HIGH SCHOOL STUDENTS’ LEARNING OF 3D GEOMETRY USING iMAT (Integrating Multitype-representations, Approximations and Technology) ENGINEERING

by

BERNARDO JORGE CAMOU

(Under the Direction of John Olive)

ABSTRACT

A review of high school geometry texts and mathematics curricula indicate that 3D geometry is usually just a small portion of most geometry courses, and the research literature indicates that it is seldom taught. This research study attempted to demonstrate that the approach called iMAT (integrating Multitype-Representations, Approximations, and Technology) could be effective for enabling students to learn 3D geometry in a meaningful and efficient way. This approach was based in the consideration of mathematics as a quasi-experimental science. Therefore, to learn 3D geometry, it is crucial to experiment using different technologies to produce several representations of the same 3D object. Each representation by itself is not enough to provide the necessary support to enable the conceptualization of the geometric object.

The researcher designed a 3D geometry unit to be taught to 140 high school students from Uruguay and the United States from June 2011 to October 2011. Four institutions and seven groups of students participated in the experiment. A mixed-methods methodology was developed with five sources of data: pretest, posttest, interviews, questionnaires, and video recording. The research questions were as follows:
1. What could high school students learn about Geometry (both 3D and 2D) in a two-week experimentation, using iMAT engineering?

2. How did that learning occur?

3. Can indicators be found that the teaching-learning iMAT approach could overcome the main epistemological and didactical obstacles for learning 3D Geometry?

A six-item pretest was administered during the first class of the unit. The researcher taught each class for 2 weeks and then administered the six-item posttest. Each class lesson was videotaped and the lesson reviewed each day by the researcher. All students were interviewed in focus groups and completed questionnaires after the posttest. Analyses of these qualitative data indicate that students learned important concepts in 3D and 2D geometry during the two-week experiment, thus suggesting that iMAT is effective for tackling the study of 3D geometry. The quantitative data were analyzed using ANOVA and multiple regression tests on the pre- and post-test results. These quantitative analyses support the qualitative conclusions. Moreover, these analyses indicated that the iMAT approach was robust with respect to different factors such as country, pretest knowledge, and gender.

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by

BERNARDO JORGE CAMOU

FORB, Instituto de Profesores Artigas, Uruguay, 1987

FORM, Université Joseph Fourier, France, 2004

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BERNARDO JORGE CAMOU

Major Professor: John Olive
Committee: Jeremy Kilpatrick
James Wilson

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
May 2012
DEDICATION

I dedicate this dissertation to someone that wanted me to be an engineer of machines, roads and bridges. I kind of disappointed him, when I decided to dedicate my life to mathematics education. Since then, I put every bit of effort and energy I had, in building bridges between mathematical knowledge and students minds, constructing roads along which everyone could discover and enjoy the wonders of mathematics and designing didactical machines to transform complexity into simplicity, darkness into brightness. I don’t know where this person who wanted me to be an engineer is now. I don’t know either, if he would know about my lifetime research work for this dissertation. I would just like to tell him: “You were right and I was right.” It was hard for me at times to continue believing that I was right. If there is something that completing this PhD degree changes in my life, it is that I cannot hesitate anymore. No matter what the future brings, how meaningless could appear, how useless might seem my efforts, I was right. Here is my work; I hope you like it; I hope other people find it useful and I hope that mathematics, for the future generations, is no longer the feared subject but on the contrary becomes the beloved subject, the place where everybody enjoys discovering the fantastic adventure of the human creation always in search for truth and beauty.
ACKNOWLEDGEMENTS

There are so many people I should be mentioning here that I wouldn’t know where to start. Among the hundreds of people that touched my life in different ways I would like to first remember my father Jorge. One day, a time long ago, when I was a teenager studying geometry I asked him: “Dad, if I have three points that are not aligned how can I construct with ruler and compass the circle that passes through those three points?” He went to his office, worked for some minutes, then he came back and told me how to proceed.

Then, I would like to thank my mother Inés who always loved to play with children with didactical games; that is, those games where we learn interesting things and have fun at the same time.

I would like to thank very much my wife Bea for supporting me (in spite of all the inherent difficulties) in this crazy adventure of completing a PhD far from home and for having made most of the Spanish interviews transcriptions. I thank my daughter Eugenia for her professional work concerning some of the pictures of this dissertation. I thank my son Gaston for video-recording several experimental classes at Gwinnett County (GA) and for teaching me about formatting (although I was a lousy student).

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Chapter 1

Problem Statement

The Story Behind the Research

Everything has a story behind it. Sometimes it is tedious to tell it or to hear the story, but some other times if we omit it, we cannot completely grasp the real sense of the present facts. In 1988, I was a young Uruguayan mathematics teacher in his 3rd year of activity and made my ninth-grade students construct using poster-board five outstanding mathematical bodies: the Platonic Solids. There was a unit of 3D geometry in the syllabus and I thought that learning about the unique five regular polyhedrons was a good natural starting point for the study of spatial geometry.

Amazed and bewildered about how these unique and beautiful geometric objects had slipped unnoticed through all my mathematics education, I asked myself a very simple question: How can we calculate the volume of the regular icosahedron and the regular dodecahedron? During my life as a student and young teacher I certainly had calculated the volume of dozens of prisms, pyramids, cylinders, cones and spheres but never made the least attempt to calculate the volume of these two regular polyhedrons. It was strange that after having studied mathematics for many years, I had never a teacher propose this problem to me. In those times the Internet didn’t exist so I was not able to browse the Web and find a document about it. I did find in a Russian handbook of mathematics (Bronshtein & Semendyayev, 2002, p.200) the following values:
\[ V_{\text{icos}} = 2.1816 \, a^3 \quad \text{and} \quad V_{\text{dode}} = 7.6631 \, a^3 \]

(“\(a\)” is the size of the polyhedron’s edge)

In the book, only those numbers were written; there was no trace of how they were found; moreover I did not know whether those numbers were rational decimals or irrational numbers rounded to four decimal places.

From the very beginning the question of calculating those volumes it was both a Didactical and Epistemological question. Why was this a Didactical question and not just a strictly mathematical question? Because, from the very beginning, I knew that this knowledge was already available at least in high academic circles. So the questions were essentially the following:

- How can I learn to calculate those values for myself?

It was typically a Didactical question since the issue was, how can I learn and, then how can I provoke others to learn a particular piece of mathematical knowledge.

- Why was this problem also an epistemological question?

Epistemology is the branch of philosophy that studies knowledge. Therefore epistemology tries to answer questions like: What is knowledge? What is the source of knowledge? What things can we know? What is the value of knowledge? Epistemology analyzes the process of construction of knowledge, its historical development and its present meaning. The Platonic Solids, of which there are just five, have been a source of inspiration for outstanding scientists throughout the centuries; they have been present in the history of mathematics for more than 2000 years. History of mathematics contributes greatly to epistemology of mathematics. Moreover, if we reflect just a little about what constitutes essential knowledge in mathematics we will readily realize that regularities, invariances and symmetries
lie in the core of any study of mathematics and science in general. Any theorem is a statement that warrants an invariance. For instance the Pythagorean theorem states that: in every right angle triangle there is a certain relationship between the hypotenuse and the legs of the triangle that holds always. The search and proofs of invariances lies in the core of mathematics; in particular in this problem it was the search of the invariant quotient between the volume of the regular icosahedron and its edge to the cube and similarly for the dodecahedron.

Either in mathematics, physics and chemistry when we study a new configuration, we normally start by studying the configurations that hold more symmetry and regularities, before moving to study more asymmetric ones. The existence of many symmetries and regularities is particularly true for the Platonic solids: the Tetrahedron, Cube, Octahedron, Dodecahedron and Icosahedron. These five bodies are rich in mathematical properties and thus they are the natural starting point, for the study of 3D geometry. Furthermore, the fact that there exists only five regular convex polyhedrons is an extra factor that motivates their thorough study and adds to their undoubtedly epistemological value. Why does this adds to their epistemological value?

There are 13 Archimedian solids and 13 Catalan solids, 92 Johnson polyhedrons, infinite type of pyramids, prisms and anti-prisms. Regarding 3D geometry we are surrounded by big numbers and infinity. The fact that the set of the most symmetric polyhedrons is formed by just five elements seems to be a good indication about where to start the study of 3D geometry.

The story continues. More than 10 years later I travelled with my wife and my two young children to Philadelphia to visit some relatives. Spending two days in New York City seemed to be the thing to do, and once we were in the Big Apple, visiting the toy-store FAO Schwartz became an obligatory trip for our children.
A plastic lattice regular icosahedron like the one in Figure 1 was standing at the store

![Plastic Icosahedron](image)

A toy or a precious mathematical object?

*Figure 1: Plastic Icosahedron*

Three years later the “toy icosahedron” had decisively helped me to answer the question:

\[
V_{\text{icos}} = \frac{15 + 5\sqrt{5}}{12} a^3 \quad \text{and} \quad V_{\text{dode}} = \frac{15 + 7\sqrt{5}}{4} a^3
\]

\(a\) is the edge’s length

Almost 15 years had passed since the beginning. The goal was achieved but it seemed pale compared with the travelled route. There was more value in having tackled an epistemological problem than in the solution itself. The journey, filled with struggle, approximations, trials and errors and multiple representations of the bodies, started to supersede the initial goal of calculating the volumes.

Thus, in the summer of 2001 sitting on the sands of an Uruguayan beach, I started to write this story thinking that perhaps the story was worth being told. For more than 12 years I have been trying to “solve a problem.” But when I finally succeeded I realized that more important than having solved the problem was to have tackled an epistemological problem.

“Problem solving” has turned into “tackling epistemological problems”. Tackling turns out to be more important than solving. When we tackle a problem the obvious goal is to solve it.
But still a problem could be very valuable, even if at present we cannot solve it. “Tackling” is valuing the whole process and not just the final and successful attempt. And what do I mean by epistemological problems?

The adjective epistemological refers to the value that the problem has, on one hand for mathematics itself, and on the other, for the individual to construct his or her own mathematical knowledge. Not all the problems have the same value for developing mathematical understanding and knowledge. There are certain problems that are crucial, because they enable us to structure knowledge and move forward. Epistemological problems refer to the kind of problems that we necessarily need to tackle because it is in this struggle where we will be able to acquire certain fundamental concepts and skills.

Lakatos (1976) writes about epistemology of mathematics: “In the deductive style, all propositions are true and all inferences valid .... the struggle is hidden, the adventure is masked. The whole story disappears”(p.142). When mathematics is presented as an ever infallible science, we hide what contributes to make mathematics meaningful; the thrill and passion within discovery and creation are concealed.

Epistemology asks: What can we learn? Where does this knowledge come from? What’s its story? Does it make mathematical sense to learn this?

And didactics asks: Does it make sense for the student to learn this? How can we ourselves learn, and how can we provoke others to learn this? What situations do we need to create in order to make this knowledge knowable?

Thus, pushed by the belief in the validity of this approach, I sought for the possibility of doing higher studies in France. Against all the odds (see Camou, 2006), I was admitted for a master’s degree in Didactics of mathematics at Grenoble, a French city in the Middle of the Alps,
birthplace of one of the most widely celebrated software for learning plane and spatial geometry: Cabri.

At that time (2003) a 3D version of their dynamic geometry software called Cabri 3D was being developed. This software enables the user to construct 3D bodies on the computer’s screen as easily as we construct 2D figures with current dynamic geometry software. Probably one of the most outstanding features of this new software is that once we have the body’s representation, we can change the point of view in every possible way, thus faithfully simulating our perception of spatial bodies.

Some time later in 2007 during a Conference in Charlotte (North Carolina) where I was presenting a small work-shop called “A didactic engineering for tackling epistemological problems”, Professor John Olive from UGA, invited me to apply for a PhD in mathematics education at the University of Georgia.

Because my wife and two children were in my country Uruguay this invitation was a huge challenge for me. Even though Dr. Olive showed me in several ways that he really meant his invitation, encouraging me every time he had a chance, there were numerous issues to be resolved, documents to be provided, exams to be taken, proofs to be shown. But finally beginning of January 2009 I was able to make it, and I arrived in Athens and enrolled as a graduate student in the Department of Mathematics and Science Education at UGA.

iMAT Engineering

What was born as just an attempt to solve some particular 3D geometry problems developed into an approach called “tackling epistemological problems” but then it was necessary to better specify the ways in which those epistemological problems were to be tackled. In this
context, I created the concept of iMAT engineering. iMAT is an acronym for “integrating Multi-
type representations, Approximations and Technology”. First I provide a brief explanation about
the word *engineering* in this context. The word “engineering” as used by Artigue (1990) and
others is a word that emerged within the French didactics of mathematics to label a systematic
didactical work that is comparable to the work of an engineer. By this, we mean that for
developing a certain sequence of learning about a specific topic, the didactical engineer needs to
take into account both theoretical and practical aspects of a particular topic of knowledge to be
taught in order to design a feasible plan. Then, the phrase “multi-type representations” is crucial
in the whole approach.

In a paper I wrote for a UGA course during 2010 called “The Geometry We teach” I
analyzed seven Geometry textbooks used in the USA. Six of those books dedicate, on average,
less than 10% of their pages to 3D Geometry. This is clearly a symptom of the weak presence of
3D geometry in secondary mathematics in the United States. To support this assertion we can
take a look at the Common Core State Standards; one will readily realize that the 3D Geometry
content is very limited, restricted mainly to just the calculation of areas and volumes of prisms,
cylinders, pyramids, cones and spheres. But this state of affairs is not exclusive to the United
States. Not only is the presence in the curriculum of 3D geometry very limited, but even when it
exists in the syllabus, teachers tend to not teach it at all. Bakó (2003) refers to this phenomenon
when she wrote that in France only 10% of the teachers teach spatial geometry; that is 9 out of
10 teachers do not teach spatial geometry in spite of the fact that it is present in the curriculum.

We live in a 3D world but when we teach geometry we mainly deal with 2D geometry
leaving 3D geometry neglected in a corner. This does not make any sense.
One of the greatest obstacles to changing this undesirable situation is the problem of representation. Whereas representations of 2D objects are in many cases straightforward, effective and faithful, representations of 3D objects are problematic, complex, time consuming and less faithful. Whereas in many cases in plane geometry it is enough to have just one representation, in 3D geometry one necessarily need multiple representations of the objects because no single type of representation capable of capturing all the features and properties of a 3D object. Thus, one needs to work with 3D solid models, 3D lattice models, Cabri 3D figures, 2D dynamic or static constructions and free hand sketches.

Each type of representation approximates the 3D object in different ways and has its virtues and its limitations. Different kinds of technologies and tools (software, manipulatives, ruler and compass) are needed to make these representations and operate with them. The iMAT approach then tries to integrate these multiple representations, making use of the available technologies. The use of multiple representations entails also the integration of different branches of mathematics and the use of approximations and estimations as the normal road towards final accuracy and exactness.

Mathematics is becoming to be considered after Lakatos (1976) a quasi-experimental science, which means that as in other sciences, the experiment is the first source of generation of mathematical knowledge. The experiment in geometry starts with the representation of the mathematical object, which in the case of 3D geometry needs to be necessarily a multi-type-representation, the only way to capture and manage the necessary information to progress in this branch of mathematics. To ignore the challenges to produce a representation of a 3D body and once the representation is obtained, to fail to recognize its weakness is probably the main epistemological obstacle for learning 3D geometry. To be oblivious of these facts and try to
teach 3D Geometry based in single, planar representations adds a didactical obstacle that reinforces the epistemological obstacle.

3D Geometry is a branch of mathematics that urgently needs to be tackled, studied, taught and learned at secondary school in ways that have not yet been done before. I am presenting iMAT engineering as one possible way. The iMAT approach has been developed and already tested with high School (2006, 2007 and 2008) and college students in both Uruguay (2008) and in the United States (2010, 2011) and I shall now attempt to further develop and validate the approach through an empirical research study with students of different citizenships and different native languages, in the context of this dissertation

Research Questions

The three major questions that I attempted to answer in this study are as follows:

1. What can high school students learn about geometry (both 3D and 2D) in a two-week experimentation, using iMAT engineering?

2. How did that learning occur?

3. What indicators are there that the teaching-learning iMAT 3.approach can overcome the main epistemological and didactical obstacles to learning 3D geometry?
Chapter 2  

Theoretical Framework

The theoretical framework that underpins iMat Engineering is social constructivism. My three main referents for the constructivism framework are Piaget, Vygotsky, and Bruner. Bartolini-Bussi (1994) illustrates well the different (but not mutually exclusive) perspectives between the two founders of social constructivism: Piaget and Vygotsky. She writes: “The Piagetian approach is based on individual schemes while the Vygotskian approach is based on social relations; for Piaget the learning process is determined from inside, for Vygotsky, it is determined from outside” (p.124).

For Piaget, there are stages of development that precede the stages of learning. A child cannot operate abstractly if he or she has not previously operated concretely.

Vygotsky (1930) introduced the ZPD (zone of proximal development), an area of knowledge where the individual in contact with a peer or a teacher can progress in ways not possible for him or her if alone. In the ZPD, learning can produce development, changing Piaget’s perspective, where development precedes learning.

*Development* refers to the changes occurring in people over time that follow an orderly pattern and that are naturally part of the growing process, whereas *learning* refers more specifically to an enduring change in the capability to behave in a certain fashion, usually acquired in a process that involves social interaction. Even though Piaget’s and Vygotsky’s perspectives could be seen as being in opposition, they can also be seen as complementary in a dialectic relation where each individual needs to construct his or her knowledge based in its own
cognitive structure but at the same time his or her constructions would be framed by the social context and interaction in which he or she is embedded. Laborde (1994) elaborates about working in small groups, which is a privileged instance where individual and social learning can occur simultaneously. The direct contact with peers that enables small groups greatly enhances learning, fostering the extension of students’ ZPD in ways that are not possible if students work just individually.

A third and also important contributor to the social constructivist paradigm is Jerome Bruner (1977). He wrote: “The foundations of any subject may be taught to anybody at any age in some form” (p. 12). I consider this statement very important for the following reason: Piaget’s stages of development were interpreted as being fixed in terms of age ranges. That led to a fixed, linear approach to curriculum design. But Bruner, always a constructivist, makes the stages flexible: they are not about ages. While respecting the individual’s cognitive structure, we can always find ways to make knowledge meaningful. For instance, if a father (with only middle school studies) and his young child go to an interactive museum of science, they should be able to comprehend that light sometimes behaves as if formed by particles and at other times as if formed by waves. Bruner says that regardless of the learners’ age or stage of development, there are always meaningful ways for them to construct their knowledge. Bruner (1973) distinguished a developmental sequence for the representation of knowledge: enactive (which involves the manipulation of physical objects), iconic (action-free images), and symbolic (using language or scientific notation). One of Piaget’s big tenets was that children, before being able to operate abstractly, need to operate concretely. I see a close parallelism between the concrete operations of Piaget and the enactive representation of Bruner.
Even though Piaget, Vygotsky, and Bruner might be as contradictory, on the contrary, they can be very complementary and together form the pillars of social constructivism (Schunk, 1991).

**Conceptual Framework**

Embedded in the social constructivist paradigm is the didactic approach. One of the closest concepts to didactics of mathematics in the Anglo-American tradition would be the notion of *pedagogical content knowledge* formulated by Shulman (1986). Didactics of mathematics is a fairly new field of study under which most of the research about mathematics education is done in Continental Europe and South America. Didactics of mathematics is the systematic study of the conditions, situations, and tools that interact in a teaching-learning process, considering epistemological features of that specific knowledge and the particular cognitive structure of the learner. Thus, didactics focuses on the construction of relationships between the learner and the knowledge—not general knowledge, but a particular piece of knowledge, situated and context dependent. In didactics, one often refers to the process of teaching-learning as a whole. Among many authors writing about didactics of mathematics are three from French didactics (which is probably one of the best articulated): Brousseau (1998), with the theory of didactic situations; Vergnaud (1990), with the theory of conceptual fields; and Chevallard (1985), with the introduction of the notion of didactic transposition.

Brousseau (1986) asserts that new knowledge needs to be nested in a fundamental and meaningful situation, capable of provoking the appearance of this new, target knowledge. The situation should contain at the same time features already known to the students and others that are new and challenging for them. The situation consists of a set of tasks, a game, an activity, or a collection of problems with the property of being both engaging for the students and essentially
connected with the target topic. Another crucial feature of the situation is that it should be able to provide feedback to the students. Thus, the students should be able to approve or dismiss solutions without necessarily having to ask the teacher; this feature of the situation enables students to establish direct contact with the knowledge. In this way, the students accomplish one of the goals of every learning situation, which is to construct their own knowledge independently of the knowledge of the teacher. Warfield (2007) has translated into English Brousseau’s theory of didactic situations, which is fundamental to comprehending the deep new perspective that didactics of mathematics brings to mathematics education. In the core of the didactic situation there exists the so called adidactic situation, which is an instance where the student confronted with the situation gets the fundamental insight to start to acquire that specific knowledge. Perrin-Glorian (2008) makes reference to this feature when she writes: “Another fundamental hypothesis is that some pieces of knowledge cannot be transmitted only by explaining them” (p.1). In other words, contradictory as it may seem, there is a moment (the adidactic core within the didactic situation) when the teacher needs necessarily to stop teaching if he or she expects his or her students to learn.

Vergnaud (1990) states that a concept can be considered as a field of four elements: (1) a set of related problems, (2) a system of representation for those problems, (3) a set of invariant operations to operate with them, and (4) a control system that assures noncontradiction. The fourth element was added by Balacheff (1995) and it relates to metacognition, which can be defined as “cognition about cognition.” In any mathematical activity, it is crucial to keep track and double check steps and results. Schoenfeld (1992) from a different perspective (problem solving) writes about the same structure of control system.
The third important notion to be mentioned is the *didactic transposition* (Chevallard, 1985). From the moment a certain piece of mathematical knowledge is discovered or created to the moment when that knowledge becomes teachable and learnable by large numbers of students, the mathematical knowledge has to undergo significant transformations that we call *didactic transposition*. Winslow (2007), referring to this phenomenon, writes: “The purpose of didactics can thus be considered as a didactical ‘conquest’ of mathematical territories, explored by mathematicians long ago, but still inaccessible to average pupils, in average classes” (p. 533).

The work of transposition is not just a mere adaptation of knowledge; it is much deeper than that. It involves creating new paths and roads to reach knowledge using often completely different tools than those used by the mathematicians who first found or constructed that knowledge. The work of transposition is also a mathematical work, and even though its authors do not appear in journals as the creators of mathematical theories, they are responsible for making accessible to many, those wonders and beautiful results of mathematics that used to be the privilege of just a few.

