggest how individual students’ conceptions can be related to the process of making their own referential relationships. Students’ conceptions were represented by “Cog A” and “Cog B” in Kaput’s (1991) diagram of referential relationship (see Figure2).

*Angela*

Angela’s referential relationships centered around algebraic notations could be explained only partially, because she was absent from school on the day of Interview 7 whose tasks were related to making referential relationships from algebraic notations to other forms of representation. However, throughout the present study, she still revealed how her conceptions of variables, rates and equations were related to making her referential relationships.

When she made referential relationships from other forms of representation to algebraic notations, she appeared to have no difficulty to identify quantities that changed in other forms of representation like the other students. However, her interpretive action to identify the changing quantities did not seem supported by her conceptions of variables. Especially, when she was asked to represent the total cost of the bike tour in Interview 8, her first equation
\[ TC = Br + Fc + Vr \] showed that she identified several quantities that changed in the given table. The sub-costs represented by \( Br, Fc, \) and \( Vr \) were, in fact, changing quantities in the changing situation, but they were not variables that she needed to represent the situation with an algebraic equation. This suggested that Angela did not necessarily identify changing quantities to identify variables in a given representation. Moreover, it implied that Angela might not clearly understand that the equation should represent the changing situation with the variables and the relationship between them.

In order to identify the operation(s) among quantities, Angela paid attention to some key word(s) such as “per” in a narrative form and operational patterns in a table and a graph. She could recognize a directly proportional pattern relatively well and her action was supported by her conceptions of rates. As described earlier, she was able to understand rates both additively and multiplicatively, and her understanding of rates helped her identify the operation(s). However, she still had some difficulties to identify the operation(s) in a linear pattern with a non-zero constant and a constant pattern. Her difficulties seemed related to her conceptions of equations as representing the operations that she carried out on the data in the given representation rather than as a representation of the relationship(s) among the data. For example, in the equation above she represented the operation of adding together the three components of the cost of the bike tour that she obtained from the table of data, ignoring the relationship of these quantities to the number of customers.

When projecting her results of interpretation and cognitive operation into algebraic notations, Angela showed some syntactical issues, which were possibly related to her conceptions of equations. For instance, she wrote some extra words or signs (e.g., “# of hours”, or “$”) in her equations and she explained that those made what her equations meant clear.
Overall, Angela’s conceptions of rates seemed to support her referential relationships from other forms of representation to algebraic notations while her conceptions of variables and equations did not seem quite as supportive.

In terms of the SOLO taxonomy, Angela’s referential relationships were, at best, multistructural as she was able to use many pieces of information in the situation but had difficulty relating them together to form a coherent representation. Her focus on the operational aspects of the representation or situation suggested that she might be working at a procedural or, at best a process level in terms of the procept model.

*Greg*

In making referential relationships around algebraic notations, Greg showed more advanced level of performance than the other students as described before. In general, his conceptions of variables and rates were supportive to making his referential relationships. However, when he created a story with the equation $C = 2n + 5$, he had some difficulties to figure out what changing situation the equation could represent. His difficulties in this task seemed related to his conceptions of variables in changing situation. As the discussion with Peggy went on, he gradually began to make sense of the trip story that he initiated, but his understanding of the story seemed operation-oriented. This also raised some caution concerning how Greg conceived changing situations represented in various forms of representation.

Regarding his understanding of algebraic notations, Greg demonstrated some operations with algebraic notations at an advanced level. For instance, he was able to infer the reverse operation embedded in an equation and write a new equation with it, and to represent his understanding of distributive property with algebraic notations. These responses could be classified as extended abstract in the SOLO taxonomy, and would support how he made sense of
algebraic notations representing a mathematical situation. In terms of the procept model (Tall et al., 2001), these responses indicated that Greg could use algebraic symbols as procepts, pivoting between process and concept.

