AN EXAMINATION OF PROSPECTIVE MIDDLE GRADES MATHEMATICS TEACHERS' CONSTRUCTION OF LINEAR EQUATIONS AND FUNCTIONS

by

SAMUEL BRADLEY CARTWRIGHT

(Under the Direction of LES STEFFE)

ABSTRACT

This study was designed to examine prospective teachers' changes in concepts, building from ratios, rates, and proportions to the construction of linear equations and functions. The 3 participants were all prospective middle grades teachers with two concentrations: mathematics and language arts. Audio-recorded interviews, audio-recorded classroom instruction, and discussions, along with field notes and assessment items such as pretests, posttests, and quizzes, were used to find patterns relating to understanding of the concepts as well as changes in conceptual understanding. Instruction and quizzes were used to track and examine changes in concepts. The posttest was used to mark the end result in terms of growth of prospective middle grades teachers' understanding of proportional reasoning and their ability or inability to construct linear equations and functions.

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DEDICATION

To my wife and son, whom I love and who have supported me unconditionally.

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CHAPTER 1

INTRODUCTION

In 10 years of teaching college-level mathematics, I have observed and noted the tendency of most students to obtain quick solutions to mathematics problems. Students give little or no attention to the mathematical reasoning about the concepts required by the problem. In a few cases, students have gone so far as to verbally chastise others for slowing down the process of finding the answer because they were focusing on mathematical reasoning. The reluctance of most students to engage cognitive processes in problem situations caught my interest. Upon taking an assignment to teach algebra concepts to prospective grades K-8 teachers, I was challenged to examine ways to generate interest and engagement in students in building conceptual understanding, thus inspiring students to engage in the mathematical reasoning required for success in advanced mathematics courses.

Background

Ratios, rates, proportions, and linear equations or linear functions are important concepts covered in the middle school curriculum. Each concept is related to the others, and if the relationships are understood, it empowers students' thinking and reasoning. For example, a ratio is a comparison between two quantities, and a proportion is a comparison between two ratios. The slope of a line can be represented by a ratio, which usually is interpreted as a rate. The National Council of Teachers of Mathematics (NCTM, 2000) recommended that the concept of ratio should be taught, with an emphasis on proportionality, as an integrative theme within the middle school curriculum. The concept of ratio should be facilitated across many areas in the mathematics curriculum to include proportions, percentages, scaling, and similarities. Students should be encouraged to think flexibly about relationships involving fractions, decimals, and percentages (NCTM, 2000).

The NCTM Standards for the Algebra Strand called for an understanding of patterns, relations, and functions, with the expectation that students will be able to use different representations of the concepts of linear equations or linear functions such as an illustration of a function in terms of tables, graphs, words, and symbolic rules. Students are expected to compare different forms of representation and establish relationships. For example, students should be able to identify a function as linear or nonlinear. They are also expected to be able to compare and contrast properties, using tables, graphs, or equations, and are expected to be able to solve problems involving ratios and rates, using linear equations (NCTM, 1989, 2000).

NCTM (2006) called for students to be able to analyze and solve a variety of problems involving linear equations, linear functions, and systems of linear equations. Students are expected to be able to recognize $y/x = k$ as a proportion and $y = kx$ as a linear equation of the form of $y = mx + b$. Here they need to see that k, the constant of proportionality, is the slope and that the graph passes through the origin. Furthermore, they need to be able to see the slope (k) not only as the slope of a straight line, but also as a constant rate of change. They need to understand the input and output of the linear function and how each changes with respect to the other. For example, if the input changes by an unknown amount simultaneously with the output, which also changes by an unknown amount, then students should be able to translate these changes verbally, graphically, algebraically, and by tabular means. Students also need to solve systems of two equations in two variables. The emphasis in thinking should be focused on pairs of lines that are parallel, the same, or intersecting (NCTM, 2006).

Understanding the concepts of rate and proportional reasoning is very powerful for prospective teachers. This power arises because, in their own way, these concepts help bridge the gap from arithmetic to algebra (Wu, 2001). According to Ketterlin-Geller and Chard (2011), who used the idea stated by Wu, students can easily move from quantitative reasoning to algebraic thinking, provided they exhibit two characteristics. First, students must have a strong conceptual understanding of numbers and number systems so that they can make estimations and computations and solve problems across number systems. Second, students must possess a strong understanding about the properties and operation of numbers and be able to apply operations correctly.

This knowledge should be well developed. Learning a concept definition through memorization does not guarantee understanding of a concept or the definition because the learner does not know the critical attributes of the concept or definition, and their characteristics might not be easily understood. An example of this difficulty in understanding is the concept of rate. Over time and through solving problems, the concept of rate might exhibit different characteristics. Once the learner sees new characteristics emerging through the problem-solving process, the definition might start to have meaning. Understanding increases because the learner has worked on the concept image and has memorized the definition, internalizing and understanding the definition (Vinner, 1991).

Rationale

Mathematics, widely perceived as the area of reasoning about objects and their relations, involves an examination and investigation of the credibility of claims about objects and relations (Carpenter, Franke, & Levi, 2003). Teaching mathematics should focus on fostering the fundamental skills in generalizing, expressing, and systematically justifying generalizations because the basis or power of mathematics lies in relations and transformations, which give rise to these patterns and generalizations (Warren, 2005).

Traditionally, teachers in elementary schools have placed little emphasis on relations and transformations as objects of study. In fact, as Malara and Navarra (2003) pointed out, classroom activities in the early years focus on mathematical products rather than on mathematical processes. Strings of numbers and operations in arithmetic are not considered as mathematical objects, but as procedures for arriving at answers (Kieran, 1990).

The NCTM (1989, 2000) called for teachers to analyze patterns of change in different contexts. They also called for students to develop a deeper understanding of mathematical ways in which changes in quantities can be represented. They recommended that students learn to interpret statements such as “the rate of inflation is decreasing” (NTCM, 2000, p. 305) and indicated that students should make use of mathematical functions to identify patterns and abnormalities in data. Recent literature promotes the approach of functions in terms of conceptual thinking, including the investigation of patterns of change (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Kaput, 1994; Monk, 1992; NCTM, 1989, 2000; Sfard, 1992; Thorpe, 1989; Vinner & Dreyfus, 1989).

A number of studies about rates, ratios, and proportions have been conducted with middle school students as well as a number of studies concerning students' covarying abilities at the college level (Carlson, Larsen, & Jacobs, 2001; Noble, Nemirovsky, Wright, & Tierney, 2001; Saldhana & Thompson, 1998). The NCTM (1989) and the National Research Council (NRC, 1989) advocated that students develop a deep understanding of mathematical methods in which changes in quantities can be represented. To make use of mathematical functions is to be able to identify patterns and abnormalities in data. Research on mathematical ways of representing changes in quantities by prospective middle school mathematics teachers could provide insight into the covariational reasoning of individuals who enter the classroom and help fulfill the requirements of the NCTM and the NRC. Further, understanding prospective middle school mathematics teachers' covariational abilities could help others who teach mathematics develop effective instruction.

Köklü (2007) stated, "Considering the difficulties students encounter in traditional calculus classrooms, understanding students' reasoning patterns about change and continuous change can suggest some alternative ways in instruction" (p. 8). A well-developed understanding of function correlates closely with success in calculus and facilitates the transition to more advanced mathematical thinking. In fact, students who are able to view variables as representing quantities whose values change dynamically along a continuum possess ready access to fundamental ideas such as rate of change and limits. These students have exhibited higher levels of achievement in mathematics (Ursini & Trigueros, 1997). Reasoning dynamically in this way might help prospective teachers engender similar thinking processes in students. In terms of co-variational reasoning, a multitude of researchers have investigated the concept of rates of change in relation to the study of calculus (Carlson et al., 2001; Hauger, 1995, 1997; Koklu, 2007; Nobel et al., 2001; Saldhana & Thompson, 1998).

Monk and Nemirovsky (1994) and Thompson (1994b) also revealed that students sometimes have difficulty representing and interpreting graphs of dynamic functions. Koklu (2007) stated that research reports centered mostly on students who have difficulty using graphs to represent dynamic functions and students who have difficulty interpreting the dynamic functions represented by graphs. These difficulties have been linked to students' weak mathematical reasoning abilities in coordinating simultaneous changes of variables. To date, research has shown that students, when trying to represent and interpret dynamic function events such as coordinating the direction of change, coordinating the amount of change simultaneously, and identifying and expressing the concavity and inflection points, have weak or limited conceptions of these topics (Köklü, 2007)

Problem Statement

In this study, I examined three prospective middle grades teachers in an algebra concepts course. I observed and interviewed these teachers to investigate and understand how they develop a conceptual understanding of ratios, rates, and proportions through their use of ratios, proportions, and rates as they construct linear functions/equations while solving problems. The classroom was used as a means to observe the communication between the prospective teachers as they discussed their problem solutions on the concepts of ratios, rates, proportions, and the development of linear functions/equations. The instructional goal was to enhance each prospective teacher's understanding of the content he or she was going to teach along with the connective understanding of the relationship between the concepts. My research questions were as follows:

A. What are the initial conceptions of prospective middle grades teachers concerning rate and proportional reasoning?
B. How do the conceptions of prospective middle grades teachers concerning rate and proportional reasoning change over time as they study algebra concepts?
C. How is prospective middle grades teachers' construction of linear equations and functions related to their conceptions of rate and proportional reasoning?

CHAPTER 2

LITERATURE REVIEW

In this chapter, I review literature that contributes to the way in which prospective mathematics teachers conceptualize mathematics. This review serves to create a framework in which I focus on prospective teachers' changes in their understanding of the concepts of ratio, rates, and proportions as they apply their proportional reasoning to the construction of linear equations in an algebra concepts course throughout the semester. Literature on students' conceptions of ratios, rates, and proportions, along with errors in students' thinking about the concepts, is reviewed. Literature on the development of the understanding of ratios, rates, and proportions over time is explored, reviewed, and examined, as well as literature on students' applications of those concepts in the construction of linear equations.

Errors in Proportional Reasoning

The research literature has shown that successful understanding of concepts is not necessarily related to age. College freshmen who are skilled in arithmetic computations might show little knowledge of proportional reasoning (Collea, 1981). Developmental differences concerning the multiplicative nature of proportional reasoning are not necessarily age related, because older students are not more apt to engage in proportional reasoning than younger students (Dixon, Allen Ahl, and Moore, 1991). Studies reviewed by Tourniaire and Pulos (1985) support the belief that many adults are poor in reasoning proportionally. Lamon (2007) estimated that number to be about 90%. One reason for this alarming statistic might be because the topic is not taught properly in middle school. It is believed that proportional reasoning should be considered a focal point of mathematics instruction in elementary schools (Lesh, Post, & Behr., 1988).

Hart (1984) found that when given ratio-related problems, students approached them intuitively within the boundary of context and without reliance on formal symbolic methods. Steffe (2009) noted that students make errors due to lack of understanding of the difference between equivalence of particular measurements in two systems of measuring a particular quantity versus the quantitative equality of the number of such units. From a long list of studies, Tourniaire and Pulos (1985) summarized many common errors students make while solving problems requiring proportional reasoning and placed those errors into four categories.

On the other hand, Karplus and Peterson (1970) found six error-coded strategies that are different from those of Tourniaire and Pulos (1985). The issue of equivalence versus equality is one of many issues. Steffe (2009)[1] asserted that when students convert units between two monetary systems, for example, this idea tends to be problematic. For a problem where 2 English pounds are equivalent to 3 U.S. dollars, he states the following:

> Converting to a unit ratio can alleviate the confusion that is usually rampant when first considering the monetary equivalence. This confusion usually starts when someone simply abbreviates pounds using "p" and dollars using "d" and writes 2p ≈ 3d thinking "two pounds is equivalent to three dollars" [note the equivalence sign ≈]. There is nothing wrong with this if it is explicitly remembered that "p" does *not* stand for the *number* of pounds, "d" does *not* stand for the *number* of dollars, and the ≈ sign is an equivalence between two systems for measuring monetary value much like "180º degrees

[1] Class Notes, EMAT 7080. Rates of Change and Their Graphs.

> Fahrenheit is equivalent to 100º Celsius" is an equivalence between two systems for measuring temperature. (p. 1)

Tourniaire and Pulos (1985) stated that sometimes students unknowingly ignore part of the question or some of the data when solving a proportion problem. Here students would use two of the three pieces of information while they solve a problem. For example, consider the problem: Billy paid 9 cents for two sticks of candy; how much would he pay for eight sticks of candy? In this case, the student might reason that eight sticks of candy cost 72 cents because that student might think implicitly that 9 cents times eight sticks is 72 cents and completely ignore the ratio that 9 cents is for two sticks of candy, not one stick.

Another error would be the use of an additive strategy as opposed to the use of a constant rate. Here the focus is on the constant difference between two numbers in a proportion—the constant difference strategy. For example, referencing the candy problem in the previous paragraph, the student might say that 9 cents is 7 more than two sticks, so in finding the missing value, that student might sum the 7 and 8 together from the second ratio to incorrectly find that eight sticks cost 15 cents.

A third error would be the combining of two strategies: additive and multiplicative. This tends to happen when numbers are used that are not integers. In this case, students might find the correct noninteger ratio and multiply by its whole number component in finding the missing value, and then add the remainder. In the candy problem, a student might reason that 9 ÷ 2 is 4 with a remainder of 1, then multiply 8 by 4 and then add 1, getting 33 instead of 36. Finally, students might use a flawed application of a correct strategy. They might set up the proportions with the numbers in the wrong places. In this case they might find the wrong unit rate and then use it correctly to find the missing value.

Teachers and Proportional Reasoning

Hillen's study. Hillen's (2005) goal was to help teachers construct or reconstruct proportional reasoning and proportional relationships in an attempt to develop their capacity to provide learning experiences for their students within her practice-based-methods course. She had four objective criteria that she measured: their fluency with the proportional representations; their ability to distinguish between quantities having multiplicative relationships; their repertoire of strategies in solving routine and non-routine problems involving proportions; and their ability to characterize proportionality over a wide range of topics in the middle-grades curriculum.

Hillen (2005) was trying to answer two questions. The first question was, "What do pre-service secondary mathematics teachers know and understand about proportional reasoning prior to participation in a course specifically focused on proportional reasoning?" (p. 11). The second question was, "What do pre-service secondary mathematics teachers know and understand about proportional reasoning immediately after participation in a course specifically focused on proportional reasoning?" (p. 11).

To generate answers for this research, Hillen (2005) engaged the teachers in five types of activities: (a) solving and discussing mathematical tasks, (b) analyzing and discussing samples of student work, (c) analyzing and discussing cases of mathematics teaching, (d) reading about and discussing issues related to mathematics teaching, and (e) discussing mathematical ideas that did not stem directly from a mathematical task that teachers solved.

The preliminary results from the pretest indicated that the 10 teachers in the treatment group all used cross multiplication in trying to solve the four missing value problems on the pretest. Only five teachers were able to solve some or all of the problems in more than one way.

Only three of the five teachers were able to solve three of the four problems in two different ways. They tended to rely on cross-multiplication procedures. However, the results also revealed that they had a limited repertoire of strategies. On the posttest, the teachers were able to use a wider range of strategies while solving the proportionality problems. Hillen (2005) noted that the teachers' frequency of using cross multiplication decreased by the end of the course.

De la Cruz's study. De la Cruz (2008) examined four middle school teachers—three in their first year and one with 22 years of experience—on the following: (a) their classroom discourse, (b) their rationale for the instructional decisions, and (c) their plans for teaching proportion concepts both before and after they were given access to research-based models on students' thinking about the concepts. Cognitive Guided Instruction (CGI) was used in the design of the workshop format and in the research findings on adolescents' proportional thinking. De la Cruz (2008) used a 2-day workshop, using written and video case studies depicting actual classroom teaching designed to increase teachers' knowledge of pedagogical content, students, curriculum, and, to a lesser degree, content. Research findings on adolescent proportional thinking were used to help the teachers better understand student thinking about the proportional reasoning concepts. In the context of the workshop (inspired by CGI), the teachers discussed (a) student solution strategies; (b) student problem-solving success and strategy choice; (c) concept-development prerequisites in building proportional reasoning; and (d) developmental theories such as those given by Piaget, Noelting, Milsailidou, and Williams; Lesh et al.; and Karplus et al. (de la Cruz, 2008, p. 120).

In her interview conducted prior to the workshop intervention to gain an understanding of each teacher's instructional style and basis for his or her instructional decisions, de la Cruz

(2008) found that none of the teachers used tasks or posed questions that required high-level thinking from their students. They asked questions that students could answer, using procedures and never asking the students to solve problems. They placed no high-level cognitive demands on the students.

After the workshop intervention, de la Cruz (2008) found that all four teachers' instruction had become more cognitively guided. They infused high-level cognitively guided questions into their classroom discourse. Three of the four teachers changed by one level from their beginning levels in terms of the Cognitive Guided Instruction (CGI) scale. One teacher, with the pseudonym Julie, was noted for being unhappy with her instruction prior to the workshop and had a strong desire to change her classroom instruction. Julie changed more than the others on the CGI scale. Three of the four teachers expressed their rationale for sequencing tasks to further develop students, whereas only one of them had done so before. The teachers sequenced tasks by using "knowledge of research-based models of students' thinking related to proportions to inform some of their instructional decisions" (de la Cruz, 2008, p. 495). All 4 of the teachers who had exhibited weak content knowledge as well as weak pedagogical content knowledge in ratio and proportions demonstrated an increase in conceptual knowledge after the workshop.

Alan Tennison's study. Alan Tennison (2010) examined three first-year mathematics teachers' conceptions about mathematics in conjunction with their teaching methods. Their conceptions were based on pre-existing beliefs about mathematical knowledge. The goal was to answer the following questions:

A. What initial conceptions regarding the teaching and learning of mathematics do secondary mathematics intern teachers reveal in their first year of teaching?

B. How do these conceptions change during their first year? a) What influence does a year-long mentoring process have on the interns' conceptions? b) What role does a university mathematics methods course have in supporting and restructuring the interns' conceptions? (p. 6)

All three interns—Matt, Emily, and John—made use of the Romberg (1992) and Stigler, Gonzales, Kawanaka, Knoll, and Serrano (1999) instrumentalist views of teaching and learning mathematics, consisting of lecture, demonstration, practice, and homework. Their reasons for using these views of teaching were not the same. Matt, as a student, had been successful with Instrumentalism, making him comfortable with using it as a teacher. Emily, who had emigrated from Bosnia when she was 16, had only 2 years of experience, as a student in an Instrumentalism classroom. She used Instrumentalism because she did not have the confidence to use another approach. John felt pressure to conform to uniform teaching already in place, Instrumentalism.

Matt's conceptions did not change. He liked structure and saw no reason to change from something that worked for him. Emily was influenced by her 16 years in Bosnia, where the national curriculum at the time stressed problem solving. She had developed a problem-solving perspective, but her problem-solving approach was viewed by Tennison as rudimentary. John had similar learning experiences to Matt. However, John had one exception; he would ask students to share their ideas and approaches and present their problem solutions and make connections to other problems or ideas.

Student teachers teach concepts based on the experiences that have affected them most. Their teaching, whether effective or not, affects their students' learning of concepts. If concepts are properly learned, they can be applied to other topics. Following this line of reasoning, it would seem that applications of proportional reasoning would support progress toward knowledge of variation and covariation. Further, it would seem that an improvement in thinking about the concept of ratios would improve thinking about the concept of rate in the context of two variables that covary in a constant ratio.

Connecting rate and proportional reasoning and the ability to construct linear equations. Proportional reasoning involves the transformation of a ratio that leaves the ratio invariant. This transformation might be either qualitative or quantitative. In the case of a problem that involves missing value and numerical comparison, the transformation involves a sense of covariation (Hart, 1994). If the ratio is conserved under the transformation, the result can be thought of as direct variation. Graphically, direct variation can be represented using the slope of a straight line. Here, the ratio *y* is to *a* as *x* is to *b* can be written $y/x = a/b$ and the creation of the multiplicative relationship gives $y = \frac{a}{b}x$ where $\frac{a}{b}$ is the constant of proportionality. However, Vergnaud (1994) argued that the linear function concept simply cannot emerge from dealing with proportion problems alone. It must be worded, analyzed, and generalized as a comprehensive concept.

Thompson (1994a) noted that many mathematics students tend to see a function as a command to calculate and that early algebra students are no more likely to see the expression $x(12(x – 5))$ as representing a number as elementary students are to see that the expression $4(12(4 – 5))$ represents anything other than something to do. When students possess a process conception of function, they can imagine the function as something that performs the sequences of operations but no longer need to actually think about the chain of operations when envisioning the result of the evaluation (Silverman, 2006). The development of covariational reasoning is related to the progression from an action to a process conception. Once learners conceive of a function as the covariation of quantities, they “can begin to imagine ‘running through’ a continuum of numbers, letting an expression evaluate itself (very rapidly!) at each number” (Thompson, 1994a, p. 26) and can, therefore, conceptualize the way in which the quantities covary.

Functions represent relationships between varying quantities. As Tall (1997) stated, “One purpose of function is to represent how things change” (p. 1). Studies have revealed that students’ underlying function conceptions play an important role in their imagining the simultaneous covariation of variables when reasoning dynamically. It has been shown consistently by many research studies that students who possess strong procedural skills such as symbol manipulations and weak conceptual structures are unable to construct images of simultaneous co-variation of two quantities in a functional relation (Carlson & Oehrtman, 2005; Monk & Nemirovsky, 1994).

The Co-variational Framework of Carlson et al. (2002) is a starting point for recognizing the mental actions necessary to characterize the mathematical behaviors involved in covariational reasoning. The relationship between the input and the output represent the quantities that are being observed for changes--either physically, in the form of visible representation of a function, or mentally, by which imagination is a conduit of the function’s construction. Functions represent the relationship among variables and are considered one of the very important components of mathematics (O’Callaghan, 1998).

A function can be either static or dynamic. If a function is static, covarying reasoning can be observed visually. On the other hand, a function that is dynamic would require an imagination to generate the behavior of the input changing the output and how they relate. This requirement exists because the function is being created through a mental construction in the individual's mind.

Framework

In this study, I adopted the Ongoing Assessment Project (Laid & Petit, 2008) Proportionality Framework (VMP OGAP). The study illustrated examples that characterized students' potential solution paths when they solve problems related to ratio, proportions, and rates. In addition, the study outlined strategies on how students might demonstrate efficiency in the application of ratios, rates, and proportional relationships when they interact with a wide range of problem situations and problem structures (Laid & Petit, 2008). Problem situations might involve ratios, rates (density), ratios ($D = RT$), rates (buy/consumer), similarities between objects, scale, probability, percent, linear equations and relationships, slope, and frequency distribution. Problem structures might involve problem types, multiplicative relationships, ratio relationships, and representations (Laid & Petit, 2008).

In the framework, when analyzing students' work, the researcher must consider three major elements: problem situation, problem structure, and evidence in student work (Laid & Petit, 2008). As students interact with different problem structures and problem situations, they might move back and forth between using proportional, transitional, and nonproportional strategies, depending on the strength of their proportional reasoning ability (Cramer, Post, & Currier, 1993; Karplus, Pulos & Sage, 1983; Laid & Petit, 2008).

Below are summaries of different strategies taken from Laid and Petit (2008).

- Students show their proportional strategies when they (a) find and apply unit ratios; (b) apply multiplicative relationships; (c) use either symbolic or graphical representation of direct variation ($y=kx$); (d) compare simplified fractions, rates, and ratios; (e) use cross multiplication after setting up a proportion; and (f) apply the correct ratio referent in a ratio problem.
- Students show their transitional proportional strategies when they (a) make use of ratio tables; (b) use builds up or down; (c) use models; (d) use proportional strategies with an error in their construction of a multiplicative relationship, in setting up the proportion correctly but misapplying cross products, and in their finding equivalent fractions/ratios.
- Students show their nonproportional strategies when they use additive reasoning; make guesses or uses random operations; misinterpret vocabulary and the related topics; use whole number reasoning; solve a nonproportional problem using proportions; and use an incorrect ratio referent.
- Some issues and errors of importance are (a) using additive strategies to find multiplicative relationships; (b) making mistakes in interpretation of the meaning of quantities; (c) making errors in equation construction; (d) making rounding errors; (e) making remainder errors; and (f) making computational errors.

(Laid & Petit, 2008).

The proportional reasoning of students can be analyzed using the matrix in Table 1. This table illustrates a matrix that has four dimensions cross listed with three categories. The dimensions are *expert, practitioner, apprentice,* and *novice*. The categories are *understanding,* *reasoning*, and *communicating*. In the present study, I used this matrix as a guide to make sense of students' reasoning as they progressed through the course

Table 1

The Dimensions Matrix

Dimensions (Categories)	**Expert** **4**	**Practitioner** **3**	**Apprentice** **2**	**Novice** **1**
Understanding	The solution shows a deep understanding of the math concepts and the procedures needed to reach it. Math concepts and procedures are applied correctly.	The solution is complete. Math concepts and procedures are applied correctly.	A solution is attempted but isn't complete. Some math concepts are used but not all of the necessary ones. Some, but not all, procedures are correct.	There isn't a solution or the solution is inappropriate. Inappropriate math concepts or procedures are used.
Reasoning	Uses an efficient strategy that leads directly to a correct solution. Can verify the solution and evaluate the reasonableness of it. Makes relevant math observations and connections.	Uses a strategy that leads to a solution. Uses effective math reasoning and procedures. All parts of the solution are correct.	Knows some of what is needed to do find a solution but doesn't find a complete solution. Does not complete all of the math procedures that the problem needs. Some parts might be right but the right answer is not achieved.	No evidence of a strategy or the strategy shown is inappropriate. There are many errors in math procedural so that a solution can't be reached.
Communication	Writing and drawing is clear, well-organized, and detailed. All steps are included. A variety of words and symbols are used accurately and appropriately. Sophisticated language is used in some parts of the solution.	Writing and drawing are clearly done. The reader might have to fill in some details. A variety of words and symbols are used accurately and appropriately.	Writing and drawing might be unclear in parts. Words and symbols are used but show errors or lack of variety.	Writing and drawing are unclear or inappropriate. Words or symbols were used inaccurately or inappropriately.

Note: Examplar's Classic Three Level Math Rubric was downloaded from http://www.exemplars.com/resources/rubrics/assessment-rubrics%205/5/2014.

Summary

This chapter presents a broad view of research, beginning with conceptions and errors relating to proportional reasoning; research tracking changes in conceptions of mathematics over time; and how traditional instruction on teaching linear equations is dominated by the procedural use of formulas and symbols. The conceptual entities all point to the importance and the impact of students having the ability to construct multiplicative relationships. This understanding is crucial to the students being able to apply their understanding of ratios, rates, and proportions to the construction of linear equations. The review serves to support the assumptions underlying the present research contentions that in the teaching of linear equations, there is (a) a need to be aware of students' initial conceptions of ratios, rates, and proportions; (b) a need for an understanding of how, after knowing (a), to track students' changes in understanding of the concepts; and (c) a need to engender the ability in teachers to apply those concepts, without using formulas, to the construction of linear equations and linear functions.

CHAPTER 3

METHODOLOGY

Introduction

I used case study methodology to examine three prospective middle grades teachers in my algebra concepts course during the fall semester 2010 at a historically Black college and university. In addition to these three students, 20 students were enrolled in the course, four of whom were middle school majors and 16 of whom were early childhood majors.

The rationale for using case study methodology (Tellis, 1997) is that case studies make use of information designed for decision making, decisions that can help provide links where cause-and-effect relationships are not easily seen (Hays, 2004). My main goal was to see the links between where the students were in their proportional reasoning and where they could be going in terms of their understanding of linear equations and functions (Merriam, 2002).

According to Patton (2002), the goal of case study research is to be able to understand the complex nature of each unique case. In the present study, I explored multiple cases over the course of a semester through conducting an in-depth data collection that involved interviews, audio-taped material, pretest, posttest, final exam, homework, and quizzes. Creswell (2007) explained that case study research includes documents, and reports a case description and case-based themes. The cases are bounded in terms of the setting and context of the problem situations used in the study in order to make sense of students' thinking as they problem solve.

Participant Selection

The following criteria were used to select participants for this present study:

A. Each participant had to be enrolled in the algebra concepts course in fall 2010.

B. Each participant had to sign a consent form to participate in the study.

C. Each participant had to be a prospective middle school teacher.

D. Each participant had to be fully participating in the class activities.

Initially, I had selected four prospective middle school mathematics teachers because they all met at least the 1st, 2nd, and 4th criteria. During the semester, one of the student participants exhibited characteristics that violated Criterion 4 because he was frequently absent or late, and a number of documents were not collected from him. Therefore, he was dropped from the study.

Description of the Algebra Concepts Course

The MATH 3510 course was entitled *Algebra Concepts* and was a required course for middle grades majors with a concentration in mathematics and early childhood. The course prerequisites were college algebra or precalculus. The goal of the MATH 3510 course was to provide students the opportunity to become familiar with concepts of middle grades mathematics, namely algebra for middle school teachers. The course aimed at preparing preservice teachers for the mathematics needs of their future students. A number of lessons were outlined and used in the algebra concepts course. These lessons included an introduction to ratios and proportions; expanding on ratios and proportions to include rates; an in-depth discussion of models of variation; a discussion of notes from Leslie Steffe[2] as they related to direct variation

[2] Class Notes, EMAT 7080. Rates of Change and Their Graphs.

and money conversion; student understanding of the imagery of variation situations; and an introduction to function machines.

Lesson 1 (August 15-19) was an introduction to the definition of ratio and proportion. The students had to use the definition of a *ratio* to discuss quantities in terms of part-part, part-whole, and whole-part relationships. The students were also expected to use similar triangles or similar shapes to solve arithmetic problems that, in general, involved similar triangles.

Lesson 2(August 22-26) *was* an expansion of the first lesson, in which the students were reintroduced to ratios and proportions. Furthermore, rates were added to the discussion, and the unit rate was introduced as a unit of speed as well as a price per unit. For example, the students had to explain which product was the better buy, using the unit to explain why.

Lesson 3 (August 20-27*)* focused on models of variation. The students were given the definitions for *direct variation, inverse variation,* and *joint variation* and had to solve problems related to these three types of variation.

Lesson 4 (August 30-September 17) was the introduction to Geometer's Sketch Pad[3]. Students followed along plotting points, making line segments, and animating points. Many students complained that the use of the software was not good for the class because I was trying to get them to do two things, and they felt that they needed to focus more on learning the actual materials than on the software.

[3] A Dynamic Geometry Software for Exploring Mathematics.

Lesson 5 (September 20-30) was centered on Steffe's notes[4]. The students were introduced to direct variation, money conversions, and change in variation problem situations. They were also introduced to story problems. The story problems in some instances expanded on previous problems.

Lesson 6 (October 1-15) included a variation exercise to see how the students looked at variation-based situations. The students were given problem situations and had to select pictures that matched the imagery they held in their heads relative to the problem situations. Then the whole class discussed the results. The idea was for students to determine the difference between linear and nonlinear situations.

Lesson 7 (October 18-29) was a return to rates in more detail, with a focus on constructing linear equations, using Steffe's rate problems[5]. At first, the students had difficulty finding rates as the rate changed from one variation to another. Furthermore, the change in variation was a challenge for them, and I had to revisit the procedure at the request of the students.

Lesson 7 (November 119) was on the function machine. The students were introduced to the function machine in terms of input and output. They had to draw a diagram illustrating a machine as they modeled linear functions. This topic was a move from linear equations to linear functions.

The posttest was administered after Lesson 5 because all the information needed had been covered. The final was administered at the end of the semester, and it included all concepts

[4] Class Notes, EMAT 7080. Rates of Change and Their Graphs.
[5] Class Notes, EMAT 7080. Rates of Change and Their Graphs.

that had been covered in the preparation of the research study and its methodology. For that reason, the final was designed to highlight both linear equations and linear functions.

Data Collection

I collected archival data, conducted pretest and posttest interviews, made direct observations, and conducted participant observations. In addition, I collected physical artifacts such as homework, quizzes, pretest, and posttest, and administered a final exam. I used the six primary data sources recommended by Yin (1994), who suggested that using multiple sources of data increases reliability of the study and validates its findings. These multiple sources allowed the researcher to triangulate evidence (Tellis, 1997). Table 2, *Representing Data Source and Data Items* summarized the data sources and data items collected.

Table 2

Representing Data Sources and Data Items

Data Sources	Data Item
Archival Data	A. Their assignments (quizzes, homework, pretest, posttest, and final exams) B. Brief questionnaire
Interviews	C. Interview after pretest D. Interview after posttest
Direct observation	E. Classroom observation
Documents	F. Handout for students (problem sets) G. Solutions to assigned problems

Data Collection Process

I conducted this study in three phases. In Phase 1, a requisition was submitted to both Fort Valley State University (FVSU) and the University of Georgia (UGA) to receive permission for carrying out the research process. Prior to the beginning of the semester, an Internal Review Board (IRB) approval from both FVSU and UGA was obtained. Then at the start of fall 2010, I obtained a signed consent form for both FVSU and UGA from each prospective early childhood and middle grades mathematics teacher enrolled in the algebra concepts course who volunteered to participate in the study. All 20 students enrolled volunteered. Next, I administered the pretest and analyzed the participants' level of proportional reasoning, using the Ongoing Assessment Project (OGAP) Proportionality Framework (Laid & Petit, 2008). I conducted interviews immediately after the pretest and analyzed the transcription to get a better understanding of the mathematical mental actions of the participants' reasoning. Each sixty minute interview focused on the preservice teachers' thinking while solving the pretest problems posed. The interviews probed the participants' reasoning and provided clarity on it.

During Phase 2, I gave the students, including the four participants, handouts on lectures, lectures and homework assignments, quizzes, and a posttest. The posttest interview followed within a day after the posttest administration. In addition, the posttest was analyzed to make sense of the participants' level of proportional reasoning using the Ongoing Assessment Project (OGAP) Proportionality Framework (Laid & Petit, 2008). As a result, I conducted interviews immediately after the posttest and analyzed the transcription to get a better understanding of the mental mathematical actions of the participants' reasoning. The interviews probed the participants' reasoning and provided clarity on their proportional reasoning skills. The final phase, which was Phase, came late in the semester. This phase involved the administration of the final exam, and the comprehensive final exam was the last artifact collected.

Participants and Setting

a) Student 1 (S1): S1 was a senior in her 4th year in the middle grades education program, and she was initially enrolled for 2 years before taking courses as an early childhood major as she waited for the middle grades program to be reinstated by the Professional Standards Commission and for National Council for Accreditation of Teacher Education (NCATE) to recertify the program. The program had lost its accreditation in 2006, and in her 3rd year, the program had been reinstated. S1 had to go all the way through the process of getting into the middle grades education program. That process included three recommendations, interviews, passing of Georgia Assessment for Certification of Education I (GACE)[6] (see http://www.gace.nesinc.com/) for basic skills, and creating a portfolio of artifacts that had a list of particulars outlined by the Georgia Department for Middle Grades Education.

S1 was an extremely organized student who kept a portfolio that included material from all classes that she had taken. The portfolio was for her personal benefit and to use in preparing to take the GACE exam for her concentrations. Prior to the Algebra Concepts (MATH3510) course, she had met the prerequisites, including Pre-Calculus (MATH 1113) and Foundations of Mathematics (MATH 2100), but she had also taken Geometry for Teachers (MATH 3400). Pre-calculus is included typically in Algebra I and Algebra II as taught in the high schools prior to her enrollment. The Foundations of Mathematics course was, in general, arithmetic in nature since some of the topics included were greatest common factors, least common multiples,

[6] GACE is required by the state of Georgia as a part of the educator certification process.

fractions, percentages, and decimals, with a focus on problem solving (a Singapore-style method was used). Geometry for Teachers was a geometry course that covered content for grades K-8 teachers.

During the algebra concepts course, S1 made sure that she got the answers for every problem she was unable to work while the problems were being discussed, and she wanted detailed explanations. That is, if S1's solution path was not correct, she wanted to discuss what she was thinking and wanted to provide an explanation outlining the steps to solve the problem correctly. S1 sat with S2 in every class, and they worked together on every problem discussed in class. Many times, they had similar answers. S1 was a hard worker, and she always gave her best in trying to solve problems that were posed, even if the problem was easier said than done for her at the time.

b) Student 2 (S2): Like S1, S2 was a senior in his 4th year in the middle grades education program. Like S1, he was initially enrolled for 2 years before he took courses as an early childhood major because he waited for the middle grades program to be reinstated. Unlike S1, S2 wanted to teach secondary mathematics, but that program was not reinstated. The FVSU College of Education wanted to press for a Master of Arts in Teaching Mathematics degree as a means of preparing students to teach secondary mathematics. Issues of the noncompeting clause came up as another local state college had applied for the MAT and was further granted the Secondary Mathematics Program. The compete clause

> Institutions wishing to offer courses and/or degree programs externally must adhere to the guidelines, criteria, and nomenclature contained in guidelines issued by the Chancellor or his/her designee. These guidelines are maintained in the Academic Affairs Handbook.

> It is desirable in most instances to have the closest qualified institution respond to off-campus credit course needs. In cases where requests for services exceed the qualifications or ability of the closest institution, attempts should be made to have such requests met by other qualified USG institutions. (This was taken from http://www.usg.edu/policymanual/section3/C338 on July 20, 2010).

S2 was content with his limited choice and majored in middle grades education. S2 was organized but not as organized as S1, and he came to class a few minutes late repeatedly. But he did his homework early. In addition, he also turned in his homework early, and before enrolling in the Algebra Concepts courses, he had completed my MATH 1111 (College Algebra), Foundations of Mathematics (MATH 2100), and Geometry for Teachers (MATH 3400) courses. S2 was a very well-organized and competent student. If he worked a problem incorrectly, he would participate in the discussions but not always press for correct answers since he would just ask about ways he could rework the problem. He took notes on discussions of problems that he was unable to answer, participated in discussions, and often went to the board to try out his solution.

c) Student 3 (S3): Like S1 and S2, S3 was a senior in his 4th year in the middle grades education program. He was also enrolled initially for 2 years before taking courses as an early childhood major, waiting for the certification of the middle grades program. S3 was more organized than S2 but less organized than S1. S3 liked solving problems and getting at the solution. He came to class on time and sat by himself. In addition, he was ready to lend a hand to others who voluntarily went to the board and were unable to solve problems posed or to show solutions for problems during classroom discussions. Like the others, S3 had taken Pre-Calculus and Foundations of Mathematics as well as Geometry for Teachers.

CHAPTER 4

STUDENT 1

S1: Case 1

This chapter provides a narrative following Student 1 (S1) from the start of the study to the end. S1's initial conceptions of ratios, rates, and proportions, and her ability to apply these concepts to the construction of linear equations by means of problem solving are the focus of this narrative. She had difficulty with many of the problems involving ratio, rates and proportions during the semester.   Her struggle with solving the Baby-Weight, the Currency and the Lady-Runner Problems evidenced her initial proportional reasoning skills as novice. She does appear to show some improvements during the course of the study; however, her reliance on formulas and experiences from pass algebra classes overshadow or inhibited that progress.