There was consensus among the mathematics education community in France during the time I completed my masters in France (2003-2004) that the three authors mentioned above (Vergnaud, Brousseau and Chevallard) were the three main foundations for the French didactics of mathematics. Even though they do not contradict each other, it is not straightforward how they relate to each other. The theory of didactic transposition tries to define the field of didactics of mathematics by separating it from pedagogy and mathematics itself. The goal of didactics is not to find general rules for teaching or the discovery of mathematical knowledge but to develop a context-dependent knowledge, embedded in particular situations, activities and problems that could be both meaningful for students and convey the target mathematical knowledge. Then the
theory of didactic situations tries to define the key features that an activity, game, or session of problem solving should hold in order to produce in students a disequilibrium, a subsequent accommodation, and a final re-equilibration. Once the situation is carefully prepared and designed, these processes can occur relatively independently from the teacher’s intervention. These situations are designed most of the time (but not exclusively) under a socio-constructivist paradigm, since the activities and games generally entail group work. The third theory of conceptual fields created by Vergnaud (who was a disciple of Piaget) focuses more on the classical constructivism that emphasizes the schemes and cognitive structures that each individual constructs in order to make sense of his or her experiences. The three theories seem to tackle the study of “the phenomena and the conditions involved for learning mathematics” under different perspectives that can be seen as complementary but not necessarily covering all the different angles under which one could tackle the study of mathematics education.

**Relations between the Frameworks and iMAT approach**

The 2-week experimental lessons were carried out in group work sessions. From my teaching experience working in teams greatly enhances learning most of the time (there could be some exceptions where there are behavioral issues). I have an untested hypothesis that working in groups with classmates could enlarge the ZPD of students. The knowledge of students is normally complementary, and the motivation to learn increases especially when a student realizes that what seems complicated for him or her is simple for a partner who is willing to share his or her understanding.
Even though the platonic solids do not appear in the Common Core State Standards, I believe that (following Bruner’s assertion) they can successfully be studied at the secondary school level with an adequate didactic transposition.

The approach of iMAT engineering, which has been already outlined but is better described in the following paragraphs, is a set of several didactic transpositions related to each other by the topics they tackle, the tools they use, and the experimental approach for learning mathematics.

One of the greatest obstacles to learning 3D geometry is related to the complexity of the representation of 3D objects. Several authors, such as Parzysz (1991) and Grenier and Tanguay (2008), recognize the need to start the study of 3D geometry with a phase of manipulating 3D models where students acquire the primitive and indispensable notions of the 3D bodies. These observations confirm the constructivist tenets from Piaget and Bruner that before working with abstract operations or symbolic representations, learners need to work with concrete operations and enactive representations. Physical manipulatives are able to furnish very faithful representations of 3D bodies and are necessarily the starting point of one’s study; however, they are operational in a small range. At the other extreme, when one represents a 3D object by a single 2D representation, there is necessarily a loss of information as stated by Chaachoua (1997), not to mention the extraordinary mathematical cognitive demands that are entailed normally to represent in a coherent way a 3D object by a drawing. To back up this assertion, I note that there is a branch of mathematics called descriptive geometry that tackles this problem using two orthogonal projections of the objects on two perpendicular planes.

Between 3D models and 2D drawings, there is a third type of representation (like the missing link) that could make up for the weaknesses of the former. Computer technology can
provide what is called “rendering.” Renderings are images on a computer’s screen that are able to simu-late 3D objects. Renderings are especially useful and widely used in architecture, video games, airplane simulators, movies, and so on. In particular, for the learning of spatial geometry, there is one software tool with the capability to simulate 3D objects on the computer screen called Cabri 3D created by Laborde and Bainville (2004). This software enables the user to create 3D objects as easily as when one creates a 2D object with a regular dynamic geometry software tool. The object can be viewed under infinitely many different points of views, thus simulating one’s perception of 3D objects. The software enables the user to operate on the objects, performing transformations, obtaining sections, making truncations, and constructing several additional geometric objects related to the original ones. But even though the software promises to be a valuable tool for advancing the study of 3D geometry, Accascina and Rogora (2006) reported, in an experimentation done with prospective high school teachers using Cabri 3D, that some misconceptions had arisen among the students while using this software exclusively. So, even though Cabri 3D is a very powerful tool for enhancing the teaching and learning of 3D geometry, it does not suffice by itself. One should not tackle the study of 3D geometry using exclusively representations provided by Cabri 3D or other renderings.

Although in planar geometry, a figure drawn with straightedge and compass or a dynamic geometry figure is a “faithful” and “powerful” representation, there is no single type of representation for 3D objects that could, by itself, provide the necessary information to work effectively in spatial geometry. This naturally leads one to move from the traditional single representation (which often is enough in 2D geometry) to a multiple representation of 3D bodies, one of the tenets of the iMAT approach.
The word *multi-representation* needs further explanation. I mean multiple types of representations. In particular, three types of representations are needed: 3D models, 3D figures (which I define as a representation on a computer’s screen capable of simulating a 3D perception of the objects), and 2D drawings. Each type of representation could eventually contain several subtypes. For instance, in the first type one can have: 3D cardboard models, lattice models using straws or plastic tubes, or other kind of manipulatives that are offered in didactic catalogues. In the second type, one we can have Cabri 3D figures or another type of 3D figures produced by similar software. In the third type, one has 2D representations such as orthogonal projections, perspective drawings, or freehand sketches either on paper or on a computer screen.

My premise is that each new concept in 3D geometry, in order to be tackled successfully, calls for at least a triple representation consisting of one of each of the types described above. In particular, during the iMAT teaching experiment the three main types to represent the polyhedra were the following: 3D cardboard models made by the students, Cabri 3D figures, and 2D iconic pictures on paper. Each representation approximates the 3D object in a different way. Each type of representation has its virtues and limitations. The different technologies enable us to construct the representations and operate with them.

If a concept is formed by a set of related problems (Vergnaud, 1990), the multiple representation of the same mathematical object enables the student to extend and deepen his or her conceptual field about the object and also fosters metacognitive knowledge about it.

In this multiple representation, the student is likely to find different ways to test his or her assumptions and hypotheses, confirming or dismissing them. The different types of representation are able to provide particular feedback, and when the several different feedbacks are consistent, the question is settled without the intervention of the teacher. When they do not
converge, the student immediately relates the inconsistency to the existence of some error, and the awareness of an error is a very good starting point for further revision or investigation. The iMAT engineering seeks, through using adequate tools and carefully designed situations, to provide students with the necessary elements that would enable them to construct on their own an integrated and meaningful knowledge about the mathematical objects in question.

The focus of the present research study was to assess the mathematical learning of students both personally and collectively. The first research question sought to assess “what” was learned by students and called at first sight for a quantitative evaluation, whereas the second and third research questions searched to find out “how” the learning occurred and “indicators” of whether the iMAT approach could be effective for learning 3D geometry. Since the indicators and the possible ways for learning were not defined in advance, the second and third question suggested a qualitative type of research; in particular, grounded theory, where the categories could emerge from the data. The individual learning was mainly studied quantitatively, and the social learning through group work was studied mainly qualitatively. The methodology to be implemented was a mixed-method methodology (Russek & Weinberg, 1993).
Chapter 3

Literature Review

Differentiating Pedagogy and Didactics

In Uruguay, teachers tend to use *pedagogy* and *didactics* interchangeably knowing that they have related but different meanings. However, to try to explain their differences is a tough question. We can start stating that *pedagogy* studies the phenomenon of teaching and learning in a more general way, whereas *didactics* is much more specific and applied to contents and situations.

According to Gerard Vergnaud (personal communication, my translation):

Pedagogy has long been the art of teaching, and didactics has an unavoidable relation with pedagogy. But didactics is born from a greater focus on the content of knowledge:

-- On the specific contents of each discipline such as mathematics, physics, language, music, and sports, and
-- On the specific content of different professions.

In fact, professional didactics has existed in France for 20 years. The greatest difference between pedagogy and didactics is the didactic focus on the concepts, their epistemology and the situations through which one can seek and encourage student activity. It is true that France has played an important role in the development of research in didactics, but the Germans have used this term since the 19th century, and the Italians adopted it very quickly. They have participated in the *écoles d'été de didactique des mathématiques* from the beginning.

The development of didactics is much more recent than the development of pedagogy. Pedagogy was born as the art of teaching and is applicable in general to all subjects or disciplines. However, from the very beginning didactics was born as discipline oriented. That is, general didactics is studied very little compared with the applied didactics of mathematics, didactics of physics, didactics of history, and so on. And among all didactics, the didactics of
mathematics is definitely the one in which the most research has been done and that has produced most theoretical constructs.

Another important remark to be made is that Vergnaud mentioned Germany and Italy as two countries that currently have didactics as a crucial research field for developing mathematics education. He did not mention, for instance, Spain or Portugal in Europe or South America, Central America, or Mexico. Until a decade ago, Spain was behind the developed countries of Western Europe with respect to research in the didactics of mathematics, but that situation drastically changed over the last few years. After the Chinese language, Spanish and English have the same number of native speakers. The population in Spanish speaking countries increases over the average of developed countries, and in all Hispanic countries didactics of mathematics is the name of the scientific discipline in charge of studying, developing, and improving the process of teaching and learning of mathematics.

Trying to relate didactics with research about mathematics education in the United States, I quote Shulman (1986), who identified the following components of the professional knowledge of teachers: content knowledge, pedagogical knowledge, and pedagogical content knowledge. This third type of knowledge “pedagogical content knowledge” seems to be a very close relative to “didactic knowledge.”.

Steinbring (1998) stated that a new kind of knowledge is needed for teachers: a kind of mixture between mathematical content knowledge and pedagogical knowledge. What seems to be stated is that for a teacher it is not enough to know his or her mathematics and have good general pedagogical skills. Maybe Steinbring, without mentioning the word, is referring to didactics as that new kind of knowledge that is needed.
Finally, I would like to share the answer that Balacheff (personal communication) gave to this same question in 2003. The core of his answer translated into English is the following:

Pedagogy studies the phenomena involved in the teaching systems by their social specificity: diffusion of knowledge in a certain socially organized context. Didactic tackles this study from the point of view of the specificity of the knowledge in question. Since we are dealing with knowledge, that specificity heavily relies on the validation process (demonstrations, proofs, experimental, or technological validations). We can note of course both the complementary and the interwoven nature of this relationship.

A concept that is worth mentioning here is metacognition. Metacognition could be defined as “cognition about cognition” and it arises differently, in most every area of study. In particular, I would like to make reference to a theoretical construct called control system introduced by Balacheff for Vergnaud’s theory of conceptual fields. In this theory a conception is defined by a set of related problems, with an associated set of representations and an invariant set of operators. Balacheff introduces a fourth element called control system, which is in charge of keeping the coherence within the conception, deleting possible contradictions. Schoenfeld (1992) wrote about the same concept but within the framework of problem solving. The control system was in charge during the problem-solving process, to keep track of all the steps, double checking, and ensuring the coherence of process and results. The whole idea of being able to control process and results entails a metacognitive capability that, even though it is not that often mentioned explicitly in didactics (and could also be brought forth by other perspectives as Schoenfeld does), is in the heart of the goals of didactics. That is, one of the main goals of didactics is to put the student in direct contact with knowledge, and that entails necessarily to foster his or her metacognitive capability, which would enable the student to independently control his or her own learning. A metacognitive attitude towards mathematics means being able
to approach a topic under several different lenses, take into account its history and epistemology, and therefore have a deep and reflective knowledge about the subject.

In an attempt to clarify the differences and connections between pedagogy and didactics, I describe my vision of the didactic triangle: knowledge–student–teacher. A teacher should be able to establish an epistemological relationship with the subject he or she intends to teach. She or he needs to build a pedagogical relationship with her or his students. And finally she or he should be able to promote a didactical relationship between her or his students and the knowledge to be learned. The vertices of the triangle were the knowledge, the student, and the teacher and its sides were the epistemology, the pedagogy, and the didactics. Figure 2 illustrates this scheme.

*Figure 2*: Triangle that links epistemology, didactics and pedagogy.
Didactics as Co-Producer of Mathematical Knowledge

One might consider mathematics education as a field just in charge of transmitting mathematical knowledge, but that is a too narrow vision of what educating others in mathematics entails. Margolinas (1998) defines didactics of mathematics as the science of production and spreading of mathematical knowledge. This characterization of didactics is wider than the one previously outlined, but its main interest resides in the fact that it includes the production of mathematical knowledge as part of didactics.

The production and spreading of mathematical knowledge is a pair that cannot be dissociated. Even the most celebrated mathematicians need to communicate their discoveries and inventions to other people. Otherwise, their work will not transcend the work of others, and they will probably no longer be considered great mathematicians. On the other hand, the most modest of all mathematics teachers needs to create mathematics to fit the context and reality of the students if he or she teaches and expects them to learn.

The mathematician Bass (1997) is deeply concerned about the need for mathematicians to receive professional preparation to develop educational duties. He writes: “Mathematical scientists, who typically spend at least half of their lives teaching, receive virtually no professional preparation or development as educators” (p. 19). He says that it is not enough to have some good isolated models of good teachers since one does not learn how to sing arias by just attending operas or learn how to cook by just eating. Bass also adds that knowing something for oneself or for communication to an expert colleague is not the same as knowing it for explanation to a student. How many times does it happen to us, that we believe that we know and understand something and then asked to explain it to someone else, we realize that in fact we know very little about it? There is a thought I wrote in Camou (2006):
If someone knows something and is able to explain it to someone, then he knows something, if someone knows something and is able to explain to many people, then he knows more, but if someone knows something and is able to explain to anybody, then he really, really knows. (p. 141)

Each time a person is teaching, he or she is creating new knowledge. This new knowledge is what will enable the student to learn and will provoke, as a rebound effect, learning on the part of the teacher. It is not possible to teach without at the same time learning.

It is true to say that learning something may enable us to then teach it, but it is also true to say that teaching enables us to learn further and deeper.

Similarly, a natural consequence of the production of mathematical knowledge is its dissemination, and then the spreading of mathematical knowledge provokes the production of new knowledge. Teaching and learning, in a similar fashion as production and spreading of mathematical knowledge, are interwoven in such an intrinsic way that one cannot exist without the other. Even in the absence of a teacher there are books, software, situations that play the teaching role. We currently say: “That experience taught me that ….” The teaching-learning process is an interactive process and to be fruitful; both parts need to be changed after the process. In Camou (2006), I refer to this phenomenon in a chapter called “The Process of Teaching-Learning as an Interaction Between Two Parts” (p. 15).

**Toward an Alternate Paradigm that Better Fits Mathematics Education**

I have extensively argued throughout these paragraphs my socio-constructivist stance for the learning of mathematics: That knowledge needs to be constructed by the individual in order that he or she can make sense of his or her surrounding world. Since the social sciences study human beings and not just physical objects, it is not simple to uncover the peculiar singularity and complexity of each person. This state of facts may lead to the belief, for instance in education, “that anything goes;” that topics can be taught in any way as long as the teacher is
sympathetic enough. And many people believe that social sciences, in spite of their name, are not real sciences. Schoenfeld (2002) said: “Every good scientific theory is a prohibition: it forbids certain things to happen” (p. 84). To illustrate, if a mathematics teacher starts presenting a new topic without giving any concrete example of what it relates to, or where it comes from, this kind of behavior is a didactical error. In mathematical texts, there are normally some mathematical errors, but if one considers them carefully, one can detect as well many didactical errors. Given a specific topic, a certain group of students, and a particular context, there are effective ways to proceed to provoke learning, and there are other ways that are ineffective to attain that goal.

Long ago there was an identification of reality with science. People believed that science was able to describe exactly our reality. For instance, the great German philosopher, Kant affirmed that the Euclidean Geometry was the structure of our physical world. Later (probably with the contribution of the non-Euclidean geometries), scientific theories appeared as the only possible models of reality, with more or less explanatory and predictive power. The appearance of relativity theory with Einstein challenged the millenary invariance of magnitudes of time and length. In systems that travel at high speeds, time is stretched and length is contracted. So the velocity of a system affects the duration of the events and the magnitudes of objects. But that is not equivalent to saying “Everything is relative” or in social sciences “Anything goes” or in mathematics education: “It doesn’t matter what problems the teacher proposes, the order in which he or she structures the activities, or the kind of rapport established between the teacher and the student, the students nowadays are so smart that one way or another they will end up finding the way to learn.” The theory of relativity is telling us at this point that some invariants could change (others do not, such as the speed of light), under certain circumstances. Scientific theories move from being absolute to being context dependent.
With Heisenberg’s Uncertainty Principle, a new constraint was introduced into science. Before Heisenberg, the scientific community shared the belief that the accuracy of measurement was limited only by the available instruments. That is, one could increase the accuracy of any measurement into an exact value as long as one could get better and better instruments. Heisenberg showed that this accuracy is limited theoretically. So the illusion of getting faultless data some day was gone. The data one works with are inherently and inevitably approximate. Now, in the social sciences, beyond the fact that the data one can get (quantitative or qualitative) have some error and that variables are context-dependent (interplaying in a rich and complex environment), researchers still should be able to construct Theory with explanatory and predictive power.

Mixed methods methodology tries to take advantage of the strengths of both quantitative and qualitative research methods. Much of quantitative research is done under a post-positivism paradigm, and much of qualitative research is done under the interpretivist paradigm. If contradiction is perceived between these two kinds of paradigms, many researchers conducting a mixed-methods research study take either an a-paradigmatic stance or a pragmatic one, thus leaving aside the possible theoretical controversy.

But others, such as Maxwell (2004), call for an alternate paradigm under which mixed-methods research could be conducted coherently. One of the stances of the alternate paradigm he uses (social scientific realism) is that causality could be addressed, not only by quantitative methods but also by qualitative ones. Maxwell writes that similarly to context, realist social scientists see the meanings, beliefs, values and intentions held by participants in a study as essential parts of the causal mechanisms. He clarifies that social phenomena are not only context-dependent but also concept-dependent in the sense that practices, rules, and relations
depend on what they mean personally and socially for the participant. This new paradigm opens the door also to analyze emotions and motivation as essential components in the learning process.
Chapter 4

Methodology

This experimental study consisted in teaching, for 2 weeks, a unit of integrated 3D geometry in four high schools: two in Uruguay and two in the United States. I was the teacher in all the classes, and the goal was to try to document thoroughly the knowledge and skills acquired by the students and the processes by which the learning occurred. The students worked in class or in the computer lab, most of the time in small group. My initial assumption was that the iMAT approach is robust enough to be little affected by citizenship, gender, or socio-economic situation. Thus, I sought similarities across the learning processes of different groups. Eventually, some differences among the groups were also analyzed but always with the intention of highlighting similarities over differences.

The content and procedural knowledge that was the focus of the experimental course included:

- The algorithm of the construction of the regular pentagon with ruler and compass using the golden ratio (see Appendices H and I).
- The construction of the regular tetrahedron, the regular octahedron, and the regular icosahedron with cardboard and then all five platonic solids with Cabri 3D.
- Counting faces, vertices, and edges of polyhedrons and the discovery of Euler’s Formula.
- Exercising and improving skills operating with square roots that would subsequently be used for calculating the golden ratio, dihedral angles, areas, and volumes (see Appendix G).
• Finding, estimating and calculating the dihedral angle of the platonic solids using different tools and trigonometry.

• Calculating the volumes of the platonic solids in an approximate and an exact way.

• Estimating and calculating the dihedral angle of a nonregular tetrahedron.

• Drawing with ruler and compass a regular octahedron and a regular icosahedron (see Appendix J).

• Truncating platonic solids with Capri 3D to obtain some Archimedean bodies, such as the “soccer ball” (truncated icosahedron).

Why a mixed methods methodology?

There are practical and theoretical reasons for using a mixed-methods methodology. Schoenfeld (2002) writes, “The more independent sources of confirmation there are, the more robust a finding is likely to be” (p. 464). The search of working with independent sources seeking for robustness in the result is also one of the reasons behind the intention to experiment with teenagers from both Uruguay and the United States in Spanish and English.

Results obtained through quantitative methods and qualitative methods have a greater potential of being complementary and enabled me to do triangulation among results obtained by different methods. Each method is biased in some way, and so the use of multiple methods could make up for the weakness of each. If convergence is identified, then the validity and the credibility of the inquiry findings are enhanced.

The search for complementarity (Greene, 2007), triangulation, and convergence are somehow the phenomena that naturally gave birth to mixed methods as a research methodology. In a certain way they suit well a pragmatic paradigm for doing research. But I would like to go
further and take an alternative paradigm stance, which is *social scientific realism* (Maxwell, 2004). This paradigm suggests that causality needs to be addressed both by quantitative and qualitative methods, not just because of their availability, but because causality itself has a quantitative and qualitative nature. Maxwell (2004) writes: “Realism provides a philosophical stance that is compatible with the essential characteristics of both qualitative and quantitative research, and can facilitate communication and cooperation between the two” (p. 1).

*Social Scientific realism* acknowledges that complete objectivity cannot be attained and that there might be more than one scientifically viable way of understanding reality. Causality is context-dependent and concept-dependent. Mental phenomena, such as emotions, beliefs, and values are part of reality, not separated from it, and thus can be part of causal explanation for an observed event. This alternative stance provides space for both qualitative and quantitative ways of thinking and understanding. Explanatory significance can be achieved quantitatively through physical or behavioral phenomena but also qualitatively (at the same level of rigor) in a process-oriented approach, which recognizes mental phenomena as real and the fundamental role of *meaning*. 

Let's remember at this point the first research question: What could high school students learn about Geometry (both 3D and 2D) in a two-week experimentation, using iMAT engineering? There are two components that I would like to highlight: one is the particular piece of geometrical knowledge that will be present in the experimentation and the other component is the group of over a hundred high school students of about the same age but from two different countries, and speaking two different languages. The students will experiment the same designed course during the same amount of time. We could affirm that initially the seven groups were going to tackle the same problems involving the same knowledge. Then, the context of each group (location, time of year, particularities of each individual and other uncontrollable factors)
makes that the initial course and target knowledge is modified. Finally the knowledge is not only context-dependent but also conceptual-dependent which means that each particular student (even if they share the same class) constructs his or her knowledge in a different way. In order to learn, knowledge has to become meaningful for the student. Each of the students is unique and different from all the others; each one has his or her personal background, history, beliefs, emotions, fears and hopes that will influence the personal way how knowledge can become meaningful to him or her. Later in this chapter we will see the faces of the over 100 students who through their activities in class, their written work and their voices furnished the data to answer the research questions in this study. There are only seven pictures but 140 realities.

Quantitative methods search evidence through the power of the numbers and qualitative methods (Mitchel, Friesen & Rose, 2007) search evidence through the power of the words. Most of the numbers and the words in this research work were provided by these young students to whom I’ll be eternally thankful.

Methods

Methods can be mixed in all the phases of the study: gathering of data, data analysis, interpretation and conclusion. That was what I did in this research: mixed the methods in all possible phases of the study, understanding that causality is best addressed if one simultaneously applies qualitative and quantitative methods.

Data Collection

The first source of data came from the administration of a diagnostic assessment (pretest) administered before starting the 2-week unit of instruction (see Appendix A).
The second source of data was the final assessment (posttest) administered at the end of the experimental lessons (see Appendix C).

The third source of data was focus group interviews at the end of the experience.

The fourth source of data was a questionnaire completed by the students anonymously at the end of the experiment (see Appendix E).

The fifth source of data was the video recordings of the experimental classes. The video camera was focused on the activities of the students and their group work.

I envisioned two dimensions for studying student learning, a personal and a social one, and two types of methodologies: quantitative and qualitative.