*Jeffery*

Jeffery showed very contrasting and uneven performances in making referential relationships from algebraic notations and making those to algebraic notations. Overall, he appeared to have less difficulty to write an equation to represent a changing situation given in other forms of representation. Especially, he seemed quite good at recognizing operational patterns to write an equation. However, his apparent fluency in making referential relationships into algebraic notations might not be supported by his conceptions of variables. Writing an equation for a changing situation without the support of his conceptions of variables seemed possible because Jeffery was able to generalize the operation(s) in the situation thanks to his numerical understanding. Evidence could be found in his process of making the other direction of referential relationships.

As described in detail before, Jeffery conceived variables as any numbers. In particular, when he interpreted algebraic notations representing a changing situation, he considered the letters representing the variables as substitutes of any numbers. Therefore, he could generate specific data pairs for the independent and the dependent variables following the operation(s) that the given algebraic notations included. This means that his process of making referential relationships from algebraic notations was literally the reverse process of making referential relationships to algebraic notations.

Consequently, Jeffery’s conceptions of variables did not hinder his process of making referential relationships toward algebraic notations, but they did in the other direction of
referential relationships. Moreover, his conceptions of variables led him to interpret a table or a graph representing a changing situation more locally rather than globally. That is, he seemed to understand specific data pairs or a simple collection of specific data pairs operationally, not conceiving the whole data pairs as a trend.

Jeffery's conception of variable can best be described in terms of Küchemann's (1978) classification of “letters as generalized numbers.” In terms of the SOLO taxonomy, his interpretation of the algebraic notation in the referential relationship from algebraic notation to narrative could be classified as pre-structural or unistructural at best, whereas his projection from narrative to algebraic notation could be classified as multistructural or even relational. In terms of the procept model, there was one occasion where Jeffery seemed to operate on the algebraic symbols at the proceptual level; that is, when he and Angela where able to produce the equation for total cost of the bike tour with a donated van without recalculating the other costs and simply deleting the “700” from their previous equation. In other referential relationships, however, Jeffery appeared to operate at the process level.

Peggy

Making sense of Peggy’s referential relationships was the most challenging for me. Moreover, her conceptions of variables and rates were quite ambiguous and unrevealing not only because her reactions during the interviews were inconsistent but also because she was quite dependent on her partner, Greg. In general, Peggy showed better performances in making referential relationships to algebraic notations from the observer’s point of view. However, her interpretation of changing situations was very operation-oriented. When she wrote an equation to represent a changing situation with a directly proportional pattern based on a given representation, her operation-oriented interpretation seemed to work although it was not clear
whether her conceptions of variables and rates supported her performance. In these cases, she generated a pattern by using her additive understanding of rates.

When it came to representing a changing situation with a linear pattern with a non-zero constant, Peggy tended to overgeneralize the same method that she used with a directly proportional pattern. The tendency seemed related to her conceptions of equations by trying to capture an operational relation in a changing situation.

Although her referential relationships from algebraic notations were not revealing during the interviews, Peggy seemed to try to do the reverse process of what she did in the other direction of her referential relationships. Her performance in this direction seemed unsuccessful because she focused on operational aspects of representing changing situations and her conceptions of variables were not supportive.

In terms of the SOLO taxonomy, the nature of Peggy’s referential relationships can best be classified as multistructural. She appeared to be able to make use of various pieces of information, but was not always successful in relating these various pieces of information to generate a coherent projection into another form of representation. Her focus on the operational aspects of her representations places her at the procedural level in the procept model. Her lack of a scheme for coordinating composite units inhibited her conceptions of both variables and rates.
CHAPTER V

DISCUSSION AND IMPLICATIONS

The present study investigated middle school students’ referential relationships centered around algebraic notations. The mathematical context was to represent changing situations where two quantities changed in relation to each other in the CMP materials. In particular, the context was introduced to students without the notion of function, which is defined as a relation between elements of two sets in most algebra textbooks. In the CMP materials, the intent of the context was to represent changing situations with two variables in various forms of representation. The present study focused on students’ referential relationships between algebraic notations and other forms of representation such as narrative, tabular, and graphical representations in order to understand how students made sense of algebraic notations. In addition, students’ conceptions of variables and rates were investigated in the referential relationships.