At the beginning of the semester, S1 was given a pretest (see Appendix A) to assess her understanding of the concepts involving, ratios, proportions, rates, and multiplicative understanding. She had to solve five problems: (a) the baby weight problem, (b) the snail problem, (c) the currency problem, (d) the gasoline problem, and (e) the runner problem. I analyzed the response to each question. The peculiarities I found were used to create an individual interview protocol to probe her thinking during the interview process. The story starts with the five pre-assessment problems in the order listed above.

The Baby Weight Problem

Betty weighed 6 pounds at birth. She gained 2 pounds every month for 8 months.

A. Create a table to represent the data over an 8-month period.
B. Draw a graph to represent the data; identify the input and the output.
C. As Betty gets older, how does her age relate to her weight?
D. Construct an equation to model the age-weight relationship.

For this problem, S1 constructed a table with multiples of 2 to 6 for the months and not 8 months as was asked in the question (see Figure 1). She constructed a table and a graph to represent the baby-weight relationship in an accurate manner. The tabular data and the graphical scaling projected different representations of the same coordinates (see Figures 1 and 2). S1 explained how she was operating while solving these problems. The explanation was given in the interview that was conducted with her, and the transcript of that interview is presented below.

Protocol I (Student 1 Pretest Interview, Lines 1-6)

I: Your table has even numbers only. Is there a reason why you did not put any odd numbers in the table and the derived graph?

S1: It says that she gained 2 pounds for 8 months, so I did not see any need for the graph to include any odd number.

I: So, how did you draw your graph?

S1: I did a linear graph with *y-axis* representing my pounds and *x-axis* representing months.

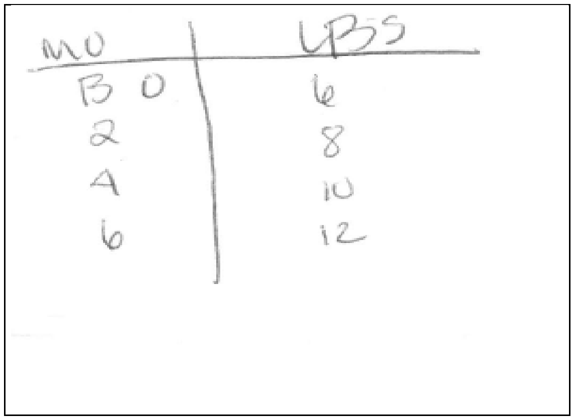


Figure 1: Table constructed by S1.

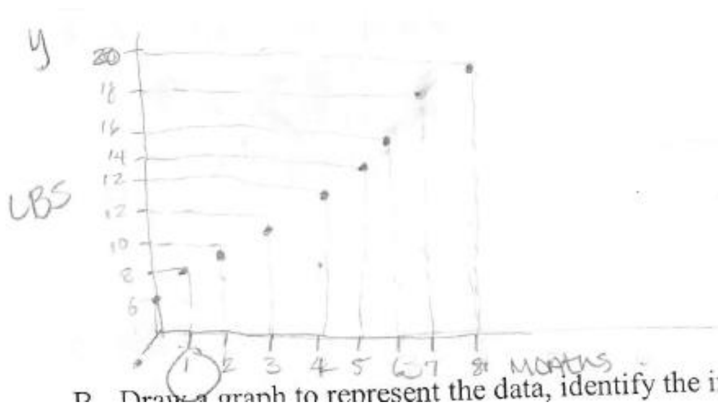


Figure 2: Graph drawn by S1.

S1's initial thinking when she was making the table indicated a lack of understanding pertaining to the concept of rate. Her table values were not correct (see Figure 1). Her plotted points were correct (see Figure 2). Perhaps she miss understood the problem. She was not able to symbolically construct and equation that would show the relationship between the baby's weight and baby's age. Her thinking on the age-weight relationship and her making use of the table to graph the data were explored.

Protocol II (Student 1 Pretest Interview, Lines 7-12)

I: Can you tell me your thinking on and ideas behind how you constructed the model to represent the age and weight relationship? I noticed that you did not generate any equation.

S1: An equation?

I: Yes. You wrote 2 pounds and *y* is age, *L* is pounds. You have *y = 2* [Figure 3].

Can you tell me what you are thinking there?

S1: I could not think of a way required to construct an equation.

S1's reasoning was additive. In the first column, she wrote out the accumulated amount of weight gained in each month, and in the second column, she represented the weight at each month. The conflation between the number of months and the accumulated weight gained in each month could have been the reason that S1 did not construct an equation (see Figure 3).

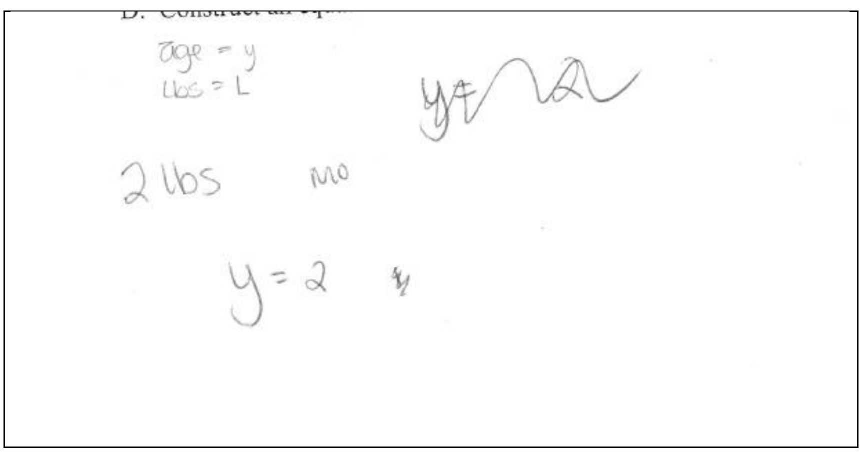


Figure 3: Sketchy response written out by S1.

Protocol III (Student 1 Pretest Interview, Lines 13–27)

I: On the same question, the graph has odd numbers. Did the table help you to draw the graph?

S1: Yes.

I: So you did use the table to draw the graph, correct?

S1: Yes and no.

I: Tell me how you did that?

S1: Okay, for the graph I knew that the question says that she gained 2 pounds for every month for 8 months. So I just numbered it 1-8 on the *x*-axis and on the *y*-axis to count by 2's for the pounds.

I: Yes, but on your *x*-axis you have 1, 2, and 3, and on your table you had 0, 2, 4, and 6. What was your idea behind that there?

S1: I drew the graph before I did the table. I did not use either one to construct an equation.

I: So your thinking was not the same on both?

S1: Not really. No.

The Snail Problem

A snail crawls the first part of 48 inches at the rate of 2 inches per hour. He crawls the rest of the way at the rate of 3 inches per hour. If it takes the snail 20 hours to crawl all 48 inches, how long did the snail crawl at each rate? Solve the problem graphically and then algebraically.

S1 divided the snail's journey into two equal parts: first part and second part (see Figure 4). She interpreted each part to cover half of the 48 inches. She did not draw a graph to give a graphical representation of the paces at which the snail moved at a 2-inch-per-hour pace and at a 3-inch-per-hour pace. She divided 24 inches by 2 inches/hour, which raises the possibility that she used 2 inches/hour as a ratio. Further, there was no indication that she had an understanding that neither 2 inches/hour nor 3 inches/hour was a constant ratio between the two varying quantities. That is, at this point, there was a small indication that she had constructed a concept of rate.

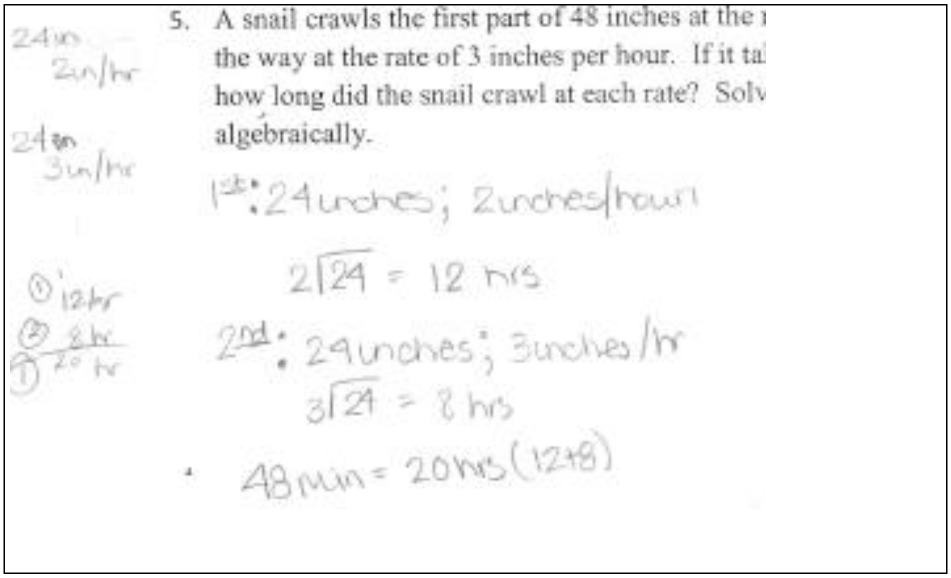


Figure 4: S1's solution to the Snail Problem.

S1 could not graph the relationship between the two rates at the point where the rate changes from 2-inches per hour to 3-inches per hour. She did try to find a time where there was a change in pace. She assumed that the snail pace changed at the halfway point along with her dividing 24 by 2, which does indicate rate thinking.

S1's Initial Conception (On the Basis of the First Two Pretest Problems)

There was little indication throughout the Baby Weight Problem and the Snail Problem that S1 had constructed a concept of rate. Her initial conception of ratio is less clear. In the graph she made (Figure 2), she did coordinate the sequence of months with the total weight of the baby at each month. This use of the ratio, 2 pounds per month, is additive as she was simply adding 2 pounds to the weight at the previous month. The table and graphical representation of the baby's weight and baby's age show that she may have misunderstood the question. For example, she may have been confused between the tracking of the baby's weight that is each month (illustrated by the graph) and every two months (illustrated by the table values).

S1 divided 24 inches by 2 inches per hour in the Snail Problem to find how many hours it took the snail to travel 24 inches. She got the correct answer with incorrect thinking. For example, she divided the snail's journey into two equal parts assumed that the trip was divided evenly without any variation. She was unable to draw the snail's path to show that there was a change in variation between the two rates. Having fixed the journey into two even distances the understanding of the 2-inches per hour and the 3-inches per hour rates were not clear and seemed missing. Therefore the division of the snail's journey into the equal distances is an indication that the 2 inches per hour was conceived of as a ratio.

The Currency Problem

Suppose 2 English pounds (£) buy the same goods as 3 U.S. dollars ($).

A. Make a conversion table to compare the two currencies.
B. What would a garment of $20 cost in terms of pounds?
C. What's worth more, the pound or the dollar, to an Englishman just off the plane on vacation in the USA?
D. If p = the number of English pounds and d = the number of U.S. dollars, write the relationship between the number of pounds and the number of dollars. Explain.
E. Graph this relationship using graph paper.

S1 did not demonstrate a clear understanding of the unit ratio: that is, 1 English pound meant 1 dollar and 50 cents. The table she constructed and used to compare the English pounds to U.S. dollars left out the unit ratio. Her table was incomplete (see Figure 5). She did not use the two to 3 ratios for each tabulated value. Instead, she added 1 to the number of pounds and 3 to the number of dollars, so she was increasing the numbers in a one-to-three relationship. Her reasoning did not follow the reasoning needed for the question being asked. It seemed as if she had no concept of a unit ratio based on a multiplicative relationship. That is, there was no evidence that she could, for example, partition each pound into three equal parts and give one part to each dollar, for a total of 2/3 pounds per dollar.

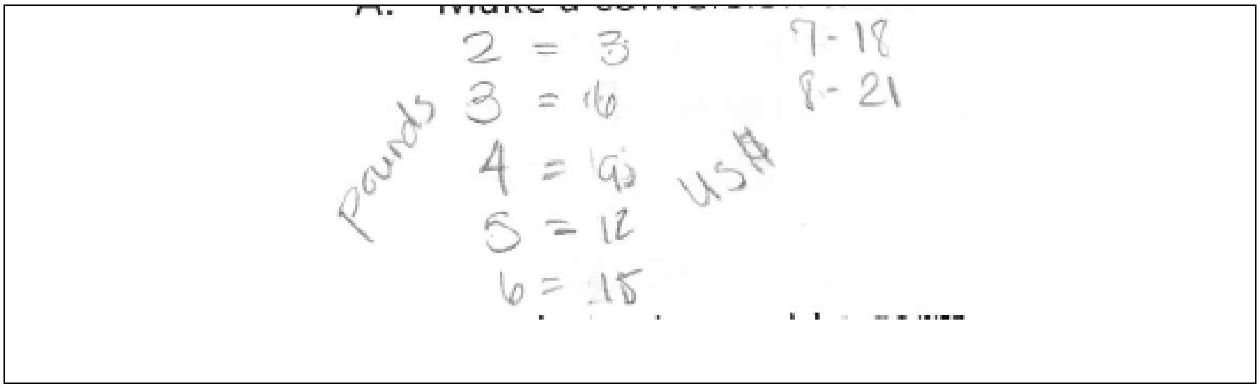


Figure 5: S1's English pounds-U.S. dollars table.

S1's table made it difficult for her to draw a graph to represent the relationship. Still, her graph was consistent with her table. That is, there was no apparent constant relationship that she was explicitly aware of between the number of pounds and the number of dollars (see Figure 6).

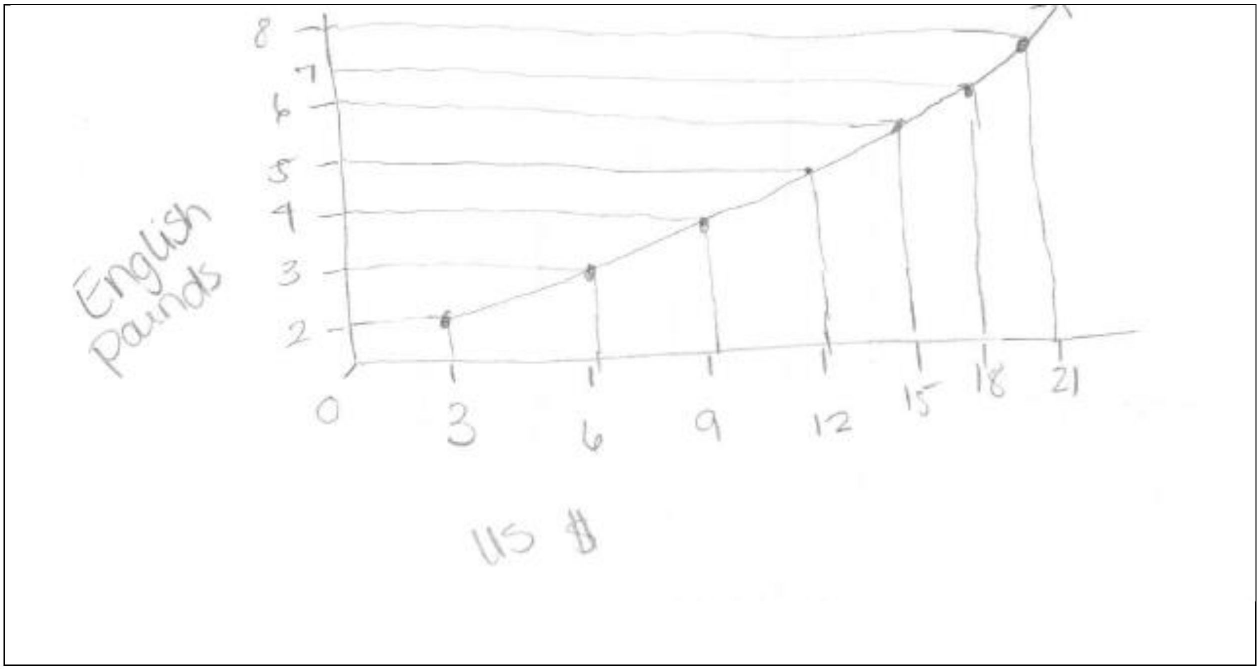


Figure 6: S1's English pounds-U.S. dollar graph.

Protocol IV (Student 1 Pretest Interview, Lines 74-83)

I: Okay. Can you graph this relationship?

S1: I graphed it.

I: How did you graph it?

S1: I graphed [it] based off the chart I had made.

I: So you used discrete values, and you made a continuous graph. So you are saying 3 U.S. dollars is equivalent to 2 English pounds. What would you think?

S1: Based on what I did at the top, yeah.

I: So you are saying equal. Was that your thinking?

S1: Yes.

At this point it seems that S1 had no apparent concept of a unit ratio, which disqualified her from using the concept of unit ratio to find a rate. Her thinking was not showing proportional reasoning; it showed inconsistency in her thoughts (see Figure 7). That is, she wrote out that $20 equals 8 pounds and 6 pounds equals $21.

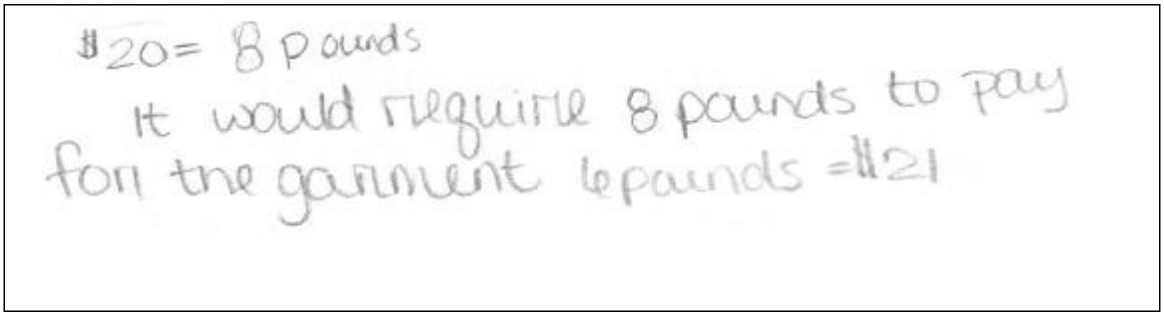


Figure 7: S1's use of symbols.

The Currency Problem, Part A

S1 did not see that the English pound was worth more than a U.S. dollar in terms of buying power (see Figure 8). Below is her response to the question, What's worth more, the pound or the dollar, to an Englishman just off the plane on vacation in the USA?

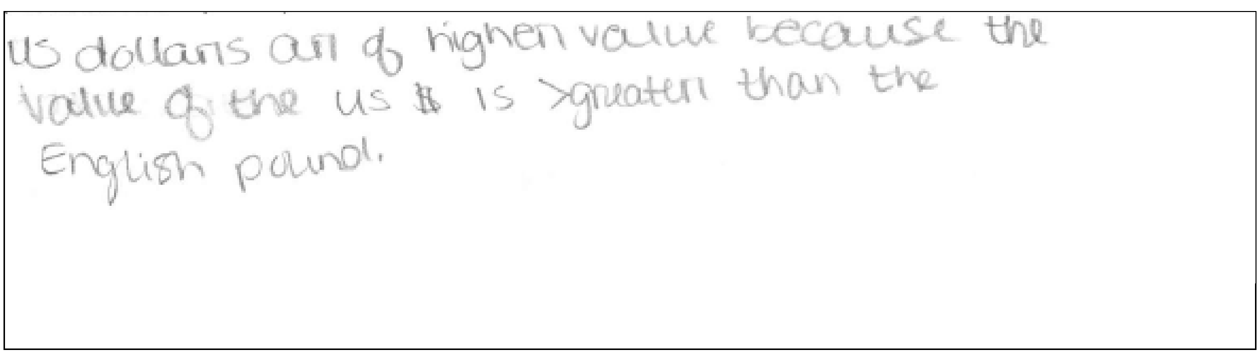


Figure 8: S1's statement that a U.S. dollar is worth more than an English pound.

Protocol V: (Student 1 Pretest Interview, Lines 53-64)

> I: So what is worth more—the pound or the dollar—to an Englishman just getting off the plane in the U.S.?

S1: The dollar is of higher value.

I: Why do you say that?

S1: Because if you have 2 pounds and 3 dollars. Thinking from an American economic mind, 3 dollars would be worth more. It's going to take more American money than English pounds. Hold on. You would lose your money because if you had more pounds. The more American money you have, the more you are going to lose in American money because 2 pounds equal 3 dollars; I am not really sure.

I: Can you give me a number of the pounds versus the number of dollars?

S1: Let's say—. It's in my head; I do not know.

The Currency Problem, Part D

It appeared that S1 became momentarily aware that a U.S. dollar was worth less than an English pound, but she was unable to articulate a rational reason for her intuitive sense of a relation between the two monetary systems. So I asked her to give me her representation of the relationship between the English pound and the U.S. dollar, where *p* represented the number of English pounds and *d* the number of U.S. dollars.

Protocol VI: (Student 1 Pretest Interview, Lines 65-73)

I: If *p* represented a number in English pounds and *d* represented a number in U.S. dollars, what is the relationship between the number of pounds and the number of dollars? Can you explain?

S1: I wrote that the number of U. S. dollars would be greater by multiples of 3, but I don't know.

I: Did you add a formula?

S1: No

I: Can you do it now?

S1: I am going to say no.

However, her work demonstrated some additive reasoning. For example, her table indicated increments of 2 for the English pound and increments of 3 for the US dollar (see Figure 6). I wanted to see if she saw the unit representation or that 1 English pound equals 1 U.S. dollar and 50 cents.

Protocol VII: (Student 1 Pretest Interview, Lines 28-61)

I: On the currency problem, your table had 2 = 3, 3 = 6, 4 = 9, 5 = 12, and you stopped at 6 = 15. Can you explain how you arrived at these numbers?

S1: Well, it says for 2 English pounds you can have the same goods for 3 dollars. So my thinking was if 2 pounds is equal to 3 dollars, then it should have increased in that same pattern over time. If there were no variables changing over time throughout.

I: Was there a way you could find what 1 pound would equal?

S1: My thinking is it would be 1 and a half U.S. dollars, because half of 3 is 1 and a half. That is my only logic on trying to figure it out.

I: On Part (d) of the question, how did you decide that 20 dollars equal 8 English pounds?

S1: What I did was from the chart I drew at the top. I knew that I would have to go over in dollars. One would have to be more than the other.

I: So were you estimating?

S1: Yes.

I: Why 8?

S1: What I did was if I knew that for the dollars I did it to where it increased by 3 each time. When I got to 8 pounds, it was equivalent to 21 dollars.

I: How did you decide that? How did you know that the U.S. dollar is bigger than the English pound?

S1: Because it says it in the problem. You said the English dollar is bigger that the U.S. pounds right?

I: The question reads worth more than a pound. So you are thinking that a pound [meant dollar] is worth more than a dollar [meant English pound], right? Can you tell me your thinking?

S1: I think that the U.S. dollar is greater than pounds. Just because if it takes me 3 American dollars, and I only get 2 pounds back, that is a difference. To me, by me being American, I am losing one over the other.

I: From your graph, what would you say [is] the relationship between the English pound and the U.S. dollar?

S1: That the number of the U.S. dollar is like for every 3 dollars you only have 2 pounds. So, if I have 6 dollars, I would only have 4 pounds. I don't know how to explain it; I just know what it looks like.

I: Okay, from the same graph, I notice that you have 0, 3, 6, 9, 12, 15, 18, and 21 in U.S. dollars. For your English pounds you have 2, 3, 4, 5, 6, 7, and 8. How did you decide which one is x and which one is y?

S1: I just used these based off the information in Number 2. From the table….

Gasoline Problem

Imagine pumping gasoline into a car.

A. What are the variables that come to mind?

B. How would you find the cost of the gasoline you pumped? Read a meter?

C. What is the relationship between the dollar amount cost and the gallons being pumped?

D. How would you determine the rate at which the gasoline is being pumped into the car?

E. If c = cost and g = the number of gallons pumped, sketch the graph to illustrate the cost of the gasoline.

The Gasoline Problem, Part A

S1 was not able to state the variables correctly on paper. Instead of itemizing the variables, she offered examples. That is, she wrote the following: (1) the current amount of gas in the car, and (2) price of the gas per gallon (a rate) (see Figure 9).

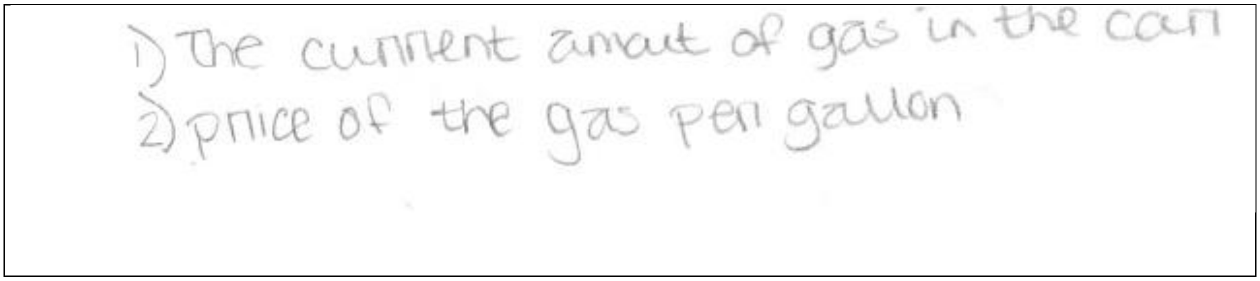

1) The current amout of gas in the carr
2) price of the gas perr gallon

Figure 9: S1's Answer to Gasoline Problem, Part A.

I could not understand why she offered these examples. I questioned her on this to get at her thinking.

Protocol VIII (Student 1 Pretest Interview 1, Lines 62-67)

> I: Okay, for the gas problem, (a) can you explain why a price of the gallon of gasoline is a variable? Why are you saying that the price of the gallon of gasoline [is] a variable?
>
> S1: 3a. What I thought about was that if I have a car and depending on how much gas I already have in the car determines how much I am going to put in the car. Looking at it now, it looks backwards. The price per gallon, I put that because those were just the two things that came to my mind. How much I have and how much I will need.

S1 was able to give a method for calculating the cost of gas to be pumped. The gas needed (to fill the car) minus the gas in the car gives a number, and that number multiplied by the price per gallon gives the total dollar amount of gas.

Protocol IX (Student 1 Pretest Interview 1, Lines 68-75)

> I: I am looking at your E [Part E of the gasoline problem]. How did you come up with the numbers you used in your graph?
>
> S1: I just picked some random numbers because there were no numbers in the problem. I just pick random numbers to show how much was in the car to determine what needs to be put in the car.

I: Can you explain what 1 - 2 means?

S1: If I had one gallon of gas it would cost 2 dollars; 2 gallons of gas would cost 4 dollars; and 3 gallons of gas would be 6 dollars, which means the cost of gas is 2 dollars per gallon.

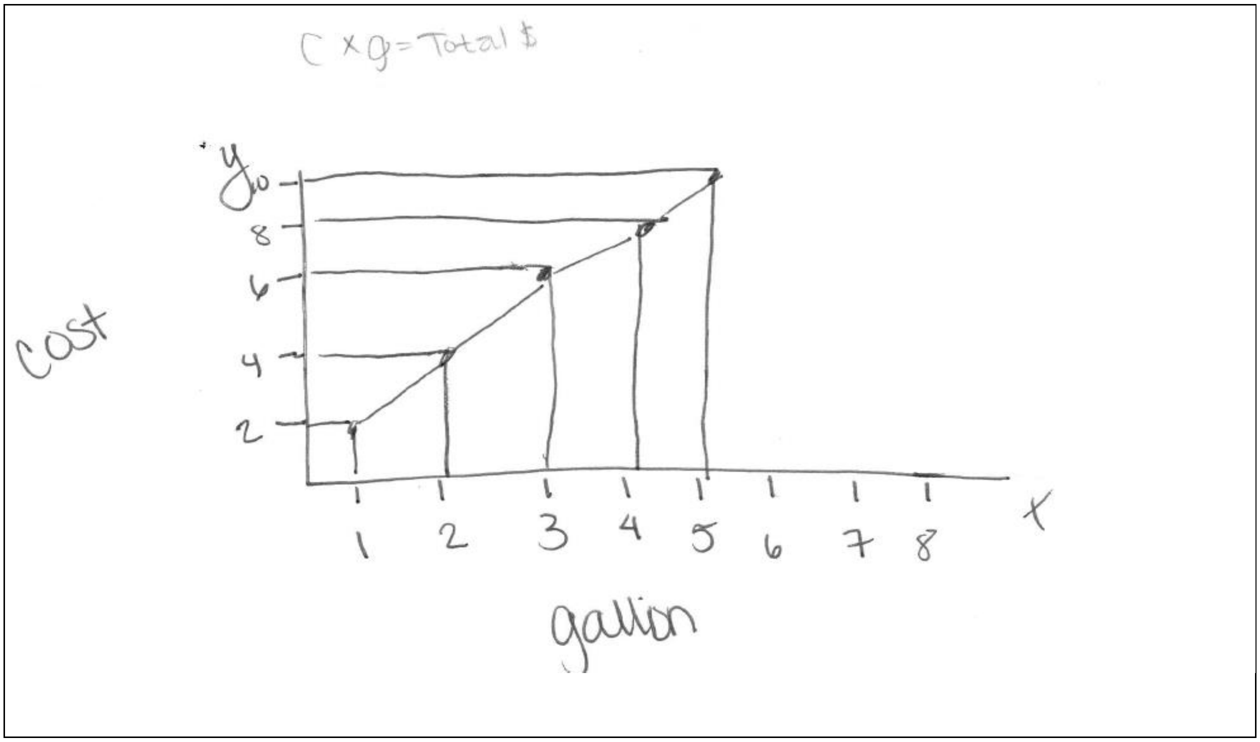


Figure 10: S1's Linear graph cost per gallon.

S1 was able to construct a multiplicative relationship to find dollar amount cost and gallons being pumped, and she was able to determine the rate at which the gasoline is being pumped into the car.

Protocol X (Student 1 Pretest Interview 1, Lines 76-79)

I: How would you determine the rate at which the gas is being pumped into the car?

S1: I did not do that one.

I: Do you have an idea now?

S1: No.

I followed up on this question for her thinking. S1 seemed to struggle with her understanding of the meaning of rate.

Protocol XI (Student 1 Pretest Interview 1, Lines 95-102)

I: For Part C, can you tell me your thinking in terms of the relationship between the dollar amount cost and the gallons being pumped?

S1: It would be price per gallon in that case.

I: Is it your price per gallon times gallons needed?

S1: Yeah.… I don't know what I was doing on that one.

I: I noticed that you left Part D blank; can you tell me you're thinking on how you solved D?

S1: No.

The Lady Runner Problem

A runner is running along and times how long she has run and how far she has run using a stopwatch and a distance meter. If one reading is 3 minutes and 3/7 miles, how long would it take her to run 8 miles if she runs at a constant pace?

S1 was not able to use ratios to solve this problem correctly (see Figure 11).

Protocol XII (Student 1 Pretest Interview 1, lines 107-113)

I: Let us talk about the runner problem.

S1: I did not do that one.

I: Can you do it now?

S1: It is going to be roughly 20 minutes.

I: How did you get 20 minutes?

S1: I took the 3/7 and divided it into the 8 and I got a decimal: 4.1. I rounded it and got that answer.

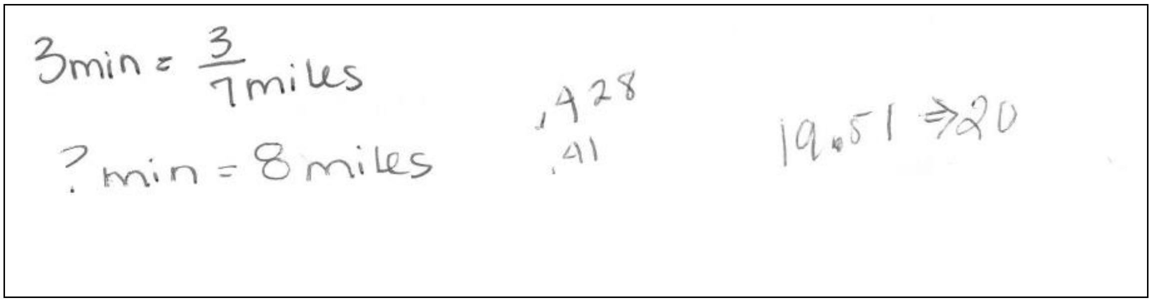


Figure 11: S1's solution to the Runner Problem.

S1 writes 3 min[utes] = 3/7 miles and then writes .428, which is 3 divided by 7, then 19.51 >> 20. The "? min[utes] = 8 miles" does indicate that she is looking for the time it would take the runner to travel 8 miles; however, it is unclear as to where the ".41" came from or if she knew how to construct the multiplicative relationship needed to solve the problem. I wanted to know more of her thinking on the runner problem. Below is what she said:

Protocol XIII (Student 1 Pretest Interview 1, Lines 84-88)

I: Okay, on Number 4, what was your thinking behind that problem? I see that you wrote "3 min equals 3/7 miles" and requested miles needed "equals 8 miles."

S1: What my thinking was if a runner runs along, how long it would take them to get there. The first reading was 3/7 miles; the question is how long it will take her to run 8 miles? So what I did was I cross-multiplied it to get …..3/7 = *x*……

S1 could not clearly represent the relationship between the distance and the time the runner ran; that is, she did not produce the ratio that the runner traveled 1 mile for every 7 minutes.

Student 1's Initial Conceptions (Last Three Pretest Problems)

S1 had difficulty with the construction of multiplicative relationships as seen in the Runner and Currency Problems. For the Gasoline Problem, S1 did construct a linear relationship; however, her understanding of the variables, when asked about them, was limited to giving rates as variables. While she was solving the Runner Problem, S1 was unable to set up the proportional relationship needed to solve the problem.

S1, in summary, was unable to set up proportional relationships consistently and make multiplicative relationships to relate the given quantities. She did not see patterns in tabular data to construct linear equations. She did not clearly exhibit a sense of variation or change in variation asked for in the Snail Problem.

Ratio and Proportions Homework (Homework 1)

Homework 1 had three problems: the Orange Juice Problem, the Log Problem, and the Chicken Problem. These problems all drew on students' ability to use their understanding of proportions to solve problems. The Orange Juice and Log problems were the ones that caused some problems for the class. S1's solutions to these problems were different from others in the class. For the Orange Juice Problem, she used a part-part relationship to solve it, whereas many of the prospective teachers used a part-whole relationship for the same problem. For the Log Problem, S1 was one of several students who did not use the number of cuts versus the time period that would have produced the appropriate ratio, when attempting to solve the problem.

The Orange Juice Problem

Frozen orange juice concentrate is usually mixed with water in the ratio of 4 parts water to 1 part concentrate. How much orange juice can be made from 18-oz can of concentrate?

S1's solution path to this problem was peculiar (see Figure 12).

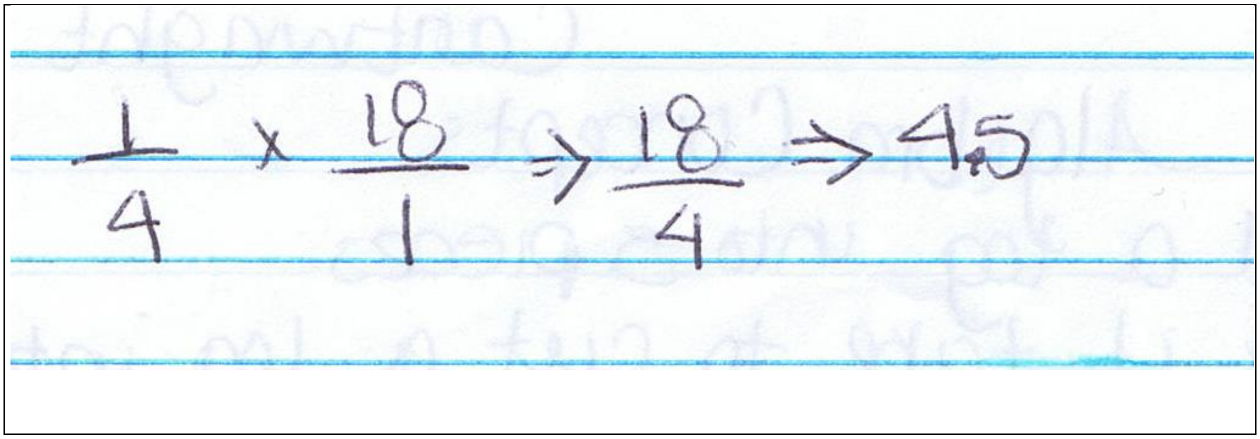


Figure 12: S1's solution to the Orange Juice Problem.

S1 set up the problem using a part-part proportional relationship. She wrote $\frac{1}{4}$ multiplied by $\frac{18}{1}$. The $\frac{1}{4}$ could be one part concentrate to four parts water and the $\frac{18}{1}$ could mean the 18 ounces of concentrate to one part water. She was unable to set up the proportional relationship between the orange juice concentrate and the water.

The Log Problem

It takes 30 minutes to cut a log into five pieces. How long would it take to cut a similar log into six pieces?

The Log Problem asked for the time it would take to cut a log into six pieces, given that it took 30 minutes get five pieces. For this problem, S1 set up the proportion as 30 minutes per five pieces with *x* minutes per six pieces (see Figure 13). She did not take into consideration the cuts. S1 got 6 minutes per piece instead of the 7.5 minutes per cut that would have generated the reasoning needed to answer the problem.