![Figure 3: Four types of possible analysis.](image)

In the matrix in Figure 3 are four possibilities, which are to study personal learning with either qualitative or quantitative methods and to study social learning (group work in the ZPD of students) through qualitative or quantitative methods. The number of participants that participated in the research was 140 students. This number provides statistical power for possibly obtaining significant results for personal learning but grouped in groups of 3 or 4 students the number of groups would have been obviously smaller, which would make attempts for
Data Analysis

Both the pretest and posttest each had 6 questions. Each question in the pretest has a related question in the posttest. Thus six \( t \)-tests were done for each pair of related questions (Moore, 2007). The fact that the six questions were independent from each other gives robustness to this procedure of data analysis, so if for some reason one of the \( t \)-tests does not produce the expected result, the five other \( t \)-tests will not be affected. Even though, the results that are obtained using these data are mainly quantitative, there are some questions in the assessments that asked for explanations, so they were coded to allow a qualitative analysis. The data collected through the assessments corresponded to Cell 1 in the matrix.

The variable in the one-sample \( t \)-test was the difference, \( \Delta \), between the score of the posttest and the pretest for each of the students. \( \Delta \) thus measures the difference of performance between the end and the beginning of the experience and so it could be taken as a good indicator of the student’s learning. It was also analyzed to what extent country of residence and gender affected the student’s performance. The third source of data was the focus group interviews that provide information of two types: They can explain some misconceptions or particular procedures that could have appeared during the assessments, and they could illuminate themes emerging from the students’ experiences during the 2-week experiment. The possibility of building upon a classmate’s comments can lead students to find new insights and thus provide a better characterization of the different aspects experienced by the students and their learning process through the iMAT engineering classes.
The fourth source of data was the questionnaire. It had three questions that required both a numerical and a textual answer and an additional three open-response questions (see Appendix C). These data enabled me to address exactly the same question through a quantitative analysis and a qualitative one. The three open-response questions sought to try to identify the activities the students considered the best for learning and which activities were considered less helpful. The questions also sought to get the students’ emotional feedback about the whole experience, so they would answer the question about how learning occurred by mentioning activities, difficulties, and feelings.

The fifth and last source of data was the videos of the experimental classes, and they were analyzed in the following three stages:

1. First, they were reviewed in real time without stopping them.
2. During a second review, key episodes are identified and selected; the key episodes were classified into categories.
3. A third and last review was to look at the selected and classified episodes taking field notes, focusing on relevant and common issues, and stopping and rewinding the video whenever necessary.

The classification, categories, and coding emerged from the data following a grounded theory approach for qualitative research. So the information from the videos was processed working mainly in Cells 2 and 3 of the matrix (Figure X), which are the qualitative analysis of personal and social learning. Cell 4 could eventually also be addressed at this point in the analysis if qualitative information could be coded and quantified, thus enabling me to study social learning also in some quantitative form.
After analyzing the data gathered through the five different sources and using both qualitative and quantitative methods, I interpreted and integrated the different findings, evaluating the possibility of finding convergence. The goal was to establish robust results, backed up by both strong qualitative and quantitative data. For certain issues where convergence could not be found, this raised new questions for possible future research.

In all the phases of the analysis the goal was to determine both what the students were able to learn and how the learning occurred during the 2-week experiment.

**Pilot Study**

A pilot study was conducted during the spring semester of 2011 in a private school in the southeastern United States in order to determine the feasibility of conducting the two-week iMAT 3D geometry unit with high school students.

**Description.** The class that agreed to do the pilot experimentation was a pre-calculus mathematics class at a private Christian school. It was a class of 15 students that met daily from 12 to 12.45 pm. In the classroom, the students had eight Macintosh laptops available for use at any time.

Monday, 18 April, 2011, I went to the school to meet the students. They seemed to be an agreeable group of teenagers. I explained to them in general terms the intended experiment to be done and the consent forms that needed to be signed, and I answered their questions. Before leaving, I performed a little act of magic and then proposed the metaphor that mathematics is also magic but in a different way. Whereas magicians do not wish their tricks to be revealed, in the mathematics class we want them to learn the tricks and figure out why they work. I also told them the similarities of Uruguay with the State of Georgia. Uruguay is at the southeast of South
America while Georgia is at the Southeast of North America. Both are between 30 and 35 degrees of latitude and have coast on the Atlantic Ocean.

The next meeting was going to be Wednesday, the 20th of April. I went to the school before the class, and it was quite a surprise to find that half of the students were not attending class because they were on a small trip somewhere. So the pretest was done with only half of the students of the class, which posed the problem of how to proceed at the next class period, which was to be on Monday the 25th of April. The class mathematics teacher initially volunteered to give the assessment to the students the following day, but then we both figured out that doing the assessment without me would be too big a variation in the research procedures. So on Monday, the other half of the group took the pretest while the first half worked on part of the activities planned for the second experimental lesson. Thus, the initial assumption that all students would be doing the pretest simultaneously could not be accomplished.

Considering the appearance of all these unexpected happenings, I realized that my initial lesson plans for 10 lessons was too ambitious or unrealistic.

**What I Learned From the Pilot Study.** The first big idea that came to my mind is that I need to be modest in the sense that even though I want to teach and make students learn plenty of nice things during those 2 weeks, I might not be able to do so for many factors that were out of my control. Some of those factors were institutional, such as the ones just mentioned about loss of students from classes. Another important factor was the student’s performance on the pretest: the results were below the expected level. I realized that maybe after giving the pretest to the students, some questions should have been given some explanation on the board. The first question, which asked the student to *construct with geometrical instruments* the first four regular polygons was reformulated to make a *proper representation* of the first four regular polygons, as
I realized that several students had made nice representations of equilateral triangles without using a compass and (probably) without using a protractor. The 2-week course was slightly shortened in content, and a couple of practice exercises were added.

Another important fact that I learned in my initial visits to the school was my need to establish rapport with the students; that is, a friendly relationship, which was absolutely necessary if I expected them to achieve significant learning.

This is not specifically part of my research questions but a good pedagogical relationship is the basis upon which I could then foster the didactic relation the students need to build themselves with mathematics in order to learn it.

The experience made me foresee the huge challenge that entails, working with different groups and different institutions, where unattended conflicts could very easily arise between what would be recommended pedagogically to keep the students in their comfort zone and what would be recommended didactically to motivate them to make the effort to learn. These possible conflicts would very probably lead to necessary changes in the plans, but I tried to keep an unaltered “core plan” that would enable me to compare achievement and learning across different groups of students.

Thus, a very important conclusion from the pilot study was to define carefully and with detail the “core plan” as to what all the students should learn, the minimum knowledge and skills all students should acquire in the worst of the scenarios.

Finally, I learned that during the experiment that I should have very clear in my mind what my plans are but at the same time I need to be flexible and act fast to adjust the plans to what the moment asks for. It may seem that being prepared to be flexible should be something that I needed to be prepared for, in advance. However, the fact of having just half the class for
the pre-assessment meant that I did in two classes what was planned to do in just one. Therefore, the class plans that I outlined before starting needed to be restructured, and hence some activity regarding some content or skill was impossible to be done in the limited time available.

All problems and irregularities that arose in the pilot study were useful to better structure the plan and widen the array of possible different strategies that could be used in pursuit of the same goal: students’ learning of 3D geometry.

The Experiment

The experiments took place between June 8th 2011 and October 19th 2011 in Uruguay and the United States in four high schools, two in each country. I identify as HS1 (High School 1) and HS2 the two high schools in Uruguay, and HS3 and HS4 the two high schools in the United States. Some days before actually starting the experimental course, I went to each school to introduce myself and to invite the students to participate in the experience. Consent and assent forms were given to the students to be signed before the starting date. The first experiment was done at Montevideo, Uruguay, in HS1, between June 8th and June 22nd. I designated this class as Group A.
Figure 4: Group A, HS1.
The second experiment was done in HS2 in Montevideo, between July 14th and August 2nd, with a group, that I call Group B.

*Figure 5: Group B, HS2.*
The third experiment took place in HS1 (Group C) between July 11th and August 4th in Montevideo.

*Figure 6:* Group C, HS1.
The fourth experiment took place also in HS1 (Group D) between July 13th and August 5th in Montevideo, Uruguay.

*Figure 7:* Group D, HS1.
The fifth and sixth experiment took place from August 15th to until August 31th in Georgia, USA in HS3. It was during 2nd and 3rd period in two junior-year honors pre-calculus courses. These were called Group E and Group F.

*Figure 8: Group E, HS3.*
Figure 9: Group F, HS3.
Finally, between October 11th and October 19th the seventh and last experiment was done in HS4, in Georgia with Group G.

Figure 10: Group G, HS4.

Seven groups participated (4 from Uruguay and 3 from the United States) but since the groups from the United States were more numerous, there were slightly more U.S. students than Uruguayans that participated in the experiments. It happened that some students that participated in the focus group interviews had missed either the pretest or the posttest or conversely, so the total number of students for the tests, the interviews, and the questionnaires was not exactly the same. But 134 students participated in the experiments: 63 from Uruguay and 71 from the United
States. The four schools that participated differed in their socio-economic level. In Uruguay, HS1 had a higher socio-economic level than HS2, and in Georgia, HS3 had higher a socio-economic level than HS4. The same 2-week course was taught in the seven groups, but since their schedules of mathematics classes were different, the order of the lessons was slightly different across the groups. The notes for each lesson of the course were made available electronically for the seven groups in order that they could review what we had done in class.

For instance, an outline of the plan that was carried out with the two groups of HS3 was:

Day 1: Pretest and construction of the pentagon with ruler and compass.

Day 2: Construction on poster-board of the regular tetrahedron and the regular octahedron.

Day 3: Construction, in the computer lab, of the 5 regular polyhedrons, with Cabri 3D; the Euler relation; rationale of the pentagon’s construction.

Day 4: Work sheet with radicals.

Day 5: Estimating, measuring, and calculating the dihedral angles of the regular polyhedrons using protractors, Cabri, and trigonometry.

Day 6: Estimating and calculating exactly with radicals the volumes of the tetrahedron, octahedron, and icosahedron.

Day 7: Measuring and calculating the dihedral angle in a nonregular tetrahedron and making a 2D representation (drawing) of a regular icosahedron.

Day 8: Truncating regular polyhedrons, with Cabri 3D, to obtain Archimedean bodies; the soccer ball as a truncated icosahedron.

Day 9: Playing with polyhedral dice and calculating probabilities; Review.

Day 10: Posttest.
The focus group interviews and questionnaires were done on subsequent days after the 2-week experimental course was over.

My subjectivity. I am a Uruguayan mathematics teacher who has been teaching mathematics mainly at the high school level for the last 25 years. I have taught all mathematics courses in the middle and high school; that is, Algebra, Pre-calculus, Calculus, Geometry, and Trigonometry. However, in the last 10 years I started to specialize in geometry, getting increasingly interested in 3D geometry.

We could argue fairly that I would be more adapted to teaching Geometry in Uruguay, since it’s where I have most experience. To somehow compensate this situation, I could point out that the first 4 experimentations were done in Uruguay and the last 3 in USA and that after each one I tried to improve it, on the understanding that a teacher should always do his very best correcting possible mistakes or taking into account the students feedback. Therefore, keeping always unchanged the core of the experimental course the teaching in the last experiences should have been slightly better. I could affirm, that regardless the mistakes that naturally I made as a human teacher, I gave absolutely my best in the 7 different groups that participated in the experimentation.

Limitations. As the experiments were advancing, I became aware of some asymmetries and limitations that I had to accept and that were completely out of my control. One concerned the mathematics class schedules of the different groups; some groups had their mathematics classes two days per week, whereas others had one 50-minute class each school day. The different class schedules implied that the plan of the course had to be adapted to each group, but always keeping the same content and approximately the same amount of time for the whole experiment. In HS2 (Group B), we could not get Cabri 3D to run in the computer lab, so the
teacher of the course, some students, and I brought our personal notebooks to enable the work with the software as planned. In HS4 (Group G), the poster-board for constructing the polyhedra was not brought by the students from home (as in all the other groups), but instead it was provided by the school and the class teacher. This situation that initially seemed beneficial for the students turned to be the other way around; the courses in that high school (HS4) are very intense (they do in one semester what students in a regular high school do in a whole year, having 2 hours of mathematics per day), and the students typically have some kind of job outside the school. As a result, the teachers seem not to require homework, with the result that most of students lost the habit of studying at home and reviewing materials to prepare for tests. This phenomenon clearly reduced the amount of student learning.

The computer lab in HS3 was excellent, enabling better work with Cabri 3D than in all the other high schools, though one can consider that this work was still good in all the schools.

Finally, in HS1 with Groups C and D, a completely unexpected and somehow delicate situation occurred. While we were in the middle of the experience, the teacher of the course made the following proposition: He asked me if I could give him the grades of the posttest so that he could use them as regular grades of his course. His rationale was the following: In the syllabus of his course, there is a unit of 3D Geometry; he considered that what had been taught so far was valuable and that the 2-week experience was covering well that part of the syllabus; hence he needed to grade his students for the unit and thought that the final assessment I was going to administer in some days would be a good grade for him to use. The situation was delicate because the consent and assent forms that had been signed by the students stated that the grades on the evaluations would not affect their grades. But this situation was kind of slightly different. If I said no, I was missing a great opportunity that the students would commit
seriously in preparing for the posttest in order to earn a good grade; I would be missing a great opportunity to foster learning among students.

Each of the seven courses were, in some respect, different; if in any new course there appeared an opportunity to improve the course in some way (always keeping the content and the amount of time unchanged) I felt obligated to take the opportunity and did not dismiss it just because I could not do the same thing in the other courses. In the signed consent forms there is another point where it says that nothing would be done that would “harm” the students. The supreme goal of an educational institution is that the students learn what is considered valuable, useful, formative, and meaningful. If I had rejected the proposition of the teacher it would have been unethical, because I would have harmed their learning since it is natural to consider that students do not learn the same amount if they prepare for a test that does not count than if they prepare for a test that does count. So, one week before the final assessment the teacher communicated to his students that the result of that test would be a grade for his course. A couple of students complained, and so the teacher told them: “Ok, if you don’t want your posttest to count as a grade it won’t count, but the next week, I would administer myself another test about the same material.” There were no more complaints.

From the moment that the teacher asked me for the grades and I agreed to give them (after, of course, communicating with the students), the experimental posttest became also a regular test for his course. The other participant teachers never asked and never were told about the results of either of the two tests. This teacher made my posttest his own test, and consequently his students studied hard for it and learned more than the students from the other groups.
The Constraints of Curriculum. Before starting the experience it was required, naturally, to obtain the authorization of the institutions and teachers to allow me to conduct the 2-week experiment. In Uruguay the syllabus of the courses included a small portion of 3D geometry, so the experiment fit quite well in the normal development of the course. In the United States, it was tougher, since most of junior and senior high school students (target population) had already taken geometry in previous years. Under these circumstances, the key to obtaining their permission was the integrated feature of the course that contains at the same time Algebra, Trigonometry, and Geometry. Consequently, it was considered to fit (not without discussion) in a Pre-Calculus course in HS3 and an Integrated Trigonometry course in HS4. I considered that 2 weeks was the minimum class time to evaluate a learning process between a pretest and a posttest and the maximum time I could reasonably borrow the classes from the mathematics teachers. Another fact that is worth mentioning is that once the experiment started and its length was set (which had to be equivalent in all seven courses), the syllabus I had to follow as a teacher was very strict since the final test was already designed. I was more bounded than in a regular course; I had to cover the experiment’s curriculum in just 2 weeks with no possible extensions.

The Pretest and the Posttest

The pretest had six questions (see Appendix A) as did the posttest (see Appendix C), and they were designed to be done in approximately 50 minutes each. These two tests are the main source of quantitative data. The results on the pretest would provide some results about mainly the geometric (but also algebraic and trigonometric) background of the students; that is the starting point of the experiment. The measurement in which I was more interested though, was the difference between the posttest and the pretest scores because that is a more accurate
quantitative indicator of the student’s learning during those two weeks. That measure is crucial for answering the research questions.

**The difference between the posttest and the pretest.** Considering the difference of scores of 134 students (63 from Uruguay and 71 from the United States) who did both the pretest and the posttest, I tested (with a t test) the null hypothesis (that there was no gain between pretest and posttest) to determine whether learning occurred at a 5% level of significance. Then I analyzed, with the same level of significance, some factors such as country of residence, gender, socio-economic level, and external motivation. External motivation was a factor that appeared in the midst of the experiment, when for reasons explained above, the final evaluation took two roles (posttest and regular course test), and therefore this was an extra and strong motivation for students to prepare for it. The goal of analyzing these factors was to test whether the iMAT engineering is robust enough to still render positive results under different circumstances.

To better analyze the students’ learning in each aspect of the course another 6 t-tests were carried out pairing each question of the pretest with the corresponding one in the posttest. It would have been easy to pair Question 1 with 1, 2 with 2, 3 with 3, and so on, but for pedagogical and didactical reasons this mechanical order was not convenient for the students, so the pairing was not that straightforward. There were three pairs of questions in which the pretest and the posttest questions were practically the same, and there were another three pairs of questions in which the pretest and the posttest questions were related but with a different level of cognitive demand (the posttest question had a higher level of cognitive demand).

**Pairing questions.** The following are the six pairings that I made for the analysis of the pretest and posttest:
• Pair 1: Question 2 in the pretest with Question 1 in the posttest. The students were given an accurate representation of a 5-point regular star and are asked the value of the angle of the star’s point and why.

• Pair 2: Question 3 in the pretest with Question 3 in the posttest. Students are shown a figure (not to scale) where they have to apply the Pythagorean Theorem, then operate with radicals and check with decimals. In the posttest, the only change was that the numbers were slightly changed.

• Pair 3: Question 5 in the pretest with Question 4 in the posttest. The question asks in both cases what a regular octahedron is, to provide information about it, and to draw a picture.

• Pair 4: Question 1 in the pretest with Question 2 in the posttest. Question 1 was to make a proper representation of the first four regular polygons, and Question 2 was to construct with ruler and compass a regular pentagon and give the rationale of the construction.

• Pair 5: Question 4 in the pretest with Question 5 in the posttest. Question 4 asked the student to estimate and calculate an angle of an isosceles triangle given the length of the sides and an accurate representation, whereas Question 5 (posttest) asked the student to estimate and calculate the dihedral angle in a nonregular tetrahedron.

• Pair 6: Question 6 in the pretest with Question 6 in the posttest. In the pretest there was a picture of a typical soccer ball, and the student was asked about the shape and number of faces, whereas in the posttest the student was asked to make a 2D representation of a regular icosahedron and about the number of hexagonal and pentagonal faces of a truncated icosahedron (soccer ball).
The Focus Group Interviews

A total of 25 focus group interviews were conducted: 13 in Uruguay and 12 in the United States. For Group A, they were done on June 23rd, 2011. There were four focus groups with five or six students in each. For Group B, there were two focus groups with six students in each, and the interviews were done August 2nd. For Group C, there were three focus groups with five students in each, and they were done August 4th. For Group D, there were four focus groups with five students in each, and they were done August 5th. For Group E, there were five focus groups with five or six students in each, and they were done August 29th and August 30th, 2011. For Group F, there were four focus groups with six or seven students in each, and they were done August 31st and September 1st, 2011. Finally, for Group G there were three focus groups of six students each, and these interviews were conducted October 19th, 2011. The length of each interview was, on average, 20 minutes. There were five basic questions that were formulated and asked in all interviews:

• Question 1. Do you think this two-week experimental course was worth it? Yes or no. Why?
• Question 2: Can you identify during these two weeks any special “light bulb” moment?
• Question 3: What do you think was the reason for using multiple-representations (posterboard, plastic tubes, Cabri figures, dice, drawings) of the same geometric objects?
• Question 4: What things did you dislike or you think could be improved for the future?
• Question 5: In an hypothetical situation that we could have one single extra class about something related with what we have studied, what would be your choice?

A total of 8 hours of audio recording was obtained from the focus group interviews, and they were all transcribed before they were analyzed.
The Questionnaires

Before starting the focus group interviews, the students were given a questionnaire (see Appendix E) to complete anonymously. A total of 140 students completed the questionnaires, 71 from the United States and 69 from Uruguay. There were 6 items in the questionnaire. In the first three, the students were asked to rate between 1 and 10 the activities done during the 2-week experience, what they learned during that time, and how they felt during these classes. The last three items were open-ended questions about describing one particular activity they liked, another they disliked, and sharing any other extra reflection or perception they experienced. The questionnaire provided both quantitative and qualitative data and was very useful to integrate the results.

The Video Recording

A total of 16 hours of video of the lessons was recorded. The initial idea was to have two cameras: one stationary that would record the whole class and the other operated manually by a person that would focus alternatively on small groups of students working together. For registering the process of the students’ learning, focusing on just a few or one single student was more appropriate for the objectives of this research than recording the whole class. However, it turned out to be problematic to find a volunteer to help with the video-recording. From the beginning, I realized that being the teacher of the experimental course was such a demanding task that I could not take care of anything to do with the recording. Just sporadically, when students were constructing the polyhedrons with poster-board, was I able to video record some small group or individual student, but in all other activities it was impossible. Very soon I realized that I depended completely on other people for recording. In Group A, the teacher of the course, Martha, volunteered to video-record; in Group B, the teaching assistant, Monica, did
the recording; in Groups C and D, again the teacher of the course, Nelson, did some video recording. For Groups E and F I got the help of my son, Gaston, the student teacher Stephanie, Kathie Daymude, and graduate students from UGA: Dario, Leighton, and Jiyoon. Finally, for Group G, I got the help of the teacher, Rhonda. Most of the classes were at least partially video-recorded. It was not easy to get volunteers to make the video-recordings, which was a big constraint since the experimental course was already scheduled and could not stop even if occasionally there was nobody to operate the cameras. Thus, the total amount of video-recording is less than the number of hours of the experimental classes, but they are very valuable.
Chapter 5
Results

The Focus Groups Interviews

**Question 1.** *Do you think it was worth this two week experimental course? Yes, No.*

*Why?*

I’ll try to summarize here the most representative answers to the question. Sometimes I’ll be making a narrative myself and other times I’ll be quoting the students directly. I will use pseudonyms and will identify just the school they belong to. Occasionally the students give their opinion without answering the question directly; sometimes I could still quote these opinions if they are judged to be valuable.

Most of the students found the course to be worthwhile and interesting. They expressed that they had fun during those two week and that they had learned a lot. Martina from HS1 said: “Aprendimos mucho; fue muy interesante” (*We learned a lot; it was very interesting*). Several students pointed out the fact that doing several different activities (constructing the 3D models, working with the computers, playing with dice) was very entertaining. Some US students expressed that during the course they got a chance to understand why things worked instead of just being told and that during the course they were more involved than if they were doing something on paper or watching something. Kaleb from HS3 said: “It helped me understand geometry in a way, I never worked before”.

There were some expressions from some US students that were almost the same as translations of comments from Uruguayan students. Felipe from HS1 said: “Estuvo bueno;
aprendi cosas que ni siquiera sabía que existían” and Helen from HS3 said: “I liked working with the different shapes because I didn’t know they even existed until the course”. Another US student recognized the value of the whole approach when he said that the class was worthwhile because there were several helpful insights for all mathematics and not just geometry.

