SECONDARY MATHEMATICS PRESERVICE TEACHERS' SENSE–MAKING OF REPRESENTATIONAL QUANTITIES AND "SUM = PRODUCT" IDENTITIES

by

GÜNHAN İ. ÇAĞLAYAN

(Under the Direction of Denise S. Mewborn)

ABSTRACT

Although multiplicative structures can be modeled by additive structures, they have their own characteristics inherent in their nature, which cannot be explained solely by referring to additive aspects. Thus, this study is about preservice teachers' understanding and sense making of representational quantities generated by magnetic color cubes and algebra tiles, the quantitative units (linear vs. areal) inherent in the nature of these quantities, and the quantitative addition and multiplication operations – referent preserving vs. referent transforming compositions (Schwartz, 1988) – acting on these quantities. I devised a set of tasks focusing on identities of the form "Sum = Product," which can also be thought of as summation formulas. Data came from videotaped individual interviews during which I asked five (2 middle school and 3 high school mathematics) preservice teachers problems related to six main mathematical ideas: modeling prime and composite numbers; summation of counting numbers, odd numbers, even numbers; and multiplication and factorization of polynomial expressions in *x* and *y*. I base my analysis within a framework of unit–coordination with different levels of units (Steffe, 1988, 1994) supported by a theory of quantitative reasoning (Schwartz, 1988; Thompson, 1988, 1993, 1994, 1995). I used a simplified version of Behr, Harel, Post, & Lesh's (1994) *generalized* *notation for mathematics of a quantity* and Vergnaud's (1983, 1988, 1994) *theorems and concepts in action* formalisms, which helped me describe the preservice teachers' understanding of linear and areal quantities and their units, and the quantitative operations taking place; and translate students' mathematical performance into a series of terminology based on a simple notation: Relational notation and mapping structures duo (Caglayan, 2007b). There was a pattern, which showed itself in all my findings. Two students constantly relied on an additive interpretation of the context whereas three others were able to distinguish between and when to rely on an additive or a multiplicative interpretation of the context. My results indicate that the identification and coordination of the representational quantities and their units at different categories (multiplicative, additive, pseudo–multiplicative) are critical aspects of quantitative reasoning and need to be emphasized in the teaching–learning process.

INDEX WORDS: Additive Reasoning, Composite Numbers, Conceptual Field, Connections, Multiplicative Reasoning, Polynomials, Preservice Teachers, Quantitative Reasoning, Relational Reasoning, Summation Formulas, Teacher's Knowledge of Mathematics, Theorems–in–Action, Unit Coordination.

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A Dissertation Submitted to the Graduate Faculty of The University of Georgia in Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

2008

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August 2008

DEDICATION

I dedicate this dissertation to all my students and to my outstanding teachers.

ACKNOWLEDGEMENTS

I wish to thank the participants of this research, without whom I would not be where I am today. They all provided very interesting and original ideas. I consider myself very lucky to have such actively participating, helpful, and kind people in my study. I am very thankful to them for being so clear during the interviews, and patiently describing why they were doing what they were doing, thanks to which I had a good opportunity to present how they make sense of the mathematics they were involved with.

I am so thankful to Dr. Denise Mewborn for accepting being my major professor; it was the most meaningful decision to be her doctoral student concerning my academic career in the mathematics education field. She was always very helpful and encouraging from the beginning, when I mentioned my ideas concerning the mathematics of the color cubes and tiles. She has always been willing to hear my ideas, provide insightful suggestions and feedback, and make herself available whenever I needed help. I learned a great deal from her concerning mathematics education research and mathematics teaching, for which I will always be very thankful. She has been a great mentor through the years of my studentship at the University of Georgia, sincerely caring for and trying to resolve my scholarly and personal problems in everyway possible. I am also very much in debt to her for all her help concerning revisions of my publications and for her support during the job search process.

I have been fortunate to be involved with the CoStar research project and to work with Dr. John Olive, Dr. Andrew Izsak, and Dr. Chandra Orrill, to whom I am very much indebted. I thank these people very much for their constant support – both financial and scholarly – and their unending faith in me as a member of the CoStar project. I am especially very much grateful for the opportunity provided by Dr. John Olive to work as part of a team to collect data, analyze it, write papers, and present research at conferences; I have learned a great deal about each of these aspects of research as well as how to work as a member of a team. It has been an amazing teamwork and Dr. John Olive has always been there for me. I am very thankful to him for this collaboration and his constant willingness to help, which made me take my job seriously, and become a productive researcher.

Many thanks also go to the amazing faculty of the mathematics education department, Dr. Jim Wilson, Dr. Dorothy White, Dr. Larry Hatfield, Dr. Pat Wilson, Dr. Leslie Steffe, and Dr. Jeremy Kilpatrick, who have been outstanding teachers and of great guidance for me. If it was not for the EMAT 6680 and 6690 classes I took from Dr. Jim Wilson, I would never come up with the idea of writing for practitioners' journals. Those classes have been of a great inspiration to me that way. I am also very lucky to be a student of Dr. Larry Hatfield. After I met him, I became a new me as a teacher I never knew before. I consider myself very lucky for having the opportunity to be a member of the mathematics education family at the University of Georgia.

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CHAPTER I

INTRODUCTION

> When the successive odd numbers are set forth indefinitely, beginning with 1, observe this: The first one makes the potential cube: the next two, added together, the second; the next three, the third; the four next following, the fourth; the succeeding five, the fifth; the next six, the sixth; and so on. (Nicomachus of Gerasa, Book II, Chapter XX).

Mathematics manipulatives have always been amazing tools for me. It was just unbelievable that you could use physical objects and then try to see the mathematics behind those objects. I have been especially developing a fondness for representing various subsets of positive integers (e.g., prime and composite), figurate numbers (e.g., triangular numbers), growing patterns (e.g., sum of consecutive odd integers), and polynomials using color tiles and cubes. I used these tools in my own learning and teaching a lot. I got the impression that my students understood mathematics better when I used these and I received good reviews.

What I like about these tools is in that they are very simple tools, yet they address a variety of mathematical objects such as square, area, cube, volume, polynomial identities, power sum formulas, prime vs. composite number, etc. I have found them useful and was therefore interested in investigating the potential they hold for eliciting students' thinking. The purpose of this present study was to examine how mathematics preservice (middle and high school) teachers understand and make sense of "sum = product" identities on representational quantities with the help of color cubes and tiles as manipulatives.

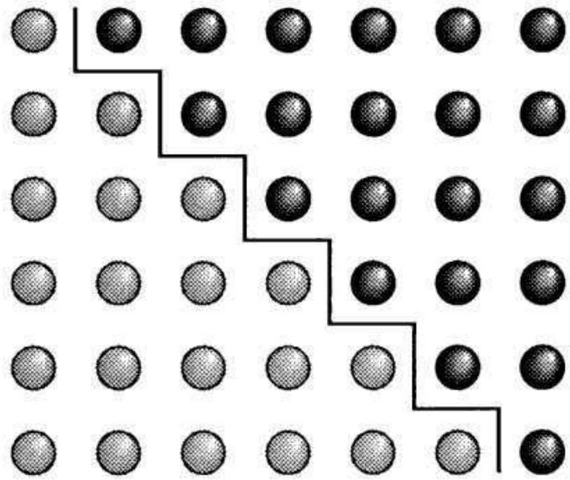
Physical objects, also often referred to as manipulatives, can serve as essential representational models in the course of experiential learning. NCTM has consistently emphasized the use of physical objects as representational tools in its publications. The Principles and Standards for School Mathematics (NCTM, 2000) states that: "Representation is central to the study of mathematics. Students can develop and deepen their understanding of mathematical concepts and relationships as they create, compare, and use various representations. Representations—such as physical objects, drawings, charts, graphs, and symbols—also help students communicate their thinking." (p. 280).

Though I have advocated the use of manipulatives, research has shown that the use of physical objects can be an obstacle to mathematical progress in some cases. Howden (1986) showed that even though students were successful mathematically at the concrete level, that was not always the case in the abstract level. Research by Suydam and Higgins (1977) on the other hand, showed that students' mathematics achievement increased through the use of mathematics manipulatives. Work by Sowell (1989) indicated that even though for K-16 students, manipulatives were effective ways of modeling and understanding mathematics, the teachers were not appreciative of their usage. As for the teachers, on the other hand, "inexperienced" ones favored their usage more often than experienced teachers (Gilbert & Bush, 1988).

Uttal, Scudder, and DeLoache state that "part of the difficulty that children encounter when using manipulatives stems from the need to interpret the manipulative as a representation of something else." (1997, p .38) I believe that a reference to any kind of physical object brings with itself the necessity to think about the object under consideration as some sort of quantity possessing a name, a value, and a measurement unit (Schwartz 1988; Thompson 1993, 1995). Attending to the quantitative nature of manipulatives may be an asset for students' success in relating the manipulatives to their written symbolic referents. The physical object itself can not be a representation of a written symbol without "meanings" projected into these "concrete objects." For instance, six $\frac{1}{6}$ fraction circle pieces together make a whole circle. One of those pieces standing alone cannot directly be mapped into the written symbol $\frac{1}{6}$. A person has to think about each piece as a quantity on its own and in relation to every other $\frac{1}{6}$ piece as well as the whole circle standing for the unit whole (Olive & Vomvoridi, 2006). A successful mapping of the "concrete" to the "abstract" depends on the manipulative itself and a "family of meanings" attached to these objects.

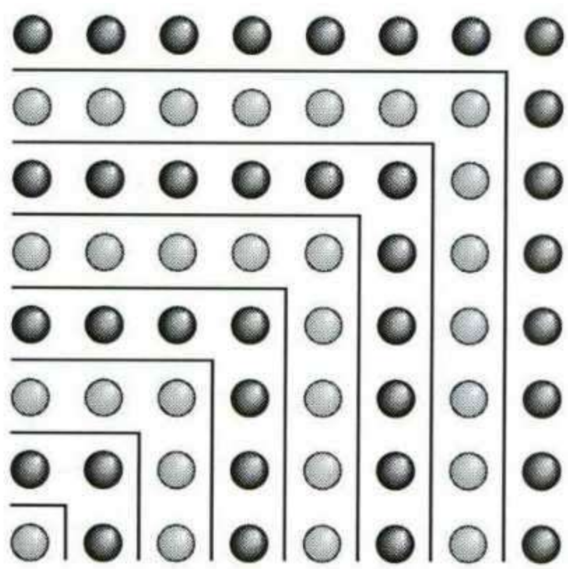
I used the color tiles and color cubes with my research participants to represent various figurate numbers and polynomial identities, where, we relied on measurement (area as a sum, area as a product) and geometry along with arithmetic of numbers. Figurate numbers, in particular, power sum identities such as $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (Figure 1.1), $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ have been known to ancient Greeks in closed form, who derived such expressions based on unity without the presence of a known measurement unit (Gardner, 1973; NCTM, 1989; Nelsen, 1993). In fact, the geometrical and physical representation of figurate numbers (by points drawn on sand or pebbles) and the study of their properties were common in the early Pythagorean era (Heath, 1921; NCTM, 1989). Pythagoreans used a point or a dot to represent 1; two dots placed apart were to represent

2, and to define at the same time a line segment connecting these dots. They represented numbers as polygonal figures (triangles, squares, pentagons, etc.) made of dots or pebbles (Heath, 1921). Archimedes (third century B.C.) expressed sum of squares in closed form (Archimedes, 1897; Dijksterhuis, 1987; Kanim, 2001) and applied this discrete sum technique to find the areas and volumes of surfaces of revolution. Nicomachus of Gerasa (first century B.C.) is credited for the sum of odd integers formula $\sum_{i=1}^{n}(2i-1)=n^2$, which he obtained via dot patterns forming symmetric L–shapes (Figure 1.2). Nicomachus (first century B.C.), Aryabhata (fifth century A.C.), and Al-Karaji (tenth century A.C.) are known for deriving the integral cubes summation expression $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$; and Al-Haytham (tenth century A.C.) for the sum of the fourth powers $\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$. Algebraic proofs of such power sum identities are very complicated and difficult to follow; their visual representations based on drawings (NCTM, 1989; Nelsen, 1993, 2000) stand as a good place to start triggering students' inductive reasoning, conjecturing, and generalizing.



$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$$

Figure 1.1. Sum of integers (Nelsen, 1973, p. 69).



$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

Figure 1.2. Sum of odd integers (Nelsen, 1973, p. 71).

The first complete investigation and the resulting properties of figurate numbers – also called polygonal numbers – were studied by Nicomachus of Gerasa (first century B.C.), which are given in the manuscript *Introductio arithmetica*. Though his work included few of his original ideas, it is commonly acknowledged that *Introductio arithmetica* stood as an artistic collection of well described, clearly presented and explained definitions and statements with a lot of illustrations based on physical forms and visual proofs (NCTM, 1989; Nicomachus of Gerasa, 1926). In our time, we make use of algebraic symbolism to represent such patterned numbers (e.g., square numbers are represented as $1^2, 2^2, 3^2, 4^2, \ldots$, or as 1, 1 + 3, 1 + 3 + 5, 1 + 3 + 5 + 7, … , or as 1, 1 + 3, 3 + 6, 6 + 10, … ; and triangular numbers are represented as 1, 1 + 2, 1 + 2 + 3, 1 + 2 + 3 + 4, … , or as 1 × 1, 1 × 3, 2 × 3, 3 × 4, … , etc.); however, properties of such polygonal numbers were treated by Nicomachus in words and by drawn or physical representations, not in algebraic symbolism: "If you add any two consecutive triangles that you please, you will always make a square, and hence, whatever square you resolve, you will be able to make two triangles of it." (Nicomachus of Gerasa, 1926, p. 247).

Nicomachus of Geresa's work contains today's well–known "sum = product" identities arising from the geometry of the figures generated by dots and via line segments connecting these dots (Heath, 1921; NCTM, 1989, pp. 54-56; Nicomachus of Geresa, 1926, pp. 230-262):

- The sum of the first n consecutive positive integers is the n^{th} triangular number

$$T_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

- The sum of the first n consecutive positive odd integers is a square number (Figure 1.2)

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

- The sum of any pair of consecutive triangular numbers is a square number

$$T_{n-1} + T_n = n^2$$

- Eight times any triangular number plus 1 is the square of an odd number

$$8T_n + 1 = (2n + 1)^2$$

- Pentagonal numbers $P_1 = 1$, $P_2 = 5$, $P_3 = 12$, $P_4 = 22$, $P_5 = 35$, … , can be represented in basic pentagonal form (Figure 1.3) or in second pentagonal form (Figure 1.4).

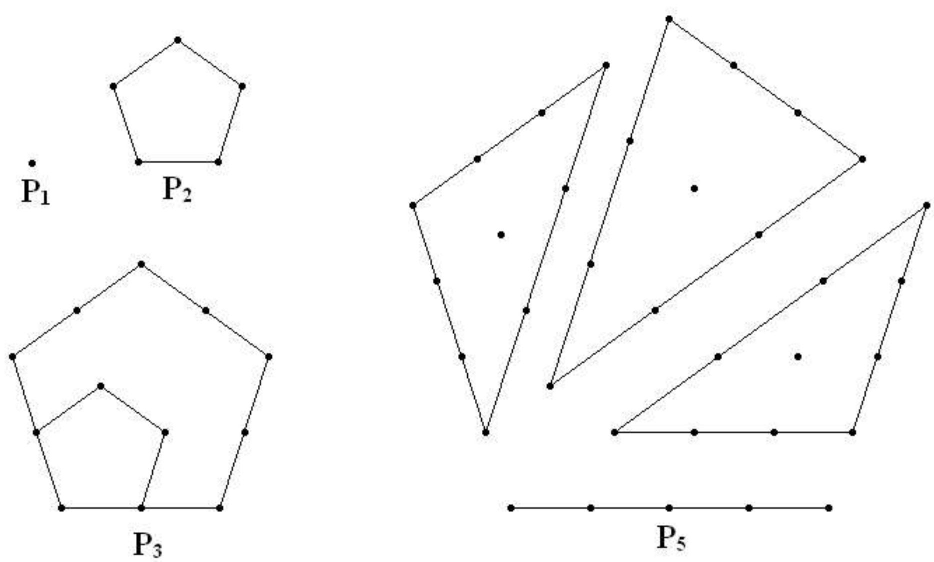


Figure 1.3. Basic pentagonal form (NCTM, 1989, p. 55)

The first form yields a "sum = sum" identity and a "sum = product" identity

$$P_n = 3T_{n-1} + n, \quad P_n = \frac{n(3n-1)}{2}$$

- The second pentagonal form produces only one identity of the form "sum = sum," $P_n = n^2 + T_{n-1}$, which is illustrated in Figure 1.4.

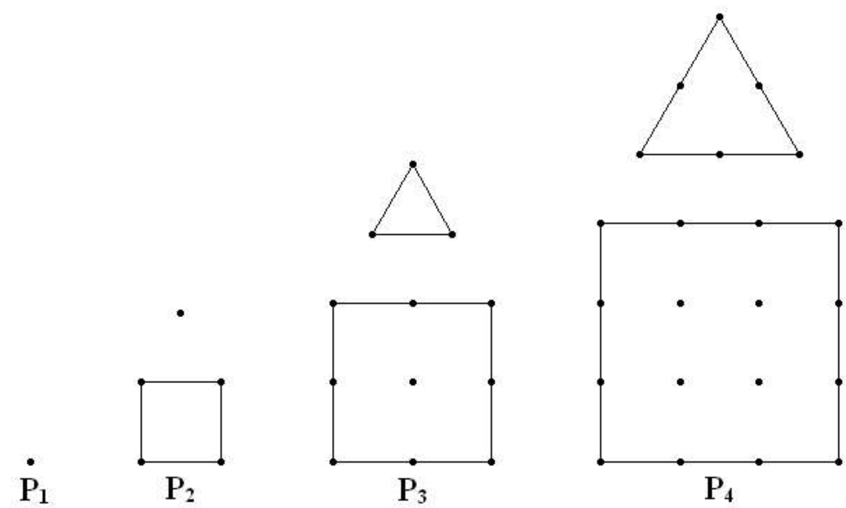


Figure 1.4. Second pentagonal form (NCTM, 1989, p. 55)

- The sum of the first consecutive positive even integers is an oblong number[1]

$$2 + 4 + 6 + \cdots + 2n = n(n+1)$$

- The n^{th} oblong number equals twice the n^{th} triangular number

$$n(n+1) = 2T_n$$

- The n^{th} cube number can be written uniquely as the sum of exactly n consecutive positive odd integers

$$1^3 = 1,\ \ 2^3 = 3 + 5,\ \ 3^3 = 7 + 9 + 11,\ \ 4^3 = 13 + 15 + 17 + 19, \cdots$$

- The sum of the first consecutive positive cube numbers is a square number, in particular, the square of the n^{th} triangular number

[1] Oblong numbers are the numbers of the form $n(n + 1)$ such as 2, 6, 12, 20, 30, etc. Nicomachus used the adjective "heteromecic" for these numbers (NCTM, 1989, p. 56).

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 = T_n^{\ 2}$$

In general, mathematics gives rise to quantities that can be represented with symbols, drawings, or physical objects. When it comes to represent these quantities, both students and their teacher should be proficient in identifying the characteristics of those quantities. Whole numbers can be expressed in terms of units of 1. For instance, the number 3 can be thought of as the collection of three singleton units. In a tiled representational situation such as one the present study investigated, three little black square tiles can be used to model the representational quantity “3.” Odd integers and their summations can be represented as tiled L−shaped figures (Figure 1.5) – which, the ancient Greek named *gnomons*; and growing squares (Figure 1.6), respectively; while even integers and their summations can be represented as rectangles with dimensions 2 by half the integer (Figure 1.7) and growing rectangles (Figure 1.8), respectively, made of one inch color tiles (Caglayan, 2006). In ancient Greek, the number that equals the sum of even integers was named a *heteromecic* number. Prime and composite numbers may have various tiled rectangular representations (Figure 1.9) as well (Caglayan, 2007a).

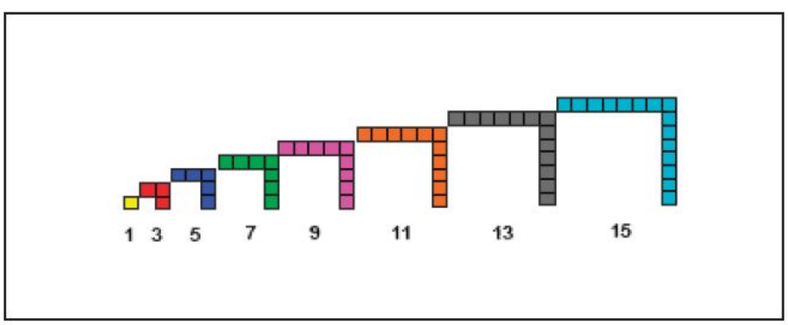


Figure 1.5. Symmetric L−shapes representing the numbers 1, 3, 5, 7, 9, 11, 13, and 15.

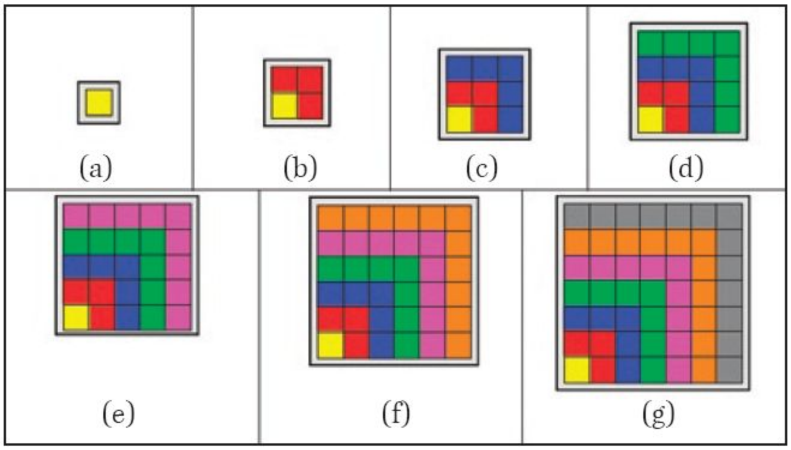


Figure 1.6. Using L–shapes to show that the sum of the first n positive odd integers is n^2.

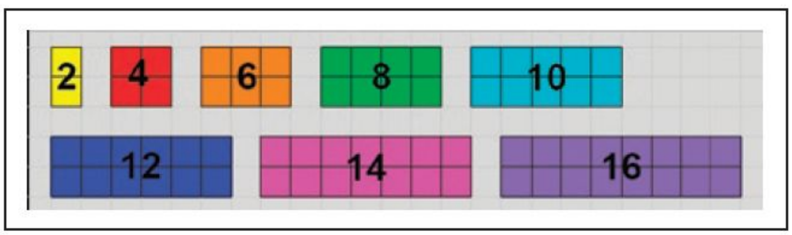


Figure 1.7. Shapes representing the numbers 2, 4, 6, 8, 10, 12, 14, and 16.

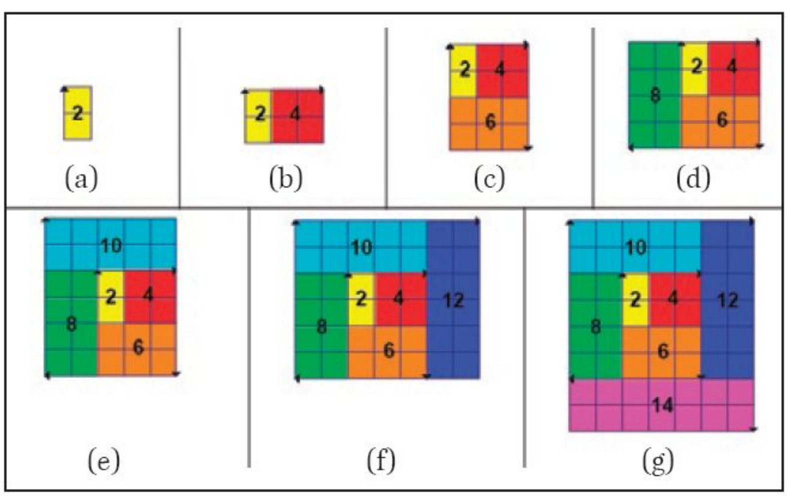


Figure 1.8. Summation representations of the first seven positive even integers.

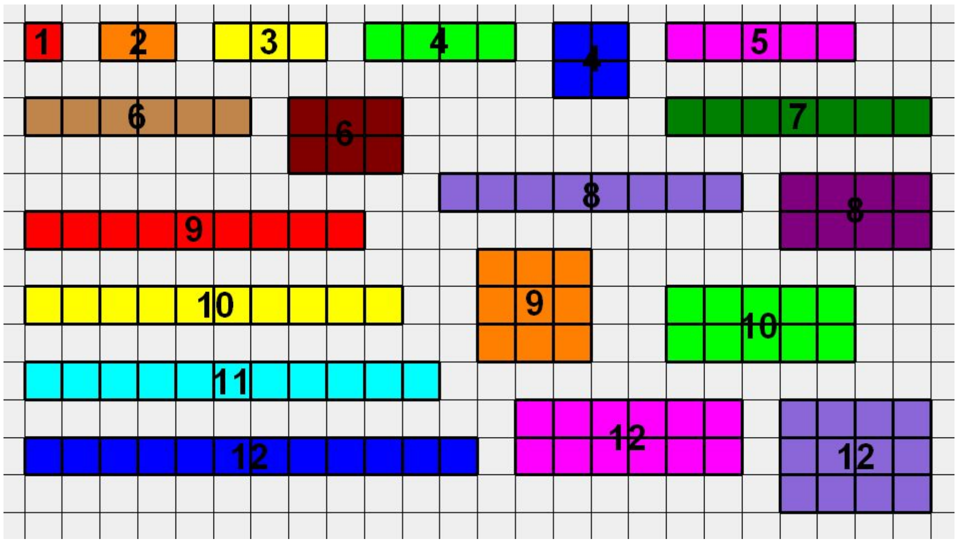


Figure 1.9. Tiled representations of prime and composite numbers.

Modeling more complicated expressions such as $2x + y + 3$ by using color tiles may not be as obvious. In the example of $2x + y + 3$, the term $2x$ is a collection of two units of x (two purple bars with the model), the term y is 1 unit of y (1 blue bar with the model), and the term 3 is a collection of three units of 1 (three little black squares with the model). Therefore, the expression $2x + y + 3$ is a collection of a collection of the individual irreducible representational units. One not only has to individually identify each representational unit (one purple bar for the x, one blue bar for the y, and one little black square for the 1), but one has to reconcile a collection of a collection of these irreducible representational units in order to demonstrate that $2x + y + 3$ can not be simplified any further because $2x$, y, and, 3 are unlike terms (representational quantities).

Representation of irreducible quantities as well as "bigger" ones "made of" these quantities is reminiscent of the "unitizing" process (Behr, Harel, Post, and Lesh, 1994;

Lamon, 1994; Steffe, 1988, 1992, 1994). All the little pieces (e.g., each one inch color cube denoting a “1” of a special number, each different size tile piece denoting a “1”, an “*x*,” a “*y*,” an “x^2”, an “*xy*,” or a “y^2”) and their various combinations (e.g., a 4 by 2 rectangle – made of 8 irreducible units of 1 – conceptualized as the unitizing of the even number 8, a $2x + y + 3$ by $x + 1$ rectangle – made of 2 irreducible units of x^2, 5 irreducible units of *x*, 3 irreducible units of 1, 1 irreducible unit of *y*, 1 irreducible unit of *xy* – conceptualized as the unitizing of the polynomial expression $2x^2 + 1xy + 5x + 1y + 3$) serve for an essential theoretical construct, which I define as Representational Unit Coordination (Caglayan, 2007c).

In its true nature, coordination is about “making various different things work effectively as a whole[2].” In the context of my study, it refers to the conception of unit structures in relation to smaller embedded units within these unit structures, or, bigger units formed via iteration of these unit structures. In the multiplicative situation, for instance, the conception of 5 as 5 units of 1 is one way of coordinating units: 5 as a (composite) unit of 1. As another example, 35 can be coordinated multiplicatively as 5 (composite) units of 7 (composite) units of 1. Power sum identities modeled by one inch color cubes require more sophisticated representational unit coordination strategies (additive and multiplicative) at the same time. In addition to these, polynomial rectangles are prone to a concatenated unit coordination type, which is called pseudo–multiplicative representational unit coordination.

[2] Cambridge Advanced Learner's Dictionary online. Retrieved December 31, 2006 from dictionary.cambridge.org

CHAPTER TWO

THEORETICAL PERSPECTIVES AND LITERATURE REVIEW

2.1. Representational Unit Coordination

Unit coordination has been previously studied by various researchers in the mathematics education field. Steffe, for instance, analyzed the coordination of different levels of units in whole number multiplication problems, which is reminiscent of a key concept in multiplication, i.e., the notion of composite units (1988). Research evidenced that the essence of multiplication lies in fact in distributive rather than repeated additive aspect (Confrey & Lachance, 2000; Steffe, 1992). In the example above, the multiplication of 5 by 7 can be thought as the injection of units of 7 (each being units of 1) into the 5 slots of 5, each slot representing a 1. In this example, the conceptualization of each singleton unit describing a unity, i.e., 1, stands for a first level of unit coordination. Moreover, 5 and 7 can be conceptualized (as composite units of 1) as 5×1 and 7×1, respectively, as a second level of unit coordination. The product 5×7, which denotes 5 (composite) units of 7 (composite) units of 1, can be conceptualized as a third level of unit coordination.

Some other researchers also studied unit coordination in a fractional situation (e.g. Lamon, 1994; Olive, 1999; Olive & Steffe, 2002; Steffe, 2002). Additionally, work on intensive (e.g., miles per hour) and extensive quantities (e.g., number of hours) reflect unit coordination as well (Kaput, Schwartz, & Poholsky, 1985; Schwartz, 1988). Olive and Caglayan's (2006, 2007) work on quantitative unit coordination and conservation also takes the unit coordination issue into account.

Steffe's Unit–Coordination construct (1988, 1994), which constitutes the main theoretical framework of this present study, is strongly related to multiplicative structures, the study of which has been conducted by mathematics education researchers since the 1980s. In his 1983 article, Vergnaud defines the notion *conceptual field* as a "set of problems and situations for the treatment of which concepts, procedures, and representations of different but narrowly interconnected types are necessary." (p. 128). In particular, he views the multiplicative structures, a conceptual field of multiplicative type, as a system of different but interrelated concepts, operations, and problems such as multiplication, division, fractions, ratios, similarity. Although multiplicative structures can to some extent be modeled by additive structures, they have their own characteristics inherent in their nature, which cannot be explained solely by referring to additive aspects.

Behr, Harel, Post, and Lesh (1994) developed two representational systems – extremely generalized and abstract – in an attempt to transcribe students' additive and multiplicative structures in which the notion "units of a quantity" plays the main role. Confrey provides splitting, "an action of creating simultaneously multiple versions of an original," (1994, p. 292) as an explanatory model for children's construction of multiplicative structures. Research on students' reconciliation of additive and multiplicative structures based on "sum = product" identities is missing in the literature.

Representational Unit Coordination, a new terminology in our field, can be defined as the different ways of categorizing units arising from the modeling of identities on representational quantities as the "area as a product" and "area as a sum" of the corresponding special rectangles made of color cubes and tiles. All the cited work above, though interesting and appealing, seem to leave the representational unit coordination construct behind. It then remains to explain how my construct, Representational Unit Coordination (RUC) is different from unit coordination in the literature.

In my study, preservice teachers started with the "area as a product" concept. In its most basic sense, area of, e.g., a rectangle, is defined as the product of its two dimensions. I am talking about the area of a rectangle, and not any other geometric figure, because the identities on representational quantities students analyzed via color cubes or tiles were always about a rectangle – prime rectangle, composite rectangle, odd rectangle, even rectangle, addition of counting numbers, odd and even integers generated as a growing rectangle, and polynomial rectangle. Coordination of these two dimensions, i.e., the arrangement of these two linear units in a particular order as an ordered pair such as (a, b) or (b, a), defines the first part of my construct: Multiplicative Representational Unit Coordination (MRUC).

The analysis of the other important concept, "area as a sum" (of a special number rectangle), is prone to several, not necessarily hierarchical levels of RUC. Additive Representational Unit Coordination (ARUC) stands for the coordination, the arrangement of (in general two or more) areal units as n–tuples such as [2, 2, 2] or [3, 3] for the composite rectangle of 6. For this RUC type, areal units being coordinated have something in common. For instance, for the composite rectangle of 6, the "2"s in [2, 2, 2] are interesting because 2 is a factor of 6, which is why this special additive type RUC is called Equal Addends Type RUC. Moreover, the coordination of less interesting (irreducible) areal units (of 1) as n–tuples such as [1, 1, 1, 1, 1, 1] for the same example, composite rectangle of the special number 6, necessitates the existence of another additive type RUC which is called Irreducible Addends Type RUC. There arose actually many more additive type RUC, which will be explained in Chapter IV in much more detail.

There is one more RUC type, in between additive and multiplicative, which I named Pseudo Multiplicative type RUC. This occurred for the "Area of the boxes of the same color as a product" in dealing with polynomial rectangles made of color tiles. For instance, the $x+1$ by $2y+3$ rectangle had 4 boxes (x by $2y$, x by 3, 1 by $2y$, 1 by 3) of the same color. Some of the preservice teachers' products were $x \cdot 2y$, $x \cdot 3$, $1 \cdot 2y$, $1 \cdot 3$; i.e., of multiplicative nature. With the relational notaion, as I will explain in detail, these linear units, namely the length and the width of each "same–color–box" can be written as $(x, 2y)$, $(x, 3)$, $(1, 2y)$, $(1, 3)$. However, the remaining preservice teachers' areas as a "product" for the same boxes were $2 \cdot xy$, $3 \cdot y$, $2 \cdot y$, $3 \cdot 1$. In other words, the first term of each "pseudo–product" was a coefficient serving as a counting number indicating how many there were of each irreducible areal unit.

In other words, "area as a product," and "area as a sum" concepts played a crucial role as I tried to establish the RUC construct meaningfully. RUC has more of a relational aspect, rather than the distributive aspect of unit coordination in the literature. The adjective "relational" refers to the ordering of the units as ordered pairs / n–tuples, for the case area as a product / sum. I have chosen this adjective because in mathematics, a binary / n–ary relation is defined to be a set of ordered pairs / n–tuples. However, as for the distributive aspect of multiplication, as Steffe describes, "for a situation to be established as multiplicative, it is always necessary at least to coordinate two composite units in such a way that one of the composite units is distributed over the elements of the other composite unit." (1992, p. 264). For instance, the coordination of 5 and 7 in "5 bags each containing 7 marbles" example has this distributive aspect. In that sense, I thought there was still so much missing that needs to be achieved, and I hoped my study would fill in that gap.

2.2. Representational Quantities: Linear vs. Areal

Quantitative reasoning is a central issue in mathematics and science. We try to learn how to call and name things. We start learning and getting to know mathematics by counting, quantifying, and operating on objects. By quantifying them, we are also assigning numerical values to these objects. We also attribute units to these objects (e.g., an hour is not the same thing as a day). Quantitative reasoning and unit coordination explain well what is going on in seemingly different areas of mathematics. When you try to solve a word problem, represent quantities with various methods (e.g., manipulatives, graphs, tables, geoboards), solve linear equations, or play mathematical games, you embrace quantitative reasoning and coordinate units whether you realize it or not. This is not emphasized by teachers or by the curriculum, though. What units are we handling? Why are we using these units in the measurement process? What are the names of the things we measure?

Thompson wrote many essays about his students' making sense of additive and multiplicative structures via quantitative reasoning (1988, 1989, 1993, 1994, 1995) and with Smith in their 2008 book chapter. Schwartz (1988), Shalin (1987), and Nesher (1988) view quantities as some sort of mathematical objects as ordered pairs of the form (number, measurement unit) whereas Thompson finds this characterization inconvenient, claiming it "confounds notions of number and quantity." (1994, p. 197). Steffe's characterization of quantity is based on unitizing or segmenting operations (1991). According to Thompson, "A quantity is not the same as a number. A person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the *possibility* of measuring it." (1993, p. 197). In another piece, he schematizes a quantity as composed of:

- An object,
- A quality of the object,
- An appropriate unit or dimension,
- A process by which to assign a numerical value to the quality (1994, p. 184).

Although he is against an ordered pair characterization of the form (number, unit), his proposed schema above still calls for an ordered – not pair but – quadruple (object, name of the quantity, measurement unit of the quantity, process to assign a magnitude for the quantity). In other words, quantities do not have to be assigned numerical values; what really matters is that a person "understands the *possibility* of measuring it" (1993, p. 197) by which to assign a value.

Time is a quantity that has the possible units measured by number of hours, number of days, number of minutes, number of light–years, etc. Time and number of hours are not the same thing. As another example, when you represent a polynomial, say, $2x + y + 3$, with tiles of different colors and sizes, you may choose to use 2 purple bars, 1 blue bar, and 3 little black squares, respectively standing for your "x"s, "y"s, and "1"s, respectively. The irreducible representational units, namely "one purple bar," "one blue bar," and "one little black square" can be thought of as the units for *something*. That something is nothing but what we call it; it can be named the variable x or the whole number 1. In this representation, we can describe $2x$ as the quantity with the name "purple bar(s)," the measurement unit "number of purple bars," and – after a measurement process – an assigned numerical value (or magnitude) "2." Naming, quantifying, attributing units for, and reasoning quantitatively about things are of paramount importance. "Quantitative reasoning is not reasoning about numbers; it is about reasoning about objects and their measurements and relationships among quantities." (Thompson, 1995, p. 204). The table below summarizes the information about quantities $2x$, y, and 3.

Table 2.1

Quantities Associated with Algebra Tiles

Quantity	Name (Referent)	Measurement Unit	Value (Magnitude)
$2x$ (2 purple bars)	Purple bar	Number of purple bars	2
y (1 blue bar)	Blue bar	Number of blue bars	1
3 (3 little black squares)	Little black square	Number of little black squares	3

Reasoning quantitatively about objects, things, brings with itself the notion of "quantitative operations" by which we make sense of these things and reason about relationships among them. In my study, "quantitative operations" are not the same as the well–known numerical operations (addition, multiplication, subtraction, division). Because "representational quantities," which are quantities, play the main role in all the activities my students worked on, I defined the quantitative analogous of the basic arithmetic operations as "representational quantitative addition operation," and "representational quantitative multiplication operation."

Thompson thinks of quantitative operations as "conceptual operations one uses to *imagine* a situation and to *reason about* a situation–often independently of any numerical calculations." (1995, p. 207). I consider the "reasoning about" part as a must, however, I do not find it necessary to "imagine" a situation, if the situation is already there, present in front of our eyes, as in my students' work with representational quantities. The conceptual operations my students induced were both quantitative and representational.

Quantitative operations can also be classified *referent preserving compositions* and *referent transforming compositions* (Schwartz, 1988, p. 41). Addition and subtraction operations are referent preserving compositions because they do not change, rather preserve the referents (names) of the quantities on which they act. Adding 3 apples and 5 apples yields 8 apples. Subtracting 4 inches from 10 inches yields 6 inches. In other words, we are *adding* (*subtracting*) like–terms. Both quantities, being extensive when composed by addition (subtraction) operation, yield a quantity that has the same unit, i.e., number of apples (number of inches). In other words, through a *referent preserving composition*, both the referent and the measurement unit remain unchanged, and we reside in the same measure space.

Multiplication and division operations, on the other hand, are referent transforming compositions because they change the referents (names) of the quantities on which they act. Two blouses can be paired with three skirts to form six different outfits: 2 blouses × 3 skirts = 6 outfits. In this case, we are composing two extensive quantities by multiplication operation and this results in an extensive quantity (a product quantity) with a totally different referent (outfit = blouse × skirt) and measurement unit (number of outfits). In other words, through a *referent transforming composition*, both the referent and the measurement unit as well as the measure space change. Product quantities such as "2 blouses × 3 skirts," "5 cm × 7 cm," "2 in × 4 in," involve measurement units of product type "blouse × skirt," "cm × cm," "in × in," that are not simply conceived by students as repeated addition (Behr et al., 1994). Schwartz considers the "repeated addition" model of multiplication as a procedural flaw (1988, p. 47).

In my study with color cubes and tiles, the linear quantities associated with the sides of the growing rectangles can be categorized as extensive quantities with basic (linear) measurement unit type (e.g., centimeters, inches, units); however, the areal quantities emerge as extensive quantities possessing product–type–units (e.g., centimeters squares, inches squared, units squared) within the rectangle itself. As part of my study, I analyzed preservice teachers' ability to make sense of the changes in referent and measurement units. Table 2.2 and Table 2.3 below summarize the quantities related with color cubes and tiles that generate growing rectangles.

Table 2.2

Representational Quantities Related with 1″ Color Cubes

Quantity	**Linear vs. Areal**	**Representational unit**	**Extensive vs. Intensive**	**Unit of measurement**	**Measure Space**
1	Linear	1" square tile of any color	Extensive	1–unit	Positive Integers (equivalently, positive multiples of linear unit 1)
1	Areal	1" square tile of any color	Extensive	(1–unit) × (1–unit)	Any prime, composite, odd, even number rectangle & symmetric L–shaped figure
a & *b* (factors of a composite	Linear	The *a* by *b* rectangle made of 1" squares	Extensive	Product (*a*–unit) × (*b*–unit)	Composite number *c*'s rectangle

number c)				with $a \mid c$ & $b \mid c$	
n (counting number)	Areal	n 1" square tile of any color	Extensive	For odd n: (1–unit) × (n–unit) For even n: (2–unit) × (($n \div 2$)–unit)	Summation of Counting Numbers: For odd n, an n by $(n+1) \div 2$ rectangle For even n, an $n+1$ by $n \div 2$ rectangle
$2n - 1$ (odd number)	Areal	Symmetric L–shaped	Extensive	(1–unit) × ((2n–1) –unit)	Summation of Odd Numbers: An n by n square
$2n$ (even number)	Areal	Rectangle with dimensions 2 by n	Extensive	(2–unit) × (n–unit)	Summation of Even Numbers: An n by $n+1$ rectangle

Table 2.3

Representational Quantities Related with Different Size & Color Tiles

Quantity	**Linear vs. Areal**	**Representational unit**	**Extensive vs. Intensive**	**Unit of measurement**	**Measure Space**
1	Linear	Square Black Tile	Extensive	1–unit	Positive Integers (positive multiples of linear unit 1)
x	Linear	Purple Bar	Extensive	x–unit	Positive Multiples of x
y	Linear	Blue Bar	Extensive	y–unit	Positive Multiples of y
1	Areal	Square Black Tile	Extensive	(1–unit) × (1–unit)	Positive Multiples of areal unit 1
x	Areal	Purple Bar	Extensive	(1–unit) × (x–unit)	Positive Multiples of areal unit x
y	Areal	Blue Bar	Extensive	(1–unit) × (y–unit)	Positive Multiples of areal unit y
xy	Areal	Big Green Rectangle	Extensive	(x–unit) × (y–unit)	Positive Multiples of areal unit xy
x^2	Areal	Big Purple Square	Extensive	(x–unit) × (x–unit)	Positive Multiples of areal unit x^2
y^2	Areal	Big Blue Square	Extensive	(y–unit) × (y–unit)	Positive Multiples of areal unit y^2

My work with color cubes and tiles relies heavily on the notion of area. In particular, area of a rectangle is defined as the sum of the irreducible areal units, which in this case is either the irreducible areal unit 1 (in the context of subsets of positive numbers) or the irreducible areal units 1, x, y, xy, x^2, y^2 (in the context of polynomial expressions). In particular, an irreducible areal unit 1 is defined as the product of the irreducible linear unit 1 by the same linear unit 1. Although they are both 1, i.e., they both have the same value, these are different quantities because they possess different units: linear vs. areal. In Figure 1.2, for instance, each symmetric L–shape is a (areal) unit of irreducible areal units of 1. Similarly, in Figure 1.5, for instance, a composite number, e.g.,12, is a rectangular (areal) unit of irreducible areal units of 1. Moreover, it can be represented as the product of linear units 1 by 12, 2 by 6, and 3 by 4. All these dimensions have something in common: That something is nothing but these dimensions, namely the 1, the 12, the 2, the 6, the 3, and the 4 are all (linear) units of irreducible linear units of 1. In other words, 1 (12−unit) or 12 (1−unit)s, 2 (6−unit)s or 6 (2−unit)s, 3 (4−unit)s or 4 (3−unit)s differ in our mental images. Verbally speaking, these expressions would barely make sense for children. However, instantiations for these expressions help them understand and make sense of the multiplicative situation. I hope that in the light of my research, we will be able to understand how preservice teachers identify and coordinate units arising from an instantiative situation, color cubes and tiles, which serve to model summation identities, prime and composite numbers, and products and factors of polynomial expressions.

2.3. Rationale and Research Questions

I was eager to study preservice teachers' sense making of linear and areal units appearing in the geometrical representations of various identities of representational quantities. I thought we need mathematics teachers in our classrooms that can identify, coordinate, and distinguish between linear and areal quantities/units associated with those identities. I thought this research was worth doing because there is a need to fill in the gap in the literature in terms of unit coordination – as outlined in the second paragraph of the first section of this chapter – and I believed that the representational unit coordination construct I was trying to develop would fill in that gap.

"Area as a product" stands for the Multiplicative Type RUC whereas "area as a sum" concept is related to the Additive Type RUC, depending on the unitizing process. This is the essence of RUC. "Area as a product" gives birth to the coordination of the (two) linear units as an ordered pair, which stands for the Multiplicative Type RUC. On the other hand, "area as a sum" yields the coordination of (in general two or more) areal units as *n*–tuples, which stands for the Additive Type RUC.

The study of multiplicative structures has been conducted by mathematics education researchers since the 1980s. In his 1983 article, Vergnaud viewed the multiplicative structures, a conceptual field of multiplicative type, as a system of different but interrelated concepts, operations, and problems such as multiplication, division, fractions, ratios, and similarity. Although multiplicative structures can to some extent be modeled by additive structures, they have their own characteristics inherent in their nature, which cannot be explained solely by referring to additive aspects. Steffe's Unit–Coordination construct (1988, 1994), the guiding theoretical framework for this study, though strongly related to multiplicative structures, encompasses only the repeated addition model and a distributive aspect for multiplication, which in my opinion are limited explanatory models for what multiplication is/could be. In fact, findings on students' understanding of multiplication in the literature are limited to a premature interpretation, too, mostly relying on addition:

- The conception of multiplication based on repeated addition (Empson, Junk, Dominguez, & Turner, 2005; Fishbein, Deri, Nello, & Marino, 1985)

- Students' frequent use of additive reasoning in the course of tasks requiring multiplicative reasoning (Hart, 1981, 1988; Karplus, Pulos, & Stage,1983; Lamon, 1993; Noelting, 1980; Resnick & Singer, 1993; Vergnaud, 1988)
- The distributive aspect argued by Steffe (1992) and Confrey and Lachance (2000)
- The correspondence principle as the basis of multiplication (Nunes & Bryant, 1996; Piaget, 1965; Vergnaud, 1983, 1988)
- Splitting as an explanatory model for multiplication (Confrey, 1994; Confrey & Smith, 1995)

Research on students' reconciliation of additive and multiplicative structures based on "sum = product" identities is missing in the literature.

My study extends prior work done by Behr et al. (1994) because identities that equate summation and product expressions are not expressed using the generalized mathematics notations in their work. My research project is a theoretical extension of Behr et al.'s framework and introduces a simplified version of generalized mathematics notations for identities that equate summation and product expressions. To be more specific, I worked with the following quantities and their representations:

- Prime & Composite Number Rectangles
- Odd & Even Number Rectangles, Counting Number Rectangles & Rectangles Corresponding to Their Summations
- Rectangles for Polynomial Expressions in x and y.

Coordination construct, though studied several times before, does not cover all possibilities. Levels of unit coordination have been used in additive, multiplicative, and fractional situations before. However, there is no prior work on unit coordination arising from the geometry of the numbers, in the form of identities, where the left hand side of the identity stands for the additive situation (area as a sum, in the geometry of the context) and the right hand side of the identity stands for the multiplicative situation (area as a product, in the geometry of the context). Both phrases, "area as a product" and "area as a sum," stand for the measure of the area of the rectangle enclosed by its sides. "Area as a product" is the conception of seeing the area as an ordered pair of linear units (Multiplicative Type RUC) whereas "area as a sum" is the conception of seeing the area as an ordered *n*–tuples of areal units (Additive Type RUC). This project, aimed at providing an extension for Behr et al.'s theory, will have a crucial impact in the mathematics education field, in that way.

All these representational quantities can be represented as the area of some rectangles. They can be written both as a sum and as a product, and even sometimes, as pseudo–products, as well[3]. In other words, the above representational quantities and expressions occupy an important place in the realm of mathematics. This is exactly where at least four different strands meet: Number Sense, Geometry, Algebra, and Measurement: A crucial domain of mathematics in which one can observe the connections. As PSSM's Connections Standard points out,

> Instructional programs from prekindergarten through grade 12 should enable all students to recognize and use connections among mathematical ideas; understand how mathematical ideas interconnect and build on one another to produce a coherent whole; recognize and apply mathematics in contexts outside of mathematics (NCTM, p. 353).

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[3] See Chapter IV for details.

Let us assume for a moment that there is a secondary school mathematics teacher who is solely focusing on number sense. S/he believes and instructs his/her students that 7 is a prime number because 7 is divisible by 1 and itself only. Let us also assume that the same teacher uses the well–known Gauss' formula to teach the identity $1+2+\ldots+n=\frac{n(n+1)}{2}$. This teacher is addressing only part of the mathematics and appealing to a particular learning style, namely memorization. Students would like to be offered different ways of approaching problem situations and hence would embrace a model (or models) that help them become mathematically proficient (Rogers, Reynolds, Davidson, & Thomas, 2001). When you can touch, color, manipulate, play with formulas and identities, now that is a moment where mathematics comes to life. Mathematics comes to life through problems which are explored and/or solved in and out of the classroom (Adams, 1997). By experience, I know that those students of mine who solely memorize formulas seemed to behave like robots. When it came to debate on not previously discussed identities and formulas in class, they failed all the time. They needed to go back home, memorize the specific day's formulas, and come back and get ready for discussing these same formulas they memorized. This is an obstacle for a person's personal development, in my opinion. In real life, you don't always face with previously experienced challenges. Providing various representations for a problem situation must be an important job of a mathematics teacher. My research project is unique in that way. As PSSM's Representation Standard points out,

> Instructional programs from prekindergarten through grade 12 should enable all students to create and use representations to organize, record, and communicate mathematical ideas; select, apply, and translate among mathematical representations

> to solve problems; use representations to model and interpret physical, social, and mathematical phenomena (NCTM, p. 359).

With color cubes and tiles, all the numbers, prime or composite, even or odd, binomial or polynomial, and the related identities will be right there in front of our eyes. 7 will be a prime number because it will have a unique rectangle. 7 will be a prime number because its unique rectangle has dimensions 1 and 7, which are nothing but the only factors of 7. 7 will be a prime number because it can be written uniquely as the product of linear units 1 and 7. Once again, 7 will be a prime number because it can be written uniquely as the sum of the areal units of 1. In my opinion, a research study embracing NCTM's Principles and Standards for School Mathematics, which emphasize connections along with representations and uses context where several strands of mathematics meet must be important and the context itself, therefore, must be a rationale for the research study on its own.

As a result of my own teaching and learning experiences, I was interested in studying how preservice secondary teachers make sense of linear and areal quantities units. Specifically, my research questions were: How do preservice secondary school teachers

- identify, describe, and interpret linear and areal units?
- represent linear and areal units?
- distinguish between areal and linear units?

2.4. Data Analytical Framework

The (analytical) theoretical framework for this research is based on *generalized notation for mathematics of a quantity* aiming at theoretical analyses and communication within the research community, developed by Behr et al. (1994) who applied these systems in the analysis of additive and multiplicative situations. In the notation for the generalized mathematics of quantity,

> The size of a unit will be notated as a number hyphen unit and enclosed in grouping symbols, (2−unit). The number of units that one has will be denoted as *n*(*b*−unit)s or 3(2−unit)s. Using this notation, symbols for units can be embedded within other symbols for units. (p. 127)

Unit coordination construct, which is the essence of the theoretical framework for my research, is strongly related to the generalized mathematics notation I introduced above. The last sentence of the above quotation reminds us of a key concept in multiplication, i.e., the notion of composite units (Steffe, 1988). The embedding of units within other units is also reminiscent of the distribution process. Given that I already explained the levels of unit coordination above, I just want to write these levels for the same example, multiplication of 5 by 7, with Behr et al.'s generalized mathematics notation in the table below.

Table 2.4

Levels of Unit Coordination vs. Generalized Mathematics Notation

First Level	(1−unit)
Second Level	5(1−unit)s and 7(1−unit)s
Third Level	5(7−unit)s or 5(7(1−unit))−unit)s

One can use Behr et al.'s notation for generalized mathematics of a quantity to represent linear equations as well. Research by Caglayan and Olive (2008, submitted) indicates that quantitative unit coordination and quantitative unit conservation are necessary constructs in dealing with a model, Cups and Tiles, representing a linear equation. In this representational model, each occurrence of the unknown in the linear equation is represented by a small circle (a cup), and the known quantities are represented by small squares (tiles). Each tile corresponds to one unit. Positive quantities are drawn in black and negative quantities (cups or tiles) in red. The unspecified rule is that the same number of tiles is hidden in each cup. The problem for the students is to solve the equation by determining how many tiles are in each cup. Cups and Tiles data can be looked at through the multiplicative unit coordination framework suggested by Behr et al. (1994). In this framework, one can think about a cup as one unit containing an unknown (to−be−found) amount of other units, tiles. In other words, one cup becomes nothing but a composite unit of the form 1(*c*−unit) or *c*(1−unit)s with the notation for the generalized mathematics of quantity. Moreover, if one has 3 cups, this can be written as 3(*c*−unit)s. Figure 2.1 below stands for an instantiation of the equation $3c+2=1c+6.$ With Behr et al.'s generalized mathematics notation, this equation takes the form 3(*c*−unit)s + 2(1−unit)s = 1(*c*−unit) + 6(1−unit)s.

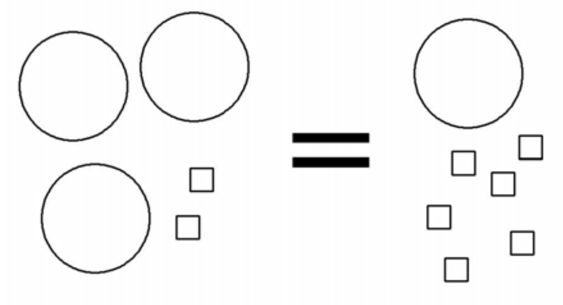


Figure 2.1. An instantiation of the equation $3c+2=1c+6$ with cups and tiles.

Notational systems proposed by Behr et al. (1994) explain the products of quantities carrying the same units (e.g., 2−apple × 5−apple, 3−cm × 4−cm) as well as those with different units (e.g., 3−apple × 4−orange, *a*−cm × *b*−inch, 2−skirt × 3−blouse); however, there is no mention of products of the form ((*a*−unit) + [*b*−unit]) × ((*c*−unit) + [*d*−unit]), nor identities of the form (*a*−unit) + (*b*−unit) + ... = [*α* unit] × [*β*−unit]. The parentheses in (*a*−unit) and [*b*−unit] are used to distinguish between different units whereas the parentheses enclosing the sum ((*a*−unit) + [*b*−unit]) just serve for grouping. The left hand side of what we call as *identity* refers to the *sum* of the odd integers, or even integers, or counting numbers, depending on the context; whereas, the right hand side refers to the *product* that equals the sum. With the area model, the sum corresponds to the area of the growing rectangle as a sum; whereas, the product corresponds to the area of the same growing rectangle as a product. With my study, I hoped that the results of my research would provide insight on how preservice teachers make sense of this complex quantitative situation.

The generalized mathematics notation corresponding to the identity $1+3+5+\ldots+(2n-1)=n\times n$ can be written as [1−unit] + [3−unit] + [5−unit] + ... + [(2*n*

– 1)–unit] = (n–unit) × (n–unit). In other words, square brackets [ ] are used for the units of the sum. Parentheses ( ) on the other hand, are used for the units that give the product. The difference lies in that the units of the product are functions depending on the number of units used in the sum. That function is nothing but the number of terms being added. As another approach, one can think of [ ] quantities as areal units whereas ( ) quantities as linear units. Similar notations can be obtained for the summation of even integers and counting numbers and polynomial multiplication and factorization expressions.

A prime number p can be written as the sum [1–unit] + [1–unit] + [1–unit] + ... + [1–unit] and as the product (1–unit) × (p–unit) or (p–unit) × (1–unit). This notation is unique for a prime number. However, for a composite number, we have more than one notation. The following table illustrates types of notations that can be used to describe the composite number 12.

Table 2.5

Generalized Mathematics Notation for Composite Number 12

12 as a Sum	**12 as a Product**
[1–unit] + ... + [1–unit]	(1–unit) × (12–unit)
1[2–unit] + ... + 1[2–unit]	(2–unit) × (6–unit)
1[3–unit] + ... + 1[3–unit]	(3–unit) × (4–unit)
1[4–unit] + 1[4–unit] + 1[4–unit]	(4–unit) × (3–unit)
1[6–unit] + 1[6–unit]	(6–unit) × (2–unit)

I also made use of Vergnaud's *theorems–in–action*, "mathematical relationships that are taken into account by students when they choose an operation or a sequence of operations to solve a problem" (1988, p. 144) as a data analysis framework. Vergnaud goes on to state "To study children's mathematical behavior it is necessary to express the theorems–in–action in mathematical terms." (p. 144). Grounded in this theory, I developed a series of terminology serving as some sort of analytical tools in an attempt to translate my students' *concepts* and *theorems in action* (e.g., mathematical behaviors, mathematical thinking, actions, statements, hand gestures, drawings, etc.). Equal Addends, Summed Addends, Representational Sets, Irreducible Linear/Areal Quantities, Representational Cartesian Products, Ordered Pair of Linear Units/Quantities, Ordered *N*–Tuple of Areal Units/Quantities, Filling in the Puzzle Strategy, Term–Wise Multiplication of Irreducible Linear Units/Quantities Strategy, Mapping Structures, Inverse Mapping Structures are examples of such terminology. In doing so, I often made use of a *Relational Notation*, a much more simplified and understandable version of the *Generalized Notation for Mathematics of a Quantity* (Behr et al., 1994).

Smith and Thompson state that "conceiving of and reasoning about quantities in situations does not require knowing their numerical value (e.g., how many there are, how long or wide they are, etc.). Quantities are attributes of objects or phenomena that are measurable; it is our capacity to measure them—whether we have carried out those measurements or not—that makes them quantities." (2008, p. 101). In mathematics, we define the Cartesian product of two sets A and B as the set of all ordered pairs in which the first component is taken from the first set, and the second component is taken from the second set. Using this analogy, one can say that a product quantity can be coordinated (composed) as an ordered pair of the form (a, b), where a and b are understood to be coming from the first set and the second set, respectively. To help visualize the situation, just as a point on the coordinate plane is associated with its x– and y–coordinates, which are coordinated as the ordered pair (x, y), a product quantity can be represented, hence coordinated, as the pair (multiplier, multiplicand). All possible orderings of the form (multiplier, multiplicand) with coordinates multiplier and multiplicand generate the binary relation under consideration. In the example of the polynomial product $(x+1)(2y+3)$, for instance, the coordination $(x, 2y)$ is not the same as (x, y) or $(x, 3)$. There are various types of product quantities modeled with polynomial rectangles. In the example of $(x+1)(2y+3)$, we have the following product quantities: (See Figure 2.2)

i. The product quantity $(x+1)(2y+3)$, which is mapped as the area of the whole rectangle (largest areal singleton) enclosed by its sides $x + 1$ and $2x + 3$ (Multiplicative Type RUC),

ii. The product quantities $x \cdot 2y$, $x \cdot 3$, $1 \cdot 2y$, $1 \cdot 3$ each being mapped as the area of the corresponding boxes of the same color (This is also a Multiplicative Type RUC, however, some interview students treated these as "pseudo–products," which necessitates a different RUC type in between Multiplicative and Additive: Pseudo–Multiplicative Type RUC),

iii. The product quantities $x \cdot y$ (there are two of them), $x \cdot 1$ (there are three of them), $1 \cdot y$ (there are two of them), $1 \cdot 1$ (there are three of them) each being mapped as the area of the corresponding irreducible areal unit (Multiplicative Type RUC). The total number of these irreducible areal units for this example is 10. In general, for any polynomial product of the form $(ax+by+c)(ex+dy+f)$, the total number of the irreducible areal units equals $(a+b+c)(d+e+f)$.

There are also Additive Type RUCs (Irreducible Addends, "Same–Color–Box" Addends, "Combined–Areal–Box" Addends[4]) arising from polynomial rectangles in x and y. In Chapter IV below, I describe all RUC types referred by students, using a simplified version of Behr's generalized mathematics notation, which I named relational notation, in more detail.

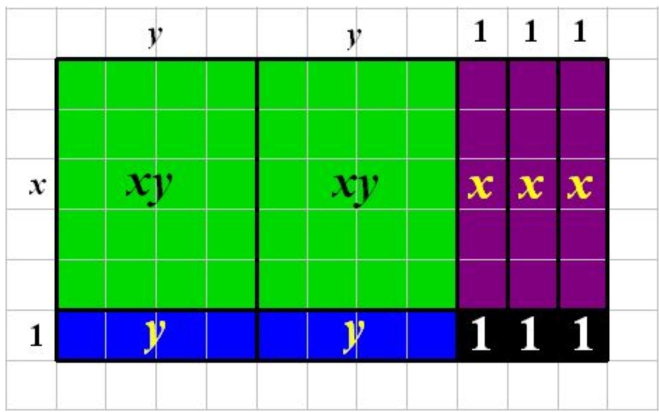


Figure 2.2. Polynomial rectangle of sides $x + 1$ by $2y + 3$.

[4] These are named as Irreducible Areal Quantities (IAQ), Same–Color–Box Areal Quantities (SCBAQ), and Combined–Areal–Quantities (CAQ), respectively, in Chapter IV.

CHAPTER THREE

CONTEXT AND METHODOLOGY

3.1. Research Design

I was interested in investigating preservice mathematics teachers' sense making of different types of units and quantities arising from the use of color cubes and algebra tiles. I was hoping to reveal the foundations underpinning these students' mathematics associated with the activities pertaining to these manipulatives. In that regard, I chose to use a qualitative design because I would have more opportunities to probe on these ideas in an attempt to reveal my research participants' mathematics.

As the concepts of "units" and "quantities" were the essential ideas guiding this research study, I used unit coordination (Steffe, 1988, 1994) and quantitative reasoning (Thompson, 1988, 1989, 1993, 1994, 1995) as the main theoretical frameworks. I also made use of Schwartz' adjectival quantities and referent preserving/transforming compositions (1988), which served as a meaningful perspective in looking at the interviews comparatively (e.g., students making use of a referent preserving composition vs. those making use of a referent transforming composition). My data analysis framework was inspired by a simplified version of Behr et al.'s (1994) *generalized notation for mathematics of a quantity* and Vergnaud's (1983, 1988, 1994) *theorems and concepts*–*in*–*action* formalisms, which helped me translate students' mathematical performance into a series of terms based on a very simple notation: *Relational notation* and *mapping structures* duo (Caglayan, 2007b). All these choices played a crucial role and stood as the essential rationale behind the idea of conducting multiple interviews.

3.2. Participants

I conducted my study with (2 middle and 3 high school mathematics) preservice teachers enrolled in the Mathematics Education Program in a university in the southeastern United States. I interviewed five people individually twice during Spring 2007 semester. Duration of each session was about 75 minutes and each interview session was videotaped using one camera.

I selected my participants from two different undergraduate level mathematics education classes. Brad, Sarah and John came from the “Concepts in Secondary School Mathematics” class of 11 enrolled preservice high−school mathematics teachers while Nicole and Robert came from the “Teaching Geometry and Measurement in the Middle School” class of 22 enrolled preservice middle−school mathematics teachers. All these five students volunteered to participate in my study. Their decision to participate or not would not affect their grade or class standing. Each participant had the right to have all of the information about him/herself returned to him/herself, removed from the research records, or destroyed. Moreover, any information collected about these students was to be kept confidential. All proper names in this study, therefore, are pseudonyms.

3.3. Data Collection

The focus of my research study is on problems on identities of the form $\Sigma = \Pi$ for prime and composite numbers along with summation of counting numbers, odd and even integers as well as products and factors of polynomials modeled with magnetic color cubes and algebra tiles. To be more specific, the 1" magnetic color cubes were used to generate sequences of growing rectangles representing identities for prime and composite numbers as well as summation of counting numbers, odd and even integers. During the interviews, I asked preservice teachers to make rectangles representing prime and composite numbers first. We then focused on patterns that generate growing rectangles for the summation of counting numbers, odd, and even integers.

As for the products and factors of polynomial expressions, we used color tiles of different colors and sizes (algebra tiles). In this model, each little black square tile represents the number 1, long purple bars represent the x, long blue bars represent the y, big purple squares represent x^2, big blue squares represent y^2, and big green rectangles represent xy. The 1, the x, and the y are called Irreducible Linear (or Areal, depending on the context) Quantities (ILQ or IAQ); whereas the x^2, the y^2, and the xy are called Irreducible Areal Quantities (IAQ). Preservice teachers constructed rectangles with specified dimensions of the form $(ax + by + c)$, where a, b, and c were natural numbers. They were asked to write their answers for the area of the polynomial rectangle as a product and as a sum (See Figure 2.2 above for the construction of the polynomial rectangle of dimensions $x + 1$ by $2y + 3$).

My rationale for collecting interview data with preservice teachers was mainly to understand how they establish $\Sigma = \Pi$ identities involving linear and areal quantities based on the color cubes and algebra tiles representational models. I also wanted to determine if they were able to reason at the different categories of linear or areal quantities (Multiplicative and Additive RUC Types) associated with growing rectangles generated by color cubes and algebra tiles (See Table 3.1 below for the interview outline I used). Moreover, we do not know about preservice teachers' mathematical knowledge on these issues. We would want to let them think about these issues before they start teaching. These are in the content of middle and high school mathematics. As researchers, we need to document information about preservice teachers' identification, interpretation and coordination of different types of representational units arising from this mathematical content because such practices are highly likely to have strong implications (e.g., curriculum writing, teacher education), which mathematics education field will benefit from.

Table 3.1

Interview Outline

Activity One – Prime and Composite Numbers Directions: • Represent prime numbers (e.g., 5, 7) and composite numbers (e.g., 15, 28) as rectangles using a different color for each new rectangle[5]. • Identify the area and the dimensions of the corresponding rectangle(s) for each number. • Use a table to organize information.	Probing questions: • What are the units associated with each prime/composite number? • What is the area of each prime/composite rectangle as a sum? As a product? • What are the length and the width of each prime/composite rectangle? • Where are the linear units? Areal units of prime/composite rectangles?
Activity Two – Summing Counting Numbers Directions: • Represent counting numbers 1, 2, 3, … using a different color for each number. • Add them so that they generate a rectangle. • Use a table to organize information.	Probing questions: • What are the units associated with each counting number? Odd integer? Even integer? • What is the area of the growing rectangle at each step as a sum? As a product? • What are the length and the width of the growing

[5] The activity I did with Nicole slightly differs from the other students in that in representing prime and composite numbers, I also used a multiplication mat, in addition to the cubes. The rest is the same for all students.

	rectangle at each step? • Where are the linear units? Areal units of the growing rectangle?
Activity Three – Summing Odd Integers Directions: • Represent odd integers 1, 3, 5, … using a different color for each number. • Add them so that they generate a rectangle. • Use a table to organize information.	Probing questions: • What are the units associated with each odd integer? • What is the area of the growing rectangle at each step as a sum? As a product? • What are the length and the width of the growing rectangle at each step? • Where are the linear units? Areal units of the growing rectangle?
Activity Four – Summing Even Integers Directions: • Represent even integers 2, 4, 6, ... using a different color for each number. • Add them so that they generate a rectangle. • Use a table to organize information.	Probing questions: • What are the units associated with each even integer? • What is the area of the growing rectangle at each step as a sum? As a product? • What are the length and the width of the growing rectangle at each step? • Where are the linear units? Areal units of the growing rectangle?
Activity Five – Polynomial Multiplication Directions: • Multiply two polynomials using a generic rectangle by placing one of the polynomials at the top, and the other, on the side of the generic rectangle. • Identify the area and the dimensions of the rectangle for a polynomial product.	Probing questions: • What is the area of each polynomial rectangle as a sum? As a product? • What are the length and the width of each polynomial rectangle? • What are the (linear) units associated with the dimensions of the polynomial rectangle? • What are the (areal) units associated with the area of the polynomial rectangle?
Activity Six – Polynomial Factorization Directions: • Build a rectangle enclosing the tiles corresponding to the polynomial expression. • Identify the dimensions (length and width) of the polynomial rectangle.	Probing questions: • What is the area of each polynomial rectangle as a sum? As a product? • What are the length and the width of each polynomial rectangle? • What are the (linear) units associated with the dimensions of the polynomial rectangle? • What are the (areal) units associated with the area of the polynomial rectangle?

In Table 3.1 above, first four tasks were used during the first interview with each student whereas the fifth and the sixth tasks are used during the second interviews. During the interviews, I asked preservice teachers to make rectangles representing prime and composite numbers first. We then focused on patterns that generate growing rectangles for the summation of counting numbers, odd, and even integers. As for the products and factors of polynomial expressions, we used color tiles of different colors and sizes (algebra tiles). The figure below illustrates examples of materials described in Table 3.1.

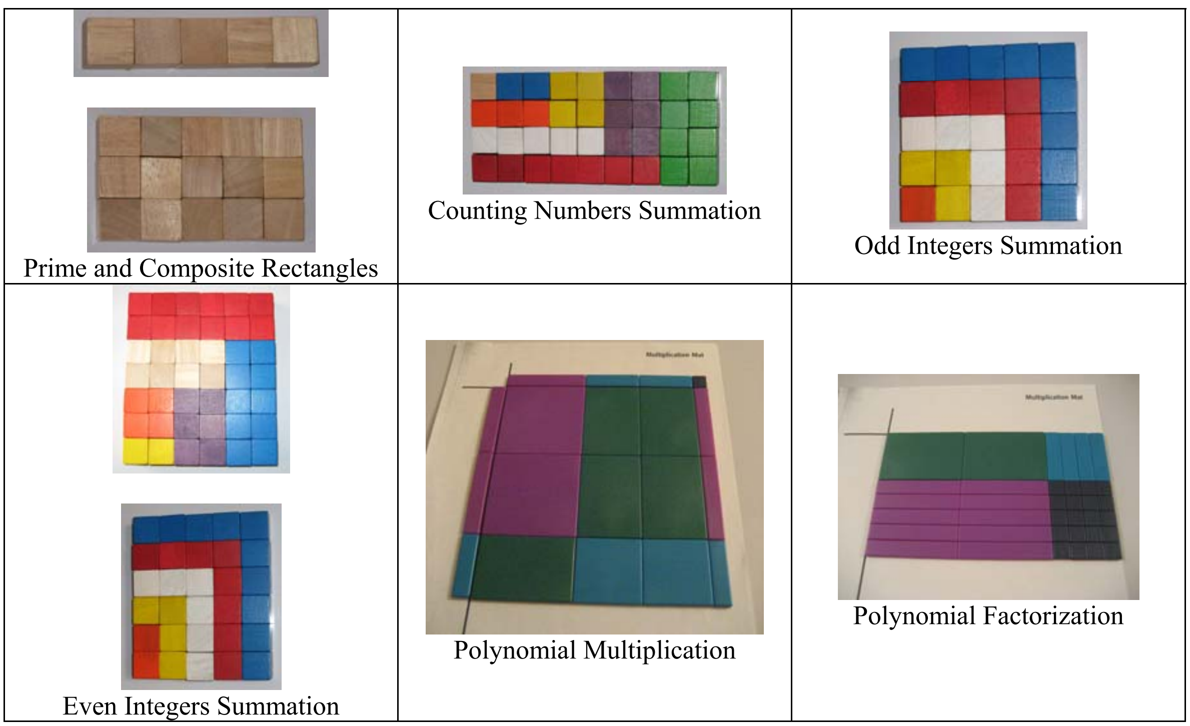


Figure 3.1. Six main activities on “sum = product” representational identities.

3.4. Analyzing Data: Relational Notation & Mapping Structures

The interviews were consecutive; no analysis was done between interviews. After the end of data collection, I reviewed each interview in order to generate possible themes for a more detailed analysis. I first generated an outline for each interview, from which I obtained a summary for each student. These interview outlines helped me identify significant events for transcription. The final write-up is organized with sections based on activities, rather than students. As for the analysis methodology, I used thematic analysis supported by retrospective and constant comparison analyses of interviews. I also benefited from *generalized notation for mathematics of a quantity* (Behr et al., 1994) and *theorems and concepts in action* (Vergnaud, 1983, 1988, 1994) framework as data analysis tools from which I developed a data analysis framework of my own: Relational notation and mapping structures duo (Caglayan, 2007b).