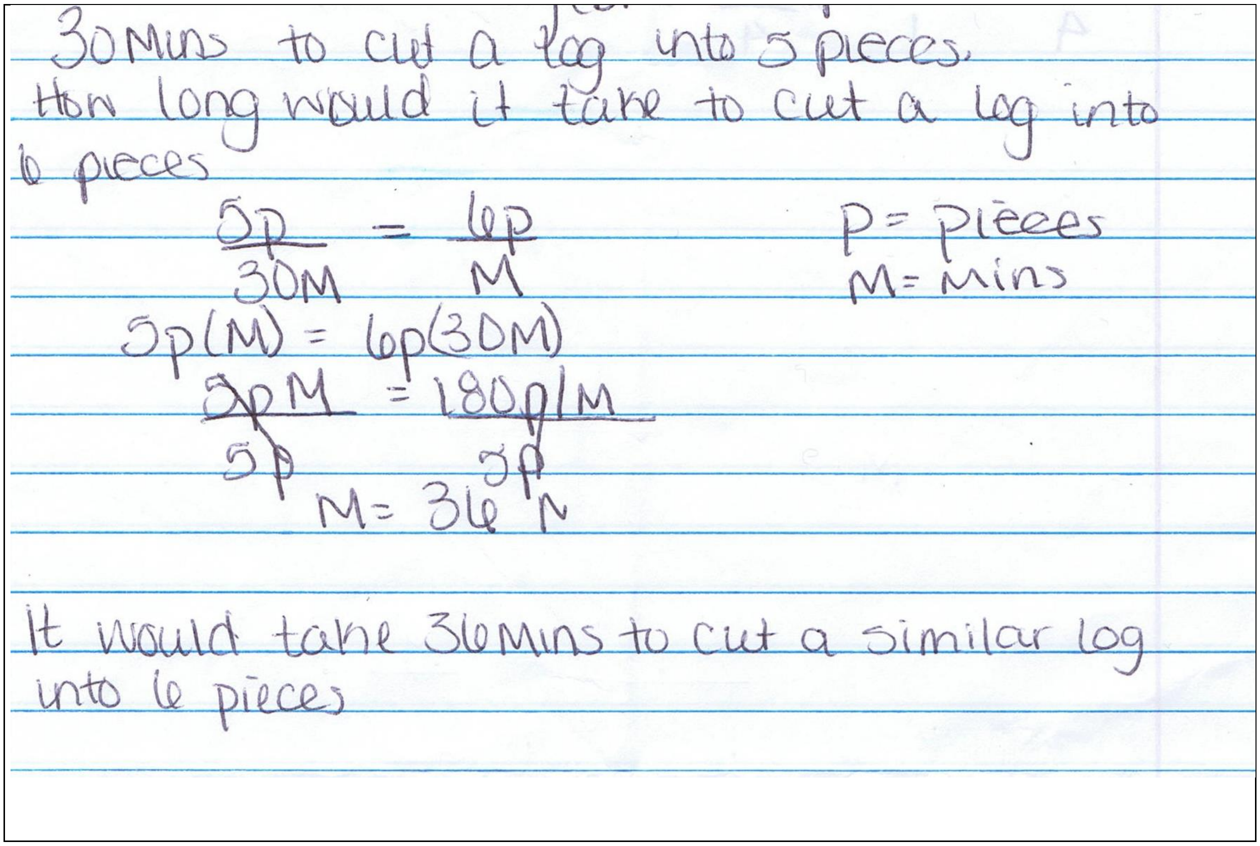


Figure 13: S1's solution to the Log Problem.

It was unclear at this point if S1 could apply the part-whole relationship looking back at the Orange Juice Problem. Even though she set up proportional relationships, they were not correct. For the Log Problem, she did not consider the number of parts produced by the cuts: she used the number of pieces instead. She did not have the understanding needed to set the portions up correctly in solving the Log Problem. There seemed to be some development in S1's multiplicative understanding; that is she did set up the proportions for the log problem, they were just not correct. The next set of homework problems were motivated by Steffe's Notes[7]. These problems came in the form of word problem and were met by opposition from students in the class. This opposition occurred because students had been accustomed to using the *Mathematics for Teachers* textbook by Sonnabend. Students wanted examples beyond the one given in the note, and these problems took some of the students, including S1, outside of their comfort zone.

Currency Problems (Homework 2 Was Motivated by Steffe's Notes)

There were two currency problems assigned for students to complete and turn in. These problems were discussed in class within the context of the notes. The first problem was the following:

> Suppose that 3 Argentine Pesos is equivalent to 4 Danish Kroner. Use the Danish Kroner as the independent variable.

A. Represent this in the form of a linear equation.

B. What is the slope of the line?

This problem was motivated by the following problems from Steffe's notes.

A. Three Argentine Pesos are equivalent to 4 Danish Kroner. Establish a multiplicative relationship between the two currencies so that any number of Kroner can be exchanged for Pesos, or vice versa. Make a variable point (*p*, *k*) on a graph, where the number of Kroner is the independent variable and develop a conversion table as well. If the number of Kroner k = -14, what is the corresponding value of the number of Pesos, **p**? What does it mean for **k** = -$14? What does the corresponding value of **p** represent?

B. Interchange the independent and dependent variables so that (k, p) is the variable point. What is the relationship between the two graphs?

S1 wrote, $\frac{3AP}{4DK} = 1$, where $A = 3$ and $D = 4$ [cf. Figure 14]. It was unclear if the 1 meant the unit ratio. There was also a mismatch between her written variables of *AP* and *DK*, which

[7] Class Notes, EMAT 7080. Rates of Change and Their Graphs.

respectively meant *A* and *D*. S1 was unable to write a multiplicative relationship that would represent an equation for Part A. For her, $\frac{3}{4}$ was the slope even though she was unable to represent an equation symbolically, graphically, or by showing a table of values. A cross multiplication with her expression would mean that $3AP = 4DK$: so the concept of equivalence versus equality was an issue here as well. For instance, the two quantities are equivalent but not equal.

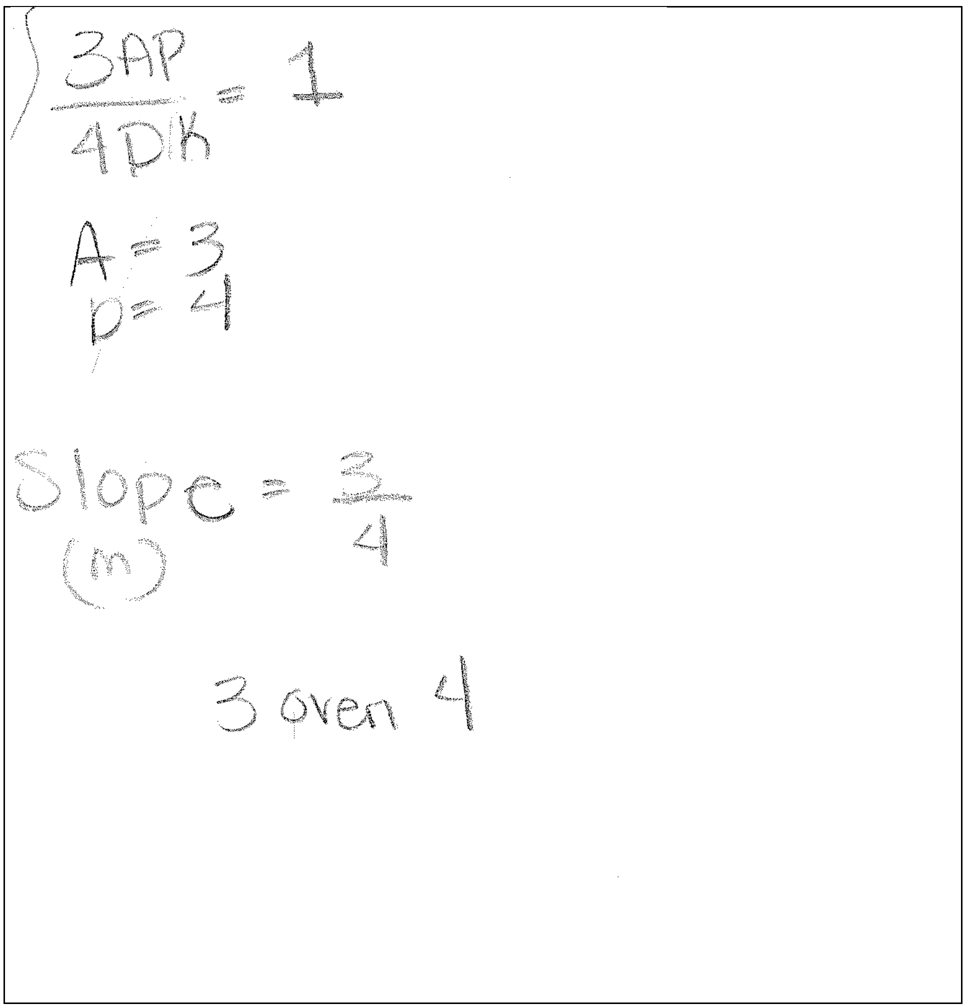


Figure 14: S1's solution to the Currency Problem.

The next problem was similar to the first problem:

In (1), the Argentine embassy decides that all Danish visitors to Argentina must hold back 10 Danish Kroner and not convert them to Argentine Pesos. Using the *xy* coordinate plane, make a graph of the number of Pesos that each Danish visitor receives with respect to all the Kroner the visitor brought to Argentina. Be sure to develop an equation.

For this problem, S1 seemed to be reliant on the point-slope [that is, $y - y_1 = m(x - x_1)$, a popular formula used in college algebra to find the equation for the line] formula as a means to generate her linear equation. Here she did indicate some idea about variation when she wrote "Right horizontal"; this was evident in her circling of the 10. This, too, was reinforced when she wrote and labeled it "slope-int. She illustrated these concepts graphically (see Figure 15).

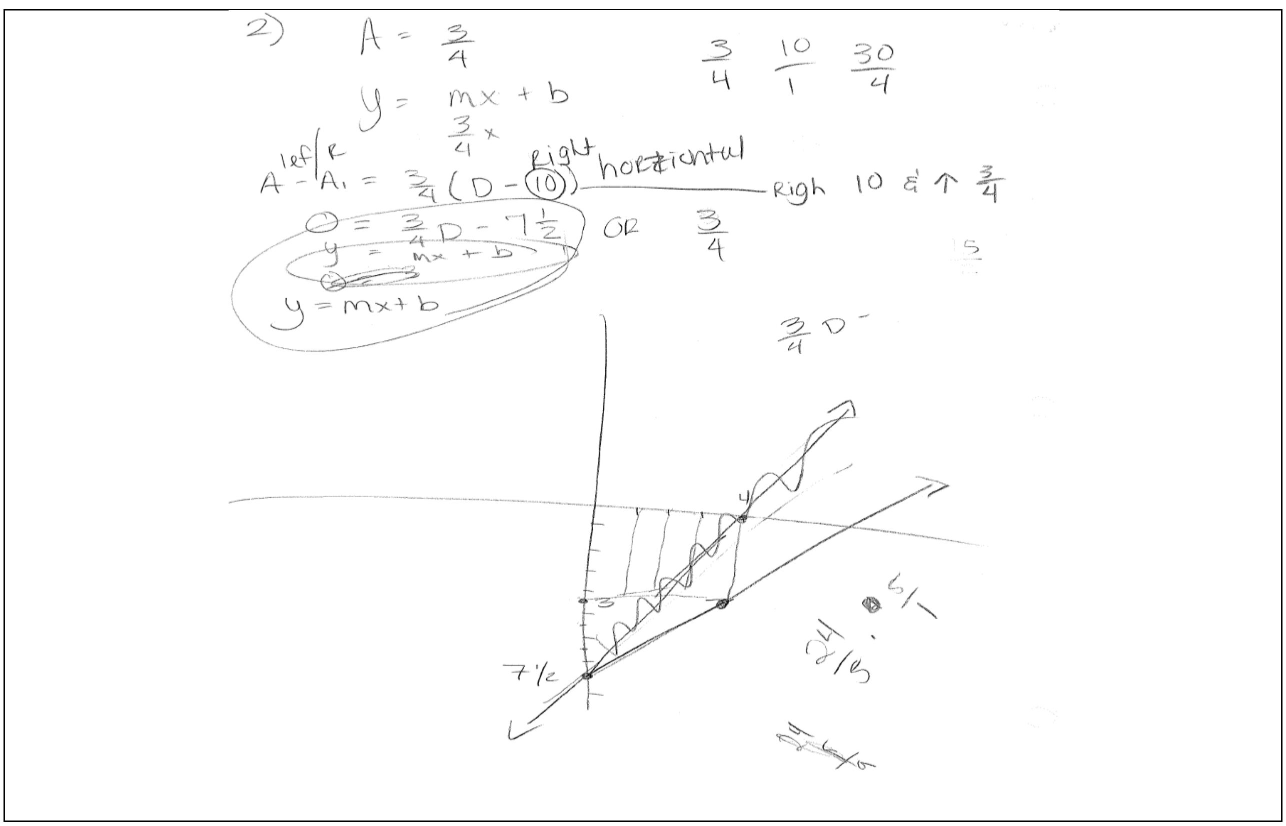


Figure 15: S1's solution to the currency problem number 2.

S 1's constructions of the linear equations were not done using her understanding of a multiplicative relationship between the Pesos and Kroner. Instead, she used two formulas: the point-slope form and the slope-intercept form. She wrote A = 3/4 and then the slope intercept formula for the equation of the line. She then used the point-slope-formula for the equation of the line and wrote out the distribution for ¾(x–10) by first writing ¾ , 10/1, and 30/4. Her graph illustrated that the –10, which she circled meant a horizontal shift right 10. From there, her graph does show a point drawn through the intercept, and her markings do illustrate up 3 units and right 4 units.

Take Home Quiz (Homework II)

> Problem 1: A race car driver drives part of a 500 mile race at 150 miles per hour and the remainder at 180 miles per hour. If it takes her a total of 3¼ hours to complete the race, how long at each rate did she drive? Solve the problem graphically and algebraically.

S1 went online to find formulas to help her solve the problems on the take-home quiz. Her attempt did not show the change from 150 miles per hour to 180 miles per hour. Even though she wrote out two separate multiplicative representations for distance traveled, there was no indication that the two rates were related (see Figure 16). She wrote $d_1 = r_1 t_1$ and $d_2 = r_2 t_2$, which did appear to be correct for the two distances covered by the race car; however, she then wrote different subscripts representing time and placed a time (3.3h) meaning 3.3 hours. Next she wrote $d = (r_1 \times t_1) + [r_2 \times (t - t_1)]$ and $500 = (150 \times 3.3) + [180 \times (3.25 - t_1)]$. The 3.3 hours seemed arbitrary since it was not given in the problem. She was unable to completely solve the problem. Her reliance on having a set formula to help her solve the problem may have made it difficult for her to do just that: problem solve. She did not know how to use the formulas she downloaded from the internet.

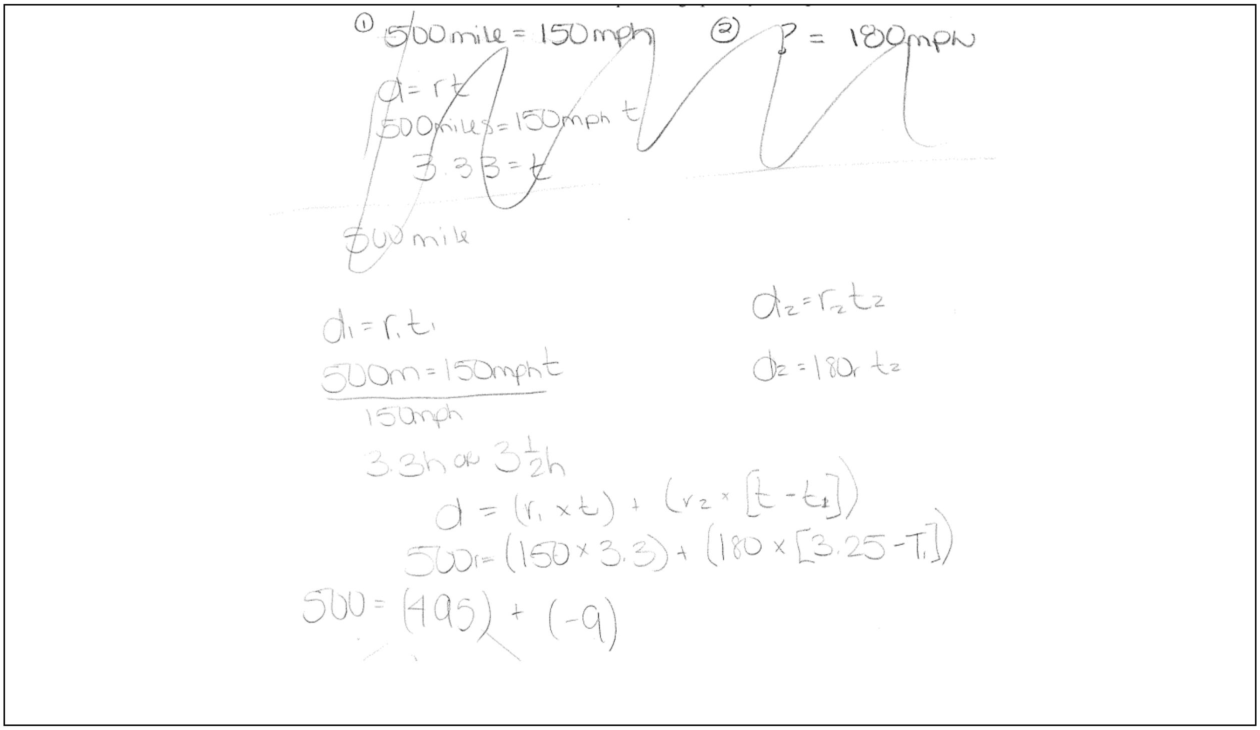


Figure 16: S1's attempt to answer the Race Car Driver Problem.

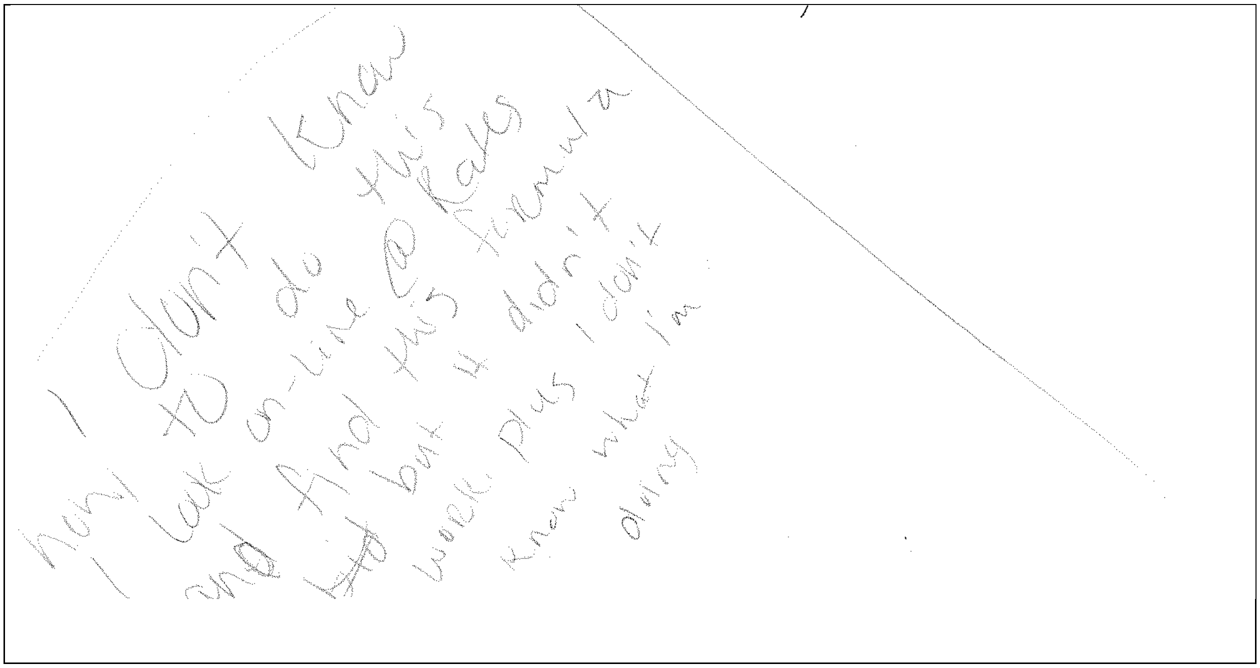


Figure 17: S1 admitting she was unable to complete the problem.

Student 1's Thinking on the Problems after Posttest

The Baby Weight Problem

This time, unlike in the pretest, S1's table was complete. Her graph was drawn correctly, a straight line and not curvilinear as before. She constructed a linear equation and explained her input as age by month and output as baby weight (see Figures 18 and 19).

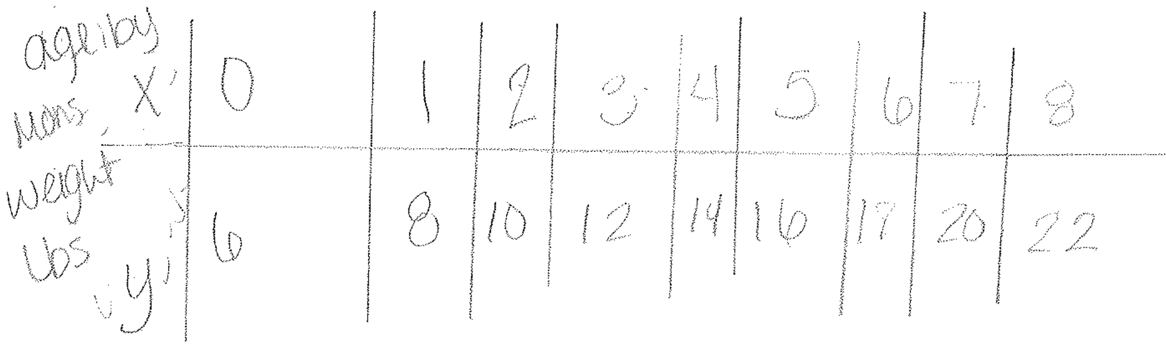


Figure 18. S1's construction of a table of values for the Baby Weight problem.

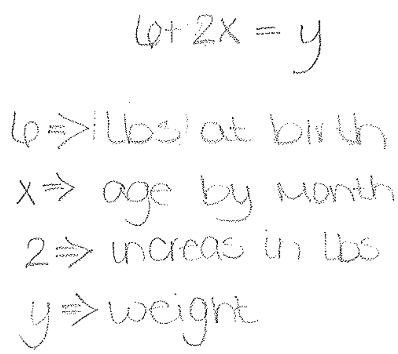


Figure 19: S1's construction of the age-weight relationship.

The Snail Problem

S1 was unable to complete the Snail problem. Still in the posttest she drew a straight line to represent the snail's trip (Figure 20). Her graph showed no change in variation for the 3-inch per hour to 2-inch per hour. She could not construct a multiplicative relationship to model either rate situation.

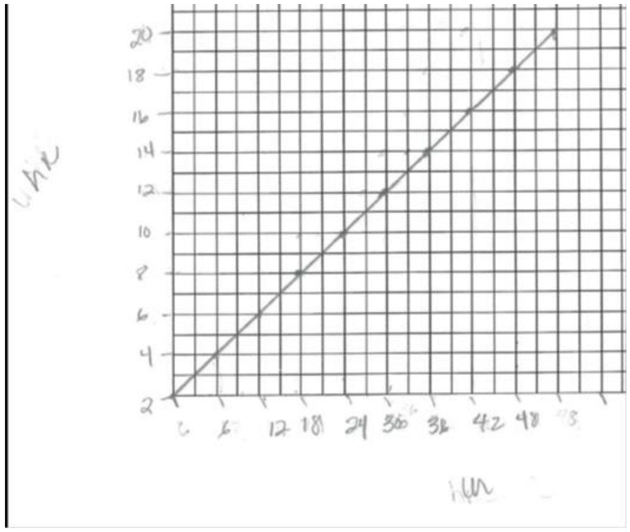


Figure 20: Graph drawn by Student 1 for the Snail Problem.

I: Okay, I noticed that you did not show how you constructed your linear graph; I am talking about Question Number 2, the Snail problem; can you explain that?

S1: No.

I: Okay, tell me why you have 6, 6, and 48?

S1: Honestly, I just did something, but I know that it is not right.

I: For what appears to be your origin, I noticed that you have a (2, 6); can you explain that?

S1: I don't know how to do that.

I: How does that graph show 2 inches per hour and 3 inches per hour?

S1: It doesn't.

I: Do you believe that you can tell me how to work this problem?

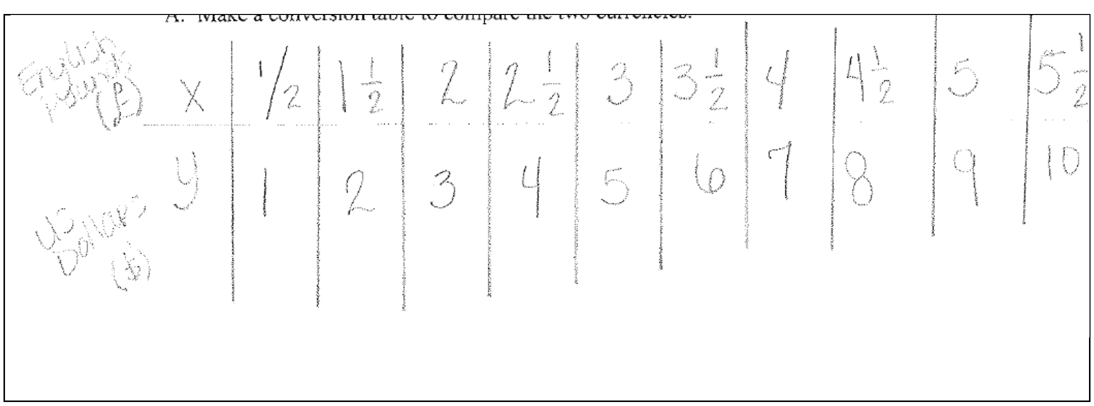
S1: No, I don't believe so.

S1 seemed to be confused as to how she had solved the problem and could not clarify what she was thinking when she generated the graph.

The Currency Problem

In the pretest, S1 did not maintain the 2 to 3 ratio. She was not able to determine what a garment of $20 would cost in terms of pounds. However, she was able to determine that the pound was worth more. In the posttest, S1 was still unable to construct a linear relationship between the number of pounds and the number of dollars (see Figure 21). However, she was able to draw a linear graph, though it was not done correctly–she had English pounds as the independent quantitative variable and U.S. dollars as the dependent quantitative variable with incorrect plotted values. Here is what she had to say:

> I: Let’s look at the currency problem; tell me why you started your table with *x* being a ½ corresponding with *y* = 1.
>
> S1: Because it says that 2 English pounds equal 3 U.S. dollars. For the English pounds, I drew 2 bars, and for U.S. dollars, I drew 3 bars. The English bar overlaps the dollar at half; that is why I have 1/2 for *x* and 1for *y*.



English Pounds (£) x	1/2	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$
US Dollars ($) y	1	2	3	4	5	6	7	8	9	10

Figure 21: Table for the Currency Problem drawn by Student 1.

I: Okay, on the same question, I noticed that x equals 1/2 and y equals 1 from your table. [It seems] that you are adding a ½ to each x value and 1 to each corresponding y value. Can you give me an explanation for your thinking here?

S1: When I started the table, I started at 2 (x) and 3 (y) because I knew that 2 pounds equal 3 dollars. That is where I started. I was writing it from that point: I went backwards. I saw that on the picture I drew, that 2 U.S dollar (bar) was 1 pound and ½ of a pound, so I based the chart off of the visual representation that I drew, and after 3, I just kept adding them.

I: Okay, what would a garment at $20 cost in terms [of] pounds? A garment cost $20 you have would cost 10.5 or 10 ½ pounds (see Figure 22). Tell me your reasoning there.

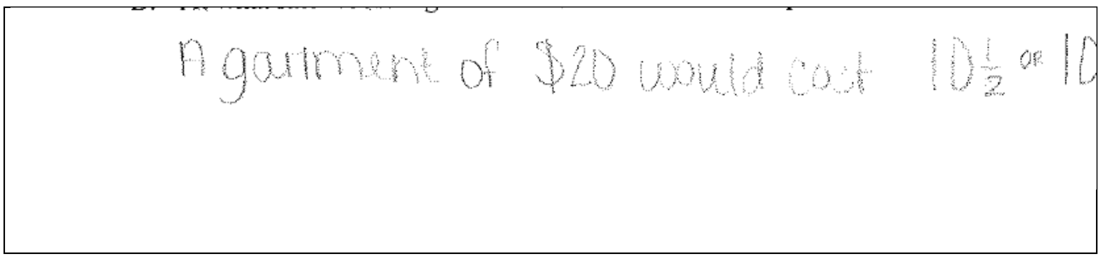


Figure 22: Student 1's conversion from dollars to pounds.

S1 explained how she came up with her table. She divided d, the dollar amount, by 2, and added 1/2 to get p, the equivalent in pounds. Her table ends with 5 ½ pounds corresponding to 10 U.S. dollars. However, she wrote that a garment of $20 would cost 10 and ½ [pounds] (see Figure 22).

S1: The reason why I did that, I noticed that for y $10 it was 5 ½ pounds, so I noticed the x was y plus ½ , so I looked at 20, and I know that ½ of 20 is 10 and that I had to add ½ to that.

I: Can you explain why you double and add? For Part C of the Currency Problem you said, "To an Englishman on vacation in the U.S., the pound is worth more because would have to double and add ½ a pound to have the equivalent to U.S. dollars."

S1: Because in the chart for $2, ½ of 2 is 1. So you have the 1 English pound, and you have to add ½ a pound just to equal 2 U.S. dollars.

I: So I'm looking at *d*; you have p = pound and d = dollar, but your equation is going $d/2 + ½ = y$.

S1: It is supposed to be *p* not *y*.

I: So instead of *y*, what should it be?

S1: *p*.

I: *p*? Okay, how would that work?

S1: The same way with the chart. If you have $4, you put 4 over 2, which is 2, and you add ½ and you get 2 ½.

I: So you are saying $p = d/2 + ½$?

S1: Yes.

I: So if you want to find *p* the pounds?

S1: Put any dollar amount in for *d*, and the total will get the pounds.

I: So if I put 4 in for dollars, what would you get?

S1: 2 ½.

I: Okay.

The Gasoline Problem

Imagine pumping gasoline into a car.

A. What is the relationship between the dollar amount cost and the gallons being pumped?
B. How would you determine the rate at which the gasoline is being pumped into the car?
C. If c = cost and g = the number of gallons pumped, sketch the graph to illustrate the cost of the gasoline.

S1's answer to Part A was to write out "cost × gallons equals total cost." Here she gives an example to illustrate her thinking. That is, $3.50 × 16 = $56.00. She wrote out the relationship of $kx = y$ and $y=3.5x$, which shows that there is a direct relationship between x and y (see Figure 23).

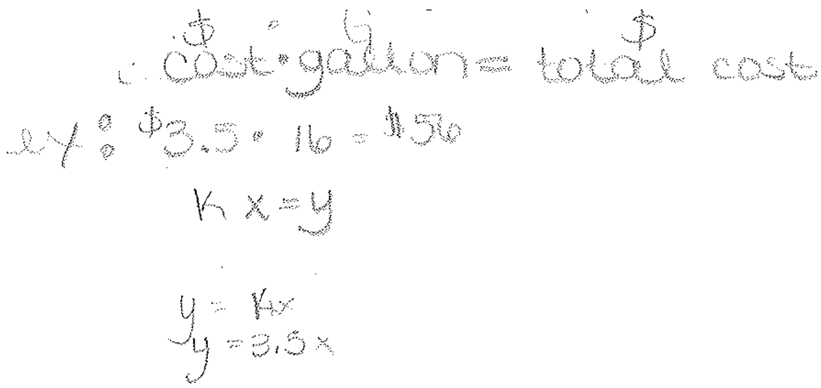


Figure 23. S1's answer to Part A of the Gasoline Problem in posttest.

I: For the gallon problem I asked, "What is the relationship between the dollar amount cost and the gallons being pumped?" And I noticed here that you wrote "cost × gallons = total cost" [and you gave the] example 3.5 ×16= 56, $kx = y$. Can you tell me your thinking there?

S1: Because the problem did not give any number, I knew that I would have to create my own numbers to show how I was thinking about the problem. When we did direct variation, I remember that k was always constant, so I [was] thinking about those problems. In this case, 3½ ($3.5) is always constant, but the x can always change, and the y could always change. So I thought about the problem in terms of when we did direct variation.

I: I noticed that you did not answer this question at all. How would you determine the rate in which the gas is being pumped into the car? Can you tell me how you would go about solving that now?

S1: No.

I: I noticed that when you constructed a graph price as the vertical for your range and gallons for your domain, you used the origin here to be (1, 3.5). Can you explain that?

S1: I made an x–y chart. I knew that at 1 gallon it was \$3½, so I just used the formula $y = kx$ and I kept the constant of k at 3½, and I just changed the x from 1 to 6, and whatever answer I got, that is what I plugged in at y, and I just graphed it based on that chart.

S1 showed progress in her thinking in that she was able to see the direct relationship between the dollar amount cost and the number of gallons of gasoline being pumped. She made use of an example to explain her thinking. She was also able to use her example and illustrate the graph here as well as in Figure 24.

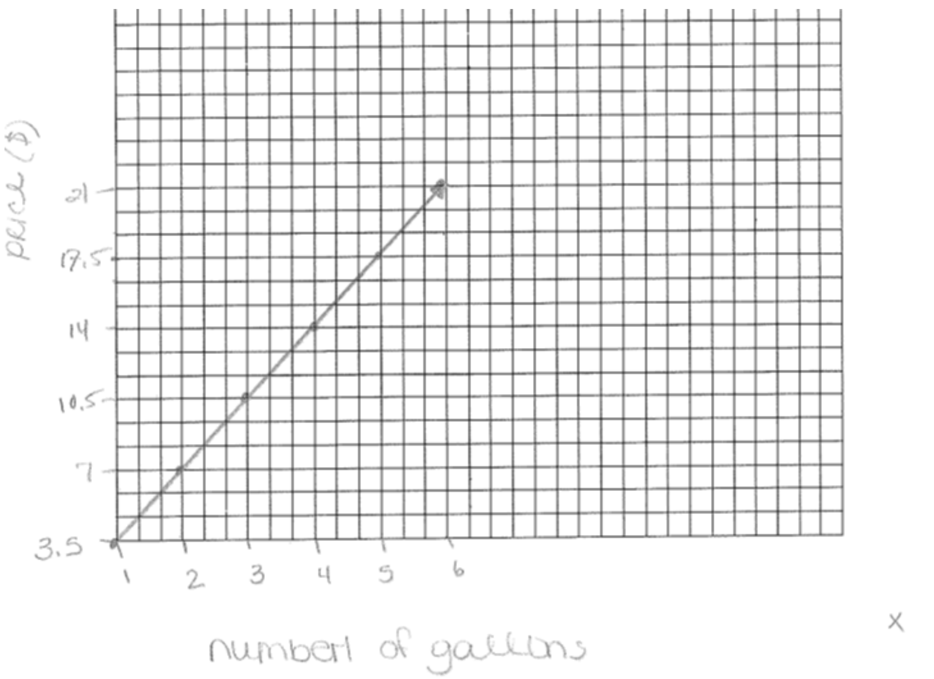


Figure 24: S1's linear representation between dollar cost and number of gallons.

The Runner Problem

S1 recognized that this was a direct variation problem, perhaps through memory of a college algebra course [if we have been learning it, then this must be direct variation. It was really a stab in the dark. I just plugged it in, *y* for time, *k* is constant, and *x* is the miles.] (see Figure 25). She worked this problem differently from what she did in the pretest. Given all of the unknowns, she found the constant *k* representing the unit ratio. Here is what she said about solving this problem:

> I: How did you come up with the equation that $y = kx$ on the runner problem?
>
> S1: That is what we have been learning the whole time about the direct variation, so the first thing that came to my mind was that if we have been learning it, then this must be direct variation. It was really a stab in the dark. I just plugged it in, *y* for time, *k* is constant, and *x* is the miles. That is how I solved. For *k* I got 7. Then I went back in and plugged in 7 for *k* and 8 for *x*, 7 × 8, and got 56 for *y*.

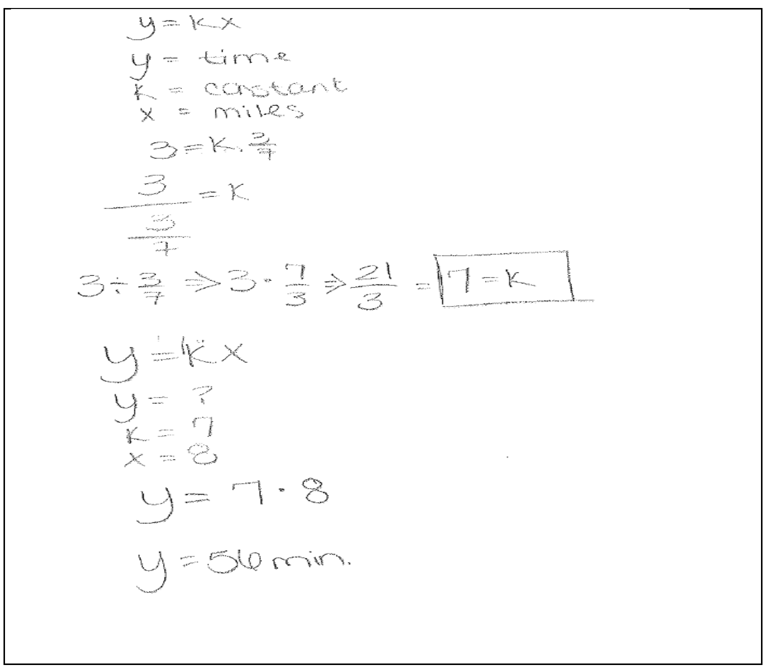


Figure 25: Student 1's solution to the Runner Problem.

Here she wrote *y*=*kx*, illustrating that it was a direct variation between *y* and *x*. She had *y* as time and *x* as miles, with the *k* being a constant. Her use of the word *plug* seemed to mean substitution. So she used a formula to help her solve the problem suggests that she was using a formula to solve the problem as opposed to using or applying proportional understanding.

Discussion S1’s Thinking about Pretest/Posttest/Homework Problems

The performance of the students (whole class) on the pretest and posttest showed that the questions had different levels of difficulty. Students (whole class) were more successful in solving the Baby-Weight, Gasoline, and Runner problems than the Snail and Currency problems. I compared S1's solutions to each pretest problem with the respective posttest problem. Changes in her conceptions of rates and linear equations were targeted. There were some changes in her thinking; however, she seems too relied on formulas as opposed to using proportional reasoning to solve problems relating to ratios and proportions. I then took a closer look at how she did across the problems to determine if any progress was made in her conception of rates and linear equations.

Pretest-Posttest by Item

S1’s understanding, reasoning, and communications, as illustrated by her problem solution to the Baby-Weight Problem, places her as an apprentice. She was able to construct a table of values for the baby’s age-weight relationship. She was also able to draw a graph to illustrate the linear relationship between the baby’s age and weight. She was not able to construct a multiplicative relationship between the baby’s age and weight. On the posttest she did show very good progress to expert level, as follows: She drew the graph correctly, constructed a multiplicative relationship for the baby’s age and weight, and created a table of values for the age-weight relationship. For this problem, she clearly progressed from an apprentice to expert. This problem was posed in a way that is typical of questions on linear equations in college algebra. Her college algebra experience and the reinforcement of this same concept from the class may have been a key to her rapid progress to expert level.