Among the approximately 140 students that participated in the interviews, there were a couple that expressed dislike but when asked to articulate this, it turned out, that it was part of a negative attitude that they had for mathematics as a whole. Maybe the best way to summarize the general feeling is to quote Logan from HS3: “I enjoyed it; it was fun to go to the computer program, make the shapes; I enjoyed the course overall” or to quote Sandra from HS4: “I’ll always remember the angle of the star; that $360^\circ$ I’ll always remember; I’ll remember that forever”. That is: students enjoyed the course and have learned from it.

**Question 2. Can you identify during these two weeks any special “light bulb” moment?**

I’ll try to summarize here the most repeated topics mentioned by the students or some single opinion that could be considered especially valuable. Two students from Uruguay from different High Schools (HS1 and HS2) mentioned that their light bulb moment was when they were using trigonometry to calculate an angle and then, when they got it, realizing that it matched the previous estimation. Not only were the students happy about this, but it also reinforced their trigonometry knowledge and their confidence about doing mathematics. Some students from both Uruguay and USA said that their light bulb moment was in the posttest when they had struggled with some exercise and then finally they got it and they knew it was right because it made sense. For other students, their light bulb moment was to see that inside a polyhedron there were other bodies or figures: the shapes within the shapes.
The soccer ball was mentioned by several students and some quotes from US students were almost identical to translations of comments from Uruguayan students or the other way around. A student from HS1 said: “A mí lo que más me llamó la atención fue la pelota de fútbol, lo del icosaedro truncado” and a student from HS3 said “It was interesting to see how the soccer ball was made because it never occurred to me [that it was a truncated icosahedron]”. Another student of HS3 identified her light bulb moment came when folding the 2D net and was able to get a 3D shape. She said: “When we were doing the poster-board shapes, I thought it was kind of cool how they all got together”. The appearance of the golden number and the fact that there exists only five regular polyhedra were also mentioned.

Several students from USA found the ruler and compass constructions very interesting. A girl from HS3 expressed: “Making a perfect pentagon; making a star and making actually a good star because when I drew one it is really, really ugly; and so I learned to make it perfectly”. Another student from HS1 said: “Mi momento de iluminación fue cuando en el pentágono descubrimos cuál es el ángulo de la estrella de 5 puntas… todo estaba enganchado” (My illuminating moment was when in the pentagon we discovered what was the angle of the five point star…everything was linked). Another student said that instead of choosing one single activity he would point out that through every activity he realized that all were connected and that he could therefore see a pattern. To end this section I will reproduce a little piece of an interview conducted in HS3.

“Triangles are everywhere” said a girl.

“What?” I asked not quite understanding what the girl meant.

“Triangles are everywhere” repeated the girl and added “That’s my light bulb moment: that triangles are involved in almost everything; they really are”.

The student’s insight was at the same time unexpected but very thoughtful.

So I decided to continue in her line of thought and asked to fill the blank in the following sentence: “The triangles are to planar 2D Geometry as ………… are to 3D Geometry”

A girl said very softly: “As important?” but then a boy said louder “I would say tetrahedrons”.

I replied immediately and enthusiastically: “Excellent!”

**Question 3.** *What do you think, it was the reason of using a multiple-representation (poster-board, plastic tubes, Cabri figures, dice, drawings) of the same geometric objects?*

I should point out first that the question was initially hard for the students because in most of the interviews I had to repeat it giving some examples of the multitype-representations used during the course. But once they understood the question they started to elaborate and were able to give very insightful answers.

Some students started saying the reason for using multiple-representations was that everyone was different and so each person understands things in different ways. This is a pedagogical argument that is valid but very soon the students moved to consider the didactical aspect of the question, analyzing specifically what they learned from each type of representation.

A girl from HS2 said: “Yo creo que te das cuenta de distintas cosas dependiendo de como lo representaste: si lo dibujás te das cuenta de una cosa; si lo tenés en la mano podés ver otras cosas (por ejemplos los volúmenes); si lo ves en la computadora podés tener cosas más exactas” (*I think that you become aware of different things depending on how you represented it: if you draw it you realize one thing; if you have it in your hands you can see other things (for instance the volumes); if you see it on the computer you can have more exact stuff.*)

The notion emerged, that with the 3D models you were able to **touch** the body and therefore, have a better perception of it. Another notion that was mentioned was the possibility to
see inside the body. I’ll quote several US students that were in the same vane and further articulate what the Uruguayan girl said. All these were expressions of US students 16, 17 or 18 years old. “I felt I got a different concept from the different representations”; “For each representation there is a different piece of information to learn and then you have a better understanding of the shape as a whole”; “Each kind of representation had something different, like a different attribute of the shape, attributed to that representation”; “Each time we did a different representation of the shape it made me think of one more way about that shape and so every time we did something different I understood the shape a little bit better”.

Very often the students expressed how much it helped to be able to touch the 3D model and also have experienced the transformation from 2D to 3D. The plastic tube models enabled them to see through and one student said that without that model he would never have picked up that an icosahedron could be broken down into 20 pyramids. The computer Cabri figures were the key representations to measure and make exact calculations with the geometric 3D object. The drawing made them aware of some properties; for instance that each vertex belongs to five edges. They talked more about what they learned about the body, from the 3D models and the Cabri figures rather than what they learned from the drawing. Hence, I specifically asked them their opinion about the drawings.

A student in the Pilot Study said that making for instance a regular icosahedron on poster-board was easier than drawing it. Later on a student in HS3 said that it was easier to draw it. This apparent disagreement was settled when we agreed with the student that constructing the 3D shape was more time consuming but it required less mathematical knowledge (just to know how to construct 20 equilateral triangles) whereas the drawing of the icosahedron might take less time but it requires more mathematical knowledge (construction of a regular hexagon and
application of a dilation). There are two main features of the representations to consider: the process of their construction and once constructed, their utility. We have just considered that each representation requires a specific set of knowledge to produce it. The students also talked about the utility of the drawings. An Uruguayan boy from HS1 touched this theme when he said about the icosahedron’s drawing: “Yo nunca había visto un icosaedro; si me lo mostraban así en el pizarrón capaz que no entendía. Lo vi en 3D y después cuando lo construimos en el pizarrón fue como que sí, lo veía, se me venía a la cabeza, de que ángulo estaba viendo el verdadero” (I never had seen an icosahedron; if it was first showed to me on the board maybe I wouldn’t understand it. I saw it in 3D and then when we constructed it on the board it was like, yes, I saw it, it came to my head from which angle I was seeing the true one).

It is very interesting how the student pointed out that if we had started with the icosahedron’s drawing probably he wouldn’t be able to make sense of it. He needed first to have in his head the image provided by the 3D shape. A 3D shape can provide infinite different images whereas the drawing is just one single image. In a drawing you cannot distinguish faces, vertices and edges by touching whereas you can distinguish these by touching on the 3D model. It is also interesting to mention how the 3D model is for the student the true icosahedron. We know that we can only make representations of the icosahedron and that the true icosahedron is an abstract mathematical object that lives in our minds. However, the fact that the student considers his model the true icosahedron shows that the 3D representation is a representation that is far more “faithful” and “closer” to the abstract object than any drawing.

A US student from HS3 was on the same line of thought when he said, referring to the drawings: “So if we had started with that one, I would have no clue ……Because all the triangles are equal whereas on paper they don’t look equal at all …….. All because of perspective”.

Another boy from HS3, when asked about what was the point of learning how to draw an icosahedron, said: “I guess with the drawings you kind of show a mastery of it…… you can now construct it out of what you know instead of just looking at it……after learning this for a whole week we were finally able to put it together and actually create it ourselves, so actually we were not just looking and seeing it, we were actually designing it”.

This student explains clearly how the drawing is the end of the process and that drawing entails a mastery of the object. In planar geometry since drawings are faithful representations (and often straightforward) of the 2D object, they can be a starting point for the study of 2D geometry. This order is clearly in the Van Hiele’s stages for learning geometry. But when we come to 3D Geometry there is a very important and outstanding inversion. The drawings cannot be the starting point of the study of 3D Geometry; to be able to produce 2D representations of 3D objects it is required, as clearly stated by the student, a certain mastery of the object. This mastery cannot be obtained starting with 2D representations. Therefore the starting point of the study of 3D Geometry is to work with 3D models, which are the most “faithful” and “closer” representations of the 3D objects.

Finally to end the comments about this question I will quote two students who in English and Spanish said practically the same thing. A student from HS1, when referring to the purpose of using multiple-representations said: “Fue para que a pesar de usar distintos materiales para construirlos, se mantienen las mismas propiedades y proporciones …..que siempre va a ser igual” and a student from HS4 said: “It was for us to understand that no matter how you do it, it still was going to be the same thing”. These students got the idea that with the use of several representations we were trying to learn about what remains unchanged, what remains the same, the invariance among the diversity. This idea of having one same thing beyond the different
representations aligns well with Vergnaud’s (1990) *Conceptual Fields* that were mentioned in the theoretical framework for this study. This notion of invariance among different representations might be the case for maybe any mathematical concept, but in 3D Geometry the use of multi-type representations is crucial, because any single representation taken on its own, is weak and the only way to reach a satisfactory level of conceptualization comes from the integrated use of several representations that approximate the mathematical object in different ways.

**Question 4. What things you disliked or you think could be improved for the future?**

There were different types of answers for this question. There was some fair criticism about the computer labs but it was completely out of the researcher’s control. Some students wished we would have spent more time in 2D drawings; another student wished we could have worked with the dice and probability; two students complained that we spent only half a class on the golden number; some students said that they got the golden number because they memorized it but that they didn’t get why it worked. A student from HS3 said: “Just it was a lot of information; you taught it well only that all by once it was hard to grasp”. Several students expressed that sometimes I went too fast and that I should have slowed down. A student from HS4 said: “Sometimes when you explain something, you just go too fast and you get excited and you never stop”. This criticism has to be taken into account of course but it relates more to the way of teaching and not much to learning using the iMAT approach. A girl from HS3 expressed that before starting each class it would have been good to briefly recall what we had done before and the plans for the new lesson; another girl expressed that maybe it could be a good strategy to take one polyhedron at a time and calculate for each one, all the stuff we studied to have things better organized.
The criticism about being too fast sometimes, raises the following question: after doing the pretest and determining the level of each group, should I have tailored a special syllabus for each group? That is, for a group with a lower mean in the pretest should I have cut out part of the 2-week syllabus? Maybe if I had proceeded in that fashion, the pace of the experimentation would have been better for some students. On the other hand, since I knew that there were many variables such as schools, language, citizenship and socio-economic status, I needed to keep the course as similar as possible in all seven groups in order to be able to consider the whole group of students as one sole sample; also keeping the same course and the same assessments enabled me to do potential comparisons.

Among some of the criticism there was also some praise. A student from HS1 said: “No hay mucha cosa para mejorar” (There is not much stuff to improve). Another from the same institution said: “Para mí estuvo todo perfecto. Redondito” (For me everything was perfect. It all came together).

**Question 5.** *In an hypothetical situation that we could have one single extra class about something related with what we have studied. What would be your choice?*

The activities that were most repeated among all the students were: to do more polyhedra with poster-board, to work more with the software Cabri 3D, and to study more about the golden number. Two students (one from Uruguay and one from USA) said that they would have liked to see how the golden number relates with the human body.

Other activities, less mentioned than the first three but still mentioned by several students, were to work more with radicals, to find more dihedral angles, to make more 2D representations or to play more with the dice. The only two boys (one from Uruguay and the
other from USA) that got full marks in the posttest (50/50) said that they would have liked to study more the regular dodecahedron, about which we only counted faces, vertices and edges. There was a strange, completely unexpected and unplanned phenomenon that happened in HS3 that may have influenced the students to choose the golden number for a possible hypothetical future class.

The room number of the computer lab at HS3, was 1.618, the approximation (to three decimal digits) for the golden number! It was a student from Group E that realized this awesome coincidence while we were in the lab deriving the existence of the golden number in the regular pentagon. Therefore, it is not surprising that many students from groups E and F became extra surprised about this outstanding number. This is a clear example of how phenomena and
knowledge are context-dependent. For the students of HS3, and only for them, 1.618 was not just the ratio between the diagonal and the side in a regular pentagon but it was also the room number of their computer lab! The particular context of HS3 modified the knowledge about the golden number; this haphazard chance probably contributed to making this knowledge unforgettable for the students in this High School. Regardless of this extraordinary coincidence that happened in HS3, for all the groups, the golden number, the poster-board polyhedrons and the Cabri 3D figures, were the students’ most preferred choices for new learning.
The Questionnaires

After the two-week experimentation and just before the focus group interviews, questionnaires were administered to the students anonymously. A total of 140 questionnaires were completed: 71 by US students and 69 by Uruguayan students. Both the English and Spanish version of the questionnaires can be found in Appendix E.

Item 1.

1) a) How would you qualify the activities done during this 2 week experimentation?

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>very bad</td>
<td>excellent</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Explain your rating.

Table 1
Median and Quartiles for Item 1

<table>
<thead>
<tr>
<th>Activities</th>
<th>Maximum Points</th>
<th>First Quartile</th>
<th>Median</th>
<th>Third Quartile</th>
</tr>
</thead>
<tbody>
<tr>
<td>Item 1</td>
<td>10</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

We see that the median for the sample of 140 students is 8 and the first quartile is 7, which means that more than 75% of the students rated 7, 8, 9 or 10 for the activities done during the experimentation.
We see in Figure 12 the Box Plot regarding the rating of Activities, which shows that the median is 8, the first quartile 7 and the third quartile 9. The minimum rating of 3 is also shown to be an outlier.

**Item 2.**

2) a) How would you qualify what you learned during this experience?

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<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
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<td>very little</td>
<td>a great deal</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Explain your rating.

Table 2

*Median and Quartiles for Item 2*

<table>
<thead>
<tr>
<th>Learning</th>
<th>Maximum Points</th>
<th>First Quartile</th>
<th>Median</th>
<th>Third Quartile</th>
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</thead>
<tbody>
<tr>
<td>Item 2</td>
<td>10</td>
<td>6</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>
Figure 13: Box Plot with the Rating of Learning

We see that the median that students gave for what they have learned is still 8, though the first quartile has decreased to 6 and the third quartile has also decreased to 8 compared with Item 1. After the quantitative analysis of the first three items we will do some qualitative analysis to try to clarify the reasons behind the ratings. Note that the two lowest ratings were again outliers.

**Item 3.**

3) a) How did you feel during the these two weeks regarding this experience?

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<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>disliked</td>
<td>liked very much</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Explain your rating.
Table 3  
*Median and Quartiles for Item 3*

<table>
<thead>
<tr>
<th>Feelings</th>
<th>Maximum Points</th>
<th>First Quartile</th>
<th>Median</th>
<th>Third Quartile</th>
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</thead>
<tbody>
<tr>
<td>Item 3</td>
<td>10</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

*Figure 14: Box Plot with the Rating of Feelings*

For Item 3 we see in Table 3 and Figure 14 that the median is again 8. The first quartile came back to 7 and the third quartile to 9 as in Item 1. Again the lower scores are outliers.

**Item 4.** If there was an activity that you particularly liked it. Can you describe which one and why?

I did not do a separate analysis for Uruguayan and US students on this item because the similarity of the answers given by the students of both countries was very remarkable. If there were some non-significant differences in the quantitative data obtained from the evaluations,
these will be mentioned in the next chapter; in the qualitative data the similarities are overwhelmingly greater than the differences.

To illustrate the students’ answers to this question, I am providing the first part of all the answers from students of Group F:

“I liked making the 3D polyhedrons”

“When we made the shapes in the computer, I thought it was cool”

“I liked the program on the computer”

“Constructing shapes with colored poster-board”

“I liked the game at the end when we rolled the dice”

“I particularly liked the 3D activity on the computer”

“I liked drawing the figures in 3D”

“I liked constructing the polyhedrons”

“Making the 3D polyhedrons”

“I enjoyed using the Cabri 3D”

“I liked creating the shapes on my own out of poster-board”

“I liked making the shapes”

“I personally liked making the shapes out of poster-board”

“I liked the cutting and pasting activities”

“Making the shapes with cardboard”

“Making the 3D shapes”
“Making the shapes”

“I really enjoyed the lesson regarding pentagons and finding the golden ratio”

“I enjoyed using the computer activity”

“I loved making the 3D representations”

“I enjoyed making the shapes by hand”

“I particularly liked working in the lab”

“I liked when we learned to draw a five point star”

“I liked creating the three-dimensional figures out of poster-board”.

We see here the preference for the hands-on activities with the poster-board followed by the work with Cabri 3D in the computer lab. In the whole sample out of 140 students, 102 answered that the activity they liked best was the construction of 3D shapes on poster-board or the work with Cabri 3D. A few students said they liked best playing with polyhedral dice and probability, making 2D drawings or working with square roots. It is a little surprising how a very concrete activity as constructing polyhedron with poster-board was the preferred activity by most students. Almost nobody considered it childish or uninteresting and, although some had chosen this activity for being different and fun, some others gave some deeper reasons for their choice.

I quote a student from Group F that wrote: “I liked creating the shapes on my own out of poster-board. It allowed me to see the figure in a 2D sense but also in a 3D sense that made much of the exercises easier”. A student from Group E wrote:

I particularly enjoyed the part when we built the octa, tetra and icosahedron. I felt like that made us involved in what we were learning and it was great to have hands-on experience because that way, I felt like I was actually learning something.
One girl mentioned that the 3D models provided her something tangible to see and made her feel that she was truly understanding what was being taught and a boy wrote that he liked to construct the regular icosahedron (of which he previously had no knowledge) and how it turned out to be, after so much work.

Students wrote that they liked the software because it enabled them to see the polyhedrons from different perspectives and some enjoyed the Cabri activity where, by cutting the vertices off of a regular icosahedron, we turned it into a soccer ball. They mentioned how that was an application to daily life.

**Item 5.** If there was an activity that you disliked it or that it was too hard to understand. Can you describe which one and why?

This question opened the door to different kinds of criticism. For instance, some students complained about the computer lab facilities, which were out of the control of the researcher. Some students wished that the experience would have lasted more than two weeks, stating that it was a lot of material; some expressed this fact in a positive way, meaning that they learned a lot but several others expressed that sometimes the class went just too fast to be able to correctly grasp the concepts. Three US students were concerned about the number of classes they were missing of the regular mathematics course and how they would be able to catch up. Some students said that sometimes the course seemed disorganized and a girl made a serious proposal (to be considered in the future) to organize the contents differently. The topics that were most mentioned as hard to understand were calculating dihedral angles, the rationale of the golden number and the volume of the icosahedron. The first two probably because it entailed a lot of integration of geometry, algebra, square roots and trigonometry, with the use of several triangles and the third topic because it involves a lot of challenging computation with square roots. These three topics probably needed more time to mature in students mind than just two weeks. It
should be mentioned also that several students wrote: “Nothing” as an answer, meaning that they liked all the activities.

**Item 6.** Any other comments, reflections, perceptions or feelings you want to add about this experience?

Many Uruguayan students left this question blank. There were many expressions about having liked the experience and also some concerns that it would have been better with more time to go over each topic. Some students were glad and thankful to have participated in the experience.

An Uruguay student from Group D (one of two groups where the posttest counted also as a regular test) wrote a complaint saying that we have changed the rules of the assent form. The most critical statement was from an Uruguayan student who wrote (translated to English): “All was badly explained; I don’t know how he became a teacher; I wouldn’t classify this as an experience but as a waste of time”.

The most positive answers came from US students. A student from Group G in HS4 wrote: “I loved the fact that the teacher was amazed by what he was teaching us. The fact that it amazes him it also amazed me”. A student from Group F in HS3 wrote: “I thought geometry was boring and stupid until this class. The activities we worked in and out of class really helped me out with math in general”. A girl from Group E in HS3 wrote: “I thought this experience was cool! Thanks! Good luck Mr. Camou :-)” and another student from the same group wrote: “It was a great experience and I’m glad I got to be a part of it”.

**Final reflections about the questionnaires.** Item 2 (about what you learned) rated slightly less than items 1 and 3. There could be several reasons for this to happen. One reason was the case of several students that were absent during some of the experimental classes; many of these students rated highly the activities but then they expressed they didn’t learn as much as
they could have, since they had to catch up. A second reason could be that the pace of the experimentation could have been at times too fast, in some lessons. A third reason that should be mentioned, is the fact that the results on the tests did not count \(^1\) for the regular course. For some students this was a relaxing situation because they did not feel the pressure of the test but the quantitative analysis shows that the groups for whom the tests counted towards their grade (C and D) scored significantly above the average for the whole sample, and I attribute the higher scores to this pressure to get a good grade in their class. One US boy from HS3 clearly brought up this very influential issue regarding learning. He said during the interviews regarding his work in class: “Like, none of it being for graded didn’t make me take it as seriously as if it was for a grade; like, I still took it seriously but I take things more seriously when it’s for a grade”.

The time of the review and preparation for the final test is an important moment when learning can occur, especially for those more elaborate exercises and concepts that require more reflection, metacognition as well as some memorization. The effort a student makes to prepare for a test that counts is very different than for a test that doesn’t count. Therefore, it seems reasonable to think that students tend to learn less in an experience where they just have the internal motivation of learning compared with the case where we add the extra motivation of the grades, to the internal normal motivation.

In spite of the above difficulties, many students expressed that they had learned a lot during the experimentation. A student from Group F wrote: “I learned so much in such a short time. So, thank you!” At some point in the questionnaire, a substantial majority of the students, both in Uruguay and USA, expressed opinions that the activities were interesting, interactive, helpful, fun or cool. Regarding how they felt, the majority expressed that they liked it, they felt entertained or they enjoyed it.

\(^{1}\) except in groups C and D
What the students wrote as explanations for their ratings is consistent with the medians for each item: 8 for the quality of the activities, 8 for the amount and quality of what they have learned and 8 about how they felt during the experimentation.

The analysis of the questionnaires seems to confirm the earlier results from the assessments and interviews but also illuminates about how the process of learning occurred during the two-week course. The feedback given by the students has been very positive even though some criticism (very harsh in one or two cases) has also arisen. To end this analysis I would like to quote a student who’s opinion is representative of what the majority expressed in several different ways: “I enjoyed the way that we learned, through models that you can touch, and those on the computer as well”.

The Video Recordings

The poster-board class. The first activity after the pretest was the construction with poster-board of the regular tetrahedron and the regular octahedron. There were no complaints at all from any students about the seemingly very concrete character of this activity. I was surprised to realize that for several US students to construct an equilateral triangle using a compass was a new skill they acquired during this activity. This confirmed what was found with the pretests: most US students have very little practice in constructions with straight edge and compass. Nevertheless, almost every student learned properly and quickly how to construct the corresponding net. We can observe in the videos the great level of the student’s engagement during the activity. The passage from the 2D net to the 3D model was exciting and also the amazing symmetry of the resulting polyhedron that could be turned around always looking the same. After gluing it, the students kept a while holding the body in their hands in an appreciative way.
In class the students constructed only the tetrahedron and the octahedron; the regular icosahedron was a task for homework.

Some student’s mistakes appeared in the videos. An Uruguayan girl wasn’t very careful in drawing the equilateral triangles; one was smaller and she realized it when she wasn’t able to properly stick together the tetrahedron. Another girl from Uruguay had drawn only seven triangles for the octahedron so when she assembled it, she found that there was a hole in the body since a face was missing. A few students from both countries forgot to draw (or cut away) the flaps and so they had some difficulties when trying to glue the faces together to form the polyhedrons.