The result of the analysis of students’ referential relationships, however, raised more issues than just a detailed description of their referential relationships. In the first section of this final chapter, I will discuss the issues that arose as questions in my mind as I analyzed the data, not all of which were addressed by the study. I believe that these issues would also provide instructional implications for teachers and mathematics educators and help them enhance students’ understanding of the mathematical content. In the latter section of this chapter, I will discuss implications for future research in this area.
Discussion

The first and very fundamental issue was whether students understood that changing situations could be represented mathematically. In terms of this issue, the CMP material motivated students to think of things that changed around them in the introduction of the unit, Variables and Patterns (Lappan et al., 2004, p. 2-3). The material mentioned some examples of things that changed including temperature, price, height, and so on. The introduction stated that sometimes there existed a relationship between two changing things and it encouraged students to think of what those possible relationships could be. Examples of possible relationships included the temperature changing as the season changed, the price of a product changing as the demand changed, and so on. Changing situations in everyday life were briefly addressed in Ms. Moseley’s class as well at the beginning of the unit.

Having discussion on changing situations could provide a good opportunity for students to think about whether changing situations could be represented mathematically and what they have to consider to represent them. Particularly, it could help students understand changing situations conceptually by discussing possible relationships between two changing things qualitatively because qualitative relationships emphasize the global tendency of changes between two variables. Moreover, it could help students relate different forms of representation later. Although I did not probe how well the participating students were aware of the issue in the present study, my general impression was that they became aware that changing situations could be represented mathematically as the class went on.

The next issue was whether students understood that a changing situation could be represented in many different ways and that various forms of representation still told the same story. Seemingly the participating students understood that a changing situation could be
represented in various forms of representation including a narrative, a table, a graph and an algebraic notation. However, it was not perfectly clear whether they appreciated the various forms of representation as tools to represent the same changing situation. Consider the changing situation of traveling by interstate highway at an average speed of 55 miles per hour. The changing situation can be represented as an equation of $d = 55t$ while $d$ and $t$ stand for the distance and the number of hours to travel, respectively. Also it can be represented in a table and a graph as follows.

<table>
<thead>
<tr>
<th>Time (hours)</th>
<th>Distance (miles)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>55</td>
</tr>
<tr>
<td>2</td>
<td>110</td>
</tr>
<tr>
<td>3</td>
<td>165</td>
</tr>
<tr>
<td>4</td>
<td>220</td>
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<tr>
<td>5</td>
<td>275</td>
</tr>
<tr>
<td>6</td>
<td>330</td>
</tr>
<tr>
<td>7</td>
<td>385</td>
</tr>
<tr>
<td>8</td>
<td>440</td>
</tr>
</tbody>
</table>

The issue was whether students could appreciate that all four different forms of representation represented the same changing situation. Particularly, students should conceive the table and the graph globally not point-wise. In other words, they should look at a set of data points on the graph and a set of pairs written in rows of the table in order to figure out the trend globally while understanding that they represented the given changing situation.

Although the various forms of representation could represent a changing situation, each representation has advantages and disadvantage. So the third issue was whether students understood that each representation had advantages and disadvantages for representing a changing situation. The CMP textbook had a section titled “Mathematical Reflections” that
asked students to discuss the advantages and disadvantages of each representation. The Teacher’s Guide of the CMP material provided the advantages and disadvantages as follows:

1. A table has the advantage of giving exact figures at a glance. However, it is often hard to see patterns or trends at a glance without doing some calculations.
2. A graph has the advantage of offering a visual image from which we can quickly see patterns in the relationship between two variables. However, it is often more difficult to read exact values from a graph.
3. A written report has the advantage of giving pieces of information that cannot be contained in a graph or table, such as the reasons a certain section of a trip took longer than another section. However, in a written report it is difficult to notice patterns or trends (Lappan et al., 2004, Teacher’s Guide, p. 35).