I have decided to extend Behr et al.'s notation in such a way as to cover identities that equate summation and product expressions of representational quantities. The fact that commutativity is an evident property of multiplication operation may not be so obvious for every student. Yes, numbers[6] commute when the binary operation under consideration is multiplication. However, though produce the same result, $a \times b$ and $b \times a$ may have different algebraic and geometric interpretations for students. In other words, although the commutative property states that order does not matter in multiplication, it may matter for some students. The relational notation I am proposing thus, naturally follows these ideas, as an ordered pair (a, b) is in general different from (b, a). They are the same only if $a = b$. Even though I use the notation (a, b) to denote an ordered pair of

[6] Matrix multiplication is not commutative, though.

linear units, for the research participants (a, b) did not have a different meaning from (b, a), therefore, in the remaining part of this manuscript, the ordered pairs (a, b) and (b, a) are to be considered equivalent.

It remains to describe a relational notation for addition, as well. In this case, once again I am not worrying about commutative property of addition for the same reason as stated for multiplication. However, I reserve square brackets [ ] for addition, which is different from the multiplicative situation. Moreover, for the multiplicative situation, we have ordered pairs while for the additive situation, we have ordered n–tuples where the integer n in general being greater than 2. For instance, the ordered n–tuple $[a_1, a_2, \ldots, a_{n-1}, a_n]$ denotes areal units $a_i, 1 \le i \le n,$ generating the area of the growing rectangle under consideration.

There must be an agreement of the ordered pair (a, b) of linear units and the ordered n–tuple $[a_1, a_2, \ldots, a_{n-1}, a_n]$ of areal units. How can we reconcile these two? At this moment, it is mapping structures that come to rescue. This will become clearer with students' work in Chapter IV, however, for the moment, I just want to describe mapping structures briefly. Without loss of generality, focusing on the $x + 1$ by $2y + 3$ polynomial rectangle example I introduced above, the multiplication operation, which behaves as a function, as a mapping, can be represented using a functional notation as $f : (x+1, 2y+3) \rightarrow 2xy + 3x + 2y + 3.$ Here, f denotes the multiplication operation that maps the linear units, x + 1 and $2y + 3$, which can be thought of as a combination of irreducible linear units, into the corresponding areal unit, namely $2xy + 3x + 2y + 3$, which is also the same as the area of the polynomial rectangle itself. In other words, f acts on the ordered pair $(x+1, 2y+3)$ of linear units and maps it into the areal unit $2xy+3x+2y+3$. This operation can also be written as an equality:

$$f(x+1, 2y+3) = 2xy+3x+2y+3.$$

Similarly, the addition operation behaves like a function, like a mapping, acting on irreducible areal units or combinations of those. For instance, the function *g*, which represents the addition operation, acts on the ordered 10–tuple [*xy*, *xy*, *x*, *x*, *x*, *y*, *y*, 1, 1, 1] of areal units and maps it into the areal unit $2xy+3x+2y+3$. Using a functional notation, this can be written as $g:[xy, xy, x, x, x, y, y, 1, 1, 1] \to 2xy+3x+2y+3,$ or with the equality:

$$g[xy, xy, x, x, x, y, y, 1, 1, 1] = 2xy+3x+2y+3.$$

In other words, though they act on different types of representational quantities, the mappings *f* and *g* agree on one thing: That one thing is nothing but the fact that their images coincide (Figure 3.1). This is the essence of what is meant by Σ = Π identity in this research project. “Area as a product” coincides with “area as a sum” at the end, thanks to these mapping structures.

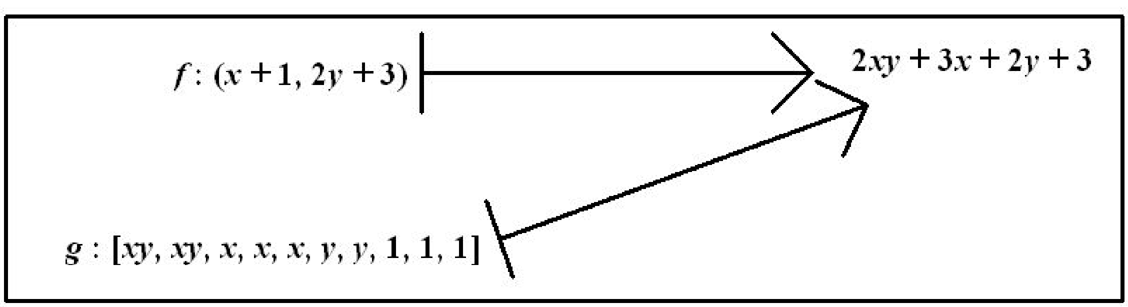


Figure 3.2. Mapping structures.

CHAPTER IV

COLOR CUBES & TILES ANALYSIS RESULTS

This chapter presents results arising from preservice teachers' behaviors based on various tasks involving magnetic color cubes and algebra tiles. In the first section, I describe these students' mathematical thinking and the strategies they used in the context of prime and composite numbers represented by rectangles made of one inch color cubes. In the second, the third, and the fourth sections, I describe their thinking for more challenging tasks: summation identities represented by one inch color cubes. The fifth and the sixth sections are about tasks requiring polynomial multiplication and factorization using algebra tiles. The seventh section is a summary of findings for each individual student. I conclude the chapter with a summary of additive and multiplicative structures.

4.1. Prime & Composite Rectangles

I started the interviews with three of the students with an activity involving rectangles formed from prime and composite numbers because I thought this activity was easy compared to others so it would serve as a warm–up activity. With the other two students, I started with an activity involving rectangles formed from counting numbers. Both activities led to the construction of rectangles made from the same number of tiles (e.g., 15, 28), but the construction of the rectangles was different because the rectangles arose from different numerical situations. This difference in initial interview tasks afforded me a good opportunity to compare the students' thinking in different yet similar contexts. After this initial activity, the remainder of the interview was the same for all six students.

My initial instruction to Brad, Nicole, Rob and Sarah was “Make a rectangle for 15.” In the same vein, I asked John to “Make a rectangle for 28.” All interview students made a 3 by 5 rectangle (John made a 4 by 7 rectangle) using wooden cubes on the desk (See Figure 4.1).

Figure 4.1. Composite rectangle representing 15.

4.1.1. Multiplicative Representational Unit Coordination (MRUC)

I define a prime rectangle as a rectangle made of a prime number of color cubes. A prime number of cubes can be arranged into one unique rectangle (disregarding rotations). Composite rectangles, on the other hand, are made from a composite number of cubes. Since a composite number has three or more distinct divisors, a composite number of cubes can be arranged into at least two different rectangles. In the context of prime and composite numbers, the multiplicative nature of representational unit coordination (RUC) appears as an ordered pair of linear units, namely, the length and width of the rectangle representing a prime or composite number. The following protocol illustrates this aspect.

> Protocol 4.1: Brad's description of composite rectangle's area as a product.
>
> B: From my perspective. Base is 3, height is 5. So it's 5 times 3... fifteen... Length is the base, width is the height.
>
> G: What is the area of this figure... as a product of what?
>
> B: Of 3 and 5, which is the length and the width.

Nicole, on the other hand, defined the length to be 3 and the width to be 5. She saw this fifteen, while pointing to the rows, as “five rows of three.” In this warm–up activity, Brad and Nicole agreed on the multiplicative RUC arising from usages “So it's 5 times 3,” “So it'd be 3 times 5.” When I look at the other students' answers, I see similar usages. Rob, for instance, made the same 3 by 5 rectangle. I then probed by asking “What is it about fifteen?” He said “Just multiplication... three rows of five each... or three columns of five each...” As for Sarah, the area of the same rectangle as a product is “5 times 3.” Finally, John attached measurement units to his answer when I asked him about the area of the 4 by 7 rectangle as a product: 4 inches by 7 inches. I will talk about the measurement units in a separate section below; however, at this elementary stage of formulating a multiplicative type RUC, I can say that all these students agreed on a standard definition of area, which can be used to describe their answers. The table below summarizes each student's answer and a corresponding relational notation that describes multiplicative type RUC.

Table 4.1.

Relational Notation Describing Students' Answers

Students	Phrases	Relational Notation
Brad	So it's 5 times 3	(5, 3)
Nicole	So it'd be 3 times 5 Five rows of three Five columns with three in each one	(3, 5) (5, 3)
Rob	Three rows of five each Three columns of five each	(3, 5)
Sarah	5 times 3	(5, 3)
John	4 inches by 7 inches	(4, 7)

Each student wrote his/her answer on an activity sheet related to each activity (See Appendix). Students' written answers were consistent with what they were saying during the interviews. The table below reflects students' written answers for the area as a product of their prime and composite rectangles. I am including once again a relational notation for each written answer for the purpose of analysis.

Table 4.2

Students' Written Work for Prime & Composite Rectangles' Areas as a Product

Students	Area of the Composite Rectangle as a Product	Relational Notation Describing the Product	Area of the Prime Rectangle as a Product	Relational Notation Describing the Product
Brad	$3 \cdot 5, 5 \cdot 3$	(3, 5), (5, 3)	$3 \cdot 1$	(3, 1)
Nicole	$3 \times 5, 5 \times 3, 1 \times 15, 15$	(3, 5), (5, 3), (1, 15),	$1 \times 7, 7 \times 1$	(1, 7), (7, 1)

	× 1	(15, 1)		
Rob	3in × 5in, 5in × 3in, 1in × 15in, 15in × 1in	(3, 5), (5, 3), (1, 15), (15, 1)	1in × 7in, 7in × 1in	(1, 7), (7, 1)
Sarah	3 × 5	(3, 5)	NA	NA
John	4in × 7in	(4, 7)	1in × 7in	(1, 7)

As can be seen in the written work, some students (Rob and John) attached units to their answers. When area of a rectangle is expressed as a product of the form 4in × 7in or 3in × 5in, the unit of measurement needs to be specified. With the relational notation, however, one does not need to attach a measurement unit to the ordered pairs because each component of the ordered pair by itself is a linear unit.

4.1.2. Additive Representational Unit Coordination (RUC)

In the context of prime and composite numbers, the additive nature of RUC appears as an ordered *n*–tuple of areal units. The areal–ness of the addends making up the total area of rectangle was spelled out by all these students either by the phrases they used, or by their written work during the interviews. A description of Brad's use of addends is found in Protocol 4.2.

Protocol 4.2: Brad's description of composite rectangle's area as a sum.

G: How about the area of this figure as a sum?

B: You can count the blocks; it's 15 so... Or let's go with the threes... adding five times... it's 3 plus 3 plus 3 plus 3 plus 3.

G: How about each one of those threes, is it a length or an area?

B: It can be both. I mean in this example, I'd say it's an area because they are cubes so it'd be like 3 by 1, 3 by 1, ...

I can use a relational notation such as [1, 1, ..., 1] to describe Brad's statement that "You can count the blocks; it's 15." In this notation, each "1" in the ordered 15–tuple is an areal unit. To be more specific, each areal "1" is an irreducible areal unit. I use the term *Irreducible Addends of Type I* for this type of additive RUC[7]. Brad's other usage "So it'd be like 3 by 1, 3 by 1, ..." in describing the addends is another type of additive RUC, which I call *Equal Addends* type RUC. In this model, the equal areal units "3" are combined additively to make an areal 15, which is the area of the composite rectangle. Using a similar strategy, the relational notation [3, 3, 3, 3, 3], an ordered quintuple, can be used to describe these areal units of the Equal Addends type RUC. Nicole's explanations on the same matter can be modeled using an additive type relational notation, as well.

Protocol 4.3: Nicole's description of composite rectangle's area as a sum.

G: How would you write the area of this rectangle as a sum? Of what?

N: It's the sum of each individual block [pointing to the blocks]... I would add up each individual square [Nicole is pointing to the blocks while speaking] or you could add 3 plus 3 plus 3 plus 3 plus 3 [She is pointing to the columns of "threes"] or even you know... 5 plus 5 plus 5 [She is pointing to the rows of "fives"] until you reach 15.

7There is another type of irreducible addends for additive RUC that I will name *Irreducible Addends of Type II*. This will be explained in the fifth section of Chapter 4 that deals with the activities on polynomial rectangles.

Nicole's usage “It's the sum of each individual block... I would add up each individual square” could also be modeled as the ordered 15–tuple [1, 1, ..., 1] of irreducible areal units, i.e., of Irreducible Addends (Type I) type additive RUC. Her repeated addition language is of Equal Addends type and can be modeled as the ordered quintuple [3, 3, 3, 3, 3] and as the ordered triple [5, 5, 5], respectively. In the relational notation guided by ordered pairs or *n*–tuples, square brackets are used to describe the set of areal units whereas ordinary parentheses are used for the ordered pair of linear units. Next I will examine Sarah's description of the area of a rectangle as it is quite different from those offered by Nicole and Brad.

> Protocol 4.4: Sarah's description of composite rectangle's area as a sum.
>
> G: How do you write the area of this rectangle as a sum?
>
> S: Well there is many different ways...
>
> G: For instance?
>
> S: Like… 14 and 1...

The fact that 15 is a composite number does not seem to affect Sarah's thinking as she does not use factors at all. I was expecting an answer like 5 + 5 + 5 or 3 + 3 + 3 + 3 + 3, i.e., area of the rectangle as the sum of “equal addends.” I define Sarah's description as *Random Addends* Type RUC. In this model, the addends, which are areal units, could be anything. In other words, the addends do not necessarily have to be of “Equal Addends” type RUC or “Irreducible Addends (Type I)” type RUC. In fact, Sarah was aware that whatever the addends were, they had to be areal units. She specified that 14 and 1 both had the same units, i.e., inches squared. Her “random” example can be modeled as an ordered pair [14, 1] of areal units via the relational notation. One does not need to have an ordered pair of areal units in this model, though. One could have things like [5, 5, 4], [7, 2, 1, 1, 1, 2], [3, 3, 6, 2] as well. The number of addends and the numerical value of each addend are completely random. The only restriction in this model is that the areal units, namely the addends, have to add up to the area of the rectangle under consideration.

Sarah was also aware that when the area is written as a product, 3×5, the 3 and the 5 both have units of inches. She said “3 inches times 5 inches… which would yield 15 inches squared, so they are different,” meaning the linear unit is different from the areal unit. I describe all students' comparison of linear and areal units in the next subsection in more detail.

> Protocol 4.5: John's description of composite rectangle's area as a sum.
>
> G: Could you suggest a way of writing 28 as a sum?
>
> J: You could add up the boxes [pointing to the unit cubes one by one. He then says “7 plus 7 plus 7 plus 7,” pointing to the rows, and puts his answer on the table]. You could do 4 plus 4 plus 4 plus 4 plus 4 plus 4 plus 4 [pointing to the columns]... I guess there is multiple ways you could do...

John's statement “You could add up the boxes” by pointing to the unit cubes one by one could warrant an Irreducible Addends (Type I) type RUC. With the relational notation, this can be modeled as the ordered 28–tuple [1, 1, 1, ..., 1] of irreducible areal units. His subsequent usages “7 plus 7 plus 7 plus 7” and “4 plus 4 plus 4 plus 4 plus 4 plus 4 plus 4” in the same conversation are of Equal Addends type RUC and can be modeled as the ordered quadruple [7, 7, 7, 7] and as the ordered heptuple [4, 4, 4, 4, 4, 4, 4] of equal areal units, respectively.

I conclude this subsection with three tables to summarize what has been discussed so far. The first table below tabulates students' language warranting additive type RUCs, the name of the corresponding additive type RUC, and a relational notation I infer from this language. The remaining two tables are slightly different from the first one in that they are based on students' written work on the activity sheet.

Table 4.3.

Relational Notation Describing Students' Answers

Students	Phrases	Name of the Additive Type RUC	Relational Notation
Brad	You can count the blocks it's 15	Irreducible Addends (Type I)	[1, 1, ..., 1]
	So it'd be like 3 by 1, 3 by 1, ...	Equal Addends	[3, 3, 3, 3, 3]
Nicole	It's the sum of each individual block	Irreducible Addends (Type I)	[1, 1, ..., 1]
	I would add up each individual square	Irreducible Addends (Type I)	[1, 1, ..., 1]
	You could add 3 plus 3 plus 3 plus 3 plus 3	Equal Addends	[3, 3, 3, 3, 3]
	or even you know... 5 plus 5 plus 5	Equal Addends	[5, 5, 5]
Rob	1 plus 1 plus 1 plus 1 plus 1	Irreducible Addends (Type I)	[1, 1, 1, 1, 1]
Sarah	Well there is many different ways... Like 14 and 1...	Random Addends	[14, 1]

John	You could add up the boxes	Irreducible Addends (Type I)	[1, 1, ..., 1]
	7 plus 7 plus 7 plus 7 [pointing to the rows]	Equal Addends	[7, 7, 7, 7]
	4 plus 4 plus 4 plus 4 plus 4 plus 4 [pointing to the columns]	Equal Addends	[4, 4, 4, 4, 4, 4]

Table 4.4

Students' Written Work for Composite Rectangles' Areas as a Sum

Students	Area of the Composite Rectangle as a Sum	Name of the Additive Type RUC	Relational Notation Describing the Sum
Brad	3 + 3 + 3 + 3 + 3	Equal Addends	[3, 3, 3, 3, 3]
	5 + 5 + 5	Equal Addends	[5, 5, 5]
Nicole	3 + 3 + 3 + 3 + 3	Equal Addends	[3, 3, 3, 3, 3]
	5 + 5 + 5	Equal Addends	[5, 5, 5]
	1 + 1 + ... + 1	Irreducible Addends (Type I)	[1, 1, ..., 1]
Rob	$3in^2 + 3in^2 + 3in^2 + 3in^2 + 3in^2$	Equal Addends	[3, 3, 3, 3, 3]
	$5in^2 + 5in^2 + 5in^2$	Equal Addends	[5, 5, 5]
	$1in^2 + 1in^2 + ... + 1in^2$	Irreducible Addends (Type I)	[1, 1, ..., 1]
Sarah	NA	NA	NA
John	$7in^2 + 7in^2 + 7in^2 + 7in^2$	Equal Addends	[7, 7, 7, 7]

Table 4.5

Students' Written Work for Prime Rectangles' Areas as a Sum

Students	Area of the Prime Rectangle as a Sum	Name of the Additive Type RUC	Relational Notation Describing the Sum
Brad	1 + 1 + 1	Irreducible Addends (Type I)	[1, 1, 1]
Nicole	1 + 1 + 1 + 1 + 1 + 1 + 1	Irreducible Addends (Type I)	[1, 1, 1, 1, 1, 1, 1]
Rob	$1in^2 + 1in^2 + 1in^2 + 1in^2 + 1in^2 + 1in^2 + 1in^2$	Irreducible Addends (Type I)	[1, 1, 1, 1, 1, 1, 1]

Sarah	NA	NA	NA
John	$7in^2$	Singleton Addend	[7]

By looking at students' work on prime and composite rectangles, I deduced that Rob and John emphasized the areal units by attaching an “inches squared” symbol to each addend. John's answer for the “area of the prime rectangle as a sum” necessitates the existence of another additive type RUC, which I define as *Singleton Addend*. In this type, as suggested by John's answer $7in^2$, there is only one addend, which stands for the areal unit corresponding to the total area of the rectangle under consideration. A relational notation denoting this additive type RUC could therefore be written as a singleton, as a unique areal unit in between square brackets, namely as [7].

4.1.3. Linear vs. Areal Units

In this subsection, I describe students' understanding and sense making of linear and areal quantities as well the meanings they project on same–valued linear and areal quantities. In our discussion about the 3 by 5 rectangle made of wooden cubes, Brad first described the area as a product and as a sum, as introduced by Protocol 4.2 above. The protocol below follows from the one above.

Protocol 4.6: Brad compares the linear and areal threes.

G: How about each one of those threes, is it a length or an area?

B: It can be both. I mean in this example, I'd say it's an area because they are cubes so it'd be like 3 by 1, 3 by 1, ...

G: How about the width of this rectangle which is also 3. Now we are comparing the threes.

B: It's more like a length. It's along a line.

Brad interpreted the 3 equal addends as areal quantities, which I inferred from his statement “I'd say it's an area because they are cubes so it'd be like 3 by 1, 3 by 1, ...” In other words, the areal “3” has some meaning attached to it. He recognized that the areal 3 is indeed an ordered pair (3, 1) of the linear units, i.e., the dimensions of the 3 by 1 rectangle. He was also aware that the other 3, namely the length of the 3 by 1 rectangle had a different nature, as can be inferred from his statement “It's more like a length. It's along a line.” I then asked him to be more explicit in distinguishing between the areal and linear threes. Brad then used the known measurement unit “inch” to describe the perimeter of the rectangle. When I asked Brad to compare the threes, he described the areal unit 3 (the one enclosed in the rectangle) as a 3 by 1 rectangle. About the linear unit 3, he said that if it (the rectangle) were drawn on a piece of paper, he would measure the perimeter as 1 inch, 2 inches, 3 inches, ... at the same time pointing to the perimeter line with his index finger.

Later on, I challenged him to make another rectangle for 15. He then made a 15 by 1 rectangle with the wooden cubes, and upon my question “what about this one, is it a length or an area?” he said that kids would see it as a line. However, he added that he thinks of it as a 15 by 1 rectangle. He said he would write this 15 as a sum as 1 plus 1 plus 1 ... I then asked him about each one of those “1”s.

Protocol 4.7: Brad unitizes the “1”s of the 15 by rectangle.

G: If I want to attach a unit to each one of those “1”s, what unit would that be?

B: It'd be an inch.

His answer “It'd be an inch” is quite interesting because I was expecting an answer like “inches squared” or “square inches” as he just wrote “15 inches squared” on the activity sheet, for the “area of its rectangle” column. Later on, when I asked him to focus on one of the cubes only and describe it, he said that one of these would be an inch by an inch and deduced it would be 1 inch squared. He went on saying the total area then would be 15 inches squared because he would be adding 15 of these 1 inch squared units. Therefore, in spite of his comment that “It'd be an inch,” I can infer that Brad used help from known measurement units, inches and square inches, to describe the linearity and areal–ness of the representational quantities under consideration.

Nicole's interpretation of linear and areal quantities in the context of the prime and composite rectangles activity started when I asked her to use a multiplication mat[8] along with the wooden cubes to represent the 3 by 5 rectangle. Nicole placed the cubes representing the length and the width on the sides, and then made her 3 by 5 rectangle in the rectangular region (See Figure 4.2).

[8] The use of a “multiplication mat” with Nicole renders this first task on prime and composite numbers to be a totally new one, as opposed to the other students with whom I did not use a multiplication mat in this activity. Other than this first task, a multiplication mat has not been used in any other task dealing with the cubes. With the algebra tiles, however, students] used a multiplication mat consistently,

Figure 4.2. Nicole's representation of length, width, and area.

Protocol 4.8: Nicole distinguishes between the linear and areal units.

G: What do you like about this representation? I mean the representation of the length and the width on the multiplication mat using the cubes? What do you see?

N: You could see like... the distinction of how it [meaning the length and width represented by the cubes] separates into the sum of the area and... even the product when you multiply this [pointing to the length] by this [pointing to the width]. That's very distinct.

G: Distinct... So how do you distinguish between these quantities [pointing to the width and the length made of cubes] and this quantity [pointing to the area of the rectangle made of cubes]?

N: This [pointing to the rectangle] is one solid rectangle so instead of trying to pull it apart to figure it [here “it” most likely refers to the area of the rectangle] out... using this… this [pointing to the length of the rectangle,

> which is 3, by Nicole's definition] is equal to one of these [here "these" most likely refers to the group of five "areal threes"] ... so I can count these [she probably means "I can count the areal threes"] without having to messed up with my rectangle... so... come to the same answer [here, "answer" probably means "the area of the rectangle"].

I hypothesize that Nicole saw the 3 × 1 rectangle as a means of additively figuring out the area of the rectangle. When placed on the left as a dimension, it stands for a linear unit; however, in the process of "finding the area of the rectangle," it behaves like an areal unit. In other words, depending on the context, the quantity with numerical value "3" could emerge as a linear unit or an areal unit for Nicole.

Later on, Nicole referred to the known measurement units to demonstrate how she made the distinction linear vs. areal. In an attempt to find the area of the 3 by 5 rectangle, she said "When you multiply the inches by inches, you'd get inches squared. This would be 15 square inches;" i.e., she not only multiplied the values of linear quantities, she multiplied the measurement units attached to those quantities as well. Her following comment "The units are still different because it's [meaning, the area of the 1 by 15 rectangle] still now inches squared" could be used to demonstrate the equivalence of areal units because both the 3 by 5 rectangle and the 1 by 15 rectangle have the same areal value "15," and same areal unit "square inches." In the same vein, Rob referred to the word "dimension" in an attempt to distinguish between linear and areal quantities based on his figure representing a 3 × 5 rectangle, as illustrated in the following protocol.

Protocol 4.9: Rob's usage of the word “dimension.”

G: Can we say that they are different... I mean area and length?

R: Yeah because area would be square inches as opposed to just inches... 'cuz it's a different dimension...

G: How do you distinguish between these linear units you described, length and width, from the areal units, I mean the area?

R: You're gonna look in a line for your width and your length whereas for area you'd [inaudible] two dimensional and you'd do [inaudible] them together.

This is how Rob compared area and length. As can be inferred from his statement “Area would be square inches as opposed to just inches...'cuz it's a different dimension,” he not only attached the known measurement units inches and square inches, but he provided a reason of his own for doing so. His reasoning involved the usage of the word “dimension” in an attempt to explain that the linear and areal quantities are of different nature. In subsequent sections, I will describe a similar usage provided by Brad who often referred to this word “dimension” in his comparison of area and length. For both students, linear and areal quantities are of one–dimensional and two–dimensional nature, respectively.

So far in this subsection, I have only talked about composite rectangles. Students dealt with areas of prime number rectangles and resulting areal units as well. Rob's description of the prime number 5 is one of those cases. Rob first made a rectangle for the number 5 (See Figure 4.3).

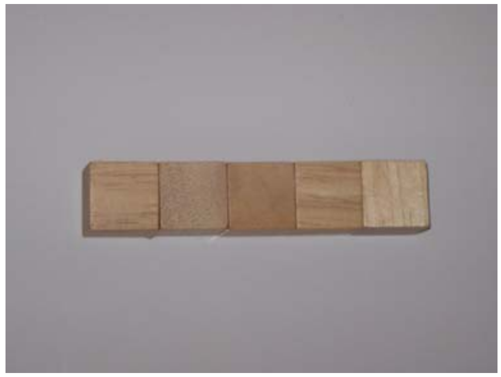


Figure 4.3. Prime rectangle representing 5.

I then asked him whether what he did looked like a linear unit.

Protocol 4.10: Rob's comparison of same–valued linear and areal quantities.

R: It's kinda... it could be linear or areal 'cuz if you wanna do a rectangle with area 5 you could also do that [pointing to the 5 by 1 rectangle he just made].

G: Okay... how am I gonna distinguish between those two then?

R: I guess... you just have to... it depends on what you are looking for... If a student was confused about it I guess you'd tell him this is just the length... you could just try to turn it to where you can only see the one dimension [See Figure 4.4]

G: If it were an area?

R: An area? Then you could just say... you could use one unit [See Figure 4.5] and say this is 1 by 1... this is 1 unit... what is the total area of all this if there is five of these all stuck together? It would be 5 by 1. [Rob fixes one dimension and iterates the other by adding ones].

Figure 4.4. Rob's hand gesture showing where the linear units are.

Figure 4.5. Rob's reference to the irreducible areal unit before iteration.

Rob's behavior is consistent with what I previously said about his reference to one and two–dimensional characteristics of linear and areal units. In effect, the one–dimensional character of the linear unit (the length of the rectangle) is evident in both his statement "just try to turn it to where you can only see the one dimension" and his hand gesture in Figure 4.4. Moreover, the two–dimensional nature of irreducible areal unit can be inferred from his usage "you could use one unit and say this is 1 by 1... this is 1 unit." In fact, this is how Rob defined the area of one square as 1 by 1, i.e., as the ordered pair (1, 1) of linear units with the relational notation. The link between the irreducible areal unit and the total area of the rectangle is provided by Rob's iteration strategy. In other words, Rob iterated his irreducible areal unit five times to obtain a 5 by 1 areal unit. This iteration strategy is reminiscent of quantitative reasoning skills of an individual who pays attention not just to the values and the units of the quantities being operated on, but to the arithmetic operation by which another quantity of the same structure is born. The 1 by 1 irreducible unit can be iterated five times and added together because each individual irreducible unit as well as the resulting "born quantity" are of the same nature, namely like areal quantities.

There is one more significant episode from the interview with Rob in which he tentatively connects his ideas about one– and two–dimensional characteristics of linear and areal quantities to the known measurement units inches and square inches. These ideas came to life when I first asked Rob to make a rectangle for the number 7. He then wrote his answers on the activity sheet. On the "Area of its rectangles as a sum" column, he wrote "$1in^2 + 1in^2 + 1in^2 + 1in^2 + 1in^2 + 1in^2 + 1in^2$"and I then asked him why he decided to write it that way. The following discussion took place.

Protocol 4.11: Rob's association of dimensionalities to known measurement units.

G: Why did you decide to write it that way?

R: Because if you just did 1in + 1in + 1in + 1in + 1in + 1in + 1in it would just be like... the line... and since this is two–dimensional... rectangles are two–dimensional figures so you need to put the square on it [meaning on inches].

Rob's ideas are more advanced than Nicole and Brad's in that he not only correctly uses the terminology related to dimensionalities and known units of measurement, but he explicitly shows how the phrases he choose to use are linked together. I term Rob's strategy as an *Association of Dimensionalities to Known Measurement Units* in an attempt to demonstrate the distinction between linear and areal quantities.

Sarah, too, started our discussion on prime and composite numbers by making a 3 by 5 rectangle for 15. I then asked her whether the dimensions of this rectangle are lengths or areas. She said that they are lengths. She then added:

Protocol 4.12: Sarah compares linear and areal fives.

S: But if you were to split it [meaning the rectangle of 15] like that [See Figure 4.6], 5 would be the area of that [pointing to one of the areal five units] rectangle.

G: How do you distinguish between this 5 [pointing the areal 5 in Figure 4.6] and the 5 which is one of the dimensions?

S: Looking at the outside... [pointing to and counting the edges] one, two, three, four, five... that would be the length.

G: Are you counting the cubes or...

S: The sides of the cubes...

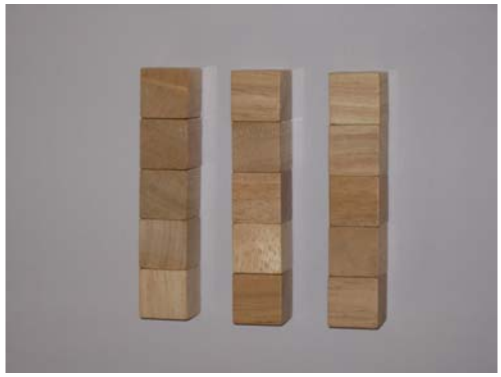


Figure 4.6. Sarah's splitting of areal 15 into areal fives.

Sarah was the first one to be explicit by using the term “sides” to demonstrate the length is a linear unit. Her use of “outside” corroborates the one–dimensional characteristic of the length, as well. Her splitting of the areal 15 into three equal areal fives supports my hypothesis above on likeness of the quantities 15 and 5 both being of areal nature. In other words, 5, if areal, needs to be part of the area of the rectangle. On the other hand, 5, if linear, has to be along a line somewhere on the sides of the rectangle made of cubes.

I conclude this subsection with John's understanding and sense–making of linear and areal units. John, when asked to write his answers on the activity sheet pertaining to the 4 by 7 rectangle of wooden cubes he made, often referred to known measurement units to distinguish between same–valued linear and areal quantities. The following protocol in which I asked him to compare linear and areal “7”s illustrates this use of units.

Protocol 4.13: John compares linear and areal sevens.

G: Are they the same "7"s? [pointing to dimension 7, and to "equal addend" 7 on the activity sheet, respectively]

J: Yeah... put inches here... [writes "in" next to dimension 7, and to each "equal addend" 7 on the activity sheet] They're both lengths...

G: What is this 7 here... or this 7 here... an area or a length? [pointing to first and third row of the 4 by 7 rectangle on the desk]

J: Um... These must be inches squared [and then changes the units of the "equal addend 7"s from in to in^2 on the activity sheet]... So I guess they're different...

G: So... where is the other 7? [meaning the dimension 7] Could you use your index finger and show me where it is? The other 7 I mean the dimension...

J: So I guess the dimension would be [moving the pen along the top edge of the 4 by 7 rectangle. See Figure 4.7] right here...

G: What do you mean... like... where is it?

J: If you take a ruler then... you'd find... this is length of this side is 7 inches [once again moving the pen along the top edge of the 4 by 7 rectangle] whereas these blocks [pointing to the first row and moving the pen along the row. See Figure 4.8] have an area of 7 inches squared.

G: So are they different? Or are they the same? How are they different and how are they the same?

J: Well... they have the same number [realizes that both quantities have the same value]... they're different because... one is inches and one is inches squared... so... basically... this one is an area and this one is a length.

Figure 4.7. John's hand gesture indicating the one–dimensionality of linear seven.

Figure 4.8. John's counting the square inch blocks adding up to areal seven.

It appears that at the beginning of the protocol, John does not realize that the “7”s are different quantities, although they have the same value. I hypothesize that at that stage of the discussion, John was not reasoning quantitatively; as he was focusing on the “value” component of a quantity, rather than taking into account both elements, namely value and unit. This is not a surprising result because John was asked to compare two

“7”s, which seem like they should be the same. Upon my further probing, however, John relied on the known measurement unit “square inches” to deduce that the seven that is part of the area should be of areal nature. This realization allowed him to conclude that the linear and areal sevens should be of different types. He supported his ideas by his statement “if you take a ruler,” suggesting that he sees the linear 7 as some sort of measurement, which indicates the one–dimensional characteristic of the linear seven. The two–dimensionality of the areal seven was not made that explicit, though.

His statement “they have the same number” indicates his realization that both quantities have the same value. From his final comment “they're different because... one is inches and one is inches squared... so... basically... this one is an area and this one is a length” I deduce that he is referring to known measurement units to distinguish between same–valued linear and areal quantities, which was the case for Brad and Nicole, as opposed to Rob and Sarah who focused on the dimensionalities. I complete this subsection by presenting a table of terminology that summarizes students' behaviors during their attempts to understand and make sense of linear and areal quantities as well as same–valued linear and areal quantities.

Table 4.6

Terminology Summarizing Students' Sense Making of Linear and Areal Units

Terminology Summarizing Students' Behaviors	**Students Fitting the Terminology**
Reference to Known Units of Measurement	Brad, Nicole, Rob, John
Equivalence of Areal Units	Nicole
Conservation of Quantitative Units	Nicole
Quantitative Reasoning	Nicole, Rob
Dimensionalities Made Explicit for Both Linear and Areal Units	Rob

Iteration Strategy Combined with the Two–Dimensional Nature of Irreducible Areal Unit	Rob
Association of Dimensionalities to Known Measurement Units	Rob
Dimensionality Made Explicit for Linear Units	Sarah, John
Splitting Strategy Combined with the Two–Dimensional Nature of Equal (Areal) Addends	Sarah
Interpretation of Linear Units as a Measurement	John

4.2. Summing Counting Numbers

I define a summed number to be the result of the summation of terms in a number sequence. Counting numbers, odd numbers, even numbers, triangular numbers, pentagonal numbers, and Fibonacci numbers constitute examples of such number sequences. Summed numbers can be represented by a sequence of growing rectangles, each made of a summed number of color cubes. The findings reported in this second section of Chapter 4 encompasses a wider range of Additive Representational Unit Coordination (ARUC) types, which are obtained from the representational subunits corresponding to each counting number as well as from those corresponding to bigger units made of these subunits. The bigger units are the growing rectangles, which represent the sum of the counting numbers.

The common direction for all the interview students was to represent the counting numbers 1, 2, 3, ... using a different color for each number and add them so that they generate a rectangle. They were also asked to write their answers on the activity sheet, which was aiming at organizing information. All students paid attention to the “add them so that they generate a rectangle” direction and came up with a similar sequence of growing rectangles. The 8th growing rectangle of the sequence, for instance, looked like this (Figure 4.9).

Figure 4.9. The 8th growing rectangle of the sequence.

4.2.1. Multiplicative Representational Unit Coordination (MRUC)

Brad made rectangles for the first 8 counting numbers using the color cubes on the white board. He used a different color for each counting number, as depicted in Figure 4.10 below.

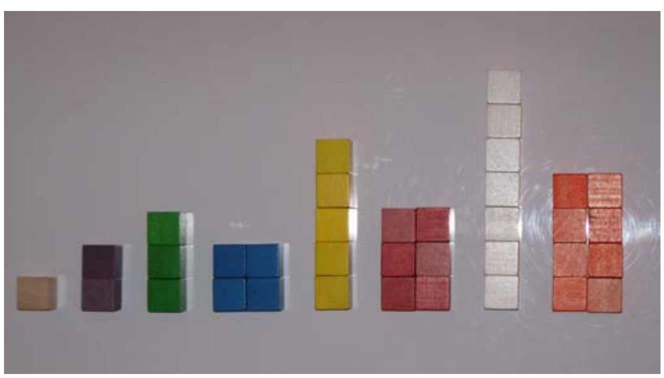


Figure 4.10. Brad's subunits representing counting numbers.

Brad said that the figures he made are all areas, and not lengths. His subunits, therefore, could be represented via relational notation as (1, 1), (2, 1), (3, 1), (2, 2), (5, 1), (2, 3), (7, 1), (2, 4); which is the essence of multiplicative RUC. He said the odd integers 1, 3, 5, 7 are prime and added "You could see that 'cuz they are straight up; they cannot make a rectangle." He probably meant they cannot be made into a rectangle other than the 1 by *n* rectangle. When asked about the even numbers, he said that 2 is prime and added "For 4, you have a square, you stick 2 at top, you get a six, eventually you'll get a taller rectangle." This language will be analyzed in more detail in the next subsection as it is of the additive type. However, for purposes of this section, I note he is relying on some sort of iteration by twos to obtain his sequence of even numbers.

I then asked him to add his counting numbers so that they generate a growing rectangle. He then produced a figure similar to the one depicted in Figure 4.9 above. His written expressions for the "area of the growing rectangle as a product" column on his recording sheet are all of multiplicative type and thus calls for the relational notations of ordered pairs (1, 1), (1, 3), (2, 3), (2, 5), (3, 5), (3, 7), (4, 7), and (4, 9) of linear units.

I asked Nicole to make her figures for the growing rectangle sequence, step by step, and at the same time write her answers on the activity sheet. I will describe the different stages of growing rectangle formation Nicole chose to use, including the figures she made, as there are some data corresponding to those stages which can be modeled using a multiplicative type RUC. At the third stage, namely when the numbers add to 6, Nicole realized there would be more than one rectangle. She decided then to go on with the "more interesting" formation, rather than the $1 \times n$ representation. In fact, toward the end of the discussion on summing counting numbers, Nicole pointed out that all rectangles representing summed numbers contain a number of cubes which is composite. Although the growing rectangles at the first and the second stages represent prime numbers, those after the third stage all represent composite numbers. I assume Nicole made a generalization based on the growing rectangle sequence after the third stage. The picture below shows Nicole's growing rectangle sequence for the first 4 stages (Figure 4.11).

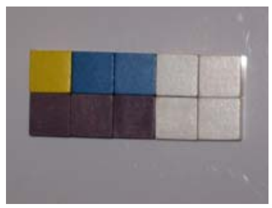


Figure 4.11. Nicole's growing rectangle corresponding to the 4th stage.

She then said "We're gonna make this prettier... rearranging..." By "this," she meant the growing rectangle at the 5th stage. In other words, she wanted to rearrange her previous figure at the 4th stage. Figure 4.12 below shows how she redesigned her previous figure and obtained her 5th growing rectangle in the sequence.

Figure 4.12. Nicole's growing rectangle corresponding to the 5th stage.

The following protocol captures the conversation that followed.

Protocol 4.14: Nicole's description of subunits and growing rectangles.

N: Stage number 5... tiles added, 5. [When it comes to write an answer for the "Dimensions of the added rectangle" column on the activity sheet, she stops and says]. Oh... I did not actually add a rectangle... do I have to?

G: Okay... Let's go on with this figure [I encourage her to continue with this formation]. What did you add? That's perfectly fine with me... What did you add and what did you change?

N: I took the 4 from here [meaning from her previous figure] and moved them down here because with 15... [hesitant] I guess I could have...

G: Let's continue with this. You changed the areal unit [meaning the 2 by 2 white rectangle representing 4] right? It was a rectangle and you changed it to a what? You changed it to an L–shape right?

N: Hm hm...

G: This one is neither a rectangle, what is it? [pointing to the red figure representing 5. See Figure 4.12 above]

N: I don't know the name for it.

I intervened a lot, trying to help Nicole realize that the added subunits, namely the L–shapes could not be expressed as a product of two linear units. Nicole was unable to give a name for these subunits, either.

Protocol 4.15: Nicole's MRUC for a particular case.

G: Okay, how about the white thing as a product?

N: I mean... I know it's still 4 times 1. But you can't see that visually.

G: Why is that?

N: Because it's not a rectangle we don't have an easy formula for describing it.

G: Okay... So... Can we say that if it's not a rectangle... if it's an irregular shape... could it still be expressed as a sum?

N: Hm hm...

G: As a product?

N: Yeah.

G: How? If it's not a rectangle, could it still be expressed as a product?

N: Not easily.

G: Tell me more about that.

N: I mean... if I was trying to use a product to find this, I would think of it [meaning the white figure] just like that [changing the white figure into a 1 by 4 rectangle. See Figure 4.13 below] and it would be 1 times 4.

Figure 4.13. Nicole's MRUC for the 1 by 4 subunit.

In other words, the rectangle concept lies at the heart of MRUC, as can be inferred from Nicole's comment “I know it's still 4 times 1. But you can't see that visually,” which can be expressed as the relational notation of ordered pair (4, 1) of linear units. She then placed the white unit cube in its original position as in Figure 4.12 above. She seemed unhappy with these irregular figures and decided to go on with the pattern she was following before. She then made the pattern corresponding to the 5^{th} and the 6^{th} stages as depicted in Figure 4.14 below.

Figure 4.14. Nicole's MRUC for all subunits at the 6^{th} stage.

I then invited Nicole to discuss the subunits corresponding to odd and even numbers on her growing rectangle. Nicole said that the odd numbers are all represented with straight lines. Later on, she added that the even numbers all have length greater than 1, which made me think that she was beginning to think in a multiplicative way, as reflected in the following protocol.

Protocol 4.16: Nicole's MRUC of even number subunits.

G: What is common about the even numbers?

N: They are all by 2. Because in all even numbers 2 is a divisor or factor.

G: How would you describe the area of this big rectangle as a product?

N: I would say three [points to the same three cubes on the left] times seven [points to the same seven cubes at the bottom]. Three inches times seven inches would give me 21 inches squared.

From this protocol, I deduce that the area of the growing rectangle, in the context of MRUC, can be expressed via the relational notation of ordered pair (3, 7) of linear units, as described by Nicole's usage of "three times seven." Nicole was able to see the multiplicative nature of the area of the growing rectangle for the special case corresponding to the 6^{th} stage; however, she could not generalize this for all growing rectangles of the sequence. As for the subunits, she specified only one of the dimensions, as can be noted from her comment "They are all by 2. Because in all even numbers 2 is a divisor or factor." This language calls for a relational notation of the form (2, ·) where the dot "·" represents the missing unspecified linear unit with value "half the even number."

Similarly, Brad was not able to generalize the linear unit corresponding to "any even number." Brad showed how he would obtain the even numbers by relying on an iteration technique as I described above. In the following paragraphs, I will show that some of the interview students were indeed able to generalize their conjectures about the dimensions for both subunits and the growing rectangles of the sequence.

As for Rob, I first asked him to make a rectangle for each counting number using a different color cube and then add them so that they generate a growing rectangle. I then asked Rob about the pattern. He said that every time you get an odd number, you can put it below the growing rectangle. The following protocol illustrates this point.

Protocol 4.17: Rob's description of the growing rectangle sequence: Bridge connection between consecutive subunits.

R: The next even number will add two rows to what... see... when we had three... [meaning when he added the odd integer three]... it was three by two [meaning the growing rectangle]. So you add two by two which is four, you get two more rows [meaning, two more cubes right next to the odd integer three] and it makes it [inaudible] to get a five. Now you have five, and for the next odd number you are gonna need a seven, which is why you add a 2 by 3 rows [meaning the two extra cubes will come from the even number 6].

Rob realizes that each even number subunit of the sequence serves as a bridge that connects the two consecutive odd number subunits. I can also infer that, just by looking at his figure, Rob knows that the difference between any two consecutive odd integers is 2, and that, “that 2” is provided by the even integer subunit that is placed between these consecutive odd integers. Rob was the only student to make use of this strategy, which I name *Bridge Connection between Consecutive Subunits*. In Thompson's words, “To reason quantitatively is to reason about quantities, their magnitudes, and their relationships with other quantities.” (Thompson, 1988, p. 164) Rob's *Bridge Connection between Consecutive Subunits* strategy has a strong indication of the quantitative reasoning described by Thompson. Rob's subunits are of a multiplicative (and additive) nature and hence can be described via a multiplicative type relational notation. Rob was also very explicit about how these quantities exist on their own as well as in relation to their neighbors–or how they serve as a bridge at each step. It is because of these bridges that all subunits and the growing rectangle made of those subunits come to exist as quantities for Rob.

I then asked Rob what is common about odd numbers and their "area as a product" because I wanted him to make a generalization for the odd numbers. He said that all the odd numbers can be represented by a rectangle whose dimensions are 1 by the odd number itself. Rob's generalization about odd number subunits could be expressed as a relational notation of ordered pair $(1, n)$ of linear units. Recall that Nicole described the odd integers simply as "long sticks," which was lacking a multiplicative nature. Brad also did not make a generalization about the dimensions of odd numbers. Brad and Nicole were alike in that they were successful in providing a multiplicative type RUC for particular cases, though. In contrast, Rob recognized the MRUC for subunits standing for both odd and even numbers. In fact, when I asked him what is common about even numbers, he said that they are all split in two columns. The following protocol captures the conversation that followed.

Protocol 4.18: Rob's MRUC concerning even number subunits.

R: They all have a width of 2 and their length is half of their amount.

G: So… how would you express all these even numbers as a product?

R: Like… 2 by one half of the number...

G: How about the units?

R: It'd be inches [meaning the dimensions would be in inches].

G: Their area as a product?

R: The area would be 2 inches by half of it [meaning the even number itself].

Unlike Nicole and Brad, Rob was able to describe a MRUC not just for particular cases but for any subunit standing for an even number in the sequence. His language of "2 by one half of the number" calls for a relational notation of ordered pair $\left(2, \frac{n}{2}\right)$ of linear units where *n* stands for any even number of the sequence. Later on, Rob noted that the growing rectangle represents a composite number because it is not just one long line. He also said that if the pattern continued the number of cubes in the rectangle would always be a composite number. I then asked Rob to focus on the summation identity at each step and explain the pattern for the "area of the growing rectangle as a product" column on the activity sheet. Below I provide one more protocol in which Rob provides a general expression for the dimensions of the growing rectangle at any stage.

Protocol 4.19: Rob's MRUC concerning the growing rectangles.

R: These numbers are used twice. Your first number is gonna increase by 1 every time... or every other time... and your second number is gonna increase by 2, every other time... [about his answers 2 × 3, 2 × 5, 3 × 5, 3 × 7, 4 × 7, 4 × 9, on the activity sheet]

G: How about these numbers [meaning the numbers in the list 2 × 3, 2 × 5, 3 × 5, 3 × 7, 4 × 7, 4 × 9] are they in inches?

R: They are in inches.

Rob's answers for the “area of the growing rectangle as a product” can be modeled with a relational notation of ordered pairs (2, 3), (2, 5), (3, 5), (3, 7), (4, 7), and (4, 9) of linear units, respectively. Rob emphasized the linearity of these units with his statement “They are in inches.” Rob's description of the linear units existing on their own as well as in relation to each other (Thompson, 1988) warrants once again my hypothesis about his quantitative reasoning. His language “These numbers are used twice” supports Thompson's description of quantities existing on their own. “Quantities existing in relation to each other” can be witnessed in his statement “Your first number is gonna increase by 1 every time....or every other time… and your second number is gonna increase by 2, every other time.” In the following paragraphs of this subsection, I will show how Sarah and Rob are alike and different.

Sarah first made the following patterns for the first eight counting numbers on the white board (Figure 4.15). She then described the areas of these subunits as a product, as depicted in the protocol below.

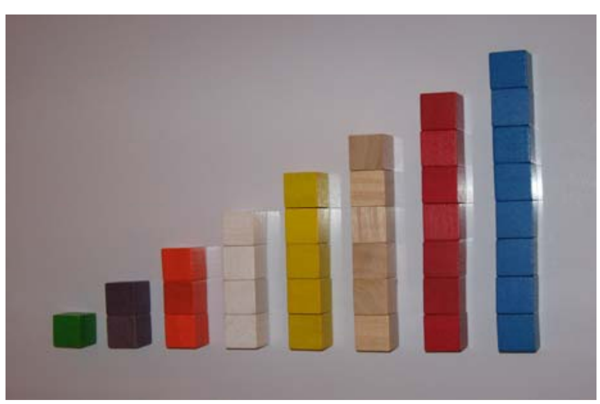


Figure 4.15. Sarah's subunits representing counting numbers.

Protocol 4.20: Sarah's MRUC concerning subunits on their own.

G: What are these? Could you describe these? Are they length or area? What do you think?

S: You could look at them as rectangles and each square as 1 for unit. And then... This would be... the area would be 1 [pointing to the green cube] and then this would be 2 times 1 [pointing to the purple rectangle] and 3 times 1 [pointing to the orange rectangle] 4 times 1 [pointing to the white rectangle] 5 times 1 [pointing to the yellow rectangle] and 6 [pointing to the wood rectangle] 7 [pointing to the red rectangle] and 8 [pointing to the blue rectangle]

G: So... you are looking them as area? Am I right?

S: Right.

G: Are these rectangles? What are the dimensions?

S: Length times width.

G: Is there a general pattern for these numbers?

S: It would just be the number of squares.

G: Great! The next step is to add them so that they generate a rectangle.

Every time she said the dimensions of a number rectangle, she pointed to both the length and the width of the rectangle with her index finger. Therefore, I infer that she is able to look at these numbers as "areas as products," namely of MRUC type. A general relational notation expression for her description "length times width" could be written as the ordered pair (length, width) of linear units. The multiplicative nature of her subunits could also be modeled via a relational notation of ordered pairs (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (7, 1), and (8, 1) of linear unit. As a brief summary, Sarah was able to identify the areas of these "long sticks" as a product, as opposed to Nicole who just described them as "long sticks," which was lacking a multiplicative nature.

In the process of adding the subunits to generate a rectangle, in particular at the 4th step, when adding the white cubes, Sarah changed the original 1 by 4 configuration to a 2 by 2 formation (Figure 4.16). When I asked why she changed the configuration, she noted that she could not add the 1 by 4 stick to the existing rectangle and retain the new shape as a rectangle (Figure 4.17).

Figure 4.16. Sarah's growing rectangle at the 4th stage.

Figure 4.17. Sarah's visual proof by contradiction.

In other words, as warranted by both her statement and figure, the rectangle concept plays a crucial role in Sarah's MRUC. Nicole and Sarah are alike that way, in that both students relied on some sort of "visual proof by contradiction" to demonstrate how the multiplicativeness of the RUC and the rectangle concept necessitate each other. At the 6th step, a similar discussion took place, and Sarah relied on a similar contrapositive visual proof. I then let Sarah complete her figure (Figure 4.18). The following protocol picks up at this point.

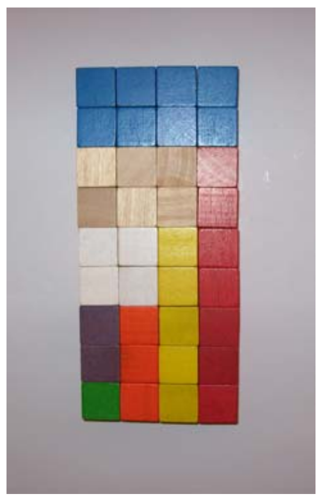


Figure 4.18. Sarah's complete figure.

Protocol 4.21: Sarah's MRUC concerning subunits in relation to each other.

G: It looks like you discovered a pattern.

S: Every even number I added, I had to split it in two.

G: And how about the odd numbers?

S: The odd numbers worked out… they were the same as the height...

G: This six [pointing to the wood rectangle in Figure 4.18]... is this an area or a length?

S: It's an area.

G: What are its dimensions?

S: 2 by 3.

G: How about this one [pointing to yellow] is it an area or a length?

S: An area.

G: As a product?

S: 5 times 1.

G: For the wood?

S: 3 times 2.

G: For the white?

S: 2 times 2.

G: The blue?

S: 4 times 2.

G: The red?

S: 7 times 1.

G: This 5 [pointing to yellow] looks like a length. Does not that bother you?

S: I see it as an area. The only reason is because of the blocks.

As can be inferred from her statements, quantities standing for odd numbers conserved their dimensions as well as their shapes. However, Sarah had to split the subunits standing for even numbers into two, which is to say that these even number subunits conserved neither their dimensions nor their shapes. Sarah was very clear in providing answers of a multiplicative nature for all these subunits. Her language maps to a relational notation of ordered pairs (2, 3), (5, 1), (2, 2), (4, 2), (7, 1) of linear units. Sarah and Rob are alike in that their RUC is of multiplicative type and in that these representational quantities are free to exist on their own. However, Sarah lacks an in–depth analysis showing how these subunits are related to each other, as opposed to Rob who provided a clear description of even number subunits serving as a bridge linking any two consecutive odd number subunits.

Finally, I describe John's MRUC treatment of the subunits and the growing rectangles. I first asked him to make a rectangle using a different color for each counting number in the sequence 1, 2, 3, 4, 5, 6. Though I did not ask him to add them yet, he came up with the following long stick formation as depicted in Figure 4.19 below. The protocol below illustrates this point.

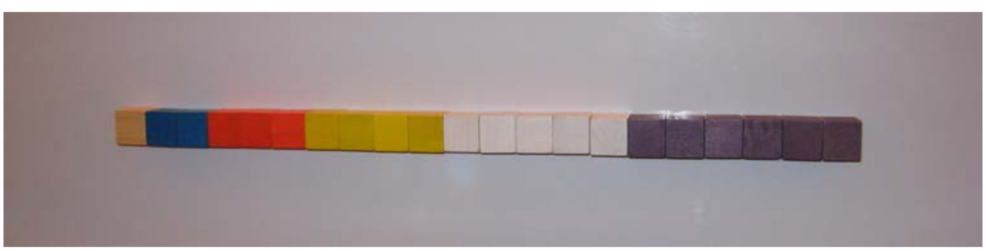


Figure 4.19. John's subunits representing counting numbers.

Protocol 4.22: John's MRUC and mapping structures concerning the subunit 5.

G: 5 for instance... is it a length or an area?

J: I guess area...

G: How do you figure?

J: Length of 1 and width of 5.

G: Tell me more about that...

J: I guess... Length of 5 and width of 1 in which case the area would be 5…

John's language "length of 1 and width of 5" could be described via a relational notation as an ordering of linear units of the form (1, 5). This usage is of multiplicative type and hence necessitates the multiplicative RUC construct. His latter comment "length of 5 and width of 1 in which case the area would be 5" calls for a slightly different analysis model, which I call "Mapping Structures." For John, I infer that the ordered pair (1, 5) that consists of linear units, is being mapped into 5, which is an areal quantity. In other words, this strategy corresponds not only to a multiplicative type RUC, but it makes the operation under consideration, namely multiplication, behave as a mapping operating on linear quantities from which a quantity of a totally different nature comes to exist: an areal quantity. All these can be summarized with a functional notation[9] such as f: (linear 1, linear 5) → areal 5, or with the equality as f (linear 1, linear 5) = areal 5, where f stands for the multiplication operation that behaves like a function or a mapping.

I then asked John to add his subunits so that they generate a rectangle. He added the first 8 counting numbers and completed his figure (See Figure 4.9 above). He said that he left 2, 3, 5, and 7 as they were before (e.g., long sticks in Figure 4.19) but changed 4, 6, and 8 to rectangles with one dimension of 2. He noted that the even numbers are rectangles of length 2 and width that goes up as 1, 2, 3, 4..., i.e., half the even number. In

[9] The equality notation will be omitted in the remainder of the manuscript.

that sense, John was the only interview student to make use of the sequence $a_n = \{n\} \equiv \{1, 2, 3, 4, ...\}$ to describe the multiplicative nature of the even number subunits. John's even number subunits can be modeled sequentially via a relational notation of ordered pairs $(2, a_n)$ of linear units. The following protocol captures the conversation that followed.

Protocol 4.23: John's MRUC concerning even number subunits.

G: Can you come up with a general formula for the even numbers on this growing rectangle... the length and the width of any even number?

J: The length will always be 2... So 2 times *n*? [hesitant]

G: What is that *n*?

J: *n* is just the number going up... 1... 2... so...

G: How about 6... is it area or length? How do you see it?

J: An area.

This is the first time a student used a variable, *n*, to describe a linear unit. Though unspecified, I believe that John would represent his even numbers as $2n$. Once again, though unspecified, his statement "*n* is just the number going up... 1... 2... so..." causes me to believe that he thinks of these linear units with values of half the even numbers in the sequence. One could assign a relational notation of ordered pairs $(2, n)$ of linear units standing for John's MRUC for any even number rectangle with the constraint that the "any even number" is of value $2n$.

Sarah, Rob and John are alike in that their RUC for the even number subunits are of multiplicative type and in that these representational quantities are free to exist on their own. Rob was able to go a step further in that he demonstrated how the representational quantities standing for odd and even number subunits exist on their own as well as in relation to each other. On the other hand, John was the only student to make use of a sequence to model the linear units describing an even number subunit. Also, John, was the only one to provide a variable, *n*, standing for his linear unit of his choice. The table below summarizes each student's answer and a corresponding relational notation that describes multiplicative type RUC.

Table 4.7

Relational Notation Describing Students' Answers

Students	**Phrases**	**Relational Notation**
Brad	NA	NA
Nicole	"...it would be 1 times 4" "I would say three times... Three inches times seven inches would give me 21 inches squared" "They are all by 2. Because in all even numbers 2 is a divisor or factor"	(1, 4) (3, 7) (2, ·)
Rob	"...it was three by two" "all the odd numbers are 1 by the odd number itself" "They all have a width of 2 and their length is half of their amount" "Like... 2 by one half of the number" "The area would be 2 inches by half of it"	(3, 2) (1, n) (2, n / 2) (2, n / 2) (2, n / 2)
Sarah	"...and then this would be 2 times 1" "...and 3 times 1" "4 times 1" "5 times 1" "Length times width" "2 by 3" "5 times 1" "3 times 2" "2 times 2" "4 times 2" "7 times 1"	(2, 1) (3, 1) (4, 1) (5, 1) (Length, Width) (2, 3) (5, 1) (3, 2) (2, 2) (4, 2) (7, 1)
John	"Length of 1 and width of 5" "Length of 5 and width of 1 in which case the area would be 5" "...even numbers are rectangles of length 2 and width that goes up as 1, 2,	(1, 5) (5, 1) (2, a_n)

	3, 4..." "The length will always be 2... So 2 times *n*?"	(2, *n*)

Each student wrote his / her answer on an activity sheet related to each activity. Students' written answers were consistent with what they were saying during the interviews. The following two tables reflect students' written answers for the area as a product of the even and odd number subunits, and the growing rectangles, respectively. I am including once again a relational notation for each written answer for the purpose of analysis.

Table 4.8

Students' Written Work for the Area of the Odd/Even Number Subunits as a Product

Students	Area of the Odd/even Number Subunit as a Product	Relational Notation Describing the Product
Brad	2 × 1, 3 × 1	(2, 1), (3, 1)
Nicole	1 × 1, 1 × 2, 1 × 3, 2 × 2, 1 × 5, 3 × 2	(1, 1), (1, 2), (1, 3), (2, 2), (1, 5), (3, 2)
Rob	1in × 2in, 1in × 3in, 2in × 2in, 1in × 5in	(1, 2), (1, 3), (2, 1), (1, 5)
Sarah	5 × 1	(5, 1)
John	2 × 1, 3in × 1in, 2in × 2in, 5in × 1in, 2in × 3in, 7in × 1in *n* odd: *n* in × 1 in *n* even: 2 in × (*n* / 2) in	(2, 1), (3, 1), (2, 2), (5, 1) (2, 3), (7, 1) (*n*, 1) (2, *n* / 2)

Table 4.9

Students' Written Work for the Area of the Growing Rectangles as a Product

Students	Area of the Growing Rectangle as a Product	Relational Notation Describing the Product
Brad	1 × 1, 1 × 3, 2 × 3, 2 × 5, 3 × 5, 3 × 7, 4 × 7, 4 × 9	(1, 1), (1, 3), (2, 3), (2, 5), (3, 5), (3, 7), (4, 7), (4, 9)
Nicole	1 × 3, 2 × 3, 2 × 5, 3 × 5, 3 × 7	(1, 3), (2, 3), (2, 5), (3, 5), (3, 7)
Rob	1in × 3in, 2in × 3in, 2in × 5in, 3in × 5in	(1, 3), (2, 3), (2, 5), (3, 5)

Sarah	3in × 5in	(3, 5)
John	3in × 1in, 3in × 2in, 5in × 2in, 5in × 3in, 7in × 3in, 7in × 4in n odd: n in × · in n even: (n – 1) in × (n / 2) in	(3, 1), (3, 2), (5, 2), (5, 3) (7, 3), (7, 4) (n, ·) (n – 1, n / 2)

As can be seen in the written work, once again Rob and John were the ones who attached some sort of units to their answers. Sarah, too, wrote inches next to her single answer for the area of the growing rectangle as a product column. Rob and John attached these known measurement units to their answers to emphasize the linearity of the quantities involved. Reasoning quantitatively, one may feel the need to specify some sort of measurement unit as this is one of the three crucial components that makes quantities what they are (Schwartz, 1988). When it comes to using a relational notation, however, one does not need to attach a measurement unit to the ordered pairs of linear units because each component of the ordered pair by itself is a linear unit.

Another point worth mentioning is John's use of a variable for a general case in his description of both even and odd number subunits and growing rectangles. His linear units for the subunits are correct; however, his linear units for the dimensions of the growing rectangles are not quite right. First of all, he specified only one of the linear units corresponding to the general case where n is an odd integer, which is equivalent to (n, ·) with the relational notation. And the correct expression for area as a product corresponding to the general case where n is even is (n + 1) in × (n / 2) in. In other words, he only had a plus/minus sign error.

I will end this subsection with a table of terminology that summarizes students' behaviors reflecting multiplicative type RUC.

Table 4.10

Terminology Summarizing Students' Behaviors Reflecting MRUC

Strategy / Terminology	**Students**
Multiplicative RUC for Special Cases of Odd/Even Number Subunits and Growing Rectangles.	Brad, Nicole, Rob, Sarah, John
Use of the phrase “more interesting.” Favoring for the “more interesting” formation representing a composite number over “long stick” formation.	Nicole, Rob
Realization of the fact that the added L–shape subunits could not be expressed as a product of two linear units.	Nicole
Inconclusiveness of MRUC from usages “long sticks,” “straight lines.”	Nicole
Mapping Structures arising from usages “Three inches times seven inches would give me 21 inches squared,” “Length of 5 and width of 1 in which case the area would be 5.”	Nicole, John
Only one of the linear units corresponding to an even number subunit is specified ≡ (2, ·)	Brad, Nicole
Both linear units corresponding to an even number subunit is specified.	Rob ≡ (2, half the number) John ≡ (2, *n*)
Both linear units corresponding to an odd number subunit is specified.	Rob ≡ (1, the odd number itself)
Bridge Connection between Consecutive Subunits. Reasoning about quantities, their magnitudes, and their relationships with other quantities.	Rob
“If the pattern continues it is always going to be a composite number.” (Rob) “They all make composite numbers.” (Nicole). At steps 1 and 2; the number is prime and after the third step, it is composite (Sarah).	Rob, Nicole, Sarah
Identify the areas of “long sticks” as a product.	Sarah
Visual Proof by Contradiction to demonstrate how the multiplicativeness of the RUC and the rectangle concept necessitate each other.	Nicole, Sarah
Reasoning about situations in terms of quantities and quantitative operations.	John
Uses a variable to describe a linear unit ≡ (2, *n*).	John
MRUC is specified for any even number subunit rectangle with the constraint that the “any even number” is of value 2*n*.	John
Their RUC for the even number subunits are of multiplicative type and these representational quantities are free to exist on their own.	Rob, Sarah, John
Representational quantities exist in relation to each other.	Rob
Sequential Description of Linear Units Corresponding to Even Number Subunits.	John

4.2.2. Additive Representational Unit Coordination (ARUC)

In the context of summation of counting numbers, the additive nature of RUC appears as an ordered *n*–tuple of areal units, as was the case for the first activity on prime and composite rectangles. The slight modification in this section is that one has to know whether the additive RUC types corresponding to areal quantities arise from the growing rectangles or from the odd/even number subunits generating the growing rectangles. This was probably the main reason students revealed a variety of ARUC types in the context of summation formulas.

Brad's language corresponding to additive type RUC occurred when he was reasoning about the even numbers. He said that 2 is prime. He added “For 4, you have a square, you stick 2 at top, you get a six, eventually you'll get a taller rectangle.” This usage calls for an additive type of RUC in which even number subunits are generated by areal units of 2, namely *Equal Addends*. A relational notation of ordered pair [2, 2] and ordered triple [2, 2, 2] of areal units (equal addends) can be used to denote Brad's way of describing how the even number subunits are related to each other. Brad obtained his even number areal subunits by iterating areal units of “2”s every time. I hypothesize that he is reasoning quantitatively in that he not only explains how the areal quantities exist on their own but also how these areal quantities are related to each other. The concepts of quantities existing on their own and quantities existing in relation to each other lie in his iteration strategy.

Later on, we discussed the “Area of the growing rectangle as a sum” column of the activity sheet (See Appendix for the activity sheets). Brad did not rely on the summation formula; rather, he preferred "equal addends," similar to what he did for the composite numbers. In other words, the colors did not matter to him. This was a surprise to me because I initially designed this project guided by color cubes with the main purpose of helping students see the different color addends. As I will show in the following paragraphs, Rob did not care about the colors and continued to use "Equal Addends" type RUC. Rob and Brad are alike in that way. Brad was adept at using visual proofs to support his "Equal Addends" type RUC for the growing rectangles. The following figures show how he decomposed the 7th and the 8th growing rectangles into equal addends. In addition to his gestures, his explanations were very clear about his focus on equal addends (Figures 4.20, 4.21, 4.21).

Figure 4.20. Brad's decomposition of the 7th growing rectangle into an ordered pair [14, 14] of (equal addends type) areal units.

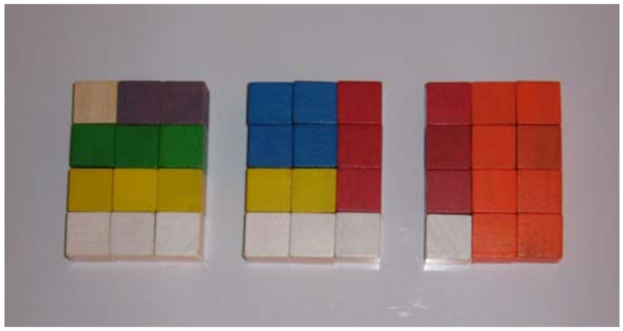


Figure 4.21. Brad's decomposition the 8th growing rectangle into an ordered triple [12, 12, 12] of (equal addends type) areal units.

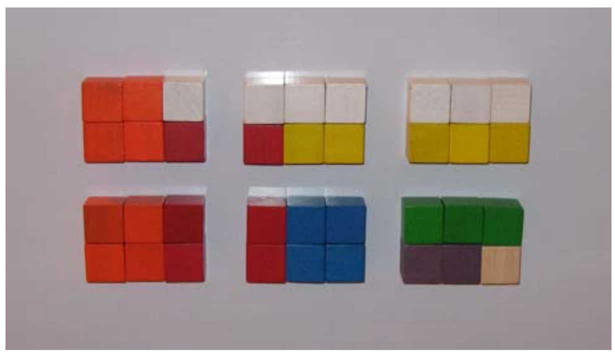


Figure 4.22. Brad's decomposition of the 8th growing rectangle into an ordered hextuple [6, 6, 6, 6, 6, 6] of (equal addends type) areal units.

Nicole's answer for the "Area of the growing rectangle as a sum" column of the activity sheet was interesting in that she treated the "growing number" as a "summed number" relying on the summation formula. In other words, Nicole presented a new additive type RUC, which I name *Summed Addends* type RUC. She also used another new additive type RUC, which I call *Recursive Addends* type RUC. She used both Summed Addends and Recursive Addends in the description of growing rectangles. As for the odd and even number areal subunits, the decomposition strategies she used are of

Irreducible Addends (Type I) and *Equal Addends* types. Nicole used a variety of ARUC types; she is unique that way. Brad and Rob, on the other hand, are alike in that they both relied solely on *Irreducible Addends* (Type I) and *Equal Addends* types in their description of both odd/even number areal subunits and the growing rectangles. Nicole used these aforementioned ARUC types only for the description of the odd/even number areal subunits.

I had interesting discussions with Sarah, too, concerning additive type RUCs. When I asked her to focus on the 5th growing rectangle she made of color cubes, for instance, she said "1 plus 2 plus 3 plus 4 plus 5" about the "area of the growing rectangle as a sum." And this answer came from a student who was sticking to a unique ARUC type, namely *Random Addends* type, in the context of Prime and Composite Rectangles. She did not hesitate a second and put her answer of *Summed Addends* type on the activity sheet very quickly. I then asked her to compare this answer of Summed Addends type with her answer for the "other 15," in the context of Prime and Composite Rectangles which was of *Random Addends* type. Recall that when we focused on the "composite number 15" for which she made a 3 by 5 rectangle made of a single color, she said that there are many different ways of expressing 15 as a sum. In particular, she provided the ordered pair [14, 1] of areal units. Now with the "summed 15," she used Summed Addends type, as can be inferred from the following protocol.

Protocol 4.24: Sarah's ARUC type concerning the subunits and the growing rectangle of the 5th stage.

S: Well... We created this rectangle [pointing to the 3 by 5 rectangle made of color cubes on the board] by adding different numbers [Figure 4.23]. The colors kinda show a way to count it.

G: How about this one as a sum? [pointing to the white]

S: 2 plus 2.

G: And many other ways?

S: Yeah.

G: For this one, eight? [pointing to the blue]

S: This is the same… four plus four... two plus two plus two plus two... one plus seven.

G: Many different ways?

S: Yeah.

G: In each one of those different ways, the addends, what units do they have?

S: Inch square.

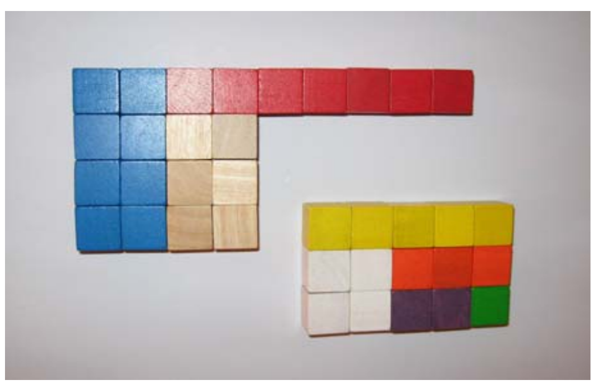


Figure 4.23. Sarah's growing rectangle of the 5th stage.

Though she continued to use Random Addends in the description of areal subunits, Sarah embraced Summed Addends in her description of the areas of the growing rectangles. Her words "The colors kinda show a way to count it" demonstrates that the color concept played the main role in her choice of Summed Addends type RUC. Therefore, I can say that, for Sarah, in the context of "Summing Counting Numbers;" 15 is made of "summed addends" whereas in the context of "Prime and Composite Numbers," it is made of "random addends."

Overall, with all five students, I observed a variety of additive type RUCs that can be described using a functional notation $\sum_{i=1}^{i=n} f(i) = g(n)$ where areal subunits "$f(i)$" are being summed from 1 to *n* (number of addends) and *i* is the stage number (ordering number for the addends). Summed Addends type RUC can be described using a functional notation such as the following: The addends of the growing rectangle are areal subunits with different shapes made of color cubes representing the "area as a sum" part of the summation formula. In the context of "Summing Counting Numbers" activity, one can write $f(i) = i, \forall i,$ for the addends corresponding to odd/even number areal subunits. As I will show below in students' written work, John also used this approach. As a brief summary, three out of five preservice teachers, namely Sarah, Nicole and John, came up with Summed Addends type RUC. Rob and Brad, on the other hand, did not care about the color shapes generating the growing rectangles. Instead, they relied on Equal Addends in expressing the area of the growing rectangle as a sum. They treated the growing rectangle as a composite number rectangle.

I will again complete this subsection with some summative tables. The first one below tabulates the additive type RUCs and a relational notation based on students' written work on areal subunits. The second table is based on the growing rectangles.