Some boys from both countries, after finishing their polyhedrons, started throwing them into the air as if they were balls of some sport. Students worked in groups, shared material and helped each other through the whole process. Two boys found that by manipulating tetrahedrons and octahedrons they could form other interesting polyhedrons.

The students started learning about the regular polyhedrons not only by looking at them from different angles but also by touching their faces, vertices and edges. They were
manipulating a model, that they had constructed themselves. They were not just watching someone else’s construction, they were touching their own.

**The computer figures.** The students went to the computer lab carrying their tetrahedrons, octahedrons and icosahedrons (constructed out of poster-board the day before). With the software Cabri 3D, we learned how to construct the five regular polyhedrons, change their size, color, surface style and move them around so that they could be visualized under different points of view. Students from all the classes similarly, were able to readily perform these activities with the software. The outstanding fact that there exists only 5 regular polyhedrons appeared naturally. The students were asked to count the number of faces, vertices and edges of each polyhedron.

![Figure 16: The five Platonic solids with Cabri 3D.](image-url)
As well as the poster-board models and the Cabri 3D figures, the teacher brought some plastic lattice models of the polyhedrons like the one in Figure 1 (page 4). Thus the students were able to choose in which representation to count faces, edges and vertices, except for the dodecahedron, for which they only had the computer figure. The number of faces of the tetrahedron, the octahedron and the icosahedron was trivial for them since they had already drawn the net for each one of these polyhedrons. For counting the number of edges, the plastic tube model was preferred since the edges had three different colors (red, yellow and green) and this facilitated the counting. An Uruguayan girl started counting the edges in the net but then another student pointed out that she was double counting some edges and so she dismissed this procedure. For the dodecahedron, they counted using the computer-generated figure and you could see several of them with their index finger on the screen going face-by-face or edge-by-edge. The 3D models were in general preferred over the computer figures for counting the three elements of each figure. However, it was still quite challenging to get the right numbers since the continuous change in the point of view (with the 3D models) gave rise to the problem of double counting or forgetting to count some element. The computer figures, initially harder to visualize, offered the possibility of leaving the body static to count all the visible elements; then reasoning about the number of hidden elements, the students just had to make an addition to have the total number of elements. The students worked in small groups and were able to get the correct number of faces, vertices and edges for each polyhedron using several different strategies and models to achieve this. They were asked to fill in a table with all the numbers, which was then discussed on the board. The teacher asked if they could find some kind of relationship between number of faces, vertices and edges (thinking about Euler’s Formula) but in most of the groups the first relationship that came up was the duality: the number of faces and vertices are
interchanged between the octahedron and the cube and between the dodecahedron and the icosahedron. Then the Euler’s Formula: \( \text{faces} + \text{vertices} - \text{edges} = 2 \) was figured out by some students in each class.

**The operations with square roots.** A sheet of exercises was given to the students who worked in small groups. The goal of the activity, as said before, was to become more skillful operating with square roots since they are crucial in order to exactly calculate areas, volumes and angles. It was brought to their attention that the numbers of the type \( a + b\sqrt{5} \) (\( a \) and \( b \) are rational numbers) form a field of numbers called a *quadratic extension field*. This has important implications. That is, we can add, subtract, multiply and divide numbers of that type and the result of those operations will be always a number of the same type. In particular the golden number: \( \frac{1 + \sqrt{5}}{2} \) and the volume of an icosahedron with a unit edge-length: \( \frac{15 + 5\sqrt{5}}{12} \) are numbers that belong to the “\( a + b\sqrt{5} \)” quadratic extension field.

US students from HS3 were specially fast and accurate when performing these operations. In all seven groups, students worked in a collaborative way, either explaining to each other or checking results. The students were sometimes surprised by the way that exercises would simplify. For instance when an Uruguayan student found that: \( \sqrt{99} + \sqrt{275} - \sqrt{704} \) was equal to 0 he said: “Qué increíble!” (*Unbelievable!*).

Students also learned to simplify (when possible) expressions like \( \sqrt{14 + 6\sqrt{5}} \) which can be written as \( 3 + \sqrt{5} \). These kinds of simplifications become very important when calculating the volume of the regular icosahedron.
The dihedral angle.

Figure 17: The Dihedral Meter for measuring Dihedral Angles.

The goal of this activity was to start to conceptualize the notion of angle between two planes by finding out the dihedral angle between two faces of a tetrahedron or two faces of an octahedron. The first step in the activity was to use the little instrument called Dihedral Meter (DM) that we see in Figure 17. The two wooden legs of the DM had to be adjusted, each one touching one face of the polyhedron (while remaining perpendicular to the edge between the two faces). Thus the angle formed by the two legs matches the dihedral angle between the faces. Then the student needed to draw (using the legs of the DM as rulers) the angle on a piece of paper and measure it with a protractor. The second step in the activity was to use the software Cabri 3D to measure the angle. We obtained with the software a more precise measure of the angle. The third step was to calculate the angle using trigonometry. At this point the students applied what they already knew, which was elementary trigonometry or the laws of cosines. If the results in the three steps matched, then the student certainly knew that everything she/he had done was correct. He/she would be constructing the concept of dihedral angle in his/her mind as well as reinforcing his/her trigonometric knowledge by applying it to something very concrete.
We could see through the video-recordings that some problems arose in the beginning of the activity. In order for the DM to match the dihedral angle, the legs of the instrument have to be placed perpendicular to the intersection of the two faces. This fact was not made explicit by the teacher (hoping that the students would find it by themselves) but several students in different classes had placed the DM incorrectly and therefore found a measure for the angle that was very far from the exact one. The manipulation to calculate the angle with Cabri was readily grasped but the last step also caused some problems. It wasn’t the fact of applying trigonometry itself; it was more the fact that before using trigonometry they had to figure out the correct triangle and do previous calculations. There was nothing particular hard in the last step; what was hard was to be able to integrate properly geometry, algebra, the operations with square roots and trigonometry. Besides the dihedral angle of the tetrahedron and the octahedron we calculated (later on in the course) the dihedral angle between two faces of a non-regular tetrahedron to reinforce all of these procedures. Most US students and some Uruguayans too, had a lot of trouble with the question about dihedral angles in the posttest. One of the reasons for the low score on this item, is probably the difficulties entailed in integrating the different kinds of knowledge.

Despite these operational challenges, there is evidence that the concept of dihedral angle was satisfactory grasped. A girl from Group F after finding the tetrahedron’s dihedral angle with Cabri and checking that it matched her DM measure, lifted both arms and said joyfully: “Oh, I got it right! Yes!” She didn’t need anybody to tell her she was right; the activities themselves were designed to provide the necessary feedback. When a student gets this kind of positive feedback from the mathematical tasks his/her self-confidence increases and this certainly enhances learning, which is the core of the Theory of Didactic Situations (Brousseau 1998).
The volumes and the truncation of polyhedrons. Very soon, after having finished the first experimentation in Group A, I realized that the calculation of the icosahedron’s volume was going to be hard for most of the students. The videos confirmed this: most of the time it was me talking and writing on the board with little participation from the students and many bored faces. Nevertheless, the possibility of being able to calculate exactly the volume of the icosahedron was a crucial factor for developing the whole iMAT approach for learning 3D Geometry. I felt that, even though I did not allow for the necessary time to make this learning happen in a constructivist way, that omitting this calculation would leave me in debt with the students; I felt that somehow the whole two-week course would be impoverished if I failed to teach this calculation. I also believed that some students were prepared to understand it well and that many others (that could only partially get it) would be in a very good position to comprehend it in the future, with their notes from class and the electronic notes I provided.

Where the students were much more active was in cutting polyhedrons to get new bodies using Cabri 3D. They loved this activity and were very good at it. We started truncating a regular octahedron in Cabri 3D by the midpoints of the edges to obtain the cube octahedron of Archimedes. Then we cut the edges of a regular icosahedron in thirds and we obtained the truncated icosahedron or soccer ball as we see in Figure 18.
Sometimes, instead of the Archimedean body appearing, there appeared some strange body; the problem was that the initial strategy needed to be modified (in fact simplified) when we advanced to the cutting. For instance, after cutting 1/3 of an edge on the icosahedron that edge is now 2/3 of the initial length; if we want to cut the other vertex of the edge we now need to find the midpoint of the 2/3-edge, which will be 1/3 of the original edge as in the initial case.

Several students were very surprised by the connection of the icosahedron and the soccer ball: the 20 faces of the icosahedron, after truncation became the 20 hexagonal faces of the soccer ball and the 12 vertices of the icosahedron became the 12 pentagons of the soccer ball. The students expressed that this was a very nice application to daily life.

**The icosahedron’s drawing.** During a course on 3D Geometry at the University of Georgia that I taught during fall 2010 with Dr. John Olive, something happened that reinforced for me the idea of the huge challenge that is entailed in making 2D representations of 3D objects. On the final exam, in one of the questions, students were asked to define a regular icosahedron and make a picture of it. Most of the students gave very good definitions but were unable to
produce a proper 2D representation of the icosahedron, even though we had extensively worked with the icosahedron during the course.

Therefore my conclusion was that in future courses, after having worked during some time with 3D models and computer figures, we should dedicate a specific span of time for teaching and helping students learn how to draw a regular icosahedron in one particular position. Thus half a class was dedicated towards the end of the course to teach the students how to construct (with ruler and compass) a 2D representation of the regular icosahedron like the one in Figure 19.

![Figure 19: Drawing of a regular icosahedron](image)
**Dice and probability.**

Groups of four students each were formed and a set of five polyhedron dice (shown in Figure 20) was given to each group. The students had to roll the five dice and record the sum of the numbers on their uppermost faces (or vertex, in the case of the tetrahedron). Each student would do this three times. Once all the students had finished, they said out loud to the class the three sums they had obtained. The minimum possible sum was $1+1+1+1+1=5$ and the maximum possible was $4+6+8+12+20=50$. We analyzed how many different outcomes there could be and we got: $4\times6\times8\times12\times20=46080$. So the probability of getting a 50 was $1/46080$ and of getting 5 as the sum was also $1/46080$. The biggest and the smallest sums were those with less probability so I gave a small prize to the biggest sum and to the smallest sum. The most extreme sum was obtained by a girl from HS4 who got 48!

*Figure 20: The five polyhedron dice*
The classes where the students played with the dice were like a party. The videos are full of smiling faces,laughs and jokes. The icosahedron and dodecahedron roll much more (because of their wider dihedral angles) than the regular cubic dice and so they often fell off the tables. Several students piled up the five dice testing their stability; other students rolled them as spinning tops in such a skillful way that the dice kept rotating for a long while. While doing all these activities, they were reinforcing the inherent properties of a tetrahedron, a cube, an octahedron, a dodecahedron and an icosahedron.

The dice were a new 3D representation of the polyhedrons. Unlike the poster-board models these could be squeezed without crushing them and they had also an extra feature: you could hear them while they were hitting the table before coming to a stop.

The qualitative data has shown so far that the students in the seven groups have found that the topics studied during the two week unit were interesting to them and that they enjoyed learning them. The feed-back given by the students in the questionnaires and interviews from the seven groups was surprisingly homogenous. Also their facial expressions caught by the video recordings showed serious and involved work and much joy when they got the solutions of some problem or were playing with the regular polyhedrons.

The Pretest and the Posttest

The pretest and the posttest were similar in length and each one had a maximum of 50 points. The difference of the scores between posttest and pretest were calculated for each student. The points for each question were different depending on the length and the difficulty of the required knowledge to give a correct answer. Thus, the questions in Pair 1 have a maximum of 6 points. In Pair 2 the maximum for each question was 9 points; for Pair 3 the maximum was 7
points; for Pair 4 the maximum score was 11; for Pair 5 the maximum was 9 points and for Pair 6 the maximum was 8 points. If we add all the maximums: 6+9+7+11+ 9+8 we get 50 points, which is the maximum for the whole test.

In the following sections, I analyze the difference between posttest and pretest for each pair of questions to see in detail the students’ performance in each topic and then I analyze the difference in the first three pairs that were formed by parallel questions and finally I analyze the difference for the whole test.

**Pair 1.**

**PRETEST**

2) a) This is a regular 5-point star. What is the angle’s measure of a point of the star?

   b) Explain how you got your answer in a)

**POST-TEST**

1) a) This is a regular 5-point star. What is the angle’s measure of a point of the star?

   b) Explain how you got your answer in a)

**Question’s Rationale.** Many flags around the world (for instance the flag of China, the European Union and USA) have regular 5-point stars. In Christmas time, we currently see
thousands of 5-point stars and everyday we see 5-point starts in various different ways. The symbol of the mythic Pythagorean School carried a 5-point star. The 5-point star has always been present in the history of mathematics, it exists in our daily lives and it is a rich mathematical object. However, it does not seem to exist in our mathematics courses. The majority of students don’t know how to make an accurate construction of the 5-point star and they cannot tell what is the angle of the star (Camou, 2006). Being aware of this situation from my previous experience, Question 2 in the Diagnostic Assessment (pretest) did not ask for a representation of the star. Rather, it provides a proper representation and it only asks about the angle of the star. The function of the picture was to give the students the opportunity to estimate the measure of such an angle by using a protractor to measure it.

After the 2-week course, the purpose of having the same question in the final assessment (posttest) was to find out whether the students had been able to move from an approximation obtained by measuring to an exact value obtained by calculation.

**The results in the pretest.** The question was graded with a maximum of 6 points. If a student wrote that the angle of the star was $36^\circ$ and the rationale was that he or she measured with a protractor he or she would get 4 points out of 6; if the answer was $34^\circ$, $35^\circ$, $37^\circ$ or $38^\circ$ he or she got 3 points out of 6 considering that it was a rather good approximation using the protractor correctly. If the answer was $36^\circ$ and the rationale was based on appropriate properties of the figure, then the score was 6.
Table 4
Quartiles, Median and Mean for Question 2 in the Pretest.

<table>
<thead>
<tr>
<th>Pretest</th>
<th>Maximum Points</th>
<th>First Quartile</th>
<th>Median</th>
<th>Third Quartile</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question 2</td>
<td>6</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>2.99</td>
</tr>
</tbody>
</table>

In Table 4 we observe that the mean is 2.99 for this question on the pre-test for the 134 students. We also observe that the median was 3. The mode (it doesn’t appear in this table) was also 3, which corresponds typically to a close approximate answer with the only rationale of having used a protractor to measure the angle.

The results in the posttest. Students who were able to state that the angle of the star was $36^0$ and were able to give a correct rationale (not necessarily perfect) based on the properties of triangles and regular pentagons got 6 point out of 6. If the value was correct and the rationale was valuable but with some flaw or incomplete they were given 5 out of 6. If the answer was $36^0$ and the rationale was having measured with the protractor they will still get 4 out of 6 (3 points for the exact value and 1 for the rationale).

Table 5
Quartiles, Median and Mean for Question 1 in the Posttest

<table>
<thead>
<tr>
<th>Pretest</th>
<th>Maximum Points</th>
<th>First Quartile</th>
<th>Median</th>
<th>Third Quartile</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question 1</td>
<td>6</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>4.59</td>
</tr>
</tbody>
</table>

We observe in Table 5 that in the posttest the median is 5, which shows significant progress from the pretest. As stated before, 5 corresponds to a correct answer of $36^0$ with a partially correct rationale based on properties of the figure. The mode for the posttest was 6; 44 students out of the sample of 134 students were able to give both the right answer and a correct explanation.
The difference between posttest and pretest.

Table 6
Mean of the Difference in Pair 1.

<table>
<thead>
<tr>
<th></th>
<th>Mean in the Pretest</th>
<th>Mean in the Posttest</th>
<th>Mean of the Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pair 1</td>
<td>2.99</td>
<td>4.59</td>
<td>1.60</td>
</tr>
</tbody>
</table>

Table 7
t-test for the difference of scores in Pair 1.

Test of Collection 1

<table>
<thead>
<tr>
<th>Attribute (numeric): Dif</th>
</tr>
</thead>
<tbody>
<tr>
<td>Attribute: Dif</td>
</tr>
<tr>
<td>Sample count: 134</td>
</tr>
<tr>
<td>Sample mean: 1.59701</td>
</tr>
<tr>
<td>Standard deviation: 2.13499</td>
</tr>
<tr>
<td>Standard error: 0.184435</td>
</tr>
<tr>
<td>Alternative hypothesis: The population mean of Dif is not equal to 0</td>
</tr>
</tbody>
</table>

The test statistic, Student's t, is 8.659. There are 133 degrees of freedom (one less than the sample size).

If it were true that the population mean of Dif were equal to 0 (the null hypothesis), and the sampling process were performed repeatedly, the probability of getting a value for Student's t with an absolute value this great or greater would be < 0.0001.

If we test the null hypothesis we get a p value smaller than 0.0001, which is the probability of obtaining this result if the null hypothesis were true. So we can affirm that the learning regarding this question has been significant at 0.05 level. We can also analyze the number of the
difference in the means occurring by chance alone. From a total of 134 students, 98 students (73%) got a positive difference, 20 students (15%) got 0 as difference and 16 (12%) got a negative difference. For instance some students were able in the pretest to state that the star’s angle was $36^0$ and were able to give a correct rationale; thus it was not possible for them, to get a positive difference since initially they had got already full marks.

Still more interesting is the number of students who were able to give the right figure of $36^0$ in the pre compared with the posttest. In all the groups a large majority could not tell in the pretest what was the measure of the angle of the star, while in the final test the situation was reversed; in all groups an overwhelming majority was able to tell correctly what was the angle of the star and in particular in three groups this was the case for 100% of the students. The percentages were as follows:

- In the pretest 93 students (69%) did not answer that the angle of the star was $36^0$ whereas 41 students (31%) did.
- In the posttest only 15 students (11%) were unable to give the right answer whereas 119 students (89%) were able to do so.

*Figure 21: Knowledge about the measure of the star’s angle in the Pretest*
In Figure 21 the orange column represents those who did not give the right answer in the pretest and in green those who answered correctly.

![Posttest Graph]

*Figure 22:* Knowledge about the measure of the star’s angle in the Posttest

Figure 22 corresponds to the final assessment; we see how the situation regarding this knowledge had changed dramatically.

*Final comments regarding this question.* During the two-week classes we learned how to construct with ruler and compass both the regular pentagon and the star and also the rationale behind the construction, which naturally involved the angles of both figures. The students did not just learn the measure $36^0$ isolated from its context. In the US classes there was a small national flag hanging from the board so the students were asked “Here in this little piece of cloth that represents your flag, how many $36^0$ angles do we have?” Several students simultaneously answered “250.”
This question assessed if the students knew the measure and the reasons behind it. The fact of having worked on the rationale of this knowledge (even if many students weren’t able to give a complete explanation) seems to highly contribute in the high percentage (89%) of right answers to part a) of the question.

It is interesting to consider how, an even incomplete or maybe erroneous rationale could still be very useful for the students to provide support or certainty regarding the measure of the star’s angle. For instance, some students wrote that since \( \frac{180}{5} = 36 \) then the star’s angle was 36. Several students wrote that when we draw from one vertex, two diagonals in a regular pentagon the angle of the pentagon (which is \( 108^\circ \)) is divided into three equal angles and therefore the angle of the star is \( 1/3 \) of 108 which is 36. This is totally true; we can object only that this property could also be justified in terms of more basic properties (as it was done in class). It is debatable how to grade these answers but those who gave the right measure of \( 36^\circ \) and used in the explanation the property about \( 1/3 \) of the angle of the pentagon got 5 out of 6 points.

A student gave an explanation regarding this question that it is worth mentioning for its originality. He counted the sides of the star, which are 10 (correct). Then he wrote \( \frac{360}{10} = 36 \). So the angle of the star is \( 36^\circ \). The starting point was correct, we cannot understand the reasons behind the operation but then the final answer is correct. This reasoning is very valuable for the student even though we cannot characterize it as “correct”. He has an explanation that leads him to the correct answer. We can only tell him that he would need to explain why (regardless of the result) the operation \( 360/10 \) is right. We can only characterize this reasoning as “incomplete” but not wrong.
In the pretest many students used the protractors and several got the right answer just by measuring accurately. In the posttest few students used the protractors (in spite of being available) because most of them knew beforehand what that angle was, and also had an explanation (more or less rigorous) for it. The process of starting from approximation and travelling through reasoned steps towards exactness was achieved for the majority of students.

Pair 2.

PRETEST

3)

a) Calculate the length AE knowing that ABCD is a square.

b) Calculate the volume of a cube which edge has the size of segment AE.

*Try to give your answers to part a) and b) in an exact form.*
3) a) Calculate the length of segment AD
b) Calculate the volume of a cube that has edge AD.

Express your results in both decimal and exact form.

Question’s rationale. The goal of this question is to test the skill level of students operating with radicals. But instead of writing the square roots explicitly in the question, they appear as a result of applying the Pythagorean Theorem (part a) which I assumed was a familiar piece of knowledge for students. In part b) students were asked to calculate the volume of a cube with the result obtained in part a). Again here, we are linking geometry (the volume of a body) with algebra (operations with square roots). In the pretest the students were asked to try to express the result in an exact way. In the posttest the figure changes very slightly but the question is basically the same. Considering the work done with the students during two weeks using radicals and decimals (see Appendix G), students were asked to give the result in both an approximate and an exact way. Having the result in two different ways enables the students to check their results and increase their confidence.
The results in the pretest. The question was graded with a maximum of 9 points.

Very few students were able to get full marks giving both results to part a) and b) with an exact irrational expression.

Table 8
Quartiles, Median and Mean for Question 3 in the Pretest

<table>
<thead>
<tr>
<th>Pretest</th>
<th>Maximum Points</th>
<th>First Quartile</th>
<th>Median</th>
<th>Third Quartile</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question 3</td>
<td>9</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>4.40</td>
</tr>
</tbody>
</table>

The mean 4.40 is less than 4.5, which is 50% of the question’s points. The third quartile is 6, which corresponds to having done the first part correctly, which corresponds to the Pythagorean Theorem, and having expressed the volume’s formula correctly without being able to operate with radicals to find the final answer.

The results in the posttest. The mean now has increased to 6.15 (see Table 9) which is clearly greater than 50% of the question’s points. Now the third quartile has been raised from 6 to 9, which means that at least 25% of the students got full marks for the question, being able not only to apply correctly Pythagoras but also being able to obtain \((1+\sqrt{5})^3\) exactly using radicals and then checking the result using decimals.

Table 9
Quartiles, Median and Mean for Question 3 in the Posttest

<table>
<thead>
<tr>
<th>Posttest</th>
<th>Maximum Points</th>
<th>First Quartile</th>
<th>Median</th>
<th>Third Quartile</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question 3</td>
<td>9</td>
<td>5</td>
<td>6</td>
<td>9</td>
<td>6.15</td>
</tr>
</tbody>
</table>

Also, we observe that the first quartile went from 3 in the pretest that corresponds to a poor performance in the question, to a 5 in the posttest that corresponds to an acceptable score.
**The difference between posttest and pretest.** In table 10 we can see the mean of the difference between posttest and pretest, which is 1.75.