The Snail Problem solution path given by S1 characterized her as an advanced novice. Her understanding and reasoning were not correct. She had some idea that there would be a change in variation from a 2-*inch* per hour rate to a 3-inch per hour rate. This was evident by her incorrectly dividing the trip into two equal parts. The correct answer was achieved with incorrect thinking. That is, her division of the trip into two equal parts gave the correct distances traveled at each pace. For that reason, by coincidence, she was able to find the correct times for the snail to travel at the two different rates. Because of her lack of understanding, she could not draw a graph showing a change from one rate to the other. On the posttest, there was no positive change. She seemed to have regressed somewhat to novice as she was not able to produce a solution. She drew a straight line that represented a linear function. However, there was no change in variation in the graph. She confessed that she did not know how to solve the problem. The Snail Problem called for an understanding of rate. The change from the 2-inch per hour to the 3-inch per hour rate was a concept that she was unable understand in a way to reason or communicate; therefore, she drew a straight-lined graph and gave up on finding a solution. S1 struggled with rate throughout the semester. She was reliant on formulas, and in the Snail Problem there was no formula that she could use. Perhaps she knew that the two rates were related but was unable to produce two multiplicative relationships that would have produced a change from one rate to the other. For that reason, she may have given up in her attempt, knowing that she was unable to reason about the rates correctly.

S1's solution to the Currency Problem indicated that her conception was at the novice level. She was unable to correctly construct a table of values to represent a relationship between English pounds and U.S. dollars and was unable to construct a multiplicative relationship between the number of English pounds and the number of U.S. dollars. She incorrectly saw U.S. dollars as worth more than English pounds. She did know that there was a linear relationship between the number of English pounds and the number of U.S. dollars. On the posttest, there were some minimal improvements in her conceptions. She was still mostly at the novice level but was moving to an apprentice level: novice because she was still unable to determine the cost rate of a $20 garment in English pounds. She was also unable to construct a multiplicative relationship between the number of English pounds and the number of U.S. dollars. However, she was able to construct (incorrectly) both a table of values and a linear graph comparing the English pound with the U.S. dollar. Changes from the pretest to the posttest on the Currency Problem were minimal. The inaccurate construction of the table of values and linear graph did not show that she was able to recognize that there is a relationship between the two quantities.

S1's solution to the Gasoline Problem shows that her conception was at the advanced novice level. This was clearly illustrated in terms of her understanding, reasoning, and communications on the pretest. She was able to construct a multiplicative relationship to find the dollar amount cost and the gasoline being pumped. However, she could not determine the rate at which gasoline was being pumped into the car. On the posttest, she showed no improvements in her conceptions. She still was unable to determine the rate at which gasoline was being pumped into the car, and she offered no solution. There was no change in conceptions by S1 from the pretest to the posttest for the Gasoline Problem. She remained between practitioner and novice. She made use of her imagination, coupled with the understanding of the ratio of miles per gallon, as a means to construct solutions to parts of the question by constructing examples. There was no noted growth in conceptions. She remained at the advanced novice level. The lack of growth could be because the concept of rate was not developing conceptually in her understanding of problems that called for a change in variation.

The Runner Problem was difficult for S1 in the pretest. Her conception was initially at the novice level. She was unable to construct a multiplicative relationship between the two quantities, time and distance; however, she did write that 3 minutes was equal to 3/7 mile. S1's conception moved from novice to expert on the posttest. She showed that she understood the part-whole relationship, she was able to reason about the proportion in terms of miles per minute, and she was able to construct the multiplicative relationship between the distance traveled and time lapsed. There was great progress here. The part-whole relationship and the creation of the multiplicative relationship between the minute laps and miles traveled seem to be achieved.

The Improvements by Problems

S1's conception of proportions progressed from the pretest to the posttest on three problems; the Baby Weight Problem, and the Gasoline Problem. These problems all required constructing a multiplicative relationship as well as graphing that relationship. There were no assigned values given in the Gasoline Problem: S1 assigned values for dollar amount cost and gallons pumped. This assignment allowed her to construct a linear graph as well as help her reason about the relationship between dollar amount cost and gallons of gasoline being pumped. In both the pretest and posttest, she was unable to determine the rate at which gasoline would be pumped into a car. With the Baby-Weight Problem, she was able to generate a table of values and construct a multiplicative relationship between the baby's age and weight.

S1's conception of rates did not progress on two problems: the Snail Problem and the Currency Problem. In the pretest, her thinking on the Snail Problem was incorrect, as follows: She divided the snail's journey into two equal parts and then reasoned about the proportions as two separate problems. That is one for each rate. On the posttest, she incorrectly drew a linear representation for the snail's journey in inches-per-hour. She offered no more solution. She did not want to attempt the problem because she thought she was unable to solve.

CHAPTER 5

STUDENT 2

S2: Case 2

At the beginning of the semester, S2 was given a pretest (Appendix A) to assess his understanding of the concepts involving ratios, proportions, and rates and multiplicative understanding. He was asked to solve five problems: (1) the Baby-Weight problem, (2) the Snail problem, (3) the Currency problem, (4) the gasoline problem, and (5) the Runner problem. His response to each question was analyzed, and the peculiarities found were used to create an individual interview protocol to probe his thinking during the interview process. The narrative starts with the five pre-test problems in the order listed above.

The Baby-Weight Problem

Betty weighed 6 pounds at birth. She gained 2 pounds every month for 8 months.

A. Create a table to represent the data over an 8-month period.

B. Draw a graph to represent the data, identify the input and the output.

C. As Betty gets older, how does her age relate to her weight?

D. Construct an equation to model the age-weight relationship.

S2 was decisive in solving the Baby-Weight Problem. He correctly constructed the table asked for in part A. His table started at birth and ended at 8 months, therefore covering the entire time period for the data. What was interesting is that on his graph, his origin was (0, 6) (see Figure 26).

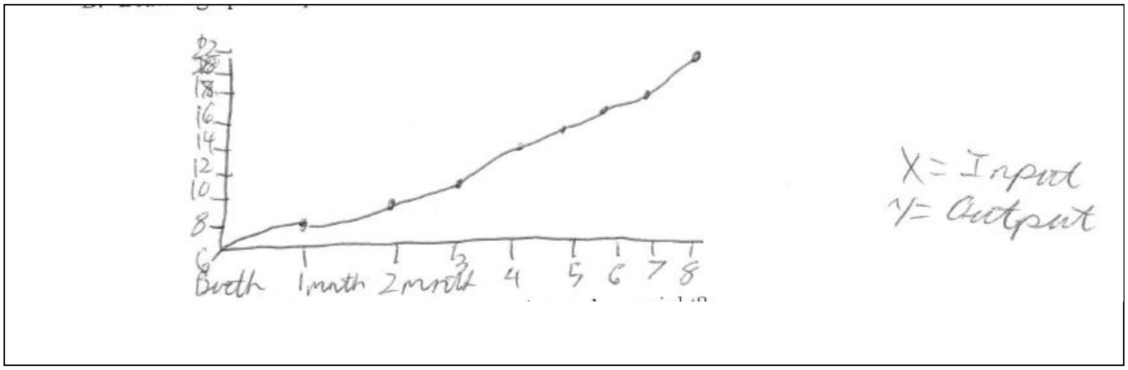
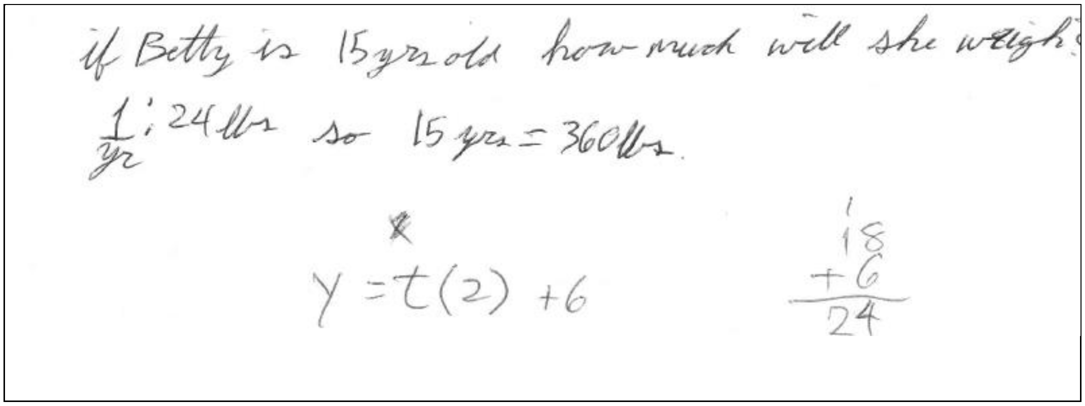


Figure 26. Graph drawn by Student 2.

S2 was able to clearly write out the ratio and construct the linear equation (see Figure 27). This is because he established a ratio between the number of months and the weight of the baby at each month. He gave an example where he predicted Betty's weight at age 15 without taking into consideration her initial birth weight of 6 pounds, so his reasoning went beyond what the question was asking. His thinking does indicate a conception of ratio and proportional reasoning. Proportional reasoning is a form of mathematical reasoning that involves a sense of covariance and multiple comparisons, and the ability to mentally store and process multiple pieces of information.



if Betty is 15yrs old how much will she weigh?
$\frac{1}{yr}$: 24lbs so 15 yrs = 360lbs.

$y = t(2) + 6$

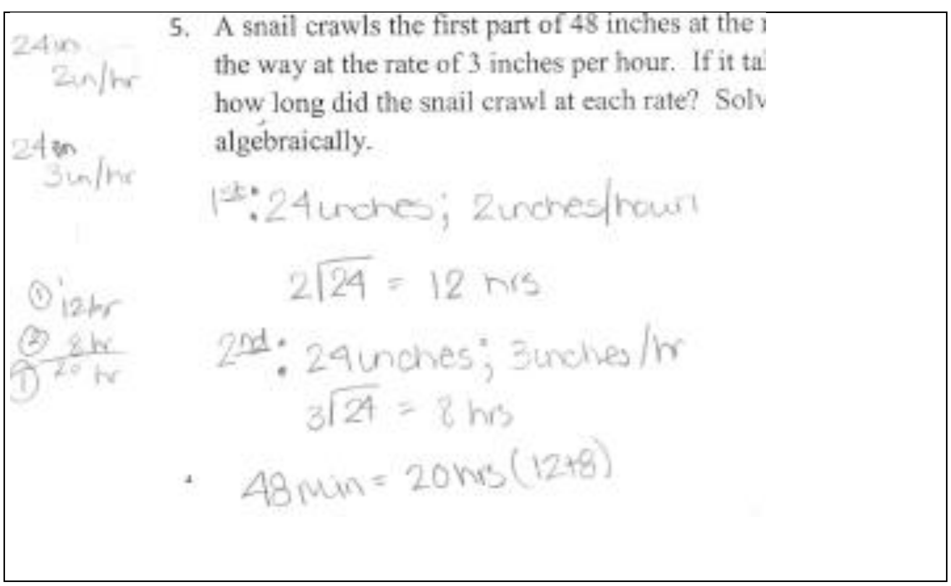
$\begin{array}{r} 18 \\ +\ 6 \\ \hline 24 \end{array}$

Figure 27. Student 2's construction of a linear equation.

The Snail Problem

> A snail crawls the first part of 48 inches at the rate of 2 inches per hour. He crawls the rest of the way at the rate of 3 inches per hour. If it takes the snail 20 hours to crawl all 48 inches, how long did the snail crawl at each rate? Solve the problem graphically and then algebraically.

S2 divided the snail's journey into two 24-inch distances. He then divided 24 by 3 and 24 by 2 (see Figure 28). His use of the unit ratios was an issue I further explored.



24 in
2in/hr
24 in
3in/hr
① 12hr
② 8 hr
① 20 hr

5. A snail crawls the first part of 48 inches at the r
the way at the rate of 3 inches per hour. If it ta
how long did the snail crawl at each rate? Solv
algebraically.

1st: 24 inches; 2 inches/hour
$2\overline{)24} = 12$ hrs
2nd: 24 inches; 3 inches/hr
$3\overline{)24} = 8$ hrs
48 min = 20hrs (12+8)

Figure 28. S1's solution to the Snail Problem.

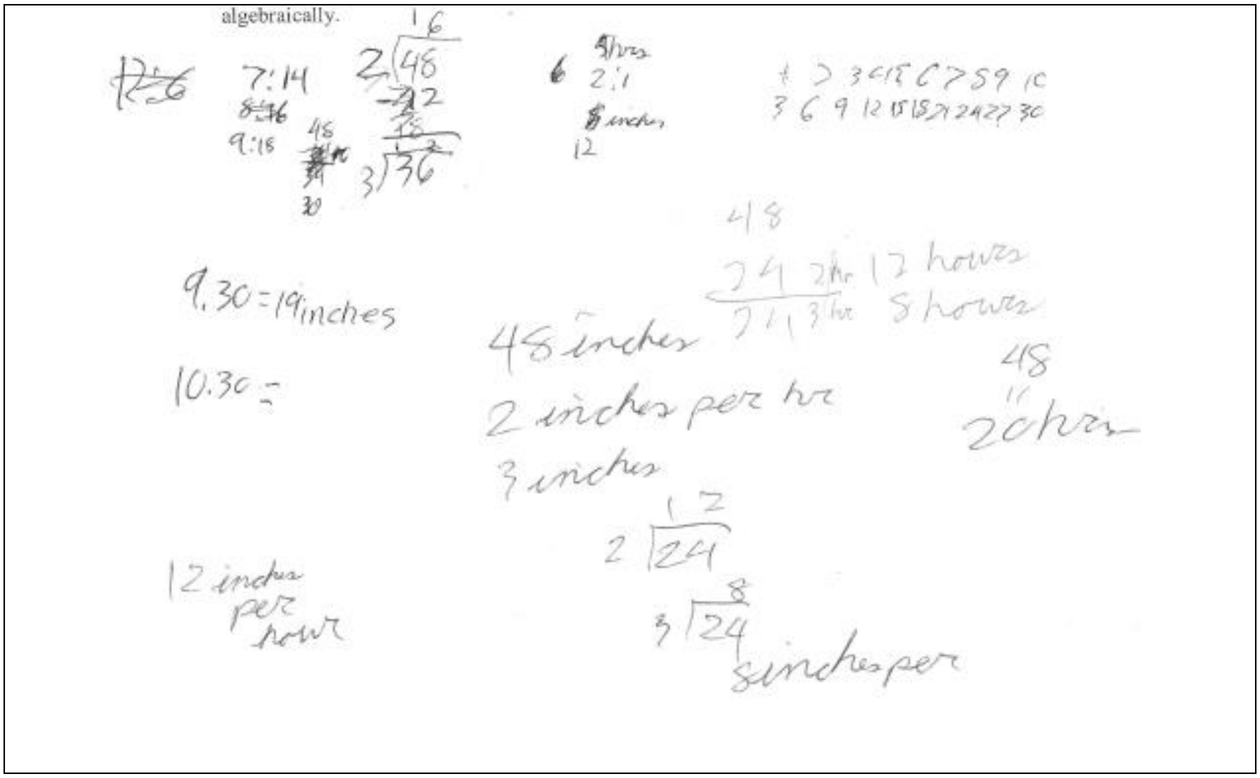


Figure 29. S2's solution to the Snail Problem.

Protocol III: (Student2 Pretest Interview- lines 53—88)

I: On the Snail Problem, can you explain you're thinking on this problem?

S2: That was probably the hardest one for me.

I: In particular, why did you say that 2 divided by 48, why did you divide by 2?

S2: Because they said that the snail was crawling 2 inches per hour and I wanted to see how many hours it took him, which is 48 inches, because he ran the first part of 48 inches at a rate of 2 inches per hour. I figured that I would divide the inches per hour into the total distance to get the hours it took him to complete.

I: What is your thinking on the first part? When you saw the first part, how long did you think it would be?

S2: I just thought about 12 inches equal a foot. I just knew that off the top of my head and added inches left; that added another foot. Since they gave it to me in inches, I just had to convert it. I divided the 2 into 48.

I: Tell me your thinking …are you looking at this, that half of the trip was 12 inches?

S2: No. The first part says he crawled the first part of 48 inches at a rate of 2 inches per hour. What I was thinking was he crawled 48 inches at a rate of 2 inches per hour, and he still had time to go.

I: I only see 9.30 … equal 19 inches, 10.30…equals then nothing; tell me your thinking behind that?

S2: When I read further into the question, it says the snail crawled the rest of the way 3 inches per hour, and I continue reading and it said if it takes the snail 20 hours to crawl 48 inches, how long did the snail crawl at each rate. So then I had to go back and look in my thinking. At first I thought the snail ran the 48 inches at 2 inches per hour. When in fact he ran 2 inches per hour for a certain amount of time and the rest of it 3 inches per hour, so I was trying to figure out how long of a distance he could run, evenly breakup 48 inches at a rate of 3 inches per hour and add into the part I broke up at 2 inches per hour evenly.

I: So you're looking at the two parts of the trip not being a first part equaling a half and the second part equaling a half?

S2: Not at first. At first I was trying to find numbers that would add up to 20. I wasn't looking at the equation fully at first. After a couple of trial and error sessions, you could see where I had 12 inches per hour, 10.3= nothing; after I sat there and looked at it for a while, I made another chart at the top. I sat and wrote out the multiplication charts. When I got to a point where it equaled 48 for both of them, I looked at what number times 2 = 24 what number times 3 = 24. That's how I got my hours. I looked for the greatest common factor. For 2 it was 12, because 2 times 12 equal 24. Then 3, it was 8. 3 times 8 equal 24. 24 plus 24 gives you 48 hours. Once I figured out 48 hours, I looked at my time to see if they would add up to 20. So 12 hours plus 8 hours would give you 20. So that's how in the end I came up with my answer.

S2 did not graph the relationship between the two rates at the point where the rate changes from 2 inches per hour to 3 inches per hour. He did try to find a time where there was a change in pace. He assumed that the snail's pace changed at the halfway point along with their dividing 24 by 2, which does indicate ratio thinking. S2 did produce markings that indicated an awareness of motion: 1-3, 2-6, 3-9, 4-12, 5-15, 6-18, 7-21, **8-24**, 9-27, and 10-30. Then below that he wrote 24, 2-12 hours and 3-8 hours. When S2 divided 24 inches by 3 inches/hour, he seemed to be thinking in terms of fixed quantities, not in terms of varying quantities. However, his markings showing the correspondences of the 1-to-3 ratio, along with his arithmetic calculations made by his assumption to divide the trip into two equal parts, indicate a discrete coordination between time, distance, and ratio rather than coordination through time and distance. Further, he made no graphical representation.

Student 2's Initial Conception up to This Point

S2's work in the Baby-Weight problem indicated that he had constructed a concept of ratio and that he could reason proportionally. However, his work in the Snail problem revealed that his concept of ratio did not entail partitioning or division. Rather, his concept of ratio seemed based on the comparison of one quantity to another under the provision that the ratio would be conserved in another comparison of the same two quantities. For example, he reasoned that if in 1 year the baby weighed 24 pounds, then in 15 years the baby would weigh 360 pounds. In that there was no indication in the Baby-Weight problem that he could find how much the baby weighed, say, in 2 1/3 months, his concept of ratio could not be said to involve partitioning or division. This observation is corroborated by his work in the Snail problem in that he divided 24 inches by 2 inches per hour and initially produced 12 inches per hour rather than simply 12 hours. S2 took this interpretation of the quotient as indicative of a conception of 2 inches per hour as 2 inches in each hour. In that in the last part of Protocol III, S2 conflated hours and inches [24 plus 24 gives you 48 hours. Once I figured out 48 hours, S2 looked at my times to see if they would add up to 20. So 12 hours plus 8 hours would give you 20. So that's how in the end S2 came up with my answer.]. The conflation corroborates my interpretation of S2's conception of ratio. Still, he seemed more advanced than S1 in that he could reason proportionally.

The Currency Problem

> The Currency Problem: Suppose 2 English pounds (£) buy the same goods as 3 U.S. dollars ($).
>
> A. Make a conversion table to compare the two currencies.
>
> B. What would a garment of $20 cost in terms of pounds?
>
> C. What's worth more, the pound or the dollar to an Englishman just off the plane on vacation in the USA?
>
> D. If p = the number of English pounds and d = the number of U.S. dollars, write the relationship between the number of pounds and the number of dollars? Explain.
>
> E. Graph this relationship using graph paper.

S2 constructed a table to compare English pounds to U.S. dollars. His table was missing a unit representation of the relationship between the English pound and the U.S. dollar, thus giving him an incomplete conversion table in that it did not show that 1 English pound was equal to 1 U.S. dollar and 50 cents. His table represented only ratios whose terms are multiples of 2 and 3 (see Figure 30) because he did not take into consideration a unit of comparison (e.g., the ratio 1 to 1.5), therefore failing to potentially produce, for example, all even and odd numbers of English pounds.

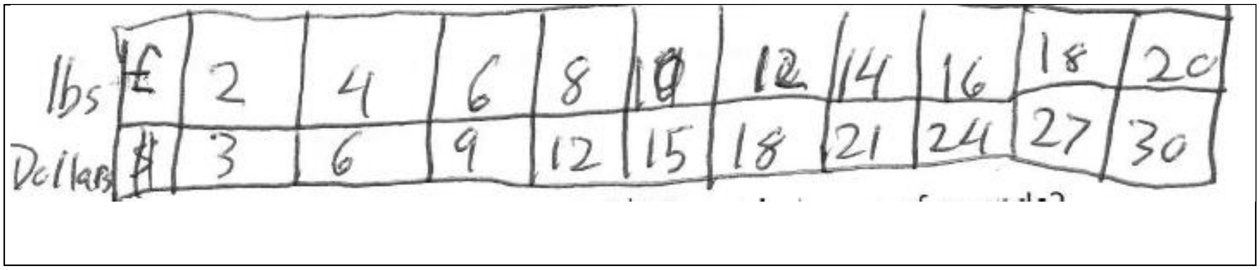


Figure 30. S2's table comparing English pounds to U.S. dollars.

He was, however, able to use the understanding of a unit ratio to find the rate which a $20 garment would cost in English pounds. Even though his table left out 1 English pound to correspond to $1.50 in U.S. currency, he was able to write it out and make the calculations that $\$20 = 13\frac{1}{3}\,lbs$ (sic) (see Figure 31).

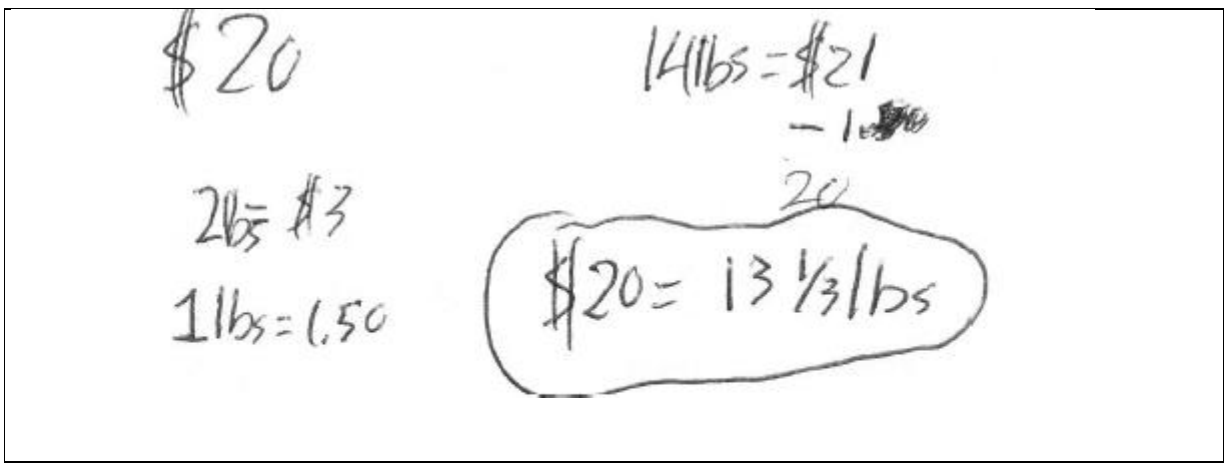


Figure 31. S2's solution to Part D of the Currency Problem.

He did see that the English pound was worth more in terms of buying power. This was indicated by his ability to construct the multiplicative relationship between the two currencies (see Figure 31) as well as noticing that £1=$1.50 (see Figure 32). In Part D it appears that Student 2 clearly understood the ratio between the buying power of English pounds and U.S. dollars.

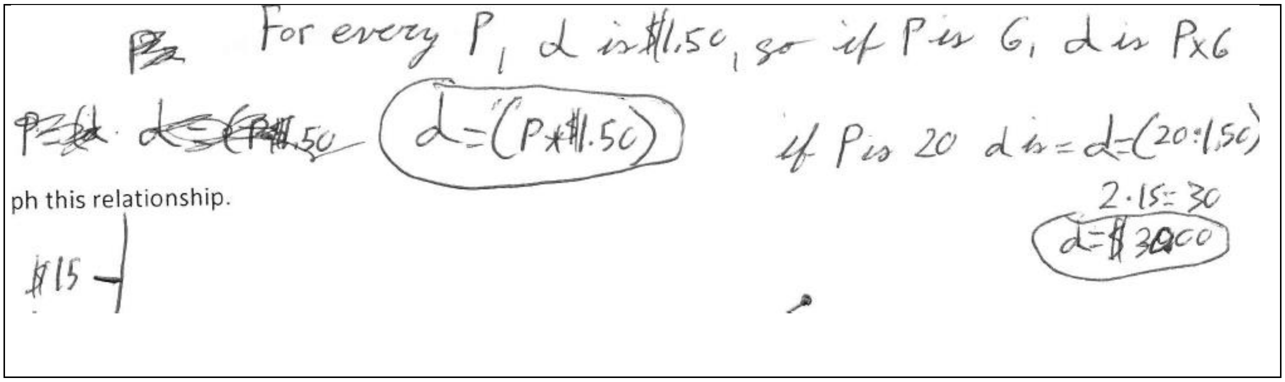


Figure 32. S2's multiplicative relation.

S2's comment, "For every p, d is \$1.50, so if p is 6, d is p×6," indicates that he has a concept of variable and that he thinks of the unit ratio as a rate. He had no problem drawing this multiplicative relationship between the buying powering of the two currencies (see Figure 33).

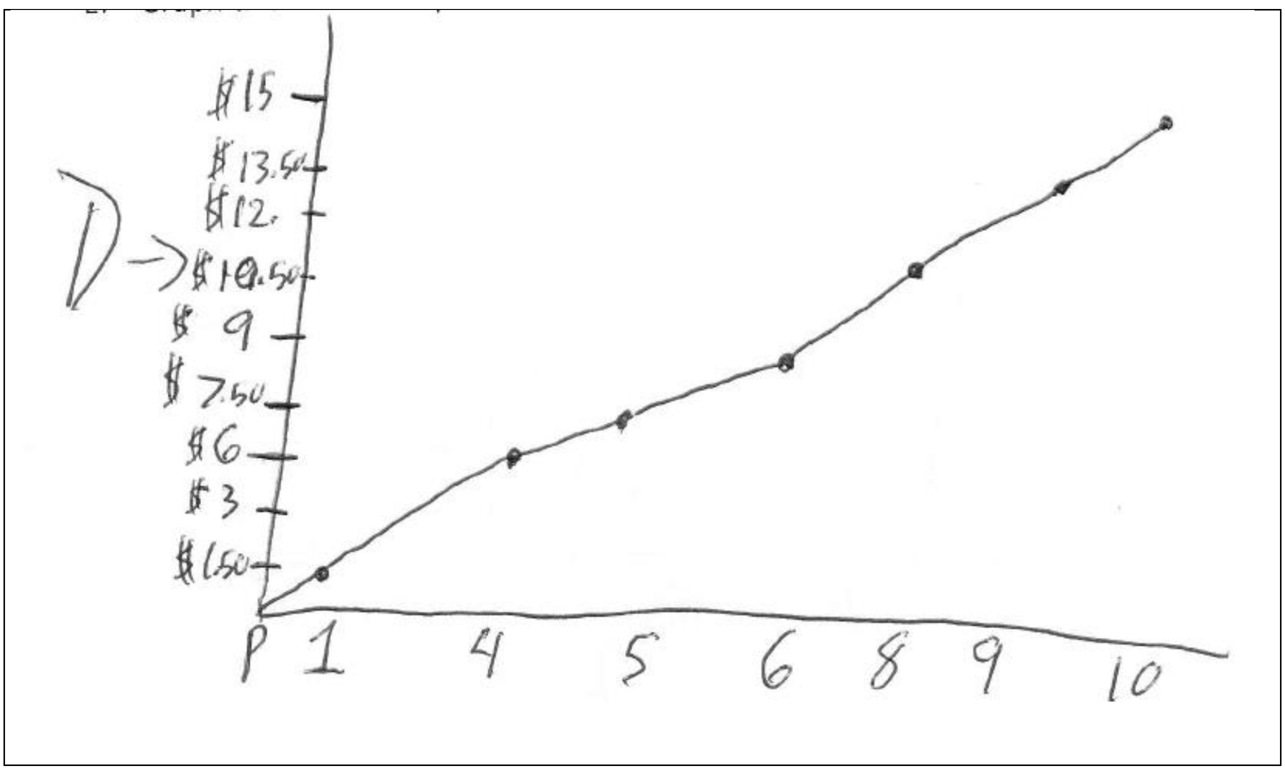


Figure 33. S2's graphical construction of the dollar-pound relationship.

The Gasoline Problem

Imagine pumping gasoline into a car.
A. What is the relationship between the dollar amount cost and the gallons being pumped?
B. How would you determine the rate at which the gasoline is being pumped into the car?
C. If c=cost and g=the number of gallons pumped, sketch the graph to illustrate the cost of the gasoline.

S2 did not state the variables. Instead he gave examples for rate. One of his examples was gallons per dollar. However, he was able to determine the cost of gas pumped (see Figure 34). He was able to show an understanding of variation using rate.

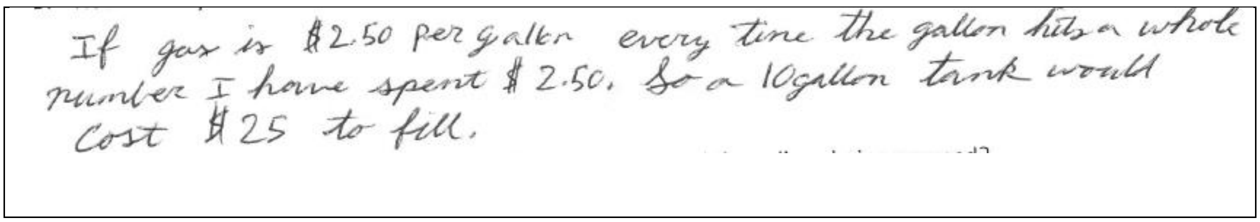
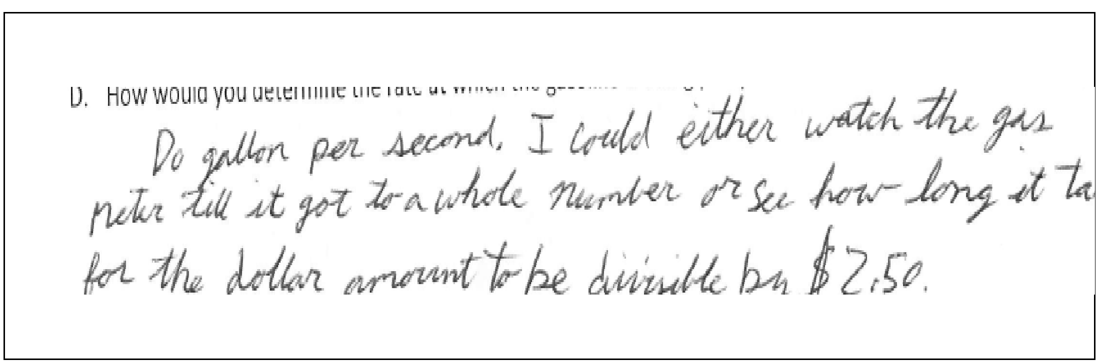

If gas is \$2.50 per gallon every time the gallon hits a whole number I have spent \$2.50. So a 10gallon tank would Cost \$25 to fill.

Figure 34. S2 shows his understanding of the direct relationship between costs per gallon.

He created a multiplicative relationship in the form of a ratio for the cost and gallons. He used a ratio of gallons to dollars (1 gal: \$2.50). S2 gave an example to clarify his reasoning (see Figure 35).


D. How would you determine the rate at which the

Do gallon per second. I could either watch the gas meter till it got to a whole number or see how long it ta
for the dollar amount to be divisible by \$2.50.

Figure 35. S2 explains how to determine the rate at which gasoline is being pumped.

S2 drew a bar graph to represent cost per gallon (see Figure 36). Even though linearity is implicit in S2's bar graph representation of the data, he used a bar chart which is used to represent qualitative data. In this case the data was quantitative and continuous.

Figure 36. S2 representing linear data using bar charts.

The Runner Problem

A runner is running along and times how long she has run and how far she has run using a stopwatch and a distance meter. If one reading is 3 minutes and 3/7 miles, how long would it take her to run 8 miles if she runs at a constant pace?

S2 did misuse the equal symbol but was able to use his understanding of ratios to solve this problem. In 3 minutes he knew the runner would be at 3/7 miles and in 7 minutes the runner would have traveled 1 mile. So he was able to use his understanding of unit to solve the problem. One mile is traveled in 7 minutes; therefore 8 miles would be traveled in 56 minutes (see Figure 37).

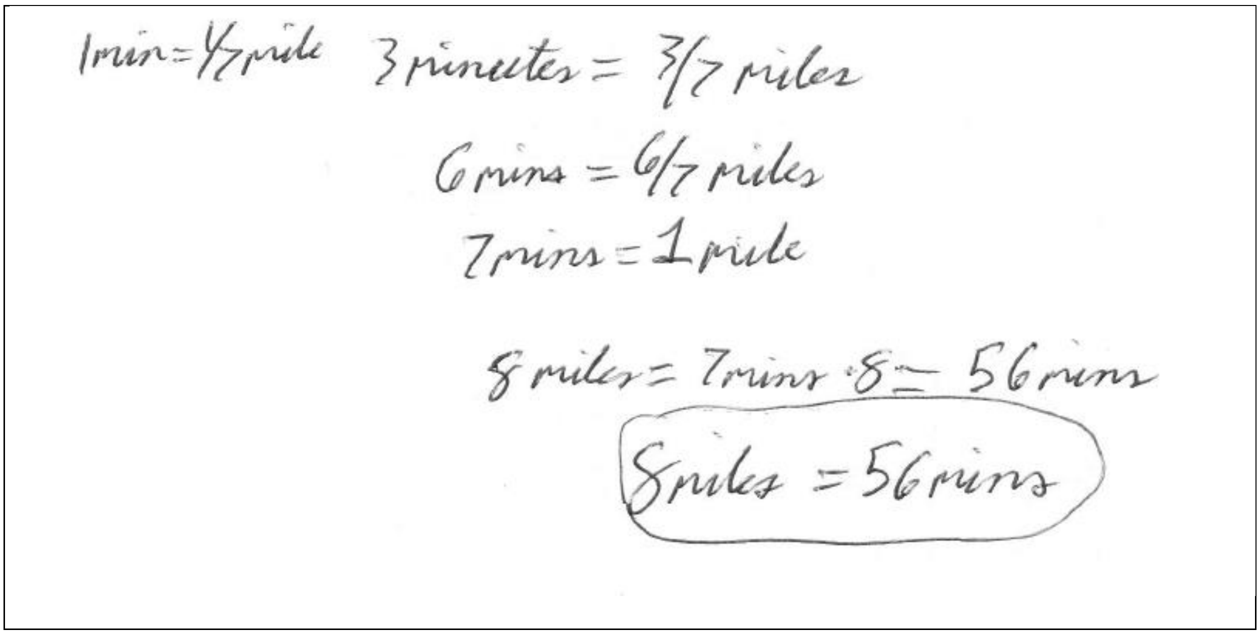


Figure 37. S2's solution to the Runner Problem.

Ratio and Proportions Homework (Homework I)

As a reminder, Homework I had three problems: the Orange Juice Problem, the Log Problem, and the Chicken Problem. S2 set up the wrong proportion in the Orange Juice Problem but worked the Log Problem correctly. These two problems were not worked well by many of the pre-service teachers, so I have highlighted S2's solutions to illustrate a common difficulty with the Orange Juice Problem as well as his distinction between the number of cuts and the number of pieces in the Log Problem.

The Orange Juice Problem

> Frozen orange juice concentrate is usually mixed with water in the ratio of 4 parts water to 1 part concentrate. How much orange juice can be made from an 18-oz can of concentrate?

S2's solution was incorrect. He set up the ratio as 4w:1c to mean the ratio between the water and concentrate. That is four parts water to one part concentrate. Then he writes 4½, stating, "because when you divide 18 by 4 you get 4 1/2." S2 divided 4 into 18, indicating that he inverted the ratio of water to concentrate in the amount of water that corresponded to the 18-oz can of concentrate.

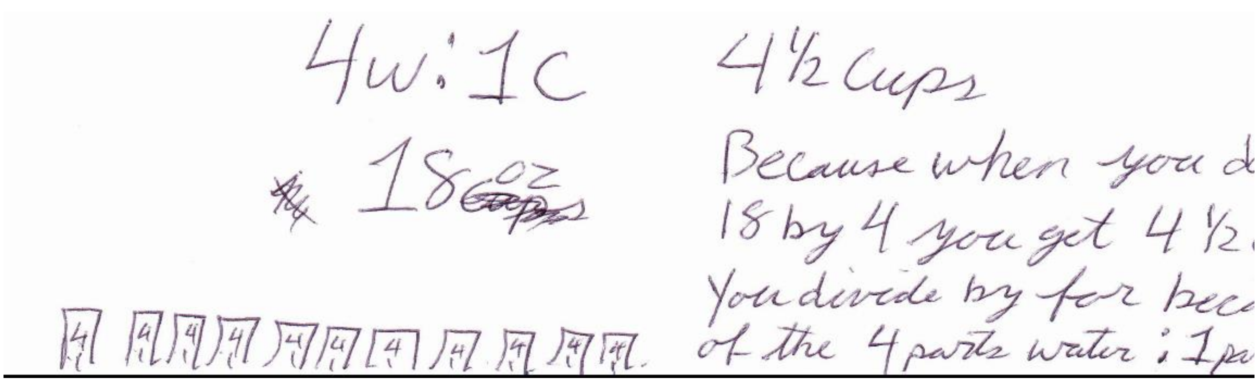


Figure 38. S2's solution to the Orange Juice Problem.

The Log Problem

> It takes 30 minutes to cut a log into five pieces. How long would it take to cut a similar log into six pieces?

S2 was among the few students who solved this problem correctly (see Figure 39).