Table 4.11

Students' Written Work for the Area of the Odd/Even Number Subunits as a Sum

Students	**Area of the Odd/even Number Subunits as a Sum**	**Name of the Additive Type RUC**	**Relational Notation Describing the Sum**
Brad	1 + 1 1 + 1 + 1	Irreducible Addends Irreducible Addends	[1, 1] [1, 1, 1]
Nicole	1 1 + 1 1 + 1 + 1 2 + 2 or 1 + 1 + 1 + 1 1 + 1 + 1 + 1 + 1 2 + 2 + 2 or 3 + 3 or 1 + 1 + 1 + 1 + 1 + 1	Singleton Addend Irreducible Addends Irreducible Addends Equal Addends Irreducible Addends Irreducible Addends Equal Addends Equal Addends Irreducible Addends	[1] [1, 1] [1, 1, 1] [2, 2] [1, 1, 1, 1] [1, 1, 1, 1, 1] [2, 2, 2] [3, 3] [1, 1, 1, 1, 1, 1]
Rob	$1\text{in}^2 + 1\text{in}^2$ $1\text{in}^2 + 1\text{in}^2 + 1\text{in}^2$ $2\text{in}^2 + 2\text{in}^2$ $1\text{in}^2 + 1\text{in}^2 + 1\text{in}^2 + 1\text{in}^2$	Irreducible Addends Irreducible Addends Equal Addends Irreducible Addends	[1, 1] [1, 1, 1] [2, 2] [1, 1, 1, 1]
Sarah	NA	NA	NA
John	2 in^2 3 in^2 4 in^2 5 in^2 6 in^2 7 in^2 $n\text{ in}^2$	Singleton Addend Singleton Addend Singleton Addend Singleton Addend Singleton Addend Singleton Addend Singleton Addend	[2] [3] [4] [5] [6] [7] [*n*]

Table 4.12

Students' Written Work for the Area of the Growing Rectangles as a Sum

Students	**Area of the Growing Rectangles as a Sum**	**Name of the Additive Type RUC**	**Relational Notation Describing the Sum**
Brad	1 + 1 + 1 2 + 2 + 2 or 3 + 3 + 3 or 1 + 1 + 1 + 1 + 1 + 1 2 + 2 + 2 + 2 + 2 or 5 + 5	Irreducible Addends Equal Addends Equal Addends Irreducible Addends Equal Addends Equal Addends	[1, 1, 1] [2, 2, 2] [3, 3, 3] [1, 1, 1, 1, 1, 1] [2, 2, 2, 2, 2] [5, 5]

	or 1 + 1 + ...	Irreducible Addends	[1, 1, ..., 1]
	3 + 3 + 3 + 3 + 3	Equal Addends	[3, 3, 3, 3, 3]
	or 5 + 5 + 5	Equal Addends	[5, 5, 5]
	or 1 + 1 + ...	Irreducible Addends	[1, 1, ..., 1]
	7 + 7 + 7	Equal Addends	[7, 7, 7]
	or 3 + 3 + 3 + 3 + 3 + 3 + 3	Equal Addends	[3, 3, 3, 3, 3, 3, 3]
	1 + 1 + ...	Irreducible Addends	[1, 1, ..., 1]
	2 + 2 + 2 + ...	Equal Addends	[2, 2, 2, ..., 2]
	4 + 4 + 4 + 4 + 4 + 4 + 4	Equal Addends	[4, 4, 4, 4, 4, 4, 4]
	7 + 7 + 7 + 7	Equal Addends	[7, 7, 7, 7]
	1 + 1 + ...	Irreducible Addends	[1, 1, ..., 1]
	4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4	Equal Addends	[4, 4, 4, 4, 4, 4, 4, 4, 4]
	9 + 9 + 9 + 9	Equal Addends	[9, 9, 9, 9]
	3 + 3 + ...	Equal Addends	[3, 3, ..., 3]
	2 + 2 + ...	Equal Addends	[2, 2, ..., 2]
	1 + 1 + ...	Irreducible Addends	[1, 1, ..., 1]
Nicole	1 + 1 + 1	Irreducible Addend	[1, 1, 1]
	or 1 + 2	Summed Addends	[1, 2]
	1 + 2 + 3	Summed Addends	[1, 2, 3]
	or 3 + 3	Recursive Addends	[3, 3]
	or 1 + 1 + 1 + 1 + 1 + 1	Irreducible Addends	[1, 1, 1, 1, 1, 1]
	1 + 2 + 3 + 4	Summed Addends	[1, 2, 3, 4]
	or 3 + 3 + 4	Recursive Addends	[3, 3, 4]
	or 1 + 1 + 1 + ... + 1	Irreducible Addends	[1, 1, 1, ..., 1]
	1 + 2 + 3 + 4 + 5	Summed Addends	[1, 2, 3, 4, 5]
	or 3 + 3 + 4 + 5	Recursive Addends	[3, 3, 4, 5]
	or 6 + 4 + 5	Recursive Addends	[6, 4, 5]
	or 10 + 5	Recursive Addends	[10, 5]
	or 1 + 1 + 1 + ... + 1	Irreducible Addends	[1, 1, 1, ..., 1]
	1 + 2 + 3 + 4 + 5 + 6	Summed Addends	[1, 2, 3, 4, 5, 6]
	or 3 + 3 + 4 + 5 + 6	Recursive Addends	[3, 3, 4, 5, 6]
	or 6 + 4 + 5 + 6	Recursive Addends	[6, 4, 5, 6]
	or 10 + 5 + 6	Recursive Addends	[10, 5, 6]
	or 15 + 6	Recursive Addends	[15, 6]
	or 1 + 1 + 1 + ... + 1	Irreducible Addends	[1, 1, 1, ..., 1]
Rob	$1in^2 + 1in^2 + 1in^2$	Irreducible Addends	[1, 1, 1]
	$2in^2 + 2in^2 + 2in^2$	Equal Addends	[2, 2, 2]
	or $3in^2 + 3in^2$	Equal Addends	[3, 3]
	or $1in^2 + 1in^2 + ...1in^2 + 1in^2$	Irreducible Addends	[1, 1, ..., 1]
	$5in^2 + 5in^2$	Equal Addends	[5, 5]
	or $2in^2 + 2in^2 + 2in^2 + 2in^2 + 2in^2$	Equal Addends	[2, 2, 2, 2, 2]
	or $1in^2 + 1in^2 + ...1in^2 + 1in^2$	Irreducible Addends	[1, 1, ..., 1, 1]
	$3in^2 + 3in^2 + 3in^2 + 3in^2 + 3in^2$	Equal Addends	[3, 3, 3, 3, 3]
	or $5in^2 + 5in^2 + 5in^2$	Equal Addends	[5, 5, 5]
	or $1in^2 + 1in^2 + ... + 1in^2 + 1in^2$	Irreducible Addends	[1, 1, ..., 1]
Sarah	$1in^2 + 2in^2 + 3in^2 + 4in^2 + 5in^2$	Summed Addends	[1, 2, 3, 4, 5]
John	$1in^2 + 2in^2$	Summed Addends	[1, 2]
	$1in^2 + 2in^2 + 3in^2$	Summed Addends	[1, 2, 3]
	$1in^2 + 2in^2 + 3in^2 + 4in^2$	Summed Addends	[1, 2, 3, 4]
	$1in^2 + 2in^2 + 3in^2 + 4in^2 + 5in^2$	Summed Addends	[1, 2, 3, 4, 5]
	$1in^2 + 2in^2 + 3in^2 + 4in^2 + 5in^2 + 6in^2$	Summed Addends	[1, 2, 3, 4, 5, 6]
	$1in^2 + 2in^2 + 3in^2 + 4in^2 + 5in^2 + 6in^2 + 7in^2$	Summed Addends	[1, 2, 3, 4, 5, 6, 7]
	$1in^2 + 2in^2 + ... + n\ in^2$	Summed Addends	[1, 2, ..., *n*]

By looking at students' written work on the summing counting numbers activity, I conclude that Rob, Sarah and John emphasized the areal units by attaching an “inches squared" symbol to each addend. John, in the description of his areal subunits, relied on *Singleton Addend* type RUC. This is similar to what he did in the first activity in his expressions of prime rectangles. John is also the only one to use a variable in his general expression for both the areal subunits and the growing rectangles.

4.2.3. Linear vs. Areal Units

When obtaining his growing rectangles, Brad emphasized that at each step, the dimensions will be in inches. He also verified that at each step, the area as a sum and the area as a product of the growing rectangle coincide. I asked him to compare the 7 in "1 + 2 + 3 + 4 + 5 + 6 + 7" with the 7 in "4 × 7." The following discussion picks up at this point.

Protocol 4.25: Brad compares the linear and areal sevens.

B: This one is 7 inches [about the 7 in 4 × 7] and this one is 7 inches squared [about the 7 in 1 + 2 + 3 + 4 + 5 + 6 + 7] because you are adding the blocks individually [at the same time pointing to the blocks].

G: Okay, so what is it about this 7 [about the 7 in 4 × 7], is it a length or an area?

B: It's a length.

G: This one [about the 7 in 1 + 2 + 3 + 4 + 5 + 6 + 7], is it a length or an area?

B: It's an area.

G: How can we convince someone or the students that this 7 [about the 7 in 1 + 2 + 3 + 4 + 5 + 6 + 7] is really an area?

B: Summations... Areas... You can add areas together to create a whole area.

G: Can't you add an area to a length?

B: No.

G: But you say that you can add an area to another area?

B: Hm hm... As long as they have the same dimensions.

First of all, Brad relies on some known measurement units (inches and inches squared) to demonstrate the linearity and the areal–ness of same–valued linear and areal quantities, respectively. In fact, as can be warranted by his statement "...and this one is 7 inches squared [about the 7 in "1 + 2 + 3 + 4 + 5 + 6 + 7"] because you are adding the blocks individually [at the same time pointing to the blocks]," he is charging the word "block" with an areal meaning. Brad and Sarah are alike that way. In a sense, Brad defined his "block" as a basic areal unit by which he obtained his areal subunit 7 as well as his bigger growing rectangle unit. His language "you are adding blocks individually" and "You can add areas together to create a whole area" indicate the areal nature of 7 in the summation 1 + 2 + 3 + 4 + 5 + 6 + 7. In fact, I have also described a similar iteration strategy used by Nicole and Rob in the first section of this chapter (See Protocols 4.3 and 4.10, Figures 4.4 and 4.5).

As can be seen in the protocol above, Brad also emphasized that the areas can be added as long as they have the same dimensions. I think he probably meant that all added areas have to be of the same units, e.g., inches squared or centimeters squared. Quantities may have different names or values; however, if they were to be operated on additively, they have to be not just of the same nature, but of the same measurement units as well

(Olive & Caglayan, 2006, 2007). In that sense, I can say that Brad was aware of a *referent preserving composition* (Schwartz, 1988), the quantitative operation addition, which serves to yield a representational quantity of the same nature and measurement units: inches squared areal quantities.

In an attempt to understand her sense making of linear and areal units, I asked Nicole some questions based on her growing rectangle figure made of color cubes. The following discussion illustrates this point.

Protocol 4.26: Nicole compares the linear and areal quantities.

G: Where are the linear units? At each stage?

N: What do you mean?

G: How would you distinguish between the linear units and the areal units at each step? I mean, how do you distinguish the dimensions from the area itself? Let's just focus on the 6^{th} stage [meaning the 3 × 7 growing rectangle]. Where are the dimensions?

N: Here [points with her right index finger to the three cubes on the left, within the rectangle itself. Figure 4.24] and here [points to and carries away for a few seconds the seven cubes at the bottom, within the rectangle itself, using her both hands. Figure 4.25]

G: So... you can also see them in the rectangle itself.

N: Hm hm...

G: Okay... so... what are the units for these dimensions?

N: So... the length would be in inches [points to the same three cubes on the left. Figure 4.26] the width would be in inches [points to the same seven cubes at the bottom] and the area of the rectangle [covers the whole rectangle in between her hands. Figure 4.27] would be in inches squared.

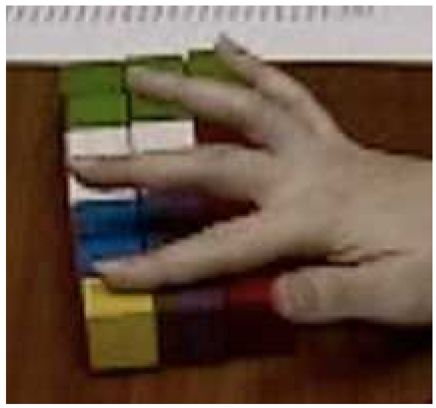


Figure 4.24. Nicole's hand gesture showing where the linear 3 is.

Figure 4.25. Nicole's hand gesture showing where the linear 7 is.

Figure 4.26. Nicole's hand gesture measuring the linear 3 as 3 inches.

Figure 4.27. Nicole's hand gesture measuring whole area in inches squared.

Nicole's hand gestures in Figures 4.24, 4.25, and 4.26 indicate that for her, the linear dimensions can exist within the areal part of the rectangle itself, which is consistent with what she was doing in the first activity about prime and composite rectangles (See Protocol 4.8 and the paragraph that follows). Therefore, I conclude that she reasons about these dimensions as quantities in relation to a particular areal quantity, namely the area of the rectangle itself.

She also interpreted the linear dimensional units as measurements, as warranted by Figure 4.26 and her statement "the length would be in inches… the width would be in inches." Moreover, she relied on a measurement interpretation for the area of the growing rectangle, as can be warranted by Figure 4.27 and her statement "and the area of the rectangle would be in inches squared." Nicole's strategy is reminiscent of what John was doing in the first activity on prime and composite rectangles in his description of linear and areal units. In that context, John supported his ideas by his statement "if you take a ruler" in an attempt to show that he interpreted the linear 7 as some sort of measurement (See Protocol 4.13 and the paragraph that follows).

When I asked Sarah about the areal subunit 5 part of the area of her growing rectangle, she said that she saw it as an area. I then tried to trouble her by saying "It looks like a length. Does not that bother you?" She replied "I see it as an area. The only reason is because of the blocks." I think she meant to say she is focusing on the upper faces of the color cubes, which is why she sees it as an area. This is reminiscent of Brad who referred to the areal–ness of the areal subunits as well as the growing rectangles using similar language "because you are adding the blocks individually." In that sense, both Brad and Sarah charge their "block" with the meaning of a "1 inch by 1 inch irreducible areal unit" by which they generate their areal subunits as well as their growing rectangle areal units.

Protocol 4.27: Sarah's projection strategy.

G: How do you distinguish between this 5 [pointing to the yellow rectangle in Figure 4.18] and the 5 which is the length of this rectangle?

S: When you do the length, it's the sides of the cubes [pointing to the edges of the rectangle] not the cubes themselves.

In other words, for the area, Sarah focuses on the upper faces of the cubes, which are of two–dimensional nature. For the length, on the other hand, she just focuses on the sides of her three–dimensional growing figure. She is projecting those sides onto the plane – that would be the whiteboard – of the three dimensional growing figure itself, as warranted by her hand gestures outlining the perimeter of the figure. I can therefore conclude that for Sarah, length is a line resulting from the projection of a vertically standing two–dimensional figure (that would be the sides) onto a plane, i.e., of linear nature. This strategy of projection was not used by any interview student except Sarah.

Next, I focus on John's understanding and sense making of linear and areal units. The protocol below captures the discussion when I ask him questions based on the figure he made corresponding to the 7th stage, i.e., the 4 by 7 growing rectangle (Figure 4.28).

Protocol 4.28: John compares same–valued linear and areal quantities.

G: This 7... [pointing to the red] is it an area or a length?

J: I guess it's an area...

G: How do you figure that?

J: I mean… Think of it as an area... It makes more sense to me.

G: The 7 which is one of the dimensions of the growing rectangle... How do you distinguish between that 7 [the dimension of the growing rectangle] and this 7 [pointing to the red rectangle]? Are they the same or different?

J: I guess they are equal to each other... they have the same area... top line 7... bottom line 7...

Figure 4.28. John's 7th growing rectangle of the sequence.

At first, John was a bit hesitant about whether the red rectangle at the bottom represents an area or a length, as can be inferred from his statement “Think of it as an area... It makes more sense to me.” When I further probed on whether the same–valued quantities, namely the dimension 7 of the growing rectangle and the red areal 7, which is part of the growing rectangle are the same or different, he inclined toward the same–ness of these quantities. In fact, his statement “I guess they are equal to each other... they have the same area... top line 7... bottom line 7” indicates that John saw the red rectangle as the length (or width) of the 4 by 7 growing rectangle. He used the phrase “bottom line” for that red rectangle, and I believe his usage of “top line” refers to the dimension of the growing rectangle itself. I can therefore deduce that Nicole and John are alike in that they both saw the dimensions as some sort of representational quantities that could take part in the areal region of the growing rectangle. These representational quantities differ, however, when Nicole and John attached some known measurement units, as can be seen in these students' written work. Therefore, for Nicole and John, these same–valued linear and areal quantities were different algebraically and alike representationally.

I believe Nicole and John's failure in seeing the difference in these same–valued linear and areal representational quantities could be due to their not paying attention to the three main components (name, value, measurement unit) of the quantities under consideration. In his study with eighth graders Thompson showed that students failed to distinguish between the name and the value of specific quantities, which resulted in these students' inabilities to explain the relationships between these quantities (1988). Two quantities may be assigned the same value; however, this does not require that they be the same quantities. Two quantities are the same only if they have the same name, value, and measurement unit (Olive & Caglayan, 2006, 2007; Schwartz, 1988).

Again, I end this subsection with a table of terminology that summarizes students' behaviors in an attempt to understand and make sense of linear and areal quantities as well as same–valued linear and areal quantities.

Table 4.13

Terminology Summarizing Students' Sense Making Of Linear And Areal Representational Quantities

Terminology Summarizing Students' Behaviors	**Students Fitting the Terminology**
Reference to Known Units of Measurement	Brad, Nicole, Sarah, Rob, John
Individual Blocks Representing “1 Inch by 1 Inch” Irreducible Areal	Brad, Sarah

Quantity	
Projection Strategy	Sarah
In the process of finding the area of the rectangle, a linear unit could behave like an areal unit. Seeing the dimensions as both linear and areal quantities.	Nicole, John
Same–valued linear and areal representational quantities are different algebraically.	Nicole, John
Same–valued linear and areal representational quantities are alike representationally.	Nicole, John
Iteration Strategy Combined with the Two–Dimensional Nature of Irreducible Areal Unit. Quantitative Unit Conservation.	Brad ("You can add areas together to create a whole area")
Interpretation of Linear and Areal Units as a Measurement	Nicole (Figures 4.26 & 4.27)

4.3. Summing Odd Integers

This third section of Chapter IV is similar to the previous one in that it encompasses many additive RUC types. The RUC types are obtained from the representational subunits corresponding to odd integers as well as from those corresponding to bigger units generated by these subunits via addition. Once again, as was the case in the previous section, the bigger representational units are the growing rectangles that represent the sum of the representational subunits, namely the odd integers.

Similar to the previous task with the summation of counting numbers, the common direction for all the interview students was to represent the odd integers (i.e., 1, 3, 5) using a different color for each number and add them so that they generate a rectangle. They were also asked to write their answers on the activity sheet to help them organize information. All students paid attention to the "add them so that they generate a rectangle" direction and came up with a similar sequence of growing rectangles made of symmetric L–shape odd integer representational subunits. John's sixth growing rectangle of the sequence, for instance, can be seen in Figure 4.29, which he described as "Each one is gonna fit in this backwards L–shape... Not only stays as a rectangle... stays as square."

Figure 4.29. The 6th growing rectangle of the sequence.

4.3.1. Multiplicative Representational Unit Coordination (MRUC)

Everyone initially made $1 \times n$ rectangles to represent odd integers. These subunits, therefore, could be represented via relational notation as ordered pairs (1, 1), (1, 3), (1, 5), (1, 7), (1, 9), (1, 11) of linear units. However, when faced with the request to make a growing rectangle to represent the sum, they did different things. Brad, for instance made this figure (Figure 4.30).

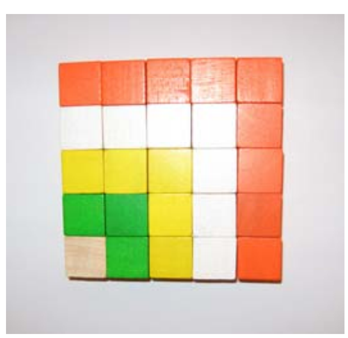


Figure 4.30. Brad's 5th growing rectangle of the sequence.

In other words, he used symmetric L–shape figures for each odd integer subunit. Though the subunits were not rectangles, he described these subunits multiplicatively, as the following protocol points out.

Protocol 4.29: Brad's MRUC concerning the odd integer subunits in the growing rectangle.

G: How about the yellow one, which represents 5, is it an area or a length?

B: I'd say two dimensional.

G: How do you figure?

B: They have an area within themselves [at the same time pointing to the yellow squares]. It's two dimensional and each block represents an area, so...

G: Okay, you are saying that it's an area, but does it bother you that it's not a rectangle, the yellow one?

B: The odd ones themselves [meaning the odd numbers] can not make a rectangle individually, I mean [inaudible] it can, it'd just be 1 by whatever it is [meaning the long bar rectangles]. So... adding them [the L–shapes] together makes a rectangle... a square...

The non–rectangular L–shape formation does not prevent him from providing a multiplicative type RUC. His statement "It'd just be 1 by whatever it is [referring to his original long sticks]" calls for a relational notation of ordered pair $(1, n)$ of linear units, where n stands for the length of the odd integer. Brad and Nicole are alike in that they both were able to express the areas of such non–rectangular subunits as a product (See Figure 4.13 and Protocol 4.15 of the previous section for Nicole's descriptions). They both agree that even though both rectangular and L–shape formations have the same area, "area as a product" notion arises only within the rectangular formation. Nicole and Sarah, in the previous section, provided a visual proof by contradiction to demonstrate how the multiplicativeness of the RUC and the rectangle concept necessitate each other (See Figure 4.17 of the previous section for Sarah's Visual Proof by Contradiction). Brad, on the other hand, is different from Nicole and Sarah in that he was content with his verbal description "It'd be 1 by whatever it is" without reference to any visual demonstration. Another surprising remark is the fact that Brad did not consider that some of these odd integers can be represented by an $a \times b$ rectangle, where a and b are different from unity. For instance, he did not consider that 9 can be represented by a 3×3 rectangle.

Nicole and Brad showed similar thinking as described in the following protocol.

Protocol 4.30: Nicole's MRUC concerning the odd integer subunits.

G: What is common about these odd integers?

N: They're all prime numbers. They only form one rectangle which is a straight line.

G: How about their dimensions? What is the area as a product for each case?

N: It's itself multiplied by 1.

G: Now add them so that they generate a rectangle.

Similar to Brad, she made an incorrect connection between odd and prime numbers. Nicole agreed with Brad about the multiplicative nature of the odd integer subunits via her statement “It's itself multiplied by 1,” which necessitates a relational notation of ordered pairs (n, 1) of linear units.

Rob's first representation of the growing rectangle at the third stage – as a growing long stick – can be modeled with a relational notation of ordered pair (1, 1 + 3 + 5) of linear units, namely of multiplicative nature. His second attempt was the L–shape formation (Figure 4.31).

Figure 4.31. Rob's odd integer subunits reorganized as L–Shapes.

Protocol 4.31: Rob's MRUC concerning the odd integer subunits.

G: How about that 5... Can you express it as a product?

R: I mean... you could say 1 inch times 5 inch but it's not like a rectangle... but I mean you can still give the answer that way...

G: But in this representation... what is this quantity... is it an area or a length? That 5... or 7...

R: It's an area...

Rob probably meant that the L–shapes, though not rectangles, still stand for areal units of 5 and 7, and hence can be expressed as products. A suitable relational notation for Rob's description is the ordered pairs (1, 5) and (1, 7) of linear units. Rob and Brad are therefore alike in that the non–rectangular formation of the subunits does not prevent them from concluding the multiplicative nature of these subunits, which was the case for Nicole and Sarah in the previous task on the summation of counting numbers. Rob, Brad, Nicole and Sarah are therefore, all alike in this aspect.

Stephenie said that she saw the long sticks representing odd integers as areas. Upon my request to add them so that they generate a rectangle, she first said it would not work. When I asked her to try it, she was very happy to see that it worked. She said that she had to change their shape as that would be the only way to make a rectangle. She added that she had not seen this pattern before (Figure 4.32).

Figure 4.32. Sarah's growing rectangle at the 5th stage.

She said that the red L–shape was an area, not a length. I then asked her how she could convince me that it is an area. She then made this figure (Figure 4.33).

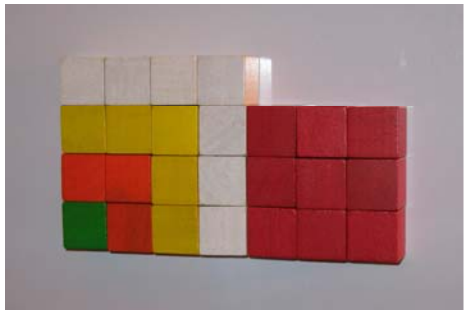


Figure 4.33. Sarah's reorganization of the red L–Shape into a rectangle.

I did not ask Sarah whether the red L–shape would be expressed as a product. I only asked her whether it is an area or not. In other words, to show that the red L–shape is an area, Sarah changed it into a well–known figure, namely a rectangle, which must be an area and not a length, as I deduced from her demonstration. Sarah established the areal–ness of the odd integer subunit as opposed to Brad, Rob, and Nicole who not only demonstrated the areal–ness but expressed the subunit's area as a product. This is a slight difference in these students' thinking.

The table below summarizes each student's answer and a corresponding relational notation that describes multiplicative type RUC.

Table 4.14

Relational Notation Describing Students' Answers

Students	**Phrases**	**Relational Notation**
Brad	It'd just be 1 by whatever it is.	(1, n)

Nicole	It's itself multiplied by 1.	(n, 1)
Rob	You could say 1 inch times 5 inch but it's not like a rectangle... but I mean you can still give the answer that way...	(1, 5)
Sarah	NA	NA
John	1 times... 11 plus 9 plus 7 plus 5 plus 3 plus 1.	(1, 11 + 9 + 7 + 5 + 3 + 1)

The following table reflects students' written answers for the "area as a product" of the growing rectangles. I include a relational notation for each written answer for the purpose of analysis.

Table 4.15

Students' Written Work for the Area of the Growing Rectangles as a Product

Students	Area of the Growing Rectangle as a Product	Relational Notation Describing the Product
Brad	2in × 2in, 3in × 3in, 4in × 4in, 5in × 5in, 6in × 6in	(2, 2), (3, 3), (4, 4), (5, 5), (6, 6)
Nicole	2 × 2, 3 × 3, 4 × 4, 5 × 5	(2, 2), (3, 3), (4, 4), (5, 5)
Rob	3in × 3in, 4in × 4in, 5in × 5in	(3, 3), (4, 4), (5, 5)
Sarah	1 × 1, 2 × 2, 3in × 3in, 4 × 4, 5in × 5in	(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)
John	2in × 2in, 3in × 3in, 4in × 4in, 5in × 5in, 6in × 6in, n in × n in	(2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (n, n)

As can be seen in the written work, everybody except Nicole attached some sort of units to their answers to emphasize the linearity of the quantities involved. Another point worth mentioning is John's use of a variable, n, for a general case in his description of stage number as n, odd integer subunit as $2n - 1$, and the area of the growing square as n in × n in. I will end this subsection with a table of terminology that summarizes students' behaviors reflecting multiplicative type RUC.

Table 4.16

Terminology Summarizing Students' Behaviors Reflecting MRUC

Strategy/Terminology	Students
Multiplicative RUC for Special Cases of Odd Number Subunits and Growing Rectangles	Brad, Nicole, Rob, Sarah, John
Generating Growing Squares via Symmetric L–Shape Odd Integer Subunits	Brad, Nicole, Rob, Sarah, John
At First Attempt, Generating Growing Long Sticks Made of 1 by *n* Odd Integer Subunits	Rob, John
Establishing the Areal–ness of the Odd Integer Subunits	Sarah, Brad, Rob, Nicole
Establishing both the Areal–ness and the Possibility to Express the Areas of Odd Integers Subunits as a Product	Brad, Rob, Nicole
Realization of the fact that the added L–shape subunits could not be expressed as a product of two linear units	Nicole
Uses a Variable to Describe Various Quantities	John

4.3.2. Additive Representational Unit Coordination (ARUC)

In the context of summation of odd integers, the additive nature of RUC appears as an ordered *n*–tuple of areal units, as was the case for the previous activities on prime/composite rectangles and summation of counting numbers. Similar to the summation of counting numbers activity, one has to know whether the additive RUC types corresponding to areal quantities arise from the growing rectangles, or from the odd number subunits generating those big rectangles.

Brad's reference to Additive RUC types concerning the odd integer subunits in general can be witnessed in the following discussion.

Protocol 4.32: Brad's symmetric addends type ARUC concerning the symmetric L–Shapes in general.

G: What is common about each odd integer?

B: What's common?

G: Yeah...

B: They are all odd... [laughing] And you have this diagonal that goes across [pointing to the main diagonal of the growing square] and you have this same number this way [pointing to the left] and this way [pointing down] (Figure 4.30).

I hypothesize that Brad unitized an odd integer L–shape subunit as 1 plus twice the same particular number (i.e., of *Symmetric Addends* type RUC). Referring back to the functional notation $\sum_{i=1}^{i=n} f(i) = g(n)$ I introduced in the previous section, where areal subunits "$f(i)$"s are being summed from 1 to n (number of addends) and i is the stage number (ordering number for the addends), *Symmetric Addends* in general can be described as follows. One has three distinct areal units pertaining to each odd integer subunit. In other words, for each symmetric L–shape, there are three addends only, i.e., $n = 3.$ One of these addends is equal to 1, and the remaining two addends are equal to each other. In other words, with the functional notation, one can write, $f(2) = 1, f(1) = f(3).$ Since *Symmetric Addends* type RUC is used to describe areal units generating the odd integer subunits, one can think of such areal units "$f(i)$"s as sub–subunits. Later on in the same interview, Brad referred to *Symmetric Addends* type RUC concerning the odd integer subunits for a particular case as well. The following protocol picks up at this point.

Protocol 4.33: Brad's symmetric addends type ARUC concerning the symmetric L–Shapes for a particular case.

G: Tell me more about that, in particular for the white one [the symmetric L–shape representing 7 in Figure 4.30].

B: The white one? Okay... This has this 1 on the diagonal, and it has 3 down, and 3 across. It's the same [meaning the same 3]. And that's being odd...

G: Does it tell you that they are odd?

B: Yes.

Brad, therefore, first focused on the areal “1” sub–subunit on the main diagonal. He then decomposed the remaining areal “6” into two with–respect–to–the–main–diagonal symmetric areal “3” sub–subunits. In other words, with the functional notation, one can describe these sub–subunits as $f(2)=1, f(1)=f(3)=3.$ An areal 7 subunit can be decomposed into *Symmetric Addends* areal 3, areal 1, and areal 3 sub–subunits. Note that $f(1)+f(2)+f(3)=3+1+3=7,$ i.e., the odd integer subunit itself. One can use a relational notation of ordered triple [3, 1, 3] of areal sub–subunits to denote this additive type RUC. Brad then used an internal connection to support his ideas concerning the symmetric addends type RUC.

Protocol 4.34: Brad's internal connection strategy: Reference to statistics.

G: How do you figure [that they are odd]?

B: 'Cuz... same thing we would have... mean or average... [trying to remember a statistical term] if you have an odd number... I guess that would be the median... if you have an odd number you can always find that middle number because you have an unequal amount of numbers... but when you take the middle number out you have an even amount of number on each side... so it's the middle number... that's the median I guess...

G: Oh I see... from statistics...

B: Yeah statistics... When you take these [all the “1”s on the diagonal] you have the same amount on each side [meaning horizontally to the left and vertically down, with respect to the main diagonal in Figure 4.30].

Brad referred to the “median” concept from statistics to explain his decomposition of the odd integer subunits into three sub–subunits. This is a very nice connection, which supports his *Symmetric Addends* type RUC. Any odd integer of the form $2n + 1$ can be written as $n + 1 + n$ as a sum, as if the “n”s are reflections of each other with respect to 1, which is the middle term. The odd integer $2n + 1$ can be thought of as being mapped into a sample of size $2n + 1$. In fact, the sub–subunit “1” on the main diagonal corresponds to the “median” from statistics, namely the middle term of the sample, from Brad's explanations. The remaining pair of “n”s, namely the “even amount of number on each side,” in Brad's words, correspond to two sub–samples of size n spread above and below the median. Brad used an internal connection strategy in an attempt to make sense of the sub–subunits generating an L–shape subunit. As pointed out in NCTM's Connection Standard, “Thinking mathematically involves looking for connections, and making connections builds mathematical understanding. Without connections, students must learn and remember too many isolated concepts and skills. With connections, they can build new understandings on previous knowledge" (NCTM, 2000, p. 273). With reference to this quote, therefore, I infer that Brad was thinking mathematically and establishing his additive type RUC meaningfully.

Our discussion on the symmetric L–shape subunits led Brad to a discovery of a new additive type RUC concerning these subunits. The following protocol picks up at this point.

Protocol 4.35: Brad's $N+(N-1)$ Type ARUC concerning the symmetric L–Shapes for a particular case.

G: Okay but it [the area of L–shape representing 5] cannot be written as a product. So...

B: Well... It might not be a rectangle but it's still a polygon. So it still can be written as an area (Figure 4.34) [Brad then decomposes the polygon into a 3 × 1 and a 2 × 1 rectangle (Figure 4.35)].

G: Which polygon is that, the yellow one?

B: [Counting the sides] I don't know what it's called...

G: It's a hexagon.

Figure 4.34. Brad's odd integer L–Shape subunit.

Figure 4.35. Brad's decomposition of his L–Shape subunit into two sub–subunits.

Though it cannot be written as a product, Brad established the areal–ness of the areal L–shape subunit via his decomposition strategy – by which he also establishes a "Rectangle Condition" for the sub–subunits. He said that he could do that because the L–shape is still a polygon, so it is an areal quantity. I define Brad's additive type RUC as of $N+(N-1)$ type for which one has two distinct areal sub–subunits pertaining to each odd integer subunit. In other words, for each symmetric L–shape, there are two addends only. With the functional notation, $f(1)=N$ and $f(2)=N-1,$ showing that the addends differ only by 1. Once again, one can think of the areal units "$f(i)$"s as sub–subunits generating odd integer subunits. Note that in general, $f(1)+f(2)=2N-1,$ the odd integer subunit itself. In the particular case where the odd integer is 5, one can write $f(1)=3,$ and $f(2)=2.$ One can use a relational notation of ordered pairs [3, 2] and [N, $N-1$] of areal units to denote this additive type RUC for the particular and the general cases, respectively.

Finally, we discussed the areas of the growing rectangles. Once again, when we discussed the "Area of the growing rectangle as a sum" column on the activity sheet, it was interesting that Brad did not rely on the summation formula; he preferred *Equal*

Addends type RUC, as if he was describing composite numbers. In fact, this was the case in the previous task on the summation of counting numbers. I was expecting him to come up with *Summed Addends* type RUC in dealing with the bigger units. From his choice of *Equal Addends* rather than *Summed Addends*, I deduce that Brad did not care about the colors. The colors did not prevent him from seeing the bigger units made of subunits of *Equal Addends* type RUC. Rob did the same thing as Brad. Perhaps these two students wanted to focus on the "composite number" itself, rather than the growing rectangle (bigger unit) made of color shapes (subunits) representing a summed number (See Table 4.19).

Rob was reasoning quantitatively in that he was trying to make sense of the areal L–shape quantities on their own and in relation to each other. The following protocol captures his quantitative reasoning on the areal subunits and growing rectangles.

Protocol 4.36: Rob's quantitative reasoning.

R: Aha! I know what is going on here [He realizes the symmetric L–shape formation]. I got an aha... Okay... every time you add an odd number it's gonna make a square [He then completes his figure at the 5th step. Figure 4.31]

G: What is common about these odd integers?

R: They are all odd... and you make a square every time... They all cross like that [hand gesture imitating the L–shape formation]. They make an L around it...

This is reminiscent of Rob's *Bridge Connection Strategy* from the previous section. Once again Rob showed how the areal subunits exist on their own as well as in relation to each other. In fact, in his first description above in the protocol, in his statement he mentioned both the subunits and the growing square. Once again in his second statement, he talked about how each L–shape subunit comes to exist and builds a growing square. In Thompson's words, "To reason quantitatively is to reason about quantities, their magnitudes, and their relationships with other quantities" (Thompson, 1988, page 164). Rob's iteration of the areal quantities too supports my hypothesis that Rob can reason about quantities, their magnitudes (values), and their relationships with other quantities. The following discussion picks up at this point.

Protocol 4.37: Rob's iteration strategy.

G: What about the "L"s in each case... Is it inch or inches squared... is it an area or a length?

R: It's kinda a weird combination of both... because these don't make a rectangle [pointing to the yellow L–shape]... I mean their area will be five inches squared because there is five one inch squared cubes.

G: About 5... for that one... can you express it as a sum... that area as a sum? [meaning the yellow L–shape in Figure 4.31].

R: Yeah... 1 inch plus 1 inch plus 1 inch plus 1 inch plus 1 inch...

G: 1 inch or 1 inch squared?

R: 1 inch squared... sorry... It [meaning the growing figure with L–shape formation] will keep going in that pattern... and... you're always gonna get as your area a square number.

Rob's statement about L–shapes "because these don't make a rectangle" upon my probing "is it an area or a length?" places the so–called "Rectangle Condition" at the heart of MRUC, as appeared before in Nicole and Brad's cases. Because of the non–rectangular formation of the L–shape subunits, Rob seems to favor an "area as a sum" rather than an "area as a product," as can be warranted by his statement "...these don't make a rectangle... I mean their area will be five inches squared because there is five one inch squared cubes." Rob iterated not only the irreducible areal sub–subunits to obtain his odd integer subunits, but he later on iterated his L–shape subunits toward the northeast direction to obtain his growing squares. Rob once again related these quantities and hence reasoned quantitatively. His first iteration can be formulated via a relational notation of ordered quintuple [1, 1, 1, 1, 1] of irreducible areal sub–subunits. This formulation is of *Irreducible Addends* type RUC. His representational iteration of the L–shape subunits yielded a sequence of growing squares; hence his growing squares are representationally of *Summed Addends* type RUC. However, algebraically, Rob favored *Equal Addends* type RUC; namely he treated the growing squares as composite numbers (See Table 4.19 summarizing students' written work below).

Sarah, on the other hand, was firm about her descriptions of the areal–ness of the L–shape subunits. She said the L–shapes were areas and not lengths. She provided various answers for the "area of the growing rectangle as a sum" column on her activity sheet. For instance, for the third stage, she said "3 + 3 + 3 or 1 + 3 + 5 or 8 + 1" and added that there are many different ways of writing the sum. Her phrase "many different ways" can be understood once again as her reference to *Random Addends* type RUC. And these are not based on the figure she made with the cubes; she was just working numerically. However, she did not find, nor did I probe her to find, an arrangement of cubes that matched her description as for 8 + 1 with the cubes. I probed on her statement "3 + 3 + 3."

Protocol 4.38: Sarah's equal addends type RUC.

G: What do you mean by 3 + 3 + 3?

S: 'Cuz it's 3 + 3 + 3 [she splits the growing square into three parts. Figure 4.36]

G: So... the colors do not bother you, right?

S: Right.

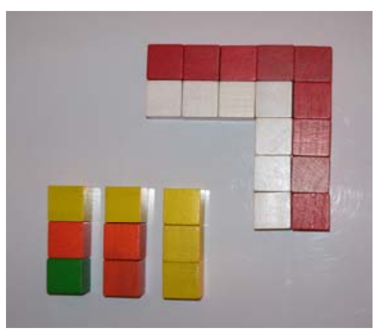


Figure 4.36. Sarah's decomposition of the 3rd growing rectangle into an ordered triple [3, 3, 3] of (equal addends type) areal units.

Although she provided *Random Addends* type RUC in the first two activities – and still stuck to that in a way, as the discussion above also shows, she decided to go on with *Equal Addends* as in "3 + 3 + 3" and with *Summed Addends* as in "1 + 3 + 5," in her written expressions. Her visual decomposition in Figure 4.36 reflects *Equal Addends* type RUC.

Sarah's additive type RUCs for this particular case at the third stage can be described using a functional notation $\sum_{i=1}^{i=n} f(i) = g(n)$ where areal subunits "$f(i)$"s are being summed from 1 to n (number of addends) and i is the stage number (ordering number for the addends) as follows. The three equal addends are all of areal nature and equal to each other, namely, $f(1) = f(2) = f(3) = 3.$ They add up to the area of the growing square at the 3rd stage. In other words, $f(1) + f(2) + f(3) = 3 + 3 + 3 = 9.$ One can use a relational notation of ordered triple [3, 3, 3] of areal subunits to denote *Equal Addends* type RUC. For the *Summed Addends*, once again $n = 3.$ Each subunit $f(i)$ is defined based on a summation formula as $f(i) = 2i - 1$ where i denotes the stage number from 1 to $n = 3.$ In other words, $f(1) = 1, f(2) = 3, f(3) = 5.$ The three summed addends are all of areal nature and add up to the area of the growing square at the 3rd stage. In other words, $f(1) + f(2) + f(3) = 1 + 3 + 5 = 9.$ One can use a relational notation of ordered triple [1, 3, 5] of areal subunits to denote *Summed Addends* type RUC. Finally, as for the *Random Addends*, there are many possibilities. One does not need to have an ordered pair or an ordered triple of areal units in this model. In this current example, one could have things like [1, 1, 3, 4], [1, 2, 1, 2, 1, 2], [1, 3, 1, 3, 1] as well. In other words, both the number of addends and the numerical value of each addend are completely random. The only restriction is that the addends add up to the area of the growing square at the 3rd stage, which equals 9. One could also have things like [1, 1, 1, 1, 1, 1, 1, 1, 1], [3, 3, 3], [1, 3, 5] as *Random Addends* type RUC. In that sense, *Irreducible Addends* type

RUC, *Equal Addends* type RUC, and *Summed Addends* type RUC are all particular cases of *Random Addends* type RUC. In fact, all the additive type RUCs interview students established in my research are particular cases of *Random Addends* type RUC. *Irreducible Addends* type RUC is a special case of *Equal Addends* type RUC, as well. In other words, Sarah could see the area as a sum in many different ways, including the “equal addends” and the “summed addends” formations. Sarah, among all the interview students, is unique that way.

John introduced $N+(N-1)$ Type Addends in his description of the odd integer subunits and supported his ideas by visually decomposing these L–shape subunits into pairs of sub–subunits. The following discussion picks up at this point.

Protocol 4.39: John's $N+(N-1)$ type additive RUC concerning the symmetric L–Shapes for three particular cases.

G: [pointing to the orange in Figure 4.29] Is it an area or a length?

J: It's an area...

G: How do you say that?

J: [He then makes a red L–shape for 3 and says] you can visualize it as the area of this rectangle added to the area of this rectangle [pointing to the 1 by 1 and the 1 by 2 rectangles, respectively. Figure 4.37]

G: You see it as the area of two rectangles... am I right?

J: Yeah... [and separates the rectangles. Figure 4.38]

G: Okay... how about this 5? [pointing to the white L–shape figure] Is it an area or a length?

J: It would be the same... It'd be an area...

G: And because? [He then makes the following figure. Figure 4.39]

J: Because area is 3 here and 2 is here so you get 5 [pointing to the rectangles he just made].

Figure 4.37. John's red L–Shape subunit representing the odd integer 3.

Figure 4.38. John's decomposition of his red L–Shape subunit into two sub–subunits.

Figure 4.39. John's decomposition of his white L–Shape subunit into two sub–subunits.

This looks like one of the RUC types I have seen before. It is actually exactly the same thing Brad did (See Protocol 4.35, Figures 4.34 and 4.35 above). Sarah also used this type twice in her written work. To demonstrate the areal–ness of the L–shape subunits, both John and Brad decomposed the subunits into two sub–subunits. These two students are alike in their thinking, as opposed to Rob, who referred to an iteration of irreducible areal units strategy. The difference in these students' thinking can be explained with the additive type RUCs under consideration (*Irreducible Addends* type RUC for Rob, and $N+(N-1)$ type RUC for Brad and John).

Both John and Brad decomposed the L–shape subunits into two *rectangles*, by which I offer a *concept–in–action* (Vergnaud, 1988), "Rectangle Condition for Arealness." In fact, this appeared to be the case in all students' work I analyzed so far. Why rectangles and not something else, in order to establish arealness? My explanation to this is that it's simply because "Rectangle Condition" lies at the heart of MRUC.

I will end this subsection with three tables. The first one summarizes additive type RUCs concerning both the odd integer (L–shape) subunits and the growing rectangle units based on interview students' verbal descriptions and gestures. The following two tables are based on these students' written work concerning the odd integer subunits and the growing rectangle units, respectively.

Table 4.17

Additive Type RUCs Concerning both the Odd Integer (L–Shape) Subunits and the Growing Rectangle Units Based on Interview Students' Verbal Descriptions and Gestures

Students	**Additive Type RUC Used in the Decomposition of the Odd Integer (L–Shape) Subunits into Sub–Subunits**	**Additive Type RUC Used in the Decomposition of the Growing Rectangles into Subunits**
Brad	Symmetric Addends $N+(N-1)$ Type Addends	NA
Nicole	NA	NA
Rob	Irreducible Addends	Summed Addends
Sarah	Random Addends	Equal Addends Random Addends
John	$N+(N-1)$ type Addends	NA

Table 4.18

Students' Written Work for the Areas of the Odd Integer Subunits as a Sum

Students	**Area of the Odd Number Subunits as a Sum**	**Name of the Additive Type RUC**	**Relational Notation Describing the Sum**
Brad	1 + 1 + 1 or 2 + 1 1 + 1 + 1 + 1 + 1 or 3 + 2 1 + 1 + ... + 1 + 1 or 4 + 3 1 + 1 + ... + 1 + 1 or 5 + 4 1 + 1 + ... + 1 + 1 or 6 + 5	Irreducible Addends $N+(N-1)$ Type Addends Irreducible Addends $N+(N-1)$ Type Addends Irreducible Addends $N+(N-1)$ Type Addends Irreducible Addends $N+(N-1)$ Type Addends Irreducible Addends $N+(N-1)$ Type Addends	[1, 1, 1] [2, 1] [1, 1, 1, 1, 1] [3, 2] [1, 1, 1, 1, 1, 1, 1] [4, 3] [1, 1, 1, 1, 1, 1, 1, 1, 1] [5, 4] [1, 1, ..., 1, 1] [6, 5]
Nicole	1 + 1 + 1 or 3 1 + 1 + 1 + 1 + 1 or 5 1 + 1 + ... + 1 + 1 or 7 1 + 1 + ... + 1 + 1 or 3 + 3 + 3	Irreducible Addends Singleton Addend Irreducible Addends Singleton Addend Irreducible Addends Singleton Addend Irreducible Addends Equal Addends	[1, 1, 1] [3] [1, 1, 1, 1, 1] [5] [1, 1, 1, 1, 1, 1, 1] [7] [1, 1, 1, 1, 1, 1, 1, 1, 1] [3, 3, 3]

	or 9	Singleton Addend	[9]
Rob	$1in^2 + 1in^2 + 1in^2$ $1in^2 + ... + 1in^2$ $1in^2 + ... + 1in^2$ or $3in^2 + 3in^2 + 3in^2$	Irreducible Addends Irreducible Addends Irreducible Addends Equal Addends	[1, 1, 1] [1, 1, 1, 1, 1] [1, 1, 1, 1, 1, 1, 1] [3, 3, 3]
Sarah	$3in^2 + 2in^2$ $5in^2 + 4in^2$	$N + (N - 1)$ Type Addends $N + (N - 1)$ Type Addends	[3, 2] [5, 4]
John	$1in^2 + 2in^2$ $2in^2 + 3in^2$ $3in^2 + 4in^2$ $4in^2 + 5in^2$ $5in^2 + 6in^2$ $(n - 1)\ in^2 + n\ in^2$	$N + (N - 1)$ Type Addends $N + (N - 1)$ Type Addends $N + (N - 1)$ Type Addends $N + (N - 1)$ Type Addends $N + (N - 1)$ Type Addends $N + (N - 1)$ Type Addends	[1, 2] [2, 3] [3, 4] [4, 5] [5, 6] [$n - 1$, n]

Table 4.19

Students' Written Work for the Area of the Growing Rectangles as a Sum

Students	Area of the Growing Rectangle as a Sum	Name of the Additive Type RUC	Relational Notation Describing the Sum
Brad	$2in^2 + 2in^2$ or $1in^2 + 1in^2 + 1in^2 + 1in^2$ $3in^2 + 3in^2 + 3in^2$ or $1in^2 + 1in^2 + ... + 1in^2 + 1in^2$ $8in^2 + 8in^2$ $4in^2 + 4in^2 + 4in^2 + 4in^2$ or $2in^2 + 2in^2 + ... + 2in^2 + 2in^2$ or $1in^2 + 1in^2 + ... + 1in^2 + 1in^2$ $5in^2 + 5in^2 + 5in^2 + 5in^2 + 5in^2$ or $1in^2 + 1in^2 + ... + 1in^2 + 1in^2$ $6in^2 + 6in^2 + ... + 6in^2 + 6in^2$ or $3in^2 + 3in^2 + ... + 3in^2 + 3in^2$ or $2in^2 + 2in^2 + ... + 2in^2 + 2in^2$ $9in^2 + 9in^2 + 9in^2 + 9in^2$ or $4in^2 + 4in^2 + ... + 4in^2 + 4in^2$ or $1in^2 + 1in^2 + ... + 1in^2 + 1in^2$	Equal Addends Irreducible Addends Equal Addends Irreducible Addends Equal Addends Equal Addends Equal Addends Irreducible Addends Equal Addends Irreducible Addends Equal Addends Equal Addends Equal Addends Equal Addends Equal Addends Irreducible Addends	[2, 2] [1, 1, 1, 1] [3, 3, 3] [1, 1, ..., 1, 1] [8, 8] [4, 4, 4, 4] [2, 2, ..., 2, 2] [1, 1, ..., 1, 1] [5, 5, 5, 5, 5] [1, 1, ..., 1, 1] [6, 6, 6, 6, 6, 6] [3, 3, ..., 3, 3] [2, 2, ..., 2, 2] [9, 9, 9, 9] [4, 4, ..., 4, 4] [1, 1, ..., 1, 1]
Nicole	1 + 3 or 2 + 2 or 1 + 1 + 1 + 1 1 + 3 + 5 or 4 + 5 or 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 1 + 3 + 5 + 7 or 4 + 5 + 7 or 9 + 7 or 1 + 1 + ... + 1 + 1 1 + 3 + 5 + 7 + 9 or 4 + 5 + 7 + 9 or 9 + 7 + 9 or 16 + 9 or 1 + 1 + ... + 1 + 1	Summed Addends Equal Addends Irreducible Addends Summed Addends Recursive Addends Irreducible Addends Summed Addends Recursive Addends Recursive Addends Irreducible Addends Summed Addends Recursive Addends Recursive Addends Recursive Addends Irreducible Addends	[1, 3] [2, 2] [1, 1, 1, 1] [1, 3, 5] [4, 5] [1, 1, 1, 1, 1, 1, 1, 1, 1] [1, 3, 5, 7] [4, 5, 7] [9, 7] [1, 1, ..., 1, 1] [1, 3, 5, 7, 9] [4, 5, 7, 9] [9, 7, 9] [16, 9] [1, 1, ..., 1, 1]
Rob	$3in^2 + 3in^2 + 3in^2$ or $1in^2 + 1in^2 + ... + 1in^2 + 1in^2$	Equal Addends Irreducible Addends	[3, 3, 3] [1, 1, ..., 1, 1]

	No Other Way $8in^2 + 8in^2$ or $4in^2 + 4in^2 + 4in^2 + 4in^2$ or $1in^2 + 1in^2 + ... + 1in^2 + 1in^2$ No Other Way $5in^2 + 5in^2 + 5in^2 + 5in^2 + 5in^2$ or $1in^2 + 1in^2 + ... + 1in^2 + 1in^2$ No Other Way	NA Equal Addends Equal Addends Irreducible Addends NA Equal Addends Irreducible Addends NA	NA [8, 8] [4, 4, 4, 4] [1, 1, ..., 1, 1] NA [5, 5, 5, 5, 5] [1, 1, ..., 1, 1] NA
Sarah	1 1 + 3 3 + 3 + 3 or 1 + 3 + 5 or 8 + 1 1 + 3 + 5 + 7 1 + 3 + 5 + 7 + 9	Singleton Addend Summed Addends Equal Addends Summed Addends Random Addends Summed Addends Summed Addends	[1] [1, 3] [3, 3, 3] [1, 3, 5] [8, 1] [1, 3, 5, 7] [1, 3, 5, 7, 9]
John	$1in^2 + 1in^2 + 2in^2$ $1in^2 + 1in^2 + 2in^2 + 2in^2 + 3in^2$ $1in^2 + ... + 3in^2 + 3in^2 + 4in^2$ $1in^2 + ... + 4in^2 + 4in^2 + 5in^2$ $1in^2 + ... + 5in^2 + 5in^2 + 6in^2$ $1in^2 + ... + (n-1)in^2 + (n-1)in^2 + n\ in^2$	Summed Addends Summed Addends Summed Addends Summed Addends Summed Addends Summed Addends	[1, 1, 2] [1, 1, 2, 2, 3] [1, 1, 2, 2, 3, 3, 4] [1, 1, 2, 2, 3, 3, 4, 4, 5] [1, 1, ..., 5, 5, 6] [1, 1, ..., $n-1$, $n-1$, n]

I will end this subsection by comparing the *Summed Addends* used by John and Nicole in their written answers for the "Area of the growing rectangle as a sum" column on the activity sheet. The slight difference lies in the functional dependence of the odd integer subunits "$f(i)$"s. For Nicole, $f(i) = 2i - 1$ whereas for John, $f(i) = i + (i - 1)$. Though the two "$f(i)$"s are identical algebraically, they seem to differ representationally as I described above. Nicole's "$f(i)$"s stand for the odd integer L–shape subunits whereas John's "$f(i)$"s are expressed as the sum of sub–subunits. In that sense, John carried his $N + (N - 1)$ *Type Addends* over and added them to obtain his growing square. John's areas of growing squares are sums of these sub–subunits, whereas Nicole's areas of the growing squares are sums of the odd integer subunits. John's summation formula based on these sub–subunits can be found in some books on visual proofs – See, for instance, Nelson (1993, p.74). Brad and Rob, on the other hand, simply relied on the fact that the areas of the growing rectangles represent composite numbers, and so they stuck to *Equal Addends* and *Irreducible Addends* type RUC only.

4.3.3. Linear vs. Areal Units

Brad paid attention to dimensionalities of areal units to establish the areal–ness of these units (See Protocol 4.28). In order to show that the L–shape subunit standing for the odd integer 5 was two dimensional, or of areal nature, he used the fact that "each block represents an area." This is similar to Rob's iteration strategy, which I introduced above in Protocol 4.37. Brad was able to iterate his irreducible areal unit to obtain a polygon, as indicated by his statement "It might not be a rectangle but it's still a polygon. So it still can be written as an area." Moreover, Brad used his decomposition strategy to obtain his two sub–subunits, which are of $N+(N-1)$ type RUC.

Nicole's growing square patterns for the fourth and the fifth stages were different from what I was expecting. As depicted in figures below, she rearranged her previous figure (Figure 4.40) when she was at the 4^{th} stage (Figure 4.41). When she was at the 5^{th} stage, she embraced her original L–shape formation (Figure 4.42).

Figure 4.40. Nicole's growing square at the 3^{rd} stage.

Figure 4.41. Nicole's growing square at the 4th stage.

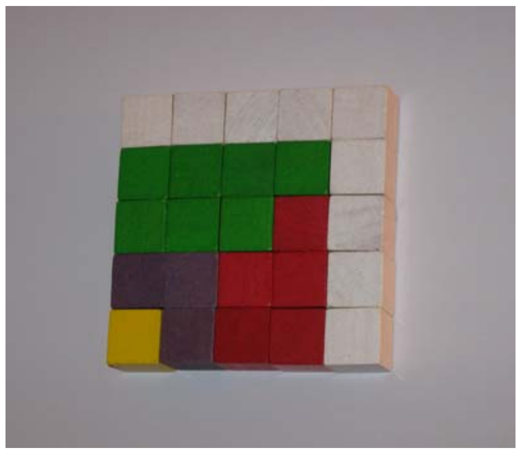


Figure 4.42. Nicole's growing square at the 5th stage.

Nicole assigned known measurement units to linear and areal quantities, as depicted in the protocol below.

Protocol 4.40: Nicole's assignment of known measurement units to quantities.

G: What are the units associated with each odd integer?

N: Inches for the length and the width; and inches squared for the area.

G: So, for each odd integer, what are the units then?

N: Each one is an inch [points to the white cube on the upper left corner in

Figure 4.42]

G: What is its unit? [About number 9 which is represented as the white L–shape]

N: 9 inches.

G: 9 inches or 9 inches squared?

N: Yeah 9 inches squared.

The white figure representing the odd integer 9 makes an L around the growing figure, which is a square. Nicole did not see this white L as part of the area of the square, as warranted by her assignment of inches as the unit of measurement. This corroborates my theory about her thinking about dimensions as behaving as both linear and areal units. In fact, the white L shape at the 5^{th} stage may be representing the dimensions of the growing square for Nicole, which caused her to favor inches instead of inches squared as a unit of measurement. She then switched to inches squared upon my probing "9 inches or 9 inches squared?" In fact, after her realization of the areal–ness of the white L–shape, she assigned an inches squared measurement unit to every other L–shape in her sequence of growing figures (7 inches squared, 5 inches squared, etc.). Her answers on the "Total Area" column of the activity sheet helped her realize that the growing figures were square numbers. The following protocol picks up at this point.

Protocol 4.41: Nicole's realization of the growing square sequence.

G: What is common about these numbers? [pointing to the "Total Area" column of the table]

N: They're all squares. And now I realize it would look a lot prettier if I had done this. You could see it better. Now you could see like... cascading... [She is rearranging her figure. She is changing her figure so that all the odd integers are represented as symmetric L–shapes. Figure 4.43]

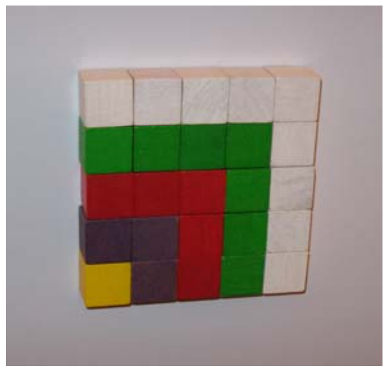


Figure 4.43. Nicole's “prettier” looking growing figure.

Nicole realized that in the “cascading” formation the odd integers follow the same pattern, symmetric L–shapes. She may have preferred this symmetry, this “commonness” of the odd integers, and that may explain why she thought of this formation as “prettier.” I think this was an “Aha!” moment for her when she realized this pattern on her own, as was the case for Rob (See Protocol 4.36 above and the paragraph that follows). The difference between these two students' thinking lies in the fact that Nicole had to refer to her written answers on the activity sheet, whereas Rob realized this formation while making his growing patterns. They are alike, on the other hand, in that they both started with non L–shape subunits generating their growing squares. The fact that the expressions on the “Total Area” column of the activity sheet were all square numbers made Nicole switch to a cascading L–shape formation is also evidence for her ability to successfully “map” algebraic symbols into a suitable representation.

Nicole then used her idea of dimensions behaving as both linear and areal quantities to develop a strategy of her own in which she reasoned quantitatively. The following discussion picks up at this point.

Protocol 4.42: Nicole's quantitative operation.

G: What is common about the odd integers in this representation now (Figure 4.43)?

N: It's like increasing the length and the width by 1 each time.

G: Where are the linear units and the areal units of the growing square in each case?

N: This is the length [points to the five cubes on the left] and this is the width [points to the five cubes at the bottom] and they are gonna be the same [Another instance for “dimensions behaving as both linear and areal units”].

G: Okay... Where are the areal units?

N: Areal units... is everything... the whole... inches squared.

G: How about this 7? What is the unit for that? [pointing to the “7” in Nicole's expression “1 + 3 + 5 + 7 + 9” on the “Area of the growing rectangle as a sum” column on the activity sheet]

N: I guess it would be inches squared.

G: How about this 5 and this 5? [pointing to Nicole's expression “W–5, L–5” on the “Dimensions of the growing rectangle” column on the activity sheet]

N: They are inches.

Nicole was thinking about the odd integers as areal quantities generating same–type quantities (growing squares) additively. Although each L–shape behaves as a linear quantity on its own (Protocol 4.40), in the process of a quantitative operation, which is addition, it behaves as an areal quantity. Nicole was aware that L–shape quantities exist on their own as well as in relation to each other via the addition operation, as suggested by her statement “It's like increasing the length and the width by 1 each time.” In Thompson's words, “A quantitative operation is a mental operation by which one conceives a new quantity in relation to one or more already–conceived quantities... It is important to distinguish between constituting a quantity by way of a quantitative operation and evaluating the constituted quantity.” (Thompson, 1994, p. 9) In other words, Nicole realized that the difference between any two consecutive odd integer is 2 = 1 + 1, where the first “1” is due to the increase in length, and the second “1” is due to the increase in width of the growing square sequence, respectively. In this way, she generated a recursively defined sequence of consecutive odd integers $\begin{cases} a_1 = 1, \\ a_{n+1} = a_n + 1 + 1 \end{cases}$ with general term $a_n = 2n - 1$. Nicole’s reference to recursion indicates how she “conceives” the new quantity $a_{n+1} = a_n + 1 + 1$ in relation to the other “already–conceived” quantities (i.e., the preceding terms of the consecutive odd integer sequence).

Rob, like Nicole, hesitated for a while at the beginning and assigned inches to the irreducible areal quantities (IAQ) generating the odd integer L–shape subunit as can be warranted by his statement “1 inch plus 1 inch plus 1 inch plus 1 inch plus 1 inch.” (See Protocol 4.37 and the paragraph that follows) I believe this is due to the fact that Rob did not realize that the L–shape is an areal quantity on its own. However, Rob was aware of the fact that the L–shape, if resulting from the iteration of irreducible areal units additively via a quantitative operation, is of areal nature. My hypotheses are warranted by his statements “but it's not like a rectangle,” “it's an area... but it's area in a different way than it was before.” Later on, Rob seemed to be more firm in his assignment of the known measurement units to the linear and areal quantities. The following discussion picks up at this point.

Protocol 4.43: Rob's association of dimensionalities to known measurement units.

G: Each added L–shape... is it a linear unit or an areal unit?

R: It's an areal unit... sure... because it's a cube...

G: At each step, the growing rectangle you obtain... is it a linear unit or an areal unit?

R: Areal.

G: Can it be expressed, at each step, the area, as a product?

R: Yes... it will always be the product of the same number. 6 inches by 6 inches...7 inches by 7 inches... 8 inches by 8 inches...

G: Do you feel the need to include a unit to distinguish between the areal and the linear units?

R: Um... I don't... but if you're trying to teach somebody you'd have to... 'cuz anytime it's two dimensional it'll be inches squared... anytime it's just a linear unit, which would be width and length, it would be just inches.

Rob not only distinguished between linear and areal quantities by assigning appropriate known measurement units, but he also referred to dimensionalities and associated those to the measurement units of his choice. Rob used this strategy in his description of linear and areal units pertaining to the first activity on prime and composite rectangles (See Protocol 4.11 and the paragraph that follows). For Rob, the linear quantities length and width are measured in inches because these quantities are of one dimensional nature. Similarly, the areal quantities are measured in square inches because of their two-dimensional nature.

Sarah, as was the case for the other interview students, was able to characterize the linear and areal quantities by reference to inches and inches squared. She also described the "area as a sum" as unit–wise equivalent to the "area as a product." The protocol below captures the relevant discussion.

Protocol 4.44: Sarah establishes the unitwise equivalence of the RHS and the LHS.

G: How about each of these? Are they inches or inches squared? [about the terms on the LHS of the identity $1+3+5+7=4\times 4$]

S: Inches squared.

G: How about this 4 and this 4? [about the terms on the RHS of the identity $1+3+5+7=4\times 4$]

S: They are inches.

G: Is it consistent? Are they equal... the LHS and the RHS?

S: When you actually do the adding or the multiplying yeah...

Sarah's words “adding” and “multiplying” refer to the quantitative operations she was performing, namely an addition on the LHS and a multiplication on the RHS. These are not the ordinary additions and multiplications, though. In Smith & Thompson's words,

> Quantitative operations (e.g., multiplicative comparison) are not the same as numerical operations (e.g., multiplication) despite the frequent similarity in terminology. Quantities that result from quantitative operations exist in two different senses, as quantities in their own right and as relationships between the two quantities. It can be conceptually demanding to reason and communicate about such quantities because we must distinguish and coordinate these two senses, and, when necessary, shift between them (2008, page 112).

If the terms on the LHS and the RHS were just numbers, then the sum on the LHS would still equal the product on the RHS. However, these are expressions Sarah obtained based on her representations. In that sense, these numbers must have some meaning, and those meanings will come from the corresponding measurement units under consideration. Because she projected those meanings to these “numbers,” these “numbers” exist as quantities for her. She was able to operate on the quantities on the LHS additively because they have the same meaning; in other words, they are like terms. Eventually, the expression on the LHS equals the expression on the RHS both numerically and unitwise. The awareness of unitwise equivalence of expressions is named quantitative unit conservation (Olive & Caglayan, 2006, 2007). At any stage of obtaining a sequence of equivalent expressions, the simplified units on both sides must be equivalent. "The simplified unit throughout the process of obtaining equivalent equations must be conserved." (Olive & Caglayan, 2007, p. 22)

With John, I played a matching game. I asked him where the "5" of the identity "$1+3+5+7=4\times 4$" was on the figure he made on the whiteboard. The following discussion picks up at this point.

Protocol 4.45: John's comparison of linear vs. areal quantities.

G: Where is that 5?

J: This is the 5 [pointing to white L shape in Figure 4.29].

G: Where is the 3?

J: [points to orange L shape]

G: Where is the 4?

J: This is the 4 [moving the pen along the edges of the 4 by 4 square]...

G: Is it clear?

J: It seems clear... the area would be 4 times 4 [He then separates the 4 by 4 square from the bigger square and writes the dimensions on the board. Figure 4.44]

G: What is 4? is it a length or area?

J: Length... Length and width [pointing to the length and width]

G: How about this 4 right here [pointing to the vertical part of the red L–shape representing number 7]?

J: That would be an area...

G: How about the 5? Is it an area or a length?

J: It's an area... yeah...

G: Do you have anything more to say on this activity? Anything to add?

J: Um... Well... I would say... I mean... If I got it right... Now it became more clear to me that this was an area rather than a length... this 4 [pointing to the vertical part of the red L–shape representing number 7]... I was not sure before... It seems more clear now... I am right! Yeah... that's it...

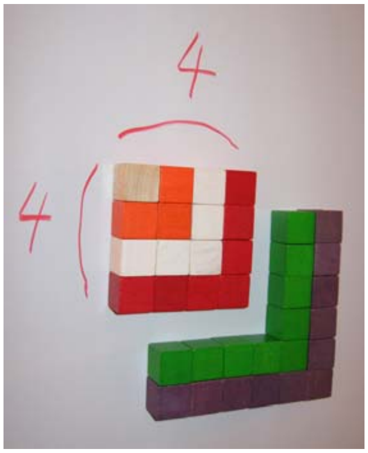


Figure 4.44. John's 4 by 4 Square Separated from the 6 by 6 Square.

In the second activity on the summation of counting numbers, John's preference often switched back and forth between same–ness and difference of same–valued linear and areal quantities. In that activity, he was hesitant as to whether the red rectangle at the bottom (Figure 4.28) stood for a linear or an areal quantity. I deduced that Nicole and John were alike in that they both saw the dimensions as some sort of representational quantities that can be part of the areal region of the growing rectangle. However, after playing the matching game as depicted in the protocol above, John did not feel any hesitation. In fact, he was firm, took a stand, and pointed out that the vertical part of the red L–shape representing the number 7 is an areal unit. He was probably remembering our comparison game where I asked him to compare the linear and areal "7"s when we were working on the growing rectangle representing the summed number 28 during the second activity. In that sense, after what happened during this interview, I infer that John was able to establish the areal–ness of the L–shape subunits on their own, as opposed to Rob and Nicole for whom these same subunits were of a linear nature on their own. Rob and Nicole established the areal–ness of these subunits by a quantitative operation. This is the slight difference between these students' thinking about L–shape subunits.

I will end this subsection on summation of odd numbers with a table of terminology that summarizes students' behaviors as they attempted to understand and make sense of linear and areal quantities as well as same–valued linear and areal quantities.

Table 4.20

Terminology Summarizing Students' Sense Making of Linear and Areal Representational Quantities

Terminology Summarizing Students' Behaviors	**Students Fitting the Terminology**
Reference to Known Units of Measurement	Brad, Nicole, Sarah, Rob, John

Iteration Strategy Combined with the Two–Dimensional Nature of Irreducible Areal Unit Generating L–Shape Subunits.	Brad, Rob
Attending to Dimensionalities of Irreducible Areal Sub–Subunits to Establish the Areal–ness of L–Shape Subunits.	Brad
Dimensions Behaving as both Linear and Areal Quantities. Seeing the Dimensions as both Linear and Areal Quantities.	Nicole
L–shapes Behave as Linear Quantities (Length and Width of the Growing Square) On Their Own. L–shapes Behave as Areal Quantities in the Process of a Quantitative (Addition) Operation.	Nicole, Rob
Assigning Inches Measurement Unit to L–Shapes in Their First Attempt.	Nicole, Rob
Association of Dimensionalities to Known Measurement Units	Rob
Establishing the Unitwise Equivalence of the RHS and the LHS of Summation Identities.	Sarah
Quantitative Reasoning	Nicole, Rob, Sarah
Quantitative Unit Conservation	Sarah
Establishing the Areal–ness of L–Shape Subunits on Their Own.	John

4.4. Summing Even Integers

When dealing with the addition of odd integers using color cubes, all interview students came up with the L–shape subunits generating growing squares. In fact, that was the only way of obtaining a sequence of growing squares representing the summation of odd integers based on the “Add them so that they generate a rectangle” direction. The rectangle condition was pertaining to the growing figures representing the summation. The uniqueness of the growing rectangle was the case for the summation of counting numbers activity, too.

The summation of even integers activity was different from the summation of counting numbers and the summation of odd integers activities in that there was not a unique growing rectangle pattern representing the summed number. As will become clearer below, students preferred either an L–shape formation (Figure 4.45), or a rectangular formation (Figure 4.46) for the even number subunits generating a growing rectangle sequence. Nicole was the only student to demonstrate both formations. Some students indicated that they remembered the L–shape formation they produced in the previous activity on the summation of odd integers. However, Sarah, who was the only student to relate the two summation formulas, provided a visual proof demonstrating how the summation identity $\sum 2i = n^2 + n$ resulted from the summation identity $\sum (2i-1) = n^2$.

Figure 4.45. Growing rectangle sequence generated via L–Shape subunits.

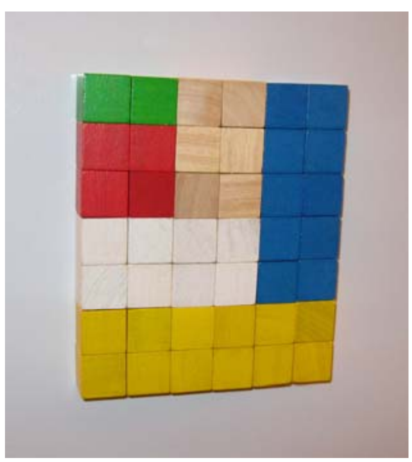


Figure 4.46. Growing rectangle sequence generated via rectangular subunits.

4.4.1. Multiplicative Representational Unit Coordination (MRUC)

Brad made even number rectangles 2, 4, 6, 8, 10, 12 as long sticks (Figure 4.47).

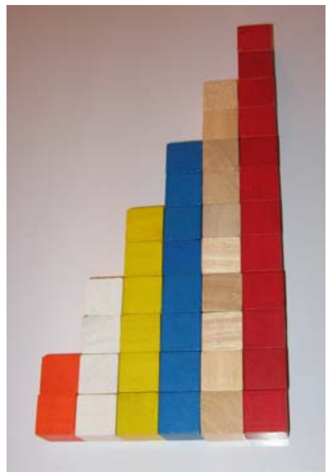


Figure 4.47. Brad's even integer subunits.

His subunits could be represented via relational notation as ordered pairs (1, 2), (1, 4), (1, 6), (1, 8), (1, 10), (1, 12) of linear units. Nicole produced the same long stick formation as Brad. As for the growing rectangle sequence, both Brad and Nicole came up with an L–shape formation. When I asked Nicole to write her answers on the activity sheet, an interesting discussion took place.

Protocol 4.46: Nicole's rectangle condition for MRUC.

N: [writes her answer “1 + 1 + 1 + 1” on the “Area of the added figure as a sum” column on the activity sheet at the second stage]

G: As a product? Is it possible to write it as a product?

N: No... Not the way I added it...