Table 10  
*Mean of the difference in Pair 2.*

<table>
<thead>
<tr>
<th></th>
<th>Mean in the Pretest</th>
<th>Mean in the Posttest</th>
<th>Mean of the Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pair 2</td>
<td>4.40</td>
<td>6.15</td>
<td>1.75</td>
</tr>
</tbody>
</table>

We can also do a *t*-test regarding this question:

Table 11  
*t*-test for the difference of scores in pair 2.

Test of Collection 1  
Test Mean: 1.75373  
Sample count: 134  
Sample mean: 1.75373  
Standard deviation: 2.98224  
Standard error: 0.257627  
Alternative hypothesis: The population mean of Dif is not equal to 0

The test statistic, Student's t, is 6.807. There are 133 degrees of freedom (one less than the sample size).

If it were true that the population mean of Dif were equal to 0 (the null hypothesis), and the sampling process were performed repeatedly, the probability of getting a value for Student's t with an absolute value this great or greater would be < 0.0001.

Similarly as in pair 1 we obtained a *p* value smaller than 0.0001 and so we can affirm that the learning regarding this question has been significant at 0.05 level. Out of the whole sample
of 134 students 91 (68%) got a positive difference between posttest and pretest in this question, 16 students (12%) got 0 as the difference and 27 students (20%) got a negative difference. The improvement was comparable with the one observed in pair 1 though slightly smaller. We can visualize these percentages in Figure 23, the first bar corresponding to negative difference, the second one to null difference and the third one to positive difference. Students have significantly improved in their skills with the algebra of square roots, which speaks in favour of the whole integrated approach. That is: in spite of being a very short course (only two weeks) and being mainly about 3D Geometry, the students still were able to improve their algebra knowledge and skills.

**Figure 23:** Negative, Null and Positive Difference in Pair 2

**Final comments regarding this question.** We spent one full period of class with a worksheet of radicals, where the idea was to operate and simplify trying to give a final exact result
using square roots. Then these types of operations were currently used for calculating exactly the volume of the regular tetrahedron, octahedron and icosahedron. Not only did the students like the work-sheet (they worked enthusiastically on it) but they also realized the usefulness of the radical operations, since they were crucial for calculating volumes and dihedral angles.

**Pair 3.**

**PRETEST**

5) What is a regular octahedron?  
Give as much information as possible including some picture.

**POSTTEST**

4)  
a) What is a regular octahedron?  
b) Give information about it including a picture.

*Question’s rationale.* The questions in the pretest and posttest are essentially the same. In the posttest, the question is split into part a) and part b) since it was expected that students in the posttest would be able to provide much more information than in the pretest. This is the first pair of questions entirely about 3D geometry. From my previous experience I assumed that students would know quite a lot about the cube, something about a tetrahedron but very little or nothing about the octahedron the dodecahedron or the icosahedron. I decided to ask about the octahedron, which is simpler than the dodecahedron and the icosahedron, and therefore I thought that there would be more chances that students initially would know something about it.

The total number of points for this question was 7, which were distributed among the definition, some properties and a proper picture.

*The results in the pretest.* The results on the pretest were very poor as expected. Many students didn’t give any answer at all and got a 0. Many students (from both United States and
Uruguay) said that an octahedron was a polygon with 8 sides and had drawn a figure that looked like an octagon; they got just 1 point. Very few students were able to say that it was a 3D figure, body or shape with eight faces and still less were able to draw any kind of proper picture. No student got 7 points in the pretest and only one got 6 points.

Table 12
*Quartiles, Median and Mean for Question 5 in the Pretest.*

<table>
<thead>
<tr>
<th>Pretest</th>
<th>Maximum Points</th>
<th>First Quartile</th>
<th>Median</th>
<th>Third Quartile</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question 5</td>
<td>7</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

The table speaks for itself: the mean is 1, the median is 1 and the third quartile is still 1!

*The results in the posttest.* The results in the posttest for this question have changed dramatically. We can observe them in Table 13.

Table 13
*Quartiles, Median and Mean for Question 5 in the Postest.*

<table>
<thead>
<tr>
<th>Posttest</th>
<th>Maximum Points</th>
<th>First Quartile</th>
<th>Median</th>
<th>Third Quartile</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question 5</td>
<td>7</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>4.45</td>
</tr>
</tbody>
</table>

The mean has been raised from 1 to 4.45, the median from 1 to 4 and the third quartile from 1 to 6. Many students got the maximum score 7. The majority of the students were now able to state that a regular octahedron was a polyhedron (or a 3D shape or figure) with 8 equal faces that were equilateral triangles; many of them also added that the octahedron had 12 edges and 6 vertices and had drawn a proper representation of it; some others mentioned other properties as the dihedral angle between two faces or the octahedron’s volume.
**The difference between posttest and pretest.** The mean difference between posttest and pretest is 3.45 and it is the greatest difference for all the six pairs of questions, which shows clearly how all the students from both countries initially knew practically nothing about this 3D Geometry question and after the two-week course the majority of them had earned a satisfactory level of knowledge about this topic.

*Figure 24: Box Plot with Mean in the Pretest*
Figure 25: Box Plot with Mean in the Posttest

Observing Figure 24 and Figure 25 we realize that the learning regarding this 3D geometry topic has been spectacular. In table 14 we see the $t$-test. Again we obtain a $p$ value smaller than 0.0001 and so we can affirm that the learning regarding this question has been significant at 0.05 level.
In the pretest 129 students (96%) got 0, 1, 2 or 3 points. “0” corresponds typically to no answer, “1” point to say that a regular octahedron is a figure with 8 sides, “2” points is to have some minimum (incomplete or flawed) information about the octahedron and “3” is to describe some correct but still insufficient feature about the octahedron. An example of a 2-point answer was to say that the octahedron was a 3 dimensional figure with 8 octagonal faces or a prism with 8 hexagonal faces. Thus an overwhelming majority had very little or no idea what was a regular octahedron at the beginning of the experimentation. Only 5 students out of 134 were able to provide a satisfactory answer about what a regular octahedron was before the experiment!

Table 14

t-test for the difference of scores in pair 3.

<table>
<thead>
<tr>
<th>Attribute (numeric): Dif</th>
<th>Test of Collection 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Attribute: Dif</td>
<td></td>
</tr>
<tr>
<td>Sample count: 134</td>
<td></td>
</tr>
<tr>
<td>Sample mean: 3.44776</td>
<td></td>
</tr>
<tr>
<td>Standard deviation: 2.03565</td>
<td></td>
</tr>
<tr>
<td>Standard error: 0.175853</td>
<td></td>
</tr>
<tr>
<td>Alternative hypothesis: The population mean of Dif is not equal to 0</td>
<td></td>
</tr>
</tbody>
</table>

The test statistic, Student's t, is 19.61. There are 133 degrees of freedom (one less than the sample size).

If it were true that the population mean of Dif were equal to 0 (the null hypothesis), and the sampling process were performed repeatedly, the probability of getting a value for Student's t with an absolute value this great or greater would be < 0.0001.
After the experiment, 94 students (70%) got “4”, “5”, “6” or “7” points being able to just give a correct definition in some cases but in many others not only that but also extra information and a correct 2D representation on paper.

Figure 26: Zero, 1, 2 or 3 points vs. 4, 5, 6 or 7 points in the Pretest

Figure 27: Zero, 1, 2 or 3 points vs. 4, 5, 6 or 7 points in the Posttest

The left columns in orange in Figure 26 represents the students that had no knowledge or insufficient knowledge (0,1, 2 and 3 points) about the regular octahedron in the pretest while in
Figure 27 the orange column represents this same situation in the posttest. The right column in green in Figure 26 represents the students who gave a satisfactory answer in the Pretest while in Figure 27 the green column represents the same situation in the Posttest.

The two figures (Figure 26 and Figure 27) show clearly how a 3D Geometric topic, of which the students initially were ignorant, became a familiar topic after the course for the majority of students.

**Final comments regarding this question.** This spectacular shift in the situation shows both that the topic was interesting for the students and that the approach for learning, during the experimentation, was adequate. The octahedron was first introduced to the students when they had drawn on poster-board eight equilateral triangles and then they folded the net into an octahedron. The following class, they made the octahedron in the computer lab with Cabri 3D. The teacher brought at that time, some plastic tube models of the octahedron to help visualization. Afterwards, the students learned how to draw on paper an octahedron, using traditional tools and finally they played with solid plastic octahedron dice. These multiple representations of the octahedron were capable of furnishing enough information (in a proper context) to allow the majority of the students to build a personal relationship with the topic. The regular octahedron became a meaningful mathematical object, thus enabling to develop a correct concept (about the octahedron) in the majority of the student’s minds.
Pair 4.

PRETEST

1) Represent properly, an equilateral triangle, a square, a regular pentagon and a regular hexagon (Ruler, compass and protractor will be provided).

POSTTEST

2) a) Construct with ruler and compass a regular pentagon.
   b) What is the ratio between the diagonal and the side of a regular pentagon?
      Justify your answer.

**Question’s rationale.** The question in the pretest asked for the construction of the first four regular polygons. The intention of this question was to find out what the students knew about the construction of polygons, knowing from previous experience that the construction of the regular pentagon was specially challenging for High School students in Uruguay (Camou, 2006). The pilot experience caused me to slightly change the question, which in the first version was: “Construct with ruler and compass the first four regular polygons”. In that previous experience in USA, I realized that many students were able to represent properly some of those figures without the use of any compass (and possibly neither a protractor). Therefore I realized, on the one hand that maybe U.S. students had little experience in construction with ruler and compass and on the other hand, that it was too narrow minded to validate only that kind of constructions when students were able to produce correct representations, showing that they recognized each figure. So I changed the wording to *construct properly*, which allowed for a wider array of strategies for producing a figure while still discouraging free-hand sketches. The difficulties that most of the students experienced in the pretest to construct a regular pentagon was a good motivation for learning the construction of the pentagon with ruler and compass.
During the course the students learned how to construct the regular pentagon (and the five-point
star) and then we went through the justification of the construction that produces the golden number, $\Phi$.

$$\Phi = \frac{1 + \sqrt{5}}{2} \approx 1.618$$

Hence, in the posttest students were asked to construct the regular pentagon with ruler and compass (Appendix K) and to provide information about the ratio between the diagonal and the side in a regular pentagon whose answer was the golden number; the justification of this, was part of the rationale for the pentagon’s construction (Appendix L). These questions were paired up since they both referred to construction of figures but in fact they were quite different and we have to be careful considering the results. Question 1 in the pretest asked only about the construction of figures without any kind of explanation whereas Question 2 in the posttest asked for a specific construction with ruler and compass and for an explanation of a property.

**The results in the pretest.** The question was worth 11 points: 2 for the representation of the equilateral triangle, 2 for the square, 3 for the hexagon and 4 for the pentagon.

Table 15
*Quartiles, Median and Mean for Question 1 in the Pretest.*

<table>
<thead>
<tr>
<th>Pretest</th>
<th>Maximum Points</th>
<th>First Quartile</th>
<th>Median</th>
<th>Third Quartile</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question 1</td>
<td>11</td>
<td>6</td>
<td>7</td>
<td>10</td>
<td>7.40</td>
</tr>
</tbody>
</table>

The mean is 7.40 and the median is 7. The score 7 corresponds typically to a student who was able to represent correctly an equilateral triangle, a square and a regular hexagon but not a regular pentagon or students who have represented correctly only the first two and then gave some skewed representations of the hexagon and the pentagon. Anyway, since the majority of the students knew how to represent the equilateral triangle and the square properly the mean and median are considerably high.
The results in the posttest. The 11 points were distributed in the following way: 7 points for the correct construction with ruler and compass and 4 points for answering the golden number and providing a correct justification.

Table 16

Quartiles, Median and Mean for Question 2 in the Pretest.

<table>
<thead>
<tr>
<th>Posttest</th>
<th>Maximum Points</th>
<th>First Quartile</th>
<th>Median</th>
<th>Third Quartile</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question 2</td>
<td>11</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>6.77</td>
</tr>
</tbody>
</table>

We see in Table in 16 that the mean has decreased to 6.77 and the median is still 7.

The majority of the students were able to correctly construct the pentagon but were unable to provide the justification of the appearance of the golden number, Φ.

Final comments regarding this question. The majority of students learned how to construct correctly the regular pentagon with ruler and compass but also the majority were unable to give a rationale for this construction. It seems that learning the reasons and arguments behind the construction was a different phase of the learning that required time for students to mature the ideas and time to review prior to the posttest. Out of the seven groups only Group C was able to get a positive mean for the difference (1.308) and coincidentally it was one of the two groups that whose posttest counted towards their course grade. Most of the students of that class were able to produce the correct rationale for the appearance of the golden number. I’m not affirming that they fully understood the rationale; I would rather say, that preparing seriously for the test, they acquired a combination of understanding and memorizing that enabled them to satisfactorily reproduce such a rationale.
Even though the difference for this pair was negative the majority of the students learned something valuable and useful: they learned how to construct the regular pentagon with ruler and compass and that the golden number somehow appears from that construction.

**Pair 5.**

**PRETEST**

4) Using trigonometry calculate the angle A and then check your result measuring it on the figure.

![Triangle Diagram](image)

**POSTTEST**

5) Calculate the dihedral angle with edge AB of the non-regular tetrahedron represented in the following figure and check your result using the provided 3D model.
Question’s rationale. One of the topics to be treated during the experimental course was the dihedral angle between two planes, in particular between two faces of a polyhedron. Thus, question 4) in the pretest was a preparation or introduction to the work with dihedral angles. Given an isosceles triangle from which we know the length of its 3 sides, calculate one angle. The figure in question 4) was in scale so that students, after calculating the angle using trigonometry, could double check their results measuring with a protractor the angle on the figure. Question 5) in the posttest was more ambitious and definitely demanded more time and higher mathematical skills to be able to correctly perform it. That is, they first have to apply the Pythagorean Theorem to a certain triangle that they had to figure out; then using that result they have to determine a second triangle where they would apply trigonometry to find the angle. Finally the double check of their work was done using the 3D model (provided by the teacher) using the DM (Dihedral Meter) tool and a protractor. Readily, I realized that the students would very hardly get positive differences between posttest and pretest due to the fact that the question in the posttest was significantly longer and more cognitively demanding. However, even though it might not be too fair to compare the posttest and pretest results for this pair, the questions are
indeed very related and question 4) in the pretest was a preparation to learn about question 5) in the posttest.

**The results in the pretest.** The question was worth 9 points. They could calculate the angle either directly using the law of cosines or splitting the triangle in two right angle triangles and applying elementary trigonometry. This second path was more available for the majority of students. We see in Table 17 that the mean was 4.858 and the median 4.5, which is exactly half of the possible points.

Table 17
*Quartiles, Median and Mean for Question 4 in the Pretest.*

<table>
<thead>
<tr>
<th>Pretest</th>
<th>Maximum Points</th>
<th>First Quartile</th>
<th>Median</th>
<th>Third Quartile</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question 4</td>
<td>9</td>
<td>1</td>
<td>4.5</td>
<td>9</td>
<td>4.86</td>
</tr>
</tbody>
</table>

The first quartile is 1 and the third quartile is 9. The fact that the first quartile and the third quartile are so far apart is somehow surprising. It shows that there is an important number of students (more than 25%) that applied trigonometry correctly to find the angle and got full marks but at the same time also, an important number of students (more than 25%) remembered practically nothing about elemental trigonometry.

**The results in the posttest.** We observe in Table 18 how the results very much decreased. The mean dropped to 2.70 and the median to 3. The third quartile descended to 3 also which means that 75% got 3 points or less corresponding to just having drawn and measured the angle (using the DM and the 3D model) or having only found the height of an isosceles triangle using the Pythagorean Theorem. We could say that 75% of the students were unable to figure out the correct triangle where trigonometry should be applied. This is a clear indication that the topic
was not well learned at all. Probably it demanded more reflection and more time to investigate than was available in two weeks. It also should be said that according to the pretest, there was a large portion of the student’s sample that had forgotten how to apply trigonometry (all students that participated in the experience had learned Trig in previous courses). And also, that in some groups some students ran out of time in the posttest. Since question 5) was the longest, they left it for the end, but then the bell rang, and there was no chance of having a five minute extension for the test.

Table 18

*Quartiles, Median and Mean for Question 5 in the Posttest.*

<table>
<thead>
<tr>
<th>Posttest</th>
<th>Maximum Points</th>
<th>First Quartile</th>
<th>Median</th>
<th>Third Quartile</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question 5</td>
<td>9</td>
<td>0</td>
<td>3</td>
<td>3</td>
<td>2.70</td>
</tr>
</tbody>
</table>

As a consequence, this pair registered a clearly negative mean for the difference between posttest and pretest scores.

*Final comments regarding this question.* I think the way these two questions were paired is debatable. Question 5) in the posttest seems to be twice the length and difficulty than question 4) in the pretest. Probably if a question about the dihedral angle was to be asked it should have been somehow broken down in maybe two smaller questions. The poor performance in this pair makes me think also that the topic of calculating angles in space with the integration of trigonometry and the algebra of square roots is altogether definitely much more challenging than what I had suspected.
Pair 6.

PRETEST

6) This is a picture of a soccer ball

a) What are the shapes of their faces?
b) How many faces of each type does it have?

POSTTEST

6) a) Make a 2D representation of a regular icosahedron.
   b) How many pentagonal and how many hexagon faces has the most popular model of soccer ball? Explain how you got your answer.

*Question’s rationale.* The maximum points for this question was 8. For this pair again the question on the posttest asks for more information than the question in the pretest. The pairing is justified partially because both questions are over the same topic, but not completely because the weight of both questions is different. In the pretest I did not expect that students would know the number of pentagonal and hexagonal faces of a soccer ball and my expectations were confirmed. During the course students realized that a soccer ball is in fact a truncated icosahedron (an Archimedean body) and therefore they could figure out the number of hexagons
and pentagons knowing the number of faces and vertices of an icosahedron. During the course the students were taught how to produce a 2D representation of a regular icosahedron using ruler and compass and that was the first part of question 6 in the posttest.

**The results in the pretest.**

Table 19  
*Quartiles, Median and Mean for Question 6 in the Pretest.*

<table>
<thead>
<tr>
<th>Pretest</th>
<th>Maximum Points</th>
<th>First Quartile</th>
<th>Median</th>
<th>Third Quartile</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question 6</td>
<td>8</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>2.90</td>
</tr>
</tbody>
</table>

The first quartile 2 and the third quartile 4 indicates that most of the students were able to state that the faces of the soccer ball were hexagons and pentagons but they couldn’t tell at all how many of each there were. The fact that the maximum was 8 shows clearly absolutely no one could tell that there were 12 pentagons and 20 hexagons in a soccer ball.

**The results in the posttest.**

Table 20  
*Quartiles, Median and Mean for Question 6 in the Posttest.*

<table>
<thead>
<tr>
<th>Posttest</th>
<th>Maximum Points</th>
<th>First Quartile</th>
<th>Median</th>
<th>Third Quartile</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question 6</td>
<td>8</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>4.19</td>
</tr>
</tbody>
</table>

We see in Table 20 that the mean has increased to 4.19 in the posttest from 2.90 in the pretest. The third quartile is 7, which means that many students were able to correctly tell the numbers of
pentagons and hexagons and some of them could give an explanation in terms of truncating a regular icosahedron. Also many students were able to provide a proper 2D representation of a regular icosahedron.

**Final comments regarding this question.** The construction of the icosahedron’s representation was worth 4 points, the correct number of hexagons and pentagons was worth 3 points and the explanation was worth only 1 point. The question in the posttest was more demanding than the one in the pretest. However, it was an accessible task for students (in the Posttest) to follow a procedure to construct a figure representing a regular icosahedron. Also the students were able to remember that a familiar object, such as the soccer ball, had the same number of hexagons as the number of faces of the icosahedron and the number of pentagons was the number of vertices of the icosahedron. It is quite remarkable that the students got a positive mean considering the different cognitive weight in the two questions of this pair. One possible reason to explain this positive result is the students’ extremely weak knowledge about 3D Geometry displayed in the pretest and another is that 3D Geometry is intrinsically so interesting that in a very small period of time they learned a lot.

**The results on the difference between the two tests.** After having studied the students’ performance for each pair we will now analyze the difference between the posttest and the pretest for the first three pairs taken together. The first three pairs were formed by parallel questions in the pretest and posttest and therefore it is pertinent to study the differences for these three pairs as it is done in Table 21.
Table 21

*Mean of the difference in the first three pairs between Posttest and Pretest.*

<table>
<thead>
<tr>
<th>Groups</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>Mean</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pretest</td>
<td>9.8</td>
<td>8.5</td>
<td>9.92</td>
<td>9.78</td>
<td>7.68</td>
<td>8.12</td>
<td>5.53</td>
<td>8.38</td>
<td>38%</td>
</tr>
<tr>
<td>Posttest</td>
<td>16.35</td>
<td>11.25</td>
<td>16.70</td>
<td>17.17</td>
<td>13.68</td>
<td>16.42</td>
<td>13.7</td>
<td>15.16</td>
<td>69%</td>
</tr>
<tr>
<td>Difference</td>
<td>6.55</td>
<td>2.75</td>
<td>6.77</td>
<td>7.39</td>
<td>6</td>
<td>8.3</td>
<td>8.18</td>
<td>6.78</td>
<td>N/A</td>
</tr>
</tbody>
</table>

The maximum possible points for the three questions in the considered three pairs was 22 in both the pretest and the posttest. The mean in the pretest was 8.38 and in the posttest was 15.16. In terms of percentage this means that in the pretest the mean was 38% of the possible points while in the posttest was 69%. This means that in the span of only two weeks the students were able to almost double what they initially knew about these topics moving from a clearly insufficient level of knowledge to a clearly satisfactory level. This constitutes strong quantitative evidence that learning occurred in the seven groups that participated in the experiment and backs up the qualitative evidence that has been extensively gathered through the questionnaires and focus groups interviews. As we can see in Table 21, the iMAT approach for learning 3D geometry has worked similarly satisfactorily regardless of the two different languages and the two different countries. I will analyze, however, some differences in the performance of Uruguayan and US students that are worth pointing out.
Comparing pretests of Uruguayan and US students. The pretests were done the first day of the experimentation so their results are naturally independent from the iMAT approach. The results of the pretest are basically a diagnosis about the high school students’ geometry knowledge and skills before starting the experience. The pretests, since they were done before the experimentation, are not only independent of the iMAT approach but also independent of the small variations that the experiment could have suffered among the seven different groups. Therefore statistical methods can be naturally applied to analyze the results.