Also the Teacher’s Guide emphasized the advantage of equations over the other forms of representation in that equations allowed students to calculate the exact value of one variable from that of the other.

In Ms. Moseley’s class, the students had a whole-class discussion over the advantages and disadvantages of each representation. Interestingly, what they pointed out as the advantages and disadvantages were quite similar to those in the Teacher’s Guide.

Nonetheless, both the Teacher’s Guide and the class discussion seemed to see the advantages and disadvantages of the representations from a local perspective as focusing on specific values for the variables in a changing situation. Although I agreed with the advantages and disadvantages mentioned above, the discussion ought to encourage a more global understanding and interpretation of each representation. That is, in acknowledging that each representation could stand for a changing situation, students should understand that each representation shows the pattern of changes in the situation and the relationship between two variables in different ways. A table depicts the pattern and the relationship numerically, a graph does so visually or geometrically, and a narrative does so linguistically. Thus, when discussing the advantages and disadvantages, the emphasis should be put on the various ways of describing
the patterns and the relationship of the changing situation, not only on the accuracy of values in
the situation.

As another way to represent a changing situation, equations have the advantage of
making it possible to predict values not included in a table or graph as both the CMP material
and Ms. Moseley emphasized. Besides the numerical advantage, equations have a very distinct
characteristic unlike the other forms of representation. Equations are very condensed algebraic
notations that represent the common essence of a collection of changing situations. For instance,
the equation \( d = 55t \) could represent the situation of traveling by interstate highway at an
average speed of 55 miles per hour, and it could represent the situation of paying the total
admission fee of $55 per person to go to an amusement park. The collection of such changing
situations shares the essence that two variables are proportionally related at the rate of 55, and
the equation \( d = 55t \) represents it by shedding the contexts.

Thus, when students write an equation to represent a changing situation, they have to
abstract the situation no matter which form of representation is used for representing the situation.
On the other hand, when they interpret an equation, they have to contextualize it with a possible
matching situation, which shares the essence shown in the given equation. However, this
characteristic may cause students to have difficulty understanding equations as a tool to represent
changing situations. In order to help their understanding, it seems necessary to provide them
with various situations. For example, students could possibly see the same relations between two
variables in two different situations (e.g., traveling by interstate highway at an average speed of
55 miles per hour and paying the total admission fee of $55 per person to go to an amusement
park), and perhaps deduce that these situations have the same relation. Moreover, they could
look for the similar situations with different rates so that they eventually connect the relation to the notion of linear functions.

In terms of an equation as an abstract representation of a changing situation, the next issue was whether students considered the equation as representing the quantitative relation between the two variables in the situation or merely a generalized operational pattern. Students are expected to abstract a relation from a given changing situation and write an equation for the situation, and conversely they are also expected to contextualize a given equation and be able to represent the situation in the other forms of representation. In fact, I found out in the present study that students were more fluent in the abstracting process than in contextualization. The reason for their uneven fluency seemed related to what they considered as the essence of a changing situation and their conceptions of equations as an abstract representation.

Especially when students write an equation to represent a changing situation, it seems possible for them to produce the correct equation by generalizing the operational pattern between two quantities. Generalizing operational patterns is differentiated from abstracting a relation from a changing situation in that it only captures the numerical aspect of each datum without acknowledging that the situation includes variables. For instance, by observing arithmetic identities such as $55 \times 1 = 55$, $55 \times 2 = 110$, $55 \times 3 = 165$, etc., students can generalize the common operation (i.e., multiplying 55 by some numbers) and represent it with the equation $55 \times x = y$ without considering $x$ and $y$ as variables in a changing situation. From the observer’s point of view, it is hard to determine from the written equation whether students have generalized the operational pattern only or abstracted a relation from the changing situation.