Nicole added her even integer subunits as non–symmetric L–shapes, which is why there was no way of writing their areas as a product. An L–shape to rectangle transformation is necessary to meaningfully establish MRUC. In fact, when filling in the same column on the activity sheet for the third stage, she said "I did not add it as a rectangle" while writing her answer "1 + 1 + 1 + 1 + 1 + 1." I deduce, therefore, that for Nicole, the areas of the figures representing even number subunits could be expressed as products only if they were of rectangular formation. This hypothesis is also warranted by my previous results concerning Nicole (See for instance Protocol 4.15, Figure 4.13, and the paragraph that follows).

Nicole produced another growing rectangle sequence generated via rectangular subunits, as well. John also represented the even numbers as long sticks (1 by n rectangles), which he then added as L–shapes as in Figure 4.45. The following table reflects students' written answers for the area as a product of the growing rectangles. I include a relational notation for each written answer for the purpose of analysis.

Table 4.21

Students' Written Work for the Area of the Growing Rectangles as a Product

Students	Area of the Growing Rectangle as a Product	Relational Notation Describing the Product
Brad	2 × 3, 3 × 4, 4 × 5, 5 × 6, 6 × 7, 7 × 8, $n \times (n + 1)$	(2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), $(n, n + 1)$
Nicole	2 × 1, 2 × 3, 3 × 4, 4 × 5, 5 × 6	(2, 1), (2, 3), (3, 4), (4, 5), (5, 6)
Rob	2in × 3in, 3in × 4in, 4in × 5in	(2, 3), (3, 4), (4, 5)
Sarah	4in × 5in, 5in × 6in, 6 × 7, 7 × 8, 8 × 9, $n \times (n + 1)$	(4, 5), (5, 6), (6, 7), (7, 8), $(n, n + 1)$
John	2 × 3, 3 × 4, 4 × 5, 5 × 6, 6 × 7, 7 × 8, $n \times (n + 1)$	(2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), $(n, n + 1)$

Rob and Sarah were the only ones to attach measurement unit “inches” to their linear quantities. Brad, Sarah, and John generalized their conjectures about their area as a product expressions as a function of stage number *n*. Though there was not an “area of the added figure as a product” column on the activity sheet, Rob was the only student to express the areas of his even number L–shape subunits as the products 2in × 2in, 2in × 3in, 2in × 4in. These expressions can be modeled via a relational notation of ordered pairs (2, 2), (2, 3), (2, 4) of linear units. Nicole, the only student to produce both patterns (L–shape and rectangular) on subunits generating growing rectangles, expressed the areas of her rectangular subunits as the products 2 × 1, 2 × 2, 2 × 3, 2 × 4, 2 × 5, 2 × 6. Like Rob, her expressions also can be modeled via a relational notation of ordered pairs (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6) of linear units.

4.4.2. Additive Representational Unit Coordination (ARUC)

Students struggled in assigning a multiplicative nature to their even integer subunits generating growing rectangles, yet they were most of the time confident in establishing an areal nature to both subunits and bigger units. They all provided a corresponding additive type RUC of their choice meaningfully for both subunits (made of sub–subunits) and bigger units (made of subunits).

At the elementary stage of formulating additivity for the even number subunits represented as long sticks (Figure 4.47), Brad relied on a splitting strategy and an iteration strategy. The following protocol reflects this point.

Protocol 4.47: Brad's splitting and iteration strategies.

G: What is common in these shapes? Does it tell you these are even numbers? How?

B: 'Cuz they have even amount of blocks.

G: Are these areas or lengths? Take this blue [long bar representing 8] for instance, is it an area or a length?

B: We'll go with area.

G: How do you convince me or yourself that it's an area?

B: 'Cuz we're dealing with blocks, each one has an area so what we're dealing with here is an area.

G: What are the dimensions?

B: It's 8 by 1.

Brad showed the commonness of his long stick configurations standing for even number subunits by referring to a splitting strategy. The commonness was not the fact that these subunits were all of the same form, namely the long stick formation. But it rather was in the fact that all of them could be decomposed into two equal parts. What Brad suggested by his statement "they have even amount of blocks" is a particular case of *Equal Addends* type RUC. In this formation, his even number subunits were split into equal addends (sub–subunits) as denoted by a relational notation of ordered pairs [1, 1], [2, 2], [3, 3], [4, 4], [5, 5], [6, 6] of areal sub–subunits. The areal–ness of these quantities was due to Brad's langauge "blocks" which was also used by other students many times in this analysis Chapter IV (See for instance Protocol 4.3 for Nicole, Protocol 4.21 for Sarah, Protocols 4.25 & 4.28 for Brad, and the following paragraphs). Brad combined his iteration strategy with the areal nature of the irreducible areal unit to establish the fact that areas can be added together to make other areas. His iteration strategy can be explained by *Irreducible Addends* type RUC denoted via a relational notation of ordered octuple [1, 1, 1, 1, 1, 1, 1, 1] of irreducible areal sub–subunits.

Brad was very quick in adding the even number subunits to generate a growing rectangle. His coordination is the L–shape formation. In this case, the L–shapes are not symmetric, though (Figure 4.48).

Figure 4.48. Brad's growing rectangle sequence.

Brad admitted that he based his response on what he did with the odd numbers in the previous task. Then we discussed the items on the activity sheet. Unlike what he did with the previous summation formula patterns, this time Brad relied on a different representational unit coordination type when he discussed "the area of the growing rectangle as a sum" column on the activity sheet. In this type, the addends are the terms being added: the even number subunits themselves (i.e., of *Summed Addends* Type RUC). Remember, for the previous cases, his coordination was of *Equal Addends* Type RUC. *Summed Addends* Type RUC for this particular case where the addends are even number subunits could be described as follows. Referring to functional notation $\sum_{i=1}^{i=n} f(i) = g(n)$ where areal even number subunits " $f(i)$ "s are being summed from 1 to n (number of addends) and i is the stage number (ordering number for the addends); one can write $f(i) = 2i, \forall i.$

His answers for the "area of the added figure as a sum" were 4in^2, 6in^2, 8in^2, 10in^2, 12in^2, which are not quite interesting as these are just singletons. With the relational notation, these *Singleton Addends* can be written as [4], [6], [8], [10], [12].

Nicole again exhibited her "dimensions behaving as both linear and areal units" behavior in discussing the summation of even integers context. The protocol below illustrates how Nicole's behavior results in her definitions of additive type RUCs.

Protocol 4.48: Nicole's additive type RUCs.

G: Each time the growing rectangle is an area or a length? (Figure 4.49)

N: It's an area.

G: How about this [pointing to yellow], this [pointing to orange] and this [pointing to purple]... can you describe it as a sum... of what?

N: It's the sum of the yellow plus the orange plus the purple.

G: How about this orange? Is it an area as well?

N: Yes... but it'd be harder to describe it as a rectangle.

G: Are you sure that it's an area?

N: I mean... you can find the area of it.

G: Is it a length?

N: No. 'cuz it's not straight.

G: So… what tells you that it's an area.

N: You can use the sum… 1, 2, 3, 4, counting... area of 4.

G: How about this little one thing [pointing to the orange cube at the bottom]... is it an area or a length?

N: Um... It depends on how you want to define an area… 'cuz it has an area of 1 square inch... whatever... but it also has just a length of 1 inch.

G: In this context?

N: I would say it's an area because it's part of the area of the whole big rectangle [covering the rectangle on the white board with her right hand].

Figure 4.49. Nicole's growing rectangle sequence based on L–Shapes.

Nicole was aware that at each stage the growing rectangle is of areal nature. She had some doubts concerning the L–shape subunits, though. The non–rectangularity of these subunits prevented her from establishing their areal–ness. She did not think of the L–shape subunits as areas; however, she thought that their areas could be found by counting the cubes. She did not think they were lengths as they were not of long stick formation. Her last comment about the irreducible orange unit in the discussion above combined with my previous results concerning Nicole led me to believe that not only dimensions and L–shapes, but also each irreducible unit block can behave as both an areal and a linear quantity. Finally, in her last comment, she established the areal–ness of these irreducible areal units supported with her hand gestures. The only additive RUC type arising from the above discussion is therefore the *Irreducible Addends* Type RUC describing the L–shaped even number subunits. Her written answers to "Area of the growing rectangle as a sum" column on the activity sheet, on the other hand, are of *Summed Addends* type RUC, similar to the previous two activities.

Nicole's statement "it's the sum of the yellow plus the orange plus the purple" indicates that she is able to decompose the biggest areal (the growing rectangle) unit into same−color−subregions. With Steffe's (1988) usage, Nicole is able to see the growing rectangle as a *composite unit* of L−shape composite units. In fact, as I will describe in the next section on the polynomial rectangles, her verbal expressions related to her decomposition of the polynomial rectangles into similar same−color−subregions (same−color−boxes) resulted in her successful interpretation of MRUC induced in such same−color−boxes.

Rob used the "2 by half the even integer" formation and based his growing rectangle sequence on it (Figure 4.46). Rob's written answers for the "area of the growing rectangle as a sum" are of *Equal Addends* Type RUC, once again. Besides their own additive RUC types, Nicole, Sarah and John were in agreement with each other in *Summed Addends* type in their description of the growing rectangle units when working on the “Summation of Counting Numbers,” “Summation of Odd Numbers,” and “Summation of Even Numbers” activities. Brad's Equal Addends type RUC agrees with Rob's RUC for “Summation of Counting Numbers” and “Summation of Odd Numbers” activities. Brad used *Summed Addends* type RUC in his assignment of an additive type RUC for the growing rectangle units made of even number subunits.

Equal Addends type RUC for Rob can be warranted by his statements during the interview as well. For instance, he says “2 inches squared, 10 times” while pointing to the “2”s of the growing rectangle of the 4th stage. He also says “10 inches squared 2 times,” “4 inches squared 5 times,” etc. And every time, he is pointing to these equal addends. I rely on Rob's statements as well as hand gestures to infer *Equal Addends* type RUC (Figure 4.59). With relational notation, these addends can be described as the ordered 10–tuple [2, 2, 2, 2, 2, 2, 2, 2, 2, 2], the ordered pair [10, 10], and the ordered quintuple [4, 4, 4, 4, 4] of areal subunits. In other words, Rob treats these growing rectangles as representations of composite numbers, rather than summed numbers.

Figure 4.50. Rob's hand gestures describing the equal addends.

John provided *Equal Addends* Type RUC as well as a new additive type RUC of his own in his description of the L–shape subunits (Figure 4.45). The protocol below illustrates this point.

Protocol 4.49: John's additive type RUCs concerning even number subunits.

G: How do you express that 8 [the red L–shape in Figure 4.45] as a sum?

J: 4 + 4... I'd say it could be 5 and 3...

G: How about the white?

J: 3 and 3... and... 4 and 2...

G: How about the green?

J: 5 and 5... and... 6 and 4...

G: How about the 6 and the 4 in the green... are they areas or lengths?

J: Areas.

I assert that for a decomposition technique to be claimed as an additive type RUC, the following conditions must be satisfied.

i. The (sub)unit under consideration is expressed as a sum of the (sub–)subunits.

ii. The arealness of the (sub–)subunits is established.

iii. When added together, the (sub–)subunits produce the original (sub)unit.

As warranted by Protocol 4.49, therefore, John makes use of *Equal Addends* Type RUC. His subunits are decomposed into sub–subunits as denoted by a relational notation of ordered pairs [4, 4], [5, 5], and [3, 3] of areal sub–subunits. John also introduces a new additive RUC of $(N+1)+(N-1)$ *Addends* type in his description of the even integer subunits and supported his ideas by decomposing these L–shape subunits into pairs of sub–subunits "5 and 3," "4 and 2," and "6 and 4." In this type of RUC, for each L–shape subunit, there are two addends (sub–subunits) only, i.e., $n=2.$ With the functional notation, $f(1)=N+1,$ and $f(2)=N-1,$ i.e., the addends differ by 2. Once again, one can think of the areal units "$f(i)$"s as sub–subunits generating even integer subunits. Note that in general, $f(1)+f(2)=2N,$ i.e., the even integer subunit itself. One can use a relational notation of ordered pair [N + 1, N – 1] of areal units to denote this additive type RUC for the general case. For the particular examples John refers in the protocol above where the even integers are 8, 6, and 10; one can write $f(1) = 5$ and $f(2) = 3$, $f(1) = 4$ and $f(2) = 2$, $f(1) = 6$ and $f(2) = 4$, respectively. One can use a relational notation of ordered pairs [5, 3], [4, 2], and [6, 4] of areal units to denote this additive type RUC for the corresponding particular examples, respectively.

I end this subsection with two tables based on interview students' written work concerning the even integer subunits and the growing rectangle units.

Table 4.22

Students' Written Work for the Areas of the Odd Integer Subunits as a Sum

Students	**Area of the Even Number Subunits as a Sum**	**Name of the Additive Type RUC**	**Relational Notation Describing the Sum**
Brad	$4in^2$ $6in^2$ $8in^2$ $10in^2$ $12in^2$	Singleton Addend Singleton Addend Singleton Addend Singleton Addend Singleton Addend	[4] [6] [8] [10] [12]
Nicole	1 + 1 1 + 1 + 1 + 1 1 + 1 + 1 + 1 + 1 + 1 1 + 1 + ... + 1 + 1 1 + 1 + ... + 1 + 1	Irreducible Addends Irreducible Addends Irreducible Addends Irreducible Addends Irreducible Addends	[1, 1] [1, 1, 1, 1] [1, 1, 1, 1, 1, 1] [1, 1, 1, 1, 1, 1, 1, 1] [1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
Rob	$2in^2 + 2in^2$ or $1in^2 + 1in^2 + 1in^2 + 1in^2$ $3in^2 + 3in^2$ or $2in^2 + 2in^2 + 2in^2$ or $1in^2 + ... + 1in^2$ $2in^2 + 2in^2 + 2in^2 + 2in^2$ or $4in^2 + 4in^2$ or $1in^2 + ... + 1in^2$	Equal Addends Irreducible Addends Equal Addends Equal Addends Irreducible Addends Equal Addends Equal Addends Irreducible Addends	[2, 2] [1, 1, 1, 1] [3, 3] [2, 2, 2] [1, 1, 1, 1, 1, 1] [2, 2, 2, 2] [4, 4] [1, 1, 1, 1, 1, 1, 1, 1]
Sarah	$4in^2 + 4in^2$ Many different ways	Equal Addends Random Addends	[4, 4] Many different ways
John	2 + 2 or 3 + 1 3 + 3 or 4 + 2 4 + 4 or 5 + 3 5 + 5 or 6 + 4 6 + 6 or 7 + 5 $n + n$ or $(n + 1) + (n - 1)$	Equal Addends $(N + 1) + (N - 1)$ Type Addends Equal Addends $(N + 1) + (N - 1)$ Type Addends Equal Addends $(N + 1) + (N - 1)$ Type Addends Equal Addends $(N + 1) + (N - 1)$ Type Addends Equal Addends $(N + 1) + (N - 1)$ Type Addends Equal Addends $(N + 1) + (N - 1)$ Type Addends	[2, 2] [3, 1] [3, 3] [4, 2] [4, 4] [5, 3] [5, 5] [6, 4] [6, 6] [7, 5] [n, n] [$n + 1$, $n - 1$]

Table 4.23

Students' Written Work for the Area of the Growing Rectangles as a Sum

Students	Area of the Growing Rectangle as a Sum	Name of the Additive Type RUC	Relational Notation Describing the Sum
Brad	$4in^2 + 2in^2$	Summed Addends	[4, 2]
	$6in^2 + 4in^2 + 2in^2$	Summed Addends	[6, 4, 2]
	$8in^2 + 6in^2 + 4in^2 + 2in^2$	Summed Addends	[8, 6, 4, 2]
	$10in^2 + 8in^2 + 6in^2 + 4in^2 + 2in^2$	Summed Addends	[10, 8, 6, 4, 2]
	$12in^2 + 10in^2 + 8in^2 + 6in^2 + 4in^2 + 2in^2$	Summed Addends	[12, 10, 8, 6, 4, 2]
Nicole	1 + 1	Irreducible Addends	[1, 1]
	2 + 4	Summed Addends	[2, 4]
	or 1 + 1 + 1 + 1 + 1 + 1	Irreducible Addends	[1, 1, 1, 1, 1, 1]
	2 + 4 + 6	Summed Addends	[2, 4, 6]
	or 6 + 6	Recursive Addends	[6, 6]
	or 1 + 1 + ... + 1 + 1	Irreducible Addends	[1, 1, ..., 1, 1]
	2 + 4 + 6 + 8	Summed Addends	[2, 4, 6, 8]
	or 6 + 6 + 8	Recursive Addends	[6, 6, 8]
	or 12 + 8	Recursive Addends	[12, 8]
	or 1 + 1 + ... + 1 + 1	Irreducible Addends	[1, 1, ..., 1, 1]
	2 + 4 + 6 + 8 + 10	Summed Addends	[2, 4, 6, 8, 10]
	or 6 + 6 + 8 + 10	Recursive Addends	[6, 6, 8, 10]
	or 12 + 8 + 10	Recursive Addends	[12, 8, 10]
	or 20 + 10	Recursive Addends	[20, 10]
	or 1 + 1 + ... + 1 + 1	Irreducible Addends	[1, 1, ..., 1, 1]
Rob	$2in^2 + 2in^2 + 2in^2$	Equal Addends	[2, 2, 2]
	or $3in^2 + 3in^2$	Equal Addends	[3, 3]
	or $1in^2 + 1in^2 + 1in^2 + 1in^2 + 1in^2 + 1in^2$	Irreducible Addends	[1, 1, ..., 1, 1]
	$4in^2 + 4in^2$	Equal Addends	[4, 4]
	or $2in^2 + 2in^2 + 2in^2 + 2in^2$	Equal Addends	[2, 2, 2, 2]
	or $1in^2 + 1in^2 + ... + 1in^2 + 1in^2$	Irreducible Addends	[1, 1, ..., 1, 1]
	$5in^2 + 5in^2 + 5in^2 + 5in^2$	Equal Addends	[5, 5, 5, 5]
	or $4in^2 + 4in^2 + 4in^2 + 4in^2 + 4in^2$	Equal Addends	[4, 4, 4, 4, 4]
	or $2in^2 + 2in^2 + ... + 2in^2 + 2in^2$	Equal Addends	[2, 2, ..., 2, 2]
	or $10in^2 + 10in^2$	Equal Addends	[10, 10]
	or $1in^2 + 1in^2 + ... + 1in^2 + 1in^2$	Irreducible Addends	[1, 1, ..., 1, 1]
Sarah	$2in^2 + 4in^2 + 6in^2 + 8in^2$	Summed Addends	[2, 4, 6, 8]
	2 + 4 + ... + 10	Summed Addends	[2, 4, 6, 8, 10]
	2 + 4 + ... + 12	Summed Addends	[2, 4, 6, 8, 10, 12]
	2 + 4 + ... + 14	Summed Addends	[2, 4, 6, ..., 12, 14]
	2 + 4 + ... + 16	Summed Addends	[2, 4, 6, ..., 14, 16]
	$2 + 4 + ... + 2n$	Summed Addends	$[2, 4, 6, ..., 2n]$
	$n^2 + n$	$N^2 + N$ Type Addends	$[n^2, n]$
John	2 + 4	Summed Addends	[2, 4]
	2 + 4 + 6	Summed Addends	[2, 4, 6]
	2 + 4 + 6 + 8	Summed Addends	[2, 4, 6, 8]
	2 + 4 + 6 + 8 + 10	Summed Addends	[2, 4, 6, 8, 10]
	2 + 4 + 6 + 8 + 10 + 12	Summed Addends	[2, 4, 6, 8, 10, 12]
	$2 + 4 + ... + 2n$	Summed Addends	$[2, 4, ..., 2n]$

Information pertaining to Table 4.22 reveals that Sarah is the only student to decompose even number subunits into random addends (sub–subunits). Sarah, John and Rob are the only students to make use of equal addends. Though the L–shape subunits stand for even numbers, Nicole favors for irreducible addends, rather than equal addends, which backs up my belief that for her, "Rectangle Condition" and MRUC necessitate each other. Put simply, this reveals a *concept–in–action* (Vergnaud, 1988) of the form "Rectangle Condition ⇔ MRUC." John provides a new additive type RUC which has not been used by anyone else. Brad does not decompose his even number subunits, which he treats as singletons.

A new additive type of RUC arises due to Sarah's expression $n^2 + n$, which she obtained just by looking at her growing figure generated via even number L–shape subunits with reference to her growing figure from the previous task on the addition of odd integers (Table 4.23). She reflected on what she did with the odd integers, remembered that the odd integers were building a growing square each time, and suggested [n^2, n] besides the subunits [2, 4, 6, ..., $2n$] of *Summed Addends* Type RUC, for the general case. I name this formation [n^2, n] as *Addends of* $N^2 + N$ *Type.*

Rob is the only student to refer to *Equal Addends* type RUC in his decomposition of the growing rectangle units, which he treats as composite numbers rather than summed numbers. Brad, who agreed with Rob in the previous two tasks on summations, favors the *Summed Addends* type RUC in the decomposition of the growing rectangle units. Nicole, Sarah, and John also favor the *Summed Addends* type RUC for these units. Nicole refers to *Recursive Addends* type RUC besides *Summed Addends* type RUC. Note that I classified Nicole's expression "6 + 6" in Table 4.23 as of *Recursive Addends* type and not of *Equal Addends* type RUC because in all other cases, she referred to *Recursive Addends* type RUC. The recursiveness of “6 + 6” lies in that the first “6” is the same as the sum “2 + 4” of the previous stage.

4.4.3. Linear vs. Areal Units

After he builds his growing rectangle sequence as in Figure 4.48, I ask Brad the sum at the n^{th} step in an attempt to make him generalize his results. By reference to his figure, he conjectures “So it'd be *N* plus *N* minus 2, plus...” without explaining what *n* stands for. I then ask him to visually prove the summation formula by referring to both his figure and written answers. The only help I provided was the fact that *n* stood for the general stage number. He gave up his initial conjecture and obtained the equality $n(n+1) = 2 + 4 + ... + 2n$ on the white board. I then asked him to verify whether this formula holds for the case $n = 5$. He verified both the product and the sum and obtained 30 = 30. He thinks that the yellow L–shape representing 6 is an area and that it is in inches squared. He explains the equality 3 × 4 = 2 + 4 + 6 by referring to the growing rectangle (Figure 4.51). He distinguishes between linear and areal quantities by assigning known measurement units inches and inches squared with reference to his edge/block iteration technique.

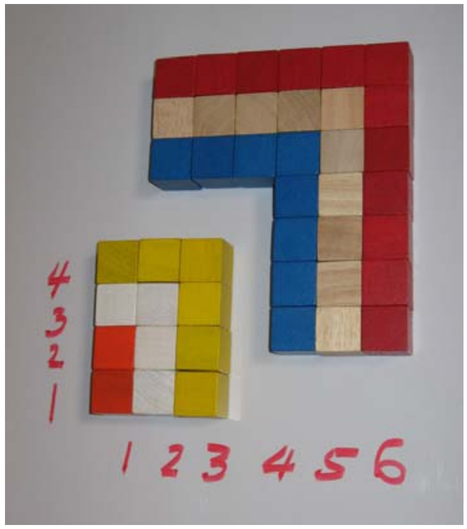


Figure 4.51. Brad's growing rectangle at the 3rd stage.

Nicole makes even number rectangles (2, 4, 6, 8, 10, 12) as long sticks (Figure 4.52). She thinks of these as linear quantities. The following protocol picks up at this point.

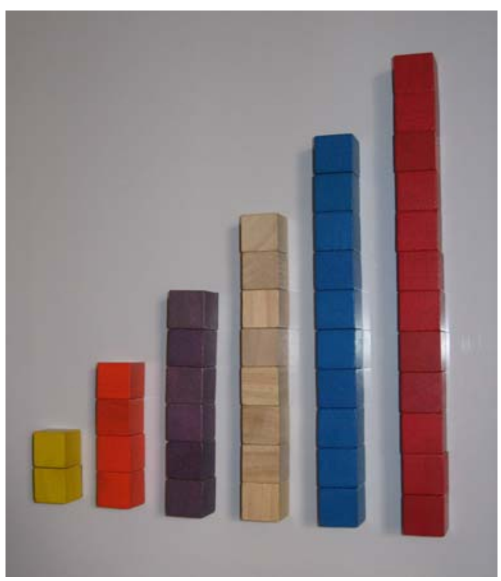


Figure 4.52. Nicole's even integer subunits.

Protocol 4.50: Nicole's even number subunits on their own.

G: Are these areas or lengths?

N: These are lengths... These are linear...

G: You made lengths right? Okay...

N: Do you want me to make areas?

G: What are the dimensions of this one [pointing to the yellow bar], for instance?

N: This is 1 by 2.

G: Okay... How about this one [pointing to the orange bar]?

N: 1 by 4.

G: You are saying 1 by 4... so it looks like it's also an area?

N: Yeah... But these are 3 dimensional shapes so...

G: Okay…

N: …but we are talking about the two dimensions of it [pointing to length and width of the orange stick] you can say the length is two [pointing to yellow stick] the length is four [pointing to orange stick] the length is six [pointing to purple stick]

G: So with these representations, these look like linear unit?

N: Hm hm [approving]

I deduce that subunits on their own stand for linear quantities for Nicole. Her question "Do you want me to make areas?" causes me to believe that if I let her continue, she was probably going to replace the $1\times n$ long sticks by $2\times\frac{n}{2}$ rectangles.

The fact that Nicole thinks of these on–their–own subunits as linear quantities prevented me from assigning a relational notation of multiplicative nature, which is why I carried this protocol over to this subsection. Her answers “1 by 2” and “1 by 4” refer just to the dimensions, and not to the areal–ness of these quantities. Nicole is aware that these are three–dimensional shapes, but at the same time she is also aware that we are talking about the two dimensions of them only. Therefore, areal–ness stands out as irrelevant in this discussion about long stick representation of even numbers. Yes the long sticks have two dimensions, length and width, however, that does not imply that these be areal quantities; they rather are linear quantities for Nicole. This line of thinking is in contrast to Sarah's thinking in which length is a line resulting from the projection of a vertically standing two dimensional figure (the sides of the cubes) onto the plane of the three dimensional figure (See Protocol 4.27 and the paragraph that follows). On their own, therefore, these even number subunits are non–areal quantities for Nicole because she focuses on their “lengths.”

I then asked Nicole to add her subunits so that they would generate a rectangle. She created an L–shape formation (Figure 4.49). The even number subunits, though of linear nature, generate areal quantities for Nicole as reflected in the following discussion.

Protocol 4.51: Nicole's subunits generating areal quantities.

G: How about this shape, the yellow and the orange together, is it an area or a length?

N: It's an area... 'cuz it's 2 by 3... area of 6.

G: How about yellow orange wood purple together?

N: It's 4 by 5.

G: Are these in inches or inches squared? [pointing to the “Area of the growing rectangle as a sum” column on the activity sheet]

N: These are in inches [writes “in” to each term under “Area of the growing rectangle as a sum”] and these are in inches squared [writes “in^2” under “Total area”].

G: You are saying these are in inches... is it because of these L–shapes?

N: Yeah.

G: You don't want to define it as in^2...

N: I wouldn't because it's pieces of it... it's not like... the whole thing yet...

I infer that for Nicole, the L–shape subunits stand for linear quantities. I expect that these L–shape subunits must generate growing rectangles of linear nature (for Nicole) because like–terms, when added together, must produce like quantities of the same nature. Therefore, I deduce that, though linear on their own, these L–shapes behave as areal quantities in the process of building a growing rectangle. In a sense, in the process of the addition operation, these L–shapes change their identity as these are linear quantities on their own (for Nicole), yet capable to generate areal quantities (growing rectangles). These hypotheses are warranted by Nicole's assignment of the measurement unit inches to her written expressions standing for the L–shape subunits in the “Area of the growing rectangle as a sum” column on the activity sheet. Nicole also described one of the growing rectangles she made both dimensionally and area–wise: “It's an area… 'cuz it's 2 by 3... area of 6.” Nicole did not reason quantitatively in the sense that she did not think about the even number subunits as quantities on their own and in relation to each other (Thompson, 1988) with the requirement that the quantitative units associated with these even number subunits in the process of the quantitative (addition) operation must be conserved at all times (Olive & Caglayan, 2006, 2007).

The following protocol illustrates a scene where Nicole came to a point where she changed her mind about the nature of the L–shape subunits.

Protocol 4.52: Nicole's change of mind about the nature of the even number L–shape subunits.

G: How about this... 5 × 6 here... this 5 here is it in inches or in^2?

N: Inches...

G: 6 and 5, are they both inches?

N: They're both inches.

G: How about these ones here... [pointing to her expression "1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1" on the "area of the added figure as a sum" column of the activity sheet] are they inches or in^2?

N: I would say just... [hesitant] as a sum... I would say they are just inches because it's not really... [intending to say "it's not really an area"]

G: It's not really?

N: Well... it's not really a length either... I guess it would be an area because you are counting the blocks [changes her mind]

For the first time, Nicole paid attention to the irreducible areal characteristics of the blocks, which generate not only the even number subunits, but everything of areal nature pertaining to the growing rectangles made of color cubes, as well. The discussion came to a turning point where Nicole felt the need to embrace linearity or arealness (but not both) of L–shape subunits. She finally put an end to the dilemma and favored arealness as she obtained her L–shapes (and eventually her growing figures) by iterating inch squared unit irreducible areal blocks. The following protocol clarifies how Nicole made sense of her preference for the arealness of these L–shape subunits.

Protocol 4.53: Nicole's sense making of areal L–Shape subunits.

G: Can we attach a unit to that? [pointing to her expression "1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1" on the "area of the added figure as a sum" column of the recording sheet]

N: Inches squared.

G: So you are changing your mind?

N: Yeah.

G: Tell me more about that... what made you change your mind?

N: It's because... We're talking about the actual area here [pointing to the "area of the added figure as a sum" column of the recording sheet]... So it can't be the length 'cuz it's curved [meaning the blue L–Shape standing for the even number subunit 10. See Figure 4.49] and so... we are talking about the area… So it would be in inches squared.

Though she had the same number of cubes standing for her even number subunits in both Figure 4.49 and Figure 4.52, there is no direct unit–wise equivalence of these same–valued quantities. For instance, for Nicole, the blue long stick 10 and the blue L–shape 10 are different quantities, in that sense. In addition, her irreducible areal unit (block) iteration strategy was valid only for the blue L–shape 10 and not for the long stick 10. Even though it looks like Nicole put an end to her dilemma, a couple of times during the interview while writing her answers on the activity sheet, Nicole said, "I did not make rectangles" about her L–shape added figures. So despite the "change of mind" turning point, she still had some concerns about the "already established arealness" of these L–shape quantities. I probed on that and asked her what she meant by "I did not make rectangles." She then came up with a visual demonstration. The protocol below illustrates her new idea.

Protocol 4.54: Nicole's growing rectangle sequence generated via rectangular subunits.

N: If I added rectangles, it would have been like this [pointing to her new demonstration. See Figure 4.53]. But I chose not to. I chose to add them as L–shapes.

G: Let's compare this 4 [meaning the orange rectangle in Figure 4.53] with the L–shape 4 [from her previous demonstration, Figure 4.49]. How about this one [meaning the orange rectangle in Figure 4.53]... Is it an area as a product of what?

N: As a product of 2 and 2 inches.

G: As a sum of what?

N: As a sum of 1, 1, 1, 1.

G: And the L–shape 4, is it an area as a sum of what?

N: 1, 1, 1, 1.

G: As a product of what?

N: You can't do it as a product.

G: Which one looks more like areal unit... this one [meaning the orange rectangle in Figure 4.53] or the L one [from her previous demonstration, Figure 4.49]?

N: This one [meaning the orange rectangle in Figure 4.53] 'cuz these are rectangles.

Figure 4.53. Nicole's growing rectangle sequence generated via rectangular subunits.

In the paragraph following Protocol 4.46, I hypothesized that the rectangle condition was a necessary condition for a figure to be established "of areal nature" for

Nicole. Based on this new data coming from Protocol 4.54, I must revise my previous hypothesis. Because Nicole defined linearity for the long sticks, I have to exclude the long sticks from the analysis of the current situation. The rectangle condition is therefore necessary but not sufficient. For Nicole, a figure made of cubes is of areal nature only if the figure makes a special rectangle such that both the length and the width of this special rectangle must be greater than or equal to 2. Nicole creates a *concept–in–action* (Vergnaud, 1988) of the form "$a \times b$ Rectangle $\left(\text{such that } a \geq 2, b \geq 2\right)$ Condition $\Leftrightarrow$ MRUC."

Because Nicole was the only student to provide both formations (Figure 4.49 and Figure 4.53), I wanted her to write her answers for this new demonstration, so I asked her to write her answers for the "Area of the added figure as a product" column on the recording sheet. Her written answers $2 \times 1, 2 \times 2, 2 \times 3, 2 \times 4, 2 \times 5, 2 \times 6$ caused her to realize that each even number has a factor of 2 and that the other factor coincides with the stage number. The following protocol elaborates on these issues.

Protocol 4.55: Nicole's incomplete mapping structures.

G: How about these numbers, are they lengths or areas? [pointing to 2×1]

N: They're lengths.

G: How do you recognize that they're lengths? Where are they?

N: Because what I am talking about is just the length of it and the width of it... not the whole entire area of it. I am talking about just the outer edges.

G: Purple one?

N: Purple one would have a width of 2, and a length of 3.

Nicole's statement “What I am talking about is just the length of it and the width of it... not the whole entire area of it. I am talking about just the outer edges” calls for a multiplicative type RUC, which can be denoted via a relational notation of ordered pair (length, width) of linear units. I deduce that a reference to *Mapping Structures* is still missing (incomplete) as Nicole does not establish the existence of an areal quantity resulting from the mapping of these linear units through the multiplication operation. Compare this thinking with Protocol 4.16 where Nicole actually referred to *Mapping Structures* by saying “I would say three [pointing to the three cubes on the left] times seven [pointing to the seven cubes at the bottom]. Three inches times seven inches would give me 21 inches squared.” John was the other interview student to make use of *Mapping Structures* via his statement “Length of 5 and width of 1 in which case the area would be 5.” (See Protocol 4.22 and the paragraph which follows) Nicole, Sarah, and John were the only interview students to map their ordered pairs of linear quantities into the appropriate areal quantity through the multiplication operation in their work with the magnetic color cubes. Though the existence of *Mapping Structures* is not completely established, Nicole was able to distinguish between the linear and areal quantities by reference to dimensionalities and known measurement units as illustrated in the following protocol.

Protocol 4.56: Nicole's association of dimensionalities to known measurement units.

G: Okay... How do you distinguish between the dimensions and the area?

N: Length... just the edges and area everything inside... so it's kinda like perimeter it's just outside... the length and the width are part of the perimeter and the area is everything within the perimeter.

G: So... the area and the perimeter... How are they different? In what ways?

N: Perimeter is in inches. It's one dimensional... Length is 1 dimensional and area is 2 dimensional... I would attach inches to perimeter, and inches squared to area.

Nicole distinguished between linear and areal quantities by assigning known measurement units inches and inches squared. She also related these measurement units to dimensionalistic properties of the corresponding linear and areal quantities. Nicole's interest in a particular linear quantity, perimeter, is worth mentioning as she is the only student to mention perimeter in a discussion. Rob used a similar strategy (without reference to perimeter) in his description of linear and areal quantities pertaining to the first activity on prime and composite rectangles and the third activity on the summation of odd integers (See Protocol 4.11, Protocol 4.43 and the paragraphs that follow). Linear quantities are on the "edges" and the areal quantities are "everything within the perimeter," according to Nicole.

Rob was the only other student to generate a sequence of growing rectangles based on rectangular subunits (Figure 4.50). The following discussion picks up at the point where Rob revealed the dimensionalistic properties of the linear and areal quantities with reference to his even number subunits and the growing rectangles.

Protocol 4.57: Rob's reference to rectangle condition and dimensionalities.

G: Now each even integer... can it be written as a sum? How? Can it be written as a product? How? For instance, this 8 [pointing to the white in Figure 4.50] Is it an area or a length?

R: Area.

G: How do you figure?

R: Because it makes a rectangle by itself... and it's two dimensional.

G: What is the length and the width? [about the 4 by 5 rectangle. Figure 4.54]

R: 4 inches by 5 inches.

G: Are they linear or areal units?

R: Linear.

G: But 4 is also in here [pointing to the red rectangle in Figure 4.54]

R: Yes... but that would be an area. But the length of the entire square [meaning rectangle] is 4 linear inches... length of the rectangle sorry...

G: Each separate even integer... you're saying that... is an area?

R: Yes.

G: How about their combination when they are added together?

R: These are all inches squared and when you put them together... still an area...

Figure 4.54. Rob's growing rectangle sequence of the 4th stage.

Rob's compound proposition “it makes a rectangle by itself... and it's two dimensional” can be thought of as two distinct statements connected via the conjunction “and.” For Rob, the arealness of the rectangular even number subunits is established only if the following conditions are satisfied:

i. Each even number subunit must be represented as a rectangle made of color cubes (Rectangle Condition)

ii. Two–dimensional characteristic of these even number subunits must be mentioned (Dimensionalities)

Rob also distinguished between same–valued linear and areal quantities by reference to the known measurement units inches and square inches. He was aware that the addition of *of–the–same–nature* even number subunits yields another *of–the–same–nature* bigger unit, namely the growing rectangle. Rob made sense of his areal quantities by attending to quantitative unit conservation (Olive & Caglayan, 2006, 2007) as opposed to Nicole who assigned different measurement units to her subunits and growing rectangles despite the fact that she obtained the latter ones by operating on the former ones additively (See Protocol 4.51 and the paragraph that follows).

Sarah started by making long bars for the even numbers 2, 4, 6, 8, and 10. She then added them to generate a growing rectangle sequence based on L–Shape subunits (Figure 4.55) similar to Brad, Nicole, and John.

Figure 4.55. Sarah's growing rectangle sequence based on L–shapes.

She said that this pattern was similar to the odd integers. The protocol below illustrates this point.

Protocol 4.58: Sarah's visual proof relating two summation formulas.

S: The only difference is that we have an extra row [She splits the extra row as in Figure 4.56]

G: So you discovered the formula I guess...

S: Yeah... It would be *n* squared plus whatever that is [pointing to the extra row she just split]... *n* [very excited]… *n* squared plus *n* yeah! [very excited]

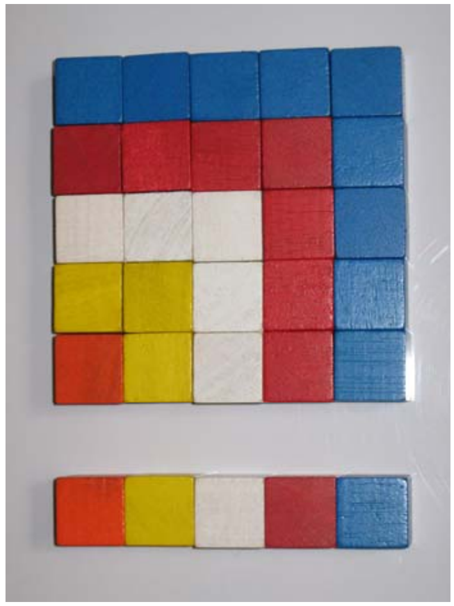


Figure 4.56. Sarah's decomposition of her growing rectangle sequence into a square and a long stick.

Sarah introduced *n* right after pointing to the extra row she just split. She therefore first visually located both the growing square and the long stick, and later on connected these objects to their dimensionalistic properties. This answer came from her, this was Sarah's idea; I did not say anything at all. In fact, I never thought about Sarah's “extra row” formulation serving as a bridge between the two summation formulas before; this was something new to me. She made the figure above and generalized it. She saw the square as an *n* by *n* square, and the bar as *n* ($N^2 + N$ Type Addends). Her algebraic generalization was based on a representation, Figure 4.56, which was a particular case for $n = 5$. Sarah was reasoning quantitatively by not only relating each L–shape even number subunit to the corresponding L–shape odd number component, but by connecting the two separate sequences of growing figures, as well.

Sarah's discovery of the literal expression $n^2 + n$ arising from the particular case $n = 5$ showed that she was able to make a conjecture and a generalization. I wanted to learn more about the meanings she would give to these quantities, as illustrated in the protocol below.

Protocol 4.59: The meanings charged into the $N^2 + N$ *addends* and mapping structures.

G: Okay... *n* squared plus *n*... tell me more about that... What units have *n* squared and *n*?

S: Well... *n* squared is *n* times *n*... so inch times inch it would be inches squared.

G: How about the extra *n*... is it an area or a length?

S: I don't know...

G: Is it in the area... that *n*?

S: Yeah... so it would be inches squared... by itself...

G: Does it make sense?

S: Yeah... well added together that has to equal an area so... since you are adding them together they have to have the same units... so it would be inches squared.

G: Okay... Where is the inches squared in *n*? In that *n*? [meaning the extra row]

S: It has to be in inches squared...

G: Okay... How do you say that? How do you figure?

S: It's just that it's n times 1... n inch and 1 inch... and then when you multiply them it'd be inches squared.

Sarah established arealness for the addend n^2 by multiplying the corresponding same–valued linear quantities. She demonstrated how the value–wise multiplication of n and n yielded "value" n^2 and the "unit–wise" multiplication of inches and inches yielded the measurement unit inches squared. When working on the first activity on prime and composite rectangles, Nicole showed a similar thinking. She multiplied the values of linear quantities as well as the measurement units attached to those quantities to produce a quantity *of–a–new–kind* (Schwartz, 1988). I infer a 2–fold Mapping Structure (concept–in–action) for Sarah's representationally coordinated quantities. Both the ordered pair of values (n, n) and the measurement units (inches, inches) are mapped into the corresponding value n^2 and the measurement unit in^2 with the multiplication operation behaving as a mapping.

A 2–fold Mapping Structure is slightly different from the ordinary Mapping Structure in that the multiplication mapping operates on both values and measurement units separately as in Sarah's statement "n squared is n times n... so inch times inch it would be inches squared." The ordinary Mapping Structures can be witnessed in Nicole and John's statements ("three inches times seven inches would give me 21 inches squared," "Length of 5 and width of 1 in which case the area would be 5") in the previous sections (cf. Protocol 4.16 & 4.25 and the paragraphs that follow). Because the relational notation of ordered pairs (or n–tuples in general) already possesses the measurement units, both types of Mapping Structures have equivalent relational notations, such as (n, 1) and (n, n) for the example above. A functional notation that describes both types of Mapping Structures Sarah refers in the protocol above can be written as $f:(n,1)\mapsto 1$ and $f:(n,n)\mapsto n^2$, where, f stands for the multiplication operation behaving as a mapping.

Sarah hesitated for a very short time in her sense making of the "extra row" when trying to determine whether it was a linear or an areal quantity. By her statement "since you are adding them together they have to have the same units... so it would be inches squared," I hypothesize that she established the arealness of this quantity by deductive reasoning. The steps Sarah followed in her deductive reasoning can be outlined as follows:

i. The big $n^2 + n$ rectangle is an areal quantity.
ii. The *addend* n^2 is an areal quantity.
iii. Therefore, the "other" *addend* n must be an areal quantity, as well.

And finally, once she established the arealness of the "extra row" quantity via deductive reasoning, Sarah validated her judgment via inductive reasoning with reference to *2–fold Mapping Structures*: She mapped both the values and measurement units associated with the linear quantities into their areal counterparts as evidenced by her statement "It's just that it's n times 1... n inch and 1 inch... and then when you multiply them it'd be inches squared."

Note that Sarah's comments and actions in Protocol 4.59 above pertained to the addends n^2 and n generating the growing rectangle. As described in the previous subsection, these addends can be modeled via a relational notation of ordered pairs [n^2, n] of areal subunits. Besides, Sarah established multiplicative structures (MRUC) embedded within additive structures (ARUC), which can be notated as $[(n,n),(n,1)]$. Although the n^2 was a growing rectangle in the context of the previous task on the summation of odd integers, Sarah interpreted it as a *subunit* in the context of the summation of even integers activity. Note that what Sarah established was the formation [n^2, n] – equivalently, the formation $[(n,n),(n,1)]$ – and not the formation (n, n + 1), denoting equivalent quantities. In fact, when she was working with the "area of the growing rectangle as a product" column on the activity sheet, after providing the answers 4 × 5, 5 × 5, 6 × 7, 7 × 8, 8 × 9, I asked her whether these looked like the expression $n^2 + n$ she discovered above. Sarah established the equivalence of these two formations as illustrated in the protocol below.

Protocol 4.60: Equivalence of [n^2, n] and (n, n + 1) formations.

G: Does this look like $n^2 + n$? [pointing to Sarah's written expressions on the "area of the growing rectangle as a product" column on the activity sheet]

S: No. Does it? I don't think so...

G: Okay... Now I am gonna ask you to factorize it...

S: Oh... n times n + 1! [She writes $n^2 + n = n(n + 1)$ and very excited] It works...

G: Does it make sense?

S: Hm hm...

G: What units would you attach to n and n + 1?

S: The n and the n + 1 are both in inches.

G: What do you think about these as teaching tools?

S: It's cool... I think it would work with summations. I did not learn the summations in high school though... I guess if you are teaching summations it should work...

Sarah's statements from Protocol 4.59 and Protocol 4.60 necessitate the existence of a theoretical construct that I name *Equivalence of Mapping Structures*. There must be an agreement of the ordered pair (n, $n + 1$) of linear units and the ordered pair [n^2, n] of areal units. These two formations can be reconciled via the equivalence of mapping structures. The multiplication operation, which behaves like a function or mapping, can be represented using a functional notation such as $f : (n, n+1) \mapsto n^2 + n.$ Here, f denotes the multiplication operation mapping the linear units n and $n + 1$ into the corresponding areal unit $n^2 + n$ that is also the same as the area of the growing rectangle.

Similarly, the addition operation behaves like a function or mapping, acting on irreducible areal quantities (unit blocks) or combinations of those. For instance, the function g, which represents the addition operation, acts on the ordered pair [n^2, n] of areal units and maps it into the areal unit $n^2 + n$. Using a functional notation this can be written as $g : [n^2, n] \mapsto n^2 + n$. In other words, though they act on different types of representational quantities, the images of the mappings f and g coincide (Figure 4.57). This is the essence of what is meant by “identity” in this research project. “Area as a product” coincides with “area as a sum” eventually because of these mapping structures.

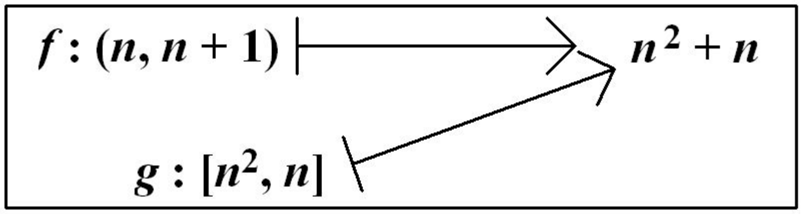


Figure 4.57. Equivalence of mapping structures.

John wrote the identities corresponding to each stage on the board. He also suggested a general formula (Figure 4.58).

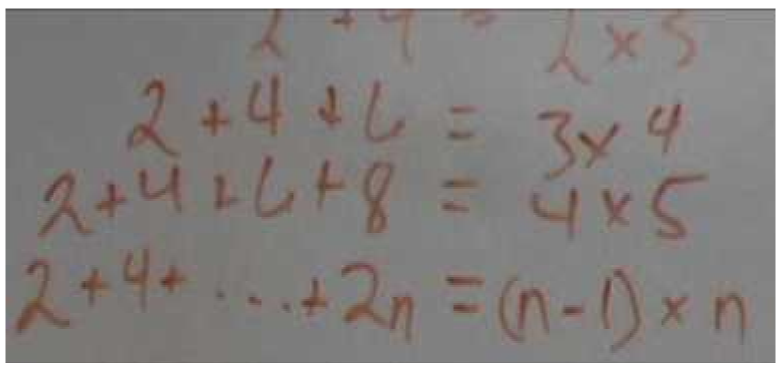


Figure 4.58. John's summation identities.

The following protocol illustrates John's conjecture in an attempt to make a generalization.

Protocol 4.61: John's conjecture.

G: This is great! You are suggesting a formula... Let's try that formula to see if it works for... for example, for $n = 4$... Let's try that...

J: n equals 4 would be this one right here [pointing to the identity $2 + 4 + 6 + 8 = 4 \times 4$]

G: This is your conjecture right? [pointing to the identity $2 + 4 + ... + 2n = (n - 1) \times n$] Now let's see if it'll work for $n = 4$. Now I want you to rewrite it for $n = 4$.

[John realizes that his conjecture is false. Figure 4.59]

J: Well... that's not gonna work... The right hand side should be n times $n + 1$ (Figure 4.60).

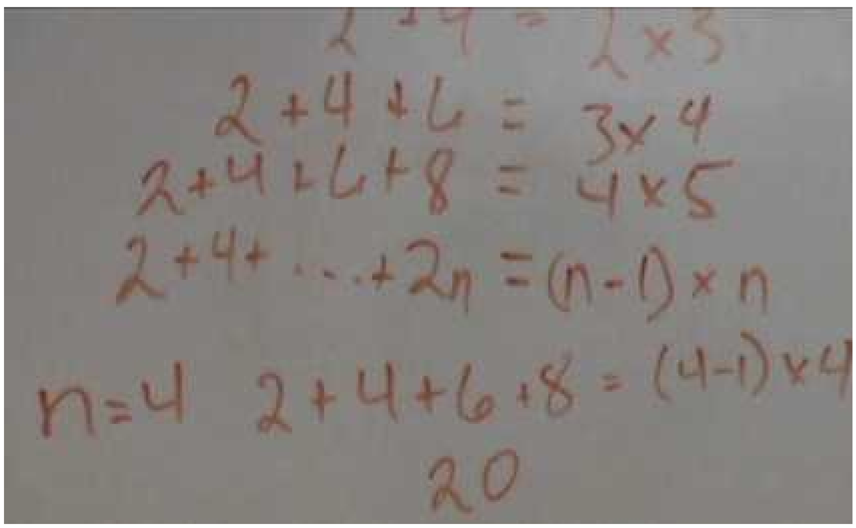


Figure 4.59. John falsifies his conjecture.

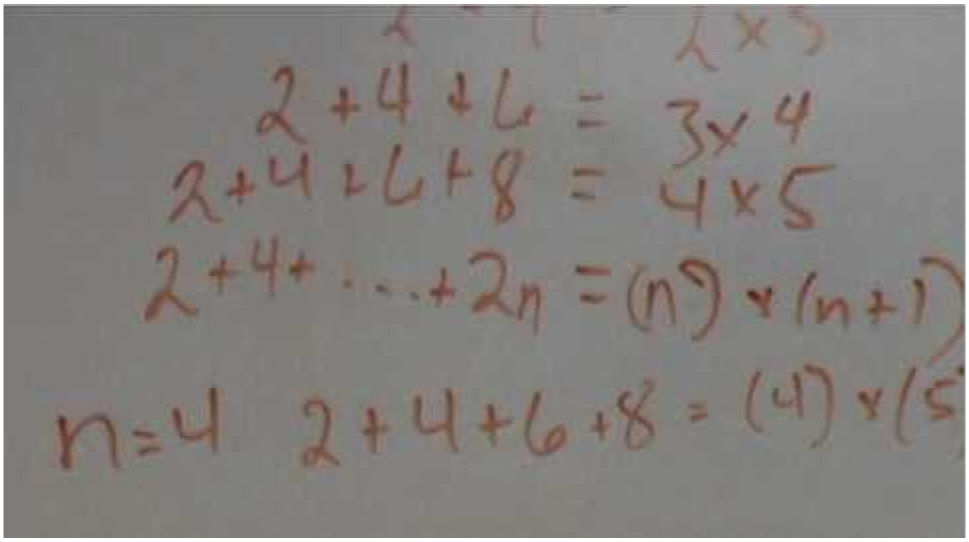


Figure 4.60. John's corrected summation identities.

We then played a comparison game. I first asked him to compare the linear and areal "*n*"s in his identity $2+4+...+2n = n(n+1)$. The protocol below elaborates on that conversation.

Protocol 4.62: John's comparison of linear and areal quantities and mapping structures.

G: How about this *n*... is it an area or a length? [pointing to the first *n* on the RHS of John's identity $2+4+...+2n = n(n+1)$]

J: This *n* is a length. [pointing to the first *n* on the RHS]

G: How about this *n*? [pointing to the *n* on the LHS]

J: It's an area.

G: How about the 2*n*?

J: It's an area.

G: How about this 8 here... [pointing to the "8" of the identity 2 + 4 + 6 + 8 =

4 × 4] is it an area or a length?

J: It's an area.

G: Where is it on the figure?

J: Right here... [pointing to the red L–shape in Figure 4.45]

G: How about the 6 and the 4 in the green (Figure 4.45)... are they areas or lengths?

J: Areas.

G: But this 6 is also the length of this rectangle [pointing to the 6 by 7 rectangle in Figure 4.45]. Are they the same or different?

J: Different.

G: How are they different?

J: These 6 cubes by itself represent an area [pointing to the horizontal part of the green L–shape in Figure 4.45] So this is... It's not just 6... It's 6 and 1.

John compared the linear and the areal quantities (as well as the same–valued linear and areal quantities) without reference to known measurement units. He simply used the phrases “lengths” or “areas” to establish the linearity or the arealness of the representational quantities under consideration.

John's language “It's not just 6... It's 6 and 1” can be explained using the *Mapping Structures* analysis model. The 2–foldness in these structures is missing as John did not mention unit–wise mapping. John only mapped the values of the linear quantities into the value of an areal quantity. Multiplicative RUC arises from his language “It's 6 and 1;” however, that is not the whole story. Multiplicative RUC is only a prerequisite for the construction of a *Mapping Structure*. In fact, John built on the Multiplicative RUC by his statement “It's not just 6,” which indicates the value of the areal quantity under consideration that exists because of the multiplication operation that behaves as a mapping acting on the ordered pair (6, 1) of linear units. I assert that for a *Mapping Structure* of multiplicative type to exist, therefore, one needs to establish the following conditions:

1. A pair ordering of the values of the linear quantities is mentioned.
2. The multiplication operation behaving as a mapping is acting on the ordered pair of these linear quantities.
3. The value of the areal quantity resulting from the mapping is indicated.

For a *2–fold Mapping Structure* to exist, on the other hand, the conditions above must hold as well as the following:

1'. A pair ordering of the measurement units of the linear quantities is mentioned.

2'. The multiplication operation behaving as a mapping is acting on the ordered pair of these linear measurement units.

3'. The measurement unit of the areal quantity resulting from the mapping is indicated.

In all the activities on color cubes, Nicole and John were the only students to make use of (ordinary) *Mapping Structures*, and Sarah was the only one to make reference to *2–fold Mapping Structures*. Conditions necessitating the existence of ordinary and 2–fold *Mapping Structures* of additive type can be established in a similar manner as in 1, 2, 3, and 1', 2', 3' above.

4. An *n*–tuple ordering of the values of the areal quantities is mentioned.

5. The addition operation behaving as a mapping is acting on the ordered *n*–tuple of these areal quantities.
6. The value of the areal quantity resulting from the mapping is indicated.

4'. An *n*–tuple ordering of the measurement units of the areal quantities is mentioned.

5'. The addition operation behaving as a mapping is acting on the ordered *n*–tuple of these areal measurement units.

6'. The measurement unit of the areal quantity resulting from the mapping is indicated.

I end this section on summation of even numbers with a table of terminology that summarizes students' behaviors in an attempt to understand and make sense of linear and areal quantities as well as same–valued linear and areal quantities.

Table 4.24

Terminology Summarizing Students' Sense Making of Linear and Areal Representational Quantities

Terminology Summarizing Students' Behaviors	**Students Fitting the Terminology**
Reference to Known Units of Measurement	Brad, Nicole, Sarah, Rob, John
Iteration Strategy Combined with the One–Dimensional Nature of Irreducible Linear Unit (unit edge) Generating the Length of the Rectangle.	Brad
Iteration Strategy Combined with the Two–Dimensional Nature of Irreducible Areal Unit Generating Even Number Rectangle Subunits.	Brad
Long Sticks Behave as Linear Quantities On Their Own.	Nicole
Quantitative Units Not Conserved.	Nicole
Same–Valued Long Stick and L–Shape Quantities Possess Different Units, hence Are Different Quantities.	Nicole
A figure made of cubes is of areal nature only if the figure makes a special rectangle such that both the length and the width of this special rectangle must be greater than or equal to 2.	Nicole

Mapping Structures Incomplete.	Nicole
Association of Dimensionalities to Known Measurement Units	Nicole
Linear quantities are on the "edges" and the areal quantities are "everything within the perimeter."	Nicole
Arealness Resulting from Rectangle Condition and Dimensionalities.	Rob
Quantitative Unit Conservation	Rob, Sarah
"Extra Row" Formulation Serving as a Bridge between the Two Summation Formulas.	Sarah
Quantitative Reasoning	Sarah
Quick Generalization of the Summation Formula During Conjectural Process.	Sarah
Establishing the Unitwise Equivalence of the RHS and the LHS of Summation Identities.	Sarah
2–Fold Mapping Structures.	Sarah
Arealness of "Extra Row" Established by Deductive Reasoning.	Sarah
Equivalence of Mapping Structures.	Sarah
Mapping Structures.	John

4.5. Multiplication of Polynomial Expressions

In this section, I will analyze data related to three multiplication types:

i. Multiplication of polynomials of the form $p(x)$ and $q(x)$, where both polynomials are elements of the set $Z[X]$. To be more specific, $p(x) = x + 1$ and $q(x) = 2x + 3$.

ii. Multiplication of polynomials of the form $p(x)$ and $q(y)$, where $p(x) \in Z[X]$ and $q(y) \in Z[Y]$. To be more specific, $p(x) = x + 1$ and $q(y) = 2y + 3$.

iii. Multiplication of polynomials of the form $p(x, y)$ and $q(x, y)$, where both polynomials are elements of the set $Z[X, Y]$. To be more specific, $p(x, y) = 2x + y$ and $q(x, y) = x + 2y + 1$.

I define a polynomial rectangle as a rectangle representing a specific polynomial made of different sized color tiles. Representationally speaking, various integer number combinations of irreducible quantities 1, x, y, xy, x^2, y^2 that are represented by different sized color tiles – also referred as algebra tiles or algebra models in the literature – are used to generate polynomial rectangles (Figure 4.61). For instance, it is not possible to represent the real coefficient polynomial $0.25+\frac{2}{3}x+\sqrt{2}y+\frac{1}{\sqrt{\pi}}y^2$ by using these tiles. To be more specific, in my study with preservice teachers, we focused on integer coefficient polynomials in one variable $Z[X]$ as well as integer coefficient polynomials in two variables $Z[X,Y]$. $Z[X]$ and $Z[X,Y]$ can be thought of as sets of integer coefficient polynomials in one variable and in two variables, respectively. Although I use the notations $Z[X]$ and $Z[X,Y]$, we only focused on polynomials with positive integer coefficients (The back sides of the algebra tiles are all red representing negative coefficients).

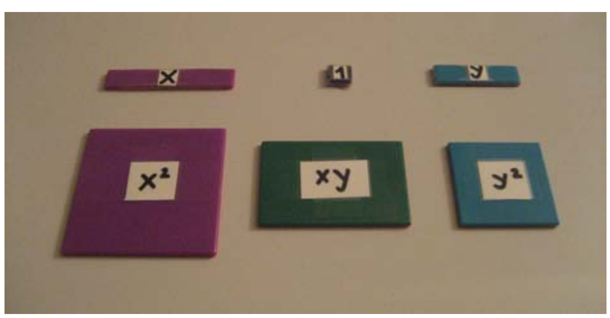


Figure 4.61. Irreducible algebra tiles.

In all student interviews, I started the discussion on polynomial rectangles by introducing the "basic" tiles. By basic tiles, I mean the purple bar representing x, the blue bar representing *y*, and the little black square representing 1. Then I asked the students to define the products *x* times *x*, *y* times *y*, and *x* times *y*, respectively, on the multiplication mat in order to familiarize them with all the different sized color tiles. They all came up with similar answers without my assistance. In this introductory part, therefore, I only write about Brad's definitions of the product tiles.

To define the product "*x* times *x*" on the multiplication mat, Brad first placed one purple bar on the left and one at the top and then located a big purple square among the algebra tiles and placed it in between them (Figures 4.62 & 4.63). Brad defined the big purple square as "*x* squared."

Figure 4.62. Algebra tiles representing *x*.

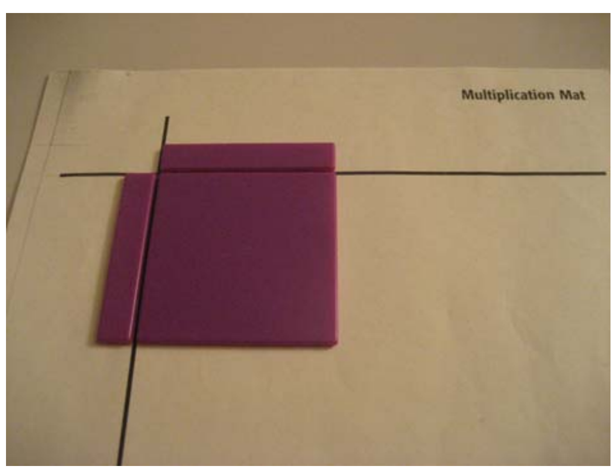


Figure 4.63. Algebra tiles representing *x* and *x* squared.

Similar to what he did with purple bars, he then placed one blue bar on the left, and one blue bar at the top; and then located a big blue square among the algebra tiles and placed them in between them (Figures 4.64 & 4.65), which he defined as "*y* squared."

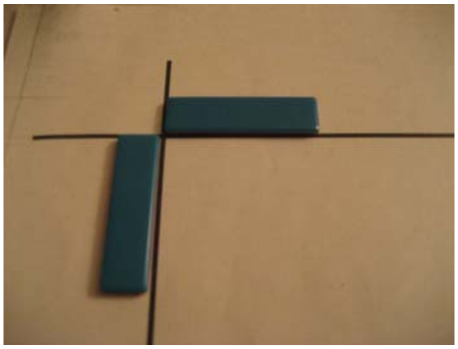


Figure 4.64. Algebra tiles representing *y*.

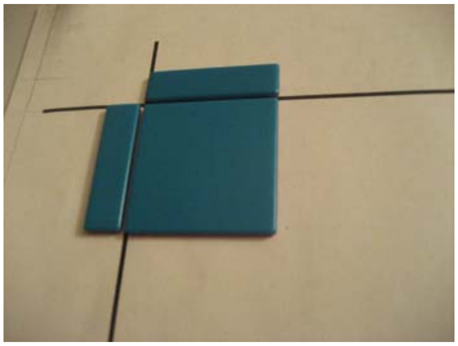


Figure 4.65. Algebra tiles representing *y* and *y* squared.

Finally, using a similar strategy, Brad defined the product tile representing *xy*, as in Figures 4.66 and 4.67 below.

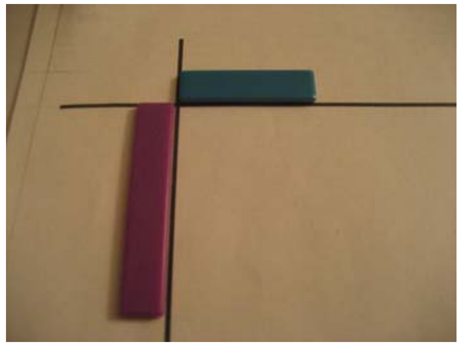


Figure 4.66. Algebra tiles representing *x* and *y*.

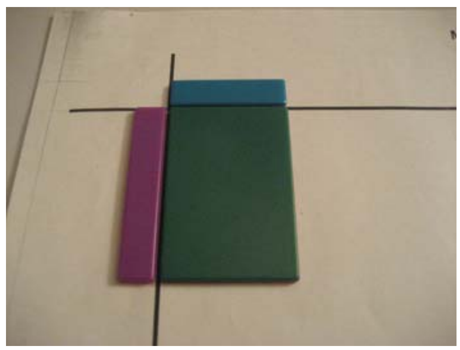


Figure 4.67. Algebra tiles representing *x*, *y*, and *xy*.

All interview students produced the same product tiles *x* squared, *y* squared, and *xy*.

4.5.1. Multiplicative Representational Unit Coordination (MRUC)

I asked students to make an $x + 1$ by $2x + 3$ rectangle using the algebra tiles. Brad's ordering of the tiles representing the $x + 1$ is interesting in that he first placed the "1" tile, and then below the "*x*" tile (Figure 4.68).

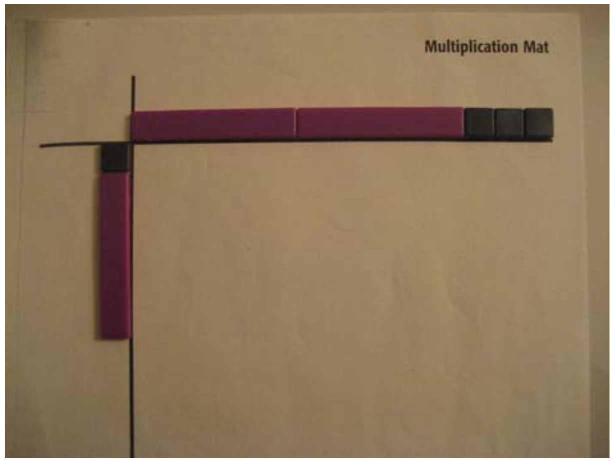


Figure 4.68. Brad's dimension tiles representing $x + 1$ and $2x + 3$.

I waited to see what Brad would do, and he seemed to be "filling in" the rectangle like a puzzle rather than doing term wise multiplication because he first placed the big purple square representing "*x* squared" at the upper left corner (Figure 4.69). If he had been thinking of term wise multiplication, then he would have placed a purple bar in the upper left corner (Figure 4.70).

Figure 4.69. Brad's areal tiles "filled in" the rectangle.

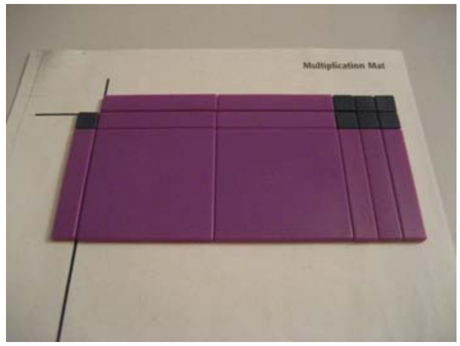


Figure 4.70. What term wise multiplication would produce.

At this elementary stage of generating a polynomial rectangle, Brad failed to think in a multiplicative way. His interpretation of “area as a product” terminology was lacking the multiplicative aspect, as illustrated in the following discussion.

Protocol 4.63: Brad's interpretation of “area as a product.”

G: What is the area of the polynomial rectangle as a product? [Brad writes $2x^2 + 5x + 3$ on the activity sheet] What did you do? How did you get that?

B: I had to multiply them [meaning $x + 1$ and $2x + 3$] together... just like making a rectangle.

G: You multiplied them algebraically in there [meaning on the activity sheet]?

B: Hm hm...

G: So, I want to... Where is the product in that case?

B: Where is the product?

G: I mean, area as a product... Where is that product?

B: The length times the width.

G: So... Does this [meaning his expression $2x^2 + 5x + 3$ he wrote on the activity sheet] look like the length times the width?

B: It's what happens when you multiply them together, yes.

There seems to be an “algebra– geometry” disconnect here as Brad was unable to locate the elements $2x^2$, $5x$, and 3 in the set of tiles he had just arranged. I infer that Brad did not see the areal tiles as quantities in this situation. He saw the area as a sum, algebraically; however, representationally, he failed to see the area as a sum or as a product. In fact, what Brad understands from “product” is the “result” of the multiplication.

Similarly, Rob disregarded term wise multiplication and was unable to relate the “resulting” areal tiles to the dimension tiles *in the process of constructing the polynomial rectangle*. Rob placed the dimension tiles and started by filling in his puzzle with two big purple squares (Figure 4.71).

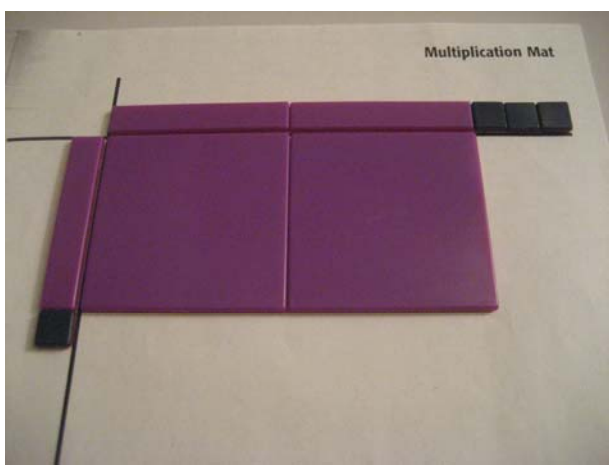


Figure 4.71. Rob's puzzle at the beginning.

After seeing that the purple squares fit well, Rob tried to place a green tile representing "*xy*" next to the second big purple square (Figure 4.72) while commenting "Any chance of fitting this there?"

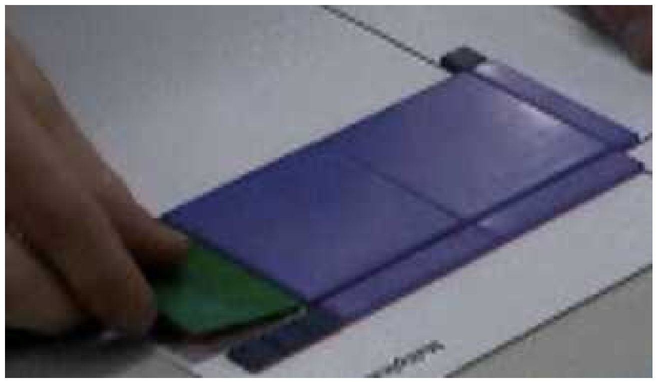


Figure 4.72. Rob's attempt on fitting the green tile.

Neither Rob nor Brad used the linear quantities on the perimeter of the figure to determine the resulting areal quantities. However, Rob was able to interpret the resulting areal tiles on their own as well as with reference to dimension tiles, in a multiplicative way, which was missing in Brad's case. The following protocol clarifies Rob's interpretation of the resulting areal tiles.

Protocol 4.64: Rob's MRUC concerning the "resulting" areal tiles.

R: This one has length [about the areal 1] but it also has width. This one [about linear 1] just has length. Length is on this edge of it... [Figure 4.73] When you put this length [pointing to the linear 1 at the top] and that length [pointing to the linear 1 on the side] together, it makes a two dimensional shape, which is this... length and width.

G: Is it how you multiply? Are you talking about the multiplication?

R: Yes.

G: What did you multiply?

R: This edge right here and this edge right here [pointing to the edges of the linear 1s at the top and on the side, respectively]

G: For these “*x*”s... [pointing to the linear “*x*”s at the top and on the side, respectively] What did you do?

R: This edge by this edge [pointing to the edges of the linear “*x*”s on the side at the top, respectively. Figures 4.74 & 4.75]

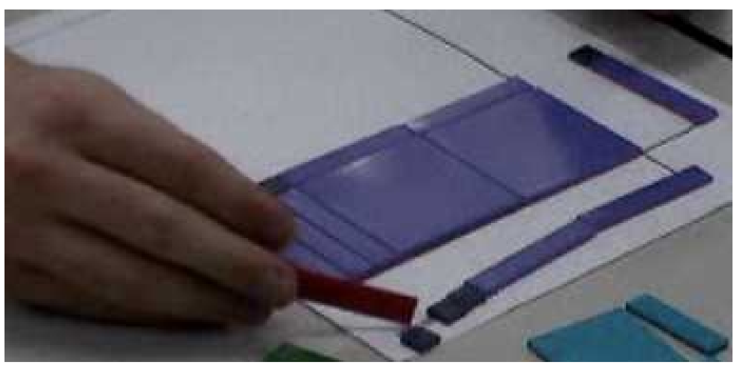


Figure 4.73. Rob's hand gesture pointing to the linear 1 at the top.

Figure 4.74. Rob's hand gesture pointing to the linear *x* on the side.

Figure 4.75. Rob's hand gesture pointing to the linear *x* at the top.

Rob's statement at the beginning, supported by his hand gestures, is the most abstract way of establishing a multiplicative RUC, as I define multiplicative RUC. In this study, I base multiplicative RUC on the "relational" aspect which is very abstract, and seemingly different from an ordinary multiplication, such as length × width. Rob's first statement "when you put this length and that length together" can be modeled with the ordering (1, 1) that corresponds to the multiplicative nature of the resulting "areal 1 unit." Rob's second statement "it makes a two dimensional shape, which is this and this... length and width" shows that he not only was aware of the resulting areal tile as suggested by the words "two dimensional shape," but he also saw the resulting areal tile as an ordered pair, as suggested by his language "which is this and this... length and width." In fact, in the notation (1, 1); the linear 1 and the linear 1 are sort of "put together," in a specific order, which calls for an ordered pair notation.

The multiplicative nature of unit coordination in this context is much different from the unit coordination described in the literature. Rob's phrase "put this length and that length together" is really about an ordering; it is like an ordered pair. RUC in my study is more of a “relational” type as opposed to the unit coordination in the literature, which is of “distributive” type (Steffe, 1992).