Table 22
US students’ pretest mean vs. Uruguayan students’ pretest mean

<table>
<thead>
<tr>
<th>Country</th>
<th>Uruguay</th>
<th>USA</th>
<th>All Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pretest</td>
<td>25.76</td>
<td>21.75</td>
<td>23.63</td>
</tr>
<tr>
<td>Number of students</td>
<td>63</td>
<td>71</td>
<td>134</td>
</tr>
</tbody>
</table>

The US students pretest mean was 21.75 out of 50 possible points which represents 43.5% of the maximum points and the Uruguayan students pretest mean was 25.76 which represents 51.5% of the available points. Many US students had no or very little experience in using a compass but during the course they learned quickly and came to appreciate very much the accuracy of the straight edge and compass constructions.
Table 23

ANOVA between US students pretest mean vs. Uruguayan students pretest mean

<table>
<thead>
<tr>
<th>Source</th>
<th>Degrees of Freedom</th>
<th>Sum of Squares</th>
<th>Mean Square</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Groups</td>
<td>1</td>
<td>538.22</td>
<td>538.217</td>
<td>13.944</td>
<td>0.0003</td>
</tr>
<tr>
<td>Error</td>
<td>132</td>
<td>5094.87</td>
<td>38.597</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>133</td>
<td>5633.08</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Alternative hypothesis: The population means of pretest grouped by Country are not equal

If it were true that the population means of pretest were equal (the null hypothesis) and the sampling process were performed repeatedly, the probability of getting a value for F greater than or equal to the observed value of 13.9444 would be 0.00028.

The ANOVA for the pretest mean for both countries produced a p value of 0.00028 so we can affirm that the pretest means for Uruguay and USA were different at a 0.05 significance level; that is, the Uruguayan students scored significantly higher than the US students in the pretest. The US students before starting the experimentation displayed a weaker level of knowledge and skills about Geometry than Uruguayan students. As an example of this in the US classes there was no student who could tell what was the formula to calculate the volume of a pyramid (area of the base times height divided by three), which was the starting point to calculate the polyhedron’s volumes.

Comparing Learning of Uruguayan and US students.

The qualitative data has shown so far the great similarities in the way of learning of the students of both countries. They are so similar that it is really hard to identify differences. I would just say at this point that the most encouraging and positive comments have come from
US students and that the worst criticism has come from an Uruguayan student. But in both countries the positive feed-backs clearly outnumbered the negative ones. The difference between posttest and pretest is an indicator of what has been learned and we will compare such difference for US and Uruguayan students to identify possibly different styles of learning for the students of each country.

Table 24

Differences in the first three pairs and in the whole tests of US and Uruguayan students

<table>
<thead>
<tr>
<th>Country</th>
<th>Uruguay</th>
<th>USA</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diff in Pairs 1,2,3</td>
<td>6.11</td>
<td>7.37</td>
<td>6.77</td>
</tr>
<tr>
<td>Whole Diff for Tests</td>
<td>6.28</td>
<td>4.22</td>
<td>5.19</td>
</tr>
</tbody>
</table>

We observe for the pairs 1,2 and 3 that were formed by parallel questions in the pretest and posttest the US students got a higher difference than Uruguayan students. Then, if we consider the whole test (the six pairs), the difference of Uruguayan students was higher than the difference of US students. Trying to interpret this phenomenon we could say that when the exercises were well specified and post and pre questions were clearly connected the US students did better but when the questions entailed several steps, or demanded writing rationales (as for the pairs 4, 5 and 6) the performance of Uruguayan students was better. The US students have expressed (in the interviews) their appreciation of how the iMAT approach enabled them to integrate the mathematical knowledge but in the tests they showed serious difficulties to do so. However they operated very well with the square roots and learned to construct accurately with ruler and compass several new figures.
There were five students who got full marks (50/50) in the posttest: one was from Group A, two from Group C, one from Group D and one from Group E. That is, four Uruguayans and one US student got full marks in the posttest. The only US student who got full marks was born in Korea, where he lived until he was 10 years old.

To end this section I would like to emphasize that in spite of being significantly different the results of the pretests for the two countries, the amount of progress and learning has not been significantly different between the students of both countries. This fact highlights the robustness of the iMAT approach, which reveals itself to be successful under different initial conditions.

**Gender**

The other factor I want to analyze at this point is gender: the performance of girls vs. the performance of boys. It is as if we have two big teams. One is formed by all the US and Uruguayan girls and the other one by all the US and Uruguayan boys. The girls obtained higher means than the boys in every single pair of questions.

**Table 24**

*Difference’s Means of boys and girls for the whole tests*

<table>
<thead>
<tr>
<th>Gender</th>
<th>Boys</th>
<th>Girls</th>
<th>All Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean of the Difference</td>
<td>3.98</td>
<td>7.18</td>
<td>5.20</td>
</tr>
<tr>
<td>Number of Students</td>
<td>83</td>
<td>51</td>
<td>134</td>
</tr>
</tbody>
</table>

The mean of the girls is 7.18 while the mean of the boys is only 3.98 – considerably less than for the girls. I was very surprised. I have observed along the years that the academic performance of girls had been improving at a higher rate than the boys, but I never suspected
such a stark difference in the mean of girls over the mean of boys. To confirm this tendency about girls outperforming boys, it should be said that out of the 5 students who got full marks (50/50) in the posttest, 3 were girls and 2 were boys.

In light of the above results, I tried to inform myself about this phenomenon in US Education and found that Sommers (2000) wrote in an article called *The War against Boys*, many interesting facts to show how, in the last few years, boys and not girls, are on the weak side of an educational gender gap. Sommers writes that in 1998 she met the president of the Board of Education of Atlanta and asked him who was faring better in Atlanta schools, boys or girls. “Girls” he replied immediately. “In what areas?” asked Sommers. “Just about any area you mention” answered the Educational Authority from Georgia. There are many indicators that lately confirm this situation. Some of them are that girls normally outnumber boys in educational institutions, they get better grades, they have less behavioral disorders and are more academically engaged. It was not part of my research questions to study learning and achievement in Geometry based on gender differences, but the results became so self-evident that as a researcher I feel it was mandatory to report this. In Uruguay there are no studies (as far as I know) about comparing academic performance of boys and girls. I can only express a conjecture based in my experience as a teacher for over 25 years: similarly as it is happening in the United States, in educational issues girls are improving at a greater rate than boys, leaving them behind in almost every area.

**When the posttest became also a grade.** As it was explained before, in two groups (C and D) the posttest counted also as a grade for the regular mathematics course. When I designed this research work I did not consider this factor. I was naïf and thought that the participating
students would work and study for the posttest in the same way they would do for a regular course’s test. I considered that it was very positive that it came up for consideration in the analysis of the posttest results, since it was a situation that could not be avoided (it was the classroom teacher’s prerogative). Fortunately, in Uruguay this divergence from the original agreement with the students did not bring up legal issues as it might have in the USA.

The teacher, in order to fulfill his/her duty ethically, needs to keep always improving his/her course and do his/her best, each time. If there exists the opportunity to do something better in Group F that wasn’t possible in Group B, the teacher cannot dismiss this opportunity just because it didn’t happen in Group B. I believe that other than the natural and internal motivation that students have to learn, the fact that the posttest was going to count as a grade was an extra motivation for students to prepare for the test and consequently learn more. This was a great opportunity to enhance student’s learning and therefore it would have been unethical to dismiss it. It was a conflict between two simultaneous roles I was playing at the time: being a teacher and being a researcher. As a researcher this situation was a deviation of the assent form and it should have been avoided but as a teacher, as I said before, I could not afford rejecting a situation that would enhance student’s learning. The results in the posttests confirm the assumption of being extra-motivated by the grades: the students in groups C and D got the highest scores in the whole sample.

There is a very interesting phenomenon regarding this situation. In Question 2 in the posttest I asked students to construct a regular pentagon, to indicate what was the ratio between the diagonal and the side of the regular pentagon and to explain why. The true rationale of the appearance of the golden number as the ratio between the diagonal and the side in a regular pentagon involves the use of similarity and Thales Theorem whereas the Pythagorean theorem
only explains the construction of the golden number with ruler and compass. None of the 103 students whose posttest did not count as a grade were able to write the true rationale that involved the use of similarity. Out of the whole sample of 134 students only 11 were able to write the true rationale and they all belonged to the two groups in which the posttest counted as a regular grade. My interpretation of this phenomenon is that while the Pythagorean theorem and its applications are well incorporated for many students, the implications and applications of the Thales theorem are very much under-developed and unknown for students. The fact of having the pressure to prepare for a regular test probably helped them in learning this new type of reasoning.

**Multiple Regression Analysis for the Different Factors.** A multiple regression analysis can provide information on how the pretest, the country of residence and the gender affected the results on the posttest. The students from Uruguay were assigned a 2 for country and the students from USA a 1; the girls from both countries were assigned a 2 for gender and the boys (also from both countries) were assigned a 1. In order to do a multiple regression analysis the factors (country of residence and gender) have to be independent from each other, which is the case for our study.

The equation of the regression line for the posttest scores is:

\[
\text{Posttest} = 1.704 + 0.717\text{Pretest} + 3.872\text{Country} + 3.22\text{Gender}
\]

The three factors altogether: Pretest, Country and Gender account for not more that 35% of the variability. That is, the level of knowledge and skills displayed in the pretest, the fact of being a boy or a girl or being a US student or Uruguayan student had some influence, but it was not at all determinant for the result in the posttest. This is another indicator of the robustness of the iMAT approach for learning 3D Geometry.
Chapter 6

Conclusions and Implications

Answering the Research Questions

1. What could High School students learn about Geometry (both 3D and 2D) in a two-week experimentation, using iMAT engineering?

Both quantitative and qualitative evidence has been found in this study that indicate learning about 3D Geometry occurred among the students during the two-week experimentation. The qualitative data gathered from the focus group interviews and the questionnaires confirmed the quantitative results from the pretest and posttest. There was convergence from all sources of data. The fact that significant learning occurred, I claim, is a consequence of two main factors: that the material studied was interesting itself and that the teaching-learning approach used was appropriate. In more technical terms we could affirm that the material was valuable under an epistemological point of view and that the approach iMAT Engineering was effective under a didactical point of view.

Almost every student before the two-week experiment had no idea of what a regular octahedron and a regular icosahedron were, and after the experiment the majority of the students had a satisfactory conception of what those two bodies are. Similarly students evolved from having no concept about what a dihedral angle was, to an initial correct conceptualization of the dihedral angle, though still the majority had problems to calculate them. The majority of the students learned how to construct a regular pentagon with ruler and compass and that the ratio
between the diagonal and the side of it was the golden number. However, the rationale of the appearance of the golden number was grasped only by a minority of students (see Appendix I). The majority of the students (in both countries) could not tell at the beginning that the angle of the 5-point star was $36^0$ (even though, Groups E, F and G had in the room a national flag with 50 stars). After the course, the majority of the students in all classes in both countries were able to state correctly that the angle was $36^0$ and many could even write a correct rationale. The majority of students improved significantly in their ability to operate with square roots. The students learned that the regular icosahedron and the soccer ball are linked by truncation and they learned how to obtain some Archimedean bodies by cutting the vertices of regular polyhedrons. They counted faces, vertices and edges of polyhedrons and discovered Euler’s formula. They participated in deduction of the exact volume of the regular tetrahedron, regular octahedron and regular icosahedron using square roots but this skill was not tested in the evaluations. The majority of the students discovered the existence of new types of dice other than the traditional cube: the tetrahedron, the octahedron, the dodecahedron and the icosahedron. They enjoyed playing with these new dice and they got an initial perception of many applications for probability that these new and fancy dice could have. The students also reinforced their trigonometric knowledge, applying it to very concrete problems that were able to be double-checked by other procedures. They acquired new knowledge strengthening at the same time previous knowledge and skills. Finally they perceived that learning mathematics is not just an activity of manipulating abstract symbols and objects but that it is also a quasi-experimental and integrated activity, where one goes back and forth from concrete to abstract.

We could argue that some of the positive results are due to the Hawthorne effect. Kilpatrick (2012) wrote in an email “The term *Hawthorne effect* is used to describe the
possibility that participants in a study may be changing their behavior just by knowing that they’re in a study and not because of any treatment being applied to them”. That is: A certain amount of the learning could have occurred due to the novelty of participating in an experiment. But to counterbalance this argument I would say that for 5 out of the 7 groups that participated in the experiment, the posttest did not count for a grade in the course. If all the students would have studied for the posttest as if it were a regular test, the difference between posttest and pretest would probably have been bigger.

2. How did that learning occur?

The starting point of the process was the construction on poster-board of the 3D models. This was a very concrete representation of the polyhedrons where, not only the bodies could be seen under infinite different points of view, but also vertices, faces and edges could be perceived by the sense of touch. Then, the plastic tube models were offered to the students; this was also a 3D representation that we can characterize as semi-concrete; in this case only the edges could be perceived by the sense of touch. The second procedure, through which the students learned to represent the polyhedrons, was using the software Cabri 3D. The process of constructing the bodies with the software is very simple; only three clicks of the mouse are needed but the resulting figure is a very faithful representation of the polyhedron. The Cabri figure is a 2D representation of a 3D object but it can be looked at from an infinite different points of view. The Cabri figure also enables the learner to easily change the size of the body, its appearance and to calculate readily and exactly several features of the polyhedron, such as area, volume and dihedral angles. Also it allows the user to perform transformations in space and truncations to create Archimedean bodies. We could characterize this representation that simulates accurately the 3D object on a 2D screen as semi-abstract. Finally students learned how to make 2D
representations of the octahedron and of the icosahedron using ruler and compass. Unlike the former representations, a drawing is just one particular view of the 3D object. It is a single and static image. In this sense it is the weakest of all representations, since it provides less information than the other ones. Moreover, to produce a proper drawing of a 3D object requires a significant amount of previous knowledge about the object. Therefore we will characterize this representation as abstract. Several students during the interviews realized and mentioned that making the drawings were only possible after having worked before with the other stronger representations. Furthermore, they implied that if we would have started the course by making those 2D representations, they would have been meaningless.

From having drawn the polyhedrons (the most abstract representations) the students went back again to concrete representations: the plastic polyhedron dice. They greatly enjoyed playing with these very concrete representations of the regular polyhedrons that could not only be seen but also touched and heard while they were bouncing on the wooden tables.

Each new representation enabled students to focus on new attributes of the body thus enabling them to double check properties and results. Each new representation provided reinforcement for what had been already studied and fostered the metacognition processes, which, as we have seen in chapter 2, is crucial for developing the students’ cognitive structures.

There was interaction between students in small groups and interaction between the students and the different representations of the same body. Maybe the imperious need to integrate the multiple-representations is best illustrated by a video, where we can see a boy watching the polyhedrons moving on his computer’s screen, with his left hand grabbing his poster-board octahedron and with his right hand on the mouse.
3. *Could indicators be found that the teaching-learning iMAT approach could overcome the main epistemological and didactical obstacle for learning 3D Geometry?*

I have already argued that the main obstacle for learning 3D Geometry is related with the intrinsic difficulties entailed in representing 3D objects. Whereas in 2D geometry very often a single proper representation is sufficient to provide the necessary information to advance in the study, in 3D Geometry the situation is radically different. All the representations: concrete, semi-concrete, semi-abstract and abstract, taken alone, are insufficient to provide the necessary information to progress in the learning of 3D Geometry. The epistemological obstacle is to associate a 3D geometric object to a single type of representation; an object and its representation, are epistemologically different entities. My hypothesis is that to progress in the learning of 3D Geometry we should associate the 3D object with a set of multi-type representations. I further contend that to try to associate the 3D object with a single representation becomes a tremendous didactical obstacle that is almost impossible to overcome and condemns to failure the study of 3D Geometry. The multi-type representations enable us to tackle an array of different but related problems, which, as Vergnaud (1990) wrote, is essential to develop a conceptual field about the mathematical object.

The multi-type representations provide the necessary environment for creating *didactical situations*. My contention in this study is that these situations, confronted *adidactically* by the students (the students’ reactions are in response to the structure of the tasks and not to the demands of the teacher) are capable of provoking their learning. The possibility for students to get feedback from the *didactical situations* is crucial according to Brousseau (1998). The test results obtained by the students in this study, their recorded voices, their evaluations and the videos furnish sufficient evidence that the iMAT approach, implemented during the
experimentation, was effective to overcome the epistemological and didactical obstacles for learning 3D Geometry.

**Implications for Curriculum**

In the former section we have commented that the previous geometry knowledge observed in the pretest for US students was weak compared with the Uruguayans’ geometry knowledge. As an illustrative example of this phenomenon, I can mention the case of one US student who was in the program of Mathematics Education at the University of Georgia. She was a graduate student in EMAT 6550 (Integrated 3D Geometry course)\(^2\) that Dr. John Olive and I have taught twice at UGA. During this course, this student commented that it was the first mathematics course in her life where she learned how to construct figures with a straight edge and compass.

Specifically with respect to 3D Geometry the lack of knowledge was similar throughout the whole sample. For instance almost nobody among US or Uruguayan students was able to tell what was a regular octahedron. Before this research work, I was already very much concerned, about how little 3D Geometry was taught in my country, Uruguay, and in the USA and probably many other countries. After having done this experimentation, I’m also very much concerned about how little Geometry (of any kind) is taught at high school level in the USA.

The study of Geometry is fundamental for teenagers for several reasons. In an article from the University of Cambridge (2002) called *Why do we study geometry? Answers through the ages* ([http://www.dpmms.cam.ac.uk/~piers/F-I-G_opening_ppr.pdf](http://www.dpmms.cam.ac.uk/~piers/F-I-G_opening_ppr.pdf)) we can read:

Over most of the last two and a half thousand years in the European or Western tradition, geometry has been studied because it has been held to be the most

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\(^2\) Dr. John Olive and myself have taught this course at UGA in Fall 2010 and 2011.
exquisite, perfect, paradigmatic truth available to us outside divine revelation. It is the surest, clearest way of thinking available to us. Studying geometry reveals – in some way – the deepest true essence of the physical world. And teaching geometry trains the mind in clear and rigorous thinking.

The above text clearly makes the point about the great epistemological value that Geometry is endowed with. But geometry has also great didactical value. In my opinion, geometry is the best tool among all branches of mathematics to discover and learn the rationale underneath properties, theorems and mathematical truths. Geometry provides plenty of opportunities to experiment and to travel that fantastic journey from the very concrete to the very abstract that constitutes learning mathematics. Geometry has the power to make simple and understandable to many, properties and theorems that otherwise would be only comprehended by just a few.

About the notorious Didactic role of Geometry, Brousseau (2003) (http://math.unipa.it/~grim/brousseau_geometrie_03.pdf) writes that Geometry offers the teachers the possibility of provoking in their students authentic mathematical activities. Geometry is capable, through the use of simple and intuitive tools, of introducing anybody to the fantastic world of mathematics; Geometry is a privileged subject where arguments can be pondered, accepted or dismissed and students can thus learn to distinguish clearly correct reasoning from incorrect reasoning based on a deep knowledge of the nature of things.

In the following lines I will analyze the geometry content in the High School Mathematics Curricula of both Uruguay and the United States. Five years ago the High School Mathematics syllabus was changed in Uruguay. In the old syllabus (plan 1976) there were five different Mathematics courses during the last two years of high school which were: Pre-calculus, Calculus, Synthetic Geometry, Analytic Geometry and Descriptive Geometry. It should be said that in their Junior year of high school students

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3 Learning Mathematics is a quasi-experimental activity (Lakatos, 1976)
were required to choose between three orientations: Humanities, Science and Biology. The students who chose Biology had only Pre-Calculus (Junior year) and Calculus (Senior year)\(^4\). The students who chose Humanities had only Pre-Calculus (Law oriented) or Pre-Calculus, Calculus and Analytic Geometry (Economy oriented). Those students who chose Science had the five mathematics courses during the last two years of high school.

I will try to summarize the present Curriculum after the Curriculum change in 2006, which we can retrieve from the website of Inspección de Matemática (http://inspeccionmatematica.blogspot.com/search/label/Programas). The 10\(^{th}\) grade mathematics course is the same for all students. Its syllabus contains geometry, functions, simultaneous 3 by 3 equations, trigonometry, statistics and probability. Together geometry and trigonometry (as it is done in the Common Core State Standards in the USA) represent 30% of the course. All of the content in geometry and trigonometry is about 2D Geometry (trigonometry of course could be applied in 3D, but teachers seldom do that) and none about 3D Geometry.

The situation for the Mathematics Course Syllabus for younger years is similar: 30% of Geometry with very little or no 3D Geometry. In 11\(^{th}\) grade there is one Mathematics course (Math I) that is mandatory for all students and a second one (Math II) that is only required for scientifically oriented students. Math I contains a Unit about Analytic Geometry, which represents 28% of the course. From that 28%, four fifths is about 2D Analytic Geometry and one fifth is about 3D Analytic Geometry. In Math II there is a

\(^4\) The Calculus course was most about limits and derivatives and very little about integrals
Unit about synthetic Geometry that represents 30% of the course (18% about 2D Geometry and 12% about 3D Geometry).

In 12th grade there are four Mathematics courses: Math I, Math II, Math III and Math IV. Math I is obligatory for all students and it is a Calculus course with no Geometry\textsuperscript{5}. Math II is 70% about Analytic Geometry (Conics) and 30% about Synthetic Geometry. In the unit about Synthetic Geometry \(\frac{3}{4}\) is about 2D Geometry and \(\frac{1}{4}\) about 3D Geometry. Math III has 60% of 2D Analytic Geometry and Math IV is a course that consists of 30% of 3D Geometry and 70% of Descriptive Geometry. Depending on the orientation chosen by the students they would take just Math I, Math I and II, Math I and III or Math I and IV.

One of the main changes from the old Curriculum is that Geometry has been spread out in several courses instead of being just concentrated in one or two. As a consequence more students (but still not all) have a chance to study at least some Geometry. The traditional synthetic Euclidean plane geometry has been a bit shortened in favor of introducing new topics (or even new geometries such as fractals). Even though the contents about 2D Geometry in the syllabus continues to clearly exceed the contents about 3D Geometry there has been a perceptible increase of the latter.

What is the situation regarding Geometry in the Mathematics Curriculum in the USA? Looking at the Common Core State Standards (CCSS) we observe that the mathematics contents (http://www.corestandards.org/the-standards/mathematics) are divided into six areas: High School Number & Quantity, High School Algebra, High School Functions, High School Modeling, High School Geometry and High School

\textsuperscript{5} Except with Art oriented students who have 10% of Geometry (includes Spirals and Fractals)
Statistics & Probability. If these six areas have the same weight in the Curriculum we are saying that Geometry represents only 1/6 (17%) of the whole Curriculum. This percentage compared with the average one in Uruguay (30%) is very low especially considering the precious role that Geometry has in developing the rational thought processes and developing appreciation of the beauty, the usefulness and the power of mathematics as a whole.

Going more in depth with the CCSS we see a phenomenon that is not exclusive to USA or Uruguay but that occurs mostly everywhere: when we see geometry in mathematics course syllabi they refer almost always to 2D planar Geometry. I will illustrate this by observing the CCSS bullets referring to congruence:

- **G.CO.1.** Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
- **G.CO.2.** Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
- **G.CO.3.** Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
- **G.CO.4.** Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
- **G.CO.5.** Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

All five bullets refer to 2D Geometry. In G-CO.1 there is the definition of angle, circle, perpendicular line and parallel line but no definitions about dihedral angles nor spheres, nor parallel planes nor orthogonal lines. In G-CO.2 we have transformations in the plane but 3D transformations are not mentioned at all. In the next bullet the reflective and rotational
symmetry of regular polygons is mentioned but no mention whatsoever about the reflective and rotational symmetry of the regular polyhedrons. Only the last few bullets of the CCSS contain some contents about 3D Geometry but there are some intriguing and conspicuous absences. The regular polyhedrons (or Platonic Solids) of which there are just five (compared with an infinite number of regular polygons) do not appear at all. Euler’s Formula (which relates the number of faces, vertices and edges in most of polyhedrons) does not appear either, even though it is a simple and powerful formula to delve into the study of 3D Geometry.6

There is also an outstanding omission regarding 2D Geometry that should me mentioned. We can read in the following bullet:

G-CO.13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

This standard requires the construction of the regular polygons of 3, 4 and 6 sides. Why are we excluding the regular polygon of 5 sides? Is it because the regular pentagon is too rich mathematically? Is it because the five-point regular star (which is constructed basically in the same way as the pentagon) has no application to real life, while students see daily this kind of star at least 50 times, just by looking at the US flag? Is it because the pentagon is the birth place of the golden number? Is it because the pentagon’s construction is able to integrate geometry, algebra and trigonometry? (See Appendices K and L).