However, when they contextualize a given equation, they seem less successful without an understanding of changing situations. Also the observer is in a better position to infer what they
abstracted in their equation. In the present study, Jeffery’s interpretation of equations supported this argument. Apparently, he did not have difficulty writing equations for changing situations given in any form of representation when two variables were directly proportional. When the situations had more complicated relation between two variables such as a linear pattern with a non-zero constant, he had some difficulty figuring out the operational patterns and writing them correctly, but he could still manage it.

When it came to contextualizing given equations such as \( d = 8t \) and \( C = 2n + 5 \), Jeffery had no idea how to interpret them. Although he was still able to figure out some pairs of values for the variables in each equation very well, he could not explain what the values meant in a possible changing situation. That is, he could complete a table and a graph by following the generalized operational patterns shown in the given equations. Nonetheless, he could not see that the pairs of values were the specific data in the pattern of a changing situation where two values in a pair had a relation. Instead, Jeffery mainly focused on the operational pattern especially when he interpreted a given equation and it also implied that he abstracted the operational patterns to write an equation in a changing situation. The reason might be that Jeffery did not understand a changing situation globally. So he paid attention to each data pair in the changing situation without realizing that it was in a trend.

Related to the previous issue, students’ conceptions of variables were also important for understanding their referential relationships. The term, variable, was used as differentiated from unknown and unspecified (generalized) quantities and it only referred to literal symbols in the context of changing situations and functions. Students’ conceptions of variables were discussed in detail in the previous chapter. However, Jeffery’s conception of variable seems worth discussing here again. He simply considered variable as “something that changes” as defined in
Ms. Moseley’s class. That is, he interpreted the literal symbols in a given equation as ‘any numbers’ that followed the operational pattern, which could correspond with the level of “letter as generalized number” in Küchemann’s (1978) heirarchy. His interpretation seemed quite consistent with his focus on the operational interpretation of the equation. This suggests that students’ understanding of changing situations should support their process of developing meaning for variables. That is, students should be continuously encouraged to think about what they have to consider to represent changing situations.

Since the changing situations discussed in the present study had linear patterns, students’ conceptions of rates became another issue. Students should understand that rates in changing situations involve two variables. As Lobato and Thanheiser (2002) pointed out, students should be able to compose and decompose rates to apply their conceptions of rates in changing situations. In the present study, students had the opportunity to discuss the constant rates in a directly proportional pattern. However, they only used the constant rate of the dependent variable to the independent variable. That is, students mainly used the fact that the rate \( \frac{y}{x} \) was constant in a linear pattern \( y = ax \). In fact, Angela and Jeffery demonstrated that they could apply the constant rate and get a constant ratio \( \frac{x_1}{x_2} = \frac{y_1}{y_2} \).

However, it seems necessary to consider another constant rate in the linear patterns. Encouraging students to think about the rate \( \frac{\Delta y}{\Delta x} \) in changing situations would be very beneficial for several reasons. First, students can observe that the rate \( \frac{\Delta y}{\Delta x} \) is constant in all linear patterns, and it is the coefficient of the term \( x \). In a directly proportional pattern, the rate \( \frac{\Delta y}{\Delta x} \) is the same
as \( \frac{y}{x} \), but it is not true for a linear pattern with a non-zero constant. Thus considering the rate \( \frac{\Delta y}{\Delta x} \) would help students understand what all the linear patterns have in common. Moreover, the rate \( \frac{\Delta y}{\Delta x} \) leads to the notion of slope of a line eventually. Also considering the rate \( \frac{\Delta y}{\Delta x} \) will help students understand changing situations globally by looking for a trend across data pairs.