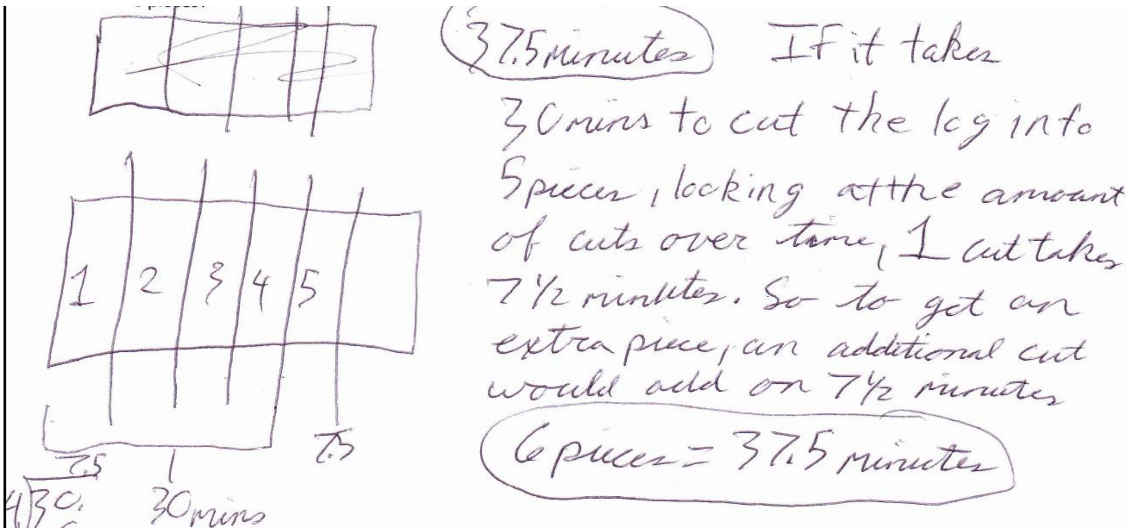


Figure 39. S2's solution to the Log Problem.

S2's graphics show that he knew that it took 30 minutes to cut the log into five pieces. He focused on the number of cuts to get at the number of pieces. That is, four cuts would create five pieces and five cuts would yield six pieces.

Currency Problems (Homework II was motivated by Steffe's Notes)

There were two currency problems assigned for students to complete and turn in. These problems were discussed in class within the context of the notes. The first problem was, as follows:

> Suppose that 3 Argentine Pesos is equivalent to 4 Danish Kroner. Use the Danish Kroner as the independent variable.
>
> A. Represent this in the form of a linear equation.
>
> B. What is the slope of the line?

This problem was motivated by the following problems from Steffe's notes.

> Three Argentine Pesos are equivalent to 4 Danish Kroner. Establish a multiplicative relationship between the two currencies so that any number of Kroner can be exchanged for Pesos, or vice versa. Make a variable point (p, k) on a graph, where the number of Kroner is the independent variable and develop a conversion table as well. If the number of Kroner k = -14, what is the corresponding value of the number of Pesos, **p**? What does it mean for **k** = -$14? What does the corresponding value of **p** represent? (p. 2)
>
> A. Interchange the independent and dependent variables so that (k, p) is the variable point. What is the relationship between the two graphs?

S2 creates a multiplicative relationship between the number of Argentine pesos and Danish Kroner (see Figure 40).

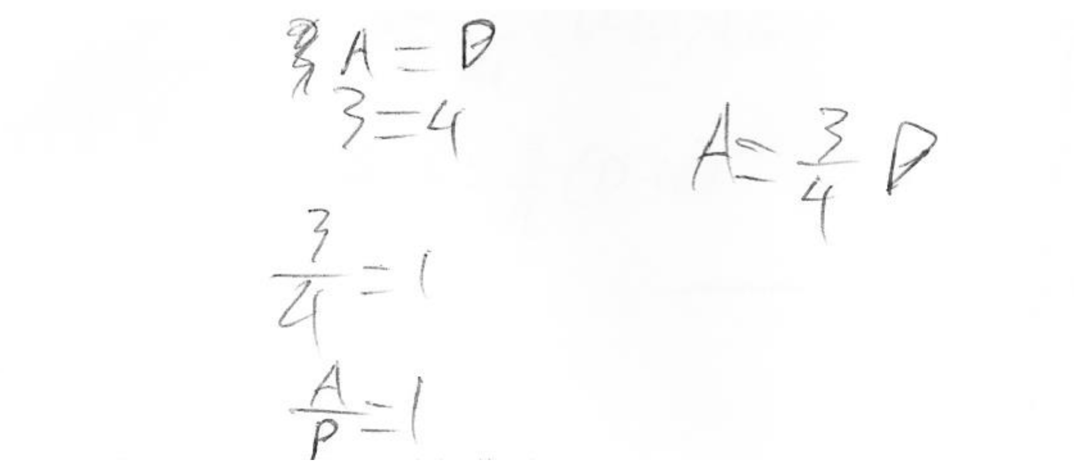


Figure 40. S2's construction of the multiplicative relationship between Pesos and Kroner.

The next problem was similar to the first problem:

> In (1), the Argentine embassy decides that all Danish visitors to Argentina must hold back 10 Danish Kroner and not convert them to Argentine Pesos. Using the x-y coordinate plane, make a graph of the number of Pesos that each Danish visitor receives with respect to all the Kroner the visitor brought to Argentina. Be sure to develop an equation.

S2 worked this problem as a continuation from the first problem. He saw the ratio, which he knew to be the slope, ¾, as a factor in the product involving the difference between Danish Kroner and the 10 Danish Kroner being held back (see Figure 41). Here he represents the relationship both graphically and symbolically. Graphically his equation seems to shift 10 places left passing through (10, 0), and from there he has the constant slope of a rise of 3 units and a run of 4 units. His understanding of variation as it related to the transformation was clear.

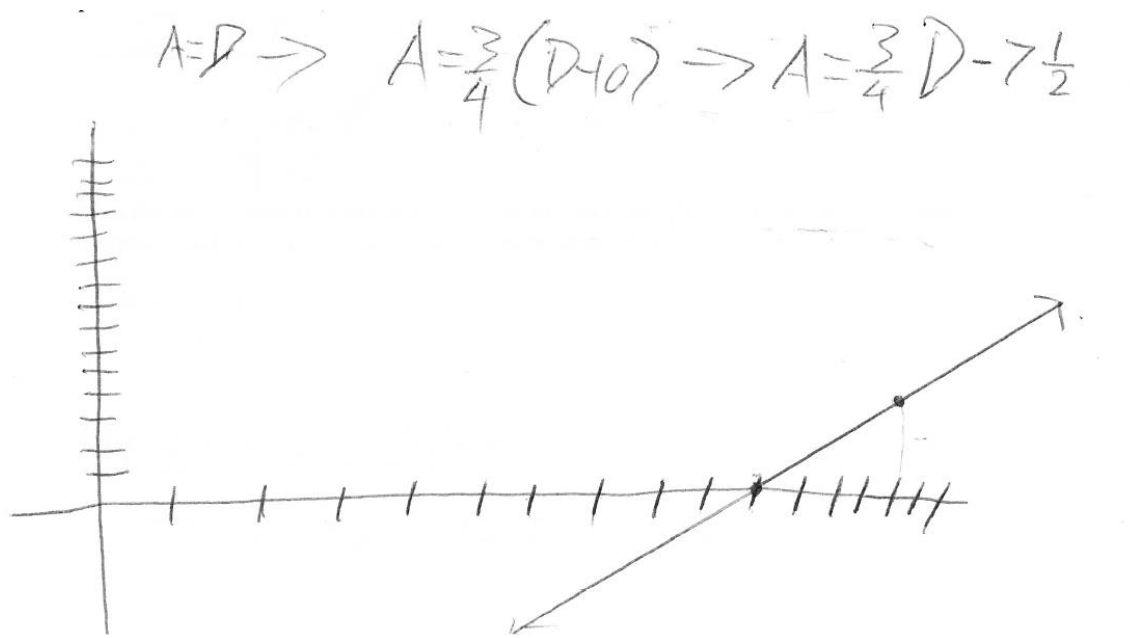


Figure 41: S2's solution to the Currency Problem, number 2.

Take Home Quiz (Homework II)

Problem 1: A race car driver drives part of a 500-mile race at 150 miles per hour and the remainder at 180 miles per hour. If it takes her a total of 3¼ hours to complete the race, how long at each rate did she drive? Solve the problem graphically and algebraically.

S2 was able to produce two proportions to represent the two different rates. However, he did not produce a graph that would show change in variation from a 150-mile-per-hour rate to the 180-mile-per-hour rate. He found that for 2 and 5/6 hours, the car driver would be traveling at 150 miles per hour and for 5/12 of an hour at 180 miles per hour (see Figure 42). His solution to the race car problem was quite insightful because he reasoned that 180 miles is to 1 hour as the total distance he traveled is to the total time he traveled. Further, the total distance he traveled at 180 miles per hour was 500 miles less the total distance he traveled at 150 miles per hour, and the total time he traveled at 180 miles per hour was the total time he traveled less the time he traveled at 150 miles per hour. This proportion exemplifies that he had constructed the

$$\frac{180}{1\,hr} = \frac{d}{t}$$

$$d = 180t$$

$$\frac{180}{1\,hr} = \frac{500-d}{3.25-t}$$

$$\frac{180}{1\,hr} = \frac{500-150t}{3.25-t}$$

$$585 - 180t = 500 - 150t$$

$$\frac{-500}{85} = 30t$$

$$t = 2\tfrac{5}{6} \text{ at } 150 \text{ miles}$$

$$\frac{500-180}{23.\tfrac{13}{12}} \quad -2\tfrac{10}{12} = \tfrac{5}{12} \text{ hrs at } 180 \text{ mph}$$

Figure 42. S2 algebraic solution to the Race Car Problem in Homework II

concept of variable quantities, co-variation between two variable quantities, and whole-to-part and part-to-whole reasoning as well as the concept of rate. S2 had constructed multiplicative relationships using ratios, as well as an understanding of linear equations.

Student 2's Thinking on the Problems after the Posttest

Baby-Weight Problem

As in the pretest, S2 was able to complete this problem correctly.

Snail Problem

S2 was able to complete this problem correctly. He solved the problem by graphing the snail's entire journey. Unlike in the pretest where he got the correct answer with the wrong thinking, he did something different–he changed his scale to adjust the rate from a 2-inch-per-hour to a 3-inch-per-hour pace (see Figure 43). Here are the solution path that he laid out and the discussion on his thinking (see Figure 44).

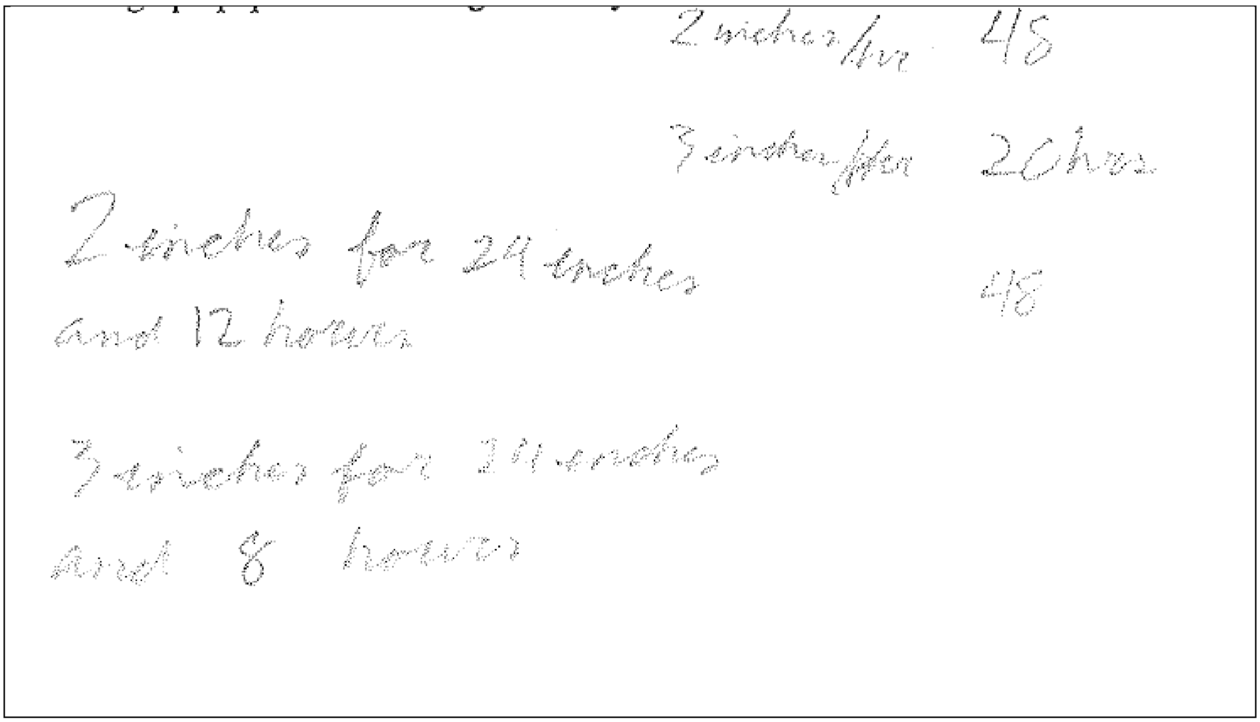


Figure 43: Student 2's solution to the Snail Problem.

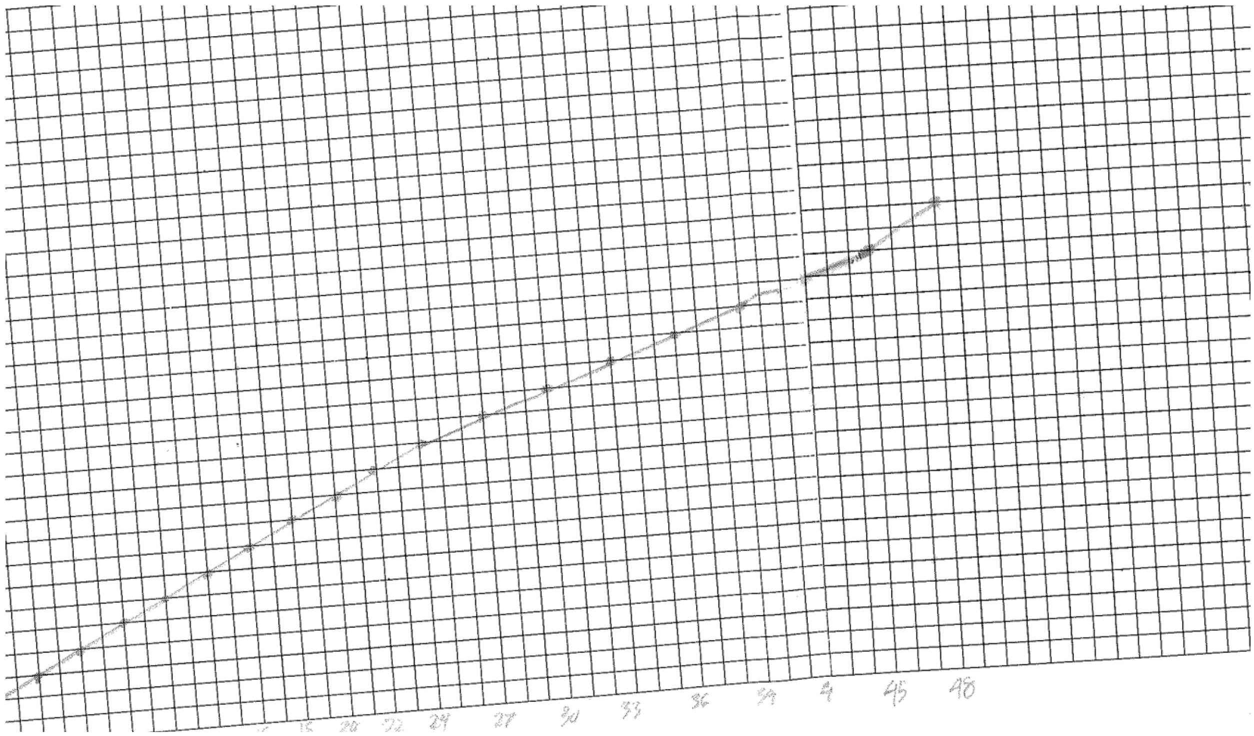


Figure 44: Student 2's graph of the snail's change in pace.

I: Tell me your thinking on how you arrived at 2 inches for 24 inches in 12 hours and 3 inches for 24 inches and 8 hours.

S2: What I did was go all the way through the race with both numbers, the 2 inches per hour and the 3 inches per hour. I went the whole way with 2 inches for 48 inches, and I went the whole way 48 inches for 3 inches. When I put the two graphs together they overlapped; they generally intersect at around 24 inches, so I just used 24 inches for each one.

I: Your graph is time against distance. Is there any reason that you put distance as input and time as output?

S2: I feel that they were looking at time put in for distance, so however far you travel, you get your output as time, so that is what I thought they were looking for. That is why I have distance as input and time as output.

I: I can see some points on the graph; can you explain to me how you plotted these points?

S2: What I did was went from 0 at a rate of 2 inches per hour, and when I got to the 24 point, that is my point of change, and when I got there I went at a rate of 3 inches per hour. That is where my point of change occurred at 24.

I: On your domain axis can you explain the change in numbers?

S2: The change in number is the change in speed or the change in distance covered. From 0 to 24 you are going at a rate of 2 inches per hour, but when you get to 24, you have your point of change going at a rate of 3 inches per hour; it goes from 20, 22, 24; once you get to 24 it goes to 24, 27, 30, 33. It goes from counting by 2's at 24 to counting by 3's.

I: Did you change the scale there?

S2: Yes I did.

I: So how many lines represent 2 inches per hour?

S2: There are two lines that represent 2 inches per hour. I drew it to scale from 0; you have two lines that is 2 and then 2 more lines that is 4; you go all the way counting by two lines until you get to 24. Once you get to 24, you count by three lines because you are going at a rate of 3 inches, so each line represents an inch.

I: How can you tell from your graph that there are two rates, 3 inches per hour and 2 inches per hour?

S2: If you look at the x-axis, you can tell simply by the number of lines used to represent the 2-inch rate and then the lines used for the 3-inch rate. If you look at the number from 22 to 24 you see two lines. If you look from 24 to 27, you see three lines. So you can see the actual change represented in the graph.

I: At the 24-inch mark, I noticed there is a visible change there; can you elaborate on that?

S2: That is the point of change where I change pace. When I had my two graphs individually that is the point there they intersected. What I did from there, that is where you see me go from 2 inches per hour to 3 inches per hour. So that is the actual change you see, is the change in rate.

S2's work in the Snail Problem confirms that he could reason proportionally. Unlike the pretest, he was able to draw a graph that showed a change from a 2-inch-per-hour rate to a 3-inch-per-hour rate even though the point of change was arrived at intuitively. Although he did not write out the solution by using the rates in the same way as in the pretest, by noticeably dividing by 2, he seemed more aware of the relationships involved in the co-variation between distance and time at each rate.

The Currency Problem

S2's solution solidly indicates that he reasoned proportionally and was able to explain his solution to the problem.

> I: Now let's move to the currency problem. On 3b can you explain to me how you got 13 1/3 pounds; take us through all the things that you considered.
>
> S2: The first thing that I did is to see if $20 could be represented visibly on my chart that I came up with in problem a. I felt that I needed to stay constant with the table that I came up with. I looked at my table; I went from 1 pound to 10 pounds. At 10 pounds I saw that would be $15, so I tried to see how far I could go to get from $15 to $20. I saw that 14 pounds would give me $21, and I knew that would be too much. I took 10 pounds and 3 pounds and added them together, and I got $19.50. What I did from there is I looked at $20-$18… looking at my chart $18 would give me 12 pounds. Then I was left with 2 pounds, so I could subtract 1 pound, and 1 pound is $1.50 and I would be left with $.50. Reducing the 1 pound and the $1.50 to get $.50, I saw that $.50 was 1/3 of 1 pound. That is how I got 13 1/3 pounds.

His reasoning about the proportions was done well on both the pretest and the posttest. On the posttest he provided more explanation and detail. Although he did not take into account the unit measure of the currencies in the pretest, where his table showed multiples of 2 and 3, he constructed a table clearly showing his understanding of the unit (see Figure 45).

£	1	2	3	4	5	6	7	8	9	10
$	1.50	3	4.50	6	7.50	9	10.50	12	13.50	15

$2P = 3d \quad \frac{2}{3} = \frac{P}{d} \quad \frac{3}{2} = \frac{d}{p}$

Figure 45. S2's table representing the relationship between English pounds and US dollars.

He correctly graphed the relationship between the number of English pounds and the number of U.S. dollars; constructed a multiplicative relationship between the number of English pounds and the number of U.S. dollars; and articulated that the relationship was direct (see Figure 46).

$d = \frac{3}{2}p$ The amount of dollars (d) varies directly with the number of pounds (p). K is constant $\frac{3}{2}$

Figure 46. S2's multiplicative relationship between English pounds and U.S. dollars.

The Gasoline Problem

S2 worked this problem much differently from what he did in the pretest. He used direct variation to solve and explain. Here is his work and dialogue on his thinking.

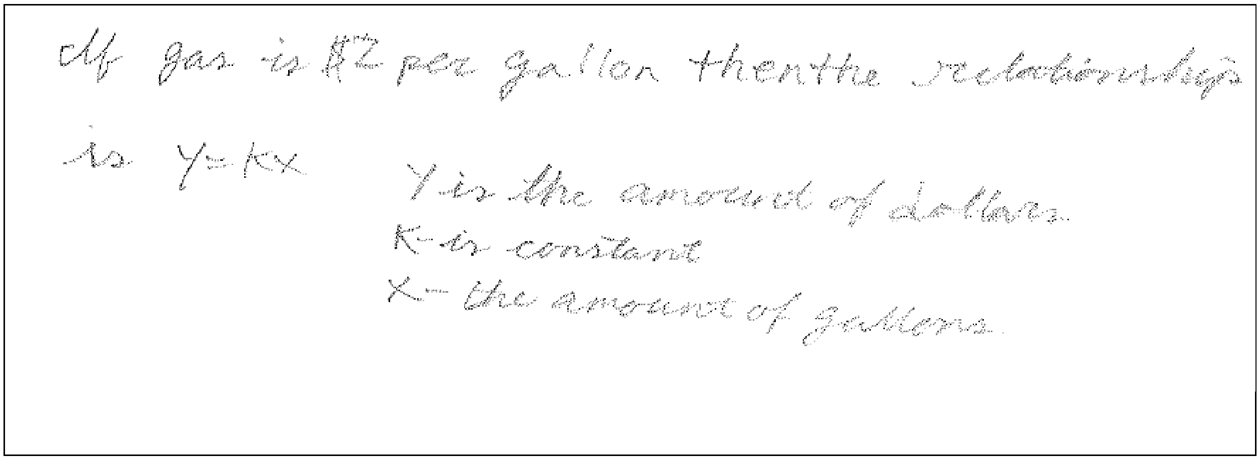


Figure 47: Student 2's relationship between dollar amount cost and gasoline pumped.

I: Now let's move to the gasoline problem, 4a; how did you come up with the relationship $y = kx$?

S2: When I think of gas, I think of a direct variation; when I am at the pump I believe that it is a direct variation. You look at your gallons and the price you pay--there is no in-between; there is no other type of variation it could be. I just gave a number for gas. I said if gas was $2 per gallon then the relationship is y=kx: y is the amount of dollars, k is constant, and x is the amount of gallons. If you have change of gallons at a constant rate depending on the number of gallons you have, you will get your dollar amount. That is how I look at it when I am at the pump, which is how I came up with the direct variation.

I: You said that k is constant; what does k represent?

S2: k represents dollars per gallon, dollars over gallons.

S2 saw that there was a direct variation between the dollar amount cost and the gallons of gasoline pumped. S2 created an example to explain his thinking. He used output and input to describe the linear function he created--input gas and output cost. S2 was able to explain the rate at which gasoline would be pumped into a car (see Figure 48).

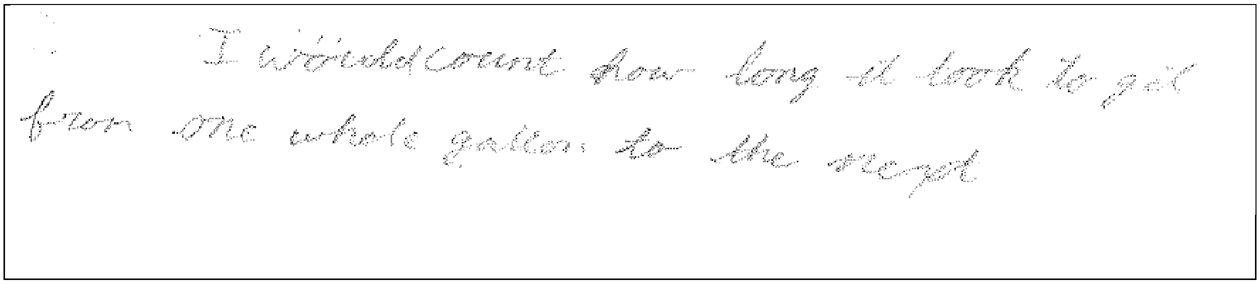

I would count how long it took to get from one whole gallon to the next

Figure 48: S 2's statement on how he determined the rate at which gasoline is pumped.

The Runner Problem

> A runner is running along and times how long she has run and how far she has run, using a stopwatch and a distance meter. If one reading is 3 minutes and 3/7 miles, how long would it take her to run 8 miles if she runs at a constant pace?

For this problem, S2 worked this problem differently than in the pretest. He again used his understanding of direct variation to solve. S2's solution showed his understanding of constant for the proportionality between time passed and distance traveled (see Figure 49).

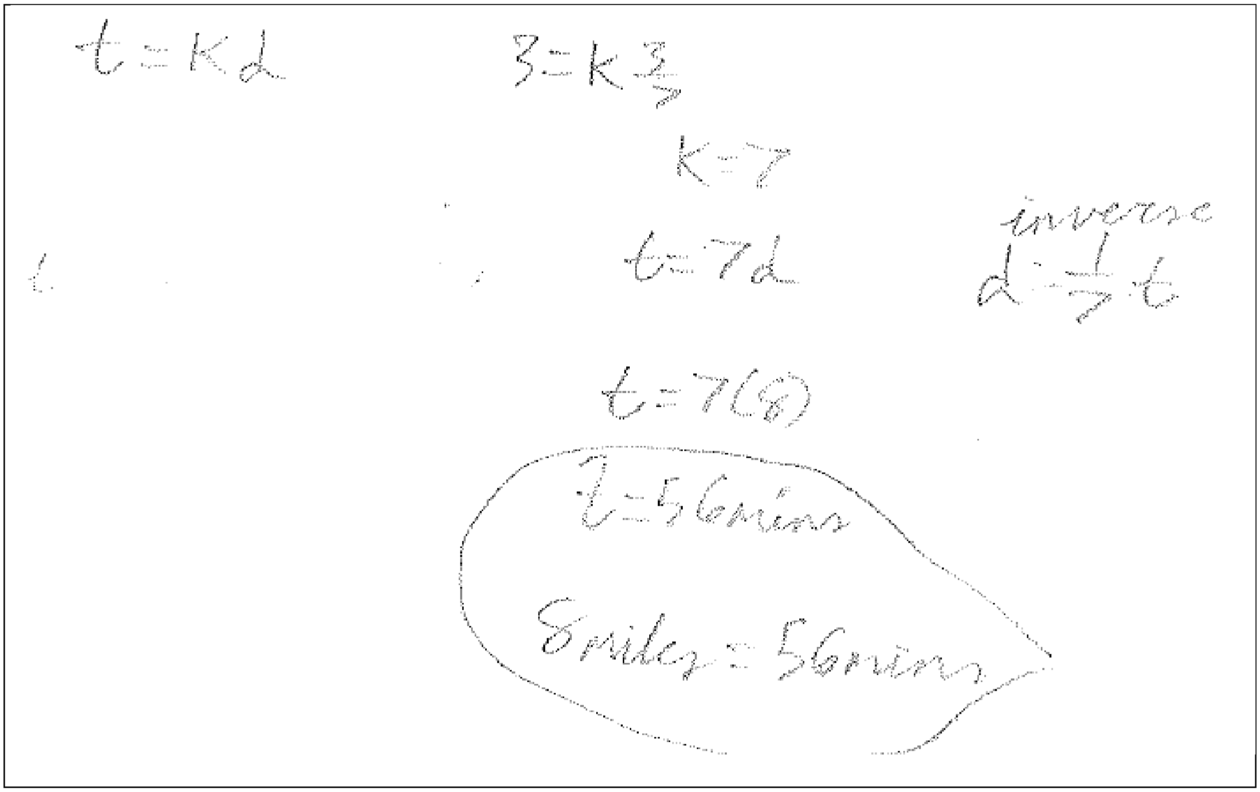


Figure 49: S2's solution to the runner problem.

I: Can you walk me through your thinking on the runner problem number 5?

S2: With the runner problem I assumed that it was a direct variation. It says a runner is running along, and it times how long she had run and how far she has run. I set it up the way it sounded to me as a direct variation. I had times equal *kd*, *y* is t for time and *x* is d for distance. If she ran for 3 minutes, 3/7 of a mile, when I did the math, k came out to be 7, so *t*=7*d*. If she runs 8 miles times that by 7, if you set it up in the problem, you have t= 7 ×8 and you get *t*=56 minutes. She runs 8 miles in 56 minutes.

I: Okay, thank you!

S2 attempted this problem differently from his initial attempt on the pretest. He saw that the relationship was a direct relationship between the time lapsed and distance traveled. He constructed a multiplicative relationship between time passed and distance traveled. He stated that *k* was the constant for the proportion.

Discussion S2's Thinking about Pretest/Posttest/Homework Problems

S2's performance for the semester was one of the best in the class on both the pretest and the posttest. He started the course with a working knowledge of ratios and proportions. In this discussion, S2's pretest and posttest by item are presented.

Pretest-Posttest by Item

On the pretest, S2 was very concise with his solution to the Baby-Weight Problem. He constructed a multiplicative relationship between the two quantities, age and weight. He drew a linear graph to illustrate the age-weight relationship, and he also was able to express the age-weight ratio as a rate. His conception of ratio and rates was indicative of an expert. On the post-test his conception was enhanced. He labeled correctly the input as month and the output as weight. His view of the linear equation was developing where he can see variables as an input-output function. He was now able to communicate his understanding in a more sophisticated way.

The snail problem was a challenge. S2's initial conception on this problem was apprentice. It was unclear if his markings of 1-3, 2-6, 3-9, 4-14, 5-15 … 10-30 and his making off of 2-12 hours and 3-8 hours indicated some understanding of the variation change. He did have the correct answer to the problem; however, his solution path was incorrect. For this problem, he divided the distance into two equal distances and calculated each rate separately. But he did produce a graph to clarify his thoughts on how he arrived at his solution. However, on the posttest, he solved the problem correctly and added a graph to show the change in variation from a 2-inch-per-hour pace to a 3-inch-per-hour pace. During the semester, S2 did demonstrate an improvement in establishing multiplicative relationships on the Homework 2's race-car problem. In that problem, he indicated by his algebraic solution that he understood how to find a change in rate by constructing a proportion that related both rates.

S2's solution to the Currency Problem corroborates that his conception was at the level of expert. He constructed a multiplicative relationship between the number of English pounds and the number of U.S. dollars and graphed the linear relationship between the two currencies, illustrating the direct relationship between them. He knew which currency was worth more. This he showed by finding the cost of a $20 garment in terms of English pounds. On his posttest he did more of the same, but this time he added in ratio understanding of the constant rate between the two quantities. S2 remained expert.

S2's solution path to the Gasoline Problem on the pretest places his conceptions of ratio and rate as a novice. He did not give the variables asked for in the problem or draw a graph showing the direct variation between the dollar amount cost and the gasoline being pumped. However, he created an example to show that he understood the multiplicative relationship between dollar amount cost and gallons being pumped. On the posttest he solved this same problem very differently. He still created an example to explain his thinking. He wrote "$y = kx$" and used his example to explain y to be the number of dollars, k to be constant, and x to be the number of gallons. He moved from novice to practitioner in his thinking.

S2's conceptual progress seemed to be enhanced by his understanding of variation, both direct and inverse. His understanding of direct variation between miles and minutes was evident by his writing 1 min = 1/7 miles and 3 minutes = 3/7 miles. His reasoning that 7 minutes = 1 mile made it easy for him to calculate that 8 miles could be run by the runner in 56 minutes.

On his posttest he showed that his conception of variation was enhanced. He wrote out two equations, which showed two different multiplicative relationships, one the inverse of the other; that is $t=7\times d$, and the other, $d=1/7 \times t$, respectively.

CHAPTER 6

STUDENT 3

S3: Case 3

S3 was a prospective middle grades mathematics teacher at the time of this study. During class he sat alone in the third row. He was active in solving problems within the classroom setting. However, he worked mostly by himself and offered help to some of the other students when they asked. The narrative begins with his solutions on the pretest, then centers on peculiarities encountered during the course of the study involving classroom instruction, and ends with the posttest interviews. The narrative starts with the five pre-test problems.

The Baby-Weight Problem

The problem is given below:

The Baby-Weight Problem: Betty weighed 6 pounds at birth. She gained 2 pounds every month for 8 months.

A. Create a table to represent the data over an 8-month period.
B. Draw a graph to represent the data; identify the input and the output.
C. As Betty gets older, how does her age relate to her weight?
D. Construct an equation to model the age-weight relationship.

For this problem, S3 constructed a table with multiples of 1 up to 8 for the months, as was asked in the question (see Figure 50). He constructed a table and a graph to represent the baby-weight relationship. His tabular data and his graphical scaling were similar representations of the same coordinates (see Figure 50). As shown in Figure 50, S3 was able to answer the entire problem correctly. S3 explained how he was operating while solving these problems.

Protocol I. (Student 1Pretest Interview--Lines 1-5)

> I: Thank you for doing this interview. Now you put birth on the table and you put 0 in the graph; can you explain?
>
> S3: 0 represents the day that the baby was born. Up here I put [a] 0 [to] represent 6 pounds at which the baby was born. And I added 2 pounds for every month because it [the baby] gained 2 pounds every month for 8 months. At 8 months the baby weighed 22 pounds.

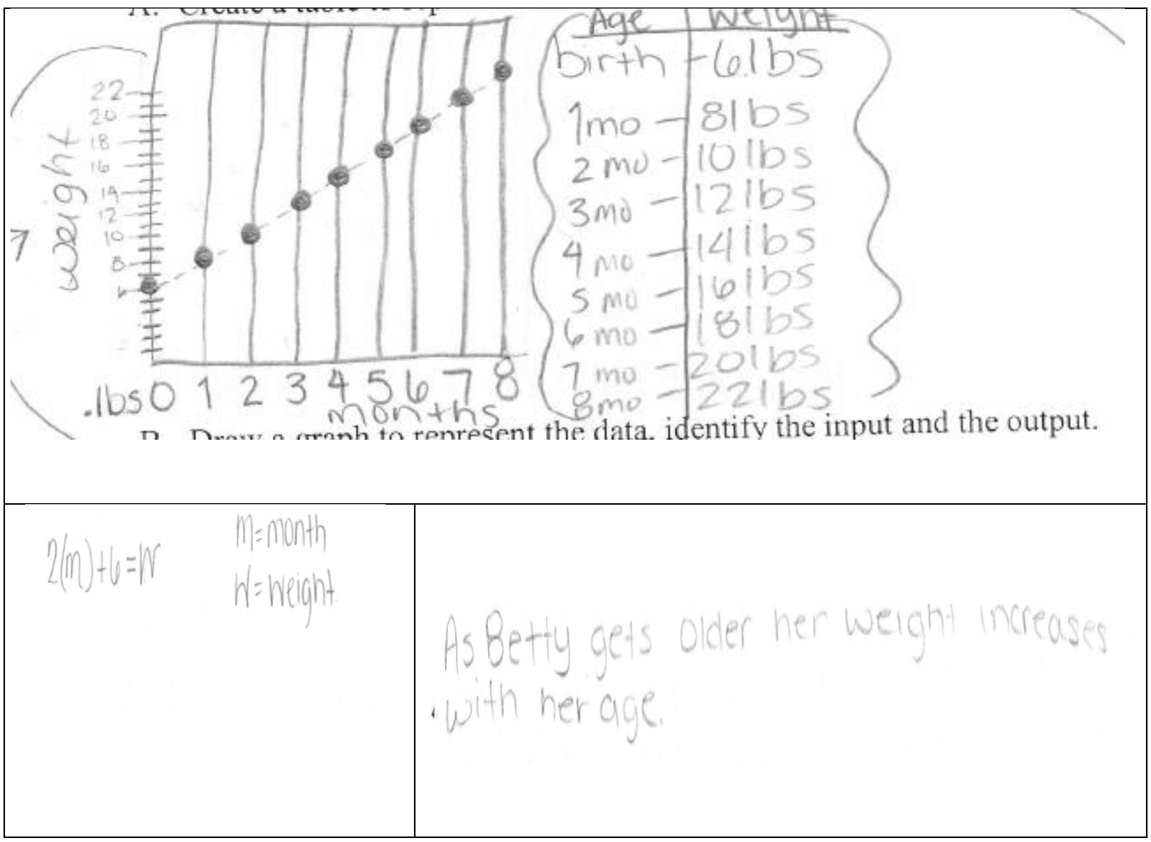


Figure 50. S3's solution to the Baby-Weight problem.

S3's initial thinking when he was making the table indicated understanding of the concept of rate and of a multiplicative relationship between months and baby weight. In the first column, he wrote out the accumulated amount of weight gained at each month, and in the second column, he represented the weight at each month. The table shows the baby gaining 2 pounds every month for 8 months. His construction of the table helped him in his construction of a multiplicative relationship between the months and the baby's weight (See Figure 50).

The Snail Problem

The problem is given below:

> The Snail Problem: A snail crawls the first part of 48 inches at the rate of 2 inches per hour. He crawls the rest of the way at the rate of 3 inches per hour. If it takes the snail 20 hours to crawl all 48 inches, how long did the snail crawl at each rate? Solve the problem graphically and then algebraically.

S3 divided the snail's journey of 48 inches into two parts by dividing 48 inches by 2 to get 24 inches (see Figure 51). After that, the journey of the snail was separated into two parts. For the first part of the journey, S3 divided that part of the journey, which is 24 inches, by 2 inches per hour. For the second part of the journey, S3 used the same procedure but divided the 24-inch journey by 3 inches per hour. After that the S3 represented the snail's trip with the help of a graph.

The graph showed a change from a 2-minute-per-hour pace to a 3-minute-per-hour pace for the trip of the snail (see figure 51). Furthermore, there was an indication that S3 had an understanding that 2 inches per hour and 3 inches per hour were constant ratios between the two varying quantities. That is, at this point, there was an indication that S3 had constructed a concept of rate.