Only the last few bullets in the CCSS High School Geometry section are dedicated to 3D Geometry. The amount of 2D geometry is overwhelmingly greater than the amount of 3D Geometry.

6 Only one of the seven Geometry Texts I analyzed (page 7) contained the Euler’s Formula
Looking at the Georgia Performance Standards (GPS) the situation is similar if not more dramatic (https://georgiastandards.org/Standards/Pages/BrowseStandards/MathStandards.aspx). In a total of 23 bullets about geometry only 2 are about 3D geometry. These two bullets refer to the area and volume of the sphere and the effect on these magnitudes when we dilate the radius; indeed interesting and important (the sphere is present everyday and gives birth to one type of non-Euclidean Geometry). But, I would like to ask: How long do we still have to wait to study with our high school students those fantastic mathematical creatures that are the regular octahedron, icosahedron and the dodecahedron? They have been around for more than 2000 years. Maybe it is time to allow them to enter the school’s curricula for the sake of those who do
love mathematics and think that it is both beautiful and useful and for the sake of those who
don’t, to give them a chance to think that mathematics is not all tedious, mechanical or
uninteresting.

Taking a look at the Mathematics Standards for California Public Schools the situation is
not very different (http://www.cde.ca.gov/be/st/ss/documents/mathstandards.pdf). There are 22
bullets dedicated to Geometry. Prisms, pyramids, cylinders and cones reappear with the sphere
but no mention whatsoever about the regular polyhedrons or the Archimedean solids. There are
two bullets about coordinate geometry and in one of them the study of transformations in space
is included. Then there are three bullets about trigonometry and the rest (the majority) are about
2D Geometry. It could be argued that in the bullet where it is indicated to use the Pythagorean
theorem, we could do that in space using polyhedrons. It is true, but we know that this happens
very seldom. Similarly in the bullets about writing geometric proofs, we could prove properties
about 3D Geometry but again in reality the geometric proofs that are worked are almost always
about 2D Geometry.

To summarize I have argued that there are two types of weakness regarding the study of
Geometry at high school level in the USA. The first type of weakness is the low percentage of
Geometry overall in the whole Mathematics curriculum. As a consequence of this, students in
general are exposed very little to Geometry while they are teenagers missing thus opportunities
to learn the most powerful tool to integrate mathematical knowledge and a great instrument to
develop their ability to elaborate rational justifications to scientific phenomena.

The second type of weakness is about the tiny percentage of 3D geometry compared to
2D Geometry that is present in all the mathematics curricula. This phenomenon is not exclusive
to the USA; it’s also present in the Uruguayan Curriculum and I suspect that it happens in most
of the countries I could think of. This is clearly related with the epistemological and didactical obstacles for learning 3D Geometry about which I have talked before.

Someone could argue: *But it is natural that we study more 2D Geometry since it comes first and we need it to then study 3D Geometry.* If we expect to master 2D Geometry to then tackle 3D Geometry we will never progress in 3D Geometry. Since any subject is infinite we will never finish the mastery of 2D Geometry. Consequently we will never start to study seriously 3D Geometry. I think we should try to teach and learn 3D and 2D Geometry simultaneously and go continually back and forth from one to the other one. Just to illustrate if we are studying the property that through the three vertices of any triangle there exists always a circumscribed circle then we should study that through the four vertices of any tetrahedron there exists always a circumscribed sphere. The 2D property can be constructed with straight edge and compass or dynamic geometry software and the 3D property can be constructed with 3D dynamic geometry software such as the one used during this research. The rationale for the 2D and the 3D properties are so similar that once we understand one of them the other one follows logically with practically no effort.

To end this section I would like to recall an anecdote that happened at the University of Georgia during Spring 2009. We were in a Geometry Seminar and one of the students asked professor Dr Jim Wilson: “Why do you think that the study of Geometry is kind of neglected at Secondary level in USA?” Dr Wilson answered: “Probably one of the reasons is that geometry is a very small research field, among professional mathematicians”. I partially agree but I would like to make a counterpoint: How could professional mathematicians be interested in Geometry research if they have hardly studied any Geometry while they were in Middle and High School?
Final Thoughts

It has been said that Geometry had reached a dead end a century ago and this fact has been given as one rationale of the low presence of Geometry in mathematics curricula. I firmly disagree on this point. If there is something that had reached maybe a dead end it might be the Synthetic Euclidean 2D Geometry. The problems about which figures can or cannot be constructed with ruler and compass originated tons of mathematics along centuries from Euclid to the recent past. but nowadays we are interested on moving on. Moving on from the unique Geometry of ruler and compass but by no way moving away from Geometry. Who wants to move away from Spherical Geometry, non-Euclidean Geometries or Fractal Geometries about which we just started to learn a little? Who wants to move away from learning 3D Geometry when we really never seriously started? Most of the Euclidean Geometry was 2D because it was based on figures that could be represented efficiently with ruler and compass. New Geometries require new tools.

This doesn’t mean that we are going to ban the ruler and compass; we are going to continue using them when it is pertinent but aware at the same time of its limitations. Circles can be efficiently represented with a compass; ellipses can’t. Dynamic Geometry is an appropriate tool to represent an ellipse; a compass is not. We cannot represent properly a sphere with a compass but we can do so with software such as Cabri 3D. A Geometry based only on ruler and compass constructions was all that we studied about Geometry in the past. We decided that that Geometry was obsolete and we seem to be taking it away altogether. We have to recognize that there are several new geometries that can use new tools (not only specialized software but also concrete materials). The fantastic and admirable building and adventure of the human spirit that started with Euclid more than 2000 years ago, continues with new Geometries, where the ruler and compass constructions continue to be valuable but now as just one chapter in a greater adventure.
I haven’t mentioned Coordinate Geometry (or analytic geometry) as one example of new geometries for one reason. It’s not that I don’t consider it valuable or useful; it is that Coordinate Geometry is an integration between algebra and geometry; it certainly has the power of solving many interesting problems but when it is included in the curriculum it should not take the place of synthetic geometry; rather it should come after.

A similar case happens with Trigonometry; Trigonometry is a very useful branch of mathematics that beautifully integrates geometry, algebra and functions. If we include Trigonometry in the Geometry section of the Curriculum, I think that it should also be included in the algebra and in the function sections. If Coordinate Geometry and Trigonometry are only included in the Geometry section, we risk spending most of the time assigned to Geometry doing Algebra.

Maybe another factor that might have some influence in making it hard for teachers to teach geometry is the system of measures. On the one hand the International Metric system is clearly simpler to operate with but on the other hand the traditional English system is present in everyday life in the USA. So maybe the teachers are torn between simplicity and applicability.

During the past two years living in Athens, Georgia I became aware of the concern among the Mathematics Education community, about lower than expected US student’s results in international mathematics evaluations such as PISA. I came to believe that probably an improvement in the teaching and learning of Geometry could translate into a measurable improvement of US student’s performances.

I think that the documented (in this research work) weakness of US students Geometry knowledge and skills could be a good opportunity to re-introduce Geometry into the Curriculum more strongly. But not the old traditional Geometry; rather a new integrated 3D-2D Geometry
that does not hesitate to delve into the rich, amazing and mostly unexplored world of 3D Geometry; a new integrated Geometry that incorporates new and powerful tools, and that is capable of integrating different branches of mathematics such as algebra, trigonometry, functions and probability. Since the USA is the mirror at which many other countries look, I surmise that the impact of a Geometry improvement in US education, would have a worldwide effect.

We live in a 3D world with 3D objects but most of the time we study 2D geometry; we seem to be creatures living in Flatland (Abbot, 1984) regarding what we normally study in Geometry. This doesn’t make any sense. Maybe long ago we had a 3D problem and we thought: we have to first break it down into some 2D problems to be able to come back to solve the original one. But we stayed on the 2D problems and never went back. Maybe we didn’t have the tools or the knowledge to go back; therefore we even forgot what the original problem was about. Nowadays, we have acquired new tools and have accumulated more knowledge and so we are starting to remember what the problems were about.

We have been so far, just in the borders of the 3D Geometry world. For the sake of new generations I hope that we can make the decision to intern ourselves into that fantastic world, full of magic, full of mysteries to unravel and adventures to be lived.

The iMAT approach is built on the consideration that learning mathematics is a quasi-experimental activity. This characterization about mathematics has deep implications. One of the implications is that the experiment is a crucial component of the process. An experimentation entails necessarily the use of different representations and approximations and all this concrete and inaccurate material transforms beautifully into the abstraction and exactness of mathematics. I have reasons to suspect that the bases of the iMAT approach are valid for learning any branch
of mathematics. My advice to those who want to learn mathematics would be: In order to reach the very abstract, start with the very concrete.

I conclude that the iMAT approach has been tested successfully, in this research work, for learning integrated 3D Geometry and I believe or hope to have given some convincing reasons about this choice. When the Didactics is based on iMAT engineering, the data in this study indicate it IS possible for high-school students to both enjoyably engage in explorations of these 3D topics and learn significant mathematics while doing so, in a fairly short time span.

At the beginning of the EMAT 6550 course at UGA (already mentioned above), I asked the students (undergraduates and graduates) what was a regular icosahedron. Only one student out of 28 knew what a regular icosahedron was. This same student wrote later on in the course (Fall 2011):

One of the biggest flaws in our educational system is the lack of emphasis on solid geometry…I think that it is imperative that students are exposed to problems involving 3D geometry because our world is made up of 3D objects…I feel that subjects involving this topic should be incorporated into secondary mathematics courses.

I was truly impressed by the eloquence of this young US student regarding the study of 3D Geometry.

In each of the seven experimentations with High School students from Uruguay and USA, I gave my very best, on anything that depended on me, to provoke significant learning among the students and to gather evidence about their learning. Whether or not this research work will be read or will be fruitful for students and teachers, no longer depends on me.
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Appendix A: Diagnostic Assessment or Pretest (English Version)

DIAGNOSTIC ASSESSMENT

**STUDENTS:**  
**SCHOOL:**  
**DATE:**

1) Represent properly, an equilateral triangle, a square, a regular pentagon and a regular hexagon (Ruler, compass and protractor will be provided).

2) a) This is a regular 5 point star. What is the angle’s measure of a point of the star?
   
b) Explain how you got your answer in a)

3) a) Calculate the length AE knowing that ABCD is a square.
   
b) Calculate the volume of a cube which edge has the size of segment AE.
Try to give your answers to part a) and b) in an exact form.

4) Using trigonometry calculate the angle $A$ and then check your result measuring it on the figure.

5) What is a regular octahedron?
   Give as much information as possible including some picture.

6) This is a picture of a soccer ball

a) What are the shapes of their faces?
b) How many faces of each type does it have?
Appendix B: Diagnostic Assessment or Pretest (Spanish Version)

EVALUACION DIAGNOSTICA

ESTUDIANTE:                                                              LICEO:                             FECHA:

1) Representa adecuadamente, un triángulo equilátero, un cuadrado, un pentágono regular y un hexágono regular (Regla, compas y semicírculo serán suministrados en caso necesario).

2) a) Esta es una estrella regular de 5 puntas.
    Cual es la medida del ángulo de una punta de la estrella?

b) Explica cómo obtuviste tu respuesta a)

3) a) Calcula la longitud AE sabiendo que ABCD es un cuadrado.
b) Calcula el volumen de un cubo cuya arista sea el segmento AE.

*Intenta dar tus respuestas para partes a) y parte b) en forma decimal y exacta.*

4) Usando trigonometría calcula el angulo A y luego comprueba tu resultado midiendo en la figura.

![Diagrama de un triángulo con medidas 7 cm, 4 cm y 7 cm]

5) Que es un octaedro regular?
Suministra toda la información posible incluyendo alguna figura.

6) a) Que es un icosaedro regular?
b) Que forma tienen las caras de una típica pelota de fútbol?
c) Cuántas caras de cada tipo tiene?
Appendix C: Final Assessment or Posttest (English Version)

STUDENTS: SCHOOL: DATE:

1) a) This is a regular 5 point star. What is the angle’s measure of a point of the star?
   b) Explain how you got your answer in a)

2) a) Construct with ruler and compass a regular pentagon
    b) What is the ratio between the diagonal and the side of a regular pentagon? Justify your answer.

3) a) Calculate the length of segment AD
    b) Calculate the volume of a cube that has edge AD.

Express your results in both decimal and exact form.
4) a) What is a regular octahedron?
   b) Give information about it including a picture.

5) Calculate the dihedral angle with edge AB of the non-regular tetrahedron represented in the following figure and check your result using the provided 3D model.

6) a) Make a 2D representation of a regular icosahedron.
   b) How many pentagonal and how many hexagon faces has the most popular model of soccer ball? Explain how you got your answer.
Appendix D: Final Assessment or Posttest (Spanish Version)

EVALUACION FINAL (tentativa)

ESTUDIANTE:                                                               LICEO:                             FECHA:

1)  a) Esta es una estrella regular de 5 puntas .
    Cual es la medida del angulo de una punta de la estrella?
 b) Explica como obtuviste tu respuesta de la parte a)

2)  a) Construye un pentagono regular con regla y compas.
    b) Cual es la medida de la diagonal de un pentagono regular en funcion de su lado?
    Justifica tu respuesta.

3)  a) Calcula la longitud del segmento AD.
    b) Calcula el volume del cubo de arista AD.

   Expresa tus resultados para ambas partes en forma decimal y exacta.
4) a) Que es un octaedro regular?  
   b) Suministra toda la posible informacion incluyendo una figura.

5) Calcula el angulo diedro de arista AB del tetraedro no regular representado por la siguiete figura. Comprueba tu resultado usando el modelo 3D suministrado.

6) a) Realiza una representacion plana de un icosaedro regular.  
   b) Cuantas caras pentagonales y hexagonales tiene el modelo mas comun de pelota de futbol? Explica como obtuviste tu respuesta.
Appendix E: Questionnaire (English Version)

FOCUS GROUP QUESTIONNAIRE

1) a) How would you qualify the activities done during this 2 week experimentation?

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very bad | excellent

b) Explain your rating.

2) a) How would you qualify what you learned during this experience?

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very little | a great deal

b) Explain your rating.

3) a) How did you feel during these two weeks regarding this experience?

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disliked | liked very much

b) Explain your rating.
4) If there was an activity that you particularly liked it. Can you describe which one and why?

5) If there was an activity that you disliked it or that it was too hard to understand. Can you describe which one and why?

6) Any other comments, reflections, perceptions or feelings you want to add about this experience?
Appendix F: Questionnaire (Spanish Version)

CUESTIONARIO PARA ENTREVISTAS GRUPALES

1) a) Como calificarias las actividades realizadas durante esta experimentación de 2 semanas?

   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
---|---|---|---|---|---|---|---|---|---|----|
Muy malas | Excelentes

b) Explica tu calificación.

2) a) Como calificarias lo aprendido durante esta experiencia?

   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
---|---|---|---|---|---|---|---|---|---|----|
Aprendí poco | Aprendí mucho

b) Explica tu calificación.

3) a) Como te sentiste durante estas dos semanas respecto de esta experiencia?

   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
---|---|---|---|---|---|---|---|---|---|----|
No me gustó | Me gustó mucho

b) Explica tu calificación.
4) Si hubo alguna actividad que te gustó en especial. Puedes describir cuál y por qué?

5) Si hubo alguna actividad que no te gustó o te pareció muy difícil? Puedes describir cuál y por qué?

6) Algun otro comentario, reflexión, percepción o sentimiento que quieras agregar respecto de esta experiencia?
Appendix G: Worksheet about Radicals

**OPERATIONS WITH RADICALS**

**PROPERTIES:** \( \sqrt{a \times b} = \sqrt{a} \times \sqrt{b} \)  
\( \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \)  
**NOT PROPERTIES:** \( \sqrt{a + b} \neq \sqrt{a} + \sqrt{b} \)  
\( \sqrt{a - b} \neq \sqrt{a} - \sqrt{b} \)

Operate and express the results in the most simple and exact form:

1) \( \sqrt{72} + \sqrt{18} \quad 2) \quad \sqrt{44} + \sqrt{99} \quad 3) \quad \sqrt{45} + \sqrt{405} - \sqrt{80} \quad 4) \quad \sqrt{99} + \sqrt{275} - \sqrt{704} \)

5) \( \sqrt{52} - \sqrt{117} \quad 6) \frac{18 - \sqrt{108}}{6} \quad 7) \frac{28 + \sqrt{980}}{7} \quad 8) \sqrt{16} + 9 \quad 9) \sqrt{841} - 400 \)

Calculate the area of the following rectangles:

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<tbody>
<tr>
<td>10)</td>
<td>6 + 2( \sqrt{3} )</td>
<td>5 - ( \sqrt{3} )</td>
</tr>
<tr>
<td>11)</td>
<td>4 + 2( \sqrt{5} )</td>
<td></td>
</tr>
</tbody>
</table>

Perform these operations in their corresponding square root extension field:

12) \( (4 + \sqrt{5})(2 - 3\sqrt{5}) \quad 13) \quad (5 - 3\sqrt{2})(5 + 3\sqrt{2}) \quad 14) \quad (7 - 4\sqrt{3})^2 \)

15) \( (1 + \sqrt{2})^2 \quad 16) \frac{19 + 5\sqrt{5}}{3 + \sqrt{5}} \quad 17) \frac{3 - 2\sqrt{2}}{1 - \sqrt{2}} \quad 18) \frac{8}{3 - \sqrt{7}} \)
19) \( \frac{22 + 3\sqrt{11}}{\sqrt{11}} \)  
20) \( \frac{36 - 18\sqrt{5}}{4 - 2\sqrt{5}} \)

Calculate the side of the following square:  
21) \( A = 14 + 6\sqrt{5} \)

Simplify the following square roots:

22) \( \sqrt{37} + 20\sqrt{3} \)  
23) \( \sqrt{6} - 2\sqrt{5} \)  
24) \( \sqrt{3} - 2\sqrt{2} \)
Appendix H: The algorithm of the pentagon’s construction with ruler and compass

HOW TO CONSTRUCT WITH RULER AND COMPASS

1) Draw a segment $[AB]$ in the middle of a page

2) Draw the ray $[Br)$; $Br \wedge AB$, $[Br)$ in the lower half-plane with border $AB$

3) Take $P$ on $[Br)$ so that $[BP] = \frac{[AB]}{2}

4) Draw the hypotenuse $[AP]$ of the triangle $ABP$ and extend it further from $P$

5) Let's take $Q$ on the ray $[AP)$ so that $[PQ] = [BP]$

6) $C(A, AQ) \subset C(B, AQ)$ determines $D$ (in the upper half-plane of border $AB$)

7) $E \in C(A, AB) \subset C(D, AB)$

8) $C \in C(B, AB) \subset C(D, AB)$
9) We have already the 5 vertices, we join them and we have the regular pentagon ABCDE.

10) If instead of drawing the sides we draw the diagonals we have the mythic 5 point star!
Appendix I: The appearance of the golden number in the regular pentagon.

The Rational of the Pentagon’s Construction and the Golden Number

In appendix J we learned how to construct a regular pentagon with ruler and compass. We also found the pentagon’s angle measure and the angle of the point of a 5 point star using the following figure:

But why this procedure for constructing the pentagon actually works?

Let's work with the green central triangle

We draw the angle bisector of \( \hat{A} \) that intersects \( OE \) in point \( J \).

We can easily observe that we have three isosceles triangles: \( AEO \), \( AOJ \) and \( AEJ \).

\( AEJ \) is similar to \( AEO \) because they have the same angles. Hence, their sides are proportional.

Suppose that \( AE = 1 \) (the side of the pentagon) and the unknown \( AO = x \) (the pentagon’s diagonal).

In terms of \( x \) and \( 1 \), what is the length of \( AJ \) and \( JE \)?
Since AEJ is isosceles, AJ=AE =1
Since AJO is isosceles, OJ=AJ=1

Since AO= OE = x and OJ=1
then JE = x−1

Since triangle AEO is similar to triangle AEJ then by the Thales Theorem we know that the sides are proportional and hence we can write:

\[
\frac{x}{1} = \frac{1}{x - 1}
\]

which is equivalent to:

\[
x^2 - x - 1 = 0
\]

This quadratic equation has two roots:

\[
x_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad x_2 = \frac{1-\sqrt{5}}{2}
\]

The second root is a negative number and thus it should be discarded, as the diagonal’s length cannot be negative, but the first root which is positive, is the solution to our problem.

**Therefore the diagonal’s length of a 1 side regular pentagon is:**

\[
\frac{1+\sqrt{5}}{2} \approx 1.618
\]

Now we can analyze the auxiliary construction which we can see again in the following picture:
If $AB=1$ and $AP=\frac{1}{2}$ then using Pythagoras $AP^2 = 1^2 + \left(\frac{1}{2}\right)^2 = 1 + \frac{1}{4} = \frac{5}{4}$

Hence $AP = \frac{\sqrt{5}}{2}$ and since $PQ = \frac{1}{2}$ then $AQ = \frac{1+\sqrt{5}}{2}$

Therefore the auxiliary construction is a procedure (using Pythagoras) to construct the diagonal of the pentagon which we have calculated above using similarity of triangles (Thales Theorem).
We have worked assuming that the side of the pentagon was 1. What if the side of the pentagon is any size “$a$”? 

If the side’s length of the pentagon is $a$ instead of 1 then (by the similarity of the two pentagons) the diagonal’s length would simply be:

$$\left(\frac{1+\sqrt{5}}{2}\right)a$$

This number is famous not only in mathematics but also in art, in architecture and even in biology.
It is named with the Greek letter $\phi$ (phi) and is known as the golden number or divine proportion.

$$\phi = \frac{1 + \sqrt{5}}{2} \approx 1.618$$

can be defined in several ways but the most illuminating, scientific and significant is that $\phi$ is the length of the regular pentagon’s diagonal in terms of its side or the ratio of the diagonal and the side in any regular pentagon.
Appendix J: The drawing of the regular icosahedron

We start by drawing a regular hexagon.

We then make a dilation centered in the centre of the hexagon and ratio $\frac{\sqrt{5} - 1}{2}$ (which is the inverse of the golden number and is approximately 0.618) to the 6 vertices of the hexagon as we can see in the following figure.
We use alternatively three points to determine an equilateral triangle that since it is visible we draw thick; then with the other three vertices we determine a second equilateral triangle which we draw thin since it is hidden.

We join each vertex in each triangle with the 3 closest in the hexagon and we draw the segments thick or thin depending which triangle we are working on.