John, too, disregarded multiplicative thinking in his first attempt at generating a polynomial rectangle. The following protocol illustrates John's strategy.

Protocol 4.65: John fills the pieces in the rectangle.

G: Could you explain what you are doing? [About the areal tiles he is placing]

J: I am fitting... I am just trying to fill in the puzzle I guess [He then finishes his figure with the “filling in the puzzle” strategy. Figure 4.76]. So... I just filled them in...

G: Are you sure that this is the correct rectangle?

J: Yeah...

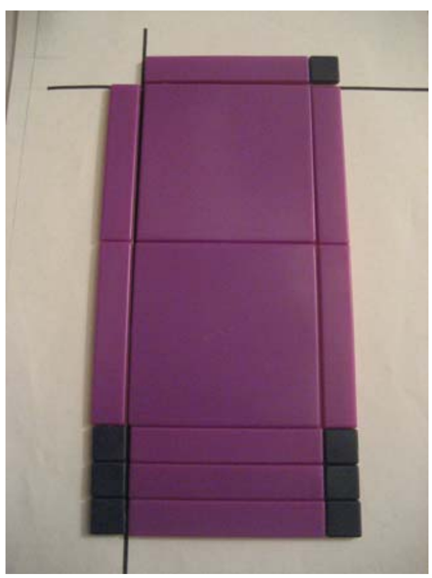


Figure 4.76. John's rectangle based on fitting.

John's behavior was similar to that of Brad and Rob in his "filling in the puzzle" strategy during the elementary stage of constructing the $x + 1$ by $2x + 3$ polynomial rectangle. In contrast, Nicole and Sarah used term wise multiplication. Obtaining the polynomial rectangle via multiplication of the irreducible linear quantities is a more meaningful and mature way than simply trying to see if the pieces will fit because the former one has the flavor of quantitative reasoning. The table below summarizes interview students' strategies on their first attempt.

Table 4.25

Strategies Used by Students in Their First Attempt

Students	**Strategy used in their first attempt**
Brad	Filling in the puzzle
Nicole	Multiplication of irreducible linear units
Sarah	Multiplication of irreducible linear units
Rob	Filling in the puzzle
John	Filling in the puzzle

John's written answers for the "Areas of the boxes of the same color as a product" column were of (almost) multiplicative nature. His answer for the $2x$ by 1 box was "$2 \cdot x$," which is of additive nature. When I look at all his answers in the same column for all the activities concerning polynomial rectangles, I see that this was the only exception. Therefore, I assume that that was just a typo and hypothesize that for John the "Areas of the boxes of the same color as a product" column are of multiplicative nature, as was the case for Nicole and Sarah. The table below illustrates students' written answers for the aforementioned column and the nature of their answers.

Table 4.26

Areas of the Same Color Boxes as a Product for the x + 1 by 2x + 3 Rectangle

Students	Students' Answers	The nature of their answer
Brad	$2 \cdot x^2, 5 \cdot x, 3 \cdot 1$	Additive
Nicole	$x \cdot 2x, x \cdot 3, 1 \cdot 2x, 1 \cdot 3$	Multiplicative
Sarah	$2x \cdot x,\ 3 \cdot x, 1 \cdot 2x,\ 3 \cdot 1$	Multiplicative
Rob	$2 \cdot x^2, 5 \cdot x, 3 \cdot 1$	Additive
John	$2x \cdot x,\ x \cdot 3,$ **2 · x**, $3 \cdot 1$	(Almost) Multiplicative

I probed Sarah to make sure that she was thinking multiplicatively in her work with the "Boxes of the same color as a product," as the following protocol describes.

Protocol 4.66: Sarah's MRUC concerning the boxes of the same color.

G: You wrote 1 times 2*x* and not 2 times *x*. I just want to know why...

S: Well... the way I was doing is... basing on lengths and... this particular length is 2*x* and this one is 1 [pointing to the linear units 2*x* at the top and 1 on the side, respectively. Figure 4.77]... so... that's why I did it that way.

G: So... 2 times *x*... would it be meaningful? Or irrelevant? What do you think?

S: Well that could give you the length of the side [pointing to the length of the areal 2*x* tile]... but it really has nothing to do with... [meaning area as a result of multiplication].

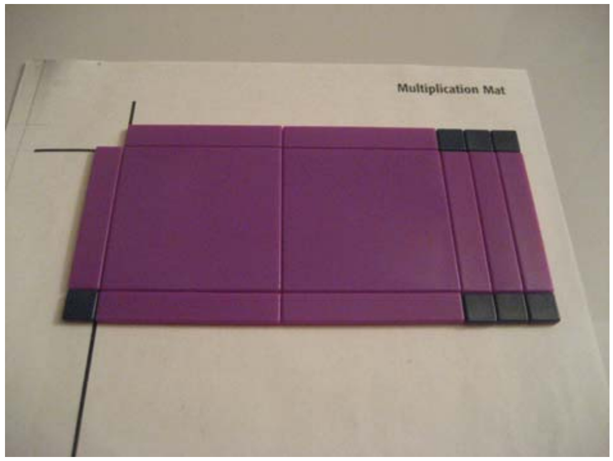


Figure 4.77. Sarah's rectangle resulting from term wise multiplication.

Sarah realized that both “1 times 2*x*” and “2 times *x*” have the same resulting value but differ in multiplicands. Because she constructed her rectangle by term wise multiplication of the irreducible linear units, she used “1 times 2*x*” – instead of “2 times *x*” which is lacking multiplicative nature. Moreover, her last comment indicates that she projected the multiplicative meaning onto the “1 by 2*x*” box, as she was unwilling to “decompose” the corresponding linear quantity “2*x*” into “2 times *x*.” This is a mature strategy because she was able to see the irreducible (linear and areal) quantities on their own as well as seeing the combined like termed (linear and areal) quantities representationally. My last findings based on Protocol 4.66 combined with Nicole and Sarah's written answers can be generalized as follows:

i. Irreducible areal quantities (IAQ) result from term wise multiplication of the corresponding irreducible linear quantities (ILQ).

ii. Term wise multiplication of the combined linear quantities (CLQ) produce corresponding areal quantities, which are also identical to the same–color–box areal quantitites (SCBAQ).

In order to explain Nicole and Sarah's *concepts–in–action* (Vergnaud, 1988) concerning the multiplicative nature of the areal quantities I developed new terminology based on set theory and Cartesian products.

● **Representational Set of Irreducible Linear Quantities (RSILQ):** A representational set of irreducible linear quantities (RSILQ) is the set of irreducible linear quantities such as 1, x, y in the context of different size color tiles (algebra tiles). The difference between a representational set of irreducible linear quantities (RSILQ) and an ordinary set is in that in a representational set of irreducible linear quantities, the irreducible linear quantities may appear as elements of the set more than once. For instance, the representational sets I_1 and I_2 of irreducible linear quantities corresponding to the $x + 1$ by $2x + 3$ polynomial rectangle can be defined as $I_1 = \{x, 1\}$ and $I_2 = \{x, x, 1, 1, 1\}$.

● **Representational Set of Combined Linear Quantities (RSCLQ):** A representational set of combined linear quantities (RSCLQ) is the set of combined irreducible linear quantities (RSCLQ) such as 5 (5 combined linear ones), $3x$ (3 combined linear "x"s), $2y$ (2 combined linear "y"s), etc. in the context of different size color tiles. In the context of the $x + 1$ by $2x + 3$ polynomial rectangle above, for instance, the representational sets C_1 and C_2 of combined linear quantities can be defined as $C_1 = \{x, 1\}$ and $C_2 = \{2x, 3\}$.

● **Representational Cartesian Product (RCP):** The (ordinary) Cartesian product of two sets A and B is the set of all ordered pairs in which the first component is taken from the first set and the second component is taken from the second set. Representational Cartesian Product (RCP) is therefore the ordinary Cartesian product defined on representational sets RSILQ and RSCLQ. For the example above, Nicole and Sarah's answers can be modeled as follows.

- RCP defined on RSILQs = RCP $_{RSILQ}$ = $I_1 \times I_2$ = $\{x, 1\} \times \{x, x, 1, 1, 1\}$ = $\{(x, x),$ $(x, x), (x, 1), (x, 1), (x, 1), (1, x), (1, x), (1, 1), (1, 1), (1, 1)\}$.
- RCP defined on RSCLQs = RCP $_{RSCLQ}$ = $C_1 \times C_2$ = $\{x, 1\} \times \{2x, 3\}$ = $\{(x, 2x), (x,$ $3), (1, 2x), (1, 3)\}$.

With the introduction of this new terminology on RSILQ, RSCLQ, and RCP based on Nicole and Sarah's concepts–in–action; a more robust definition of *Mapping Structures* will be given in the next section on *Factorization of Polynomials*.

Brad's lack of thinking multiplicatively and lack of operating on the RSILQ, RSCLQ, RCP levels *in the process of constructing the polynomial rectangle* can be explained by his *filling in the puzzle* strategy. Rob and John, too, made their rectangles using the same strategy. However, Rob later referred to term wise multiplication based on the *complete polynomial rectangle with dimension tiles placed around,* as opposed to John who failed to verbally describe the multiplicative RUC but yet succeeded in providing (almost) multiplicative type written answers for the "boxes of the same color as a product" column on the activity sheet. Brad's verbal description of his filling in the puzzle strategy in his construction of the $x + 1$ by $2y + 3$ (next task below) was very similar – almost identical – to John's construction of $x + 1$ by $2x + 3$ (Protocol 4.65). Brad first placed the dimension tiles on the multiplication mat (Figure 4.78) and thought out loud: "Well... I am just trying to figure out which figures go in here... I am just taking these [the two big green rectangles] 'cuz they fit in here." (Figure 4.79)

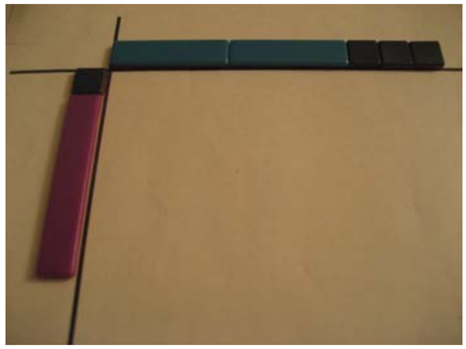


Figure 4.78. Algebra tiles representing dimensions "1 + x" and "2y + 3."

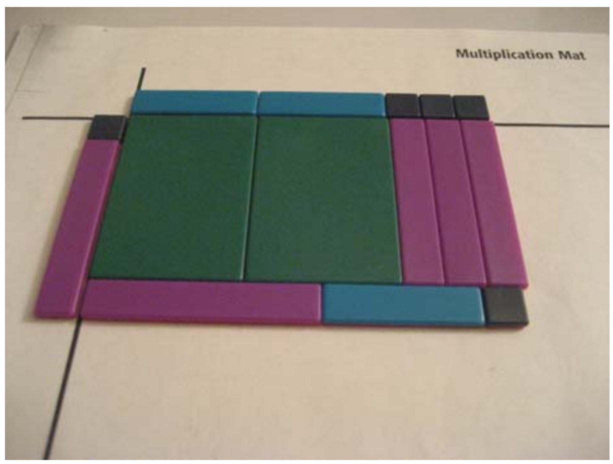


Figure 4.79. Algebra tiles filled in the puzzle.

In other words, similar to what he did above with the $x + 1$ by $2x + 3$ rectangle, this time Brad placed two big green rectangles each standing for "xy" at the top left corner (Compare with Figure 4.69), which is a strong indication that he was not using term wise multiplication – because if he was, then he should have first placed two blue (areal) bars right below the two (linear) blue bars resulting from the multiplication of linear 1 by linear 2y. The table below summarizes the phrases used by Brad, John, and Rob indicating filling in the puzzle strategy.

Table 4.27

Phrases Supporting Filling in the Puzzle Strategy

Students	Phrases
Brad	Well... I am just trying to figure out which figures go in here... I am just taking these [the two big green rectangle] 'cuz they fit in here.
Rob	Any chance of fitting this [the green tile] there?
John	I am fitting... I am just trying to fill in the puzzle I guess. So... I just filled them in...

Brad's first row seemed to "fit" well; however, the second row did not quite fit well (Figure 4.79). This created a perturbation for Brad, and he began to use (almost) term–wise multiplication. He changed his figure accordingly and concluded "That fills it in" (Figure 4.80). Note that the rows still need be interchanged to correspond to the linear units on the edges of the mat.

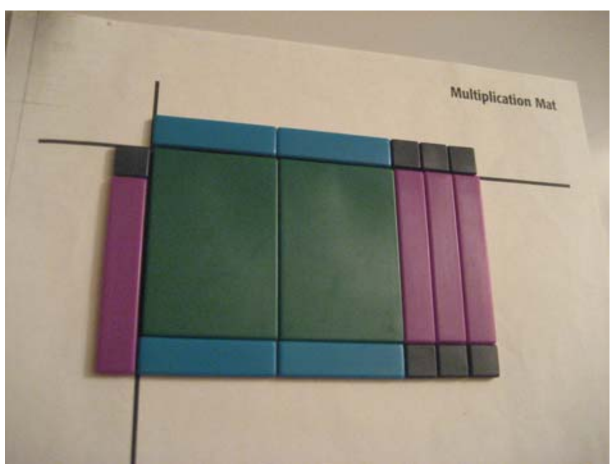


Figure 4.80. Brad's "revised" rectangle based on term wise multiplication.

It seems that something happened as Brad started to act on and think about the algebra tiles and the meanings projected onto them. "To know an object is to act on it." (Piaget, 1972, p. 8) While doing "term wise multiplication," he pointed to both the dimension tiles and the resulting areal tile. I infer that his learning resulted from his actions, combined with a desire for reasoning quantitatively. Bert van Oers defined action as “an attempt to change some object from its initial form into another form.” (1996, p.97) I infer that in Brad's interpretation, the dimension tiles transformed into something more meaningful from some sort of organizers. They were no longer purposelessly standing color figures anymore. I infer that the “action” was Brad's willingness to project some meanings onto the “previously useless” dimension tiles.

In John's work with the $x + 1$ by $2y + 3$ rectangle, something similar happened: He produced a polynomial rectangle with blue squares, blue bars, and black squares only (i.e., the polynomial rectangle was independent of x). Brad's first row was misplaced, and his second row was based on random fitting (Figure 4.79). John first placed the dimension tiles on the multiplication mat. Filling in the puzzle strategy appeared in that John started with blue squares instead of green rectangles, which indicated that what he was doing was definitely not term wise multiplication (Figure 4.81).

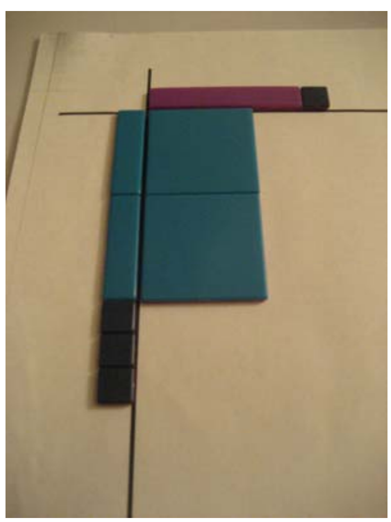


Figure 4.81. John's fitted “*y* squared” areal tiles.

Below the two blue squares, he placed 3 blue bars (Figure 4.82).

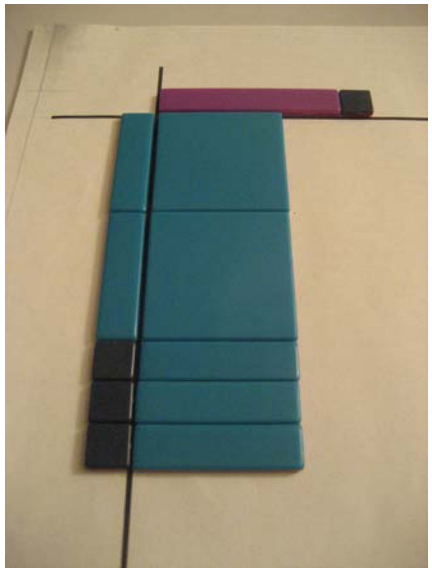


Figure 4.82. John's fitted "*y*" areal tiles below.

Right next to the blue square at the top, he placed 3 blue bars (Figure 4.83).

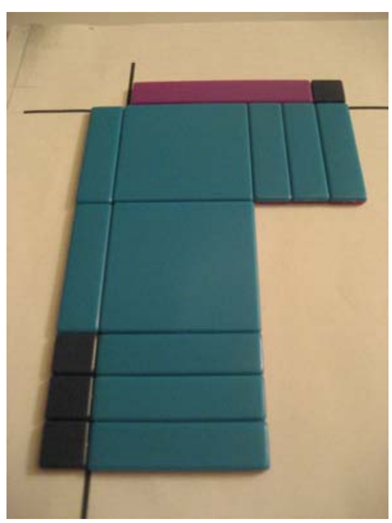


Figure 4.83. John's fitted "*y*" areal tiles on the second column.

Figure 4.83 stands as visual evidence that John was not using multiplication. In fact, John said “I am making the rectangle by parts.” Therefore, John's statement validates my previous hypothesis that the “Filling in the Puzzle” strategy seems to be related to an “area as a sum” strategy, namely calling for an additive nature. This contrasts with the “Term Wise Multiplication of Irreducible Units” strategy, which naturally defines the irreducible areal quantities (IAQ) as products, namely of multiplicative nature (See Tables 4.25 & 4.26 above and the paragraph that follows).

John was aware that there was something wrong. He decided to revise his figure by removing the three blue bars in the second column and suggested replacing them with a blue square. The following protocol illustrates this point.

Protocol 4.67: John's struggle with the puzzle.

J: These two [blue squares] fit here but this one [he locates another blue square among the tiles and tries to fit it right next to the blue square at the top] is too long for here [Figure 4.84]. Likewise can't put another one of these [he then removes the same blue square and tries to fit it right below the blue squares on the first column] here [Figure 4.85] it's too long... So... I'll use as many of these [blue squares] as I can to simplify...

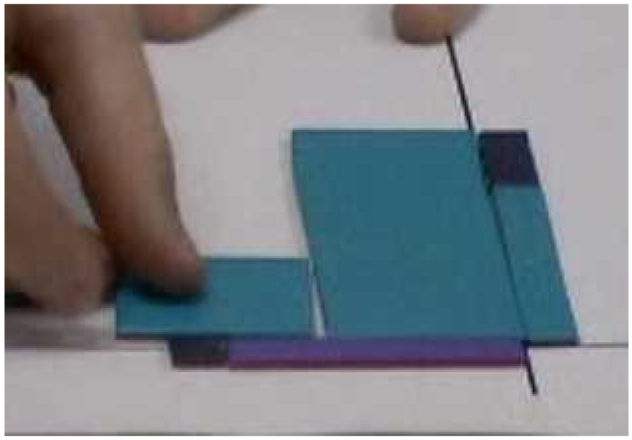


Figure 4.84. John's attempt to fit the "*y* squared" areal tile.

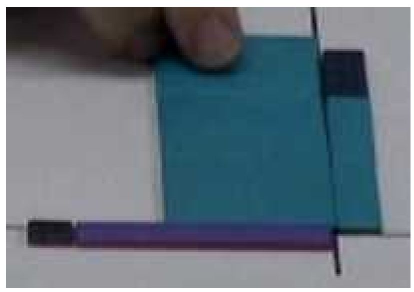


Figure 4.85. John's attempt to fit the "*y* squared" areal tile below.

He did not like his last attempts and shifted back to his previous figure (Figure 4.83). He then went on with the "Filling in the puzzle strategy" again by placing three more blue bars right below the three blue bars at the top (Figure 4.86).

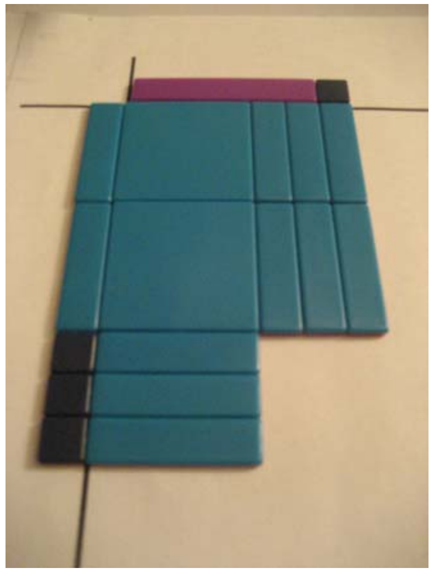


Figure 4.86. John fitted three more "*y*" areal tiles in the second column.

Finally, he placed 9 black squares right below the previous three blue bars hence completing his puzzle (Figure 4.87).

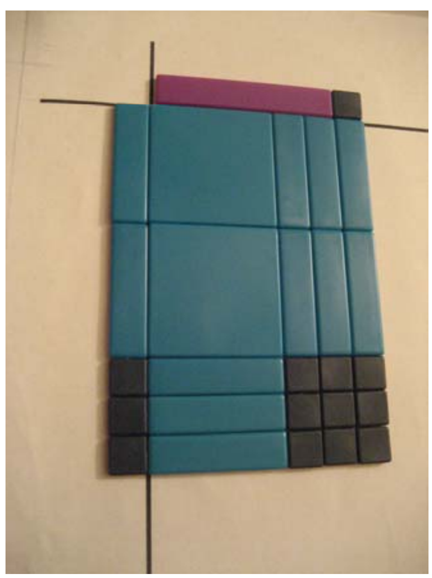


Figure 4.87. John's complete rectangle made of blue and black tiles only.

John disregarded the term–wise multiplication of the irreducible dimension tiles, which is why he obtained a different rectangle. The area of the correct rectangle is $2xy + 3x + 2y + 3$ whereas John's rectangle had an area of $2y^2 + 9y + 9$. I then asked him to write his answers in the table. Although he wrote $(x + 1)(2y + 3)$ for the "Area of the polynomial rectangle as a product" on the activity sheet, the dimensions of his rectangle actually were $y + 3$ and $2y + 3$. And in fact, John relied on the dimensions of his rectangle when answering the "Area of the boxes of the same color as a product" column of the activity sheet. He also relied on the properties of his rectangle when answering all the columns except the "Area of the polynomial rectangle as a product" column on the activity sheet. I infer that there was some kind of a disconnect here, perhaps caused by his "Filling in the Puzzle" strategy. By "disconnect," I mean John did not attend to the "irreducible linear units" of the dimensions. He rather saw the dimensions as some kind of "organizers" that traced the borders of his puzzle, which is the essence of this disconnect.

Even though his figures were lacking a multiplicative character, John's written answers for the "Area of the boxes of the same color as a product" column on the activity sheet were products of the corresponding linear quantities *based on his rectangle* with dimensions $y + 3$ by $2y + 3$. His answers $2y \cdot y$, $3 \cdot y$, $3 \cdot 2y$, $3 \cdot 3$ can be modeled with a relational notation of ordered pairs $(2y, y)$, $(3, y)$, $(3, 2y)$, $(3, 3)$ of linear units, respectively. The following protocol demonstrates how John was able to refer to the dimensions $y + 3$ and $2y + 3$ of *his rectangle* while totally disregarding the dimension tiles placed around representing $x + 1$ and $2y + 3$.

Protocol 4.68: John's successful description of MRUC based on his rectangle.

G: How about that... why is it $2y$ times y... and not 2 times y^2?

J: Because I am doing the areas of the smaller boxes [meaning, same–color–box subunits] by length times width.

G: You see that product as length times width?

J: Yeah... If you take this rectangle, it would be y times $2y$ [moving his index finger along the dimensions of the $2y$ by y "same–color–box"]

G: How about this... The same? [About his answer "$3 \cdot y$" on the table]

J: Yeah... [pointing to the dimensions of the 3 by y "same–color–box"]

G: How about this 3 times $2y$... The same? [About his answer "$3 \cdot 2y$" on the table]

J: Yes.

This exchange is a reminiscent of Rob, who also disregarded the dimension tiles placed around the polynomial rectangle with dimensions $x + 1$ by $2x + 3$ *in the process of constructing the polynomial rectangle*. In the process of obtaining their polynomial rectangles, although both John and Rob relied on the filling in the puzzle strategy while disregarding the existence of the dimension tiles placed around, they were both successful in projecting a multiplicative meaning onto the resulting areal tiles. John was able to think in a multiplicative way in his descriptions of the areas of the boxes of the same color, as opposed to Rob who revealed a multiplicative RUC only for the irreducible areal tiles. *After the completion of the polynomial rectangle*, John was able to see the areas of the boxes as products based on the areal tiles only without reference to the dimension tiles, as opposed to Rob who could do both with the irreducible areal tiles. He emphasized the arealness of these irreducible quantities with reference to the dimension tiles placed around.

Nicole and Sarah were successful in projecting the multiplicative meanings onto both irreducible areal tiles and boxes of the same color *both in the process of and after the completion of* their polynomial rectangles with reference to the dimension tiles placed around and in their ability to operate on the RSILQ, RSCLQ, RCP levels. In fact, upon my instruction “Make an $x + 1$ by $2y + 3$ rectangle using the algebra tiles,” Nicole and Sarah used term wise multiplication of irreducible linear quantities. These two students spoke aloud while placing each areal tile resulting from term wise multiplication, which indicates that an RSILQ was available to them. Nicole and Sarah's induction of RCP on the two RSILQs via sense–making resulted in meaningfully organized polynomial rectangles (Figure 4.88).

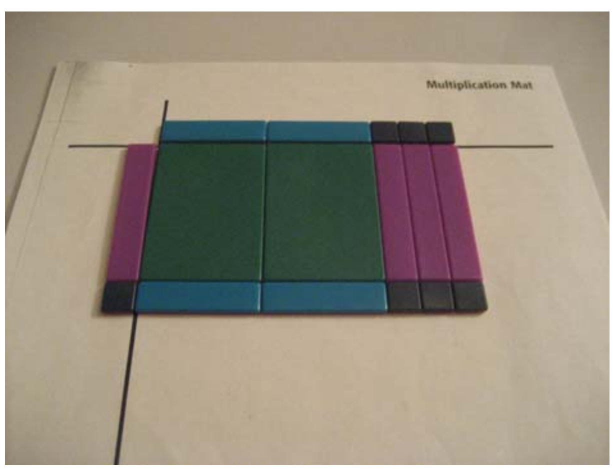


Figure 4.88. Nicole and Sarah's complete rectangle via RCP defined on RSILQ.

Nicole and Sarah's written answers for the “area of the boxes of the same color as a product” column on the activity sheet were areas defined as the product of the corresponding combined linear quantities (CLQ) (i.e., multiplicative in nature). In other words, these students are able to operate on the (RCP defined on) RSCLQ levels as well. With relational notation, therefore, Nicole and Sarah's answers can be modeled as follows.

- RCP defined on RSILQs = RCP $_{\text{RSILQ}} = I_1 \times I_2 = \{x, 1\} \times \{y, y, 1, 1, 1\} = \{(x, y), (x, y), (x, 1), (x, 1), (x, 1), (1, y), (1, y), (1, 1), (1, 1), (1, 1)\}$.
- RCP defined on RSCLQs = RCP $_{\text{RSCLQ}} = C_1 \times C_2 = \{x, 1\} \times \{2y, 3\} = \{(x, 2y), (x, 3), (1, 2y), (1, 3)\}$.

The table below illustrates students' written answers for the “area of the boxes of the same color as a product” column for the $x + 1$ by $2y + 3$ polynomial rectangle and the nature of their answers.

Table 4.28

Areas of the Boxes of the Same Color as a Product for the $x + 1$ *by* $2y + 3$ *Polynomial Rectangle*

Students	**Students' Answers**	**The nature of their answer**
Brad	$3 \cdot x, 2 \cdot y, 2 \cdot xy, 3 \cdot 1$	Additive
Nicole	$x \cdot 2y, x \cdot 3, 1 \cdot 2y, 1 \cdot 3$	Multiplicative
Sarah	$2y \cdot x,\ 3 \cdot x, 1 \cdot 2y, 1 \cdot 3$	Multiplicative
Rob	$2 \cdot xy, 3 \cdot x, 2 \cdot y, 1 \cdot 3$	Additive
John (Based on His Rectangle)	$2y \cdot y,\ 3 \cdot y, 3 \cdot 2y, 3 \cdot 3$	Multiplicative

On the third and final task on the multiplication of polynomials $2x + y$ and $x + 2y + 1$, all the students who previously used the filling in the puzzle strategy *in the process of constructing the polynomial rectangle* shifted to term wise multiplication of irreducible units. Brad did not say anything about fitting or filling. He did term wise multiplication of irreducible linear quantities (ILQ) carefully and placed the resulting irreducible areal quantities (IAQ) in their correct locations. He knew what he was doing and finished the whole rectangle in less than a minute (Figure 4.89). Rob's filling in the puzzle strategy evolved: Rob still used phrases “fitting” and “filling;” however, this time, his “filling in the puzzle” was based on the term wise multiplication of irreducible units, and he paid attention to dimension tiles. He said he already knew which one was going to “fit” where. Rob was “fitting” with reference to the dimension tiles. Brad and Rob's complete rectangles were identical (Figure 4.89).

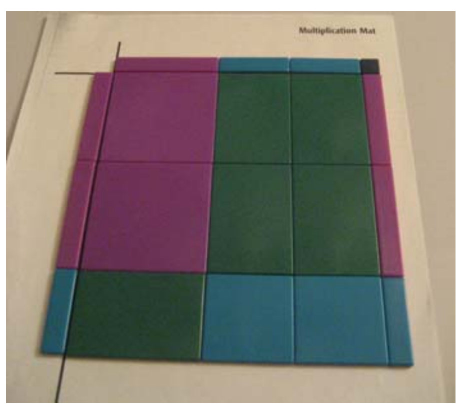


Figure 4.89. Brad's complete rectangle based on term wise multiplication.

John first made his polynomial rectangle solely based on filling in the puzzle strategy; however at some point he gave up, similar to Brad in the previous activity. In that sense, his thinking eventually evolved and led him to the RSILQ level. I include John's first attempt based on filling in the puzzle strategy for analysis purposes.

Protocol 4.69: John's filling in the puzzle strategy & change of mind.

J: First, I am going to put two x^2 rectangles down because they fit here... I can see (Figure 4.90) ... And then next I am gonna put as many of these down as I can [he locates six blue squares among the tiles and tries to fit them right below the two purple squares. Figure 4.91] and it so just happens to fit [i.e., still working with the "filling the puzzle" strategy. But then suddenly he gives up this strategy] Well actually... it'd make more sense... [he first removes the six blue squares from the figure. He then locates four green rectangles among the tiles and places them right below the two purple squares. Figure 4.92] Put one of these [green rectangles] here since this is *y* times *x*... [Figure 4.93] And this will be *y* squared... and then *y* squared [Figure 4.94] And then *x* times 1 plus *x* times 1... *y* times 1... [places the areal tiles on the last row. Figure 4.95]

Figure 4.90. John's "fitted" purple squares.

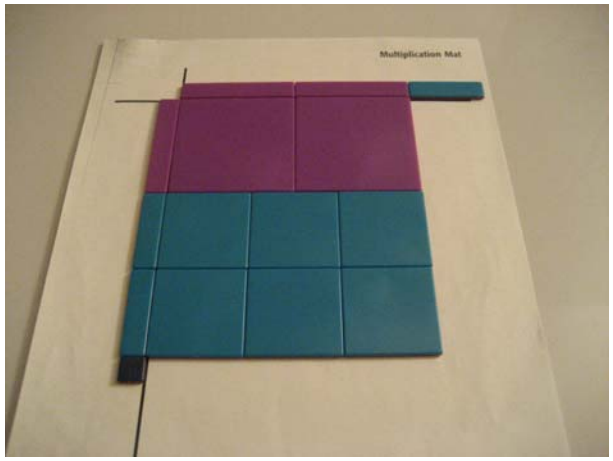


Figure 4.91. John's “fitted” blue squares.

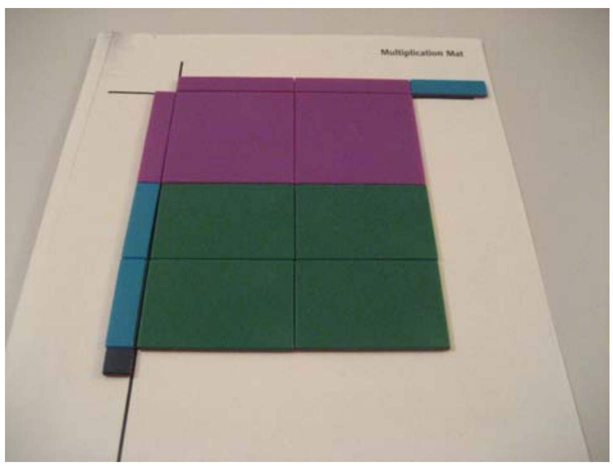


Figure 4.92. Turning point.

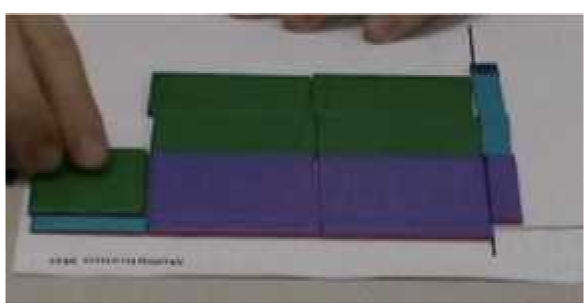


Figure 4.93. Term wise multiplication of linear *x* by linear *y*.

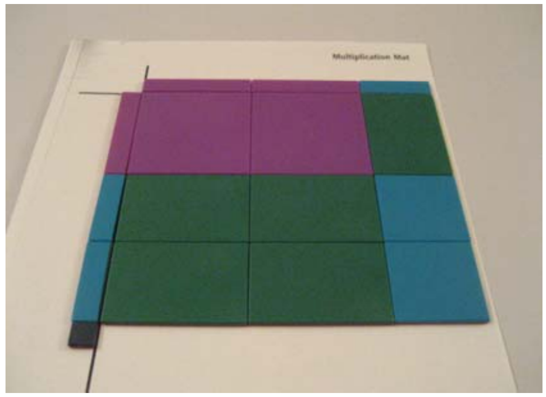


Figure 4.94. Term wise multiplication of linear *y* by linear “*y*”s.

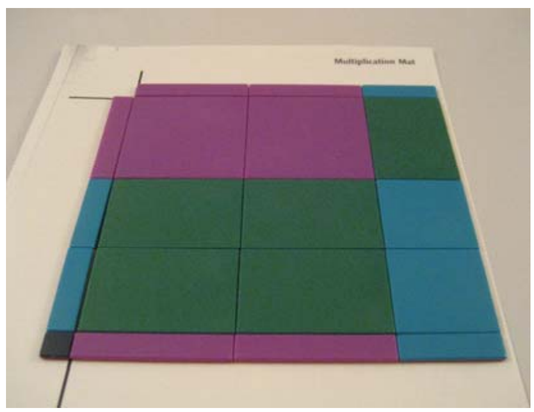


Figure 4.95. John's completion of the last row via term wise multiplication.

At that “change of mind” turning point where he said “Put one of these [green rectangles] here since this is *y* times *x*,” John gave up the “filling in the puzzle” strategy and decided to go on with “multiplication of irreducible linear units” strategy. This was a crucial moment for me because John learned this by himself. I think he realized that the irreducible areal quantities (IAQ) somehow had to be a product, a result, of the irreducible linear quantities (ILQ). John was keeping track of the mathematical practices he experienced in the previous two tasks on the multiplication of polynomials. His "actions" combined with his previous experience carried John to that "turning point." John's learning occurred because he assumed the situation was supposed to "make sense" in terms of the relationship between the dimension tiles and the resulting areal tiles. In Bert van Oers' words, "the learning of mathematics as a meaningful activity refers both to the process of technically mastering mathematics as a historically developed activity and to the process of attaching personal meaning to the actions, methods, and results involved." (1996, p.94)

John's answers for the "Areas of the boxes of the same color as a product" were of multiplicative nature. Therefore, I can infer that both RSILQ and RSCLQ levels were available to John, and he was able to induce a RCP on these representational sets. With relational notation, therefore, John's verbal descriptions *after the turning point* and his written answers can be modeled via representational Cartesian products (RCP) as follows.

- RCP defined on RSILQs = RCP $_{\text{RSILQ}} = I_1 \times I_2 = \{x, x, y\} \times \{x, y, y, 1\} = \{(x, x),$ $(x, y), (x, y), (x, 1), (x, x), (x, y), (x, y), (x, 1), (y, x), (y, y), (y, y), (y, 1)\}$.
- RCP defined on RSCLQs = RCP $_{\text{RSCLQ}} = C_1 \times C_2 = \{2x, y\} \times \{x, 2y, 1\} = \{(2x, x),$ $(2x, 2y), (2x, 1), (y, x), (y, 2y), (y, 1)\}$.

Because John gave up on his filling in the puzzle strategy at the turning point, I wanted to test what he would do with the previous task (the $x + 1$ by $2y + 3$ rectangle) for which he obtained a totally different *y–dependent–only* polynomial rectangle. He re–worked that problem and came up with the correct rectangle. The following protocol illustrates his reference to term wise multiplication of irreducible linear dimension tiles.

Protocol 4.70: John is reworking the $x + 1$ by $2y + 3$ rectangle (Figure 4.96).

G: Why did you place the greens now?

J: I placed the greens because I know the green is x times y [pointing to the linear x at the top and linear y on the side, respectively]... and this area right here [pointing to the green rectangle at the top] is x times y... as is this one [pointing to the green rectangle at the bottom]

G: So... are you actually doing multiplication?

J: Yeah... to find that specific spot [pointing to the areal tiles in his last figure]

G: Which one makes more sense? This one? Or what you did previously?

J: This one...

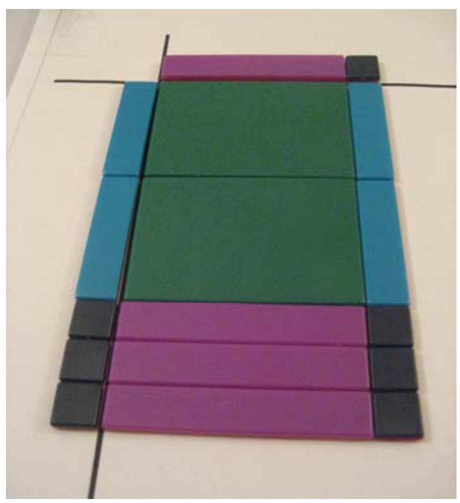


Figure 4.96. John's $x + 1$ by $2y + 3$ rectangle via term wise multiplication.

John's statements corroborate that RSILQ levels were available to him and that he was able to operate on these RSILQs with a RCP by which he obtained irreducible areal tiles. John's language "to find that specific spot" validates my previous theories about the multiplication operation behaving as a “mapping.” In other words, the ordered pair of irreducible linear units (x, y) from the dimensions were being mapped into the area of the rectangle as an irreducible areal unit: the xy “spot.” John's behavior can be modeled with reference to *Mapping Structures*. In fact, this is not something new for him. John was one the few students to make use of *Mapping Structures* in his work with the summation activities modeled with color cubes.

I end this subsection on *Multiplicative RUC* with a description of Nicole and Sarah's work on the multiplication of polynomials $2x + y$ and $x + 2y + 1$. Nicole, when placing the dimension tiles, followed the “x tile followed by the y tile followed by the 1 tile” ordering. As was the case with all the previous problems concerning algebra tiles, she actually did each term wise multiplication carefully by pointing to the corresponding irreducible linear tiles, and placed the resulting irreducible areal tile accordingly. RSILQ levels were available to her and she was able to map the RCP defined on the RSILQs as in the following rectangle (Figure 4.97).

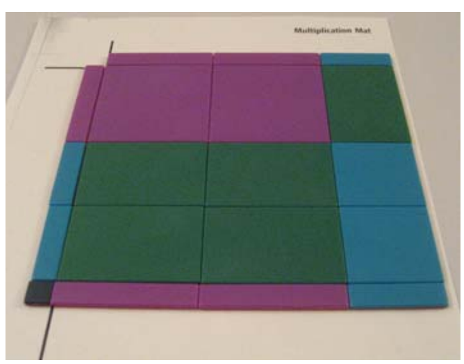


Figure 4.97. Nicole's $2x + y$ by $x + 2y + 1$ rectangle via term wise multiplication.

Then we discussed the “area of the boxes of the same color as a product” column on the activity sheet. Her answers, once again, were areas defined as the product of two quantities, i.e., multiplicative in nature. This is in contrast to Rob and Brad's written answers for the same problem, which are additive in nature. Nicole was very clear in her descriptions of the RSILQs and RCP generated via the ordered pairs of elements from RSILQs. The following protocol illustrates this point.

Protocol 4.71: Nicole's reference to RCP defined on RSCLQs.

G: You are saying $2x$ times x [About Nicole's expression $(2x) \cdot (x)$, which she wrote on the table]. And why not 2 times x^2? [Trying to challenge her]

N: Because I just saw these as a pair together... [pointing to the linear $2x$] same things you can group them together. So it's just *A* [pointing to the linear $2x$ at the top] times *B* [pointing to the linear x on the side]. Because when you look at this whole thing, this whole purple area [pointing to the $2x$ by x “same−color−box”] as one area... so you look the length as one number, $2x$, instead of 2 times x.

G: So what is the difference between this and the other way [I am asking her to compare the expressions $(2x) \cdot (x)$ and $(2) \cdot (x^2)$] representationally?

N: I did it [About her expression $(2x) \cdot (x)$, which she wrote on the table] in terms of the length times the width. Now it [About the expression $(2) \cdot (x^2)$ via which I am trying to challenge her] would be... talking about... how many of these [pointing to the purple squares] you have. [inaudible] other case, length times width gives the area of the whole thing [pointing to the

2x by x “same–color–box”].

G: So, that's the difference?

N: Yeah.

G: Please do the same for this one... [pointing to her expression $(2x) \cdot (2y)$, which she wrote on the table] why not 4 times xy?

N: Again I did this [pointing to the 2x by 2y “same–color–box”] as one area... I did it as length times width. Now this [about the expression 4 times xy I am trying to challenge her with] means I have 4 of them [meaning 4 green rectangles] and each one is an xy.

Nicole showed a mathematically fruitful performance in creating a Representational Cartesian Product, in her comparison of “2x times x” vs. “2 times x^2.” I did not have any contribution, nor intervention during her performance. The “pair” in Nicole's statement “I just saw these as a pair together” refers to the pair of linear “x”s in “2x” and not to the ordered pair $(x, 2x)$ of linear quantities. In other words, at the initial stage of defining an RCP, she first identified the elements of the RSCLQs. Her later usage “So it's just A [pointing to the linear 2x at the top] times B [pointing to the linear x on the side]” signaled signaled the onset of a Representational Cartesian Product (RCP), another *concept*–(or *definition*–)*in*–*action* (Vergnaud, 1988). In this way, she established the existence of her concept–in–actiont: Nicole first picked an element, “A” from the first RSCLQ and then picked another element, “B” from the other RSCLQ. She then formed pairs (A, B) – another concept–in–action – of combined linear quantities (CLQ) by which she generated her RCP. In fact, what she referred to is an abstract definition of a

Representational Cartesian Product (RCP) as $\{(A, B) \mid A \in C_1, B \in C_2\}$ where $C_1 = \{2x, y\}$, $C_2 = \{x, 2y, 1\}$ with the set builder notation. A and B can be anything as long as they are coming from the first set C_1 and the second set C_2, respectively. Nicole did not describe the expressions $(2) \cdot (x^2)$ and $(4) \cdot (xy)$ as representationally multiplicative, as opposed to Brad and Rob. Although $(2) \cdot (x^2)$ and $(4) \cdot (xy)$ are representationally additive, as Nicole explains in Protocol 4.71 above, for Brad and Rob these expressions are of multiplicative nature. The algebraic symbols $(2) \cdot (x^2)$ and $(4) \cdot (xy)$ can be deduced only within an additive context, according to Nicole and Sarah, *representationally*.

Like Nicole, Sarah used the "x tile followed by the y tile followed by the 1 tile" ordering when placing the dimension tiles. She then used component wise multiplication and placed the resulting areal tile accordingly (Figure 4.98). She thought aloud and pointed to the irreducible linear tiles at the top and on the side for each multiplication. She also said the name of the resulting areal tile, e.g., "this is x times x, this is x times y." The "multiplicative nature" of the "areal tiles" seems once again to be warranted by her statements in the following protocol.

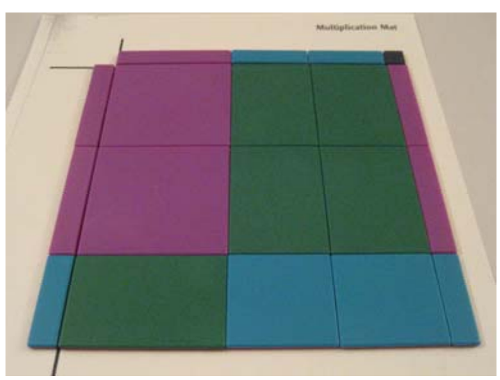


Figure 4.98. Sarah's $2x + y$ by $x + 2y + 1$ rectangle via term wise multiplication.

Protocol 4.72: Sarah's reference to RCP defined on RSILQs.

S: This is [pointing to and placing the areal x squared tile] x [pointing to the linear x tile on the side] times x [pointing to the linear x tile at the top]. This one is also x times x [in a similar manner]. This one is x times y [pointing to and placing the green tile representing the areal unit xy]. And x times y [in a similar manner].

G: Where is the x times y?

S: y [pointing to the linear y tile at the top] and x [pointing to the linear x tile on the side]. And x times y [in a similar manner]. And x times y [in a similar manner]. And this is x times y [in a similar manner]. And then this is y [pointing to the linear y tile at the top] times y [pointing to the linear y tile on the side]. And y times y [in a similar manner]. This is x [pointing to the linear x tile on the side] times 1 [pointing to the linear 1 at the top]. And x times 1 [in a similar manner]. And y [pointing to the linear y tile on the side] times 1 [pointing to the linear 1 at the top].

She did not say "x squared," nor "y squared." She rather said "this is x times x," "and then this is y times y," i.e., multiplicative in nature. Her language "y and x" is also indicative of an ordered pair (y, x) of linear units. Like Nicole, therefore, RSILQ levels were available to Sarah, and she was able to define a RCP on these representational sets. With relational notation, therefore, Sarah's verbal descriptions can be modeled via a representational Cartesian product (RCP) defined on RSILQs as follows.

- RCP defined on RSILQs = RCP $_{\text{RSILQ}} = I_1 \times I_2 = \{x, x, y\} \times \{x, y, y, 1\} = \{(x, x),$

$(x, y), (x, y), (x, 1), (x, x), (x, y), (x, y), (x, 1), (y, x), (y, y), (y, y), (y, 1)\}$.

I then asked her to outline the boxes of the same color with her finger. The multiplicative nature of the areas of these "boxes" was prevalent, as reflected in the protocol below.

Protocol 4.73: Sarah's reference to RCP defined on RSCLQs.

S: This one is $2x$ [pointing to the linear $2x$ on the side] times x [pointing to the linear x on the top]. This one is $2y$ times $2x$ [pointing to the corresponding linear tiles in a similar manner]. This one is $2x$ times 1 [pointing to the corresponding linear tiles in a similar manner]. This one is x times y [pointing to the corresponding linear tiles in a similar manner]. This would be y times $2y$ [pointing to the corresponding linear tiles in a similar manner]. And this would be y times 1 [pointing to the corresponding linear tiles in a similar manner].

G: So the product... each time you were doing the same thing... tell me more about that... I just want to make sure that I understand that...

S: I was using the area as a length times width where... this is a length or... and this would be the width... and basing it of like that... otherwise I could have added the insides [pointing to the areal tiles]... the way I did it was length times width.

In other words, Sarah was aware that what she was doing was term wise multiplication of the combined linear quantities (CLQ) and not addition. Her statement "otherwise I could have added the insides" combined with her gestures indicate that there are only two possibilities. The areas of the "same–color–boxes" could be modeled either via multiplication, or addition, *representationally*. But since she was asked about the areas of these boxes as products, the other option, namely additive RUC is irrelevant as she responded "the way I did it was length times width." This is in contrast to Brad and Rob's written answers and verbal descriptions of these "boxes" indicating an additive nature. Like Nicole, therefore, RSCLQ levels were available to Sarah, and she was able to define a RCP on these representational sets. With relational notation, therefore, Sarah's verbal descriptions can be modeled via a representational Cartesian product (RCP) defined on RSCLQs as follows.

- RCP defined on RSCLQs = RCP $_{\text{RSCLQ}} = C_1 \times C_2 = \{2x, y\} \times \{x, 2y, 1\} = \{(2x, x), (2x, 2y), (2x, 1), (y, x), (y, 2y), (y, 1)\}$.

The table below illustrates students' written answers for the "area of the boxes of the same color as a product" column for the $2x + y$ by $x + 2y + 1$ polynomial rectangle and the nature of their answers.

Table 4.29

Areas of the Boxes of the Same Color as a Product for the $2x + y$ *by* $x + 2y + 1$ *rectangle*

Students	**Students' Answers**	**The nature of their answer**
Brad	$2 \cdot x^2, 5 \cdot xy, 2 \cdot y^2, 2 \cdot x, 1 \cdot y$	Additive
Nicole	$2x \cdot x, y \cdot x, 2x \cdot 2y, y \cdot 2y, 2x \cdot 1, y \cdot 1$	Multiplicative
Sarah	$2x \cdot x, 2y \cdot 2x, 2x \cdot 1, x \cdot y, y \cdot 2y, y \cdot 1$	Multiplicative
Rob	$2 \cdot x^2, 5 \cdot xy, 2 \cdot y^2, 2 \cdot x, 1 \cdot y$	Additive
John	$2x \cdot x, 2x \cdot 2y, 2x \cdot 1, y \cdot x, y \cdot 2y, y \cdot 1$	Multiplicative

4.5.2. Additive Representational Unit Coordination (ARUC)

ARUC types arising from the data are essentially derived from two main components: 1) Students' written expressions, 2) Hand gestures combined with verbal descriptions in the process of constructing the rectangle representing the multiplication of the polynomials. Once again, as was the case for the activities involving color cubes, I will categorize ARUC types into two main categories: 1) Addends (Areal subunits) describing the polynomial rectangle unit, 2) Addends (Areal sub–subunits) describing the areal subunits. To be more specific, the area of the polynomial rectangle itself can be thought of as the biggest areal unit composed of areal sub–units. Similarly, each areal sub–unit can be decomposed into areal sub–subunits. Since the areal subunits serve as a bridge in between the biggest areal unit and the areal sub–subunits, I start the discussion on ARUC types by defining these areal subunits.

- **Boxes of the Same Color Type Addends (Subunits):** Also called Same−Color−Box Areal Quantities (SCBAQ), these are rectangular subregions of the same color within the polynomial rectangle itself. Though of multiplicative nature, these areal subunits are prone to be interpreted as of pseudo–multiplicative type (See the next subsection). The rectangularity is the essence of these "same−color−boxes." These are rectangles, not L–shapes or hexagons; in other words, they are multiplicative in nature. For instance, in Sarah's $x + 1$ by $2x + 3$ polynomial rectangle (Figure 4.77), there are four "same−color−boxes:" First, the x by $2x$ purple box; second, the x by 3 purple box; third, the 1 by $2x$ purple box; and fourth, the 1 by 3 black box. As is the case for all ARUC type addends, the sum of the "same−color−box" addends (subunits) equals the area of the polynomial rectangle under consideration. One can use the following relational notations of ordered n–tuples of “same–color–box” type addends to generate the corresponding polynomial rectangle:

- The quadruple $[2x^2, 3x, 2x, 3]$ generating the $x + 1$ by $2x + 3$ polynomial rectangle (Figure 4.77).
- The quadruple $[2xy, 3x, 2y, 3]$ generating the $x + 1$ by $2y + 3$ polynomial rectangle (Figure 4.88).
- The hextuple $[2x^2, 4xy, 2x, yx, 2y^2, y]$ generating the $2x + y$ by $x + 2y + 1$ polynomial rectangle (Figure 4.98).

- **Like Areal Tiles Combined Type Addends (Subunits):** Also called Combined Areal Quantities (CAQ), these are not necessarily closed subregions of the same color within the polynomial rectangle itself. Though of additive nature, these areal subunits are prone to cause confusion and to be mistaken for the *Boxes of the Same Color Type Addends*. For instance, when they were asked about the “boxes of the same color as a product,” Brad and Rob produced *Like Areal Tiles Combined Type Addends* instead of *Boxes of the Same Color Type Addends* at least once in each task (See the next subsection). There is no rectangularity requirement, nor closed figure condition. Representationally speaking, disconnected areal subunits can be added together to form one *Like Areal Tiles Combined Type Addend* as well. Once again, with reference to Sarah's figures, these addends can be observed to exist as follows:
  - In the $x + 1$ by $2x + 3$ polynomial rectangle (Figure 4.77), there are three Like Areal Tiles Combined Type Addends: First, the purple

subunit with an area of $2x^2$; second, the combined disconnected purple subunits with an area of $5x$; and third, the black subunit with an area of 3. With the relational notation, the quadruple [$2x^2$, $5x$, 3] generates the $x + 1$ by $2x + 3$ polynomial rectangle.

- In the $x + 1$ by $2y + 3$ polynomial rectangle (Figure 4.88), there are four Like Areal Tiles Combined Type Addends: First, the green subunit with an area of $2xy$; second, the purple subunit with an area of $3x$; third, the blue subunit with an area of $2y$; and fourth, the black subunit with an area of 3. With the relational notation, the quadruple [$2xy$, $3x$, $2y$, 3] generates the $x + 1$ by $2y + 3$ polynomial rectangle [Note: One gets the impression that this coincides with the Boxes of the Same Color Type Addends; however, the difference lies in the multiplicative versus additive interpretation].
- In the $2x + y$ by $x + 2y + 1$ polynomial rectangle (Figure 4.98), there are five Like Areal Tiles Combined Type Addends: First, the purple subunit with an area of $2x^2$; second, the combined disconnected green subunits with an area of $5xy$; third, the purple subunit with an area of $2x$; fourth, the blue subunit with an area of y; fifth, the blue subunit with an area of $2y^2$. Once again, as is the case for all ARUC type addends, the sum of the Like Areal Tiles Combined Addends (subunits) equals the area of the polynomial rectangle under consideration. With the relational notation, the quintuple [$2x^2$, $5xy$, $2x$, $2y^2$, y] generates the $2x + y$ by $x + 2y + 1$

polynomial rectangle.

When decomposing each areal subunit into areal sub–subunits, all interview students preferred the *Irreducible Addends Type ARUC*, which I was expecting.

- **Irreducible Addends Type ARUC (Sub–subunits):** These are the smallest possible areal quantities (sub–subunits) that generate either areal subunits described above. For instance, each one of the "boxes of the same color" type addends in the hextuple [$2x^2$, $4xy$, $2x$, yx, $2y^2$, y] can be decomposed into irreducible addends as follows (Figure 4.98):
  - The ordered pair [x^2, x^2] of irreducible areal sub–subunits generating the [$2x^2$] purple box.
  - The quadruple [xy, xy, xy, xy] of irreducible areal sub–subunits generating the [$4xy$] green box.
  - The ordered pair [x, x] of irreducible areal sub–subunits generating the [$2x$] purple box.
  - The areal singleton [yx].
  - The ordered pair [y^2, y^2] of irreducible areal sub–subunits generating the [$2y^2$] blue box.
  - The areal singleton [y].

As another example, each one of the "Like Areal Tiles Combined" type addends in the quintuple [$2x^2$, $5xy$, $2x$, $2y^2$, y] can be decomposed into irreducible addends as follows (Figure 4.98):

  - The ordered pair [x^2, x^2] of irreducible areal sub–subunits generating the [$2x^2$] purple subunit.

- The quintuple [xy, xy, xy, xy, xy] of irreducible areal sub–subunits generating the green [$5xy$] combined disconnected subunits.
- The ordered pair [x, x] of irreducible areal sub–subunits generating the [$2x$] purple subunit.
- The ordered pair [y^2, y^2] of irreducible areal sub–subunits generating the [$2y^2$] blue subunit.
- The singleton [y].

The table below summarizes the ARUC types used by the interview students in the multiplication of polynomials activities.

Table 4.30

ARUC Types Used by Interview Students

Students	Subunits Generating the Polynomial Rectangle	Sub–Subunits Generating the Subunits
Brad	Like Areal Tiles Combined Type Addends	Irreducible Addends
Nicole	Boxes of the Same Color Type Addends	Irreducible Addends
Rob	Like Areal Tiles Combined Type Addends	Irreducible Addends
Sarah	Boxes of the Same Color Type Addends	Irreducible Addends
John	Boxes of the Same Color Type Addends	Irreducible Addends

4.5.3. Pseudo – Multiplicative Representational Unit Coordination (PMRUC)

The name I chose for this RUC type may seem to be misleading at first. One could ask “Why Pseudo–Multiplicative and not Pseudo–Additive?” The reason behind this choice is the following. I asked the interview students their understanding and interpretation of the phrase “Area of the Boxes of the Same Color as a Product.” If one solely focuses on the “same color,” then one may end up providing multiple answers for this phrase. However, the key aspect of my question was not only the "Same Color" part but also the phrase "As a Product." In that sense, students had to pay attention to this whole phrase and to what it would mean representationally, based on the figures they were generating. Brad and Rob interpreted this phrase as referring to "Like Areal Tiles Combined Type Addends." But then this is something hard to reconcile with the "As a Product" part of the instruction, because not all "Like Areal Tiles Combined Type Addends" can be expressed as products. In other words, not all "Like Areal Tiles Combined Type Addends" are of multiplicative nature. Besides, Brad and Rob's answers were of additive nature, although the initial instruction "area as a product" asked them to think in a multiplicative way. This dilemma necessitated the existence of a new RUC type in between additive and multiplicative, not quite additive nor quite multiplicative, totally based on students' (mis)interpretation, which I named *Pseudo–Multiplicative type RUC (PMRUC)*[10].

Students' answers for the "Area of the boxes of the same color as a product" instruction can be classified into two main categories (See Tables 4.26, 4.28, and 4.29 above): Multiplicative (Nicole, Sarah, John) and Additive (Brad, Rob). Representationally speaking, I define Brad and Rob's answers "$3 \cdot x$, $2 \cdot y$, $2 \cdot xy$, $3 \cdot 1$" in Table 4.28, for instance, as *pseudo–products*, rather than products because the first term of each "pseudo–product" is a coefficient serving as a counting number indicating *how many* there are of each irreducible areal quantity (IAQ). Though written as a "product," Brad and Rob's expressions are of additive nature. The additive nature of these answers

[10] PMRUC arises to be the main issue in the present study. In another study dealing with 8th grade students' understanding of polynomial multiplication modeled by algebra tiles, students had a problem interpreting the missing square at the most left corner where the dimension tiles meet (Caglayan & Olive, manuscript in preparation).

becomes apparent via the relational notation of ordered triples and ordered pairs [x, x, x], [y, y], [xy, xy], [1, 1, 1] of irreducible *areal* quantities (IAQ). The “3” in Brad and Rob's expression “$3 \cdot x$” is a unit–less constant which can be thought of as the *cardinality* of the set of the ordered triple [x, x, x].

Let A be any set. The symbol $card(A)$ denotes the number of elements, i.e., the *cardinality* of the set A. In particular, in the context of my dissertation, the function *card* acts on *representational sets of areal quantities*. For the ordered triples and ordered pairs given by Brad and Rob for instance, one can write the following: $card$ ([x, x, x]) = 3, $card$ ([y, y]) = 2, $card$ ([xy, xy]) = 2, $card$ ([1, 1, 1]) = 3. In other words, Rob and Brad's pseudo–products corresponding to the areas of the “boxes of the same color” were guided by cardinal numbers defined by the cardinality function. The table below summarizes cardinalities and relational notation describing Brad and Rob's pseudo–products for the three tasks on the multiplication of polynomials.

Table 4.31

Pseudo–Products and Cardinalities Based on Brad and Rob's Answers for the “Area of the Boxes of the Same Color as a Product” Instruction

Pseudo–Products	**Relational Notation**	**Cardinalities**
$2 \cdot x^2$ $5 \cdot x$ $3 \cdot 1$	$[x^2, x^2]$ $[x, x, x, x, x]$ $[1, 1, 1]$	$card\ ([x^2, x^2]) = 2$ $card\ ([x, x, x, x, x]) = 5$ $card\ ([1, 1, 1]) = 3$
$2 \cdot xy$ $3 \cdot x$ $2 \cdot y$ $3 \cdot 1$	$[xy, xy]$ $[x, x, x]$ $[y, y]$ $[1, 1, 1]$	$card\ ([xy, xy]) = 2$ $card\ ([x, x, x]) = 3$ $card\ ([y, y]) = 2$ $card\ ([1, 1, 1]) = 3$
$2 \cdot x^2$ $5 \cdot xy$ $2 \cdot y^2$ $2 \cdot x$ $1 \cdot y$	$[x^2, x^2]$ $[xy, xy, xy, xy, xy]$ $[y^2, y^2]$ $[x, x]$ $[y]$	$card\ ([x^2, x^2]) = 2$ $card\ ([xy, xy, xy, xy, xy]) = 5$ $card\ ([y^2, y^2]) = 2$ $card\ ([x, x]) = 2$ $card\ ([y]) = 1$

Note that the "cardinality function" *card* maps a *representational set of areal quantities* into its cardinal number. Each ordered n–tuple (based on Brad and Rob's pseudo–products) in Table 4.31 can be thought of as disjoint subsets of a particular *Representational Set of Irreducible Areal Quantities (RSIAQ).*

- **Representational Set of Irreducible Areal Quantities (RSIAQ):** A representational set of irreducible areal quantities (RSIAQ) is the set of irreducible areal quantities (IAQ) such as $1, x, y, x^2, y^2, xy$ in the context of different size color tiles (algebra tiles). The difference between a representational set of irreducible areal quantities (RSIAQ) and an ordinary set is in that in a representational set of irreducible areal quantities, the irreducible areal quantities may appear as an element of the set more than once. For instance, for the $x + 1$ by $2x + 3$ polynomial rectangle, Brad and Rob's RSIAQ can be written as the disjoint union of the three *representational subsets* of RSIAQ as follows:
    - $\text{RSIAQ}_{x+1 \text{ by } 2x+3} = \{x^2, x^2\} \cup \{x, x, x, x, x\} \cup \{1, 1, 1\}$.

Brad and Rob's RSIAQs for the other two polynomial rectangles can be written as disjoint unions as follows:

- $\text{RSIAQ}_{x+1 \text{ by } 2y+3} = \{xy, xy\} \cup \{x, x, x\} \cup \{y, y\} \cup \{1, 1, 1\}$.
- $\text{RSIAQ}_{2x+y \text{ by } x+2y+1} = \{x^2, x^2\} \cup \{xy, xy, xy, xy, xy\} \cup \{y^2, y^2\} \cup \{x, x\} \cup \{1\}$.

As for Nicole, Sarah, and John's answers, one can imitate a table similar to Table 4.31 as follows.

Table 4.32

Products and Cardinalities Based on Nicole, Sarah, and John's Answers for the "Area of the Boxes of the Same Color as a Product" Instruction

Products	Relational Notation	Cardinalities
$2x \cdot x$	$(2x, x)$	$card\ (\ (2x, x)\ ) = 1$
$3 \cdot x$	$(3, x)$	$card\ (\ (3, x)\ ) = 1$
$1 \cdot 2x$	$(1, 2x)$	$card\ (\ (1, 2x)\ ) = 1$
$3 \cdot 1$	$(3, 1)$	$card\ (\ (3, 1)\ ) = 1$
$x \cdot 2y$	$(x, 2y)$	$card\ (\ (x, 2y)\ ) = 1$
$x \cdot 3$	$(x, 3)$	$card\ (\ (x, 3)\ ) = 1$
$1 \cdot 2y$	$(1, 2y)$	$card\ (\ (1, 2y)\ ) = 1$
$1 \cdot 3$	$(1, 3)$	$card\ (\ (1, 3)\ ) = 1$
$2x \cdot x$	$(2x, x)$	$card\ (\ (2x, x)\ ) = 1$
$2y \cdot 2x$	$(2y, 2x)$	$card\ (\ (2y, 2x)\ ) = 1$
$2x \cdot 1$	$(2x, 1)$	$card\ (\ (2x, 1)\ ) = 1$
$x \cdot y$	(x, y)	$card\ (\ (x, y)\ ) = 1$
$y \cdot 2y$	$(y, 2y)$	$card\ (\ (y, 2y)\ ) = 1$
$y \cdot 1$	$(y, 1)$	$card\ (\ (y, 1)\ ) = 1$

Compare the third columns of Table 4.31 and Table 4.32. The cardinality function is well–defined only when there is a representational set of *areal* quantities with *at least one* element to act on. In fact, in Nicole, Sarah, and John's cases, all such representational sets are *areal–singletons* in the context of the "Area of the Boxes of the Same Color as a Product." This makes sense because this is how Nicole, Sarah, and John define each "same–color–box" via a "single" product, as opposed to Brad and Rob who failed to provide "single" products. Nicole, Sarah, and John's "single" products $(2x, x)$, $(3, x)$, $(1, 2x)$, $(3, 1)$; $(x, 2y)$, $(x, 3)$, $(1, 2y)$, $(1, 3)$; $(2x, x)$, $(2y, 2x)$, $(2x, 1)$, (x, y), $(y, 2y)$, $(y, 1)$ each define a "single" box of the same color, namely a "single" areal quantity, i.e., an *areal–singleton*. And in fact, it is because of this singularity that the *card* function maps each areal–singleton into the number "1." Nicole, Sarah, and John's answers call for the definition of a new representational set, analogous to Brad and Rob's RSIAQ.

- **Representational Set of Same–Color–Box Areal Quantities (RSSCBAQ):** A representational set of same–color–box areal quantities (RSSCBAQ) is the set of same–color–box areal quantities defined by the "Area of the Boxes of the Same Color as a Product" of the polynomial rectangle resulting from the multiplication of polynomials. The difference between a RSSCBAQ and a RSIAQ is in that repetition of "elements" is *not* allowed in a RSSCBAQ. RSSCBAQ for each polynomial multiplication example above can be written uniquely as disjoint unions of *areal–singletons* as follows:
    - $\text{RSSCBAQ}_{x+1 \text{ by } 2x+3} = \{ (2x, x) \} \cup \{ (3, x) \} \cup \{ (1, 2x) \} \cup \{ (3, 1) \}$.
    - $\text{RSSCBAQ}_{x+1 \text{ by } 2y+3} = \{ (x, 2y) \} \cup \{ (x, 3) \} \cup \{ (1, 2y) \} \cup \{ (1, 3) \}$.
    - $\text{RSSCBAQ}_{2x+y \text{ by } x+2y+1} = \{ (2x, x) \} \cup \{ (2y, 2x) \} \cup \{ (2x, 1) \} \cup \{ (x, y) \} \cup \{ (y, 2y) \} \cup \{ (y, 1) \}$.

One could argue that the areal–singletons $\{ (2y, 2x) \}$ and $\{ (x, y) \}$ in the third RSSCBAQ are of the same color; so in fact repetition of elements *is* allowed in a RSSCBAQ. However, the areal–singletons $\{ (2y, 2x) \}$ and $\{ (x, y) \}$ are different "elements" because they define two distinct "same–color–box" products. Note that Nicole, Sarah and John constructed these "same–color–box" products via reference to different pairs of combined linear quantities (CLQ) as can be observed in the multiplicative nature of the relational notation of ordered pairs $(2y, 2x)$ and (x, y) of combined linear quantities.

I complete this subsection with a table listing phrases used by Brad and Rob related to PMRUC in their verbal descriptions for the "Area of the Boxes of the Same Color as a Product" instruction; and Sarah's proof by contradiction invalidating PMRUC.

Table 4.33

Brad and Rob's PMRUC Type Phrases

Students	Phrases	Ordered *n*–Tuples
Brad	*xy* plus *xy*... *x* plus *x* plus *x*... *y* plus *y*... 1 plus 1 plus 1...	[*xy*, *xy*] [*x*, *x*, *x*] [*y*, *y*] [1, 1, 1]
Rob	This is an *xy* [pointing to the green rectangle] thing... rectangle... and you have two of them... so 2 times *xy*" There is three of them [about his written expression "3 · *x*"]	[*xy*, *xy*] [*x*, *x*, *x*]

Protocol 4.74: Sarah's comparison of "2*y* · *x*" vs. "2 · *yx*"

G: Why did you write 2*y* times *x*, and not, 2 times *yx*? [About her answer 2*y* · *x* on the "Area of the Boxes of the Same Color as a Product" column on the recording sheet]

S: The same as before... side times side gives the area. This side is 2*y* [pointing to the linear unit 2*y* at the top], and this side is *x* [pointing to the linear unit *x* on the side]. But if I were basing it of the inside [pointing to the green rectangles] that would be *xy* plus *xy*... which would be the next question... or 2 times *xy*.

G: OK so... What does the "2" in "2 times *xy*" refer to?

S: Well this is one *xy* [pointing to the green rectangle] and because there's two of them... so it would be the quantity of...

G: So... Are they different? I mean the 2*y* times *x*, and the 2 times *yx*? Representationally, are they different?

S: Yes.

G: Because of what you said?

S: Yeah. The way that I write it [she wrote $(2y) \cdot (x)$ on the table] would be this is all one [pointing and outlining the “green box” made of two green rectangles]. And if you were to do 2 times *xy*, you are looking at the two separate tiles [pointing to the green rectangles].

Sarah was aware that the 2*y* by *x* “same–color–box” was an areal–singleton. In fact her statement “The way that I write it [she wrote $(2y) \cdot (x)$ on the table] would be this is all one” and her hand gestures pointing and outlining the “green box” made of two green rectangles indicate an areal–singleton of cardinality of 1. To be more specific, her language “this is all one” describes the subset $\{ (x, 2y) \}$ of $\text{RSSCBAQ}_{x + 1 \text{ by } 2y + 3}$ of singular character, namely the fact that *card* ($(x, 2y)$) equals 1. In other words, her last comment clearly shows the meaning of 2 times *xy*. For Sarah, “2 times *xy*” is “additive” in nature, i.e., she was aware that “2 times *xy*” is different from “2*y* times *x*” with the latter one being multiplicative in nature. She was also able to “see” the green “box” as the subset $\{xy, xy\}$ of $\text{RSIAQ}_{x + 1 \text{ by } 2y + 3}$. She was aware that the subset $\{xy, xy\}$ is a representational set of irreducible areal quantities with cardinality 2. In summary, Sarah was able to interpret the green “box” in two different ways, relying on nothing but her interpretation, which is reminiscent of Gestalt Psychology Principle *Figure and Ground*.

Gestalt is a German word meaning *a unified or meaningful whole*. Gestalt psychologists are interested in discovering meanings projected in objects – figure – in relation to the surroundings – ground (Koffka, 1935; Köhler, 1947; Wertheimer, 1920). A classical example reflecting figure–ground visual perception is *Rubin's Vase*, devised by Danish psychologist Edgar Rubin who pioneered the seminal work on the figure–ground principle (Rubin, 1915). According to the figure–ground principle, “the human perceptual system separates stimuli into either figure elements, or ground elements. Figure elements are the objects of focus, and ground elements compose an undifferentiated background.” (Lidwell, Holden, & Butler, 2003, p.80)

Figure 4.99. Rubin's Vase[11].

Rubin's Vase presents a dilemma as to which part of the “single picture” corresponds to the main figure and which part corresponds to the (back)ground (Figure 4.99). Lidwell et al. (2003) outlined a set of visual cues that serve to separate figure from background:

- The figure has a definite shape whereas the ground is shapeless.
- The ground continues behind the figure.
- The figure seems closer with a clear location in space, whereas the ground seems farther away and has no clear location in space.

11Rubin's Vase. Retrieved September 7, 2007 from www.inkycircus.com/photos/uncategorized/turn_your_head.jpg

- Elements in the lower regions of a design are more likely to be perceived as figures whereas elements in the upper regions are more likely to be perceived as ground (p.80).

Gestalt psychologists believe that we do not learn from the objects themselves but rather the relations between those objects (Wertheimer, 1982). In his book “Productive Thinking,” Max Wertheimer described a learning strategy developed by a five–year old girl who was asked how she would find the area of a parallelogram. The girl then cut the right triangle from one side and carried it over to the other side and came up with a rectangle whose area could easily be found. Max Wertheimer called this learning strategy based on the little girl's reasoning about the geometric figures in relation to each other *productive thinking*. In that sense, Sarah was able to think productively in her successful figure–to–ground transitions. Sarah's statement “Side times side gives the area... This side is 2*y* [pointing to the linear unit 2*y* at the top], and this side is *x* [pointing to the linear unit *x* on the side]” as an explanation for the “Area of the boxes of the same color as a product” indicates that Sarah was paying attention to the combined linear quantities (CLQ) and defining the area of the “same−color−box” multiplicatively as an ordered pair of these linear quantities. In other words, in the context guided by the instruction “Area of the boxes of the same color as a product,” the dimension tiles multiplicatively defining the “same−color−box” stand as the *figure* whereas the areal–singleton (the “same−color−box”) behaves as the (back)*ground*.