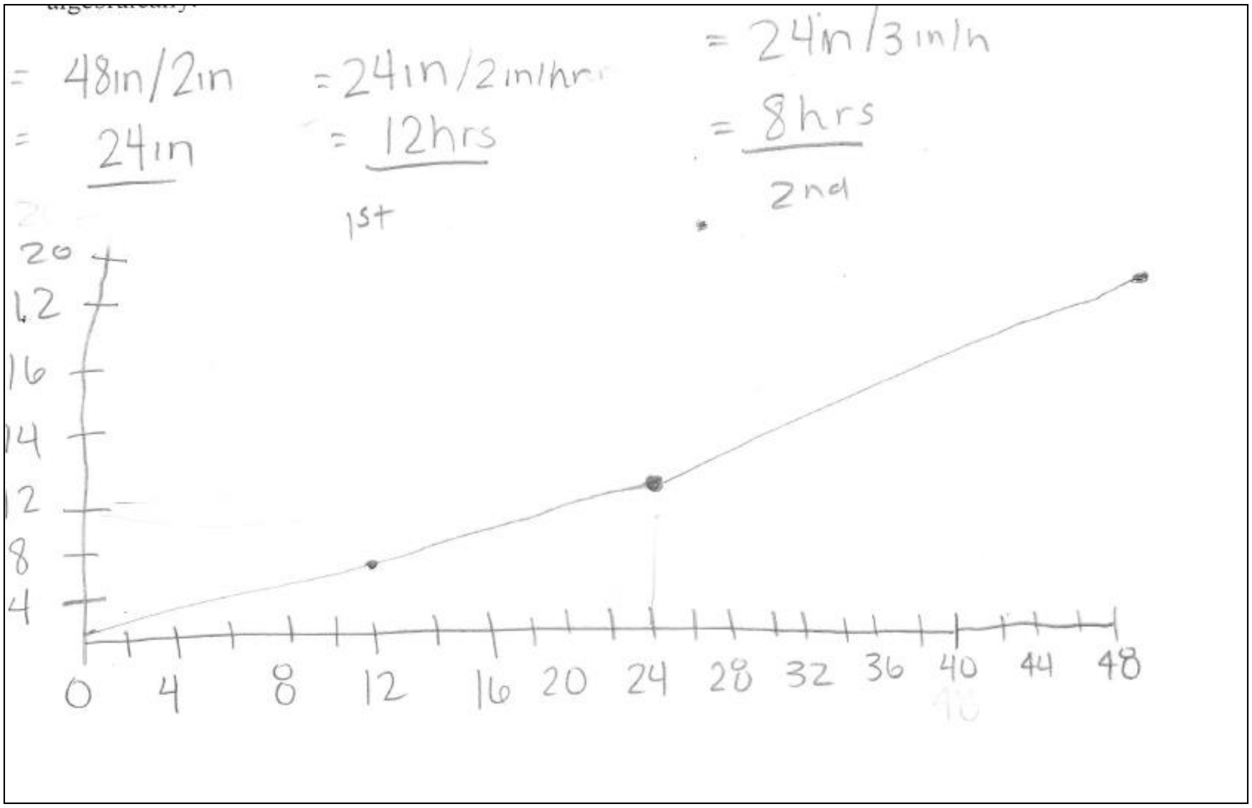


Figure 51. S3's solution to the Snail Problem.

S3 divided the snail's journey into two parts, labeled 1st and 2nd. The points (24, 12) and (48, 20) indicate 24 inches in 12 hours and 48 inches in 20 hours, with a change in variation shown at the (24, 12) point. The graph is a solid indication that S3 understood that there was a change in pace at the 24-inch/12-hour mark.

Protocol II. (Student 3 Pretest Interview--Lines 51-57)

> I: Clarify your thinking because I noticed that you had divided the 48 inches by 2 inches.
>
> S3: When it says the first part of 48 inches, I automatically said the first part is 48 inches; you probably have to divide by 2 inches, which will give you 24 inches……the first part ….it took him 12 hours to crawl the first part….24* inches divided by 2 inches per hour, it would take him 12 hours to crawl the first part; 24 divided by 3 inches would give you 8 hours to crawl the second time.
>
> I: Were you thinking a part of the rate was a half of the distance?
>
> S3: Yes, that's how I did it. It took the snail 20 hours to crawl.

Student 3's Initial Conception (First Two Pretest Problems)

S3's solution path for both the Baby-Weight Problem and the Snail Problem illustrated that he had constructed a concept of rate. The initial conception of ratio is very clear. For the Baby-Weight problem, S3 was able to represent the relationship between the baby's age and weight graphically, and he was able to write the multiplicative relationship as well. His thinking on the age-weight relationship and his use of the table to graph the data were explored in Protocol II, which corroborates that S3's reasoning was multiplicative in nature. In the first column of the table in Figure 50, he wrote out the accumulated amount of age gained at each month, and, in the second column, he represented the weight at each month. This representation for the problem indicates that he has clear understanding. The table shows that the baby is gaining 2 pounds every month for 8 months. When the baby was born, she had 6 pounds of weight, and at 8 months the baby weighed 22 pounds. With the help of this data, S3 constructed a linear relationship through which he could easily estimate the baby's weight in each of the months. For the Snail Problem, S3 divided the snail's journey of 48 inches into two parts by dividing 48 inches by 2 to get 24 inches, based on the apparent assumption that the snail crawled one half of the distance at each rate. After that, the journey of snail was separated into two parts.

For the first part of the journey, S3 divided the first 24 inches by 2 inches per hour to establish the time it took to crawl the 24 inches, indicating a rate concept. Likewise, for the second part of the journey, S3 divided the 24-inch journey by 3inches per hour. After that, S3 represented the snail's trip with the help of a graph. The graph showed a change from a 2-minute-per-hour pace to a 3-minute-per-hour pace, indicating a solid conception of the quantitative relations involved in the problem, as well as an understanding that 2 inches per hour and 3 inches per hour were constant ratios, or rates, between the two varying quantities.

The Currency Problem

The problem is given below:

Suppose 2 English pounds (£) buy the same goods as 3 U.S. dollars ($).

A. Make a conversion table to compare the two currencies.
B. What would a garment of $20 cost in terms of pounds?
C. What's worth more, the pound or the dollar, to an Englishman just off the plane on vacation in the USA?
D. If p = the number of English Pounds and d = the number of U.S. dollars, write the relationship between the number of pounds and the number of dollars? Explain.
E. Graph this relationship using graph paper.

Figure 52 represents S3's construction of a table of values to compare English Pounds to U.S. dollars. The table also showed that S3 was able to represent the relation between the English pounds and U.S. dollars. The table also showed the unit comparison of "English pounds per dollar." The value, 2 pounds equal to 3 dollars, indicates a unit comparison of English pounds per dollars, given his thinking. So from the table, S3 could produce or generate all cases.

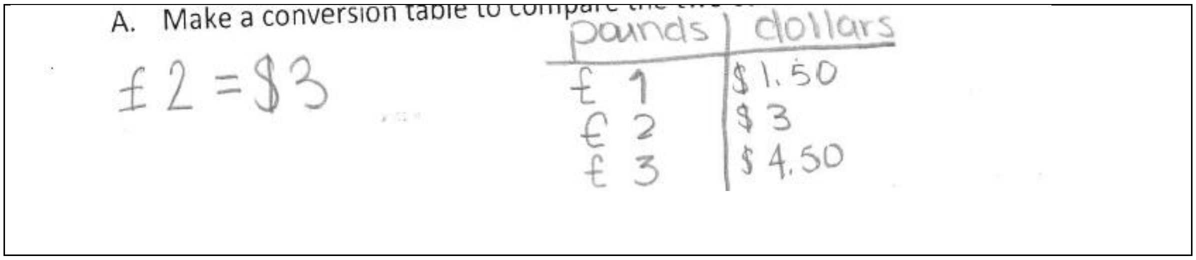


Figure 52. S3's table comparing English pounds to U.S. dollars.

S3 was able to convert between the number of English pounds and the number of U.S. dollars. His thinking was multiplicative. For example, he wrote 1 English pound equaling $1.50 and then he wrote out that $20 = £13.33. That 1 English pound equals $1.50 is a rate that is corroborated by what a $20 garment would cost in English pounds (see Figure 53).

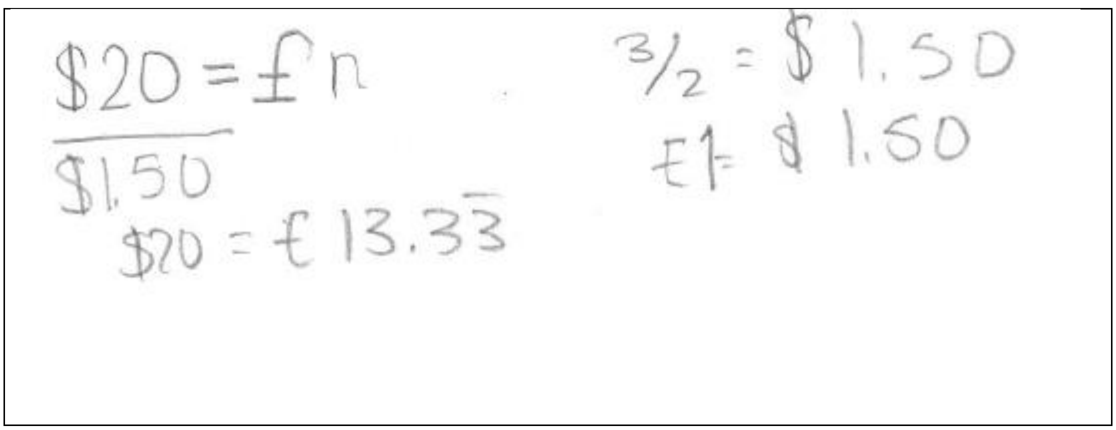


Figure 53. S3's use of symbols.

S3 also knew that the English pound was worth more in terms of buying power. He wrote, "If [a] tourist from England come[s] to the U.S. with one English pound, the value of that would be $1.50 in U.S. dollars." His comment shows that he understood the relationship as a multiplicative relationship (see Figure 54).

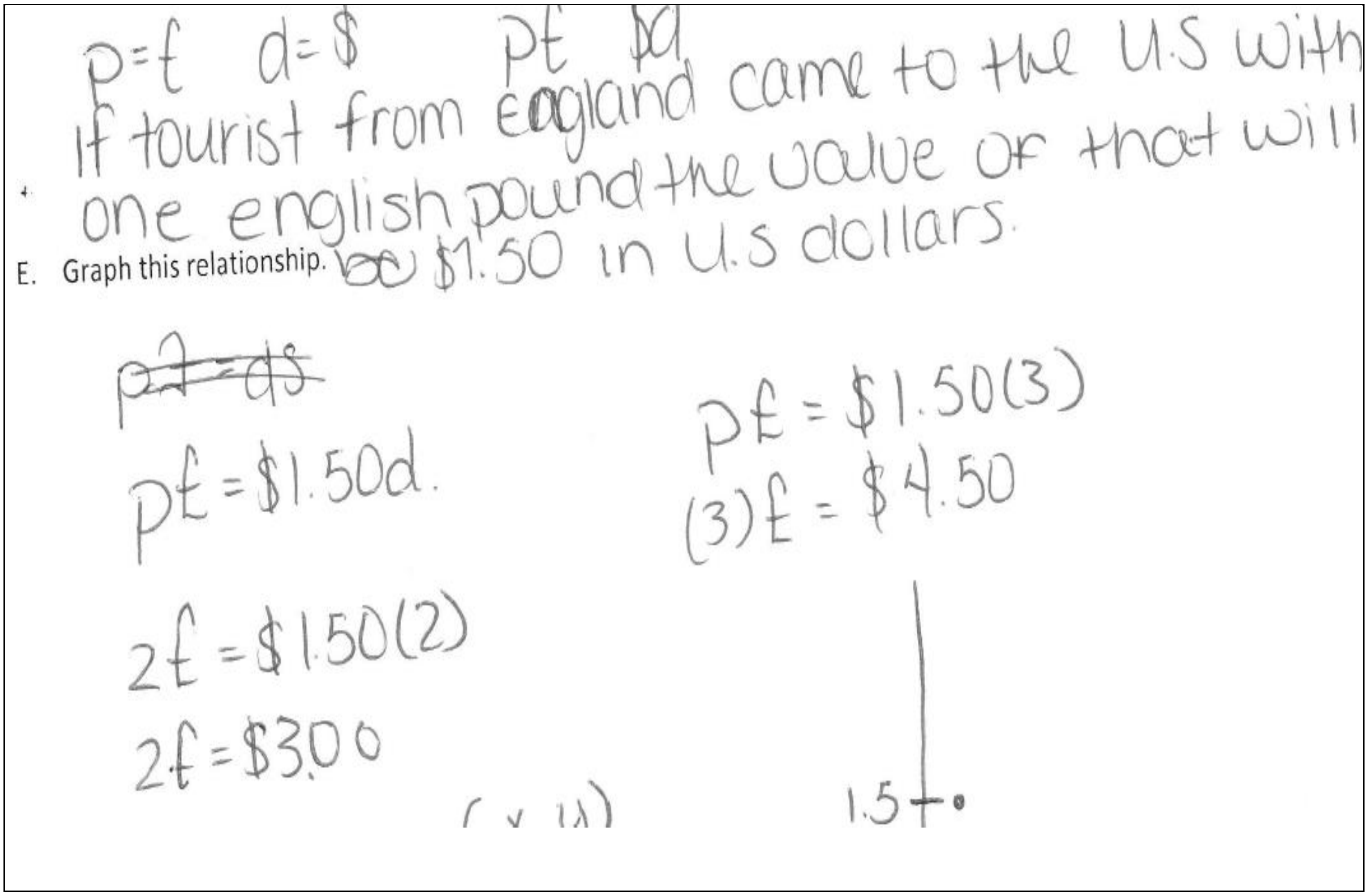


Figure 54. S3 comparing the buying power of the English pound versus the U.S. dollar.

When S3 attempted to make a graph of the relationship, he only attempted to plot the unit ratio. He could generate pairs of values as shown in Figure 54, but seemed to not be aware of the linear relationship between the corresponding values of the two currencies. That is, there was no apparent linear relationship between the number of English pounds and the number of U.S. dollars that he was explicitly aware of (see Figure 6). It would seem that S3 could have easily plotted the complete graph with the help of the table. But he only graphed one point to represent the relationship between the buying power of the English pound and the U.S. dollar (see Figure 55). S3 may have plotted only one point because on the x-axis of the graph, 1 represented all of the values for the pound. If so, then by repeatedly adding 1.5, which is $1.50 for every pound, S3 could have produced the results for other values on the graph.

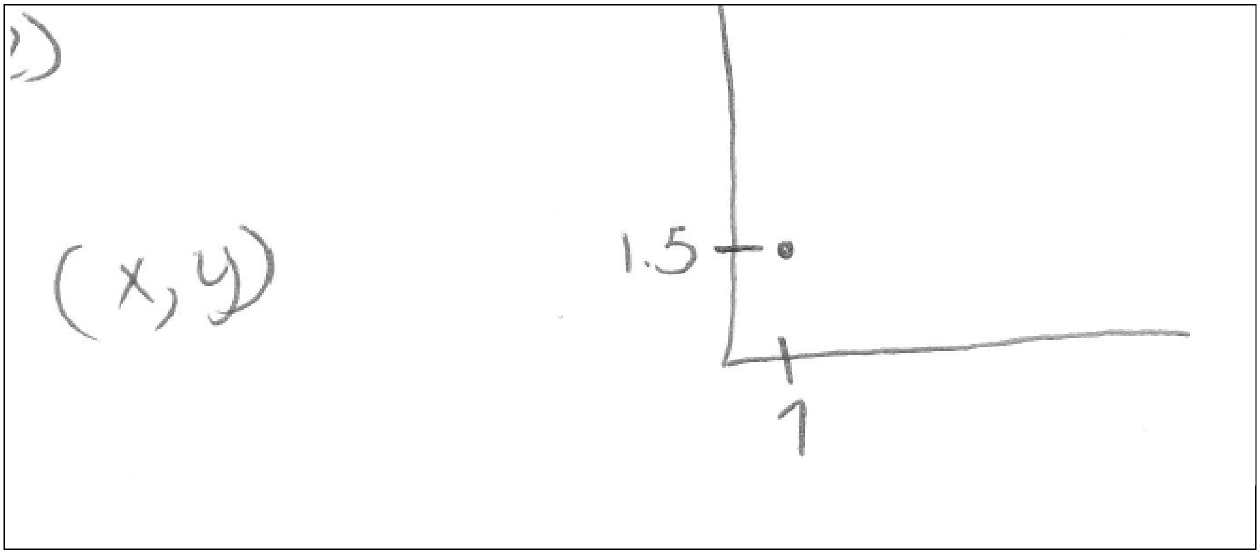


Figure 55. S3's attempt to graph the English pound—U.S. dollar relationship.

S3 showed that he had a clear concept of a unit ratio. He used the unit ratio to generate corresponding values given a certain number of pounds, which indicates that he may have constructed a rate concept and proportional reasoning. Further, according to S3, the worth of English pounds is more than U.S. dollars. The one thing lacking in the work of S3 is that he graphed only one point. He did not graph the linear relationship between the number of English pounds and the number of U.S. dollars, although in the following protocol, he seemed aware that he could have done so.

Protocol III. (Student 3 Pretest Interview--Lines 6-17)

I: Can you explain why you think the English pound is more than the U.S. dollar?

S3: I believe it weighs more because every English pound you get 1.50 on the American dollar. So if 1 English pound represents a dollar, in the USA it would be worth more.

I: On the same problem, you plotted 1, 1.5 on your graph; can you tell me what the relationship is between the English pound and the U.S. dollar?

S3: I guess I would say for every English pound you would get $1.50. 2 English pounds is equal to 2 U.S. dollars. So I made a table, and I put 1 English pound equal $1.50, so for 2 English pounds, so on and so on. So for every English pound you will add 1.50.

I: Can you explain why you only gave one point?

S3: I did one point because for every English pound. At the bottom of the graph for the *x*-axis I put 1, which will equal pounds. If you look at the table for 2a [referencing the currency problem], you can see you will add 1.5 which is $1.50 for every pound as you go up.

The Gasoline Problem

Imagine pumping gasoline into a car.

A. What is the relationship between the dollar amount cost and the gallons being pumped?
B. How would you determine the rate in which the gasoline is being pumped into the car?
C. If c=cost and g=the gallons being pumped, sketch a graph to illustrate the cost of the gasoline.

S3 did not state the variables on paper. Instead of clearly stating the variables, S3 gave a general statement of how to find the total cost of gasoline. Therefore, S3 was able to establish the cost of gas pumped (see Figure 56). Even though he did not have to explicitly write an equation to demonstrate his understanding of multiplicative relationship, he wrote $gal \times P = C$.

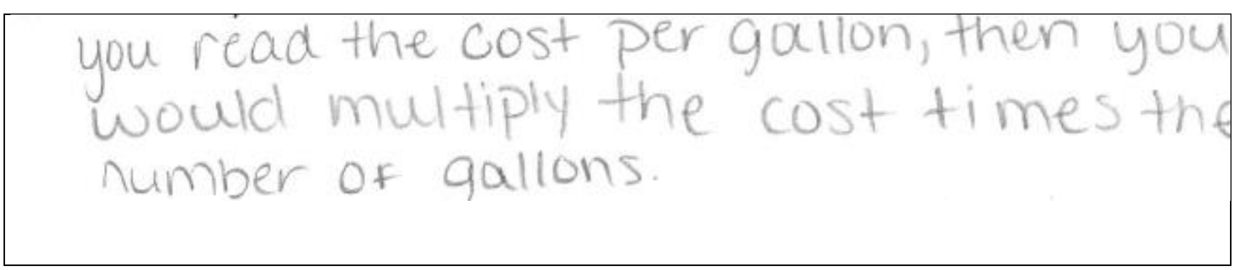

you read the cost per gallon, then you would multiply the cost times th
number of gallons.

Figure 56. S3’s use of rates to state the variables.

To determine the rate at which the gas is being pumped into the car, S3 used the multiplicative relationship ($gal \times P = C$) to get $P = \frac{C}{gal}$. So to him, the price was the rate. S3 took the dollars per gallon as a rate. So in the multiplicative relationship, *P* is the price and C is the cost, so gallons times the price of gas will equal the cost one will have to pay. For the problem, S3 used some random numbers. He writes "*C* times *G*" as the cost times the gallons. Through this, S3 was able to reason in a multiplicative way to find out how much someone would pay based on the price.

S3 constructed a linear graph to model the relationship between the cost and the number of gallons without specifically plotting points. Therefore, his lack of the use of corresponding numbers indicated that S3 had used his general concept of the quantitative relationship to construct the graph. He selected two numbers to plot one point and then created a linear graph with no indication on how it was established. According to S3, depending on the price of the gas, the graph will change. His graph did seem to represent a direct variation between dollar amount cost and gallons pumped (see figure 57). But it was problematic as there is no indication that it would pass through the origin.

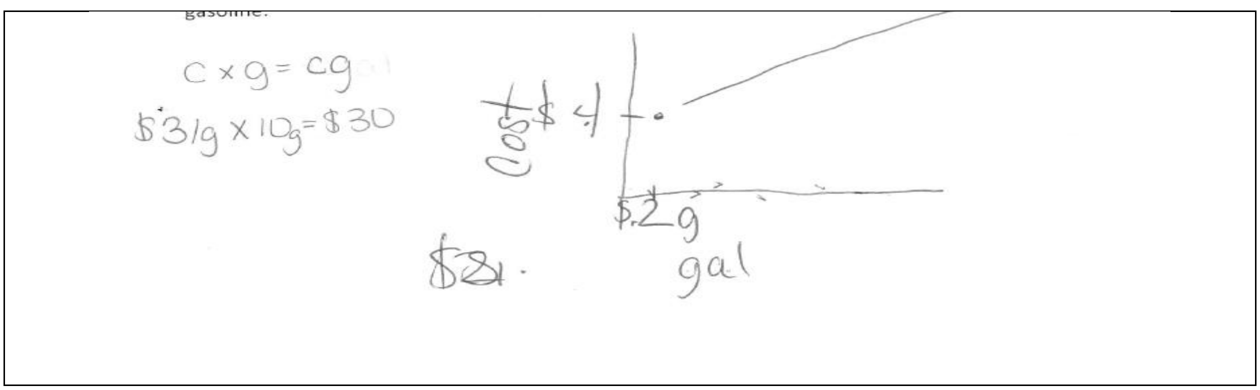


Figure 57. S3's cost by gallons graph.

Protocol IV. (Student 3 Pretest Interview--Lines 18-45)

I: On the gasoline question, you indicated that dollars per gallon was a variable; can you explain?

S3: I did dollar per gallons. Cause when you are at the gas pump, they give it to you like 3.95 per gallon. So that's why I did that; that's what came to mind.

I: On the same question, you have gallons times *P* equals *C*; can you explain more about this relationship?

S3: *P* is the price and *C* is the cost, so gallons times the price of gas will equal the cost you will have to pay depending on how many gallons or how much money you spend to put gasoline into the car.

I: On the same question, you wrote *C* times *G* equals *CG* and 3 dollars per gallon. How did this information help you to sketch the graph? How did you decide on the numbers you used?

S3: I just used some random numbers. *C* times *G* is the cost times the gallons. And you will find out how much you would pay based on the price. On this one, I put 3 [dollars] per gallon and [if] you pumped 10 gallons [this would cost] 30 dollars.

I: In 3c [referencing the gasoline problem] you have gallons times *P* = *C*; you're saying gallons times price equals cost, right?

S3: Yes.

I: But in your answering the question you wrote *C* times gallons equal cost gallons; is there a relationship between those two expressions?

S3: I guessed [on] this one. The *CG* kind of represent the *P*, so I probably took that back and put the *CG* in the place of the *G*. And put the *G* over and make that the *P*. *C* times *P* is equal to *T*. *CG* is representing the cost. I would take the *C* and put the *P*.

I: I see that the graph is a straight line. How do you know this since you only plotted one point on the graph?

S3: Actually depending on the price of the gas, the graph will change

1. In this case on this graph, If you're getting any gas, your cost would be 0; if the gas here on the ground is 2 dollars per gallon so if you are paying 2 dollars per gallon and you were paying 2 dollars, the cost would be 4 dollars. This graph shows the cost. I just drew a line to show a constant.

I: What if there are no gallons?

S3: It would be 0 dollars; if you didn't pump no gallons then the car would start at 0.

The Lady-Runner Problem

> A runner is running along and times how long she has run and how far she has run using a stop watch and a distance meter. If one reading is 3 minutes and 3/7 miles, how long would it take her to run 8 miles if she runs at a constant pace?

Student 3 was able to use ratios to solve this problem correctly. S3 writes that 3 minutes is equal to 3/7 mile and took that as if it took 1 minute to run 1/7 of a mile, so if it took the lady runner 3 minutes to run 3/7 of a mile, it would take the runner one minute to run 1/7th of a mile. S3 stated that if the runner was running a minute per seventh of a mile, it would take the lady 7 minutes to run 1 mile. S3 demonstrated that it would take 56 minutes for the lady runner to run 8 miles if she runs at a constant pace. This shows that the concepts were clear to S3 and S3 was aware of the multiplicative relationship that is needed to solve the problem. Therefore, S3 could clearly represent the rate relationship; that is, the runner travelling 1 mile for every 7 minutes (see Figure 58).

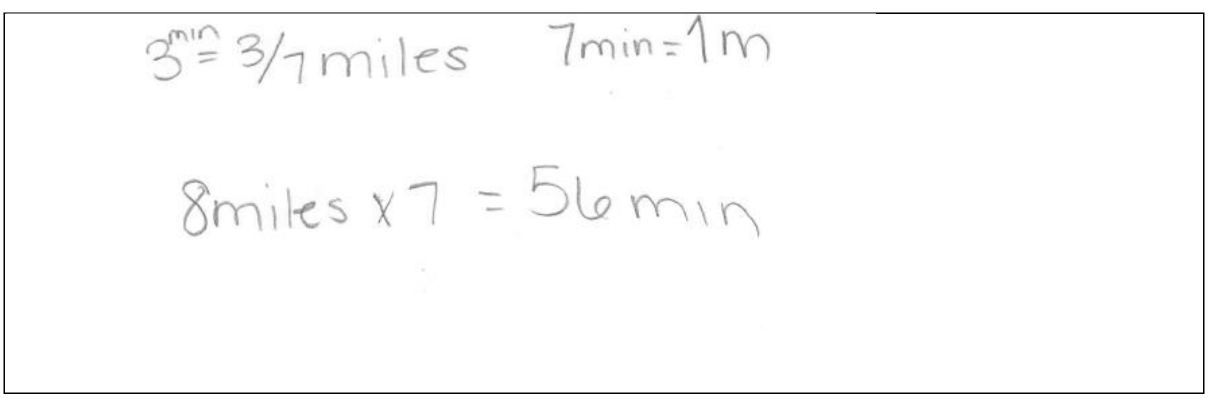


Figure 58. S3's solution to the Lady Runner Problem

Protocol V. (Student 3 Pretest Interview--Lines 46-50)

I: On the lady runner problem can you explain that it takes 7 minutes to run 1mile?

S3: Clearly here it says it takes 3 minutes to run 3/7 of a mile; I took that as if it took 1 minute to run 1/7 of a mile, so if it took her 3 minutes to run 3/7 of a mile, it would take her 1 minute to run….so if she was running a whole mile, it would take her 7 minutes to run 1 mile. If she was running a minute per 1/7 of a mile, it would take her 7 minutes to run 1mile.

Student 3's Initial Conceptions Up to This Point (Last Three Pretest Problems)

For the currency problem, S3 constructed a table to compare the English pounds to the U.S. dollars. The table showed that S3 was able to represent the relation between English pounds and U.S. dollars. This is because it showed the unit comparison, "English Pounds per U.S. Dollars." So from the table S3 could produce or generate all cases. Here it seemed as if S3 has a clear concept of a ratio. His clear understanding of a unit ratio enabled him to compute the cost of a $20 garment in English pounds.

For the gasoline problem, S3 was not able to state the variables on paper. Instead, he stated a general relationship. Therefore, S3 was able to establish the cost of gas pumped. Even though he was not asked to explicitly write an equation to demonstrate his understanding of multiplicative relationship, he wrote $gal \times P = C$. To determine the rate at which the gas is being pumped into the car, S3 used the multiplicative relationship ($gal \times P = C$) to get $P = \frac{C}{gal}$. So to him, the price was a rate. S3 took dollars per gallon as a rate. On the other hand, in the multiplicative relationship, P is the price and C is the cost, so gallons times the price of gas will equal the cost you will have to pay, depending on how many gallons or how much money you spend to put gasoline into the car.

S3 constructed a linear graph to model cost per gallon. However, he seemed to pick numbers arbitrarily to plot one point and then created a linear graph with no indication of how it was constructed. Still, according to S3, depending on the price of the gas, the graph will change, which indicated an awareness of slope. The straight line nature of the graph demonstrated the constant nature of the relationship between the two quantitative variables.

S3 seemed to know that the relationship between cost and gallons was linear, as can be seen from his graph. His graph was problematic as the graph did not pass through the origin and there was no intercept to represent cost for 0 gallons of gasoline.

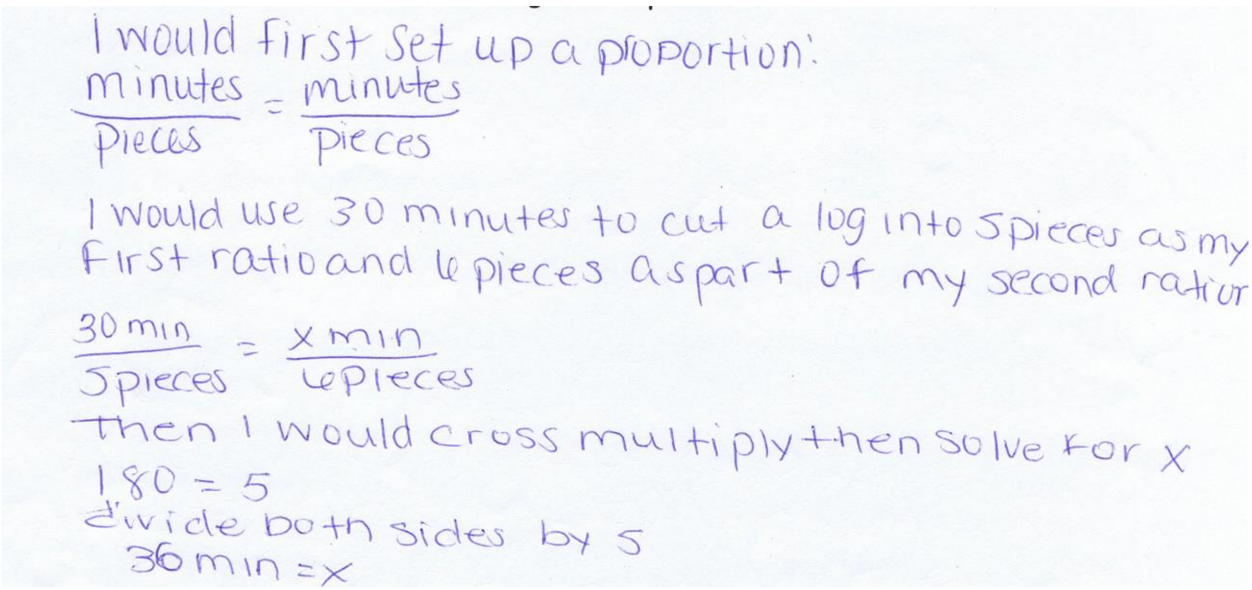
In the Lady Runner problem, S3 was able to use ratios to solve the problem correctly. S3 wrote that 3 minutes is equal to 3/7 mile and took that as if it took 1 minute to run 1/7 of a mile, so if it took the lady runner 3 minutes to run 3/7 of a mile, it would take the runner 1 minute to run 1/7 mile. S3 stated that if the runner was running a minute per seventh of a mile, it would take the lady 7 minutes to run 1 mile. S3 demonstrated that it would take 56 minutes for the lady runner to run 8 miles if she runs at a constant pace. This shows that the concepts were clear to S3 and that S3 was also aware of the multiplicative relationship needed to solve the problem. Therefore, S3 could clearly represent the rate--that is, the runner travelling 1 mile for every 7 minutes.

Ratio and Proportions Homework (Homework I)

The Log Problem

This problem drew on the students' ability to use their understanding of proportions for problem solving. S3's solution to this problem was different than others in the class.

The Log Question: It takes 30 minutes to cut a log into 5 pieces. How long would it take to cut a similar log into 6 pieces?



I would first set up a proportion:

$$\frac{\text{minutes}}{\text{pieces}} = \frac{\text{minutes}}{\text{pieces}}$$

I would use 30 minutes to cut a log into 5pieces as my first ratio and 6 pieces as part of my second ratio

$$\frac{30\text{ min}}{5\text{pieces}} = \frac{x\text{ min}}{6\text{pieces}}$$

Then I would cross multiply then solve for x

180 = 5

divide both sides by 5

36 min = x

Figure 59. S3's solution to the Log Problem.

Although S3's explanation seems flawless, it is not correct. S3 did not consider the number of cuts. Had he drawn a diagram to help him solve the problem, it is possible that he would have noticed that the time it takes to cut a log into five pieces depends on making cuts. Tourniaire and Pulos (1985) stated that sometimes students unknowingly ignore part of the question or some of the data when solving a proportion problem, and this seemed to be the situation with S3. I have no doubt, given his solution, that he would have solved the problem correctly had he considered cuts rather than pieces.

Ratio and Proportions Homework (Homework I)

Currency Problems

There were two currency problems assigned for students to complete and turn in. These problems were discussed in class within the context of the notes that students had to complete. The first problem:

> Suppose that 3 Argentine Pesos are equivalent to 4 Danish Kroner. Use the Danish Kroner as the independent variable.
> A. Represent this in the form of a linear equation.
> B. What is the slope of the line?

S3 illustrated that 3 Argentine Pesos are equivalent to 4 Danish Kroner and used the ratios between the two currencies to write the multiplicative relationship (see Figure 60). A cross multiplication with his expression would mean that P=3/4 D. He stated also that the slope was ¾.

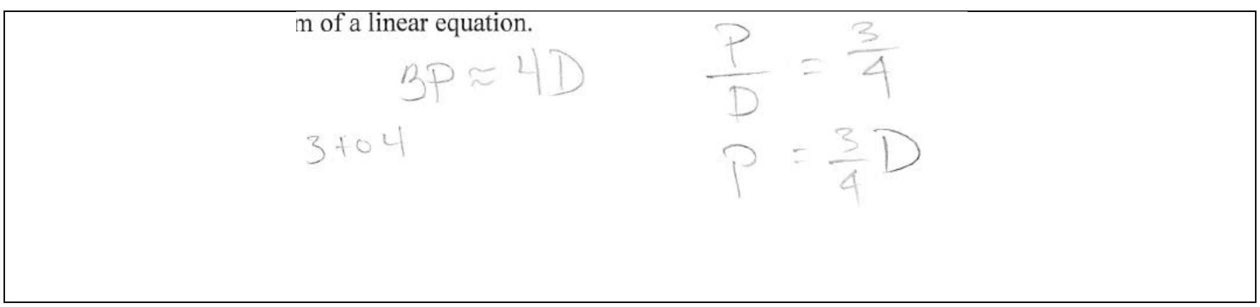


Figure 60. S3's solution to part A.

The next problem was a continuation of the first problem:

In (1), the Argentine embassy decides that all Danish visitors to Argentina must hold back 10 Danish Kroner and not convert them to Argentine Pesos. Using the x-y coordinate plane, make a graph of the number of Pesos that each Danish visitor receives with respect to all the Kroner the visitor brought to Argentina. Be sure to develop an equation.

For this problem, S3 seemed to be reliant on the point-slope formula as a means to generate his linear equation. Here he indicated some idea about variation when he wrote, "Right horizontal shift." This was evident in his circling of the 10. He constructed a graph using a transformation of the previous linear representation of P=3/4 D by shifting its graph 10 units right. However, what was not clear was the variation of a rise of 3 units and a run of 4 units. There were no marks on the vertical axis or beyond the point (0, 10) on the horizontal axis that would indicate the slope.

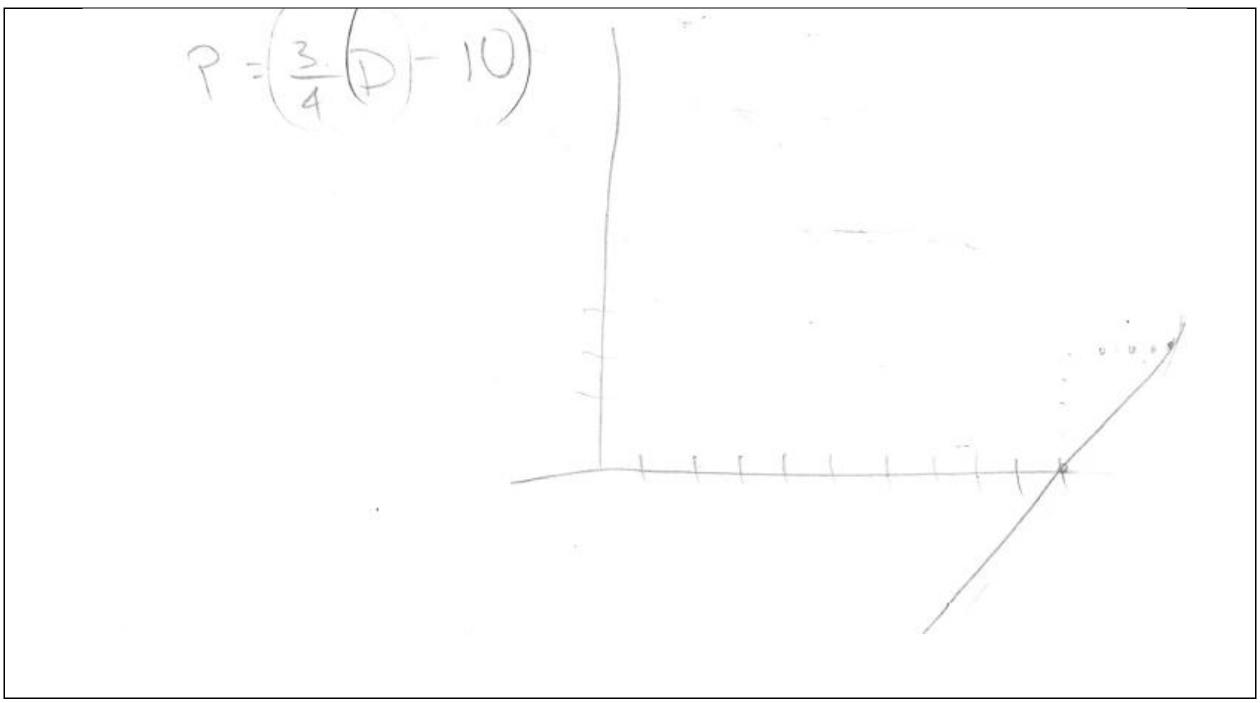


Figure 61. S3's solution to problem 2.

Homework II: Suppose that 2 English pounds is equivalent to $\frac{14}{5}$ European Euros. Use the Euros as the independent variable.

A. Construct a linear equation to represent this relationship (use fractions rather than decimals).
B. What is the slope of the line (use fractions rather than decimals)?
C. Sketch the graph of the linear equation in part A and describe the variation.