Finally, the last issue was whether students acknowledged that each representation involved some syntax. In fact, not every representation has a strict syntax, but there are some conventions that make mathematical communication clearer and help understanding better. The narrative representation has the least syntax to represent changing situations. It has to describe the relation of a changing situation linguistically by using phrases including ‘y per x’ or ‘as x increases by a, y increases by b’. The tabular representation also has less syntax, but it seems customary to write the independent variable in the left column or upper row and the dependent variable in the right column or lower row. Although this convention is not always recommended, it seems worth considering because it would help students understand variables and the relation between two variables in a changing situation.

The graphical representation involves the most syntax in order to represent changing situations without biases. In a graph, the convention is that the independent variable goes to the \( x \)-axis and the dependent variable goes to the \( y \)-axis. This syntax is similar to that of the tabular representation, but it seems more universally respected. It also leads to the consistency in the order that the values of dependent variable follow those of the independent variable in the table and the coordinate pairs. Scale is very important syntax in a graph, and the students in this study seemed to understand that the marks on each axis had to be evenly spaced to present the graph fairly. However, when students wrote the numbers for every other grid line, they were
sometimes confused. As shown in Figure 18, Jeffery had an uneven mark, especially in the first mark. The reason may have been because he counted the y-axis as the first mark. To prevent the confusion, students should consider the scale as the distance between marks. Another syntax-related issue was the way Greg and Jeffery wrote the numbers for the marks on the axes, and the way to write numbers caused some confusion for readers. Having discussion over the issue seemed helpful for students to think about how they could present their graphs without biases and confusion.

Writing equations involves some syntax and some of the syntax is related to students’ conceptions of equations as a representation of changing situations. Angela over-generalized the syntax of omitting the multiplication sign between numeral and letter and omitted the addition sign on one occasion. After a discussion, she realized that omitting both signs would cause confusion for readers, and therefore omitting the addition sign was not allowed. Both Angela and Peggy wrote some extra words or signs in an equation. It seemed to relate to their conceptions of equations because they said that they wanted to provide additional information about the context. Because they did not fully understand the abstractness of equations, they thought that providing more contextual information would be better.

Writing the letter for the dependent variable alone on one side of equations would be another syntax in writing equations. The standard form of writing an equation in ‘y-form’ is to put the dependent variable y on the left (e.g., \( y = 2x + 3 \)). This syntax is, however, opposite to other syntax in a table and a graph, which is to write the dependent variable following the independent variable in a table and a coordinate pair. Discussion on the syntax is encouraged in that students need to be aware of what the independent variable and the dependent variable are in each representation.
So far, I have raised some issues related to the mathematical topic of representing a changing situation and students’ understanding of it. These issues are closely related to one another and they suggest some implications that would help mathematics teachers and mathematics educators understand what students thought about the content and how to help them. In the following section, I will raise theoretical issues in the present study and make some suggestions for future studies in this area.

Implications for Future Research

In the present study, Kaput’s (1991) referential relationship was the main theory that guided the analysis of the set of data. It provided a tool to analyze how students interpreted one form of representation and how to relate the interpretation cognitively in order to project them into another form of representation. Thus, Kaput’s (1991) model helped me describe the process of students’ referential relationship in going from one form of representation to another.

However, a single theory could not explain students’ referential relationships sufficiently. That is, it could not explain how their relationships were different according to who made the relationship and the types of representation, and what made the differences with Kaput’s (1991) model. In order to explain how their relationships were different in nature, the SOLO taxonomy was used to decide how each action that students took in making referential relationships was related to another. In general, Angela and Peggy made their referential relationships at the multistructural level at most. Greg made his referential relationships, at least, at the relational level. Jeffery’s referential relationships from algebraic notations to other forms of representation corresponded to the unistructural level, while his referential relationships to algebraic notations made at the relational level. In particular, the discrepancy that Jeffery made in two directions of referential relationship was mainly due to his conceptions of variables. Because Jeffery focused
on operational aspect of variables and considered variables as generalized numbers, he was not successful to make sense of the given algebraic notations within changing situations. Yet, he managed to make his referential relationships toward algebraic notations.