In an attempt to answer my question “Why did you write 2*y* times *x*, and not, 2 times *yx*?” she provided a verbal proof by contradiction. She very quickly interchanged the functions of the dimension tiles and the “same−color−box.” In other words, *figure* became *ground*, and *ground* became *figure*, as indicated by her statement "But if I were basing it of the inside [pointing to the green rectangles]." She started with the "inside" assumption, namely she focused on the "areal quantities," which became *figure* by her assumption in an attempt to lead her to a contradiction. She then noted "that would be *xy* plus *xy*... which would be the next question... or 2 times *xy*." That would be *xy* plus *xy* which is *not* a multiplication. What she needed was the area of the "same−color−box" as a *product*; in other words, her assumption led her to a contradiction. Sarah was able to see both the *figure* (the areal–singleton inside) and the *ground* (the dimension tiles outside) in relation to each other via her productive thinking. Nicole also can be described as thinking productively, for the same reason (cf. Protocol 4.71 above). In Rob and Brad's cases, on the other hand, the areal tiles were standing for *figure*, and there was no (back)*ground*.

4.5.4. Linear vs. Areal Units

Brad reasoned quantitatively in distinguishing linear quantities from areal quantities in the context of polynomial multiplication. He paid close attention to the measurement units and referents (Schwartz, 1988) of same–valued linear and areal quantities to demonstrate that they stand for different quantities, as reflected in the protocol below:

> Protocol 4.75: Brad's reference to measurement units and referents of the same–valued quantities.
>
> G: Is this a *y*, this thing? [pointing to the blue tile (linear unit *y*) from the top

dimension. See Figure 4.80]

B: Hm hm.

G: How about this one? [pointing to the blue tile (areal unit *y*) enclosed in the rectangle]

B: Yes.

G: Are they the same? Or different? How are they the same? How are they different?

B: They're the same... This is part of the width [about the blue tile standing for linear unit *y*] and this is part of the area [about the blue tile standing for areal unit *y*]. So... If I were to attach units I guess they would not be the same. If this was inches [about the blue tile standing for linear unit *y*], this would be inches squared [about the blue tile standing for areal unit *y*]. 'cuz what we are putting here [meaning inside the rectangle] is an area now, not just a line.

G: Oh... tell me more about that. Do you see this [pointing to the dimension tiles at the top] just as a line? What do you mean?

B: You are only looking at the edge [pointing to the *y* side of the blue dimension tile at the top].

G: So... How about this *x* [pointing to the purple tile (linear unit *x*) from the left dimension] and this *x* [pointing to the purple tile (areal unit *x*) inside the rectangle], are they the same or different?

B: That's a line [pointing to the purple tile (linear unit *x*) from the left dimension] and that's an area [pointing to the purple tile (areal unit *x*) inside the rectangle]. They are the same, they are both "*x*"s. But they are different as in the dimensional part.

...

G: Is this *x* the same as this *x*? [I am asking him to compare the linear unit *x* with the areal unit *x*. See Figure 4.89]

B: They're different 'cuz this one is one dimensional and this one is two dimensional. When I look at this [the linear unit *x*], it's actually a line; and when I look at that [the areal unit *x*], it's actually a rectangle. [He is using the word "rectangle" for the areal unit *x*, and that was quite interesting].

G: And the "*y*"s?

B: Same as the "*x*"s.

According to Brad, the two "*y*"s are the same in that they are both *y* and they are both part of "something." In other words, they both represent something (Figure 4.80). However, at some point, Brad felt the need to attach measurement units to these same–valued quantities. In fact, at that point, he realized that the two quantities have the same "value" but different units, which was enough for him to claim that they stand for different quantities. He referred to dimensionalities to distinguish between these same–valued quantities. For Brad, the linear *x* and the linear *y* quantities were represented by the "edges" of the corresponding dimension tiles, whereas the areal counterparts were "actually rectangles." Remember, neither in the process of constructing his polynomial rectangles nor in his descriptions of "Area of the boxes of the same color as a product" did Brad care about the dimension tiles. For the first time in the context of polynomial multiplication by algebra tiles, the dimension tiles captured his attention, and Brad was successful in projecting a one–dimensional meaning to these quantities *in comparison with* their two–dimensional counterparts. I am therefore still doubtful whether the linear quantities *on their own* existed in Brad's world. Brad came up with a strategy he discovered on his own, in an effort to establish the *one–dimensionality* of the dimension tiles.

Protocol 4.79: Brad's "standing–up positioned" linear quantities strategy and dimensionalistic mapping structures.

G: How about this thing, is it a length or an area? [pointing to the green tiles (the areal units each representing *xy*). See Figure 4.80].

B: It's an area.

G: How about this 1? [pointing to black square (the areal unit representing 1)]. I want you to compare this "1" [pointing to black square (the linear unit representing 1)] and this "1" [pointing to black square (the areal unit representing 1)]. Are they the same or different?

B: This is a one–dimensional thing [He is now trying to place the dimension tile in a "standing–up position" in order to convince me that it really is a one–dimensional thing (i.e., a linear unit). Figure 4.100]. When you multiply them [meaning the linear unit 1 on the top and the linear unit 1 on the side] together, they become two dimensional [i.e., the areal unit 1].

G: Could you do the same demonstration for the *y* please? That was quite interesting, nobody did that before...

B: It'd be just a line across [Figure 4.101] you'd see that it's still a *y*, but it's a line. When you multiply it out here, it becomes flat, and then it's an area... 'cuz it came from one dimensional to two dimensional.

G: I never thought about that... great strategy!

Figure 4.100. Brad's “standing–up positioned” linear 1.

Figure 4.101. Brad's “standing–up positioned” linear 2*y*.

Brad's statements “This is a one–dimensional thing,” “It'd be just a line across you'd see that it's still a *y*, but it's a line” while trying to place the dimension tiles in a “standing–up position” indicate that he emphasized the one–dimensional character of these things (i.e., linear quantities). In particular, his statement “you'd see that it's still a *y*, but it's a line” indicates that he was able to “read” through the picture and deduce that the “standing–up positioned” thing still stands for the “linear” quantity with referent “*y*.” He did not refer to the known measurement unit “inches” to do so, though. He rather worked at a more abstract level of dimensionalities.

Students often referred to dimensionalities in their sense making of linear and areal units as well as in the comparison of the same–valued linear and areal quantities. Brad and Rob were the only students to *act on* dimensionalities via *Mapping Structures*. It is interesting to note that Brad had not previously referred to Mapping Structures, but in his first experience with *Mapping Structures* he worked on the very abstract *dimensionality* level. Although in his first statement “When you multiply them [the linear unit 1 on the top and the linear unit 1 on the side] together, they become two dimensional” the abstract one–dimensionalities were not made explicit, he was very explicit in his statement “When you multiply it out here, it becomes flat, and then it's an area... 'cuz it came from one dimensional to two dimensional.” I assert that for a *Dimensionalistic Mapping Structure* of multiplicative type to exist, therefore, one needs to establish the following conditions:

i. A pair of the abstract one–dimensionalities is mentioned.

ii. The multiplication operation behaving as a mapping is acting on the ordered pair of these abstractified one–dimensionalities.

iii. The two–dimensionality resulting from the mapping is indicated.

I ask the reader to compare Dimensionalistic Mapping Structures with the definitions of ordinary and 2–Fold Mapping Structures given in the fourth section of this chapter. A functional notation describing Brad's statements can be written as f: (one–dimensional, one–dimensional) → two–dimensional, where, f stands for the multiplication operation behaving as a mapping. Dimensionalistic mapping f of multiplicative type maps the abstract pair of one–dimensionalities onto the abstract two–dimensionality. Besides an association of units (at a more abstract level) to dimensionalities, Rob also made use of dimensionalistic mapping structures in his sense making of linear and areal quantities, as the following protocol illustrates.

Protocol 4.77: Rob's association of dimensionalities to units and dimensionalistic mapping structures.

G: So what is a unit for this term and this term? [pointing to the terms (x + 1) and (2x + 3) Rob wrote under the “Area of the polynomial rectangle as a product” column of the recording sheet for the x + 1 by 2x + 3 polynomial rectangle.]

R: Just unit [writes “unit” under (x + 1)]

G: And how about 2x + 3?

R: [writes “unit” under (2x + 3)] and then unit times unit equals unit squared [writes “unit^2” below]

G: Why did you do that?

R: 'cuz whenever you multiply these [pointing to x + 1 and 2x + 3] out you'll get this [pointing to $2x^2 + 5x + 3$] that's square units... It'll be like... A line... That line [pointing to one of the edges of the rectangle] together with that line [pointing to the other edge of the rectangle] makes that [pointing to the rectangular region]. One dimensional times one dimensional makes two dimensional.

…

[Rob compares the linear y with the areal y in the $2x + y$ by $x + 2y + 1$ polynomial rectangle. Figure 4.102]

R: Like we said earlier, this is [about the areal y] the product of this edge [pointing to the edge of the linear y on the side] times this [pointing to the edge of the linear 1 at the top] one unit edge.

G: Are you saying that it's a product?

R: Product of two one–dimensional lines.

G: Very clear!

Figure 4.102. Rob's $x + 2y + 1$ by $2x + y$ polynomial rectangle.

Rob's assignment of "unit" for the linear quantities $x + 1$ and $2x + 3$ may be thought of as an indication of a level of abstraction. He then attached "unit^2" to the product $(x + 1)(2x + 3)$ in a consistent manner. Rob and Sarah were the only students to work on such a level of abstraction (cf. Protocol 4.80 for Sarah's reference to phrases "unit" and "unit^2"). Rob then connected these abstract "units" to the dimensionalities. Moreover, MRUC is evident in his language "That line together with that line" which can be denoted as the relational notation of ordered pair (that line, that line) of linear quantities $x + 1$ and $2x + 3$. "That" in Rob's phrase "makes that" corresponds to the areal quantity $(x + 1)(2x + 3)$. In that sense, Rob constructed a *Mapping Structure,* which can be modeled with the functional notation f: (that line, that line) → that, where, f stands for the multiplication operation behaving as a mapping. Besides his *Mapping Structure* at an abstract level solely based on phrases "that line" and "that," Rob also referred to a *Dimensionalistic Mapping* f of multiplicative type that maps the ordered pair (one–dimensional, one–dimensional) onto "two–dimensional." Rob's statement "One dimensional times one dimensional makes two dimensional" can be written as f: (one–dimensional, one–dimensional) → two–dimensional, where, f once again stands for the multiplication operation behaving as a mapping. All these *Mapping Structures* arise very naturally and are based on relational type phrases in Rob and Brad's case above.

Rob was also very successful in distinguishing between same–valued linear and areal quantities as well as in providing an iteration strategy on his abstract "units" as illustrated in the following protocol.

Protocol 4.78: Rob's iteration of abstract "units" (Figure 4.103)

G: For this problem, what is this and this? [pointing to the linear and areal "*x*"s respectively]

R: This is *x*, areal *x*... [pointing to the areal *x*]. 1 by *x*... It's 1 times *x* area... so it would be... it has an area of *x* square units.

G: What do you mean by *x* square units?

R: Like... this is a square unit [pointing to a black square]... This is [pointing to the areal *x*] made up of *x* square units... whatever the number *x* is.

G: How about this one? [pointing to the linear *x*]

R: That's just length *x*.

G: It's not *x* by 1?

R: No... It's just length... we could draw it with a pencil if we needed to... um... there is no... you don't need this area value... you just need its length... you just need this edge [pointing to the edge of the linear *x*] as opposed to like... I can take these away... and this away... [taking the dimension tiles away from the rectangle] we know... we just need to know the length... so this would be *x*, 1, 2, 3... [pointing to the dimension tiles at the top] and this would be *x*, 1... [pointing to the dimension tiles on the side]

R: This is 1 unit length [about the linear 1] and this is 1 square unit [about areal 1] just because of what it's being used for.

G: How are they the same? How are they different?

R: They are the same piece of plastic. This one has length [about the areal 1] but it also has width... This one [about linear 1] just has length. Length is on this edge of it [pointing] When you put this length [pointing to the linear 1 at the top] and that length [pointing to the linear 1 on the side] together, it makes a two dimensional shape, which is this... length and width.

Figure 4.103. Rob's $x + 1$ by $2x + 3$ polynomial rectangle.

Rob's statement "it has an area of x square units" indicates that Rob's areal units are "square units" or "unit^2" with the exponential notation. He used an iteration strategy to explain what "x square units" mean. He went back to the basics, attending to the irreducible areal unit from which he deduced that "the quantity x square units" is made of x "square units." With reference to Schwartz's (1988) referent–value–unit trinity, I infer that Rob's areal "measurement unit" is the abstract notion of "square units" and x stands for the value of his areal quantity. His referent, namely the name of his areal quantity is the "areal x." The following table summarizes Rob's referent–value–unit trinity for the areal x.

Table 4.34

Rob's Quantitative Reasoning (Referent–Value–Unit Trinity)

Referent (Name)	**Value**	**Measurement Unit**
Areal *x*	*x*	Square Units ($unit^2$)

Rob's statement "When you put this length [pointing to the linear 1 at the top] and that length [pointing to the linear 1 on the side] together, it makes a two dimensional shape, which is this... length and width" calls for a Mapping Structure, which can be modeled with the functional notation *f*: (this length, that length) → two–dimensional shape, where, *f* stands for the multiplication operation behaving as a mapping as before.

Rob often referred to his iteration strategy in the $x + 1$ times $2y + 3$ polynomial multiplication problem in much more detail.

Protocol 4.79: Rob's iteration strategy in more details.

G: What do you mean by 2 times *xy*?

R: This is an *xy* [pointing to the green rectangle. Figure 4.104] thing... rectangle... and you have two of them... so 2 times *xy*.

G: How about the ones in here... are they areas or lengths? [pointing to Rob's expression "1 + 1 + 1" on the "area of the boxes of the same color as a sum" column on the recording sheet]

R: They have one unit square of area.

G: How about the "*x*"s in here? [pointing to Rob's expression "*x* + *x*" on the "area of the boxes of the same color as a sum" column on the recording sheet]

R: They have x unit square of area.

G: How about this x in here? [pointing to "x" in Rob's expression "$3 \cdot x$" on the "area of the boxes of the same color as a product" column on the recording sheet]

R: This has x unit square of area but there is three of them.

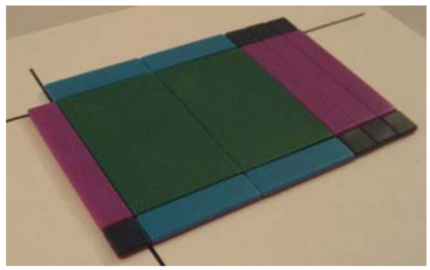


Figure 4.104. Rob's $x + 1$ by $2y + 3$ polynomial rectangle.

Rob used an iteration strategy for representational quantities of PMRUC type ($2 \cdot xy$ and $3 \cdot x$) and ARUC type ($1 + 1 + 1$ and $x + x$), as indicated in the protocol above. The basic irreducible areal units he induced in his iteration are the areal–singletons [xy], [1], [x], and [x], generating the areal quantities $2 \cdot xy$, $1 + 1 + 1$, $x + x$, and $3 \cdot x$, respectively. He explicitly stated his abstract areal unit "square units" for the areal–singletons [1], [x], and [x] respectively. For Rob, $3 \cdot x$ stands for "3 of them [areal "x"s]," i.e., additive in nature. Similarly, his statement "you have two of them... so 2 times xy" indicates the additive meaning Rob projects onto his written expression $2 \cdot xy$. Even though Rob was able to "map" the irreducible linear units onto the corresponding irreducible areal unit in the rectangular region, when he focused on the "same–color–boxes" (the areal units, which are not irreducible) his answers were of an additive nature. I therefore deduce that Rob's iteration strategy played the main role in preventing him from deducing a multiplicative character (MRUC) for the "same–color–boxes" and in leading him to favor additivity (PMRUC) whenever possible.

Sarah also used the same abstract units "unit" and "$unit^2$." In fact, referring to the "linear units" column on the activity sheet, she said "these are units unless otherwise specified" and put "$unit^2$" on the areal units column on the recording sheet. She even demonstrated that these are iterable; however, she did not assign an additive character to the "same–color–boxes" when she was asked to describe their area as a "product." In other words, Sarah knew what she was doing as to when to use an iteration strategy (ARUC) or when to express the same–color–boxes as products (MRUC).

Sarah used unique language in her comparison of and distinction between linear and areal quantities, as reflected in the following protocol.

Protocol 4.80: "Using As" & "Basing As" vs. "Would Be" & "Is."

G: Let's first look at this *y* [pointing to the linear *y*] and this *y* [pointing to the areal *y*]. Are they the same "*y*"s? (Figure 4.88)

S: Yes.

G: How about this [pointing to the linear *x*] *x* and this [pointing to the areal *x*] *x*? How are they the same? How are they different?

S: They're the same.

G: How about this [pointing to the linear 3] 3 and this [pointing to the areal 3] 3? How are they the same? How are they different?

S: They're the same. Are they not the same? Yeah... I guess we are *basing this* *as* a length and this *is the* area [pointing to the linear and areal 3, respectively]

G: Tell me more about that... what do you mean?

S: We are *using these* [pointing to the dimension tiles at the top] *as* a length... and *these* [pointing to the dimension tiles on the side] *as* a length... and *these* [pointing to the areal tiles enclosed in the rectangle] *would be* the area inside here...

G: This $2y$ [pointing to the linear $2y$] and this $2y$ [pointing to the areal $2y$]... are they the same or different? How are they the same? How are they different?

S: I have been *basing these* [pointing to the dimension tiles at the top] *outside tiles as* my length... So this [pointing to the linear $2y$ at the top] I have been *using as* length and this [pointing to the areal $2y$] *is* actually an area. This area [pointing to the areal $2y$] has this particular [pointing to the linear $2y$] length... or width... [inaudible]

G: How about this [pointing to the linear $x + 1$] and this [pointing to the areal $x + 1$]?

S: The same thing [meaning, her previous answer is still valid]. I was *basing this* [pointing to the linear $x + 1$] *as* the length and this *is* an [pointing to the areal $x + 1$] area.

Sarah's italicized phrases “using this/these as” and “basing this/these as” above indicate that she thought of the “dimension tiles” as some sort of instruments that serve to represent the length and the width. These dimension tiles are *not* the length and the width itself; rather, Sarah was "using these as" length and width. I define these linear representational quantities based on Sarah's phrases as *Instrumentalized Linear Quantities* (ILS) as these do not stand for the length or the width; they rather stand for *instruments* by which Sarah established linearity. I also hypothesize that, since she did *not* say "We are *using* these *as* area" about the areal tiles, she thought that the areal tiles that are enclosed in the rectangle *are* actually the areal parts of the rectangle itself. According to Sarah, the "collection" [2 green rectangles, 3 purple bars, 2 blue bars, and 3 black squares] *is* actually the area of the polynomial rectangle itself, which leads us naturally to the famous *principle of identity* "A is A". Whitehead and Russell defined a relation to be *reflexive* "when it holds between a term and itself." (1912, p. 22) In that sense, Sarah established *Areal Reflexivity* for the areas, namely a relation that holds between an element (the areas) and itself (the areal tiles).

According to Spradley, "Relational theory of meaning is based on the premise that the meaning of any symbol is its relationship to other symbols." (1979, p.97). In a relational type analysis such as domain analysis, a semantic relation serves for linking two symbols (folk terms). For instance, "*is a kind of*" is a semantic relation that connects the two symbols "Oak" and "Tree" via the proposition "Oak *is a kind of* tree." In general, for any pair (X, Y) of symbols, an open proposition based on the "is a kind of" semantic relationship can be written as $P(X,Y)$: "X is a kind of Y." In this example, "tree" stands for the general folk term, which Spradley calls "cover term" and the special type of tree, namely "oak" stands for "included term." In this way, a domain based on the "is a kind of" semantic relationship is generated via the cover term "tree" along with the included terms such as “oak,” “pine,” “palm,” etc. According to Spradley, there are nine *universal type semantic relationships* (Table 4.35).

Table 4.35

Universal Semantic Relationships (Spradley, 1979, pp.110–111)

1– Strict Inclusion	*X* is a kind of *Y*
2– Spatial	*X* is a place in *Y*, *X* is a part of *Y*
3– Cause – Effect	*X* is a result of *Y*, *X* is a cause of *Y*
4– Rationale	*X* is a reason for doing *Y*
5– Location for Action	*X* is a place for doing *Y*
6– Function	*X* is used for *Y*
7– Means – End	*X* is a way to do *Y*
8– Sequence	*X* is a step (stage) in *Y*
9– Attribution	*X* is an attribute (characteristic) of *Y*

An *informant expressed semantic relationship*, on the other hand, is a semantic relationship serving to generate cover terms and included terms (hence a domain) provided by the research participants, other than universal types. Based on Spradley's definitions, Sarah's semantic relationship phrases can be categorized as follows:

Table 4.36

Sarah's Universal and Informant Expressed Semantic Relationships

Phrases	**Semantic Relationship**
Using ... As ...	Universal
Basing ... As ...	Universal
... Is ...	Informant Expressed
... Would Be ...	Informant Expressed

Spradley named the universal type open proposition "*X* is used for *Y*" in Table 4.35 above as a *Function*. In fact, that is the only semantic relation given in that table of the *passive voice* type. Sarah's phrases "Using/Basing As" can be categorized as in the *active voice*. Therefore, before generating a domain based on Sarah's cover terms and included terms, it will be useful to convert Sarah's phrases in active voice into open propositions in passive voice, including the cover terms *Y* and the included terms *X*. Table 4.37 below lists Sarah's converted propositions as well as the ones based on "is" and "would be," which do not need conversion.

Table 4.37

Open Propositions Based on Sarah's Phrases

Sarah's Original Phrases	**Open Propositions**
Basing this as a length.	*X* based as *Y*. (Passive Voice)
This is the area.	*X* is *Y*.
Using these as a length.	*X* used as *Y*. (Passive Voice)
And [using] these as a length.	*X* used as *Y*. (Passive Voice)
These would be the area inside here.	*X* would be *Y*.
Basing these outside tiles as my length.	*X* based as *Y*. (Passive Voice)
This I have been using as length.	*X* used as *Y*. (Passive Voice)
And this is actually area.	*X* is *Y*.
I was basing this as the length.	*X* based as *Y*. (Passive Voice)
And this is an area.	*X* is *Y*.

Sarah referred to four open propositions, two of which can be categorized as *Universal* semantic relations and two as *Informant Expressed* semantic relations. I want to give names to these open propositions before generating a domain.

- $P(X,Y)$: "*X* used as *Y*."

- *Q*(*X*,*Y*): "*X* based as *Y*."
- *R*(*X*,*Y*): "*X* is *Y*."
- *S*(*X*,*Y*): "*X* would be *Y*."

In this propositional notation, *Y* denotes the cover term, and *X* denotes the included term. For Sarah's open propositions *P* and *Q*, the cover term is always "length" whereas for the open propositions *R* and *S*, the cover term is always "area." Sarah made reference to three distinct included terms "this," "these," and "these outside tiles" (Table 4.37). However, because these included terms are *possessive pronouns*, it would make more sense to work with the *actual representational quantities* to which she makes reference via her hand gestures. The table below contains Sarah's cover terms and included terms for each open proposition and the terminology describing Sarah's behavior.

Table 4.38

Included Terms as Representational Quantities

Propositional Notation	Open Proposition	Included Term (*X*)	Cover Term (*Y*)	Terminology
***Q*(*X*, *Y*)**	*X* based as *Y*.	Linear 3	Length	Instrumentalized Linear Quantities (ILQ)
***R*(*X*, *Y*)**	*X* is *Y*.	Areal 3	Area	Areal Reflexivity (AR)
***P*(*X*, *Y*)**	*X* used as *Y*.	Linear $2y + 3$	Length	Instrumentalized Linear Quantities (ILQ)
***P*(*X*, *Y*)**	*X* used as *Y*.	Linear $x + 1$	Length	Instrumentalized Linear Quantities (ILQ)
***S*(*X*, *Y*)**	*X* would be *Y*.	Areal $2xy + 2y + 3x + 3$	Area	Areal Reflexivity (AR)
***Q*(*X*, *Y*)**	*X* based as *Y*.	Linear $2y + 3$	Length	Instrumentalized Linear Quantities (ILQ)
***P*(*X*, *Y*)**	*X* used as *Y*.	Linear $2y$	Length	Instrumentalized Linear Quantities (ILQ)
***R*(*X*, *Y*)**	*X* is *Y*.	Areal $2y$	Area	Areal Reflexivity (AR)

$Q(X, Y)$	X based as Y.	Linear $x + 1$	Length	Instrumentalized Linear Quantities (ILQ)
$R(X, Y)$	X is Y.	Areal $x + 1$	Area	Areal Reflexivity (AR)

A domain based on Sarah's open propositions P, Q, R, S with cover terms and included terms can be generated as in the table below (Table 4.39).

Table 4.39

Sarah's Domain Analysis Table

<table>
<tr><td rowspan="8">Instrumentalized Linear Quantities
(ILQ)
&
Areal Reflexivity
(AR)
Domain</td><td rowspan="4">Length (Cover Term)</td><td>Linear 3 (Included Term)</td></tr>
<tr><td>Linear $2y + 3$ (Included Term)</td></tr>
<tr><td>Linear $x + 1$ (Included Term)</td></tr>
<tr><td>Linear $2y$ (Included Term)</td></tr>
<tr><td rowspan="4">Area (Cover Term)</td><td>Areal 3 (Included Term)</td></tr>
<tr><td>Areal $2xy + 2y + 3x + 3$ (Included Term)</td></tr>
<tr><td>Areal $2y$ (Included Term)</td></tr>
<tr><td>Areal $x + 1$ (Included Term)</td></tr>
</table>

Data from Tables 4.38 and 4.39 based on Spradley's Domain Analysis support my previous results on ILQ and AR. Sarah used the semantic relations “is,” and “would be” in relating the representational areal quantities to “area” (Areal Reflexivity) as opposed to the semantic relations “used as,” and “based as,” in relating the dimension tiles to “length” (Instrumentalized Linear Quantities). Usage of some phrases may be an indication of what people actually mean, especially when they describe a situation with manipulatives. I infer that for Sarah, the two–dimensional tiles representing the length or the width do *not* stand for the actual length or the width as supported by Sarah's choice of the phrases “using as,” “basing as.” The two–dimensional tiles representing the area, on the other hand, *do* stand for the actual area of the corresponding box, as can be inferred from Sarah's phrases “is,” and “would be” (Principle of Identity).

Both Nicole and Sarah established the multiplicative nature of *both* the irreducible areal quantities (IAQ) and the same–color–box areal quantities (SCBAQ) via verbal descriptions. Nicole and Sarah (and John in some cases) first defined their areal representational sets RSIAQ and RSSCBAQ via *Mapping Structures of multiplicative type* acting on the corresponding linear representational sets RSILQ and RSCLQ, respectively. They then acted on these areal representational sets via *Mapping Structures of additive type* to obtain expressions for “area as a sum.” The following excerpt illustrates Nicole's reference to additive type Mapping Structures acting on these areal type representational sets.

Protocol 4.81: Nicole's reference to mapping structures of additive type.

G: How do you describe this [pointing to the $2x^2$ “box”] as a sum? (Figure 4.105)

N: This is [pointing to the $2x^2$ “box”] $x^2 + x^2$. And this is [pointing to the $2x$ “box”] x plus x. This is [pointing to the $3x$ “box”] x plus x plus x... And [pointing to the 3 “box”] 1 plus 1 plus 1.

G: Is the $2x^2$ an area or a length?

N: It's an area.

G: How about this $2x$ [pointing to the areal $2x$ “box”]? Is it an area or a length?

N: It's an area because it's part of the rectangle. It has a length of 1, and a width of $2x$.

G: How about x? Is it an area or a length? [pointing to one of the areal "x"s of the same $2x$ "box"]

N: The x is an area.

G: The other x?

N: It's also an area.

G: Why?

N: Because this [pointing to one of the areal "x"s of the same $2x$ "box"] x itself is a product of x times 1. It has a length of 1, and a width of x.

G: How about this 3 [pointing to the 3 "box" inside the rectangle] now... Is it an area or a length?

N: It's an area... It's an area because it's a part of the whole rectangle. Each one has a length of 1 and a width of 1 [On the last column of the activity sheet, she writes $(x + 1)(2x + 3) = 2x^2 + 5x + 3$, without doing any computation].

G: How about this thing on the right hand side... is it an area or a length?

N: It's an area.

G: How about the first term in parentheses on the left hand side... is it an area or a length?

N: $x + 1$? It's a length.

G: How about the other one?

N: It's also a length.

G: Is it consistent in terms of units... I mean... are the units equal?

N: Hm hm... Length times width is area... you know... inches times inches is inches squared... that makes sense...

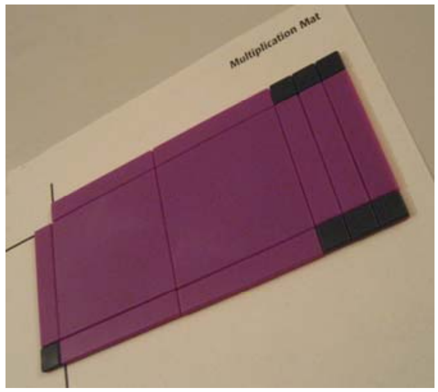


Figure 4.105. Nicole's complete rectangle resulting from RCP defined on RSILQs.

Every time she described a same–color–box (areal singleton) as a sum, Nicole not only made reference to an additive type Mapping Structure, but she also touched upon her previous experience with the irreducible areal quantities (IAQ) of these areal singletons. In other words, Nicole kept track of what she has done previously. Her statement “It's an area [the $2x$ by 1 box] because it's part of the rectangle. It has a length of 1, and a width of $2x$” indicates that she saw the box as a product, modeled as $(1, 2x)$ with the relational notation. But she was also aware that the same “box” can be written as a sum, as can be warranted by her statement “And this is [pointing to the $2x$ “box”] x plus x.” In addition, she referred to her previous experience by saying “Because this [pointing to one of the areal “x”s of the same $2x$ “box”] x itself is a product of x times 1. It has a length of 1, and a width of x.” In other words, she “remembered” that each areal x in fact is resulting from a multiplication. I can therefore schematize Nicole's thinking as follows:

- On one hand we have:

- MRUC principle: The 1 by $2x$ purple same–color–box can be expressed multiplicatively as $(1, 2x)$.
- Mapping Structures (multiplicative): When the multiplication operation behaving as a function f acts on the ordered pair $(1, 2x)$ of combined linear quantities, one obtains an areal singleton $[2x]$.
- Functional Notation: One can write f: $(1, 2x) \rightarrow 1 \times 2x$.

- On the other hand, we have:
  - MRUC principle: Each 1 by x irreducible areal quantity (there are two of them) can be expressed multiplicatively as $(1, x)$.
  - Mapping Structures (multiplicative): When the multiplication operation behaving as a function f acts on each ordered pair $(1, x)$ of irreducible linear quantities (there are two of them), one obtains the irreducible areal singleton $[x]$.
  - Mapping Structures (additive): When the addition operation behaving as a function g acts on the representational set of irreducible areal quantities $\{x, x\}$, one obtains an areal value $x + x$.
  - Functional Notations: One can write f: $(1, x) \rightarrow x$. Then the function g acts on the resulting areal quantities which can be denoted as g: $[x, x] \rightarrow x + x$.

A function followed by another function as above (Nicole's History Keeping Strategy) is reminiscent of the *composition of functions* concept, whose representational version can be described as follows. Consider the last equality $g\ [x, x] = x + x$. The right hand side is the final value; one does not worry about that for the moment. Instead, when one substitutes $f(1, x)$ for each "areal x" in square brackets, one obtains $g\ [f(1, x), f(1,$

$x)] = x + x$. Using the *representational composition* symbol "∘" one obtains $g \circ f((1, x), (1, x)) = x + x$. In fact, in Protocol 4.81 above, Nicole established the equivalence of $f(1, 2x) = 1 \times 2x$ and $g \circ f((1, x), (1, x)) = x + x$. I define these functions f and g as *Representational Mappings*, which are illustrated in the figure below (Figure 4.106).

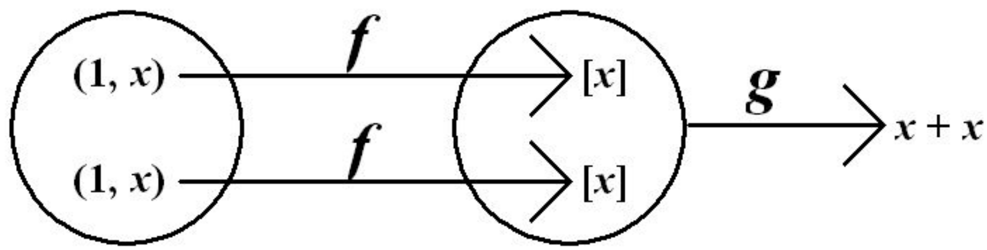


Figure 4.106. Nicole's representational mappings.

The multiplicative representational mapping f acts on the ordered pair $(1, x)$ of linear quantities and sends it to the areal quantity $[x]$. The additive representational mapping g, on the other hand, acts on the "representational set" of areal quantities and sends this "set" to the value $x + x$. In that sense, f can be thought of acting on ordered pairs (domain elements) whereas g acts on representational sets (i.e., a set behaves like a domain element for the representational mapping g). The d*omain of f* is a representational set of ordered pairs whereas the *domain of g* is a set of representational sets. More robust definitions of these representational mappings based on RCP, RSILQ, RSCLQ, RSIAQ, RSSCBAQ will be given in the next section on *Factorization of Polynomials*.

In her work with the polynomial rectangles, Nicole constantly referred to mapping structures, and she knew which RUC type (MRUC of ARUC) made more sense, as illustrated in the following protocol.

Protocol 4.82: Nicole's sense making of the same–valued linear and areal quantities with reference to MRUC, ARUC, and mapping structures.

G: How are they different, this $2y$ [pointing to linear $2y$] and this $2y$ [pointing to areal $2y$] here? (Figure 4.107)

N: Here [pointing to linear $2y$] we are just focusing on one aspect of the *y*... just the width... Here [pointing to areal $2y$] we are focusing on both aspects of the *y*... the length and the width...

G: Great explanations! I am gonna ask you the same thing about this 3 in here [pointing to linear 3] and this 3 in here [pointing to areal 3].

N: Here [pointing to linear 3] again we are focusing only on one aspect of these squares... just the length... The length is along the bottom line of all three of them. Here [pointing to areal 3] we are focusing on two aspects of the squares... just the length and the width...

...

N: Here we have a linear unit, the length [pointing to her expression in parenthesis on the LHS, $x + 1$]; and here we have a linear unit, the width [pointing to her expression in parenthesis on the LHS, $2y + 3$]; when you multiply them you get an areal unit [pointing to her expression in parenthesis on the RHS, $2xy + 3x + 2y + 3$] which is the whole rectangle. Every term here [pointing to her expression “$2xy + 3x + 2y + 3$”] is an area.

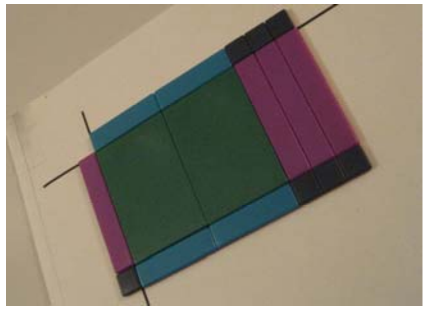


Figure 4.107. Nicole's $x + 1$ by $2y + 3$ rectangle.

Nicole distinguished between the same–valued ($2y$) linear and areal quantities via the "aspects" inherent in the nature of these objects. *Number of Aspects*, a concept–in–action, can be thought of as an index assigned to these same–valued quantities. If this index is 1, then the object represents a linear quantity; and if the index is 2, then the object represents an areal quantity. Nicole's "Number of Aspects" index can be modeled via a *Bi–valued Function* consisting of two range elements only. The domain of such a function comprises the following:

- Irreducible Linear Quantities (ILQ)
- Combined Linear Quantities (CLQ)
- Irreducible Areal Quantities (IAQ)
- Same–Color–Box Areal Quantities (SCBAQ)

Note that a Bi–valued Function is not defined on Combined Areal Quantities (CAQ) in the context of polynomial rectangles as there is no rectangularity requirement nor close figure condition for these quantities (Disconnected areal subunits can be added together to form Combined Areal Quantities as well). A Bi–valued Function maps the quantities ILQ, CLQ, IAQ, and SCBAQ to either 1, or 2, as depicted in figure below (Figure 4.108).

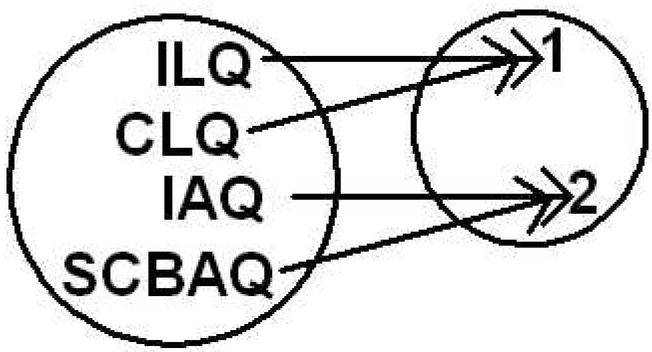


Figure 4.108. Nicole's *Number of Aspects* assignment via Bi–valued Function.

Finally, Nicole made use of a multiplicative mapping structure as indicated in her last statement. Her representational mapping f can be thought of as acting on the ordered pair $(x + 1, 2y + 3)$ of combined linear quantities (CLQ) that is mapped onto the "biggest" areal singleton $[2xy + 3x + 2y + 3]$. With the functional notation, this process can be notated as $f: (x + 1, 2y + 3) \rightarrow [2xy + 3x + 2y + 3]$.

Nicole constantly referred to Bi–valued Functions in her comparison of same–valued linear and areal quantities. When I asked her to compare the linear units with their areal counterparts [linear x vs. areal x, linear $2x + y$ vs. areal $2x + y$] in the next problem, she said that for the linear units she was looking at the one dimension only, and for the areal units she was looking at the two dimensions, the length and the width (Figure 4.109). She then wrote an identity on the last column of the activity sheet:

Protocol 4.83: Nicole's reference to cardinalities in CAQs.

G: Is the right hand side equal to the left hand side? [about her identity ($2x$ +

$y)(x + 2y + 1) = 2x^2 + 5xy + 2y^2 + 2x + y]$

N: Yes.

G: How about in terms of units... is the right hand side equivalent to the left hand side?

N: Yes, both would be cm^2.

G: How about the $5xy$ here... what unit does that have?

N: cm^2.

G: How about the x in $5xy$? The x of $5xy$? what unit...

N: It's just cm.

G: How about y?

N: Just cm.

G: How about 5?

N: It does not have a unit... It's just how many of the xy you have.

G: How about the $2y^2$ there, does it also have cm^2 as unit?

N: Hm hm...

G: How about the 2 in there?

N: It does not have a unit... It's just how many you have...

G: How do you know that the left hand side has the unit of cm^2?

N: Because this is the length which is cm, this is the width which is cm, and when you multiply them together that gives cm^2.

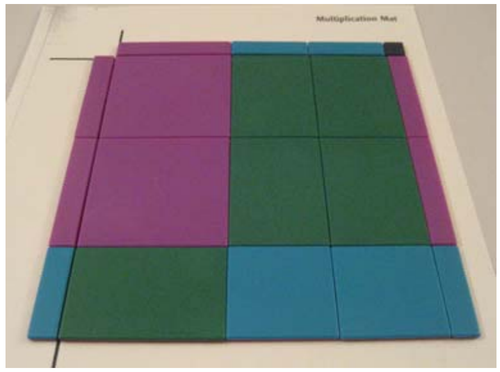


Figure 4.109. Nicole's $2x + y$ by $x + 2y + 1$ rectangle.

Nicole established the fact that the coefficient “5” of the combined areal quantity (CAQ) $5xy$ is of dimensionless nature. For Nicole, “5” is a unit–less constant that can be thought of as the *cardinality* of the set of ordered quintuple $[xy, xy, xy, xy, xy]$. Moreover, because Nicole established the areal–ness of each $[xy]$ singleton by assigning the measurement unit cm to *both* linear x and linear y, Nicole's ordered quintuple can be notated as $[(x, y), (x, y), (x, y), (x, y), (x, y)]$ as well. She established the dimensionless–ness of the “2” in the combined areal quantity (CAQ) “$2y^2$” in a similar manner. In other words, “2” is a unit–less constant that is the same as the *cardinality* of the set of ordered pair $[y^2, y^2]$. Finally, Nicole made use of a multiplicative representational mapping f acting on the ordered pair (cm, cm) of linear measurement units. With the functional notation, this can be notated as f: (cm, cm) $\rightarrow$ cm^2.

Though he obtained a totally different polynomial rectangle for the second task, John's written answers and verbal descriptions were consistent in that he was always referring to his y–dependent–only polynomial rectangle. Because the initial instruction was to make a polynomial rectangle with length $x + 1$ and width $2y + 3$, at some point he had to write an identity in the last column of the activity sheet. In fact, he wrote the identity "$(x + 1)(2y + 3) = 2y^2 + 3y + 6y + 9$" as his answer. John's written answer warrants the disconnect theory as well, in that John was unable to write an area as a product expression (LHS) *based on his rectangle*. If he was able to refer to his rectangle, then the correct identity would be "$(y + 1)(2y + 3) = 2y^2 + 3y + 6y + 9$" instead of "$(x + 1)(2y + 3) = 2y^2 + 3y + 6y + 9$." The following protocol takes this issue into account and reflects how John reconciled the equivalence of *x*– and *y*–dependent LHS with the *y*–dependent–only RHS.

Protocol 4.84: John establishes LHS–RHS equivalence.

G: Are they equal? [about the LHS and the RHS of his identity "$(x + 1)(2y + 3) = 2y^2 + 3y + 6y + 9$"]

J: I mean... they're equal... they have to be equal...

G: Do you want to verify?

J: Do you want me to multiply that [the LHS] out? [I then ask him to do it on the board. Here is the first step of his verification. Figure 4.110]

G: Is there something wrong?

J: No... It's just that... we don't know what *x* is... so... if you knew what *x* was you'd probably... *x* probably equals... [He looks at his figure] It looks like *x* equals *y* plus 2 [He then substitutes $x = y + 2$ and completes his verification. Figure 4.111]

G: So it works with the condition that...

J: With the condition that *x* equals *y* plus 2.

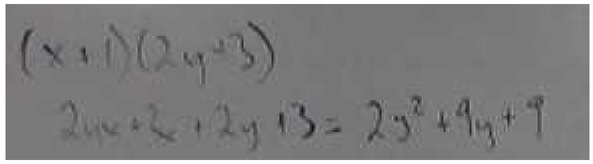


Figure 4.110. John's first step of verification.

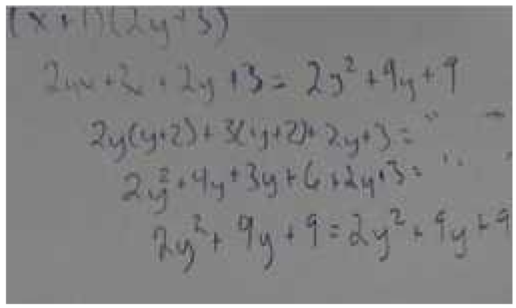


Figure 4.111. John's second step of verification.

At the beginning of the conversation, John was so certain about his equality that he did not feel the need to question it. Upon my request to verify his findings, he obtained "$2yx + 3x + 2y + 3 = 2y^2 + 9y + 9$." At this point, he realized that the RHS is *y*–dependent–only whereas the LHS has "*x*"s and "*y*"s, and deduced that he somehow had to get rid of the "*x*" on the LHS. He then referred to his figure made of tiles. He actually *measured* the *x* at the top of his figure using the "*y*" and the "1" tiles. In order to get rid of the "*x*" on the LHS, he substituted $x = y + 2$, based on his measurements. In other words, *John made sense of the dimension tiles for the first time*[12]. The dimension tiles do not stand as irreducible quantities whose term wise multiplication yields the corresponding irreducible areal quantity, though. They rather stand as some sort of measurement tools helping John establish the LHS–RHS equivalence of his written identity. Note that all this

[12] I carried this protocol over this section for analysis purposes. I must remind the reader that John made sense of the dimension tiles when working on the third polynomial multiplication task. See Protocol 4.69 and Figures 4.90–95 above.

is happened before the “Change of Mind” turning point (See Protocol 4.69). I hypothesize that John's sense making of the dimension tiles as some sort of measurement tools prepared the way for him to give up the *Filling in the Puzzle Strategy* and to embrace *Term Wise Multiplication of Irreducible Linear Tiles* in the third task on the multiplication of $2x + y$ by $x + 2y + 1$. The following protocol picks up at the point where John made sense of the *product quantities* (Behr et al., 1994) with reference to mapping structures and his comparison of the same–valued linear and areal quantities.

Protocol 4.85: John's comparison of the same–valued linear and areal quantities.

G: What is this? [pointing to the areal *x* at the bottom in Figure 4.95]

J: Just this one? That's *x*...

G: What is this one? [pointing to the linear *x* at the top]

J: It's the length of *x*...

G: Are they the same or different? How are they the same... how are they different? [about the linear *x* at the top and the areal *x* at the bottom.

J: Well... This [pointing to the linear *x* at the top] just represents the length of *x* whereas this [pointing to the areal *x* at the bottom] represents the area of *x*... so... this [pointing to the linear *x* at the top] would be *x* centimeters whereas this [pointing to the areal *x* at the bottom] would be *x* centimeters squared... they are different... one is an area one is a length...

G: How about this *y* here [pointing to the linear *y* at the top] and this *y* here [pointing to the areal *y* at the bottom]...

J: Same thing... I mean... same thing as this one... [meaning what he said for the previous comparison question holds]... this is length of y [pointing to the linear y at the top] and this is area of y [pointing to the areal y at the bottom]...

G: What do you mean by area of y?

J: Well it's not just... this [pointing to the linear y at the top] represents the length whereas this [pointing to the areal y at the bottom] represents the area... you gotta look at this [pointing to the linear y at the top] and this [pointing to the linear 1 on the side] to find this one (Figure 4.112).

G: Tell me more about that... You are pointing to those two...

J: This right here [pointing to the areal y] is 1 [pointing to the linear 1 on the side] times y [pointing to the linear y at the top]. So we get y centimeter squared whereas this is just representing the length of y [pointing to the linear y at the top]

G: How about this $2x + y$ and this $2x + y$... [pointing to the linear $2x + y$ at the top and areal $2x + y$ at the bottom, respectively] Are they the same or different? How are they the same... how are they different?

J: They're different... for the same reason as before... This is a length [pointing to the linear $2x + y$ at the top] so it's represented by centimeters if these are centimeters... and this is an area... this [pointing to the areal $2x + y$ at the bottom] is represented by this [pointing to the linear $2x + y$ at the top] times this [pointing to the linear 1 on the side]... I mean... Well... If this was taken away [removing the black tile from the corner] this would just be the other length [pointing to the areal $2x + y$]... like... representing a length as well... whereas... yeah I guess that is the difference [placing the black tile back at the corner] yeah it is... 'cuz this [pointing to the areal $2x + y$ at the bottom] is this [pointing to the linear $2x + y$ at the top] times this [pointing to the linear 1 on the side]

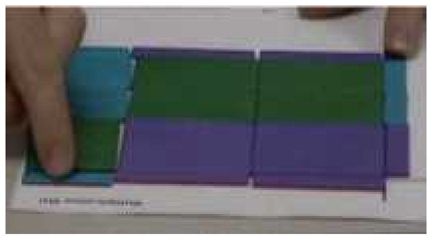


Figure 4.112. John's hand gestures indicating mapping structures.

In his comparison of the same–valued linear and areal quantities, John constantly used the word "represents" for both linear and areal tiles. This is in contrast to Sarah who thought that the dimension tiles "are used as" length and the areal tiles "are" the areas (See Protocol 4.80 and the following paragraphs). John and Sarah are therefore alike in that they both agree that the dimension tiles do not stand for the "actual" linear quantities and that these rather "represent" or "are used as" length (Instrumentalized Linear Quantities). John thought the same for the areal tiles as opposed to Sarah for whom the areal tiles actually "are" areas (Areal Reflexivity).

As warranted both by Figure 4.112 and his statement "you gotta look at this [pointing to the linear y at the top] and this [pointing to the linear 1 on the side] to find this one," John made use of multiplicative type mapping structures, which can be modeled via the functional notation f: $(y, 1) \rightarrow y$. John's language "this and this to find this one" is reminiscent of Rob's language "when you put this length and that length together it makes a two dimensional shape, which is this... length and width." (See Protocol 4.64) Both statements stand as a strong indication of both multiplicative type RUC as well as multiplicative type mapping structures.

John's comparison of the ($2x + y$)–valued linear and areal quantities is interesting in that he makes use of some sort of visual proof by contradiction. Sarah also frequently made use of this strategy (cf. Protocol 4.74). John said that if the black square on the side was not there, then the "$2x + y$"s at the top and the bottom would be the same; they would just be lengths. In other words, it is the "black square" that causes the difference. John's reasoning could be explained as follows. The quantity ($2x + y$, 1) is of multiplicative nature. If he removes the black square, he is left with ($2x + y$, ...)–just one term, instead of an ordered pair. Once he removes one of the terms of the ordered pair, the ordered pair becomes a linear unit, just a number. In other words, while ($2x + y$, 1) is of multiplicative nature, when 1 is removed, he is left with the *scalar* ($2x + y$). John's statement "If this was taken away [removing the black tile from the corner] this would just be the other length" is reminiscent of the difference between a *vector* and a *scalar*. The quantity ($2x + y$, 1) can be thought as an ordered pair as well as a vector on the plane, whereas $2x + y$ alone describes a scalar, a number on the real number line.

4.6. Factorization of Polynomial Expressions

In this section, I analyze data related to two factorization problems:

- Factorization of a polynomial of the form $p(x)$ over the set $Z(X)$. To be more specific, the students factored $p(x) = x^2 + 5x + 6$.

- Factorization of a polynomial of the form $q(x, y)$ over the set $Z(X, Y)$. To be more specific, the students were asked to factor
  $q(x, y) = 2x^2 + 7xy + 3y^2 + 5x + 5y + 2.$

4.6.1. Multiplicative Representational Unit Coordination (MRUC)

In the polynomial multiplication task analyzed in the previous section, the dimension tiles were always placed on two sides of the rectangle, and in both cases students either relied on an additive approach (filling in the puzzle strategy) or on a multiplicative approach (term–wise multiplication of the irreducible linear tiles). I added this task on the factorization of polynomials to the interview outline because I was trying to understand whether students would be able to realize the multiplicative nature of the irreducible areal tiles as well as the boxes of the same color *without the presence of the dimension tiles initially*. In that sense, this task required quantitative reasoning at a more advanced level. Some students were simultaneously placing the irreducible linear tiles corresponding to the irreducible areal tiles generating the polynomial rectangle, which was an indication of *reverse reasoning*. Other students preferred first completing their rectangles, then placing the dimension tiles around the edges.

The common direction for the first problem was "Make a rectangle for the expression $x^2 + 5x + 6$, then factor the expression using the algebra tiles." In less than a minute all students produced the correct rectangle (Figure 4.113).

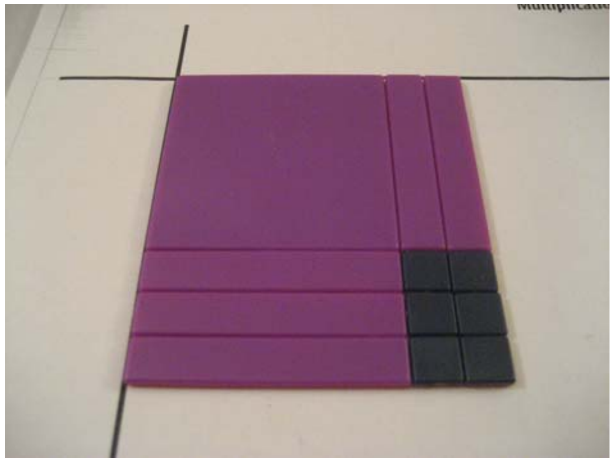


Figure 4.113. Rectangle representing polynomial $x^2 + 5x + 6$.

However in this first problem, none of the students relied on the *Term Wise Factorization of Irreducible Areal Quantities into Irreducible Linear Quantities* strategy. All students rather first completed their rectangle and then placed the dimension tiles representing $x+2$ and $x+3$ around two adjacent edges. This behavior involves some reverse reasoning to factor the polynomial rectangle unit into combined linear quantities (CLQ), which are the dimensions of the rectangle itself. I name this *Term Wise Factorization of Polynomial Rectangle Areal – Singleton into Combined Linear Quantities* strategy. Relying on this strategy, all students generated their rectangle first, as in Figure 4.113 above, and then "representationally factored" their rectangle (the $x+2$ by $x+3$ areal singleton) into two representational sets of combined linear quantities (RSCLQ) as in Figure 4.114.

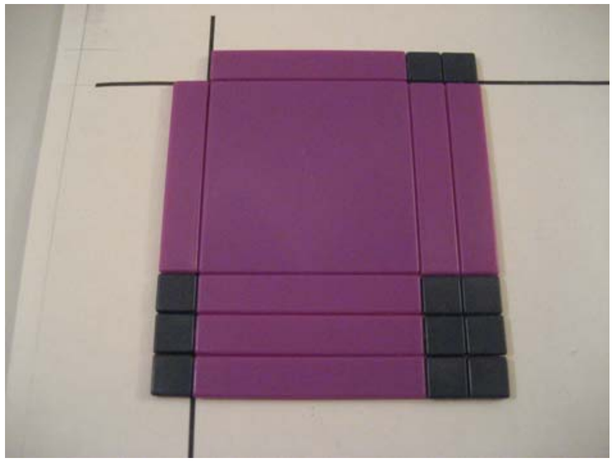


Figure 4.114. Representationally factored polynomial into RSCLQs.

After making similar figures as above, all students established the dimension tiles (RSCLQs) as the factors of the polynomial under consideration. The following discussion with Nicole illustrates this point.

Protocol 4.86: Nicole's MRUC concerning the RSCLQs.

N: This expression here $x^2 + 5x + 6$ [pointing to the expression $x^2 + 5x + 6$ written on the activity sheet] has two factors which are $x + 2$ [pointing to the corresponding dimension tiles on the side] and $x + 3$ [pointing to the corresponding dimension tiles at the top]

G: How about $x + 2$, is it a length or an area?

N: $x + 2$ is a length.

G: How about $x + 3$, is it a length or an area?

N: $x + 3$ is a length. When you multiply them together you get an area.

Referring to a *Term Wise Factorization of Polynomial Rectangle Areal – Singleton into Combined Linear Quantities* Strategy, Nicole decomposed the biggest areal singleton, namely the polynomial rectangle $x^2 + 5x + 6$, into the ordered pair $(x + 2, x + 3)$ of combined linear quantities (CLQ). Moreover, MRUC is evident in the relational representation $(x + 2, x + 3)$. The ordered pair $(x + 2, x + 3)$ of combined linear quantities can be thought of as the elements of the two RSCLQs combined via the addition operation, where, $RSCLQ_1 = \{x, 2\}$ and $RSCLQ_2 = \{x, 3\}$. Compare this with the RSILQs, which can be notated as $RSILQ_1 = \{x, 1, 1\}$, and $RSILQ_2 = \{x, 1, 1, 1\}$. To be more specific, the addition operation "+" first acts on both sets to yield the combined linear quantities (CLQ) $x + 2$ and $x + 3$. Juxtaposition then "acts" on this pair of CLQs to form the ordered pair $(x + 2, x + 3)$ of combined linear quantities. Juxtaposition, in that sense, can be thought of as a synonym for a Representational Cartesian Product (RCP) acting on the pair of linear singletons $\{x + 2\}$ and $\{x + 3\}$. This results in the Representational Cartesian Product $\{x + 2\} \times \{x + 3\}$, equivalently, the ordered pair $(x + 2, x + 3)$ of combined linear quantities.

In the second task on the factorization of the polynomial $2x^2 + 7xy + 3y^2 + 5x + 5y + 2$, all students exhibited the same strategy (Term Wise Factorization of Polynomial Rectangle Areal – Singleton into Combined Linear Quantities). In other words, they were all able to representationally factor the expression $2x^2 + 7xy + 3y^2 + 5x + 5y + 2$ as the product of $2x + y + 1$ and $x + 3y + 2$ (Figure 4.115).

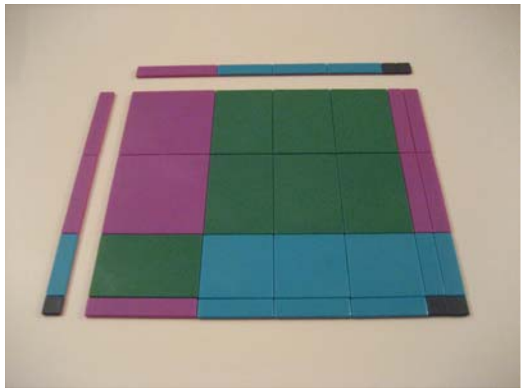


Figure 4.115. Representationally factored polynomial into RSCLQs.

Equivalently, these students were able to decompose the biggest areal singleton, the rectangle itself, into the ordered pair $(2x + y + 1, x + 3y + 2)$ of combined linear quantities (CLQ). In a sense, students were able to define the *inverse* of an RCP on the biggest areal singleton to obtain the ordered pair $(2x + y + 1, x + 3y + 2)$ of combined linear quantities (CLQ), which can be thought of as a juxtaposition of the RSCLQ levels. To be more specific, the RSCLQ levels for this task are $RSCLQ_1 = \{2x, y, 1\}$ and $RSCLQ_2 = \{x, 3y, 2\}$. Compare this with the RSILQs, which can be notated as $RSILQ_1 = \{x, x, y, 1\}$ and $RSILQ_2 = \{x, y, y, y, 1, 1\}$. In the *Linear vs. Areal Units* subsection below, I will show that Sarah established the reversibility notion not just with the RSCLQ levels, but with the RSILQ levels as well.

4.6.2. Additive Representational Unit Coordination (ARUC)

Once again, ARUC types arising from the data are essentially derived from two main components: 1) Students' written expressions, 2) Hand gestures combined with verbal descriptions in the process of constructing the rectangle representing the multiplication of the polynomials.

- **Boxes of the Same Color Type Addends (Subunits):** In Rob's polynomial rectangle representing $x^2 + 5x + 6$, there are four "boxes" of the same color: First, the x by x purple box; second, the x by 2 purple box; third, the 3 by x purple box; and fourth, the 3 by 2 black box (Figure 4.114). In the polynomial rectangle representing $2x^2 + 7xy + 3y^2 + 5x + 5y + 2$, on the other hand, there are nine "boxes" of the same color: First, the $2x$ by x purple box; second, the y by x green box; third, the 1 by x purple box; fourth, the $2x$ by $3y$ green box; fifth, the y by $3y$ blue box; sixth, the 1 by $3y$ blue box; seventh, the $2x$ by 2 purple box; eighth, the y by 2 blue box; and ninth, the 1 by 2 black box (Figure 4.115). The sum of the "boxes of the same color" addends (subunits) equals the area of the polynomial rectangle (unit) under consideration. The following relational notations of ordered n–tuples of "boxes of the same color" type addends generate the corresponding polynomial rectangle:
    - The quadruple $[x^2, 2x, 3x, 6]$ generating the polynomial rectangle representing $x^2 + 5x + 6$ (Figure 4.114).
    - The 9–tuple $[2x^2, yx, x, 6xy, 3y^2, 3y, 4x, 2y, 2]$ generating the polynomial rectangle representing $2x^2 + 7xy + 3y^2 + 5x + 5y + 2$ (Figure 4.115).
- **Like Areal Tiles Combined Type Addends (Subunits):** With reference to Rob's figures, the following addends are used. In the polynomial rectangle representing $x^2 + 5x + 6$ (Figure 4.114), there are three Like Areal Tiles Combined Type Addends: First, the purple subunit with an area of x^2; second, the combined disconnected purple subunits with an area of $5x$; and third, the black subunit with an area of 6. In the polynomial rectangle representing $2x^2 + 7xy + 3y^2 + 5x + 5y +$

2 (Figure 4.115), there are six Like Areal Tiles Combined Type Addends: First, the purple subunit with an area of $2x^2$; second, the combined disconnected green subunits with an area of $7xy$; third, the blue subunit with an area of $3y^2$; fourth, the combined disconnected purple subunits with an area of $5x$; fifth, the combined disconnected blue subunits with an area of $5y$; and sixth, the black subunit with an area of 2. With the relational notation, the above addends can be modeled as follows:

- The triple $[x^2, 5x, 6]$ generating the polynomial rectangle representing $x^2 + 5x + 6$ (Figure 4.114).
- The hextuple $[2x^2, 7xy, 3y^2, 5x, 5y, 2]$ generating the polynomial rectangle representing $2x^2 + 7xy + 3y^2 + 5x + 5y + 2$ (Figure 4.115).

When decomposing each areal sub–unit into areal sub–subunits, all interview students preferred the *Irreducible Addends Type ARUC*, which was also the case for the polynomial multiplication task analyzed in the previous section. The notation is straightforward and similar to the one I used in the previous section; therefore, I omit it here.

The table below summarizes the ARUC types used by the interview students in the factorization of polynomials activities.

Table 4.40

ARUC Types Used by Interview Students

Students	Subunits Generating the Polynomial Rectangle	Sub–Subunits Generating the Subunits
Brad	Like Areal Tiles Combined Type Addends	Irreducible Addends
Nicole	Boxes of the Same Color Type Addends	Irreducible Addends

Rob	Like Areal Tiles Combined Type Addends	Irreducible Addends
Sarah	Boxes of the Same Color Type Addends	Irreducible Addends
John	Boxes of the Same Color Type Addends	Irreducible Addends

4.6.3. Pseudo – Multiplicative Representational Unit Coordination (PMRUC)

In the first task on the polynomial factorization, Brad and Rob provided *pseudo–products* "$1 \cdot x^2$, $5 \cdot x$, $6 \cdot 1$" for the "Area of the boxes of the same color as a product," whereas Nicole, Sarah, and John came up with *products* "$x \cdot x$, $x \cdot 3$, $x \cdot 2$, $3 \cdot 2$." Brad and Rob's pseudo–products are of additive nature, which becomes apparent via the relational notation of singletons and ordered *n*–tuples $[x^2]$, $[x, x, x, x, x]$, $[1, 1, 1, 1, 1, 1]$ of irreducible *areal* quantities (IAQ). The cardinality function *card* maps Brad and Rob's pseudo–products onto the cardinalities of these *representational sets of areal quantities*. One can notate these as $card\ ([x^2]) = 1$, $card\ ([x, x, x, x, x]) = 5$, $card\ ([1, 1, 1, 1, 1, 1]) = 6$. The table below summarizes cardinalities and relational notation describing Brad and Rob's pseudo–products for the two tasks on the factorization of polynomials.

Table 4.41

Pseudo–Products and Cardinalities Based on Brad and Rob's Answers for the "Area of the Boxes of the Same Color as a Product"

Pseudo–Products	**Relational Notation**	**Cardinalities**
$1 \cdot x^2$	$[x^2]$	$card\ ([x^2]) = 1$
$5 \cdot x$	$[x, x, x, x, x]$	$card\ ([x, x, x, x, x]) = 5$
$6 \cdot 1$	$[1, 1, 1, 1, 1, 1]$	$card\ ([1, 1, 1, 1, 1, 1]) = 6$
$2 \cdot x^2$	$[x^2, x^2]$	$card\ ([x^2, x^2]) = 2$
$7 \cdot xy$	$[xy, xy, xy, xy, xy, xy, xy]$	$card\ ([xy, xy, xy, xy, xy, xy, xy]) = 7$
$3 \cdot y^2$	$[y^2, y^2, y^2]$	$card\ ([y^2, y^2, y^2]) = 3$
$5 \cdot x$	$[x, x, x, x, x\ ]$	$card\ ([x, x, x, x, x]) = 5$
$5 \cdot y$	$[y, y, y, y, y]$	$card\ ([y, y, y, y, y]) = 5$
$2 \cdot 1$	$[1, 1]$	$card\ ([1, 1]) = 2$

As was the case in the previous section, Brad and Rob's pseudo–products can be modeled as the disjoint subsets of a particular *Representational Set of Irreducible Areal Quantities* (RSIAQ). The following disjoint unions describe these students' behaviors.

- RSIAQ $_{x+2 \text{ by } x+3} = \{x^2\} \dot{\cup} \{\text{x, x, x, x, x}\} \cup \{1, 1, 1, 1, 1, 1\}$.
- RSIAQ $_{2x+y+1 \text{ by } x+3y+2} = \{x^2, x^2\} \dot{\cup} \{xy, xy, xy, xy, xy, xy, xy\} \cup \{y^2, y^2, y^2\} \cup \{x, x, x, x, x\} \cup \{y, y, y, y, y\} \cup \{1, 1\}$.

Nicole, Sarah, and John, on the other hand, provided products that can be modeled as *areal singletons* whose disjoint union generates a *Representational Set of Same–Color–Box Areal Quantities* (RSSCBAQ) for each task:

- RSSCBAQ $_{x+2 \text{ by } x+3} = \{ (x, x) \} \cup \{ (2, x) \} \cup \{ (x, 3) \} \cup \{ (2, 3) \}$.
- RSSCBAQ $_{2x+y+1 \text{ by } x+3y+2} = \{ (2x, x) \} \dot{\cup} \{ (y, x) \} \dot{\cup} \{ (1, x) \} \dot{\cup} \{ (2x, 3y) \} \dot{\cup} \{ (y, 3y) \} \dot{\cup} \{ (1, 3y) \} \dot{\cup} \{ (2x, 2) \} \dot{\cup} \{ (y, 2) \} \dot{\cup} \{ (1, 2) \}$.

The disjoint subsets that model Brad and Rob's pseudo–products are shown in the following diagrams.

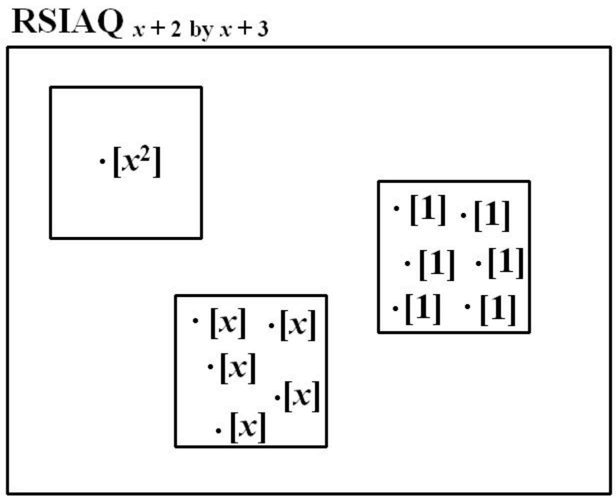


Figure 4.116. Disjoint subsets of RSIAQ $_{x + 2 \text{ by } x + 3}$ for Rob and Brad.

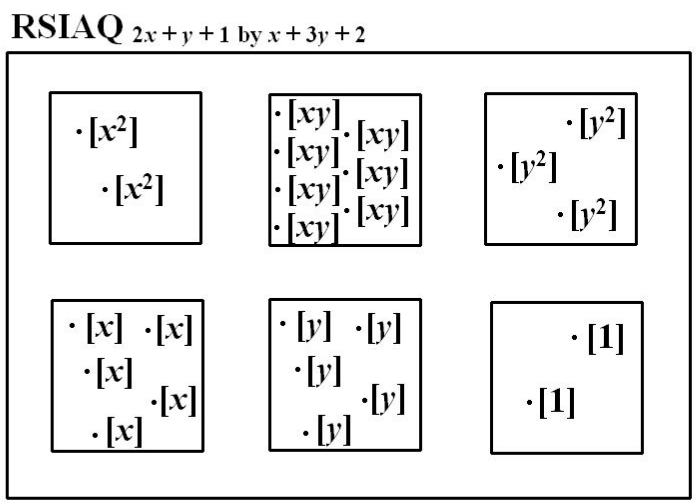


Figure 4.117. Disjoint subsets of RSIAQ $_{2x + y + 1 \text{ by } x + 3y + 2}$ for Rob and Brad.

Figures 4.116 and 4.117 above reveals the following:

- *card* (RSIAQ $_{x + 2 \text{ by } x + 3}$) = 1 + 5 + 6 = 12.
- *card* (RSIAQ $_{2x + y + 1 \text{ by } x + 3y + 2}$) = 2 + 7 + 3 + 5 + 5 + 2 = 24.

The disjoint subsets that model Sarah, Nicole, and John's products (areal singletons) are shown in the following diagrams.

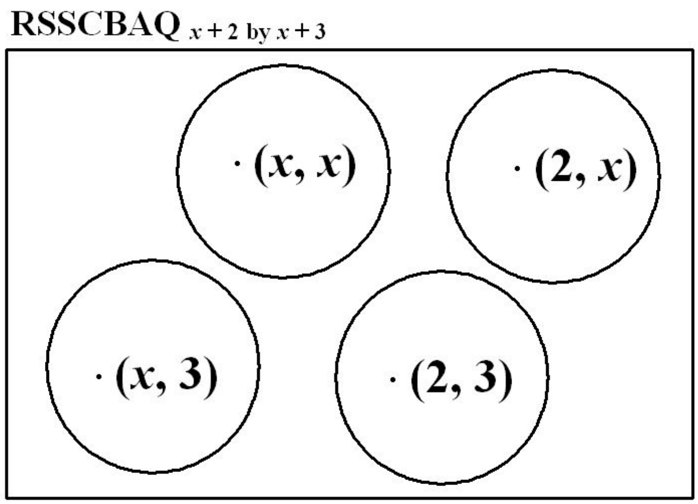


Figure 4.118. Disjoint subsets of RSSCBAQ $_{x + 2 \text{ by } x + 3}$ for Sarah, Nicole, and John.

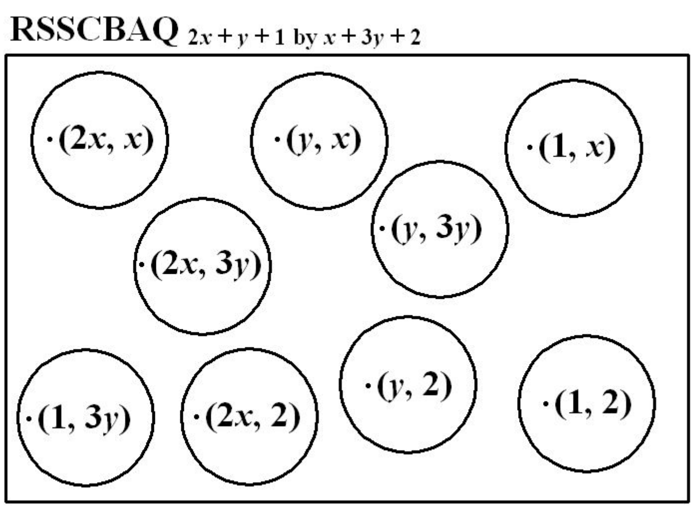


Figure 4.119. Disjoint subsets of RSSCBAQ $_{2x + y + 1 \text{ by } x + 3y + 2}$ for Sarah, Nicole, and John.

Figures 4.118 and 4.119 above reveals the following:

- *card* ($\text{RSSCBAQ}_{x+2 \text{ by } x+3}$) = 1 + 1 + 1 + 1 = 4.
- *card* ($\text{RSSCBAQ}_{2x+y+1 \text{ by } x+3y+2}$) = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 9.

In all derivations above, I used the additive property of the cardinality function. When we compare the diagrams in Figures 4.116 and 4.117 with those in Figures 4.118 and 4.119, we see that, not just the cardinalities of disjoint subsets, but the cardinalities of the union sets are different for the two groups of students, as well.

I complete this subsection with Sarah's proof by contradiction invalidating PMRUC.

Protocol 4.87: Sarah's proof by contradiction invalidating PMRUC.

G: How about this 7*xy*... [this is one of the terms on the LHS of the identity on the last column of the activity sheet] now... *y* is a unit... *x* is another unit... 7 is another unit... it must be units cube...

S: Well... It's more like if you were to take seven of them [meaning, seven green rectangles] and stick them... [We were out of green rectangles, but we had many purple bars. Sarah then wanted to demonstrate her idea with purple bars. She placed 7 purple bars below her polynomial rectangle and did a little demonstration. Figure 4.120] It would be 7*x* [pointing to the length] times *y* [pointing to the width]. It's more like 7 times... like... if you were to split it up... 7*x* times *y*.

G: So... that does not bother you, right? You are sure that the whole thing is in units squared?

S: Right.

G: How about for $3y^2$? [this is one of the terms on the LHS of the identity on the last column of the activity sheet. I am trying to trouble her once again]. y^2 is units squared and 3 is a unit, so, $3y^2$ must be units cube...

S: y^2 is a square but inside [she is referring and pointing to the $3y^2$ “box”] you are adding three of them [i.e., the 3 here behaves as a coefficient, it's unitless] 3 is not necessarily a length... it's more of a quantity.

G: 3 is more of a quantity... tell me more about that...

S: It's the same with the 7*xy*... 'cuz it's more like you are adding *xy* plus *xy* plus *xy*... 7 times... [i.e., the 7 in 7*xy* is additive in nature] whereas *xy* is units squared and you add them all... you get 7*xy* [which is also in units squared].