Here S3 seemed confused about how to find the conversion factor (see Figure 62). His representation for this problem showed as ($\frac{2}{1}$ English pounds = $\frac{14}{5}$ Euros), which is a statement of the equivalence between the values of the currencies. Unlike the previous problem involving Argentine Pesos and Danish Kroner, the presence of a fraction served as a constraint in setting up a proportional relation between the two currencies. Not setting up a proportion between the two currencies brings into question his meaning of the proportion in the previous currency problem in that his meaning of P and D seemed ambiguous. Rather than refer to the number of Argentine Pesos and the number of Danish Kroner, my interpretation is that the letters stood in for the value of the currencies as the words did in this case. Therefore, he was not able to construct the linear equation in order to show the relationship between the number of English pounds and the number of Euros (See Figure 62).

$$\frac{2}{1}\text{ pounds} = \frac{14}{5}\text{ Euros}$$
$$1\text{ pound} = \frac{7}{5}\text{ Euros}$$
$$\left(\frac{7}{5}\right)n\text{ pounds} = b\text{ Euros}$$

Figure 62. S3's solution to part A.

Because of his confusion between equivalence versus equal, there was no possibility that he could determine the rate of exchange and interpret this rate as the slope. His graph for part C had English pounds shown as the independent variable (see Figures 63 and 64), and he was not sure about the slope (see Figure 64).

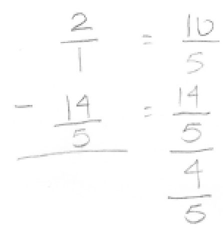


Figure 63. S3 trying to determine slope.

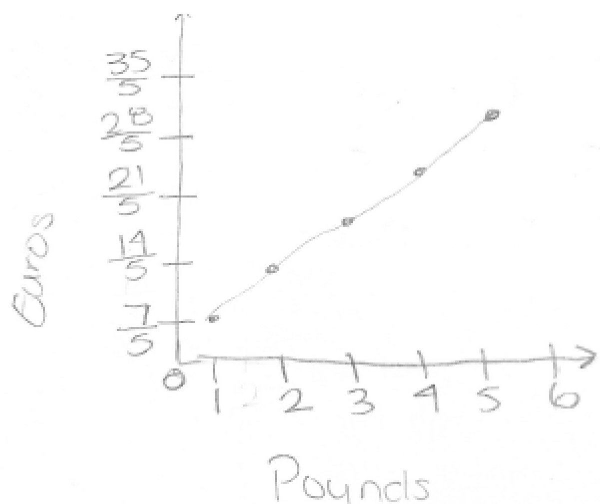


Figure 64. S3's Euro-English pound graph.

On the second problem for Homework II, his problem solution showed a continued regression because he did not establish the linear equation in the first part of the problem.

Homework II: In (1), the English embassy decides that all visitors from Europe to Great Britain must hold back 5 Euros and not convert them to English pounds.

A. Construct a linear equation to represent this relationship (use fractions rather than decimals).

B. Using the x-y coordinate plane, make a graph of the number of English pounds that each European visitor receives with respect to all of the Euros the visitor brought to Great Britain.

C. Describe the variation.

Thus, unlike the problem we did in class, he could not use a transformation based on the previous problem to help with constructing the graph asked for in part B. Even though the graph correctly started from (0, 5), the slope was not correct. In fact, the slope of the line he constructed showed a rise of 1 and run of 1 from (0, 5). This is shown by the markings made on the graph sketched by S3. The variation of a rise of 5 and a run of 7 cannot be seen in the graph (see Figure 65).

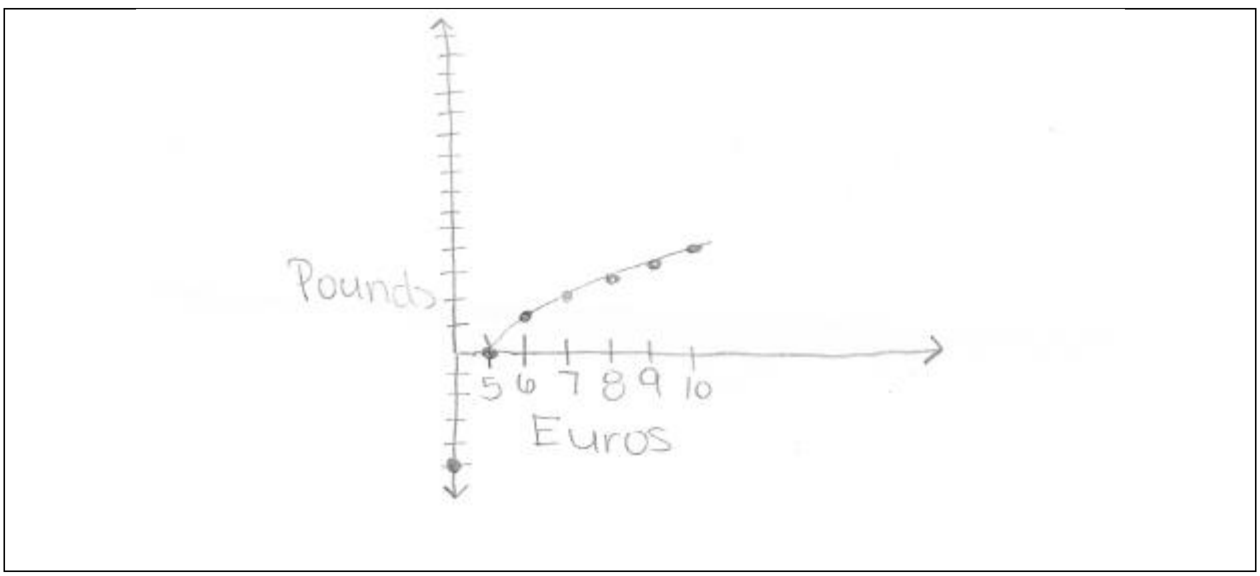


Figure 65. S3's graph for part B.

The statement by S3 in Figure 66 explained the role that holding back 5 Euros played in making the graph.

Since each immigrant have to hold
back 5 Euros you would subtracy
5 Euros from the amount
that they have before
you convert them to pounds

Figure 66. S3's explanation for part C.

Homework II (Race Car Problem): A racecar driver drives part of a 500-mile race at 150 miles per hour and the remainder at 180 miles per hour. If it takes her a total of 3 l/4 hours to complete the race, how long at each rate did she drive? Solve the problem graphically and algebraically.

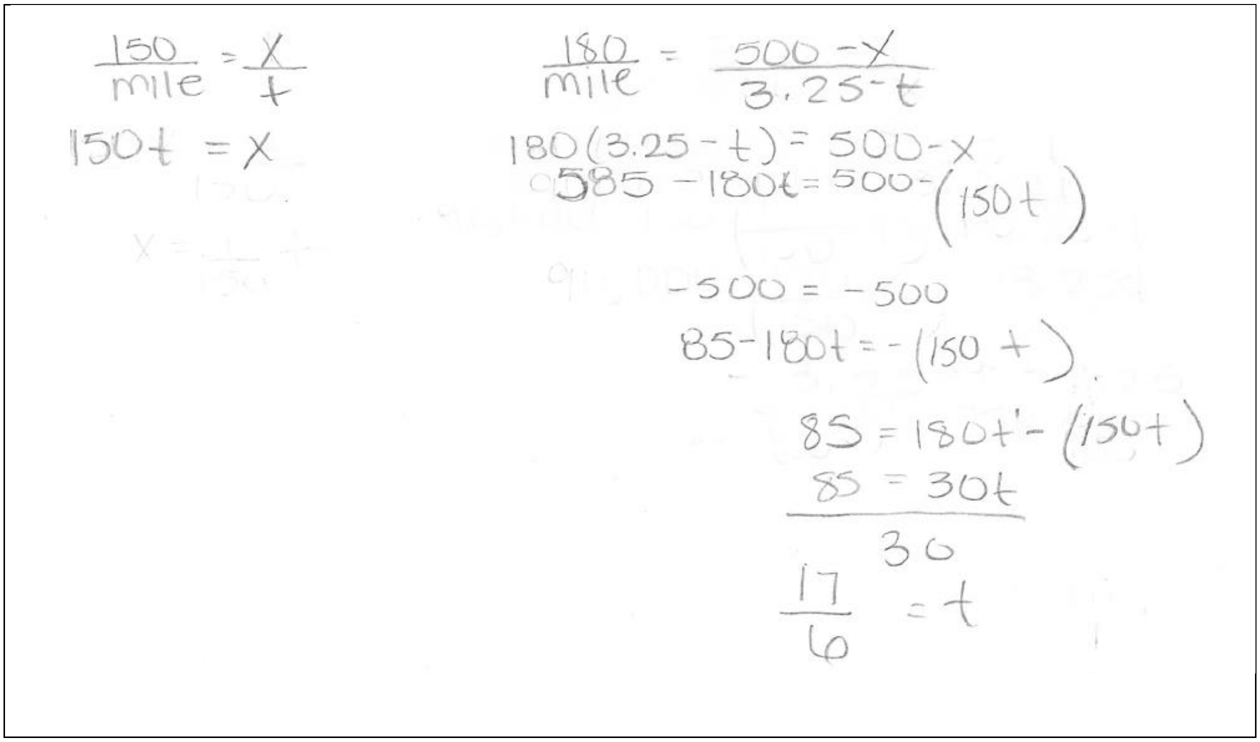


Figure 67. S3's solution to the Race Car Problem.

S3 found the time the driver drove at 150 miles per hour, but did not find the time she drove at 10 miles per hour, nor did he find the respective distances driven. Still, his use of proportionality to find the time driven at 150 miles per hour does indicate progress in his reasoning, which evolved out of instruction. S3 also did not attempt to graph the rate graphs.

However, with the Work-Rate Problem stated below, he attempted to draw a graph but drew an incorrect one (see Figure 68). He did again use ratios to generate the work pay rates. He found that Mandy worked 31.34 hours at the \$16 per hour and 23.57 hours at the \$30 per hour rate (see Figure 67).

Homework III (Work-Rate Problem): Mandy decides to take two jobs to pay her debt. The first pays her at a rate of \$30 per hour, and the second pays her at the rate of \$16 per hour. If she works for 60 hours and earns \$1400, how long did she spend on each job? Graph the rates of change and find the equations based on this graph.

$$\$30 - \$16 = \$14$$

$$\$30n + \$16n = 1400$$
$$\$46n = 1400$$
$$n = 30.43 \text{ hours}$$

$$\frac{30}{1} = \frac{d}{t}$$

$$\frac{30}{1} = \frac{1400 - 16t}{60}$$

$$1800 = 1400 - 16t$$

$$\frac{400 = 16t}{16}$$

$$= 25$$

$$\frac{16}{1} = \frac{1400 - 30t}{60 - t}$$

$$960 - 16t = 1400 - 30t$$

$$-960 \quad +30t = +30 \quad -960$$

$$14t = 440$$

$$t = 31.43$$

$$\$16 \quad \frac{\overset{60}{31.43} \text{ hrs}}{}$$

$$\$30 \quad 28.57$$

Figure 68. S3's solution to the Work-Rate Problem.

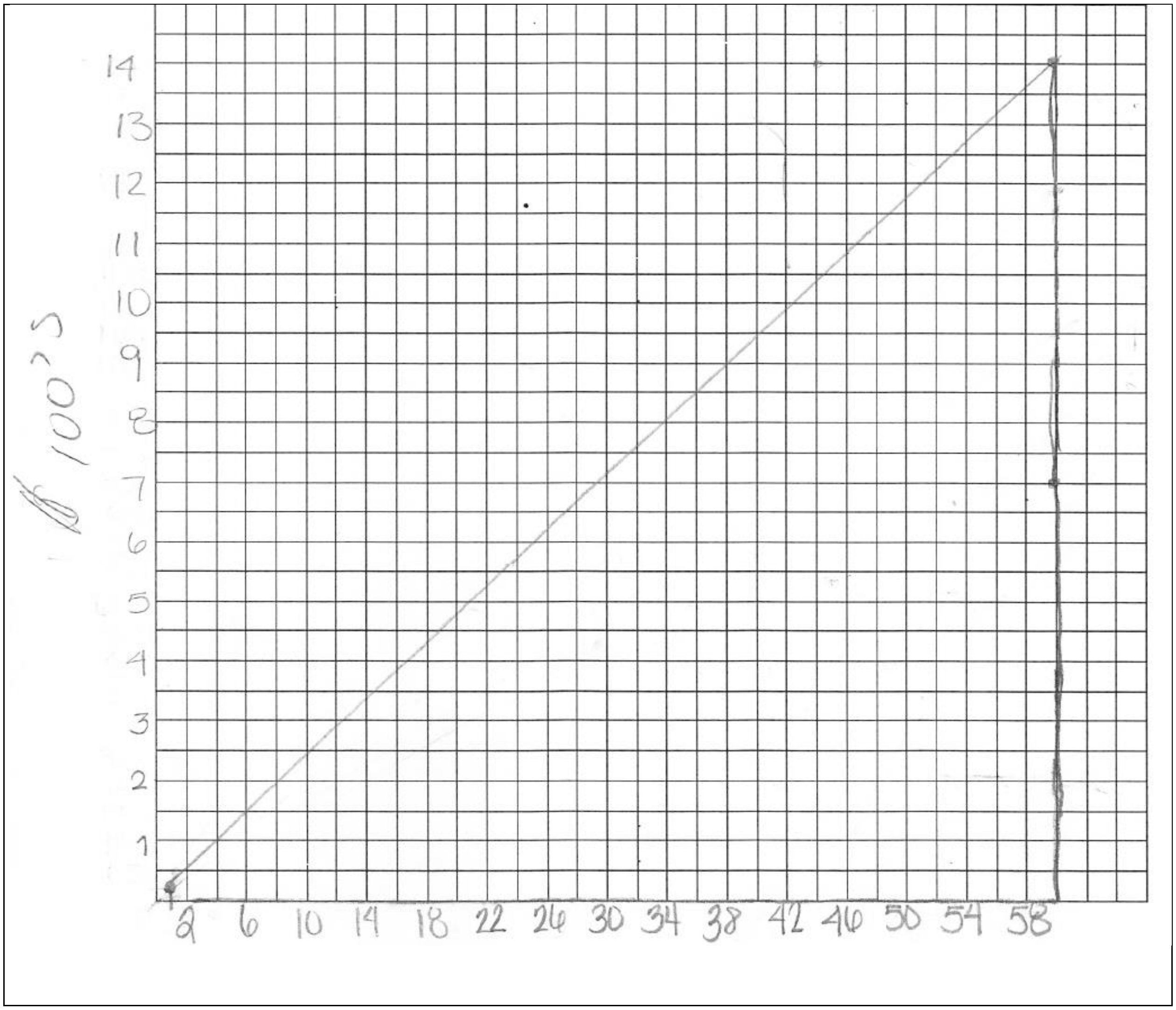


Figure 69. S3's graph to the Work-Rate Problem.

Student 3's Thinking on the Problems on the Posttest

At the end of the study, S3 exhibited results very similar to the pretest. He was however, more confident with describing his thinking on each of the problems posed.

The Baby-Weight Problem

On the Baby-Weight Problem, S3 explained his thinking, as follows:

Protocol VI. (Student 3 Posttest Interview--Lines 1-19)

I: Could you walk me through your equation on question number 1 to model the age-weight relations?

S3: Okay, right here I have 2(m)+6=weight because Betty gained 2 pounds, and when she was born she weighed 6 pounds, so you have to add that, and she gained 2 pounds, so 2 times the month plus 6 equals the weight.

I: On the graph you did to represent the weight, what was your input and output?

S3: Weight and age.

I: Can you explain; well tell me your thinking on variation? How are y and x varying?

S3: By 2, when you say varying what are you asking?

I: Can you elaborate?

S3: Here you rise 2 and run 1, like here from the 6 you go up to which is to 8 and you go over 1, which is the first month, so the first month here Betty weighed 8 pounds at month 1. Then you go up 2 and over 1 again, so the second month Betty weighed 10 pounds.

I: Do you think that beyond the 8-month period that this would be a direct variation?

S3: Well I'm sure Betty didn't gain 2 pounds each month her whole life, but according to what was given, yes.

The Snail Problem

His solution to the Snail Problem did not change. He answered it in a similar way (see Figure 70). He drew a graph that also indicated the change in variation from a 2-inch-per-hour rate to a 3-inch-per-hour rate (see Figure 71). He explained his thoughts on the problem as follows:

Protocol VII. (Student 3 Posttest Interview--Lines 21-35)

I: Let's look at the snail problem. I need your thinking here; it looks like you divided 48 inches by 2. Are you looking at the first part at being the first part of the trip and the second part at being the second part of the trip?

S3: Yes

I: Why?

S3: When they said the first part, I was looking at it as being half of the travel, so what I did was divide 48 by 2, and the first part the snail traveled 2 inches per hour, so I divided 24 inches by 2 inches per hour, and I came up with 12 hours, and I did the same thing with the second part, and I came up with 8 hours to travel the second part, and it came up to be 20 hours, and that's the total time it took the snail to travel the whole 48 inches.

I: Can you explain the variation of how this snail is traveling?

S3: In the first part, the variation I went up 1 over 2, and the rate changed at 24 inches 12 hours, and then you would go up 1 over 3 because the speed of the snail changed during the trip at that point.

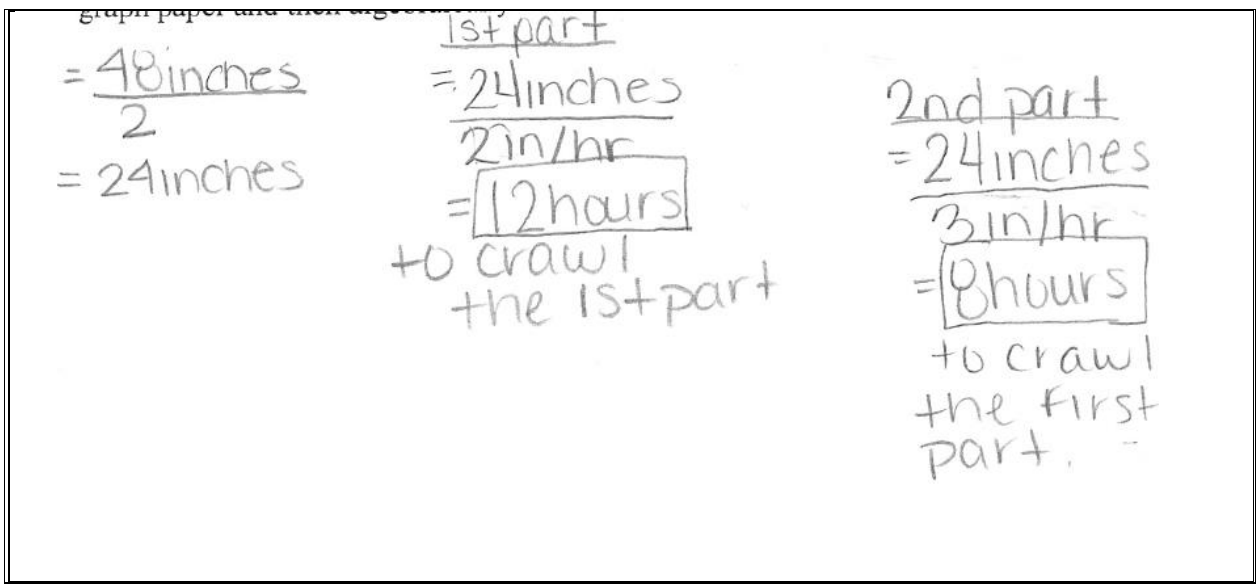


Figure 70. S3's problem solution for the Snail Problem.

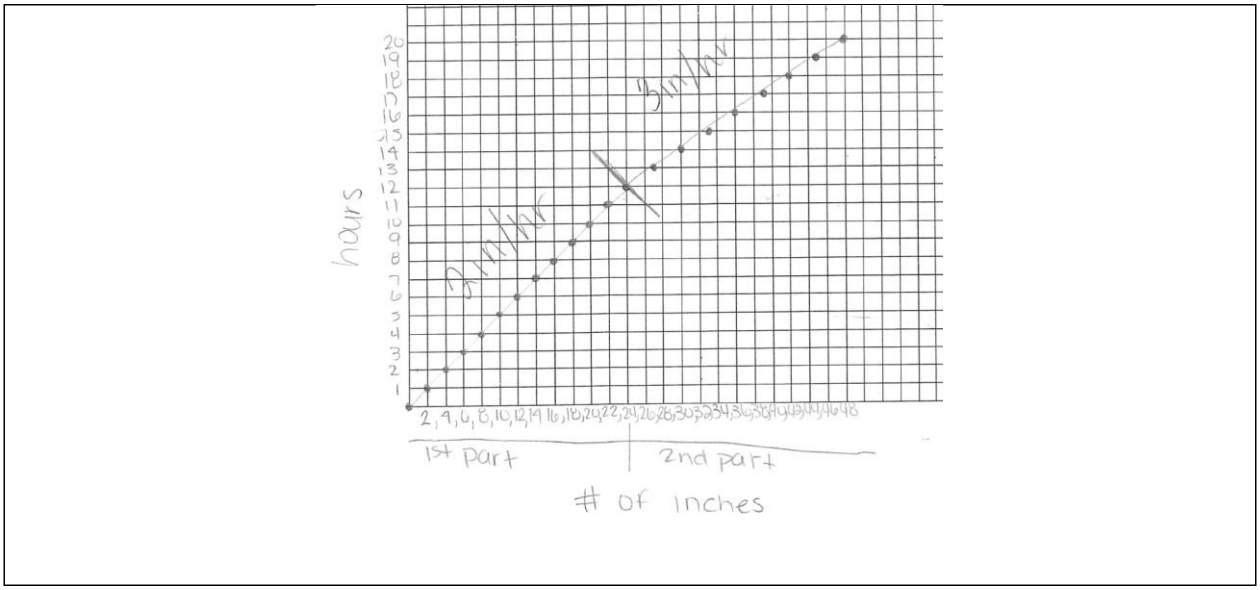


Figure 71. S3's posttest graph for the snail problem.

S3's graph on the posttest was different on the Posttest (see Figure 71). On the pretest the change in pace appear to show an increase in the pace from a 2-inches per hour to a 3-inches per hour. It looks like he has the snail traveling 1 hour every two inches for 24 inches and 1 hour for every 3 inches for 24 inches, it looks like a decrease in pace for the same change in variation.

The Currency Problem

With the Currency Problem, S3 was able to work this entire problem with less difficulty compared to the pretest. In the pretest he was not able to sketch the graph, but now he was able to sketch the graph (see figure 72). His thoughts on solving this problem do show some difference in thinking.

Protocol VIII. (Student 3 Posttest Interview--Lines 36-45)

> I: Let's look at the currency problem. I notice that you have 1 pound in your left column and 5 for your dollar column; okay, why did you stop at 5 pounds and 7.50 at your dollar?

S3: Well it wasn't a particular reason why I stopped; I just stopped on the table at that point, but it goes on and on because you're adding 1.50 to each pound to get the dollar amount.

I: How would you describe the variation for Part D of the Currency Problem as you're talking about dollars per pound?

S3: I went up 1.5 and over 1.

(S3 used (1, 1.5) as his center.)

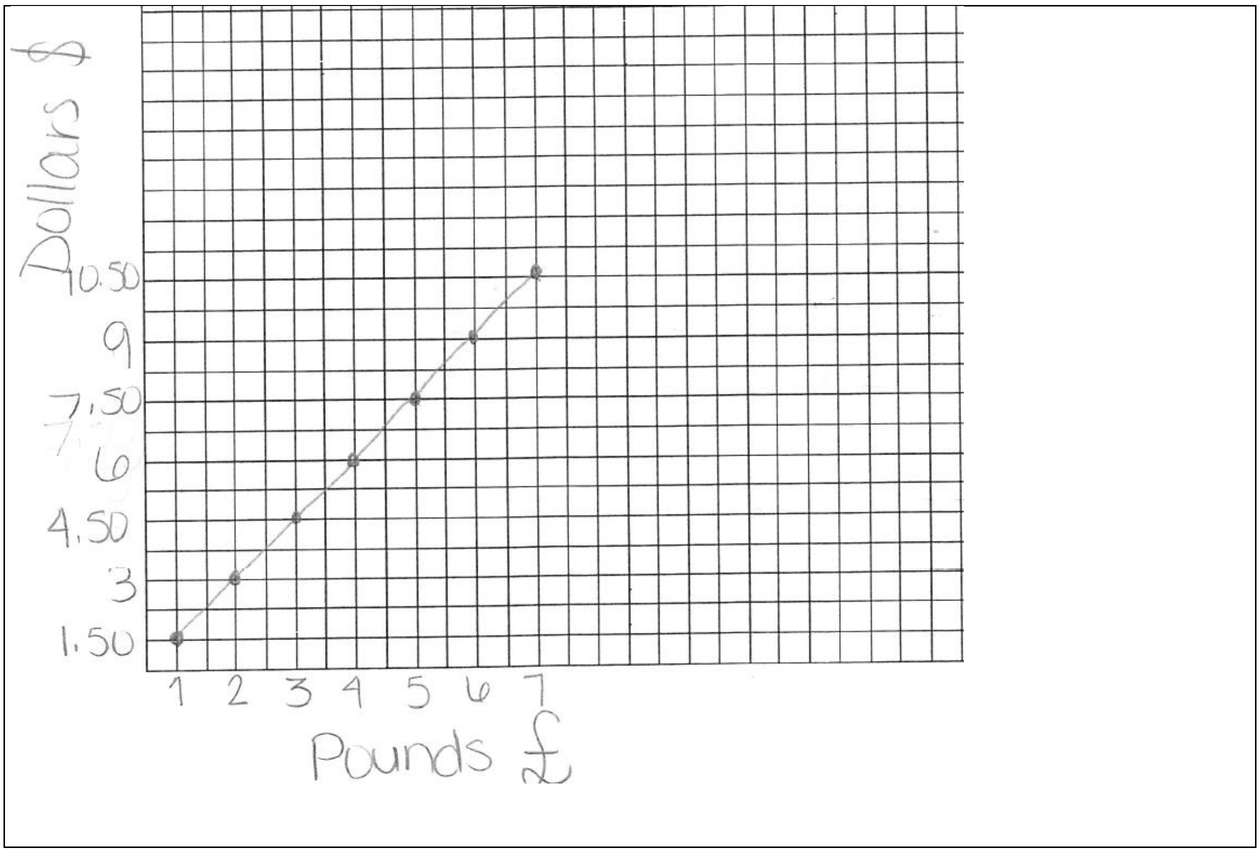


Figure 72. S3's posttest English pound-U.S. dollar relationship.

The Gasoline Problem

For the Gasoline Problem, S3 was able to construct the graph to represent the rate of dollars per gallon. His thoughts on the problem are explained.

Protocol IX. (Student 3 Posttest Interview--Lines 47-54)

> I: On your gasoline problem, you have cost per gallon; can you walk me through your graph?
>
> S3**:** Okay, they didn't give me any kind of numbers to use. So I said that gasoline was $1.50 per gallon. This means that if I was getting no gas, I'll be at 0; so for 1 gallon of gas, I went up 1.50 and over 1. I used gallons of gas on my *x*-axis and cost on the *y*-axis. So if I was getting a gallon, I would go up 1.5 and over 1, and if I wanted another gallon, I would go up another 1.50, which would be at 3, and over 1, which would be at 2 gallons.

The Runner Problem

The solution process for S3's answer for the Runner Problem did not change much from the pretest. He seemed more confident answering the questions and giving his explanation for his thinking. He explained his thoughts on the problem.

Protocol X. (Student 3 Posttest Interview-Lines 56-61)

> I: On the runner problem you have 3minutes= 3/7 mile, 7min= 1 mile, and 8miles×7minutes= 56 minutes.
>
> S3**:** Okay, it was given that it took the runner 3 minutes to run 3/7 mile, and I know that 7/7 mile equals 1 mile, so I automatically put that it would take 7 minutes to run 1 mile. And it asked how long it would take to run 8 miles. So it was 7 minutes per mile, so I put 8 miles times 7 minutes, and I came up with 56 minutes.

Discussion S3's Thinking about Pretest/Posttest/Homework Problems

For the Baby-Weight Problem, S3 is at the level of expert. He constructed a table with multiples of 1 up to 8 for the months, as was asked in the question; he constructed a table and a graph to represent the linear relationship between baby's age and weight; his tabular data and his graphical scaling were similar representations of the same coordinates (see Figure 50). S3 was able to answer the entire problem correctly and explained how he was operating while solving these problems. In the posttest, S3 remained at the expert level.

S3 divided the snail's journey into two parts by dividing 48 inches by 2 to get 24 inches. For the first part of the journey, S3 divided the first part of the journey, that is 24 inches, by 2 inches per hour. For the second part of the journey, S3 used the same procedure but divided the 24-inch journey by 3 inches per hour. After that, S3 represented the snail's trip with the help of a graph. The graph showed a change from a 2-inch-per-hour pace to a 3-inch-per-hour pace for the trip of the snail. Furthermore, there was an indication that S3 had an understanding that 2 inches per hour and 3 inches per hour were constant ratios between the two varying quantities--that is, rates. Representing the change from a 2-inch-per-hour pace to the 3-inch-per-hour pace indicates that the graph represented a change in a constant variation. There is strong indication to support his conception of rate at the level of expert in terms of both his initial conception and final conception. The main difficulty with his work on the Snail Problem resided in that fact that he did not correlate his algebraic solution of the time and distance traveled before he changed pace with his graphical work.

On the Currency Problem, S3's initial conception is beginner expert in terms of his solution path. He constructed a table of values to compare the English pounds to the U.S.

dollars. It showed the unit comparison of "English Pounds per Dollar." He applied his understanding of a unit ratio to determine the cost of a $20 garment in English pounds. However, he was unable to correctly draw the graph to represent the linear relationship. S3 plotted only one point because on the x-axis of the graph, 1 represented the pound. The graph showed that S3 put 1 pound, which is equal to 1.5. So by adding 1.5, which is $1.50 for every pound, S3 could have developed the result for other values on the graph, which may be why he plotted only one point. S3's solution path on the posttest showed him as an expert. He completely answered the problem and was able to accurately draw the graph showing the direct variation between the number of English pounds and the number of U.S. dollars.

Gasoline Problem

S3's solution on the pretest shows him as a novice. That is, instead of clearly stating the variables, he wrote out examples of rates to represent variables. His thinking was multiplicative in nature. This is because, even though he did not explicitly write an equation to demonstrate his understanding of a multiplicative relationship, he wrote $gal \times P = C$ to determine the rate at which the gas is being pumped into the car. He uses this thinking (that is, $gal \times P = C$) to get $P = \frac{C}{gal}$. So to him the price was a rate. For this problem, S3 used some random numbers to explain his reasoning. He wrote "*c* times *g*" as the cost times the gallons, in this case conflating the price and the cost. Through this, S3 was able to reason in a multiplicative way to find out how much someone would pay based on the price. S3 explained that for 3 dollars per gallon, if someone pumped 10 gallons, that would cost 30 dollars. S3 constructed a linear graph to model cost per gallon. The use of numbers to assign values to the two given unknowns indicated that S3 used an example to explain his understanding of working with quantitative unknowns. He chose numbers arbitrarily to plot one point and then created a linear graph with no indication of how it was constructed. Still, the graph was an indication that he had constructed the concept of rate because, according to S3, depending on the price of the gas, the graph will change. His graph represented a direct variation between dollar amount cost and gallons pumped. His graph was problematic in that there is no indication that it would pass through the origin since there was no intercept to represent cost for 0 gallons of gasoline. On the posttest, he was still at the beginner expert level because he was able to construct the graph to represent the relationship for dollars per gallon.

S3 was expert at solving the Runner Problem on both the pretest and the posttest. His understanding of the rate allowed him to solve this problem. S3 wrote that 3 minutes is equal to 3/7 mile and took that as if it took 1 minute to run 1/7 mile, so it would take the lady runner 3 minutes to run 3/7 mile. He demonstrated that it would take 56 minute for the lady runner to run 8 miles if she runs at a constant pace.

CHAPTER 7

CONCLUSION

Overview

This chapter provides an overview of the findings of the study and implications for future research. The purpose of the study was to examine the conceptions of middle grades mathematics teachers as they related to rate and proportional reasoning. The research questions for this study were as follows:

1. What are the initial conceptions of middle grades teachers concerning rate and proportional reasoning?

2. How do the conceptions of middle grades teachers concerning rate and proportional reasoning change over time as they study algebra concepts?

3. How are prospective middle grades teachers' construction of linear equations and functions related to their conceptions of rate and proportional reasoning?

Three students who were in their junior year in a middle grades education program participated in this study. Their conceptions were observed from a number of different documents, as well as interview transcriptions. A pretest, posttest, final exam, and homework were analyzed along with interview transcriptions of the pretest interview and posttest interviews.

Findings

This section is composed of three parts. In the first part, I compare and contrast the three students' initial conceptions based on the pretest. In the second part, I examine the students' mathematical behavior on course-assigned readings and homework to explore how their conceptions of rate and proportional reasoning changed over the course of the semester in the algebra concepts course. In the third part, I reflect on the posttest and compare it with homework examining the students' ability to construct linear equations and linear functions.

Initial Conceptions

Baby-Weight Problem

S1's initial conceptions hovered between apprentice and novice when evaluated across the dimensions of understanding, reasoning and communications. This fluctuation was evident in her solution to the Baby-Weight and Currency Problems because she was unable to create a table of values. S1's table for the Baby-Weight Problem did not contain a tabular value that allowed her to produce a multiplicative relationship to represent the age-weight relationship. Even though she was able to make a clear drawing with ratio points, the tabular data and the graphical representation of the data did not match.

S2's initial conceptions were more developed than those of S1. He was able to construct multiplicative relationships, as evidenced in the Baby-Weight and Currency Problems. He was also able to produce tables that showed clear patterns in both the independent and dependent variables. He was functioning at the practitioner or expert level on these two problems. He was able to illustrate his understanding by using three different representations: graphical, tabular, and symbolic or multiplicative.

S3's solution illustrated that he was functioning at the expert practitioner level. This functioning was evident in his ability to construct multiplicative relationships and table values and to generate linear relationships. He showed this ability in his solving of the Baby-Weight Problem.

It was clear that all three students had different conceptions of proportions and rates. Only S2 and S3 had a multiplicative understanding of rate in Thompson's (1994a, p. 26) term. That is, they could conceive a function as a covariation of quantities by means of evaluating the multiplicative relationship they constructed at different input values.

Snail Problem

The Snail Problem was not completed by any of the students. S1 was not able to show evidence that she conceived of the relations that were involved in establishing a change between the two rates. The snail's journey was simply divided into two parts in a mechanical or routine way. The change from 2 inches per hour to 3 inches per hour seemed to be done through an educated guess and not by multiplicative reasoning. S1's conceptions of rate for the most part were novice, with some movement towards apprentice at best.

S2 did know that there was a change from 2 inches per hour to 3 inches per hour in the Snail Problem. However, his solution did not show a multiplicative connection that would locate a change in variation between the two rates. His scratch marks showing a 1-to-1 correspondence 1 to 3, 2 to 6… 9 to 27, and 10 to 30 shows potentially a connection with 24 that is, 2 inches/hour for 12 hours and 3 inches/hour for 8 hours.

S3 was not able to generate two related multiplicative relationships in his solution of the Snail Problem. However, he did graphically represent a change in variation from a 2-inch-per-hour to a 3-inch-per-hour rate for the snail. He also divided the trip into two parts. Symbolically, he was novice in that regard.

All three students seemed to divide the snail's journey into two equal parts. S1 worked the problem from that vantage point and completed it with correct answers using incorrect thinking. Even though S2 did something similar, his ratio marking illustrated that he was mapping out the snail's journey but was limited in that he was unable to graphically represent the snail's journey. S3 was able to graphically represent the snail's journey. However, he was not able to construct the relationships that would have produced linear representations that would have permitted him to symbolically find the changes in rate.

Currency Problem

S1 reinforced similar conceptions with her solution to the currency problem. Her table showed additive reasoning. She was not able to create a multiplicative relationship between the number of English pounds and the number of US. dollars. Her conceptions of rate seemed more like an apprentice. In terms of her understanding, she showed some semblance of correctness, perhaps at the apprentice level. That is, her writing out 2 = 3, 3 = 6, …, 6 = 15 meaning 2 pounds = 3 dollars showed an equivalent relationship that would produce a pattern between English pounds and U.S. dollars. However, her writing of $20.00 = 8 pounds showed an inappropriate solution with errors in her reasoning: This was simply novice.

S2 was able to represent the relationship between the English pounds and the U.S. dollars as an expert. He was able to construct a multiplicative relationship, draw a graph, and create a table and reason about the proportions in finding the worth of $20 U.S. dollars in English pounds.

Like S2, S3 was also at expert level, as was evident in his solution. He was able to construct a multiplicative relationship between the two currencies, create a table of values between them, and even determine the worth of $20 in English pounds.

S1 seemed to be making errors, perhaps because she lacked an understanding of the equivalence in measurements between the two quantities. S2 and S3 were able to use their ratio understanding to develop relationships between the two currencies. On the pretest, it appeared that these two students approached this problem intuitively (Hart, 1984).

Gasoline Problem

The Gasoline Problem allowed S1 to use her imagination to create a multiplicative relationship between the dollar amount cost and the gallons being pumped. She created a multiplicative relationship for the price of *x* gallons. She moved back and forth between practitioner and apprentice in her understanding and reasoning. She drew a linear representation for the price of *x* gallons.

S2's initial conception of ratio and rate was novice. He was unable to use his imagination to generate a multiplicative relationship between the dollar amount cost and the gallons of gasoline being pumped. He did not give the variables asked for in the problem, nor did he draw a graph showing the direct variation between the dollar amount cost and the gasoline being pumped.

S3 was able to use his imagination in solving the Gasoline Problem. He functioned as an apprentice. He constructed a linear graph to model cost per gallon. However, he seemed to pick numbers arbitrarily to plot one point and then created a linear graph with no indication of how it was constructed. He did illustrate an understanding of the ratios, but he did not explain in detail. However, the straight line nature of the graph demonstrated the constant nature of the relationship between the two quantitative variables.

S1 and S2 could not use their imagination to meaningfully construct the relationship between the dollar amount cost and the amount of gasoline being pumped. However, S3 showed signs that he could visualize the relationship as a direct variation. That is, S3 used his imagination as a conduit to visually represent the relationship as a linear graph (Carlson et al., 2002). S3initially had a process conception of a function in that he could envision the end result but was unable to perform the necessary sequence of operations needed to correctly plot and draw the linear graph (Silverman, 2006).

Lady Runner Problem

S1's handling of the Lady Runner Problem was clearly novice in that she was unable to create a multiplicative relationship from the given rate of minutes per mile or miles per minute. The part-whole relationship was not evident, and she did not show a direct relationship between miles traveled in terms of minutes.

S2 was expert in his solution presentation. He was able to reason with the proportions and solve the problem.

S3’s part-whole relationship seemed expert in that he solved the Lady Runner Problem using a multiplicative understanding. He wrote 3 minutes = 3/7 mile and 7 minutes = 1 mile, then 8 miles × 7 = 56 minutes.