By analyzing students’ conceptions of variables as well as their conceptions of rates, I could understand the evident differences in students’ referential relationships. The analysis suggested that students’ conceptions of variables and rates affected both the process and the nature of their referential relationships.

Finally, the procept model (Tall et al., 2001) also helped me understand students’ referential relationships. Originally the procept model was designed to explain how flexibly students could use symbols for switching between process and concept, and it seemed less relevant to students’ referential relationship at the first sight. However, students who made more fluent referential relationships showed that their symbol uses were closer to the proceptual level. For instance, Greg, who was able to make referential relationships at the relational level in the SOLO taxonomy, demonstrated that he could operate on symbols without revisiting the problem contexts but still maintaining the conceptual link with those contexts. Greg effortlessly generated the equation \( N = C \div 4.25 \), representing the relation between the total lunch cost and the number of people by using the reverse operation, and he also symbolically represented his understanding of the distributive property in the bike tour scenarios. On the other hand, Jeffery could not maintain a conceptual link with the context when operating with the algebraic equations, remaining at a procedural or process level. These results implied that students’ referential relationships supported their use of symbols (from procedural to proceptual) and their use of symbols could predict the fluency of their referential relationships.
Hence, instead of using a single theory to investigate students’ referential relationships, I used Kaput’s (1991) model as the main theory and the SOLO taxonomy and the procept model as complementary theories. The analysis of students’ conceptions of variables and rates also helped me understand students’ referential relationships. Although I investigated students’ referential relationships with several theories, I remain open to the possibility that other combinations of theoretical tools could add to their explication.

In relation to the body of literature in this area, the present study belongs to the group of studies that approached symbolism as representing mathematical situations outside of a symbol system. This study focused on the relations among the multiple representations as did studies by Coulombe and Berenson (2001) and Friedlander and Tabach (2001). However, this study is different in that the referential relationships between algebraic symbolism and the other forms of representations were systematically investigated. The underlying assumptions of this study include that algebraic symbolism is the most abstract form of representation, and therefore students’ sense-making of algebraic symbolism could be enhanced by the relations they make to the other forms of representation. Moreover, the present study adds to the current body of literature because it used more systematic methodology and complementary theories to investigate students’ referential relationships.

To study students’ referential relationships further, it would be beneficial to study their referential relationship in different mathematical contexts. Possible contexts include representing changing situations with quadratic patterns and exponential patterns. It is also necessary to study referential relationships among narrative, tabular, and graphical representations as well as the referential relationships between each of these and algebraic notations. It would be especially meaningful to investigate students’ referential relationships
between algebraic notations and unconventional forms of representation, or representations that students generated, in that the investigation could explain what it means for students to represent changing situations and understand how they conceive certain mathematical concepts.


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Ohlsson, S. (1988). Mathematical meaning and applicational meaning in the semantics of fractions and related concepts. In J. Hiebert & M. Behr (Eds.), *Number concepts and
operation in the middle grades (pp. 53-92). Reston, VA: National Council of Teachers of Mathematics.


### Appendix A: EXAMPLE OF INTERVIEW GRAPH

Interview with Greg & Peggy on Feb 12 2004

<table>
<thead>
<tr>
<th>Time</th>
<th>Description</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>8:13:27</td>
<td>Interview begins with providing an equation “d = 8t” and asking Peggy to read the equation. She reads, “d equals to 8 times t.” Then I ask them to make up a story. Peggy tells that d can be a distance traveled at 8 miles per hour. She clarifies that d stands for a distance, 8 means 8 miles per hour, and t stands for the time. Greg agrees as saying that he thought as Peggy told.</td>
<td>Interview tasks are to generate other forms of representation from symbolic representation.</td>
</tr>
<tr>
<td>8:15:10</td>
<td>Then with the equation I ask them to make a table for every hour up to 5 hours. Greg makes a table for every half hour and with two columns of distance and time. He sets up the time first and writes the distance for the time. Peggy makes a table for every half hour first and changes into for every one hour.</td>
<td>Greg makes his table with a half hour interval, which is my next question.</td>
</tr>
<tr>
<td>8:18:05</td>
<td>I ask Peggy to explain why she changed into every hour. Peggy tells that she simply forgot and remembered my direction. For the distances, Peggy tells that she added 8 for each hour since the speed is 8mph. Then I ask about the distance for hour 4 and Peggy realizes her mistake and corrects it.</td>
<td>* Lesson graph had the same format with interview graph.</td>
</tr>
</tbody>
</table>
Appendix B: KATRINA’S HEIGHT DATA
(Lappan et al., 2004, p. 27)