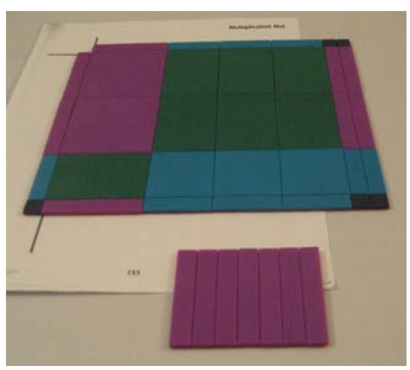


Figure 4.120. Sarah's demonstration.

Sarah originally defined the purple bar as the representation of an *x* by 1 rectangle. Later she redefined this purple bar as an *x* by *y* rectangle, where the *x* became the shorter side, and the *y* became the longer side. She did this, and she did it very quickly, for the purpose of demonstrating that "7*xy*" must have units squared as units, and that the "7" stands for a coefficient serving to count how many there are of each irreducible "*xy*" areal quantity. In a similar way she verbally explained that the "3" of "$3y^2$" is a unitless constant behaving the same as the "7" of the "7*xy*."

4.6.4. Linear vs. Areal Units

As was the case for the polynomial multiplication tasks analyzed in the previous section, Nicole made reference to a Bi–Valued Function in her sense–making of the same–valued linear and areal quantities. Rob and Sarah, on the other hand, operated on an abstract level of assigning measurement units to the same–valued linear and areal quantities, namely by referring to "units" and "units squared" rather than inches or centimeters, as in the previous tasks. Brad, once again, referred to his "Dimension Tiles in a Standing–Up Position Strategy" in his comparison of the same–valued linear and areal quantities (Figure 4.121).

Figure 4.121. Brad's "standing–up positioned" linear *x*.

On the second task on polynomial factorization, my instruction was “Make a rectangle for the expression $2x^2 + 7xy + 3y^2 + 5x + 5y + 2$ first, then factor the expression $2x^2 + 7xy + 3y^2 + 5x + 5y + 2$ using the algebra tiles.” Sarah was the only student to *simultaneously* place the *pair of irreducible linear tiles* corresponding to *each* irreducible areal tile generating the polynomial rectangle, which was an indication of *reverse reasoning*. In contrast, Nicole, John, Rob, and Brad first completed the rectangle and then placed the dimension tiles around it. Sarah first collected all the pieces she thought she would need. At the first stage, she placed the purple square representing the *x* squared on the upper left corner. She then placed the pair of irreducible dimension tiles accordingly. She said “We start with that... [about the purple box] the *x* times *x*.” (Figure 4.122)

Figure 4.122. First stage of Sarah's reverse reasoning.

In a similar manner, she placed the second *x* squared areal tile, and then one linear *x* tile at the top, right next to the previous linear *x* tile (Figure 4.123).

Figure 4.123. Second stage of Sarah's reverse reasoning.

She then placed two green rectangles below the purple squares, and at the same time, she placed one blue bar right below the x tile on the side (Figure 4.124).

Figure 4.124. Third and fourth stages of Sarah's reverse reasoning.

She continued this pattern, making sure that each time she placed a box in the area, she also placed the relevant irreducible linear tile(s) on the side and/or at the top. In that sense, Sarah worked with both the irreducible areal quantities (IAQ) and irreducible linear quantities (ILQ) *at the same time*. Sarah was the only student to associate each irreducible areal quantity (IAQ) with its dimensions, namely the corresponding pair of irreducible linear quantities (ILQ), in a polynomial factorization problem, *in the process of* generating the polynomial rectangle under consideration. In this way, Sarah established the multiplicative nature of the irreducible areal quantities (IAQ). She was able *both* to generate the correct polynomial rectangle (Figure 4.125) and to induce a

Representational Cartesian Product (RCP) of *Representational Set of Irreducible Linear Quantities* (RSILQ) via *reverse reasoning*. As before, I denote this Representational Cartesian Product as RCP $_{\text{RSILQ}}$.

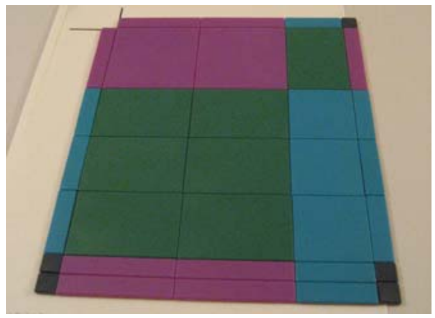


Figure 4.125. Sarah's complete rectangle via reversively induced RCP $_{\text{RSILQ}}$.

Sarah's behavior concerning her reverse reasoning and her induction of a RCP $_{\text{RSILQ}}$ calls for the notion of *Invertible Mapping Structures* (IMS). Starting from the beginning, Sarah's first action (Figure 4.122) can be notated with the functional notation as $f^{-1}: [x^2] \to (x, x)$. Her second action (Figure 4.123) can be modeled with the same functional notation. Her third and fourth actions (Figure 4.124) are notated as $f^{-1}: [xy] \to (x, y)$. The table below lists the inverse functional notation and equality for Sarah's actions on each irreducible areal quantity (IAQ).

Table 4.42

Sarah's Inverse Mapping Structures and Induced RCP$_{RSILQ}$

IAQ	Total Number of IAQ	Functional Notation	Functional Equality	Resulting Pair of ILQ	Induced RCP
$[x^2]$	2	$f^{-1}: [x^2] \to (x, x)$	$f^{-1}\ [x^2] = (x, x)$	(x, x)	$\{x\} \times \{x\}$
$[xy]$	6	$f^{-1}: [xy] \to (x, y)$	$f^{-1}\ [xy] = (x, y)$	(x, y)	$\{x\} \times \{y\}$

[x]	4	$f^{-1}: [x] \to (x, 1)$	$f^{-1}\ [x] = (x, 1)$	$(x, 1)$	$\{x\} \times \{1\}$
[yx]	1	$f^{-1}: [yx] \to (y, x)$	$f^{-1}\ [yx] = (y, x)$	(y, x)	$\{y\} \times \{x\}$
[y^2]	3	$f^{-1}: [y^2] \to (y, y)$	$f^{-1}\ [y^2] = (y, y)$	(y, y)	$\{y\} \times \{y\}$
[y]	2	$f^{-1}: [y] \to (y, 1)$	$f^{-1}\ [y] = (y, 1)$	$(y, 1)$	$\{y\} \times \{1\}$
[x]	1	$f^{-1}: [x] \to (1, x)$	$f^{-1}\ [x] = (1, x)$	$(1, x)$	$\{1\} \times \{x\}$
[y]	3	$f^{-1}: [y] \to (1, y)$	$f^{-1}\ [y] = (1, y)$	$(1, y)$	$\{1\} \times \{y\}$
[1]	2	$f^{-1}: [y] \to (1, 1)$	$f^{-1}\ [y] = (1, 1)$	$(1, 1)$	$\{1\} \times \{1\}$

Now based on Sarah's actions, I give a more robust definition of the Mapping Structures involving the Representational Sets of Irreducible Linear Quantities (RSILQ), the Representational Sets of Combined Linear Quantities (RSCLQ), the Representational Sets of Irreducible Areal Quantities (RSALQ), the Representational Sets of Same–Color–Box Areal Quantities (RSSCBAQ), and the induced Representational Cartesian Products (RCP). The notations $\text{RCP}_{\text{RSILQ}}$ and $\text{RCP}_{\text{RSCLQ}}$ will be reserved for the Representational Cartesian Products (RCP) defined on the pair of Representational Sets of Irreducible Linear Quantities (RSILQ) and the pair of Representational Sets of Combined Linear Quantities (RSCLQ), respectively.

Because I already introduced the multiplication operation behaving as a function *f*, which is the essence of the Mapping Structures concept, in the previous sections, I want to start the discussion with the inverse of *f*, namely the function f^{-1}. As can be warranted by Sarah's actions and descriptions above, f^{-1} acts on the Representational Set of Irreducible Areal Quantities (RSIAQ) and maps this set onto the Representational Cartesian Product defined on two Representational Sets of Irreducible Linear Quantities ($\text{RCP}_{\text{RSILQ}}$) in a one–to–one correspondence (Figure 4.126).

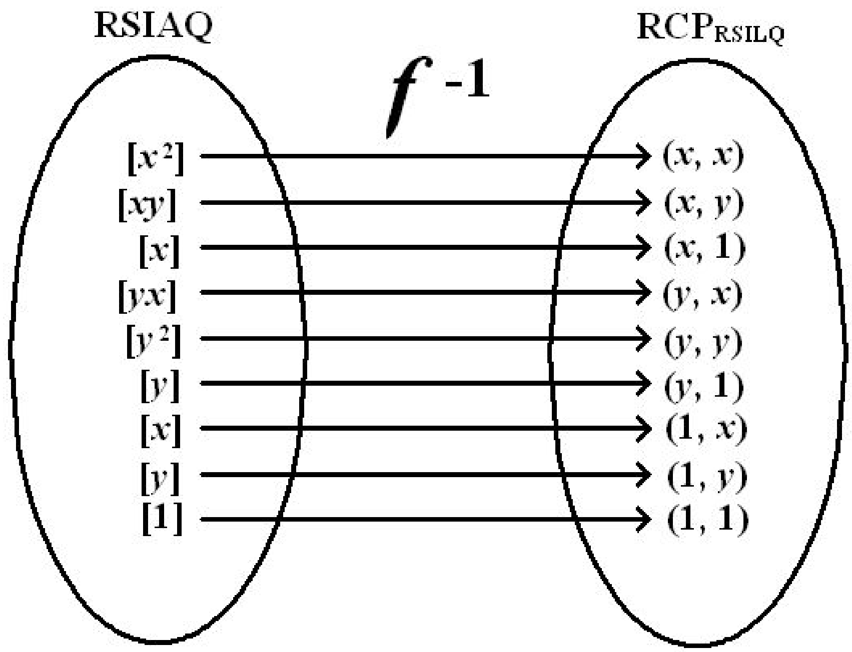


Figure 4.126. Sarah's inverse mapping structures.

Figure 4.126 above is only a representative of the true RSIAQ and RCP_{RSILQ} as both sets are of cardinality 24. Remember, the repetition of elements in a representational set is allowed in my work. To be more specific,

- RSIAQ = { [x^2], [x^2], [xy], [xy], [xy], [xy], [xy], [xy], [x], [x], [x], [x], [y^2], [y^2], [y^2], [y], [y], [x], [y], [y], [y], [1], [1] }
- RCP_{RSILQ} = { (x, x), (x, x), (x, y), (x, y), (x, y), (x, y), (x, y), (x, y), (x, 1), (x, 1), (x, 1), (x, 1), (y, y), (y, y), (y, y), (y, 1), (y, 1), (1, x), (1, y), (1, y), (1, y), (1, 1), (1, 1) }
- *card* (RSIAQ) = *card* (RCP_{RSILQ}) = 24.
- f^{-1}: RSIAQ → RCP_{RSILQ}.

The function f^{-1} is onto (surjective) and one–to–one (injective); therefore, it is also a bijective mapping, namely invertible. Its inverse, $(f^{-1})^{-1}$, is *f* itself, which I defined in the previous section. The induced Representational Cartesian Product defined on two Representational Sets of Irreducible Linear Quantities (RCP_{RSILQ}) can be observed to exist as follows:

- $RCP_{RSILQ} = RSILQ_1 \times RSILQ_2$ where "×" denotes the Cartesian Product.
- $RSILQ_1 = \{x, x, y, 1\}$ and $RSILQ_2 = \{x, y, y, y, 1, 1\}$.
- $RCP_{RSILQ} = \{x, x, y, y, 1\} \times \{x, y, y, y, 1, 1\}$.

The decomposition of the induced RCP_{RSILQ} into RSILQs can be observed in the following figure.

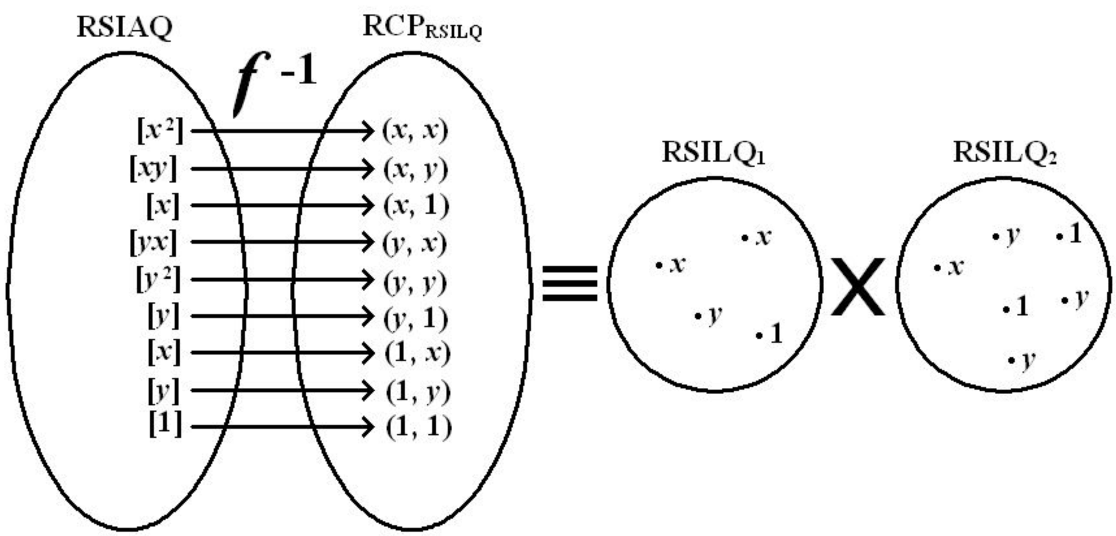


Figure 4.127. Sarah's decomposition of the RCP into RSILQs.

For a polynomial multiplication problem, on the other hand, everything stays the same except that one uses the function f instead of the function f^{-1} and the arrows become inverted in Figure 4.127 above. In the previous section, I showed that students constructing their rectangle via Term Wise Multiplication of Irreducible Linear Quantities (ILQ) Strategy made use of this function *f*. Both *f* and f^{-1} are bijective mappings and are therefore invertible.

Sarah was the only student to refer to both types of Mapping Structures associated with *f* and f^{-1}. In fact, after constructing her polynomial rectangle via reverse reasoning as I described above, Sarah then made use of Mapping Structures associated with the function *f* in her description of the Same–Color–Box Areal Quantities (SCBAQ). The following discussion illustrates this point.

Protocol 4.88: Sarah's reference to mapping structures defined on the representational sets of combined linear quantities (RSCLQ).

G: How many different boxes of the same color do you see this time?

S: [counting and at the same time pointing to the same–color–boxes] One, two, three, four, five, six, seven, eight, nine.

G: Now let's write the products again... [meaning the areas of the same–color–boxes as a product]

S: Well this is gonna be 2*x* times *x* [pointing to the corresponding dimension tiles]. This one's gonna be *x* times *y* [pointing to the corresponding dimension tiles]. This is 1 times *x* [pointing to the dimensions of the box]. This is 2*x* times 3*y* [pointing to the corresponding dimension tiles]. This one is *y* times 3*y* [pointing to the dimensions of the box]. This one is 3*y* times 1 [pointing to the corresponding dimension tiles]. This one is 2*x* times 2 [pointing to the corresponding dimension tiles]. This one is *y* times

2 [pointing to the corresponding dimension tiles]. And this is 2 times 1 [pointing to the dimensions of the box].

In this discussion, Sarah started by first inducing a Representational Cartesian Product (RCP) on the two Representational Sets of Combined Linear Quantities (RSCLQ). To be more specific,

- $RSCLQ_1 = \{2x, y, 1\}$ and $RSCLQ_2 = \{x, 3y, 2\}$. Applying the Cartesian Product "×" she obtained:
- $RSCLQ_1 \times RSCLQ_2 = \{2x, y, 1\} \times \{x, 3y, 2\} = \{ (2x, x), (2x, 3y), (2x, 2), (y, x), (y, 3y), (y, 2), (1, x), (1, 3y), (1, 2) \} = RCP_{RSCLQ}$. She then defined a multiplication operation behaving as a bijective mapping from RCP_{RSCLQ} **onto and into** RSSCBAQ as follows:
- f: $RCP_{RSCLQ} \rightarrow$ RSSCBAQ.

The table below lists the functional notation and equality for Sarah's actions on each ordered pair of Combined Linear Quantities (CLQ) that is mapped onto and into the corresponding Same–Color–Box Areal Quantity (SCBAQ).

Table 4.43

Sarah's Mapping Structures Acting on RCP_{RSCLQ}

CLQ Pairs	Total Number of CLQ Pairs	Functional Notation	Functional Equality	Resulting SCBAQ
$(2x, x)$	1	f: $(2x, x) \rightarrow [x^2]$	$f(2x, x) = [x^2]$	$[x^2]$
$(2x, 3y)$	1	f: $(2x, 3y) \rightarrow [6xy]$	$f(2x, 3y) = [6xy]$	$[6xy]$
$(2x, 2)$	1	f: $(2x, 2) \rightarrow [4x]$	$f(2x, 2) = [4x]$	$[4x]$
(y, x)	1	f: $(y, x) \rightarrow [yx]$	$f(y, x) = [yx]$	$[yx]$
$(y, 3y)$	1	f: $(y, 3y) \rightarrow [3y^2]$	$f(y, 3y) = [3y^2]$	$[3y^2]$
$(y, 2)$	1	f: $(y, 2) \rightarrow [2y]$	$f(y, 2) = [2y]$	$[2y]$

(1, *x*)	1	*f*: (1, *x*) → [*x*]	*f* (1, *x*) = [*x*]	[*x*]
(1, 3*y*)	1	*f*: (1, 3*y*) → [3*y*]	*f* (1, 3*y*) = [3*y*]	[3*y*]
(1, 2)	1	*f*: (1, 2) → [2]	*f* (1, 2) = [2]	[2]

The diagrams below show Sarah's construction of RCP_{RSCLQ} by applying a Cartesian Product "×" on the two Representational Sets of Combined Linear Quantities $RSCLQ_1$ and $RSCLQ_2$.

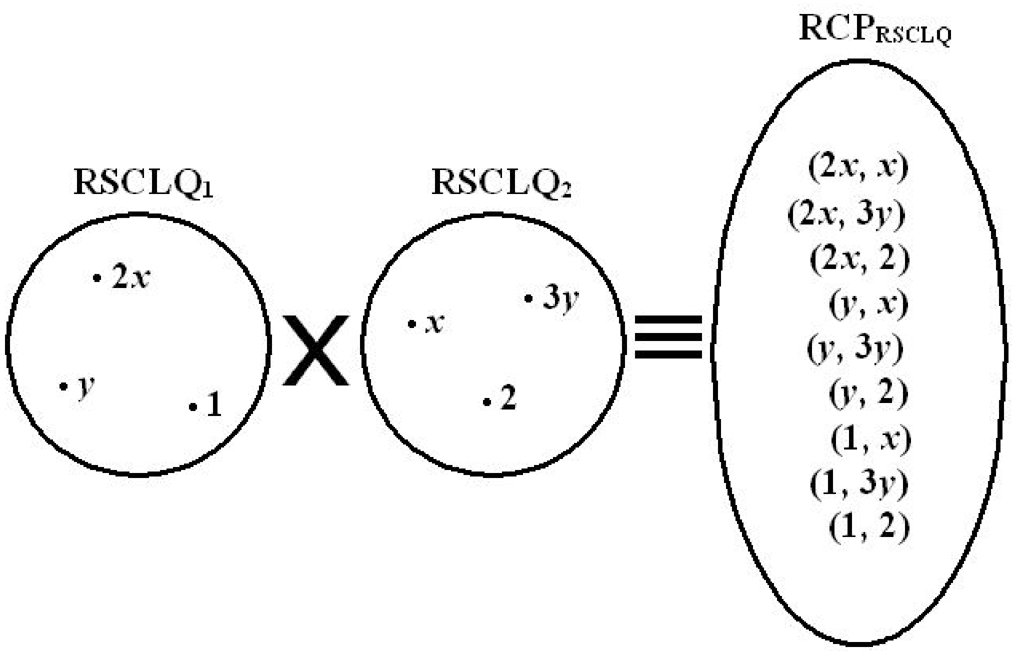


Figure 4.128. Sarah's definition of the RCP on the RSCLQs.

Finally, bijectivity of the mapping *f* becomes clearer with the following diagram.

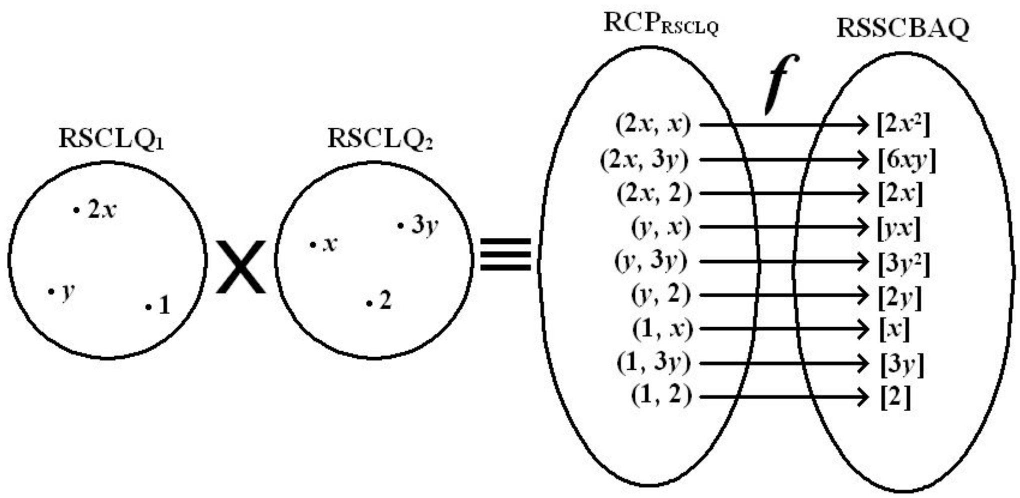


Figure 4.129. Sarah's bijective mapping structures.

Note that whereas Figures 4.126 and 4.127 were just representatives of the "true" RCP_{RSILQ} and RSIAQ, Figures 4.128 and 4.129 above show the true RSSCBAQ and RCP_{RSCLQ}. In fact, *card* (RSSCBAQ) = *card* (RCP_{RSCLQ}) = 9 both pictorially and algebraically. This is not surprising as repetition of elements is allowed in RCP_{RSILQ} and RSIAQ, and not allowed in RSSCBAQ and RCP_{RSCLQ} as established in the previous section.

I end this subsection with John's reference to Mapping Structures in his work with the factorization problem $2xy + 10x + 4y + 20$. Note that this is an extra problem I posed only to John in an attempt to make sure that he saw the Same–Color–Box Areal Quantities (SCBAQ) as products, like Nicole and Sarah, rather than pseudo–products, like Brad and Rob. My instruction for John was "Make a rectangle for the expression $2xy + 10x + 4y + 20$ first, then factor the expression $2xy + 10x + 4y + 20$ using the algebra tiles." He very quickly produced the correct polynomial rectangle (Figure 4.130).

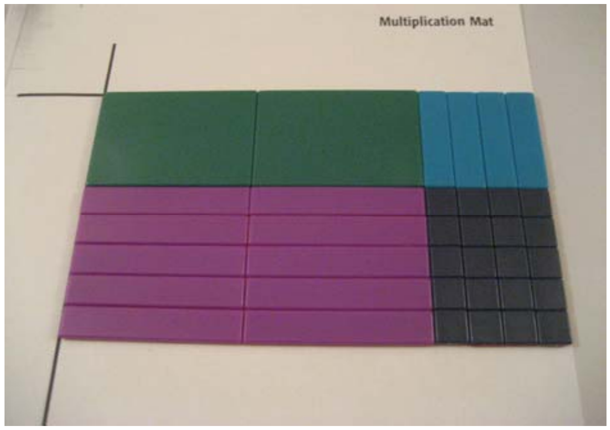


Figure 4.130. John's rectangle representing the polynomial $2xy + 10x + 4y + 20$.

I then asked him to focus on the boxes of the same color. The following protocol picks up at this point.

Protocol 4.89: John's reference to mapping structures.

J: Green box as a product is $2x$ times y... this purple one is $2x$ times 5... this blue one is 4 times y... and the last one is 4 times 5...

G: Why is it $2x$ times 5... and not $10x$ times 1... or 10 times x?

J: Because I am taking the product length times width... so... the length for the purple box is $2x$ and the width is 5... so that's why I do $2x$ times 5... [He then writes the identity $(2x + 4)(y + 5) = 2xy + 10x + 4y + 20$ on the last column of the activity sheet]

G: Are they consistent in units? [the LHS and the RHS of the identity]

J: Yeah... centimeters times centimeters is centimeters squared... so... yeah...

G: How about $4y$... what units does it have?

J: Centimeters squared.

G: How about the 4 of $4y$?

J: 4 is centimeters.

G: Why?

J: Because that is the length. Length is represented by centimeters [He means the length of the blue box].

G: And what units does y have... y of $4y$?

J: It's also centimeters... that's the width... [He means the width of the blue box]

G: How about 20... what units does it have?

J: Centimeters squared.

G: How do you say that?

J: Because 20 is... I got 20 by the length times the width... so... that's an area... centimeters squared...

G: How about this $2xy$ in here [pointing to the $2xy$ in John's expression $(2x + 4)(y + 5) = 2xy + 10x + 4y + 20$]... What units does each one have... 2 and x and y separately... what is 2... what is x... what is y?

J: I guess... y is definitely centimeters... $2x$ is represented by x plus x... I guess you could put a centimeters on x... not on 2... that's kinda tricky...

G: Tell me more about that...

J: Well... $2x$ is the sum of the length... so... i can't do... 2 times x... i can't put a unit on both of those [meaning 2 and x]... it'd be $2x$ centimeters squared... there is only $2x$ centimeters. So you need to think of $2x$ as $x + x$ [In other words, the 2 is additive] not 2 times x [In other words, John uses the word "times" only when two quantities are multiplied]. 2 is just a constant [In other words, a unitless quantity, in our context]

G: Your explanation is very good...

John made his rectangle *without the presence of the dimension tiles*. However, this did not prevent him from inducing a Representational Cartesian Product (RCP) defined on the two Representational Sets of Combined Linear Quantities (RSCLQ). His first sentence "Green box as a product is 2*x* times *y*... this purple one is 2*x* times 5... this blue one is 4 times *y*... and the last one is 4 times 5" necessitates a bijective function f^{-1} of multiplicative type mapping the Representational Set of Same–Color–Box Areal Quantities (RSSCBAQ) onto and into and induced Representational Cartesian Product defined on the two Representational Sets of Combined Linear Quantities (RCP_{RSCLQ}). His first statement "Green box as a product is 2*x* times *y*" can be notated with the functional notation as $f^{-1}: [2xy] \to (2x, y)$. His second statement "this purple one is 2*x* times 5" can be modeled with the functional notation as $f^{-1}: [10x] \to (2x, 5)$. His third statement "this blue one is 4 times *y*" can be notated as $f^{-1}: [4y] \to (4, y)$. Finally, his fourth statement "and the last one is 4 times 5" can be notated as $f^{-1}: [20] \to (4, 5)$. The table below lists the inverse functional notation and equality for John's statements about each same–color–box areal quantity (SCBAQ).

Table 4.44

John's Inverse Mapping Structures and Induced RCP_{RSCLQ}

SCBAQ	Total Number of SCBAQ	Functional Notation	Functional Equality	Resulting CLQ Pair	Induced RCP
$[2xy]$	1	$f^{-1}: [2xy] \rightarrow (2x, y)$	$f^{-1}\ [2xy] = (2x, y)$	$(2x, y)$	$\{2x\} \times \{y\}$
$[10x]$	1	$f^{-1}: [10x] \rightarrow (2x, 5)$	$f^{-1}\ [10x] = (2x, 5)$	$(2x, 5)$	$\{2x\} \times \{5\}$
$[4y]$	1	$f^{-1}: [4y] \rightarrow (4, y)$	$f^{-1}\ [4y] = (4, y)$	$(4, y)$	$\{4\} \times \{y\}$
$[20]$	1	$f^{-1}: [20] \rightarrow (4, 5)$	$f^{-1}\ [20] = (4, 5)$	$(4, 5)$	$\{4\} \times \{5\}$

The diagrams below illustrate John's four statements, which call for a bijective function f^{-1}, mapping the Representational Set of Same–Color–Box Areal Quantities (RSSCBAQ) onto and into an induced Representational Cartesian Product defined on two Representational Sets of Combined Linear Quantities (RCP_{RSCLQ}).

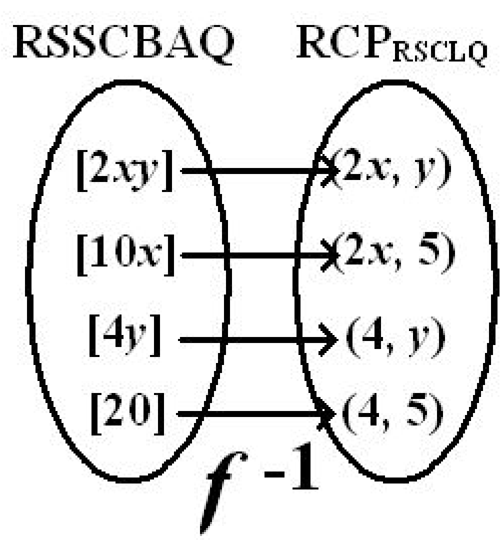


Figure 4.131. John's inverse mapping structures.

Figure 4.131 above shows the true RSSCBAQ and RCP_{RSCLQ} as both sets are of cardinality 4. Remember, the representational sets RSSCBAQ, RSCLQ, and RCP_{RSCLQ} are uniquely defined; therefore, one does not have to worry about whether the repetition of elements is allowed or not. To be more specific,

- RSSCBAQ = { [2*xy*], [10*x*], [4*y*], [20] }.
- RCP_{RSCLQ} = { (2*x*, *y*), (2*x*, 5), (4, *y*), (4, 5) }.
- *card* (RSSCBAQ) = *card* (RCP_{RSCLQ}) = 4.
- f^{-1}: RSSCBAQ → RCP_{RSCLQ}.

Note that when one makes reference to *inverse* Mapping Structures, one has to *induce* a RCP. In fact, this is the case with the tasks on polynomial factorization. Compare with the RCP one naturally defines with reference to the *ordinary* Mapping Structures when working on the tasks on polynomial multiplication. This is the slight difference between the two RCPs, which depends on whether one uses the bijective mapping f^{-1} or *f*. In John's work above, the *induced* Representational Cartesian Product defined on two Representational Sets of Combined Linear Quantities (RCP_{RSCLQ}) can be observed to exist as follows:

- RCP_{RSCLQ} = $RSCLQ_1$ × $RSCLQ_2$ where "×" denotes the Cartesian Product.
- $RSCLQ_1$ = {2*x*, 4} and $RSCLQ_2$ = {*y*, 5}. The induced RCP_{RSCLQ} can be decomposed more explicitly as follows:
- RCP_{RSCLQ} = {2x, 4} × {*y*, 5}.

The *decomposition* of the *induced* RCP_{RSCLQ} into RSCLQs is illustrated in the following diagrams.

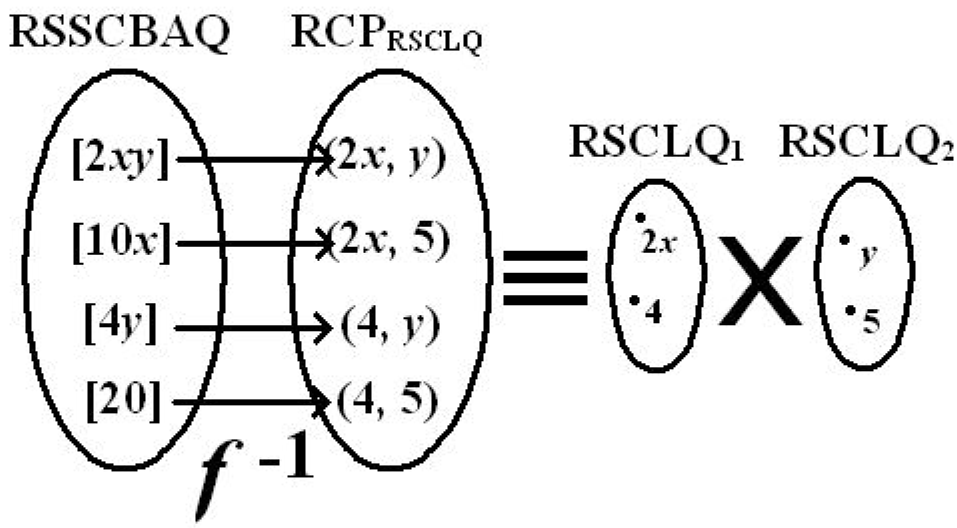


Figure 4.132. John's decomposition of the induced RCP into RSCLQs.

4.7. Summary of Findings for Each Individual Student

4.7.1. Brad

Brad's interpretation of prime–composite rectangles and growing rectangles representing summed (counting, odd, even) numbers made of one inch color cubes was multiplicative (MRUC), upon the "Area as a Product" instruction. As for the "Area as a Sum" instruction, on the other hand, he provided Equal Addends and Irreducible Addends (ARUC). The fact that he decomposed the growing rectangles representing summed numbers into Equal Addends (rather than Summed Addends) was indicative of the fact that he saw these counting numbers as composite numbers (and not as summed numbers, despite he obtained his growing rectangle sequence by adding different color counting number subunits). Brad's additive decomposition of the subunits was slightly different, though. For instance, he decomposed the odd integer L–shape subunits into Symmetric Addends (sub–subunits).

During the polynomial multiplication tasks, he preferred the *Filling in the Puzzle* strategy in the process of constructing his polynomial rectangles, which was the indication that what he was doing was addition, and not multiplication. Since a *Term–Wise Multiplication of Irreducible Areal Quantities* strategy was non–existent for him, representational Cartesian products (RCP) were not available, either. In fact, his additive thinking caused him to (mis)interpret the structure inherent in the Same–Color–Box Areal Quantities (SCBAQ) when he was asked to express the area of these areal quantities as products. His answers were of the form (a coefficient) times (an irreducible areal quantity) instead of the form (a combined linear quantity) times (a combined linear quantity), the former indicating a *pseudo–product*, a concatenation of multiplicative meaning (PMRUC). For Brad, the cardinality function *card* has a value equal to the coefficient (dimensionless constant) of the pseudo–products, which corroborates the additivity of the situation.

Brad was the only student to come up with *Standing–Up Positioned Dimension Tiles* strategy in an effort to emphasize the one–dimensionality associated with the dimension tiles (Figures 4.100, 4.101, 4.121). He often touched upon both one– and two–dimensionalities in his sense making of the linear and areal quantities. He was one of the few students to *act on* dimensionalities via mapping structures, by which to construct Dimensionalistic Mapping Structures (DMS). He acted on these very abstract dimensionality levels by indicating the pair of abstract one–dimensionalities, the multiplication operation behaving as a mapping acting on the ordered pair of abstract one–dimensionalities, and the two–dimensionality resulting from the mapping.

4.7.2. Rob

Rob, a very strong algebra student, had no problem in switching back and forth the linearity and the arealness of the quantities (in all tasks, guided by cubes or tiles) under consideration. In the case of the activities modeled with color cubes, the areas of the growing rectangles as products were true products (and not pseudo – products) for Rob. As for the expressions of the areas of the growing rectangles as sums for the same activities, on the other hand, he made use of a limited ARUC types (Equal Addends and Irreducible Addends only). Though he generated his growing rectangle sequence for the summation activities by relying on the instruction "Add them so that they (counting, odd, even number subunits, respectively) generate a rectangle at each step," he failed to realize the summed numbers at each stage, and to produce Summed Addends. Rob still was able to express the areas of such rectangles as products (MRUC), however, his treatment of the growing rectangles as representations of composite numbers rather than summed numbers prevented him from establishing the summation formulas $\sum_{i=1}^{n} f(i) = g(n)$ for the particular examples we studied. Like Brad, he made use of Equal Addends and Irreducible Addends.

Rob's Bridge Connection between Consecutive (Areal) Subunits strategy was indicative of the fact that he was able to reason about these areal quantities (and their magnitudes) on their own and their relationships with each other. In that sense, he was able to reason quantitatively (cf. Protocols 4.17 & 4.36). In the case of summing odd integers activity, in particular, his quantitative reasoning led him to realize that every added L–shape subunit built on a growing square sequence. Rob was the only student to associate dimensionalities to known measurement units.

In Rob's work on polynomial multiplication activities, representational Cartesian products (RCP) [acting on the representational sets of irreducible linear quantities (RSILQ), or the representational sets of combined linear quantities (RSCLQ)] were non–existent, which is indicative of the fact that this student was not reasoning multiplicatively. He constantly stuck to the *Filling in the Puzzle* strategy, an additive scheme, by which he generated the representational (areal) quantities of his polynomial rectangles. It was then of no surprise that his interpretation of the areas of the representational sets of same –color–box areal quantities (RSSCBAQ) as products was lacking a multiplicative flavor; his expressions were pseuo–products. The first term of each pseudo–product for Rob were some sort of coefficients behaving like counting numbers that show how many there are of each irreducible areal quantitiy (IAQ). In other words, cardinality function *card* manifested itself (with values different from 1) in Rob's case in agreement with his additive thinking (in the polynomial multiplication and factorization tasks requiring a multiplicative approach).

4.7.3. Nicole

One of the most important results concerning Nicole was her indecisiveness about the (linear–areal) character of the odd/even L–shape subunits generating the growing rectangle sequence. Despite that fact that these odd/even number L–shape subunits represent areal subregions of the growing rectangle sequence – hence areal in reality –

that was not the case for Nicole; her concept of area was computation rather than measure. In fact, she necessitated a *Rectangle Condition* to establish the arealness of these quantities. In fact, in her work with the summing even integers activity, after representing even number L–shape subunits generating the growing rectangle sequence; she provided a different formalism in which all even number subunits were $2\times\frac{n}{2}$ rectangles. This is another important result concerning Nicole, because she was the only student to represent even integer subunits in both ways, as L–shapes and as rectangles, respectively. Her comparison of the two formations led her to conclude that the even number subunits in the latter ones looked "more areal because they were rectangles." One other important result pertaining to Nicole's work with the summation activities is the variety of additive type RUCs by which she established the "Area as a sum equals Area as a Product" identities. Besides the Summed Addends, in particular, she was the only student to produce Recursive Addends.

Unlike Rob and Brad who constantly stuck to the *Filling in the Puzzle* strategy, which indicates these students' additive reasoning; Nicole relied on the *Term–Wise Multiplication of the Irreducible Areal Quantities* strategy by which she established the MRUC. Her proficiency in MRUC resulted in a RCP defined on pairs of RSILQs and pairs of RSCLQs, respectively. She was very careful, and successful, in the process of identifying the elements of the pairs of RSILQs (and RSCLQs). In particular, her statements in Protocol 4.71 were *pure mathematical*, establishing the existence of the RCP. She did not make any mistake in her expressions of the "Area as a Product of the Boxes of the Same Color." Her expressions were products – and not pseudo products – of the form (a combined linear quantity) times (another combined linear quantity). For

Nicole, each Same–Color–Box (SCB) was an areal singleton of cardinality 1, unlike Rob and Brad for whom the cardinality function was taking values (equal to the coefficient of the pseudo–product representing the area of the SCB) different from 1. Nicole was unique in her ability to distinguish between same–valued linear and areal quantities with reference to a Bi–Valued Function, an index taking values 1 and 2 as for the *Number of Aspects*, respectively (cf. Protocol 4.82 & Figure 4.108).

4.7.4. John

John was the only student to emphasize the mapping structures *from the beginning*, starting from the task on prime and composite rectangles all the way through the end, polynomial factorization. The following table illustrates his statements involving the linear quantities juxtaposed by the connective "and," the strongest indicator of MRUC and mapping structures. Other than the ones in the table below, his statements "To find that specific spot" and "I got 20 by the length times the width" necessitate mapping structures. His consistent reference to the mapping structures in the comparison of the same–valued linear and areal quantities stood as an evidence for his multiplicative reasoning. John also came up with a vectorial approach to distinguish between same–valued linear and areal quantities. In this approach, the $(2x+y)-$valued areal quantity corresponds to the plane vector $(2x+y,1)$ whereas the $(2x+y)-$valued linear quantity corresponds to the scalar $2x+y$ on the real number line.

Table 4.45

John's Reference to Connective "And" & Mapping Structures

Statements	**MRUC**	**Mapping Structures**
Length of five **and** width of one in which case the area would be 5.	(5, 1)	$f:(5,1)\mapsto 5$
It's not just 6… It's 6 **and** 1.	(6, 1)	$f:(6,1)\mapsto 6$
You gotta look at this [pointing to the linear y at the top] **and** this [pointing to the linear 1 on the side] to find this one (See Figure 4. 112).	(y, 1)	$f:(y,1)\mapsto y$

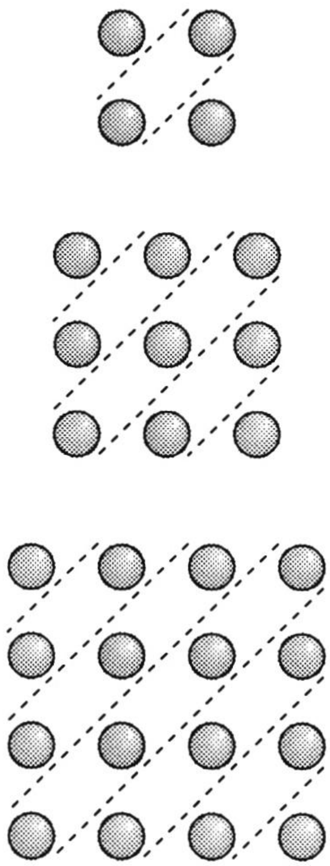
Unlike Nicole, John had no problem in establishing the arealness of the (symmetric or nonsymmetric) L–shape (odd or even number) subunits. For this purpose, he used a wide range of ARUC types, which are summarized in the table below.

Table 4.46

John's Decomposition of the Subunits into Sub – Subunits

Context	**Decomposition into Sub–Subunits**	**Relational Notation for the Sub–Subunits**
Symmetric L–Shapes (Odd Number Subunits)	$N+(N-1)$	(N, 1) & ($N-1$, 1)
Nonsymmetric L–Shapes (Even Number Subunits)	$N+N$	(N, 1) & (N, 1)
Nonsymmetric L–Shapes (Even Number Subunits)	$(N+1)+(N-1)$	($N+1$, 1) & ($N-1$, 1)

As suggested by the third column of Table 4.46, John was able to establish a rectangle condition for the sub–subunits, indicating their areal character. John was the only student to make use of the ARUC type $N+(N-1)$ in his derivation of the odd integers' summation identity $\sum_{i=1}^{n} i+(i-1)=n^2$, which can be found in some textbooks on visual proofs (Figure 4.133).



$$1 + 2 + 1 = 2^2$$

$$1 + 2 + 3 + 2 + 1 = 3^2$$

$$1 + 2 + 3 + 4 + 3 + 2 + 1 = 4^2$$

$$1 + 2 + \cdots + (n-1) + n + (n-1) + \cdots + 2 + 1 = n^2$$

Figure 4.133. Sums of odd integers (Nelsen, 1993, p. 74).

In the process of constructing polynomial rectangles via algebra tiles, John was reasoning additively in the first two tasks (*Filling in the Puzzle* strategy). In his work with the $(x+1)\times(2y+3)$ polynomial rectangle, spontaneous learning occurred and he shifted from *Filling in the Puzzle* (FIP) strategy to *Term–Wise Multiplication of Irreducible Linear Quantities* (TWMILQ) strategy. John was unique in that he was the only student to use *both* strategies (Figure 4.134). At times, he was also able to make sense of the dimension tiles as some sort of measurement tools (e.g., he provided the $x = y + 2$ relation for his "false" identity $(x+1)(2y+3) = 2y^2 + 9y + 9$ in an attempt to reconcile the LHS and the RHS).

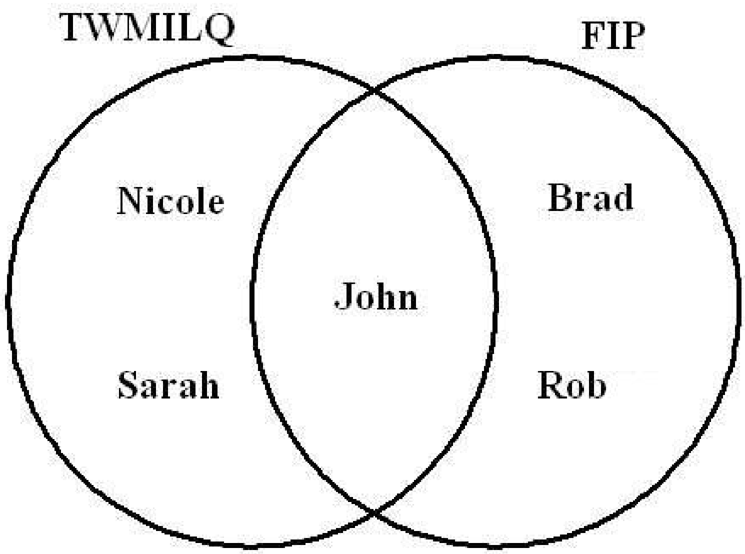


Figure 4.134. Strategies used by the preservice teachers.

Like Nicole and Sarah, John interpreted the same–color–box areal quantities (SCBAQ) as "areal" in nature by providing true products of the form (a combined linear quantity) times (another combined linear quantity) contrary to Brad and Rob who failed to assign this multiplicative interpretation. The cardinality function *card*, therefore, was taking the same value "1" for each SCBAQ in John, Sarah, and Nicole's case. For these three students, therefore, SCBAQs can be thought as 1–valued areal–singletons under the cardinality function *card*. All these three students came up with contradictory verbal proofs invalidating PMRUC for the SCBAQs.

4.7.5. Sarah

Sarah was very consistent in processing her ideas, especially when making internal connections. For instance, she was able to provide a summation formula for the even numbers relating her growing rectangle sequences for odd integers and even integers, respectively. She obtained her formula $2+4+\ldots+2n=n^2+n$ in the conjectural process, from her figure representing the 5$^{\text{th}}$ stage (n = 5), indicating an advanced level of thinking. She was able to see that both n^2 and n on the RHS were to represent areal quantities. MRUC was evident from her statements necessitating relational notations (n, n) and (n, 1) for these quantities. As described in Chapter 4 in details, Sarah established LHS–RHS equivalence in various tasks with reference to such ARUC and MRUC interpretation, as above.

In regard to algebra tile models, Sarah was the only student to exhibit a complete multiplicative understanding *in the process of* constructing a polynomial rectangle for the polynomial factorization tasks. The difference between Sarah and John is that John induced the RCP after completing his rectangle (without the dimension tiles placed around) whereas Sarah induced her RCP *in the process of* generating the polynomial rectangle (by placing the dimension tiles around), indicating a reference to *Inverse Mapping Structures* (IMS). In that sense, Sarah relied on the *Decomposition of Irreducible Areal Quantitites into Pairs of Irreducible Linear Quantities* (DIAQPILQ) strategy, which can be thought as the inverse of the *Term–Wise Multiplication of Irreducible Linear Quantities* (TWMILQ) strategy. Both strategies corroborate Sarah's multiplicative understanding at a sophisticated level. These strategies require that multiplication operation behave as invertible functions (bijections) f and f^{-1}, which are inverse function of each other $(f \circ f^{-1} = f^{-1} \circ f = I)$. In this notation, f and f^{-1} go with the *Term–Wise Multiplication of Irreducible Linear Quantities* (TWMILQ) strategy and the *Decomposition of Irreducible Areal Quantitites into Pairs of Irreducible Linear Quantities* (DIAQPILQ) strategy, respectively. Table below summarizes the meanings (multiplicative vs. additive) projected on the irreducible (IAQ) and same–color–box areal quantities (SCBAQ) by the interview students for the cases “in the process of” and “after the completion of” the polynomial rectangles in the polynomial multiplication and factorization tasks.

Table 4.47

Preservice Teachers’ Multiplicative (⊗) vs. Additive (⊕) Interpretation of the Areal Quantities

<table>
<tr><th>Task</th><th>During vs. After</th><th>AQ Type</th><th>Brad</th><th>Rob</th><th>John</th><th>Nicole</th><th>Sarah</th></tr>
<tr><td rowspan="4">Polynomial Multiplication</td><td rowspan="2">In the Process of Constructing the Polynomial Rectangle</td><td>IAQ</td><td>⊕</td><td>⊕</td><td>⊗</td><td>⊗</td><td>⊗</td></tr>
<tr><td>SCBAQ</td><td>NA</td><td>NA</td><td>NA</td><td>NA</td><td>NA</td></tr>
<tr><td rowspan="2">After the Completion of the Polynomial Rectangle</td><td>IAQ</td><td>⊕</td><td>⊕</td><td>⊗</td><td>⊗</td><td>⊗</td></tr>
<tr><td>SCBAQ</td><td>⊕</td><td>⊕</td><td>⊗</td><td>⊗</td><td>⊗</td></tr>
<tr><td rowspan="4">Polynomial Factorization</td><td rowspan="2">In the Process of Constructing the Polynomial Rectangle</td><td>IAQ</td><td>⊕</td><td>⊕</td><td>NA</td><td>NA</td><td>⊗</td></tr>
<tr><td>SCBAQ</td><td>NA</td><td>NA</td><td>NA</td><td>NA</td><td>NA</td></tr>
<tr><td rowspan="2">After the Completion of the Polynomial Rectangle</td><td>IAQ</td><td>⊕</td><td>⊕</td><td>⊗</td><td>⊗</td><td>⊗</td></tr>
<tr><td>SCBAQ</td><td>⊕</td><td>⊕</td><td>⊗</td><td>⊗</td><td>⊗</td></tr>
</table>

Spradley’s relational type *Domain Analysis* arose as a powerful data analysis tool in the investigation of one of the protocols regarding Sarah. I was able to code Sarah’s phrases via Universal (Table 4.35) and Informant Expressed Semantic Relationships. Sarah’s semantic relations “is” and “would be” were indicative of the areal–ness of the representational areal quantities (Areal Reflexivity) whereas the semantic relations (of passive voice) "used as" and "based as" served for linking the dimension tiles to length (Instrumentalized Linear Quantities). There was a one–to–one correspondence between Sarah's semantic relations and the (linear or areal) character of the representational quantities.

As described before, in their comparison of the same–valued linear and areal quantities, Nicole and Sarah often referred to verbal contradictory proofs in an attempt to invalidate the PMRUC approach, which was spelled out by some students (Brad and Rob). Besides these verbal proofs by contradiction, just as Nicole provided a *pure mathematical* approach involving Cartesian products (Protocol 4. 71), Sarah relied on *productive thinking* (Protocol 4. 74) by which she established a *figure* (the $(2x+y)$ – valued areal–singleton inside) and a *ground* (the $(2x+y)$ – valued dimension tiles outside). For Rob and Brad, in contrast, the areal tiles representing the $(2x+y)$ – valued areal–singleton inside were standing for *figure*, and there was no such thing as *ground*.

CHAPTER V

DISCUSSION: MAPPING STRUCTURES CONCEPTUAL FIELD

The discussion in the previous chapter revealed how the mathematical concepts arising from students' work on color cubes and tiles are intricate and inseparable from each other. Conceptual Field Theory (CFT) (Vergnaud, 1983; 1988, 1994) aims to present the complexity inherent in the nature of “simple” tasks on additive and multiplicative reasoning. Research indicates that the Multiplicative Conceptual Field (MCF) is very complex and has many concepts of mathematics in its structure, other than multiplication itself (Behr, Harel, Post, & Lesh, 1992; Harel & Behr, 1989; Harel, Behr, Post, & Lesh, 1992). “Additive reasoning develops quite naturally and intuitively through encounters with many situations that are primarily additive in nature” (Sowder, Armstrong, Lamon, Simon, Sowder, & Thompson, 1988, p. 128). Building up multiplicative reasoning skills, on the other hand, is not obvious; schooling and teacher guidance are essential to acquire a profound understanding and familiarization with multiplicative situations (Hiebert & Behr, 1988; Resnick & Singer, 1993).

According to Steffe (1988), children who are on a unit coordination pathway start by constructing singletons representing unities from which they achieve more sophisticated unit coordination schemes (e.g., composite units, iterable units). “As an adult, I can say that multiplication of whole numbers is an operation that is based on repeated addition” (Steffe, 1988, p. 128). “It is the shift from operating with singleton units to coordinating composite units that signals the onset of multiplication" (Singh, 2000, p. 273). In all activities concerning cubes and tiles, my preservice teachers were able to refer to singleton units – irreducible areal quantities (IAQ) – in their expressions of the area of the rectangle. In particular, as for the prime rectangles, the area as a sum was solely based on the sum of such singleton units (represented by wooden color inch cubes). As for the area of the composite rectangles as a sum, on the other hand, they were successful in referring to both singleton units – irreducible areal quantities (IAQ) – and composite units – combined areal quantities (CAQ).

As for the summation activities, students still established the "onset of multiplication" (Singh, 2000, p. 273) in quite different formalisms. Everybody obtained the area of the growing rectangles with reference to the number *n* of various figures[13] (subunits) of different colors for each situation. The area of the growing rectangle is expressed as repeated quantitative additions (e.g., Irreducible Addends, Equal Addends) and as other functional type quantitative additions (e.g., Summed Addends, Recursive Addends, Random Addends). In all such cases, students referred to name–unit–value trinity. In the functional type in particular, students relied on the number *n* of Summed Addends to establish a multiplicative formalism. For instance, the *n* in the product expression $n \times (n+1)$ is the number of different color even integer subunits being added. With the relational notation, this corresponds to the ordered pair $(n, n + 1)$ of linear quantities, which is equivalent to the ordered *n*–tuple $[2, 4, \ldots, 2n]$ of areal quantities.

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[13] Summing Counting Numbers: $1 \times n$ and $2 \times \frac{n}{2}$ rectangle subunits for odd and even integer subunits, respectively.
Summing Odd Integers: Symmetric L–shape subunits (irregular hexagons).
Summing Even Integers: Non–symmetric L–Shapes (Nicole, Brad, John, Sarah) and $2 \times n$ rectangle subunits (Nicole, Rob).

This was quite different from the conception of multiplication based on repeated addition (Fishbein et al., 1985). In that sense, my students establishing the onset of the multiplication based on functional type quantitative additions looked quite different from the distributive aspect argued by Steffe (1992), and Confrey and Lachance (2000) as well.

Steffe (1994) found that "For a situation to be established as multiplicative, it is necessary to at least coordinate two composite units in such a way that one of the composite units is distributed over the elements of the other composite units" (p.19). My results in the previous chapter indicate that this definition is inadequate to reflect what multiplication is/could be. For instance, in the polynomial multiplication tasks, Sarah, Nicole and John, who relied on the "Term Wise Multiplication of Irreducible Linear Quantities" Strategy, referred to Mapping Structures in generating their polynomial rectangle. The dimensions of the polynomial rectangle, namely the Combined Linear Quantities (CLQ), still possessed some sort of composite units (namely the Irreducible Linear Quantities) inherent in their structure; however, in the process of "multiplication," a relational aspect was evident, rather than distributive aspect argued by Steffe.

I want to illustrate the inadequacy of Steffe's (1994) distributive aspect for multiplication with my preservice teachers' work with the color cube activities as well. In my study, students established the existence of areal units, areal subunits, and areal sub–subunits, which is in agreement with Steffe's three levels of unit–coordination. However, the structure of these units is different in that students emphasized the arealness and the quantitative character in an attempt to launch identities of the form "Area as a Sum =

Area as a Product" based on the growing rectangles they created. In nearly all activities, students used the following three level structure:

- Singleton areal sub–subunits,
- Composite areal subunit of areal sub–subunits,
- Composite areal unit of composite areal subunits of areal sub–subunits.

When one refers back to the 5×7 multiplication example in the introduction of Steffe's Unit–Coordination framework in the first chapter, one can deduce the following.

On the 1st level of unit–coordination (Steffe, 1994), students make sense of unity as singleton units, each singleton unit corresponding to the number "1." This corresponds to the areal sub–subunits in the present study. As for the color cube activities, areal sub–subunits were most of the time (irreducible) areal singletons except some particular cases. For instance, Brad's areal sub–subunits for the Summation of Odd Integers Activity were of the form $N + 1 + N$, by which I established the terminology Symmetric Addends. As another example, John's decomposition yielded an $N + (N - 1)$ structure. As for the algebra tiles activities, on the other hand, there were three different singleton unit types: An x–singleton, a y–singleton, and a 1–singleton. These can be thought as the singleton (irreducible) members of the Representational Set of Irreducible Areal Quantities (RSIAQ).

On the 2nd level of unit–coordination, students make sense of 7 or 5 as 7 composite unit of singleton units, and 5 composite unit of singleton units, respectively, each singleton unit once again corresponding to the number "1." The "composite unit" notion corresponds to the areal subunits in my study. The $1 \times n$ rectangles representing the odd integers (in the Summation of Counting Numbers Activity), the $2 \times \frac{n}{2}$ rectangles representing the even integers (in the Summation of Counting Numbers Activity), the symmetric L–shapes representing odd integers (in the Summation of Odd Integers Activity), the nonsymmetric L–shapes representing even integers (in the Summation of Even Integers Activity), the $2 \times n$ rectangles representing the even integers (in the Summation of Even Numbers Activity) are established as "composite" areal subunits (quantities) of areal sub–subunits (quantities) by the preservice teachers. As for the polynomial rectangles activities, on the other hand, same–color–boxes stood as the subunits. As described before, these subunits are the members of the Representational Set of Same−Color−Box Areal Quantities (RSSCBAQ). Students interpreted these areal quantities in two different fashions, though. I made use of the cardinality function and dimensional analysis in an attempt to establish which students reasoned about these quantities additively and which students multiplicatively. Sarah, Nicole, and John were able to think about these quantities both additively and multiplicatively whereas Rob and Brad only came up with additive reasoning. Pseudo–Multiplicative Representational Unit Coordination (PMRUC) is the name I assigned for the exhibition of additive reasoning with these boxes, which was lacking a multiplicative flavor. In fact, PMRUC is the exact same thing as Steffe's 2^{nd} level unit–coordination. I must also indicate that it took John quite some time to realize that it was possible to express the area of a same–color–box as a product of two (combined) linear quantities.

On the 3^{rd} level of unit–coordination, students make sense of 5 as the 5 composite unit of 7 composite unit of singleton units, each singleton unit once again corresponding to the number "1." The "composite unit of composite unit of singleton units" notion corresponds to the biggest areal units in my study. By "biggest areal units," I mean the growing rectangle sequence in the summation activities, and the polynomial rectangle itself in the algebra tiles activities. In Steffe's 3rd level of unit–coordination, the composite units (addends) are all "7"s, namely equal addends, whereas in my study, students provided a variety of addends. In their work with summation activities, students came up with Equal Addends as well as Summed Addends, Recursive Addends, and Random Addends. However, once again the same students, Brad and Rob, were the ones to provide Equal Addends in their expression of the area of the growing rectangle in the summation activities. As an example, the area of the growing square at the 5th stage in the Summation of Odd Integers Activity, can be thought as 5 composite unit of 5 composite unit of 1, for Rob and Brad (Equal Addends). For Sarah, Nicole, and John who all provided Summed Addends, this can be written as 5 composite unit of 5 non−equal L−shape unit of 1 (Summed Addends).

I also want to illustrate how my students' RUC was different from Steffe's 3rd level UC for the algebra tiles activities with a specific example they studied during the interviews. The task was to multiply $2x + y$ by $x + 2y + 1$ using algebra tiles. All students produced the same rectangle (Figure 5.1) but their interpretation of the same−color−box areal quantities (SCBAQ) differed.

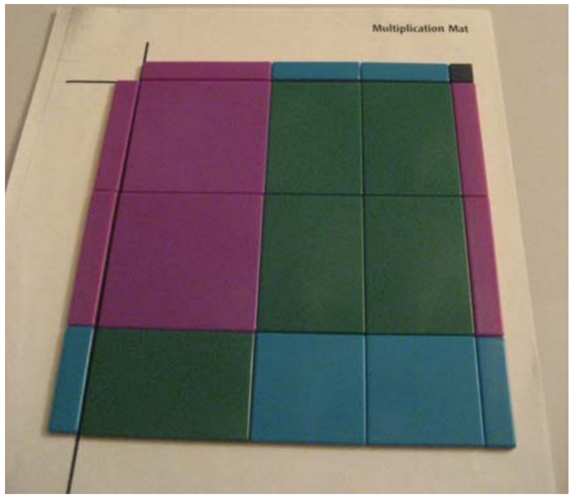

Figure 5.1. The $2x + y$ by $x + 2y + 1$ polynomial rectangle.

For Brad and Rob, the area of the polynomial rectangle was 5 SCBAQ composite unit of irreducible singletons, where each SCBAQ was interpreted additively whereas for Nicole, Sarah, and John, it was 6 SCBAQ areal singleton units, where each SCBAQ was interpreted multiplicatively. The table below illustrates the difference in these students' thinking.

Table 5.1

Preservice Teachers' Multiplicative vs. Additive Interpretation of the Same–Color–Box Areal Quantities (SCBAQ)

Color of the SCBAQ	**Dimensions of the SCBAQ**	**Brad, Rob**	**Nicole, Sarah, John**
Purple	$2x \times x$	2 composite unit of areal x^2–singletons	1 unit of $2x \times x$ areal singleton
Green	$y \times x$	5 composite unit of areal *xy*–singletons	1 unit of $y \times x$ areal singleton
Green	$2x \times 2y$		1 unit of $2x \times 2y$ areal singleton
Blue	$y \times 2y$	2 composite unit of areal	1 unit of $y \times 2y$

		y^2–singletons	areal singleton
Purple	$2x \times 1$	2 composite unit of areal x–singletons	1 unit of $2x \times 1$ areal singleton
Blue	$y \times 1$	1 composite unit of areal y–singleton	1 unit of $y \times 1$ areal singleton

Though Steffe's Unit–Coordination was the essential theoretical framework, I also felt the need to use sub–frameworks in order to respond to my research questions. Only the names, or only the measurement units, or only the values of quantities involved in a mathematical situation do not suffice to adequately reflect the nature of those quantities. My research participants and I delved further into the nature of the linear and areal quantities involved in the color cubes and algebra tiles activities in order to expose the units associated with them. Just as a point on an xyz –space is associated with its x–, y–, and z–coordinates, that is, coordinated as an ordered triple (x, y, z), a quantity is born only when it is correctly represented as the triple (name, unit, value), thus coordinated properly. For instance, in a mathematical situation involving a pile of oranges, the coordination (oranges, weight of oranges in lb, 12) is not the same as (oranges, cost of oranges in $, 24) or (oranges, number of oranges, 36). Schwartz (1988) used the term *referent* in a way similar to how I used *name* and called such quantities *adjectival quantities* (p. 41). He stated that all quantities have referents and that the "composing of two mathematical quantities to yield a third derived quantity can take either of two forms, referent preserving composition or referent transforming composition." (p. 41). Referent preserving compositions (e.g., addition and subtraction) yield quantities *of the same kind* whereas referent transforming compositions (e.g., multiplication and division) yield quantities *of a new kind*.

The fact that some students (Brad and Rob) relied on the *Filling in the Puzzle Strategy* and some others (Nicole, Sarah, John) relied on the *Term–Wise Multiplication of Irreducible Linear Quantities Strategy* in the process of constructing polynomial rectangles suggest that all these students were aware that they were dealing with areal quantities, however, the latter students were able to operate with *both* referent preserving and transforming compositions whereas the former ones took the referent preserving composition into account only. Brad and Rob were generating their polynomial rectangles by *adding* the irreducible areal quantities (IAQ), which were already areas; there was no such thing as the creation of a quantity of a *new* kind. Nicole, Sarah, and John, on the other hand, first *multiplied* the corresponding pair of irreducible linear quantities (ILQ), wherefrom obtained the corresponding irreducible areal *of–a–new–kind* quantities (IAQ). They then *added* these *new* quantities. For these students, each quantitative multiplication operation (referent transforming composition) was immediately followed by a quantitative addition operation (referent preserving composition).

The findings in the paragraph above can be slightly modified for my research participants' sense making of the Same–Color–Box Areal Quantities (SCBAQ). When I asked them to express the area of these SCBAQ as products, Brad and Rob provided pseudo–products, which possessed an additive character in their structure. In that sense, once again I can conclude that these two students were referring to a referent preserving composition, the quantitative addition operation, operating on the irreducible areal singleton constituents of the SCBAQ. As for John, Sarah, and Nicole, on the other hand, I can conclude that, because their (both written and verbal) expressions were products of the corresponding pairs of combined linear quantities (CLQ), they were making use of a referent transforming composition: the quantitative multiplication operation. I name this strategy as *Term−Wise Multiplication of Combined Linear Quantities Strategy*. Each pair of combined *linear* quantities (CLQ), possessing a linear character, is being *transformed* into a quantity (Same−Color−Box *Areal* Quantity) of a totally *new* (*areal*) kind via a referent transforming composition.

Besides the writings of Schwartz, I made use of Thompson's work on quantitative reasoning (Thompson, 1988, 1989, 1993, 1994, 1995) as a theoretical sub−framework, in order to explain my students' sense making of the linear and areal quantities. According to Thompson, "to reason quantitatively is to reason about quantities, their magnitudes, and their relationships with other quantities." (1988, p. 164). As described in the previous chapter in detail, there were instances where all students were reasoning quantitatively, with Thompson's point of view. In particular, Rob exhibited a quantitative reasoning in his work with the summation of odd integers activity where he established the existence of the (areal L−shape) subunits as quantities on their own and as quantities in relation to each other by which he made sense of his growing square sequence. Brad, in a similar pathway, was able to demonstrate his quantitative reasoning in providing a link between each L−shape areal subunit and the corresponding Symmetric Addends (areal sub−subunits). Moreover, he provided a beautiful internal connection by referring to statistical concepts of mathematics in doing so. Sarah's quantitative reasoning showed itself in her visual proof connecting the two summation formulas $\sum_{i=1}^{n}(2i-1)=n^2$ and $\sum_{i=1}^{n}2i=n^2+n=n\times(n+1)$ where she obtained the latter one with reference to the former one in the conjectural process at the 5th stage. She was able to apply the quantitative addition operation on the areal quantities n^2 and n from which she obtained an (areal) *of−the−same−kind* quantity. In the meantime, she was aware that the areal quantities n^2 and n resulted (were born) from a quantitative multiplication operation acting on the ordered pair (n, n) and the ordered pair (n, 1) of linear quantities, respectively. She was also able to establish a LHS−RHS equivalence meaningfully by demonstrating how the summed quantity $\sum_{i=1}^{n} 2i = n^2 + n$ and the product quantity $n \times (n+1)$ stood for the same areal quantity. I used a relational notation $[(n,n),(n,1)] \equiv (n,n+1)$ describing this equivalence.

Thompson (1988) established several "cognitive obstacles" (p. 167) to students' quantitative reasoning. The most important cognitive obstacle was that students' "failure to distinguish between a quantity and its measure hindered their ability to explicate relationships." (p. 168). Another cognitive obstacle was that "Multiplicative quantities of any sort (products, ratios, rates) were commonly misidentified or given an inappropriate unit." (p. 168). Olive and Caglayan (2006, 2007) found that "quantitative unit coordination" and "quantitative unit conservation" are essential constructs for overcoming these cognitive obstacles when students reason quantitatively about word problem situations. The present study established "mapping structures" as one such crucial construct to overcome cognitive obstacles to students' quantitative reasoning in a representational situation (e.g., in comparing same−valued linear and areal quantities, in expressing the area of a same−color−box as a product).

In contrast to the view of "multiplication as a repeated addition," there is research advocating the "correspondence principle" as the basis of multiplication (Nunes & Bryant, 1996; Piaget, 1965; Vergnaud, 1983, 1988). According to this principle, multiplication is the conception of determining how many in total of certain objects (e.g., oranges) exist if there corresponds a fixed number of these objects per each counterpart object (e.g., picnic basket) with a certain number. Vergnaud (1983) defined an invariant relation $x = f(y)$ to represent the correspondence principle, where the invariance lies in the "fixed number of oranges per each picnic basket." Park and Nunes (2001) found that children make sense of multiplication problems via the correspondence principle, not as a repeated addition (p. 771). My study shows that in the context of prime and composite rectangles, preservice teachers' multiplicative reasoning was based on both the correspondence principle and multiplicative representational unit coordination along with mapping structures. They were also able to form product quantities (e.g., 3 inches × 5 inches) reflecting their quantitative reasoning. Preservice teachers also were able to employ a more sophisticated version of repeated addition (quantitative addition operation), a *referent preserving composition* (Schwartz, 1988), that involves an iteration of quantitative composite units (e.g., 5 inches squared + 5 inches squared + 5 inches squared) or quantitative irreducible units (e.g., 1 inch squared + 1 inch squared + … + 1 inch squared) by which they deduced the identity "area as a sum = area as a product."

Goodrow and Schliemann[14] found that "working with multiplicative functions on the coordinate grid can support students in their transition from additive to

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[14] Retrieved August 12, 2007 from http://www.earlyalgebra.terc.edu/our_papers/2003/goodrow_schliemann_pme2003.pdf

multiplicative reasoning.” In fact in my research, this was the case for John who shifted from “Filling in the Puzzle Strategy” to the “Term Wise Multiplication of Irreducible Linear Quantities Strategy” in the second task on polynomial multiplication. From Goodrow and Schliemann's point of view, the polynomial rectangle John generated can be thought of as a coordinate grid where a point would correspond to an irreducible areal quantity (IAQ). Just as a point on the coordinate system is made of an ordering of two values (the x and the y values), an irreducible areal quantity (IAQ) becomes to exist as an ordered pair of the corresponding irreducible linear quantities (ILQ). John's constant reference to “multiplication operation behaving as a mapping” corresponds to the phrase “multiplicative function” in Goodrow and Schliemann's findings. In that sense, my findings are in agreement with theirs.

Many research studies show that children do not use multiplication in multiplicative tasks or word problems; they rather refer to repeated addition (Fishbein et al., 1985; Kouba, 1989; Mitchelmore & Mulligan 1996; Peled, Levenberg, Mekhmandarov, Meron, & Ulitsin, 1999). Some other researchers claim that the essence of multiplication lies in the distributive rather than repeated additive aspect (Confrey & Lachance, 2000; Steffe, 1992). My research indicates that students' additive approach in a multiplication task concatenates multiplicative meaning and it becomes something else– neither addition, nor multiplication. In the polynomial multiplication tasks, for instance, although my instruction was “Express the area of the boxes of the same color as a product,” Brad and Rob constantly referred to pseudo–products, as opposed to Nicole, Sarah, and John who provided Same–Color–Box Areal Quantities (SCBAQ) as products of the corresponding pairs of Combined Linear Quantities (CLQ). I believe that the latter students' successful answers were mainly due to the fact that they were able to reason quantitatively (Thompson, 1988, 1989, 1993, 1994, 1995), paying attention to the referent–value–unit trinity (Schwartz, 1998), and attending to the mapping structures involved in these multiplication tasks. My research, therefore, suggests “Mapping Structures” and “Relational Aspect” duo as the main extension to multiplicative reasoning.

Pseudo–Multiplicative Representational Unit Coordination (PMRUC), which I established to be of additive nature, is categorized as a multiplicative approach by many researchers. Empson, Junk, Dominguez, and Turner, for instance, concentrated on the “relationship between two sharing quantities as an index of the development of fractions as multiplicative structures” (2005, p. 23). Although they claim to “distinguish between children’s use of multiplication in strategies that involved coordinating quantities that were more additive in nature, and the multiplicative coordination of quantities,” I do not see a “multiplicative sense” in either strategy; they both represent additive reasoning (Empson et al., 2005, pp. 24–25). I agree with them in their categorization of the expression “$\frac{1}{n}+\frac{1}{n}+\cdots\frac{1}{n}$ (T times)” as an additive coordination strategy; however, I am not convinced that the expression “$(T)(\frac{1}{n})$” represents a multiplicative coordination (See Table 5, p. 25). My research indicated that both of these expressions are of additive nature; they both stand for repeated (quantitative) additions.

The English National Numeracy Strategy (DfEE, 1999) advocated teaching the multiplication concept as repeated addition to foster students' understanding. The Japanese Association of Mathematical Instruction (JAMI), on the other hand, claimed that “repeated addition is a way to calculate multiplication, not a meaning of it” (Yamonoshita & Matsushita, 1996, p. 291). Confrey and Smith (1995) showed the inadequacy of repeated addition for describing multiplicative situations. In another piece, Confrey also introduced an alternative model, splitting, which is “an action of creating simultaneously multiple versions of an original” (Confrey, 1994, p. 292), as more suitable than repeated addition as an explanatory formalism for multiplication. According to Vergnaud, “multiplicative structures rely partly on additive structures; but they also have their own intrinsic organization which is not reducible to additive aspects” (Vergnaud, 1983, p. 128). In this sense, my findings on Brad and Rob’s PMRUC are in agreement with Yamonoshita & Matsushita (1996), Confrey & Smith (1995), Vergnaud (1983), and many other research studies demonstrating students’ frequent use of additive reasoning in the course of tasks requiring multiplicative reasoning (Hart, 1981, 1988; Karplus et al.,1983; Lamon 1993; Noelting, 1980; Resnick & Singer, 1993; Vergnaud, 1988). At a primitive level where there is no multiplicative sense making at all, the expression $5x$ is nothing but an abbreviation, a shortcut for the repeated addition expression $x + x + x + x + x$. Even the whole number multiplication notation 6×5 represents 6 fives; i.e., “add five repeatedly six times.” When there is a lack of quantitative reasoning, the “operation” we call “multiplication” does not possess a multiplicative meaning; that is not multiplication. A name–value–unit trinity has to be invoked in order to make sense of multiplication or to call what we are doing multiplication. Representations play a critical role, in that sense.