With this problem, S1 was novice as compared with S2 and S3. This difference showed the developmental differences between the three (Collea, 1981). Adults can have underdeveloped reasoning with proportions (Tourniaire & Pulos, 1985).

Impact of Homework

Each of the three student teachers had different challenges that may have affected them differently. They started off with different initial conceptions of ratios, proportions, and rates.

Their initial conceptions of proportions were key in helping them evolve to a multiplicative understanding of how a family of equations of the form $y = kx$ could change in terms of a horizontal transformation such as $y = kx+b$.

S1 initially exhibited a novice understanding of ratios and proportions. As the semester began, she did seem to understand both the ratio and proportion definitions and could solve basic problems by simply performing the computations. Applications needing to show a progression towards understanding presented a challenge. That was evident with both the Orange Juice Problem and the Log Problem. For the Orange Juice Problem, she did not see the one part concentrate and the four parts water as forming five equal parts to make the pitcher of orange juice. In the Log Problem, she did not see all aspects of what was taking place in the problem, so she used the correct procedure to solve the problem with incorrect proportions. S1 did show the slope and was able to draw a graph in some of the problems; however, she used formulas learned in college algebra. On homework problems involving rate, she resorted to using formulas she downloaded from the Internet. She resisted change and did not use ratios and proportions to build a multiplicative understanding that was needed to generate or construct a linear equation or linear function. This resistance and her need for a quick solution made it difficult for her to build her understanding from a novice proportional thinker to an expert one.

S2 initially was expert in ratio and proportional reasoning. His multiplicative understanding enhanced his ability to construct linear equations and linear functions. He was not reliant on formulas as S1 was. He seemed to embrace learning of the concepts and developing his proportional understanding. S2 was not perfect, however. Even though he made mistakes with the Orange Juice Problem that is, the part-whole relationship with the water and concentrate he made only a few mistakes with his reasoning with other similar problems. In the currency problems, he showed a progression from constructing a multiplicative relationship to understanding transformations involved when the situation called for holding back amounts to make the conversions. His evolution at this point illustrated how he was freed from formula usage as he began to construct linear equations based on his proportional understanding. As the problems became more intense, he got better in his use of proportions. As in the Race Car Problem, he generated the proportions, used cross multiplication to generate linear equations to solve, and found where the rate changed.

S3 started off as expert, just like S2. He, too, progressed well with his proportional reasoning during the semester. In the case of the Race Car Problem, he struggled to generate linear equations that would show the change in rate, but the drawbacks he faced were limited. With the exception of the Log Problem, where he was unable to find the correct time it would take to get the number of pieces, he progressed well in developing the multiplicative understanding needing to construct linear equations and linear functions.

Reflections

For this study, ratios and proportional understanding were needed for students to be able to construct linear equations without resorting to set formulas where they made substitutions without much regard for understanding the underlying connectivity of the information that they were processing. The two students with a good working knowledge of ratios and proportions seemed to develop or express their multiplicative understanding well and did not seem to be reliant on using formulas. They seemed to develop a knack for building on their multiplicative understanding, as illustrated in the Homework Currency Problem and the Race Car Problem. The two students, S2 and S3, hovered around the expert level in terms of their initial conceptions of ratios and proportions, and each seemed to have the multiplicative understanding needed to construct linear equations. S1, on the other hand, hovered around novice and apprentice on the pretest. Her multiplicative understanding grew slowly, and she did not show that she constructed linear equations independently of formulas. Story-type problems that were used to help with constructing linear equations proved to be problematic for her.

The story-type problems were designed to support the students' progress in constructing linear equations that were supposed to show growth, starting with a family of linear equations whose graphs passed through the origin and progressing to linear equations that were transformations of the equations of that family. The student S1, who had difficulty with ratio and proportions, was hampered when solving the problems because of a lack of understanding about how the ratio was situated as it related to the particular transformation. She simply used the point-slope form of the equation that she recalled from college algebra.

In observing how prospective middle grades teachers construct linear equations and functions as it related to their conceptions of rate and proportional reasoning, I reflected on the assignments given during the semester and used their posttest and homework to analyze their construction of linear equations. In examining the posttest, it was clear that all three prospective teachers grew in their conceptions of ratio and proportions since the start of the semester. S1, who had a novice understanding of ratios, proportion, and rates, was able to improve in her reasoning about the concepts. However, she regressed when solving the Snail Problem. She did less work in the posttest than in the pretest. She still thought of the snail's journey as two equal parts, meaning that she could not assimilate information in the problem that would have helped her solve the problem (Tourniare & Pulos, 1985). Even though she generated a multiplicative relationship between the baby's age and weight, there is no strong indication that she understood rate in terms of a movement from one ratio point to the next. She resorted to using a formula when pressed to answer the Race Car Problem. Thus, the improvement in her understanding of the concept with the Baby Weight Problem did not change the challenges exhibited in the Race Car Problem that showed a reliance on a formula taken from the Internet.

S2 also continued to improve. He used his understanding of ratio and proportions to modify his thinking. For example, he took ideas from Steffe's notes[8] to work problems differently. In the Runner Problem, he used the idea of direct variation; that is, $p = nx$. The conceptual growth in ratio and proportion understanding can also be clearly seen within his

[8] Class Notes, EMAT 7080. Rates of Change and Their Graphs.

solution to the Race Car Problem. From the two ratios, he constructed the proportions and generated linear equations that helped him find a point where there was a change in rate. Like S2, he had strong initial conceptions of ratios and proportions. Unlike S2, who presented his work using acquired knowledge from reading, lectures, and problems done in classroom discussions, S3 was not as consistent. He had a good understanding of ratio and proportions, but he did struggle with these concepts when solving the Race Car Problem. The Log Problem also presented a challenge to him because he did not understand all of the information given in the problem (Tourniare & Pulos, 1985).

Implications of the Study

Students need to be situated in mathematics classrooms that are conducive to mathematical thinking. Concepts like ratios and proportions that should have been covered and mastered early in children's mathematics education need to be a part of their curriculum focus as they progress on throughout their academic life. These concepts should continually evolve throughout middle and high school as students use them in the study of linear equations and functions. In this study of three college students who were preservice middle school mathematics teachers, however, one of them was yet to construct proportional reasoning and rates, which seriously constrained her construction of linear equations as well as her ability to reach a correct solution in problematic situations. The two other students, who had constructed proportional reasoning at the start of the study, were yet to construct rich and powerful conceptions of linear equations as representative of the solutions of a variety of quantitative situations. Those two students were able to construct linear equations in the context of working problematic situations, but the necessity for them to actually produce the equations in context essentially disallowed pedagogical discussions concerning the teaching of linear equations. That is, because the class as a whole struggled with the concepts and time spent of disusing the concepts made that impossible.

Given the essential role that ratio, rate, and proportional reasoning played in the construction of linear equations for the three students involved in this study, student construction of linear equations needs to be carefully studied throughout the middle school and teachers need to be prepared for it. In order to investigate whether there are general methods of reasoning that undergird ratio, rate, and proportional reasoning. For students who are able to engage in ratio, rate, and proportional reasoning, given the performance of the two students in the study who were able to reason in the context of linear equations, studies are needed that are designed to investigate how such reasoning is leveraged in quantitative reasoning as it evolves into graphical and analytic representations of that reasoning.

Mathematics should be taught in a meaningful way. The concepts of ratios, proportions, and rates are used in everyday life, and they need to be taught in quantitative context, perhaps using story problems to help build understanding. Paying for a taxi cab, paying a phone bill, or even paying tuition are just a few examples of the use of ratios and rates in quantitative contexts. Direct variation is fundamental, and it is one of the basic ideas that should evolve into a family of linear equations passing through the origin, as well as serving as an example of how two quantities vary in a direct way. For example, paying tuition and clocking and timing speed each has a classic form: some function equal to the sum of a product of a rate and a count with a fixed amount (i.e., $f(x)$ = (constant rate of change) x + initial amount). These issues need to be addressed. Resources such as time, money, and effort need to be put into creating activities that connect concepts in a meaningful way that encourages students to work independently of formulas.

More research needs to be done with preservice middle school teachers on ratio, rates and proportions with concepts by context; with the concepts by using animated computer programs; with the concepts through creating class materials.

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APPENDICES

Appendix A

Pretest

PREASSESSMENT: STUDENT PROFILE

Name ____________________________

Professor ____________________________ Time: __________

Directions: Please complete all sections as directed.

Circle One

Middle Grades **Early Childhood**

I am **18—22 years old** **23+ years old**

I. High School algebra courses studied: Circle each completed:

Algebra I   Algebra II   Trigonometry   Geometry   Calculus   Other - ________

II. Please list any college mathematics courses taken:

III. Major challenge you think you will have in studying algebra concepts:

IV. Grade you anticipate earning in this course: _________

V. Circle your intentions for studying for this course:

Full Commitment   Half Commitment   Less than Half Commitment

VII. Describe your attitude towards mathematics ___________________________

STUDENT # ________

Show all work.

Baby Weight Problem

1. Betty weighed 6 pounds at birth. She gained 2 pounds every month for 8 months.

A. Create a table to represent the data over an 8-month period.

B. Draw a graph to represent the data, identify the input and the output.

C. As Betty gets older, how does her age relate to her weight?

D. Construct an equation to model the age-weight relationship.

Snail Problem

2. A snail crawls the first part of 48 inches at the rate of 2 inches per hour. He crawls the rest of the way at the rate of 3 inches per hour. If it takes the snail 20 hours to crawl all 48 inches, how long did the snail crawl at each rate? Solve the problem graphically using graph paper and then algebraically

Currency Problem

3. Suppose 2 English pounds (£) buy the same goods as 3 U.S. dollars ($). Make a conversion table to compare the two currencies.

A. What would a garment of $20 cost in terms of pounds?

B. What's worth more, the pound or the dollar, to an Englishman just off the plane on vacation in the U.S.A.?

C. If p = the number of English pounds and d = the number of U.S. dollars, write the relationship between the number of pounds and the number of dollars? Explain.

D. Graph this relationship using graph paper.

Gasoline Problem

4. Imagine pumping gasoline into a car.

A. What is the relationship between the dollar amount cost and the gallons being pumped?

B. How would you determine the rate at which the gasoline is being pumped into the car?

C. If c=cost and g=the number of gallons pumped, sketch the graph to illustrate the cost of the gasoline.

The Runner Problem

5. A runner is running along and times how long she has run and how far she has run using a stopwatch and a distance meter. If one reading is 3 minutes and 3/7 mile, how long would it take her to run 8 miles if she runs at a constant pace?

Appendix B

Models and Functions

A. Determine whether the tables in Activities 1 and 2 represent function. Assume that the input is in the left column.

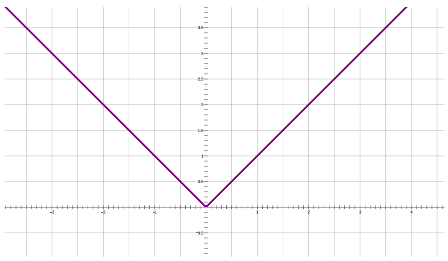
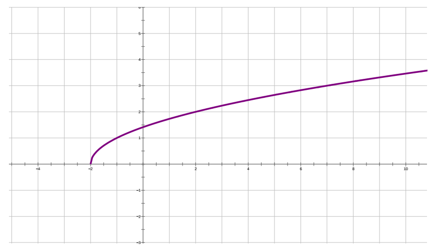
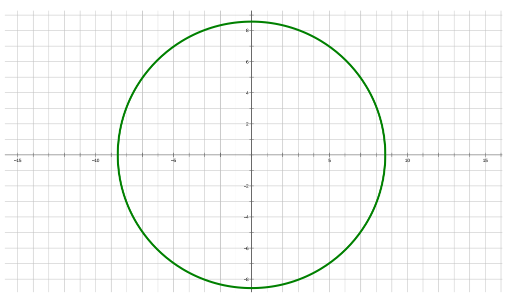
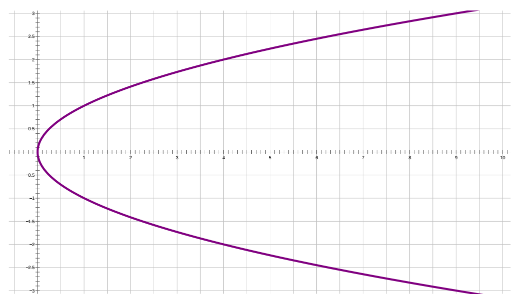
Activity 1

Year	iPod sales (millions of units)
2002	0.14
2003	0.336
2004	2.016
2005	6.451

Activity 2

Military rank (4 years of service)	Basic monthly pay in 2005 (dollars)
Second Lieutenant	2948
First Lieutenant	3541
Captain	3823
Major	4388
Lt. Colonel	4961
Colonel	5784
Brigadier General	7119
Major General	8459

B. Which of the following graphs represent functions? (The input axis is horizontal.)

A. 	**C.** 
B. 	**D.** 

C. **Darkness Hours** **H(d)** is the number of hours of darkness in Anchorage, Alaska, on the ***d***th day of the year. Write the following statements in function notation.

A. On the 121st day of the year, Anchorage has 7.5 hours of darkness.

B. The duration of darkness in Anchorage on the 361st day of the year is 18.5 hours.

C. In Anchorage there are only 4.5 hours of darkness on the 181st day of the year.

D. ***Sales*** A fraternity is selling T-shirts on the day of a football game. The shirts sell for $8 each.

A. Complete the table. Revenue is defined as the number of units sold times the selling price.

Number of shirts sold	**Revenue (dollars)**
1	
2	
3	
4	
5	
6	

B. Construct a revenue graph by plotting the points in the table.

C. How many T-shirts can be purchased with $25?

D. If an 8% sales tax were added, how many T-shirts could be purchased with $25?

E. **Weight** A baby weighing 7 pounds at birth loses 7% of her weight in the 3 days after birth and then, over the next 4 days, returns to her birth weight. During the next month, she steadily gains 0.5 pound per week. Sketch a graph of the baby's weight from birth to 4 weeks. Accurately label both axes.

F. For the equation below, find the input of the function corresponding to each output value given.

$f(x) = 7.2x + 3$; $f(x) = 6.6$, $f(x) = 3$

G. It cost a company $19.50 to produce 150 glass bottles.

A. What was the average cost of production of a glass bottle?

B. Assuming C(x) is the total cost for producing x units, write an expression for average cost per unit.

C. **Milk Consumption** Per capita milk consumption in the United States between 1980 and 1999 can be modeled by

M(t) = -0.219t + 45.23 gallons per person per year and consumption of whole milk for the same period can be modeled by **W(t)** = $0.01685t^2 - 3.49t + 188$ gallons per person per year

In both models, t is the number of years since 1900. Use the models to construct a model giving the per capita consumption of milk other than whole milk, and estimate this per capita consumption for the year 2000.

D. **Transplants** The number of kidney and liver transplants performed in the United States between 1992 and 1996 can be modeled by K(x) = 9.09 + 1.7 In x kidney transplants L(x) = 2.42 + 9.2 ln x liver transplants where x is the number of years since 1900.

A. Construct a model for the number of kidney and liver transplants between 1992 and 1996.

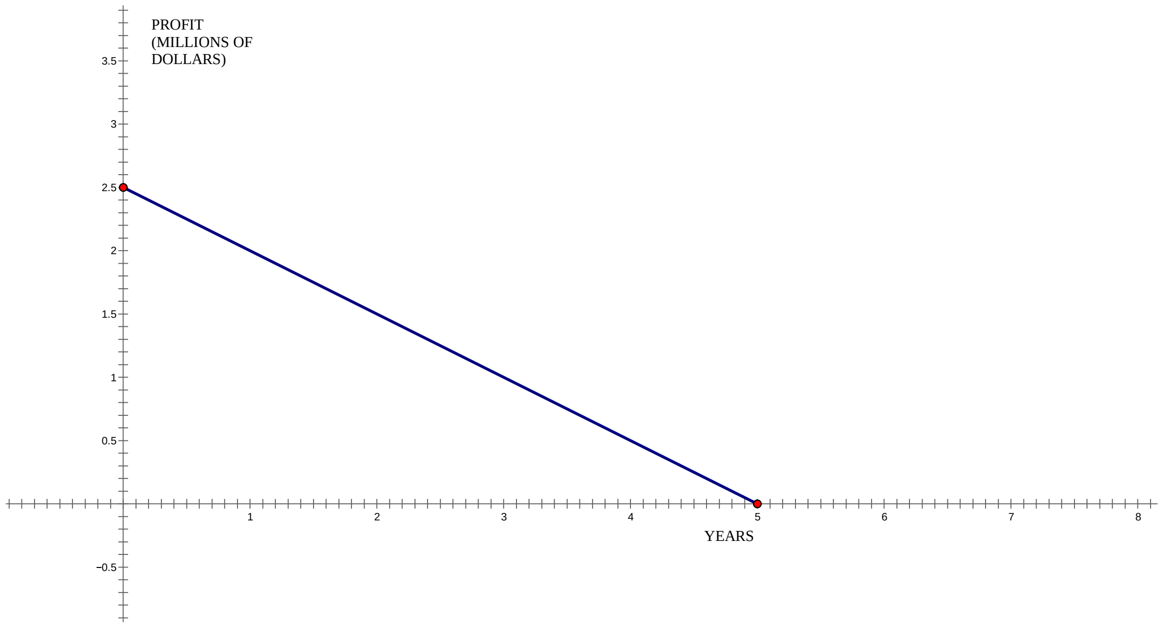
B. Use the model to estimate the number of kidney and liver transplants in 1995.

These problems are taken from: Latorre, D. R.; Kenelly, J. W.; Reed, I. B.; Biggers, S. (2008). *Calculus Concepts: An Approach to the Mathematics of Change* (pp. 19-23). Boston, MA: Houghton Mifflin Company.

Appendix C

Lesson Two: Linear Functions and Models

1. **Profit** The accompanying graph shows a corporation's profit, in millions of dollars, over a period of time.



A. Is the profit function increasing or decreasing? Is the slope of the graph positive or negative?

B. Estimate the slope of the graph, and write a sentence explaining the meaning of the slope in this context.

C. What is the rate of change of the corporation's profit during this time period?

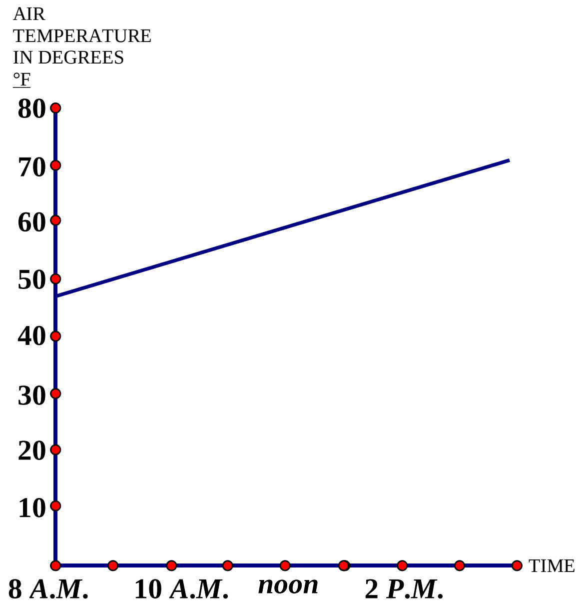
D. Identify the points where input is zero and where output is zero, and explain their significance to this corporation.

2. **Temperature** The air temperature in a certain location from 8 A.M. to 3 P.M. is shown in the accompanying graph.

A. Is the temperature function increasing or decreasing? Is the slope of the graph positive or negative?

B. Estimate the slope of the graph, and explain its meaning in the context of temperature.

C. How fast is the temperature rising between 8 A.M. and 3 P.M.?



1. **Organ Donors** The number of organ donors in the United States between 1988 and 1996 can be modeled by D(t) = 382.5t + 5905 donors ***t*** years after 1988.

A. Is the rate of change positive or negative? Is the function increasing or decreasing?
B. According to the model, what is the rate of change of the number of organ donors?
C. Sketch a graph of the model. What is the slope of the graph?
D. Find the point at which input is zero, and explain its significance in context.

2. **Bankruptcy** Total Chapter 12 bankruptcy filings between 1996 and 2000 can be modeled by B(t) = -83.9t + 1063 filings t years after 1996

A. Is the rate of change positive or negative? Is the function increasing or decreasing?

B. What is the rate of change of the number of Chapter 12 bankruptcy filings?

C. Sketch a graph of the model. What is the slope of the graph?

D. Find the point at which input is zero, and explain its significance in context.

3. **Revenue** The revenue for International Game Technology was $2128.1 million in 2003 and $2484.8 million in 2004. Assume that revenue was increasing at a constant rate.

4. Find the rate of change of revenue.

5. By how much did revenue increase each quarter of 2004?

6. Assuming that the rate of increase remains constant, complete the following table.

7. Find the equation for revenue in terms of the year.

Year	**Revenue** **(millions of dollars)**
2003	
2004	
2005	
2006	

8. **Car Value** Suppose you bought a Honda Civic Hybrid in 2005 for $24,000. In 2007 it was worth $18,200. Assume that the rate at which the car depreciates is constant.

9. Find the rate of change of the value of the car.

10. Complete the following table:

Year	**Value**
2005	
2007	
2008	
2009	
2010	

11. Find an equation for the value in terms of the year.

12. How much will the value of the car change during a 1-month period? Round your answer to the nearest dollar.

13. **Sales** A house sells for $97,500 at the end of 2000 and for $112,000 at the end of 2007.

A. If the market value increased linearly from 2000 through 2007, what was the rate of change of the market value?
B. If the linear increase continues, what will the market value be in 2010?
C. In what year might you expect the market value to be $100,000? $150,000?
D. Find a model for the market value. What does your model estimate for the market value in 2005? What assumption did you make when you created the model?

14. **Births** Thirty-two percent of U.S. births that occurred in 1995 were to unmarried women. The percentage of U.S. births to unmarried women in 2002 was 34.
    A. Find the rate of change of the percentage of births, assuming that the percentage of births to unmarried women increased at a constant rate.
    B. Estimate the percentage of births to unmarried women in 2003.

15. **Credit** Consumer credit in the United States was $1719 billion in 2000 and $2040 billion in 2003. Assume that consumer credit increases at a constant rate.

A. Find the rate of increase.
B. On the basis of the rate of increase, estimate consumer credit in 2006.
C. Is the assumption that consumer credit increases at a constant rate valid? Explain.
D. Assuming a constant rate of change, when will consumer credit reach $3 trillion?

16. **Break-even Point** You and several of your friends decide to mass-produce "I love calculus and you should too!" T-shirts. Each shirt will cost you $2.50 to produce. Additional expenses include the rental of a downtown building for a flat fee of $675 per month, utilities estimated at $100 each month, and leased equipment costing $150 per month. You will be able to sell the T-shirts at the premium price of $14.50 because they will be in such great demand.

A. Give the equations for monthly revenue and monthly cost as functions of the number of T-shirts sold.
B. How many shirts do you have to sell each month to break even? Explain how you obtained your answer.

These problems are taken from: Latorre, D. R.; Kenelly, J. W.; Reed, I. B.; Biggers, S. (2008). *Calculus Concepts: An Approach to the Mathematics of Change* (pp. 36-39). Houghton Mifflin Company.

Appendix D

Lesson Four: Instantaneous Rates of Change

Rates

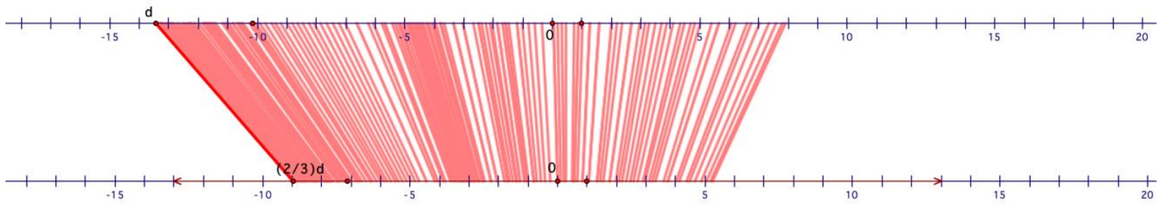
To establish the ratio 13/4 dollars/gallon as a rate, imagine reading the meters that measure the number of gallons and the number of dollars as a gasoline pump is pumping gasoline. If you stopped the pump at any time and formed the ratio between the number of dollars and the number of gallons at that particular time, you would get a decimal equivalent of 13/4 dollars/gallon. In fact, as the pump is pumping the ratio between the measurements of the two quantities is a constant. That is, the two quantities vary in a constant ratio. In this case the constant ratio is considered as a rate.

A rate is a measure of the change in one quantity with respect to the measure of the change in another quantity. For example, for each increase of one gallon, we have a corresponding increase of $3.25. This is a unit rate because an increase of one unit [one gallon] of gasoline corresponded to an increase of $3.25. In the shift from ratio to rate, there is a shift from fixed quantities to varying quantities. A quantitative variable is the potential measurement of a varying quantity.

Quantitative Variable: The potential result of measuring a varying quantity.

It is customary to use a letter to symbolize the measurement of a varying quantity and to call this letter a variable. For example, we might let **x** = the volume of water in a container that is being filled with water. **x**, however, is not the variable—it is only a symbol for the quantity of water that is in the container at any moment that it is being filled.

We can take advantage of the dynamic capabilities of GSP to represent the relationship between two quantities that vary in a constant ratio. We will start by using two number lines to represent the measurements of two variables. Let's use **p** = (2/3)**d** and let **d** be the independent variable. Let's also use two number lines, one to represent the number of dollars and the other the number of pounds. The following figure is a copy of a dynamic representation of the co-variation between the number of dollars and the number of pounds.



Problems

1 a. Make your own dynamic representation of **p** = (2/3)**d** using GSP like the figure above. If **d** = -$14, what is the corresponding value of **p**? What does it mean for **d** = -$14? What does the corresponding value of **p** represent?

b. Using the same situation, graph the rate using the coordinate axes.

2. Repeat both parts of (1) except use 3 Argentine Pesos is equivalent to 4 Danish Kroner. Use the Danish Kroner as the independent variable.

3. What is the slope of the line in (1)? In (2)? In (2), calculate the slope of the line using 4 Danish Kroner and 7 Danish Kroner. What did you need to do?

4. In (1), suppose that the British embassy gives each U.S. citizen 6 pounds if they convert their dollars into pounds for an incentive to shop in Britain. Now work 1(b) for this situation.

5. In (2), the Argentine embassy decides that all Danish visitors to Argentina must hold back 10 Danish Kroner and not convert them to Argentine Pesos. Using the x-y coordinate plane, make a graph of the number of Pesos that each Danish visitor receives with respect to all of the Kroner the visitor brought into Argentina. Be sure to develop an equation.

6. In (5), suppose that the Argentine embassy also decides that they will give each Danish visitor 6 Pesos as an incentive to shop in Argentina. Now, repeat number 5 assuming that each Danish visitor also holds back 10 Kroner.

7. Water is being pumped out of a dam into an irrigation tank at the rate of 5 gallons per second. The meter that records the number of gallons that is pumped into the irrigation tank and the number of seconds that the pump running is on, is turned on 7 seconds after the operator came to work. At that time, the irrigation tank already has 23 gallons of water in it. Make a rate graph that represents the relation between the total number of gallons of water that is in the irrigation tank at any time relative to the time the operator came to work. Then develop an equation for the graph.

8. A jogger runs 3.1 miles in 25 minutes. He starts out at a 7-minutes-per-mile pace, but tires along the way, and drops back to a 9-minutes-per-mile pace.

A. Use GSP to find graphically how many minutes elapsed before he changes pace and how many miles he has run.

B. Now develop the equations of the two rate graphs and explain how you did it.

C. Solve the two equations simultaneously for the time and the distance. Be sure your algebraic solution corresponds to your graphical solution.

9. A snail crawls the first part of 48 inches at the rate of 2 inches per hour. He crawls the rest of the way at the rate of 3 inches per hour. If it takes the snail 20 hours to crawl all 48 inches, how long did the snail crawl at each rate? Solve the problem graphically and then algebraically.

10. A racecar driver drives part of a 500-mile race at 150 miles per hour and the remainder at 180 miles per hour. If it takes her a total of 3 1/4 hours to complete the race, how long at each rate did she drive? Solve the problem graphically and algebraically.

11. Mandy decides to take two jobs to pay her debt. The first pays her at a rate of $30 per hour and the second pays her at the rate of $16 per hour. If she works for 60 hours and earns $1400, how long did she spend on each job? Graph the rates of change and find the equations based on this graph.

12. Water boils at 100º Celsius and 212º Fahrenheit and water freezes at 0º Celsius and 32º Fahrenheit. Develop a relationship between the two scales for measuring temperature and graph the relationship using GSP.

13. Rate situations provide a nice context to develop meaning for the product of two signed numbers. Consider the following rate situation: Grain is elevated out of a grain bin into a huge truck at the rate of 7 bushels per minute [+7 bushels per minute]. A meter that records the number of bushels of grain that comes out of the elevator and the time the elevator is on is attached to the elevator, much like what you experience at a gasoline station when a meter records the number of gallons of gasoline and the price of the gasoline as you pump gasoline.

    a. If we take 12:00 Noon as the reference time, make a graph that represents the number of bushels ***b*** elevated into the truck at time ***t*** relative to Noon. How do you think about the number of bushels elevated into the truck at any time ***t*** after the reference time? At any time ***t*** before the reference time? [Note: take the number of bushels in the truck at the reference time as your reference number of bushels—as a constant number of bushels. Let the reference number of bushels be 0 because we are interested only in the number of bushels elevated into the truck relative to the reference time, not in the total number bushels in the truck.]

    b. Now, use the number of bushels of grain in the grain bin at the reference time as your reference number of bushels—or constant number of bushels. With respect to this reference number of bushels, the grain is being elevated out of the bin at a rate of –7 bushels per minute. Make a graph that represents the number of bushels ***c*** elevated out of the grain bin [***c*** must be thought about with respect to the reference number of bushels in the grain bin] at time ***t***. What is the relationship between ***c*** and ***b*** of part (a)?

    c. Use your reasoning in (a) and (b) to establish multiplication of directed measurements [i.e., what is (+7)(+3)? (+7)(-3)? (-7)(+3)? (-7)(-3)?]

Appendix E

Lesson Three: Describing Change: Rates

Problem Set 2: Rates of Change and Their Graphs

Unit Ratios

In this problem set, we continue to investigate the construction of the straight line and its equations. In Problem Set II, we worked with rate problems, but we did not systematically develop the concept of rate by starting with ratios. Consider the following monetary exchange. Two British pounds buys as much as 3 U.S. dollars. We first develop the concept of a unit ratio by asking how many pounds are there per dollar?

If the top two red rectangles represent the 2 pounds and the bottom three yellow rectangles represent the dollars, if we distribute the 2 pounds equally among the 3 dollars, the question is how much of a pound is there per dollar. After making the partition of each pound into three equal parts and distributing one part to each dollar, we find that there is 2/3 of a pound for each dollar. This is called a unit ratio because each dollar is considered as a unit. So we converted the ratio of 2 pounds for 3 dollars to two-thirds pounds for 1 dollar. The latter is called a *unit ratio* and it is expressed as 2/3 pounds per dollar. If the slash is introduced, it is abbreviated as 2/3 pounds/dollar.

Now, if a traveler wants to convert 4 dollars into pounds, using the unit ratio, the number of pounds is found as follows: 2/3 pounds/dollar x 4 dollars. It is crucial to *not* think as follows: "The dollars cancel out leaving pounds." Thinking like that simply impedes understanding. Rather, if the traveler has 4 dollars, for each dollar she is given 2/3 pounds, so she has 2/3 pounds four times, or (2/3)x4 pounds.

In general, if **d** = the number of dollars the traveler has and **p** = the number of pounds she will receive for the **d** dollars, then **p** = (2/3)**d.**

Converting to a unit ratio can alleviate the confusion that is usually rampant when first considering the monetary equivalence. This confusion usually starts when someone simply abbreviates pounds using "p" and dollars using "d" and writes 2p ≈ 3d thinking "2 pounds is equivalent to 3 dollars" [note the equivalence sign ≈]. There is nothing wrong with this if it is explicitly remembered that "p" does *not* stand for the *number* of pounds, "d" does *not* stand for the *number* of dollars, and the ≈ sign is an equivalence between two systems for measuring monetary value much like "180º degrees Fahrenheit is equivalent to 100º Celsius" is an equivalence between two systems for measuring temperature.

After students can establish unit ratios, the proportion **p** is to **d** as 2 is to 3 can be established by asking questions like, "How many pounds can be exchanged for 5 dollars?" Here, **p** is to 5 as 2 is to 3. In the case of the question, "How many dollars can be exchanged for 7 pounds?" we would need to establish how many dollars per pound. In this case, the proportion would be **d** is to **7** as 3 is to 2 [why?]. Either of these proportions can be rewritten using 3**p** = 2**d**. [Note that this equation is much different than the expression of equivalence 2p ≈ 3d when "p" and "d" simply stand in for "pounds" and "dollars", respectively, but *not* the number of pounds and the number of dollars.].

If students can establish unit ratios, rather than establishing the proportion **p** is to **d** as 2 is to 3 [or **d** is to **p** as 3 is to 2], establishing the equation **p** = (2/3)**d** **[or d =** (3/2)**p]** not only makes conversions "easier," it also opens the way for students to construct the straight line. Remember, 2/3 is a unit ratio—the number of pounds per dollar and 3/2 is a unit ratio—the number of dollars per pound. Consider the following situation.

Imagine we have a rather large barrel of pennies to count, large enough so that counting them is very impractical. We do have, however, a scale that we can use to weigh the barrel of pennies or any part of them. *Devise a way to find out about how many pennies are in the barrel by means of using the scale before you read further.*

We could first establish a unit ratio be finding how many pennies per pound. We could then weigh the barrel of pennies with the intention of multiplying the number of pounds the barrel of pennies weigh times the unit ratio to find the approximate number of pennies in the barrel. So, let's define our unknowns. If **k** = the number of pennies per pound, **p** = the number of pounds the barrel of pennies weigh, and **n** = the number of pennies in the barrel, then the following equation represents a multiplicative relationship between **n** and **p**:

$$\mathbf{n} = \mathbf{k} \times \mathbf{p}.$$

In words, the number of pennies in the barrel is equal to the number of pennies/pound times the number of pounds the barrel of pennies weigh. Note that **n, p** and **k** are unknowns prior to actually weighing.

The idea of a quantitative unknown is essential in starting to take students from arithmetic to algebra. For example, we know that there are a certain number of pennies in the barrel and that we could measure them by counting them. Or we know that the barrel of pennies has a certain weight and we could measure the number of pounds by actually weighing the barrel. These outcomes of measuring are unknown before we actually measure, and this is what we mean by a quantitative unknown.

Quantitative unknown: The potential result of measuring a quantity before actually measuring it.

Complete the Following Exercises

A. If 20 gallons of gasoline costs 65 dollars, develop a multiplicative equation between the number of gallons and the number of dollars using a unit ratio. Explain how you found the unit ratio. Use fractions rather than decimals. Be sure to define the unknowns. Now, develop the inverse unit ratio and its equation.

B. If 20 candy bars are shared among 65 people, develop a multiplicative equation between the number of candy bars and the number of people using a unit ratio. Explain how you found the unit ratio. Use fractions rather than decimals. Be sure to define the unknowns. Now, develop the inverse unit ratio and its equation.

C. If a runner goes 3 miles in 20 minutes, develop a multiplicative equation between the number of miles and the number of minutes using a unit ratio. Explain how you found the unit ratio. Use fractions rather than decimals. Be sure to define the unknowns. Now, develop the inverse unit ratio and its equation.

Appendix F

Student Consent Form

Student Consent Form

An Investigation of prospective middle grades teachers proportional and rate reasoning as they develop their understanding of linear equations and variational reasoning

I ______________________________, agree to participate in a research study titled "An Investigation of prospective middle grades teachers proportional and rate reasoning as they develop their understanding of linear equations and variational reasoning," conducted by Samuel Cartwright from the Department of Mathematics and Science Education at the University of Georgia (478 825 6997) under the direction of Dr. Leslie Steffe, Department of Mathematics Education, University of Georgia (706 542-4553). I understand that my participation is voluntary. I can refuse to participate or stop taking part without giving any reason, and without penalty. I can ask to have all of the information that can be identified as mine, removed from the research records, or destroyed.

PURPOSE: The purpose of this research is to investigate the connections made between proportions, rates, and linear equations as prospective middle grades teachers reasoning evolve from proportional and rate reasoning into variational and covariational reasoning respectively.

PROCEDURES: I also understand that I may be asked to participate in at least two semi structured sixty-minute task-based audio-taped interviews. The video will not include any visual of the participants: it will focus only on the prospective middle grades teachers task related work. For example, seat work or board work showing their problem solutions. The interviews will be focused on mathematical thinking as it relates to my problem solutions on either classroom discussions or assessment documents where there were unexpected peculiarities within the algebra concepts course. Participation in these interviews will take place in a way that does not take away from my regular school duties. Each interview will be approximately one (1) hour in length. The extent and timing of all data collection will be negotiated between the researchers and me.

BENEFITS: There is no direct benefit from participating in this study.

DISCOMFORTS & RISKS: There are no risks or discomforts anticipated as a result of my participation in this study. With the exception of interviews, all data will be collected during the regular scheduled algebra concepts course class time. This will ensure that it does not interfere with any of my other scheduled courses.

CONFIDENTIALITY: Any reports of this research will use pseudonyms. No information that identifies me will be shared with others without my written permission. Any data gathered, audiotapes as well as copies of students' written work will be used for research purposes only. All tapes will be destroyed three years after the project is completed (May 1, 2012). No information that identifies me will be shared with school officials.

FURTHER QUESTIONS: The researchers will answer any further questions I have about this research, now or during the course of the project. The primary contact person is Mr. Samuel Cartwright, who can be reached at (478) 825 6997 or at cart29@uga.edu or cartwris@fvsu.edu.

CONSENT: My signature below indicates that the researchers have answered all of my questions to my satisfaction and that I consent to participate in this study. I have been given a copy of this form.

Participant's signature____

Researcher's signature__________________________________ Date: __________

Samuel Cartwright, Tel. 478 825 6997, Email - cart29@uga.edu or cartwris@fvsu.edu.
Additional questions or problems regarding your rights as a research participant should be addressed to The Chairperson, Institutional Review Board, University of Georgia, 612 Boyd Graduate Studies Research Center, Athens, Georgia 30602-7411; Telephone (706) 542-3199; E-Mail Address IRB@uga.edu

Appendix G

Homework I

Show all work for full credit.

1. Frozen orange juice concentrate is usually mixed with water in the ratio of 4 parts water to 1 part concentrate. How much orange juice can be made from 18-oz can of concentrate?

2. Last year at the Laundromat Users convention, 3,224 people ate 1,003 chickens at the Saturday afternoon picnic. This year, 3,820 people are expected to attend. How many chickens should be ordered?

3. It takes 30 minutes to cut a log into 5 pieces. How long would it take to cut a similar log into 6 pieces?