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>Height (inches)</th>
</tr>
</thead>
<tbody>
<tr>
<td>birth</td>
<td>20</td>
</tr>
<tr>
<td>1</td>
<td>29</td>
</tr>
<tr>
<td>2</td>
<td>33.5</td>
</tr>
<tr>
<td>3</td>
<td>37</td>
</tr>
<tr>
<td>4</td>
<td>39.5</td>
</tr>
<tr>
<td>5</td>
<td>42</td>
</tr>
<tr>
<td>6</td>
<td>45.5</td>
</tr>
<tr>
<td>7</td>
<td>47</td>
</tr>
<tr>
<td>8</td>
<td>49</td>
</tr>
<tr>
<td>9</td>
<td>52</td>
</tr>
<tr>
<td>10</td>
<td>54</td>
</tr>
<tr>
<td>11</td>
<td>56.5</td>
</tr>
<tr>
<td>12</td>
<td>59</td>
</tr>
<tr>
<td>13</td>
<td>61</td>
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<td>14</td>
<td>64</td>
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<td>15</td>
<td>64</td>
</tr>
<tr>
<td>16</td>
<td>64</td>
</tr>
<tr>
<td>17</td>
<td>64.5</td>
</tr>
<tr>
<td>18</td>
<td>64.5</td>
</tr>
</tbody>
</table>

a. Make a coordinate graph of Katrina’s height data.
b. During which time interval(s) did Katrina have her largest “growth spurt”?
c. During which time interval(s) did Katrina’s height change the least?
d. Would it make sense to connect the points on the graph? Why or why not?
e. Is it easier to use the table or the graph to answer parts b and c?
Appendix C: THE WOMEN’S OLYMPIC 400-METER DASH DATA
(Lappan et al., 2004, p. 32)

The following table shows the winners and the winning times for the women’s Olympic 400-meter dash since 1964.

<table>
<thead>
<tr>
<th>Year</th>
<th>Name</th>
<th>Time (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1964</td>
<td>Celia Cuthbert, AUS</td>
<td>52.0</td>
</tr>
<tr>
<td>1968</td>
<td>Colette Besson, FRA</td>
<td>52.0</td>
</tr>
<tr>
<td>1972</td>
<td>Monika Zehrt, E. GER</td>
<td>51.08</td>
</tr>
<tr>
<td>1976</td>
<td>Irena Szewinska, POL</td>
<td>49.29</td>
</tr>
<tr>
<td>1980</td>
<td>Marita Koch, E. GER</td>
<td>48.88</td>
</tr>
<tr>
<td>1984</td>
<td>Valerie Brisco-Hookis, USA</td>
<td>48.85</td>
</tr>
<tr>
<td>1988</td>
<td>Olga Bryzguina, USSR</td>
<td>48.65</td>
</tr>
<tr>
<td>1992</td>
<td>Marie-Jose Perec, FRA</td>
<td>48.83</td>
</tr>
</tbody>
</table>

a. Make a coordinate graph of the (year, time) information given in the table. Be sure to choose a scale that allows you to see the differences between the winning times.

b. What patterns do you see in the table and graph? For example, do the winning times seem to be rising or falling? In which year was the best time earned?