Through drawings or interesting stories children at all levels (elementary, middle, high school) may be taught the multiplicative meaning rooted in quantitative reasoning. The following is an example.

- Angela receives invitation from six different guys (Bob, Carlos, Dan, Elliot, Federico, Greg) for her high school graduation prom. She does not immediately say yes because she has a condition. Her condition is that each prom–partner candidate should try each one of the 5 different color (Orange, Pink, Red, White, and Yellow) prom outfits Angela picks (All six guys should try the same 5 outfits). Angela also wants to take their pictures wearing these outfits (1 photo for each outfit for each guy). How many photos will Angela have to shoot before her prom–partner decision?

The teacher could assign this problem as a homework assignment for a future class discussion. As a hint, s/he could guide the students by asking them to draw pictures, or take actual photos perhaps. Via drawings with different colors for instance students could generate 30 different (boy, outfit) pairs, where each "pair" could be "matched with" or "mapped as" a photo. In other words, each "photo" corresponds to a boy × outfit "product–quantity." (Table 4. 49)

Table 5.2

Boy × Outfit Product Quantities

×	**Orange**	**Pink**	**Red**	**White**	**Yellow**
Bob	(B, O)	(B, P)	(B, R)	(B, W)	(B, Y)
Carlos	(C, O)	(C, P)	(C, R)	(C, W)	(C, Y)
Dan	(D, O)	(D, P)	(D, R)	(D, W)	(D, Y)
Elliot	(E, O)	(E, P)	(E, R)	(E, W)	(E, Y)
Federico	(F, O)	(F, P)	(F, R)	(F, W)	(F, Y)

Greg	(G, O)	(G, P)	(G, R)	(G, W)	(G, Y)

Research shows that multiplicative reasoning is indispensable for proportional reasoning, and in particular, in the context of fractional situations, decimal, ratio, rate, proportion, and percent problems (Kieren, 1995; Lamon, 1994; Thompson, 1994). According to Vergnaud, "understanding multiplicative structures does not rely upon rational numbers only, but upon linear and *n*–linear functions, and vector spaces too" (Vergnaud, 1983, p. 172). Confrey's explanatory model for multiplication is based on the idea of splitting. "Splitting can be defined as an action of creating simultaneously multiple versions of an original, an action often represented by a tree diagram" (Confrey, 1994, p. 292). In this model, multiplication and division manifest themselves as inverse operations. The figure below represents the splitting structure of "3" in which movement to the right means "multiply by 3" and movement to the left means "divide by 3."

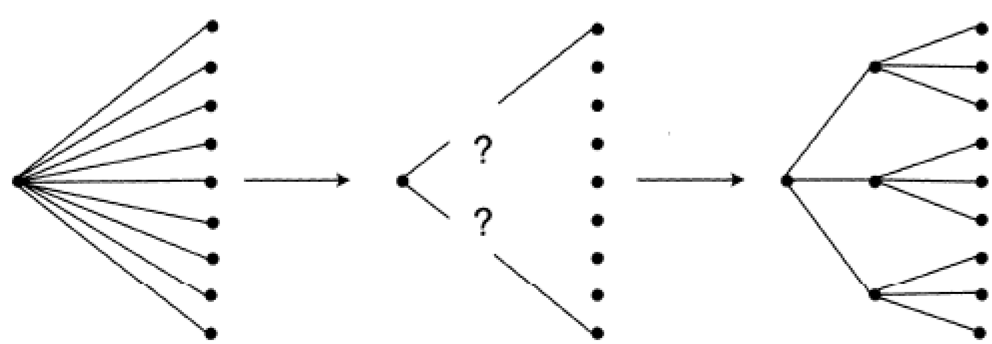

Figure 5.2. Splitting structures (Confrey & Smith, 1995, p. 70).

In my study, multiplicative reasoning in the context of polynomial factorization manifests itself as a reversible reasoning with reference to (multiplication operation behaving as) a function or a mapping rather than division. Although polynomial factorization is intuitively thought to be an inverse operation for polynomial multiplication, my students did not refer to ideas of division; they rather worked with mapping structures. To be more specific, on the second task on the factorization of the polynomial "$2x^2 + 7xy + 3y^2 + 5x + 5y + 2$" using algebra tiles, Sarah was the only student to *simultaneously* place the *pair of irreducible linear tiles* corresponding to *each* irreducible areal tile generating the polynomial rectangle, which was an indication of *reverse reasoning*. In contrast, Nicole, John, Rob, and Brad first completed their rectangle, then placed the dimension tiles around it. Though she was reasoning reversibly with reference to Inverse Mapping Structures, Sarah was not doing division. Sarah was the only student to associate each irreducible areal quantity (IAQ) with its dimensions, namely the corresponding pair of irreducible linear quantities (ILQ), in a polynomial factorization problem, *in the process of* generating the polynomial rectangle under consideration. In this way, Sarah established the multiplicative nature of the irreducible areal quantities (IAQ). She was able *both* to generate the correct polynomial rectangle (Figure 4.125) and to induce a *Representational Cartesian Product* (RCP) of *Representational Set of Irreducible Linear Quantities* (RSILQ) via *reverse reasoning*. In other words, she was referring to mappings, namely invertible functions represented as sets of ordered pairs of some linear quantities.

CHAPTER 6

SUMMARY AND CONCLUSIONS

I divide this final chapter into four sections. The first one is an overview of the entire study, which consists of my rationale, research questions, how this study is situated in the literature, a brief summary of research methodology, and a brief summary of findings. In the second section, I present a set of conclusions that derive from the findings. In the third section, I reflect on this study as a whole in an effort to offer implications for curriculum development and teacher education. Finally, I complete this chapter with suggestions for future research in this area.

6.1. Summary

I have been very interested in conducting a research study solely based on preservice teachers' understanding of representational quantities modeled with magnetic color cubes and algebra tiles. I was interested in knowing about how they make sense of such representational quantities, the quantitative units (linear vs. areal) inherent in the nature of these quantities, and how they reconcile the quantitative addition and multiplication operations (referent preserving vs. referent transforming compositions) acting on these quantities. For that purpose, I devised a set of tasks focusing on identities of the form "Sum = Product."

My work drew upon a unit–coordination (Steffe, 1988, 1994) theoretical framework, in an attempt to understand how preservice teachers coordinate, identify, and describe quantitative (linear and areal) units arising from summation expressions and polynomial rectangles. I also made use of Thompson's (1988, 1989, 1993, 1994, 1995) *quantitative reasoning* and Schwartz's (1988) *adjectival quantities* and *referent preserving vs. transforming compositions* as subframeworks in an effort to explain my preservice teachers' understanding of representational quantities. My data analysis framework has been inspired by a simplified version of Behr et al.'s (1994) *generalized notation for mathematics of a quantity* and Vergnaud's (1983, 1988, 1994) *theorems and concepts–in–action* formalisms, which helped me translate students' mathematical performance into a series of terminology based on a very simple notation: *Relational notation* and *mapping structures* duo (Caglayan, 2007b).

My rationale for attempting such a study has several reasons. The study of multiplicative structures has been conducted by mathematics education researchers since the 1980s. In his 1983 article, Vergnaud viewed the multiplicative structures, a conceptual field of multiplicative type, as a system of different but interrelated concepts, operations, and problems such as multiplication, division, fractions, ratios, and similarity. Although multiplicative structures can to some extent be modeled by additive structures, they have their own characteristics inherent in their nature, which cannot be explained solely by referring to additive aspects. Steffe's Unit–Coordination construct (1988, 1994), the guiding theoretical framework for this study, though strongly related to multiplicative structures, encompasses only the repeated addition model and a distributive aspect for multiplication, which in my opinion are limited explanatory models for what multiplication is/could be. In fact, findings on students' understanding of multiplication in the literature are limited to a premature interpretation, too, mostly relying on addition:

- The conception of multiplication based on repeated addition (Empson et al., 2005; Fishbein et al., 1985)
- Students' frequent use of additive reasoning in the course of tasks requiring multiplicative reasoning (Hart, 1981, 1988; Karplus et al.,1983; Lamon 1993; Noelting, 1980; Resnick & Singer, 1993; Vergnaud, 1988)
- The distributive aspect argued by Steffe (1992) and Confrey and Lachance (2000)
- The correspondence principle as the basis of multiplication (Nunes & Bryant, 1996; Piaget, 1965; Vergnaud, 1983, 1988)
- Splitting as an explanatory model for multiplication (Confrey, 1994; Confrey & Smith, 1995)

Research on students' reconciliation of additive and multiplicative structures based on "sum = product" identities is missing in the literature.

Second, the coordination construct, though studied several times before, does not cover all possibilities. Levels of unit coordination have been used in additive, multiplicative, and fractional situations before (Behr et al., 1994, Lamon, 1994; Olive, 1999; Olive & Steffe, 2002; Steffe, 1988, 1994, 2002). However, there is no prior work on unit coordination arising from the geometry of the numbers, in the form of identities, where the left hand side of the identity stands for the additive situation (area as a sum, in the geometry of the context) and the right hand side of the identity stands for the multiplicative situation (area as a product, in the geometry of the context). Both phrases, "area as a product" and "area as a sum," stand for the measure of the area of the rectangle enclosed by its sides. "Area as a product" is the conception of seeing the area as an ordered pair of linear units (Multiplicative Type Representational Unit Coordination) whereas "area as a sum" is the conception of seeing the area as an ordered *n*–tuple of areal units (Additive Type Representational Unit Coordination).

Third, my study extends prior work done by Behr et al. (1994) because identities that equate summation and product expressions are not expressed using generalized mathematical notation in their work. My research project is a theoretical extension of Behr et al.'s framework and introduces a simplified version of generalized mathematical notation for identities that equate summation and product expressions.

The fourth rationale for collecting interview data with preservice teachers was mainly to understand how they establish $\Sigma = \Pi$ identities involving linear and areal quantities based on the color cubes and algebra tiles representational models. I also wanted to determine how they were able to reason in the different categories of linear or areal quantities (Multiplicative and Additive Representational Unit Coordination) associated with growing rectangles generated by color cubes and algebra tiles. Moreover, we do not know much about preservice teachers' mathematical knowledge on these issues–their identification, interpretation and coordination of different types of representational units arising from this mathematical content.

The present study investigated three main research questions: How do preservice secondary school teachers

- identify, describe, and interpret linear and areal units?
- represent linear and areal units?

- distinguish between areal and linear units?

Data came from individual interviews during which I asked five (2 middle and 3 high school mathematics) PSTs problems related to six main mathematical ideas: modeling prime and composite numbers; summation of counting numbers, odd numbers, even numbers; and multiplication and factorization of polynomial expressions in x and y. I selected my participants from two different undergraduate level mathematics education classes.

Magnetic color cubes were used to generate sequences of growing rectangles representing identities for prime and composite numbers as well as summation of counting numbers, odd and even integers. During the interviews, I asked preservice teachers to make rectangles representing prime and composite numbers first. We then focused on patterns that generate growing rectangles for the summation of counting numbers, odd, and even integers. As for the products and factors of polynomial expressions, we used color tiles of different colors and sizes (algebra tiles).

After the end of data collection, I reviewed each interview in order to generate possible themes for a more detailed analysis. I transcribed significant events of these interviews. As for the analysis methodology, I embraced thematic analysis supported by retrospective as well as constant comparison analyses of interviews. I also made use of Verganud's *theorems–in–action* and *concepts–in–action* models (1983, 1988, 1994) as an (explanatory) analytical theory to explain students' sense making of the linear and areal quantities involved in the color cubes and algebra tiles activities. Vergnaud's conceptual field theory asserts that "one needs mathematics to characterize with minimum ambiguity the knowledge contained in ordinary mathematical competences."

(1994, p. 44) He claimed that one has to make use of mathematics itself (e.g., mathematical concepts, definitions, theorems) to analyze students' understanding of mathematical situations. Grounded in this theory, I developed a series of terminology to serve as analytical tools in an attempt to describe my students' *concepts*– and *theorems–in–action* (e.g., mathematical behaviors, actions, operations, hand gestures, etc.). In doing so, I often made use of a *Relational Notation*, a simplified version of the *Generalized Notation for Mathematics of a Quantity* (Behr et al., 1994).

In their work with the summation activities, PSTs established the arealness of the subunits with reference to the reference preserving composition (Schwartz, 1988), which was the quantitative addition operation. They often provided arguments such as "areas, when added together, produce areas." They established the arealness of the subunits under consideration by reference to known measurement units (e.g., inches squared, centimeters squared) as well as abstracted units (e.g., units squared) both in their written work and verbal descriptions as well. As for the linear and areal quantities in general, two students were able to operate with referent preserving compositions only, whereas the other three students were able to operate with both referent preserving and referent transforming compositions (Schwartz, 1988). PSTs also treated the quantitative multiplication and addition operations as functions or mappings when expressing the area of their growing rectangles made of magnetic color cubes and algebra tiles as sums and products. Their behavior necessitated the existence of a conceptual field of relational type, which I called "Mapping Structures."

Three main types of Representational Unit Coordination (RUC) arose as the PSTs made growing rectangles for special numbers such as prime and composite numbers, summation expressions for odd, even, and counting numbers as well as polynomials in *x* and *y* using magnetic color cubes and algebra tiles. In a Multiplicative type RUC, I used a relational notation of the form (*a*, *b*) where *a* and *b* stood for the corresponding linear units of the growing rectangle represented by the dimension tiles. I also observed more than one additive type of RUC that can be described using a functional notation $\sum_{i=1}^{n} f(i) = g(n)$ where the areal quantities "$f(i)$" are being summed from 1 to *n* (number of addends) and *i* is the stage number (ordering number for the addends). There is one more RUC type, in between additive and multiplicative, which I named Pseudo Multiplicative type RUC. There was a pattern, which showed itself in all my findings. The same group of students constantly relied on an additive interpretation of the context whereas the other students were able to distinguish between and when to rely on an additive or a multiplicative interpretation of the context.

6.2. Conclusions

Unit–Coordination (Steffe, 1988, 1994) theory serves as an eminent framework describing students' understanding and making sense of linear and areal quantities arising from prime–composite rectangles, summation activities, and polynomial multiplication and factorization problems. My participants coordinated linear and areal quantities in a somewhat different fashion than described by Steffe, whose scope is limited and can be applied to numbers only. In my study, students established the existence of linear units, areal units, areal subunits, and areal sub–subunits, which is in agreement with Steffe's three levels of unit–coordination. However, the structure of these units is different in that students emphasized the different dimensions (linearity and arealness), the quantitative character, and the quantitative operations taking place, in an attempt to establish identities of the form “Area as a Sum = Area as a Product” based on the growing rectangles they created.

Unit–coordination, though not offered as a component of the definition of a quantity before (Schwartz, 1988[15]; Thompson, 1994[16]), stands out as a fundamental dimension for a quantity. Representational Unit Coordination is a reconciliatory act unifying quantitative reasoning and unit–coordination and plays an essential role in students’ making sense of the mathematics arising from their performance with the representational quantities.

Quantitative addition and quantitative multiplication operations stand out as the most relevant compositions of referent preserving and referent transforming type, respectively, when representing mathematics with physical objects. Being able to establish the existence of representational quantities on their own as well as in relation to each other brings with it the notion of “quantitative operation,” which is the origin of a successful comprehension of the representational quantities. Such an understanding necessitates also students’ awareness of when a given referent is preserved, or when a quantity *of a new kind* is born. The quantities on the left–hand–side of the identity “Area as a Sum = Area as a Product” have to be of the same (areal) character, must be composed solely by the quantitative addition operation (i.e., a referent preserving composition), and the “summed quantity” at each stage has to be of the same (areal) character as well. Similarly, the quantities on the right–hand side have to be of the same (linear) character, and must be composed solely by the quantitative multiplication

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[15] A quantity is an ordered pair of the form (number, unit).
[16] A quantity is an ordered quadruple of the form (object, quality, unit, process to assign a value).

operation (i.e., a referent transforming composition). Though a "product quantity" comes to exist at each stage on the right–hand–side, yet this new type of quantity has to be of the same (areal) character as the "summed quantity" on the left–hand side. A coordination of quantities as well as a coordination of coordinated quantities, and an awareness of the types of quantitative operations (referent preserving or referent transforming compositions) taking place are necessary when reasoning quantitatively about a situation.

Big concepts of discrete mathematics such as set theory, relations[17], functions, graphs[18] of equations and functions, along with analytic geometry, probability, logic, and algebra of propositions are based on this powerful connective word "and," which also frequently appears as its orthographic synonym comma ",". The mapping structures conceptual field, as developed by the participants of this research study, is grounded in the usage of "and" and its symbolic representation comma ",". "And" requires the quantitative multiplication operation to behave as a function, a mapping acting on linear quantities, by which such quantities are transformed into an areal quantity. In another context, "and" manifests itself in the juxtaposition of the areal quantities in an additive manner, by which such quantities are preserved in referent, thanks to the quantitative addition operation behaving as a function. Through these mapping structures eventually, the left–hand side and right–hand side of the representational identities equal each other. They do not necessarily have to be equal to each other, yet they equal each other thanks to these mapping structures grounded in "and."

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[17] In particular, ordered pairs or *n*–tuples.
[18] In particular, points on plane or space.

"And" also emerges as a helpful asset in comparing and distinguishing between same–valued linear and areal quantities. In a mathematical situation involving representational quantities such as in figure below, the length of the whole rectangle has a value (or magnitude) of 6, which is the same as the value (magnitude) of the area of the horizontal part of the green L–shape (Figure 6.1). They both have the same value, wherefrom they stand as same–valued linear and areal quantities. The linear quantity, however, is simply 6, whereas the areal quantity is "not just 6… it's 6 **and** 1." The "it's 6 **and** 1" part of "it's not just 6… it's 6 **and** 1" establishes the existence of an areal quantity as well as a multiplicative representational unit coordination (MRUC). The "it's not just 6" part of "it's not just 6… it's 6 **and** 1," on the other hand, indicates the value of the areal quantity under consideration that exists because of the multiplication operation that behaves as a mapping acting on "6 **and** 1," namely the ordered pair (6, 1) of linear quantities. For a mapping structure of multiplicative type to exist, one needs to establish the following conditions:

4. **The "and" condition**: A juxtaposed pair ordering of the values of the linear quantities is mentioned.
5. **The MRUC condition**: The multiplication operation behaving as a mapping is acting on the ordered pair of these linear quantities.
6. **Existence of a mapping structure**: The value of the areal quantity resulting from the mapping is indicated.

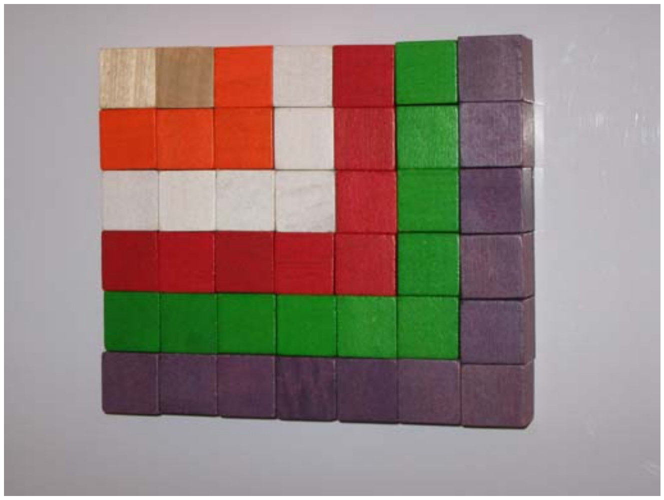

Figure 6.1. Linear and areal representational quantities.

Concepts–in– action and *theorems–in–action* are powerful instruments to illustrate and explain the continuing progress of students' mathematical proficiency in a certain conceptual field (e.g., multiplicative structures, relational structures, mapping structures, quantitative structures). They also present a way to analyze, compare, and transform students' knowledge intrinsic in their mathematical performance (e.g., hand gestures, actions, operations) into the actual known and written algebraic identities and mathematical theorems. In that sense, these tools help teachers and researchers get a better sense of how students make sense of, reconcile, and shift among physical observables at different cognitive levels (e.g., algebraic expressions, their various representations, etc.). Using concepts– and theorems–in–action, teachers and researchers can come up with better strategies to diagnose what students do understand or fail to understand, to reveal the source of their misconceptions and conceptual flaws, and to help them see the internal and external connections. In this way, students are provided with a set of more interesting, better prepared activities, and mathematically fruitful situations, which help them strengthen their knowledge, and increase their mathematical proficiency.

Vergnaud (1988) claimed that "a single concept does not refer to only one type of situation, and a single situation cannot be analyzed with only one concept" (p. 141). He argued that teachers and researchers should study conceptual fields rather than isolated concepts. He then went on to define a conceptual field as "a set of situations, the mastering of which requires mastery of several concepts of different natures" (p. 141). In the same vein, the participants of this present study gave birth to the *conceptual field of mapping structures* (CFMS), which can be described as

- A set of representational quantities (and relationships) whose existence are to be established as quantities on their own, and as quantities in relation to each other.
- A set of mathematical situations involving different kinds of representational quantities, which require referent preserving and referent transforming compositions (i.e., quantitative addition operation and quantitative multiplication operation) acting on these representational quantities.
- A set of (multiplicative and additive representational) unit–coordination strategies in establishing the existence of quantitative units, subunits, and sub–subunits.
- A set of explicit knowledge based on students' written symbols, algebraic expressions, drawings, figures.
- A set of implicit (intuitive) knowledge (concepts and theorems in action) based on students' actions, operations, hand gestures, and verbal expressions.

6.3. Implications

6.3.1. Curriculum Development

The phrase "Summation Formulas" is ambiguous and somewhat misleading because although it stands for an identity of the form "Sum = Product," one gets the impression that one deals with a summation only. The main implication arising from my study is to replace this terminology with "Sum–Product Identities." In this way, students will first start by focusing on the words constituting this phrase, namely the fact that they are dealing with **identities** based on **sums** and **products**. Moreover, "summation formulas" in current textbooks focus only on subsets of integers. As the participants in this research demonstrated, identities based on polynomial expressions too stand as sum–product identities. A general and more meaningful statement "Sum–Product Identities" makes more sense, and is more suitable in addressing subsets of Z as well as polynomial expressions defined on Z.

The curriculum materials (e.g., textbooks, activity books, online modules, manipulatives, teacher guides) should emphasize the necessity of attending to the nature of the quantities, their units, and the quantitative operations taking place on each side of the sum–product identities.

Under the direction "Students will use sequences and series," Georgia Performance Standards state

a. Use and find recursive and explicit formulas for the terms of sequences.
b. Recognize and use simple arithmetic and geometric sequences.
c. Find and apply the sums of finite and, where appropriate, infinite arithmetic and geometric series.

d. Use summation notation to explore finite series (Georgia Performance Standards Mathematics 4, p. 4)[19].

GPS has reference to the summation notation, however, there is no mention of the existence of an identity, nor the parameters involved, nor how the parameters are related to each other. Students should be helped to make sense of a sum–product identity such as $\sum_{i=1}^{n} a_i = f(n)$ by asking the meaningful questions such as the following:

- What does i represent?
- What does n represent?
- What does a_i represent?
- What does $\sum_{i=1}^{n} a_i$ represent?
- What does the expression $f(n)$ on the RHS represent?
- Is it a product of two numbers?
- Is it a product of two quantities?
- Does it depend on n or i?
- What are the units associated with a_i, $\sum_{i=1}^{n} a_i$, and $f(n)$?

None of the participants of this present study were introduced to sum–product identities in high school. Another important implication of my study, therefore, is in that the use of magnetic color cubes, algebra tiles, color tiles (or paper cutouts perhaps) as representational tools in teaching summation formulas and polynomial multiplication and factorization provides students with opportunities to make better sense of and to explore

[19] Retrieved March 8, 2008 from http://www.georgiastandards.org

and discover algebraic connections between the sum–product identities and concrete operations. Activities that incorporate such manipulatives provide teachers with an easily accessible concept–building activity for developing sum–product identities for the summation of consecutive positive counting, even, and odd integers, and polynomial multiplication and factorization. These representations could also be used to model consecutive square numbers, cube numbers, triangular numbers, pentagonal numbers, Fibonacci numbers, as well as polynomials of higher degree.

Sum–Product Identities do not solely apply to the mathematics context my research participants and I investigated. Focusing on the big picture, one can find Sum–Product Identities (or LHS–RHS Identities in general) in various contexts such as linear or quadratic equations based on word problems (Caglayan & Olive, 2008; Olive & Caglayan, 2006, 2007), identities based on derivatives of functions $\frac{d}{dx} f(x) = f'(x)$ or integral expressions of the form $\int f(x)\, dx = F(x)$. The findings of my study imply that such content be written and guided by a framework based on Representational Unit Coordination, which pushes students to reason quantitatively, at the same time paying attention to the relevant mappings and quantitative operations taking place.

Finally, one of the PST in my study chose to stand the algebra tiles on edge, rather than laying them flat on the desk. This "misuse" of the tool may have some implications in the design of new tools accompanying the algebra tiles. Instead of using a flat multiplication mat, one could develop a three dimensional rectangular box shaped multiplication mat having sliders on the side and at the top, which will allow the dimension tiles *only* in a standing up position. In this way students and teachers will be able to make sense of the linearity of the dimension tiles. Moreover, they will be able to easily distinguish between the same valued linear and areal quantities that are represented by the same color algebra tiles.

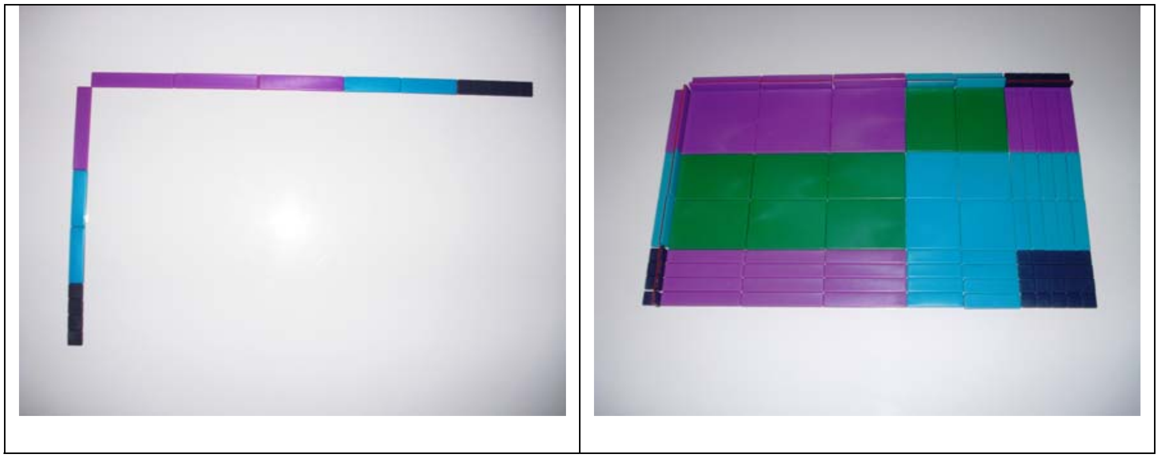

Figure 6.2. Standing up positioned $(x+2y+4)\times(3x+2y+5)$ dimension tiles.

6.3.2. Teacher Education

Vergnaud defined *theorems–in–action* as "mathematical relationships that are taken into account by students when they choose an operation or a sequence of operations to solve a problem. To study children's mathematical behavior it is necessary to express the theorems–in–action in mathematical terms" (1988, p. 144). Concepts– and theorems–in–action framework in this present study helped me produce a set of terminology closely related to mathematical terms (e.g., Representational Cartesian Products, Representational Sets, Summed Addends, Polynomial Rectangles, etc.) to describe what my participants were doing. These notions, as a mathematics teacher, helped me make sense of what my students were doing and delve into their understanding of the mathematical situations. There were many instances from the protocols analyzed in the previous chapter in which my students were "performing mathematically." This mathematical performance included – but was not limited to – verbal descriptions, hand gestures, color cube and tile generated figures.

Teacher education programs should provide opportunities for students to explicitly engage in quantitative reasoning in a manner that leads to using all three levels of unit coordination. This necessitates a focus on discrete mathematics content with a particular emphasis on sets, relations, Cartesian products, mapping structures, which by definition encompass levels of unit coordination and quantitative reasoning in their structure. In particular, at first, summation activities and polynomial multiplication and factorization can be thought of as totally irrelevant to set theoretical aspects, quantitative reasoning, or unit coordination. However, when preservice teachers want to make sense of what they are doing, they will naturally end up performing mathematically, and face set theoretical aspects. Such courses could benefit from my findings on students' mathematical performance on the irreducible/combined linear and areal quantities, the operations taking place, and their interpretation of the areas of areal quantities (multiplicative vs. additive) as well.

6.4. Suggestions for Future Research

In 2006, at the preliminary stage of my dissertation research process, in implementing summation of odd integers and even integers activities with high school students, it was established that most students prefer to manipulate the colored tiles or paper cutouts first, before transferring the representation to the 1−inch grid paper (Caglayan, 2006). Some students found that placing the cutouts directly onto 1−inch grid paper also aided in keeping the visual patterns intact. Possible extensions to these activities include altering the placement of the rectangular counting integer or even integer representations, L–shaped odd integer (or even integer) representations onto the previous figures. However, alternate arrangements, although yielding the same results, are more difficult for students to interpret. In summation of odd integers and even integers activities, for instance, extension questions can include the development of alternate geometrical interpretations of both n^2 and $n(n + 1)$ and their significance in the visualization of sum–product identities.

Research by Caglayan and Olive (2008) explored the writing and solving of equations in one unknown – involving both positive and negative quantities – using a representational metaphor of cups (that hold an unknown number of tiles) and tiles. They found that addition and multiplication operations are the most meaningful and relevant operations when using drawn representations of cups and tiles – there is no way to represent subtraction. The participants of this present study, on the other hand, established that the only relevant operations taking place are addition, multiplication, and inverse multiplication. Subtraction was irrelevant since they worked with positive quantities only, and they were asked to add or multiply these quantities. There was no way to represent division; yet, they established an operation, which can be thought as "inverse multiplication," with reference to mapping structures. Possible extensions to this present study, therefore, could be to include both positive and negative quantities – the back sides of the algebra tiles are all red and stand for the negative quantities. Possible research questions would be: What operations will exist for the students? Will addition, multiplication, inverse addition, and inverse multiplication be the only relevant quantitative operations? Or will it be possible to represent subtraction and division operations as well?

I am suggesting the study of students' sense making of linear and areal quantities (or quantities in general) in different mathematical situations. Such mathematical situations could include visual proofs of more challenging summation formulas such as sum of consecutive squares, sum of triangular numbers, sum of consecutive cubes represented as growing rectangles; or identities based on pattern blocks. Unit–coordination theory may stand as a suitable perspective in studying these different contexts, too. One may have to take into consideration volumic units/quantities as well.

A research study guided by a possible research question such as "How do college students/preservice teachers understand and make sense of different types of quantities involved in integral expressions?" could shed some light on teacher education and curriculum development. We should get students' attention to what integral expressions such as $\int_a^b f(x)\,dx$ or $\iint_A f(x,y)dxdy$ stand for. The symbols $\int_a^b$ and $\iint_A$ tell us to "add," which can be viewed as a quantitative operation of referent preserving type (Schwartz, 1988). We could gather information from students via interviews during which we can ask them questions such as

- What does $f(x)$ stand for? What does $f(x,y)$ stand for?
- What does *dx* mean? What does *dxdy* mean?
- Is it an extensive or intensive quantity? Is it a linear quantity? Is it an areal quantity? Is it a volumic quantity?
- What does $f(x)\,dx$ mean? What does $f(x,y)\,dxdy$ mean?

- Is it a product of two numbers? A product of two quantities?
- What is the nature of the resulting quantity? Is it an extensive or intensive quantity? Is it a linear quantity? Is it an areal quantity? Is it a volumic quantity?
- What do the symbols $\int_a^b$ and $\iint_A$ do?
- How are they related to $f(x)\,dx$ and $f(x,y)\,dxdy$?

We could also work with Riemann Sums and Areas in order to get their attention to the "representational quantities" involved in such contexts.

Any study guided by representational models invites students to produce some connections between such representations (e.g., manipulatives, drawings, algebraic symbols, graphs, tables, etc.) A study attempting to understand how students interpret and connect the quantities (existing on their own and in relation to each other) in different levels of such representations could be useful. Quantities will be the same; however, will the students' or preservice teachers' understanding of them stay the same? What do we mean by "same" or "different?" How successful will our students be in reconciling such "same" quantities represented via different models? How will our students make sense of quantitative operations (e.g., referent preserving, referent transforming compositions) in multiple representational models?

Other than the idea of studying connections among representational models, one could also benefit from studying connections between mathematics and another branch of science such as physics or chemistry. Physics and chemistry stand as rich contexts with underlying concepts such as extensive vs. intensive quantities, unit structures, referent preserving vs. transforming compositions (quantitative operations), etc. We could benefit from research on students' interpretation and comparison of the quantities manifested in experiments, and quantities recorded in graphs, datasheets, or tables.

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APPENDIX A: RECORDING SHEETS SHOWING BRAD'S WRITTEN ANSWERS

TABLE FOR ACTIVITY ONE – PRIME & COMPOSITE RECTANGLES

Number	Dimensions (Width & Length) of Its Rectangle(s)	Area of Its Rectangle(s) as a Sum	Area of Its Rectangle(s) as a Product	Area of Its Rectangle(s)	Number of Its Rectangle(s)	Its Divisors (Factors)	Number of Its Divisors (Factors)	Prime? Composite? Neither?
15	3×5	3+3+3+3+3 5+5+5	$3 \cdot 5$ $5 \cdot 3$	15 in^2	2	5 3 15 1	4	Composite
3	3inches by 1 inch	$1 \text{ in}^2 + 1 \text{ in}^2 + 1 \text{ in}^2$	3inches by 1 inch	3 in^2	1	NOT DISCUSSED		

TABLE FOR ACTIVITY TWO – SUMMING COUNTING NUMBERS

Stage Number	Number of Tiles Added	Dimensions (Width & Length) of the added (small) rectangle	Area of the added (small) rectangle as a product	Area of the added (small) rectangle as a sum	Dimensions (Width & Length) of the growing (big) rectangle	Area of the growing (big) rectangle as a product	Area of the growing (big) rectangle as a sum	Total Area
1					1 by 1			1 in^2
2	2	2 by 1	2 in × 1 in	1 in^2 + 1 in^2	1 by 3	3 in × 1 in	1 in^2 + 1 in^2 + 1 in^2	3 in^2
3	3	3 by 1	3 by 1	1 in^2 + 1 in^2 + 1 in^2	2 by 3	2 in × 3 in	2 in^2 + 2 in^2 + 2 in^2 3 in^2 + 3 in^2 and one	6
4	4				2 by 5		2 + 2 + ... 5 + 5 1 + ...	10
					3 by 5		3 + ... 5 + ... 1 + ...	15
					3 by 7		7 + 7 + 7 3 + ... 1 + ...	21
					4 by 7		2 + 2 + 2 + ... 4 in^2 + 4 in^2 + 4 + ... 7 + ... 1 + ...	28
					4 by 9		4 + ... 9 + ... 3 + ... 2 + ... 1 + ...	36

2
3
4
5
6
7
8

TABLE FOR ACTIVITY THREE – SUMMING ODD INTEGERS

Stage Number	Number of Tiles Added	Area of the added figure as a sum	Dimensions (Width & Length) of the growing (big) rectangle	Area of the growing (big) rectangle as a product	Area of the growing (big) rectangle as a sum	Total Area
1						1 in²
2	3	1 in² + 1in² + 1in² 2 in² + 1 in²	2 in by 2 in	2 in x 2 in	2 in² + 2 in² 1 in² + 1in² + . . .	4 in²
3	5	1 in² + 1 in² + . . 3 in² + 2 in²		3 in x 3 in	3 in² + 3 in² + 3 in² 1 in² + 1in² + . . .	9 in²
4	7	1 in² + . . . 4 in² + 3 in²		4 in x 4 in	4 in² + 4 in² + . . 2 in² + 2 in² + . . 1 in² + 1in² + . . 8 in² + 8 in²	16 in²
5	9	5 in² + 4 in² 1 in² + . . .		5 in x 5 in	5 in² + . . . 1 in² + . . .	25 in²
6	11	6 in² + 5 in² 1 in² + . . .		6 in x 6 in	6 in² + 6 in² + . . . 3 in² + . . 2 in² + . . 9 in² + . . . 1 in² + . . . 4 in² + . . .	36 in²
7						
8						
X	2x-1					X²

TABLE FOR ACTIVITY FOUR – SUMMING EVEN INTEGERS

Stage Number	Number of Tiles Added	Area of the added figure as a sum	Dimensions (Width & Length) of the growing (big) rectangle	Area of the growing (big) rectangle as a product	Area of the growing (big) rectangle as a sum	Total Area
1	2				↑ Relies on the summation formula!	
2	4	4 in² ✓	2 in by 3 in ✓	2 x 3	4 in² ✓ + 2 in² ✓	6 in² ✓
3	6	6 in²	3 in by 4 in	3 x 4	6 in² + 4 in² + 2 in²	12 in²
4	8	8 in²	4 by 5	4 X 5	8 in² + ...	20 in²
5	10	10 in²	5 by 6	5 x 6	10 in² + ...	30 in²
6	12	12 in²	6 by 7	6 x 7	12 in² + ...	42 in²
7				7 x 8		

⋮

(n) n(n+1) n

TABLE FOR ACTIVITY FIVE – POLYNOMIAL RECTANGLES

Dimensions of the Rectangle	Linear Units	Area of the polynomial rectangle as a product	Areal Units	Areas of the Boxes of the same color as a product	Areas of the Boxes of the same color as a sum	Area of the polynomial rectangle as a sum	Does Sum Equal Product?
Length: x+1 Width: 2x+3		$2x^2+5x+3$				$2x^2+5x+3$ $(x^2+x^2+x+x+x+x+x+1+1+1)$	
Length: x+1 Width: 2y+3		$2xy+2y+3x+3$ $(x+1)(2y+3)$		$3\cdot x$ $2\cdot y$ $2\cdot xy$ $3\cdot 1$	$3(x)+2(y)+2(xy)+3(1)$	$2xy+2y+3x+3$	
Length: 2x+y Width: x+2y+1		$(2x+y)(x+2y+1)$		$2(x^2)$ $1(y)$ $5(xy)$ $2(x)$ $2(y^2)$	$2(x^2)+2(y^2)+5(xy)+1(y)+2(x)$	$2x^2+2y^2+5xy+y+2x$	
Now Backwards: Make a rectangle for the expression x^2+5x+6 then factor the expression		$(x+3)(x+2)$		$(x)(x)$ $2(3)$ $3(x)$ $2(x)$		x^2+5x+6	$(x+3)(x+2)=x^2+5x+6$
Make a rectangle for the expression ~~$3x^2+7xy+2y^2+14x+13y+15$~~ then factor the expression		$(2x+y+1)(x+3y+2)$		$2(x^2)$ $2(1)$ $3(y^2)$ $7(xy)$ $5(y)$ $5(x)$	$2(x^2)+3(y^2)+5(y)+5(x)+2(1)+7(xy)$		$(2x+y+1)(x+3y+2)$ $=$ $2x^2+3y^2+5y+5x+7(xy)$

APPENDIX B: RECORDING SHEETS SHOWING NICOLE'S WRITTEN ANSWERS

TABLE FOR ACTIVITY ONE – PRIME & COMPOSITE RECTANGLES

Number	Dimensions (Width & Length) of Its Rectangle(s)	Area of Its Rectangle(s) as a Sum	Area of Its Rectangle(s) as a Product	Area of Its Rectangle(s)	Number of Its Rectangle(s)	Its Divisors (Factors)	Number of Its Divisors (Factors)	Prime? Composite? Neither?
15	Length: 3 width: 5 Length: 1 width: 15	3+3+3+3+3 5+5+5 1+1+1+1+1+1+1	3x5 5x3 1x15 15x1	15in² 15in²	2	3,5,1,15	4factors	Composite
7	Length: 1 width: 7	1+1+1+1+1+1+1	1x7 7x1	7in²	1	1,7	2	Prime

TABLE FOR ACTIVITY TWO – SUMMING COUNTING NUMBERS

Stage Number	Number of Tiles Added	Dimensions (Width & Length) of the added (small) rectangle	Area of the added (small) rectangle as a product	Area of the added (small) rectangle as a sum	Dimensions (Width & Length) of the growing (big) rectangle	Area of the growing (big) rectangle as a product	Area of the growing (big) rectangle as a sum	Total Area
1	1	1 - L 1 - w	1x1	1				
2	2	L - 1 w - 2	1x2	1+1	L - 1 w - 3	1x3	1+1+1 or 1+2	3
3	3	L - 1 w - 3	1x3	1+1+1	L - 2 w - 3	2x3	1+2+3 or 3+3 or 1+1+1+1+1+1	6
4	4	L - 2 w - 2	2x2	2+2 or 1+1+1+1	L - 2 w - 5	2x5	1+2+3+4 or 3+3+4 or 1+1+1....	10
5	5	L - 1 w - 5	1x5	1+1+1+1+1	L - 3 w - 5	3x5	1+2+3+4+5 or 3+3+4+5 or 6+4+5 or 10+5 or 1+1.....	15
6	6	L - 3 w - 2	3x2	2+2+2 3+3 1+1+1+1+1+1	L - 3 w - 7	3x7	1+2+3+4+5+6 or 3+3+4+5+6 or 6+4+5+6 or 10+5+6 or 15+6 or 1+1+1	21

There was also an irregular formation here. Mention that in your analysis. That's related to a different kind of RUC [different from what's written here]

TABLE FOR ACTIVITY THREE – SUMMING ODD INTEGERS

Stage Number	Number of Tiles Added	Area of the added figure as a sum	Dimensions (Width & Length) of the growing (big) rectangle	Area of the growing (big) rectangle as a product	Area of the growing (big) rectangle as a sum	Total Area
1						
2	3	1+1+1 3	W-2 L-2	2x2	1+3 or 2+2 or 1+1+1+1	4
3	5	5 or 1+1+1+1+1	W-3 L-3	3x3	1+3+5 or 4+5 or 1+1+...	9
4	7	7 or 1+1...	W-4 L-4	4x4	1+3+5+7 or 4+5+7 or 9+7 or 1+...	16
5	9	9 or 1+1... 3+3+3	W-5 L-5	5x5	1+3+5+7+9 or 4+5+7+9 or 9+7+9 16+9 or 1+1+1	25

She is also suggesting irregular shapes other than L-shapes //

RUC - composite number type /// [EQUAL ADDENDS]

RUC - summed addends type ///

TABLE FOR ACTIVITY FOUR – SUMMING EVEN INTEGERS

Stage Number	Number of Tiles Added	Area of the added figure as a sum	Dimensions (Width & Length) of the growing (big) rectangle	Area of the growing (big) rectangle as a product	Area of the growing (big) rectangle as a sum in	Total Area in^2
1	2	1+1	1x2	2x1	1+1	2
2	4	1+1+1+1	2x3	2x3	2+4 or 1+1+1+1+1+1	6
3	6	1+1+1+1+1+1	3x4	3x4	2+4+6 or 6+6 or 1+1+....	12
4	8	1+1+1.....	4x5	4x5	2+4+6+8 or 6+6+8 or 12+8 or 1+1....	20
5	10	1+1+1.....	5x6	5x6	2+4+6+8+10 or 6+6+8+10 or 12+8+10 or 20+10 or 1+1+1...	30
6	12					

TABLE FOR ACTIVITY FOUR – SUMMING EVEN INTEGERS

Stage Number	Number of Tiles Added	Area of the added figure as a SUM	PRODUCT	Dimensions (Width & Length) of the growing (big) rectangle	Area of the growing (big) rectangle as a product	Area of the growing (big) rectangle as a sum	Total Area
1		2x1	2x1				
2		2x2	2x3				
3		2x3	4x3				
4		4x2	4x5				
5		2x5	6x5				
6		6x2	6x7				

TABLE FOR ACTIVITY FIVE – POLYNOMIAL RECTANGLES

Dimensions of the Rectangle	Linear Units	Area of the polynomial rectangle as a product	Areal Units	Areas of the Boxes of the same color as a product	Areas of the Boxes of the same color as a sum	Area of the polynomial rectangle as a sum	Does Sum Equal Product?
Length: x+1 Width: 2x+3		$(x+1)\cdot(2x+3)$		$x\cdot 2x$ $x\cdot 3$ $1\cdot 2x$ $1\cdot 3$	(she says in words) "x^2 plus x^2" "x plus x" "x plus x plus x" "1 plus 1 plus 1"	x^2+x^2+5x+3 $2x^2+5x+3$	$(x+1)\cdot(2x+3)=$ $2x^2+5x+3$
Length: x+1 Width: 2y+3		$(x+1)\cdot(2y+3)$		$x\cdot 2y$ $x\cdot 3$ $1\cdot 2y$ $1\cdot 3$		$2xy+3x+2y+3$	$(x+1)\cdot(2y+3)=$ $2xy+3x+2y+3$
Length: 2x+y Width: x+2y+1	L: 2x+y cm W: x+2y+1 cm	$(2x+y)\cdot(x+2y+1)$	$(2x+y)\cdot(x+2y+1)$ cm^2	$(2x)\cdot(x)$ $(y)\cdot(x)$ $(2x)\cdot(2y)$ $(y)\cdot(2y)$ $(2x)\cdot(1)$ $(y)\cdot(1)$	x^2+x^2 xy $xy+xy+xy+xy$ y^2+y^2 $x+x$ y	$2x^2+5xy+2y^2+2x+y$	$(2x+y)\cdot(x+2y+1)=$ $2x^2+5xy+2y^2+$ $2x+y$
Make a rectangle for the expression x^2+5x+6 then factor the expression				SAYS IN WORDS AT 59:30			$(x+2)\cdot(x+3)=$ x^2+5x+6
Factor the expression $3x^2+7xy+2y^2+13y+14x+15$							

TABLE FOR ACTIVITY FIVE – POLYNOMIAL RECTANGLES

Dimensions of the Rectangle	Linear Units	Area of the polynomial rectangle as a product	Areal Units	Areas of the Boxes of the same color as a product	Areas of the Boxes of the same color as a sum	Area of the polynomial rectangle as a sum	Does Sum Equal Product?
Length: $x+1$ Width: $2x+3$							
Length: $x+1$ Width: $2y+3$							
Length: $2x+y$ Width: $x+2y+1$							
Now Backwards: Make a rectangle for the expression x^2+5x+6 then factor the expression							
Make a rectangle for the expression $2x^2+7xy+3y^2+5x+5y+2$ then factor the expression $2x^2+7xy+3y^2+5x+5y+2$		$(2x+y+1)\cdot$ $(x+3y+2)$		$(2x)(x)$ $(x)\cdot(y)$ $(x)\cdot(1)$ $(2x)\cdot(3y)$ $(y+1)\cdot(3y)$ $(2x)\cdot(2)$ $(y)\cdot(2)$ $(2)\cdot(1)$	x^2+x^2 xy x $(xy)+(xy)\cdots$ $y^2+y^2+y^2+y+y+y$ $x+x+x+x$ $y+y$ $1+1$		$(2x+y+1)\cdot$ $(x+3y+2)=$ $2x^2+7xy+3y^2+$ $5x+5y+2$

APPENDIX C: RECORDING SHEETS SHOWING ROB'S WRITTEN ANSWERS

TABLE FOR ACTIVITY ONE – PRIME & COMPOSITE RECTANGLES

Number	Dimensions (Width & Length) of Its Rectangle(s)	Area of Its Rectangle(s) as a Sum	Area of Its Rectangle(s) as a Product	Area of Its Rectangle(s)	Number of Its Rectangle(s)	Its Divisors (Factors)	Number of Its Divisors (Factors)	Prime? Composite? Neither?
15	3×5 5×3 1×15 15×1	15 in^2 3 in^2 + 3 in^2 + 3 in^2 + 3 in^2 + 3 in^2	3 in × 5 in 5 in × 3 in	15 in^2	4 or 2	3 5 1 15	4	Composite
		5 in^2 + 5 in^2 + 5 in^2 1 in^2 + 1 in^2 + ... 1 in^2	1 in × 15 in 15 in × 1 in					
7	1×7 7×1	1 in^2 + 1 in^2 + 1 in^2 + 1 in^2 + 1 in^2 + 1 in^2 + 1 in^2	1 in × 7 in 7 in × 1 in	7 in^2	2 or 1	1 7	2	prime

TABLE FOR ACTIVITY TWO – SUMMING COUNTING NUMBERS

Stage Number	Number of Tiles Added	Dimensions (Width & Length) of the added (small) rectangle	Area of the added (small) rectangle as a product	Area of the added (small) rectangle as a sum	Dimensions (Width & Length) of the growing (big) rectangle	Area of the growing (big) rectangle as a product	Area of the growing (big) rectangle as a sum	Total Area
1								
2	2	1×2	1 in × 2 in	1 in² + 1 in²	1 × 3	1 in × 3 in	1 in² + 1 in² + 1 in²	3 in²
3	3	1×3	1 in × 3 in	1 in² + 1 in² + 1 in²	2×3	2 in × 3 in	2 in² + 2 in² + 2 in² 3 in² + 3 in² | 1 in² + 1 in²…	6 in² 2×3
4	4	2×2	2 in × 2 in	2 in² + 2 in²	2×5	2 in × 5 in	5 in² + 5 in² | 1 in² + … 2 in² + 2 in² + 2 in² + 2 in² + 2 in²	10 in² 2×5
5	5	1×5	1 in × 5 in	1 in² + 1 in² + …	3×5	3 in × 5 in	3 in² + 3 in² + 3 in² … 5 in² + 5 in² + 5 in² 1 in² + …	15 in² 3×5
6								21 in² 3×7
7								28 in² 4×7
8								36 in² 4×9

TABLE FOR ACTIVITY THREE – SUMMING ODD INTEGERS

Stage Number	Number of Tiles Added	Area of the added figure as a sum	Dimensions (Width & Length) of the growing (big) rectangle	Area of the growing (big) rectangle as a product	Area of the growing (big) rectangle as a sum	Total Area
3	5	$1 in^2 + \ldots$	$3 in \times 3 in$	$3 in \times 3 in$	$3 in^2 + 3 in^2 + 3 in^2$ $1 in^2 + \ldots + 1 in^2$ no other way	$9 in^2$
4	7	$1 in^2 + \ldots$	4×4	$4 in \times 4 in$	$4 + 4 + 4 + 4 in^2$ $1 in^2 + \ldots$ (16 times) $8 in^2 + 8 in^2$ $2 in^2 + 2 in^2 \ldots$	$16 in^2$
5	9	$1 in^2 + \ldots$ $3 in^2 + 3 in^2 + 3 in^2$	5×5	$5 in \times 5 in$	$5 + 5 + 5 + 5 + 5 in^2$ $1 in^2 + \ldots$ (25 times) No other way	$25 in^2$

TABLE FOR ACTIVITY FOUR – SUMMING EVEN INTEGERS

Stage Number	Number of Tiles Added	Area of the added figure as a sum	Dimensions (Width & Length) of the growing (big) rectangle	Area of the growing (big) rectangle as a product	Area of the growing (big) rectangle as a sum	Total Area
2	4	sum $2in^2 + 2in^2$ $1in^2 + \ldots$ / Product $2in \times 2in$	2×3	$2in \times 3in$	$2in^2 + 2in^2 + 2in^2$ $3in^2 + 3in^2$ $1in^2 + \ldots$	$6in^2$
3	6	Product $2_{in} \times 3_{in}$ / $3in^2 + 3in^2$ $2in^2 + 2in^2 + 2in$ $1in^2 + \ldots$	3×4	$3in \times 4in$	$2in^2 + \ldots$ (6 times) $4in^2 + 4in^2 + 4in^2$ or $1in^2 + \ldots$ (12 times)	$12in^2$
4	8	Product $2_{in} \times 4_{in}$ / $2in^2 + \ldots$ (4 times) $4in^2 + 4in^2$ $1in^2 + \ldots + 1in^2$	4×5	$4in \times 5in$	$5in^2 + \ldots + 5in^2$ (4 t) $4in^2 + \ldots + 4in^2$ (5 t) $10in^2 + 10in^2$ $2in^2 + \ldots + 2in^2$ $1in^2 + \ldots$	$20in^2$

TABLE FOR ACTIVITY FIVE – POLYNOMIAL RECTANGLES

Dimensions of the Rectangle	Linear Units	Area of the polynomial rectangle as a product	Areal Units	Areas of the Boxes of the same color as a product	Areas of the Boxes of the same color as a sum	Area of the polynomial rectangle as a sum	Does Sum Equal Product?
Length: x+1 Width: 2x+3	$ units	(x+1)(2x+3) unit unit unit²	square units (units²)			2x²+5x+3 square units	(x+1)(2x+3)= 2x²+5x+3 units² ✓
Length: x+1 Width: 2y+3	units	(x+1)(2y+3)	↓	2(xy)=G 3x=P 2y=B 1×3=DB	xy+xy=G x+x+x=P y+y=B 1+1+1=DB	2xy+3x+2y+3 units²	(x+1)(2y+3)= 2xy+3x+2y+3 ✓
Length: 2x+y Width: x+2y+1	↓	(2x+y) × (x+2y+1)	↓	(2)(x²)=BP (5)(xy)=G (2)(y²)=BB (2)(x)=LP (1)(y)=LB	x²+x²=BP xy+xy ...=G y²+y²=BB x+x=LP y=LB	2x²+2y²+5xy+ 2x+y units²	(2x+y)(x+2y+1) 2x²+2y²+5xy+ 2x+y
Now Backwards: Make a rectangle for the expression x²+5x+6 then factor the expression	↓	(x+2)(x+3)	↓	(1)x²=BP (5)x=LP 6×1=DB	x²=BP x+x+x+x+x=LP 1+1+1+1+1+1=DB	x²+5x+6 units²	(x+2)(x+3)= x²+5x+6
Make a rectangle for the expression 3x²+7xy+2y²+14x+13y+15 then factor the expression 2x²+7xy+3y²+5x+5y+2	↓	(2x+y+1) (x+3y+2) unit	↓	(2)(x²)=BP = (7)(xy)=G = (3)(y²)=BB = (5)(x)=LP = (5)(y)=LB = 2(1)=DB =	x²+x² xy+xy ... y²+y²+y² x+x+x+x+x y+y+y+y+y 1+1	2x²+7xy	

APPENDIX D: RECORDING SHEETS SHOWING SARAH'S WRITTEN ANSWERS

TABLE FOR ACTIVITY ONE – PRIME & COMPOSITE RECTANGLES

Number	Dimensions (Width & Length) of Its Rectangle(s)	Area of Its Rectangle(s) as a Sum	Area of Its Rectangle(s) as a Product	Area of Its Rectangle(s)	Number of Its Rectangle(s)	Its Divisors (Factors)	Number of Its Divisors (Factors)	Prime? Composite? Neither?
15	Length: 3 in width: 5 in	14+1	3 x 5					

TABLE FOR ACTIVITY TWO – SUMMING COUNTING NUMBERS

Stage Number	Number of Tiles Added	Dimensions (Width & Length) of the added (small) rectangle	Area of the added (small) rectangle as a product	Area of the added (small) rectangle as a sum	Dimensions (Width & Length) of the growing (big) rectangle	Area of the growing (big) rectangle as a product	Area of the growing (big) rectangle as a sum	Total Area
5	5	5 by 1				3 × 5 in × in	1 + 2 + 3 + 4 + 5 $1n^2 + 1n^2 + 1n^2 + 1n^2 + 1n^2$	

TABLE FOR ACTIVITY THREE – SUMMING ODD INTEGERS

Stage Number	Number of Tiles Added	Area of the added figure as a sum	Dimensions (Width & Length) of the growing (big) rectangle	Area of the growing (big) rectangle as a product	Area of the growing (big) rectangle as a sum / product	Total Area
1			1 x 1		1 = 1	
2			2 x 2		1+3 = 2 x 2	
3	5	3 in² + 2 in²	3 in x 3 in		3 + 3 + 3 (in²) 1 + 3 + 5 (in²) = 3 x 3 8 + 1	
			4 x 4		1+3+5+7 = 4 x 4	
5	9	5 in² + 4 in²	5 in x 5 in		1+3+5+7+9 = 5 x 5	
			.		.	
			.		.	
			.		.	

TABLE FOR ACTIVITY FOUR – SUMMING EVEN INTEGERS

Stage Number	Number of Tiles Added	Area of the added figure as a sum	Dimensions (Width & Length) of the growing (big) rectangle	Area of the growing (big) rectangle as a product	Area of the growing (big) rectangle as a sum	Total Area
4	8	~~8~~ 4 in² + 4 in² and many different ways	4 in × 5 in	4 in × 5 in =	2 in² + 4 in² + 6 in² + 8 in²	
5		↓		5 in × 6 in =	2+4+6+8+10	
				6 × 7 =	. . . . +12	
				7 × 8 =	. . . . . +14	

8 × 9 = ... +16

. . . . $n^2 + n$ = ... +2n

TABLE FOR ACTIVITY FIVE – POLYNOMIAL RECTANGLES

Dimensions of the Rectangle	Linear Units	Area of the polynomial rectangle as a product	Areal Units	Areas of the Boxes of the same color as a product	Areas of the Boxes of the same color as a sum	Area of the polynomial rectangle as a sum	Does Sum Equal Product?
Length: x+1 Width: 2x+3		(x+1)·(2x+3)		(2x)·(x) (3)·(x) (1)·(2x) (3)·(1)	x²+x² x+x+x x+x 1+1+1	2x²+5x+3	= (x+1)(2x+3)
Length: x+1 Width: 2y+3		(x+1)(2y+3)		(2y)·(x) (3)·(x) (1)·(2y) (1)·(3)	xy+xy x+x+x y+y 1+1+1		2xy+2y+3x+3 = (x+1)(2y+3)
Length: 2x+y Width: x+2y+1	units	(2x+y)·(x+2y+1)	(units)²	(2x)·(x) (2y)·(2x) (2x)·(1) (x)·(y) (y)·(2y) (y)·(1)	x²+x² xy+xy+xy+xy x·1+x·1 x·y y²+y² y	2x²+5xy+2x+y+y²	= (2x+y)(x+2y+1)
Now Backwards: Make a rectangle for the expression x^2+5x+6 then factor the expression		(x+3)(x+2)		x·x x·3 x·2 3·2	x² x+x+x x+x 1+1+1+1+1+1=6	x²+5x+6 =	(x+3)(x+2)
Make a rectangle for the expression $3x^2+7xy+2y^2+14x+13y+15$ then factor the expression 2x²+7xy+3y²+5x+5y+2		(x+3y+2)·(2x+y+1)		2x·x 3y·1 x·y 2x·2 1·x y·2 2x·3y 2·1 y·3y		2x²+(7x)y+3	y²+5x+5y+2 = (x+3y+2)(2x+y+1)

APPENDIX E: RECORDING SHEETS SHOWING JOHN'S WRITTEN ANSWERS

TABLE FOR ACTIVITY ONE – PRIME & COMPOSITE RECTANGLES

Number	Dimensions (Width & Length) of Its Rectangle(s)	Area of Its Rectangle(s) as a Sum	Area of Its Rectangle(s) as a Product	Area of Its Rectangle(s)	Number of Its Rectangle(s)	Its Divisors (Factors)	Number of Its Divisors (Factors)	Prime? Composite? Neither?
28	4in × 7in	1+2+3+4+5+6+7 7in+7in+7in+7in	4×7=28	28	3	1 & 28 2 & 14 4 & 7	6	Composite
7	1in × 7in	7in²	7	7	1	1 & 7	2	prime
1	1 × 1	1	1	1	1	1 & 1	1	neither

TABLE FOR ACTIVITY TWO – SUMMING COUNTING NUMBERS

Stage Number	Number of Tiles Added	Dimensions (Width & Length) of the added (small) rectangle	Area of the added (small) rectangle as a product	Area of the added (small) rectangle as a sum	Dimensions (Width & Length) of the growing (big) rectangle	Area of the growing (big) rectangle as a product	Area of the growing (big) rectangle as a sum	Total Area
2	2	2 * 1	$2 in^2$	$2 in^2$	3in 1in	$3 in^2$	[illegible] $2 in^2$ =	$3 in^2$
3	3	3in * 1in	$3 in^2$	$3 in^2$	3in 2in	$6 in^2$	$1in^2 + 2in^2 + 3in^3$	$6 in^2$
4	4	2in * 2in	$4 in^2$	$4 in^2$	5in 2in	$10 in^2$	$1in^2 + 2in^2 + 3in^2 + 4in^2$ =	$10 in^2$
5	5	5in * 1in	$5 in^2$	$5 in^2$	5in 3in	$15 in^2$	$1in^2 + 2in^2 + 3in^2 + 4in^2 + 5in^2$ =	$15 in^2$
6	6	2in * 3in	$6 in^2$	$6 in^2$	7in 3in	$21 in^2$	"$1in^2$ . . " $+ 6in^2$ =	$21 in^2$
7	7	7in * 1in	$7 in^2$	$7 in^2$	7 4in	$28 in^2$	"$1in^2$ . . . " $7in^2$ =	$28 in^2$
n	n	n odd: nin * 1in n even: 2in * $\frac{n}{2}$in	$n in^2$	$n in^2$	n odd: nin * n even: n-1in * $\frac{n}{2}$in		"$1in^2 + 2in^2 + ... + n in^2$"	

TABLE FOR ACTIVITY THREE – SUMMING ODD INTEGERS

Stage Number	Number of Tiles Added	Area of the added figure as a sum	Dimensions (Width & Length) of the growing (big) rectangle	Area of the growing (big) rectangle as a product	Area of the growing (big) rectangle as a sum	Total Area
2	3	1 in² + 2 in²	2 in × 2 in	4 in²	1 in² + 1 in² + 2 in²	4 in²
3	5	2 in² + 3 in²	3 in × 3 in	9 in²	1 in² + 1 in² + 2 in² + 2 in² + 3 in²	9 in²
4	7	3 in² + 4 in²	4 in × 4 in	16 in²	"1 in² + . . . 3 in²" + 3 in² + 4 in²	16 in²
5	9	4 in² + 5 in²	5 in × 5 in	25 in²	"1 in² + . . . + 4 in²" + 4 in² + 5 in²	25 in²
6	11	5 in² + 6 in²	6 in × 6 in	36 in²	"1 in² + . . . + 5 in²" + 5 in² + 6 in²	36 in²
n	2n-1	(n-1) in² + n in²	n in × n in	n² in²	1 in² + 1 in² + . . . + (n-1) in² + n in²	n² in²

TABLE FOR ACTIVITY FOUR – SUMMING EVEN INTEGERS

Stage Number	Number of Tiles Added	Area of the added figure as a sum	Dimensions (Width & Length) of the growing (big) rectangle	Area of the growing (big) rectangle as a product	Area of the growing (big) rectangle as a sum	Total Area
2	4	2+2 or 3+1	✓	✓	2+4	6
3	6	3+3 or 4+2	✓	✓	2+4+6	12
4	8	4+4 or 5+3	✓	✓	——— +8	20
5	10	5+5 or 6+4	✓	✓	——— +10	30
6	12	6+6 or 7+5	✓	✓		
n	2n	n+n or (n+1)+(n-1)	✓	✓	——— +2n	n(n+1)

TABLE FOR ACTIVITY FIVE – POLYNOMIAL RECTANGLES

Dimensions of the Rectangle	Linear Units	Area of the polynomial rectangle as a product	Areal Units	Areas of the Boxes of the same color as a product	Areas of the Boxes of the same color as a sum	Area of the polynomial rectangle as a sum	Does Sum Equal Product?
Length: x+1 Width: 2x+3	cm	(x+1)(2x+3) ~~2x~~ 5x+3	cm²	2x² + 3x + 2x + 3 (2x · x)+(x·3)+(2·x)+(3·1)	(x²+x²)+(x+x+x)+(x+x)+(1+1+1)	~~[illegible]~~ 2x² + 3x + 2x + 3	(x+1)(2x+3) = 2x²+3x 2x²+5x+3 = 2x²+5x+3 yes
Length: x+1 Width: 2y+3	" "	(x+1)(2y+3)	" "	(2y·y)+(3·y)+(3·2y)+(1·3)	(y²+y²)+(y+y+y)+(y+y+y+y+y+y)+(1+1+....+1)	2y² + 3y + 6y + 9	(x+1)(2y+3) = [illegible] 2y²+3y+6y+9
Length: 2x+y Width: x+2y+1	" "	(2x+y)(x+2y+1)	" "	(2x·x)+(2x·2y)+(2x·1)+(y·x)+(y·2y)+(1y·1)	(x²+x²)+(xy+xy+xy+xy)+(x+x)+(xy)+(y²+y²)+(y)	2x² + 4xy + 2x + xy + 2y² + y	(2x+y)(x+2y+1) = 2x²+5xy+2x+y+2y
Now Backwards: Make a rectangle for the expression x²+5x+6 then factor the expression	" "	(x+2)(x+3)	" "	(x · x)+(x ·3)+(2·x)+(2 · 3)	(x²)+(x+x+x)+(x+x)+(1+1...+1)	x² + 3x + 2x + 6	x²+5x+6 = x²+5x+6 x²+5x+6 = (x+2)(x+
Make a rectangle for the expression 3x²+7xy+2y²+14x+13y +15 then factor the expression	" "	(x+3y+2) · (2x+y+1)	" "	(x · 2x)+(x·y)(x · 1)+(3y·2x)(3y·y)+(3y·1)+(2·2x)+(2·y)+(2·1)		x² + xy + x + 6xy + 3y² + 3y + 4x + 2y + 2	
		(2x+4)·(y+5)		(2x·y)+(2x·5)+(4 ·y)+(4·5)			(2x+4)·(y+5) = 2xy + 10x + 4y + 20

APPENDIX F: GLOSSARY OF TERMINOLOGY

- Additive Type Representational Unit Coordination (RUC): An additive type RUC is used by a person when dealing with the "area as a sum" part of an identity describing growing rectangles made of color cubes or algebra tiles.
- Boxes of the Same Color (or Same−Color−Boxes): These are rectangular subregions of the same color within the polynomial rectangle itself. Though of multiplicative nature, these areal quantities are prone to be interpreted as of pseudo−multiplicative type.
- Cartesian Product: The Cartesian product of two sets A and B is the set of all ordered pairs in which the first component is taken from the first set and the second component is taken from the second set.
- Color Cubes: These are one−inch color cubes students used in the investigation of prime & composite rectangles, along with summed counting, odd, and even numbers.
- Color Tiles: These are different size color tiles students used in the investigation of polynomial rectangles in x and y.
- Composite Rectangle: A rectangle made from a composite number of cubes. Since a composite number has three or more distinct divisors, a composite number of cubes can be arranged into at least two different rectangles.

- Extensive quantity: An extensive quantity is a quantity that can be measured directly. Lengths, areas, volumes, cardinalities, and ordinalities are examples of extensive quantities (Thompson, 1988, p. 164).
- Growing Rectangle: In the context of prime and composite numbers, this represents a rectangle that is generated either by irreducible addends, or by equal addends. In the context of summation identities, it is generated via various types of addends such as symmetric or nonsymmetric L–shape subunits, rectangular subunits, equal addend subunits, and irreducible addend subunits. As for the polynomial expressions, a growing rectangle is generated by algebra tiles via irreducible areal quantities, same–color–box areal quantities, or combined areal quantities.
- Intensive Quantity: An intensive quantity is a quantity that is the ratio of two extensive quantities, which can not be directly measured. Speeds, densities, temperatures, pressures, and pitches are examples of intensive quantities, as is a multiplicative comparison (ratio) of two quantities (Thompson, 1988, p. 164).
- Irreducible Areal Quantities (IAQ): These are the most basic area type units generating subsets of positive integers in the context of color cubes, and combined areal quantities (CAQ) or same−color−box areal quantities (SCBAQ) in the context of polynomial expressions in x and y.
- Irreducible Linear Quantities (ILQ): These are the most basic length type units generating subsets of positive integers (in the context of color cubes) and combined linear units (in the context of polynomial expressions in x and y).

- Measure Space: The set of all possible quantities that can be generated by iterating the measure – unit of measurement – under consideration.
- Multiplication Mat: This is an organizer that helps separate the linear quantities (dimension tiles) from the areal quantities within the rectangular region.
- Multiplicative Type RUC: A multiplicative type RUC is used when dealing with the “area as a product” part of an identity describing growing rectangles made of color cubes or tiles.
- Prime Rectangle: A rectangle made of a prime number of color cubes. A prime number of cubes can be arranged into one unique rectangle (disregarding rotations).
- Pseudo−Multiplicative Type RUC: This is an RUC type somewhere in between additive and multiplicative, which occurred for the “Area of the Boxes of the Same Color as a Product” in dealing with polynomial rectangles made of color tiles. The first term of each “pseudo−product” is a coefficient serving as a counting number indicating how many there are of each irreducible areal quantities (IAQ).
- Quantitative Reasoning: To reason quantitatively is to reason about quantities, their magnitudes, and their relationships with other quantities. Quantitative reasoning is to reason about situations in terms of quantities and quantitative operations (Thompson, 1988, p.164).
- Referent Preserving Composition: Through a referent preserving composition, both the referent and the measurement unit remain unchanged.

- Referent Transforming Composition: Through a referent transforming composition, the referent and the measurement unit as well as the measure space change.
- Relation: A binary relation is any set of ordered pairs. In general, an n–ary relation is a set of ordered n–tuples.
- Relational Notation: Relational notation is used to describe linear units as ordered pairs in ordinary parentheses ( ) and areal units as n–tuples in square brackets [ ].
- Representational Cartesian Product: This is the ordinary Cartesian product defined on representational sets (See the definition of representational sets below).
- Representational Quantities: Quantities arising from generating rectangles via color cubes and tiles in the context of subsets of positive integers (prime, composite, odd, even) as well as summation formulas (e.g., counting numbers, odd and even numbers) along with polynomial expressions in x and y.
- Representational Set of Combined Linear Quantities (RSCLQ): A representational set of combined linear quantities is the set of combined linear quantities (CLQ) such as 5 (5 combined linear ones), $3x$ (3 combined linear "x"s), $2y$ (2 combined linear "y"), in the context of different size color tiles. For instance, the representational sets A and B of combined linear quantities (CLQ) corresponding to the $x + 1$ by $2y + 3$ polynomial rectangle can be defined as $A = \{x, 1\}$ and $B = \{2y, 3\}$.
- Representational Set of Irreducible Areal Quantities (RSIAQ): A representational set of irreducible areal quantities (RSIAQ) is the set of irreducible areal quantities

(IAQ) such as 1, x, y, x^2, y^2, xy in the context of different size color tiles (algebra tiles). The difference between a representational set of irreducible areal quantities (RSIAQ) and an ordinary set is in that in a representational set of irreducible areal quantities, the irreducible areal quantities may appear as an element of the set more than once.

- Representational Set of Irreducible Linear Quantities (RSILQ): A representational set of irreducible linear quantities is the set of irreducible linear quantities such as 1, x, y in the context of different size color tiles. The difference between a representational set of irreducible linear quantities (RSILQ) and an ordinary set is in that in a representational set of irreducible linear quantities (RSILQ), the irreducible linear quantities (ILQ) may appear as an element of the set more than once. For instance, the representational sets A and B of irreducible linear quantities (ILQ) corresponding to the $x + 1$ by $2y + 3$ polynomial rectangle can be defined as $A = \{x, 1\}$ and $B = \{y, y, 1, 1, 1\}$.
- Representational Set of Same–Color–Box Areal Quantities (RSSCBAQ): A representational set of same–color–box areal quantities (RSSCBAQ) is the set of same–color–box areal quantities defined by the "Area of the Boxes of the Same Color as a Product" of the polynomial rectangle resulting from the multiplication of polynomials. The difference between a RSSCBAQ and a RSIAQ is in that repetition of "elements" is not allowed in a RSSCBAQ.
- Representational Unit Coordination (RUC): Representational Unit Coordination can be defined as the different ways of categorizing units arising from the modeling of identities on representational quantities as the "area as a product" and

"area as a sum" of the corresponding special rectangles made of color cubes or tiles.

- Singleton: A set with one element.
- Summed Number: I define a summed number to be the result of the summation of terms in a number sequence. Counting numbers, odd numbers, and even numbers constitute examples of such number sequences. Summed numbers can be represented by a sequence of growing rectangles, each made of a summed number of color cubes.
- Theorems–in–Action: "Mathematical relationships that are taken into account by students when they choose an operation or a sequence of operations to solve a problem. To study children's mathematical behavior it is necessary to express the theorems–in–action in mathematical terms." (Vergnaud, 1988, p. 144).