A Perspective on Conics in the Real Projective Plane

by

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(Under the direction of Theodore Shifrin)

Abstract

In this paper we will introduce the reader to projective geometry and what it means for two figures to be projectively equivalent. This leads to a discussion on the projective generation of conics. Special attention is given to the relationship between the projective group and conics, as well as the concept of projective duality.

Index words: Projective Group, Perspectivities, Projective Plane, Conics

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Dedication

To

my parents David and Susan,

my life partner and best friend Justin,

my friends and teachers.

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I am indebted to my advisor, Ted Shifrin, for his support, guidance, and friendship over the years. In particular, I would like to thank him for introducing me to the beautiful subject of projective geometry. His contributions have been invaluable to my studies and growth as a scholar.

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Chapter 1

Introduction

In the eighteenth century a German philosopher by the name of Immanuel Kant stated that geometry is the study of the properties of the physical space that we live in. However, after Kant's death in 1804, Klein, Gauss, Lobachevski, and Bolyai proved that he could not be more wrong with the development of non-Euclidean geometries. In particular, Klein redefined geometry to consist of sets of objects and relations among them. Then, the relation-preserving mappings form a group of automorphisms.

In this paper we will start by giving an introduction of a special type of non-Euclidean geometry called projective geometry. In this geometry, we study projections and properties that are invariant under a projection. We shall start our investigation of projective geometry with an algebraic approach. In particular, we will be working in $\mathbb{P}^1$ and $\mathbb{P}^2$. In the treatment of projective geometry we include some classic theorems as well as the concept of projective duality. The theorem which says any projective transformation is the composition of at most two perspectivities was motivated by exploration in Geometer's Sketchpad. I used Geometer's Sketchpad to prove this theorem by construction. The main reference for this chapter is *Abstract Algebra: A Geometric Approach* by Ted Shifrin.

In the third chapter we direct the focus to projective conics and show the projective generation of conics. We describe the relationship between conics and projective transformations. Along with reading and understanding the proofs, I did great deal of exploration on conics in Geometer's Sketchpad. Geometer's Sketchpad files are available upon request by sending me an email at joebrown83@gmail.com. We close this chapter with a treatment of the dual conic and the question of classifying conics based on the geometry of the projective transformation. Main references for this chapter are *Abstract Algebra: A Geometric Approach* by Ted Shifrin, and *Geometry: A Comprehensive Course* by Dan Pedoe.

Chapter 2

Preliminaries

2.1 Introduction to projective n-space

Definition 2.1.1. We define projective n-space, denoted $\mathbb{P}^n$, to be the set of all lines through the origin in $\mathbb{R}^{n+1}$. That is, if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+1} - \{\mathbf{0}\}$ then we say that $\mathbf{x} \equiv \mathbf{y}$ if $\mathbf{y} = c\mathbf{x}$ for some $c \in \mathbb{R} - \{0\}$. Then $\mathbb{P}^n$ is the set of all $\equiv$ equivalence classes. The topological structure of $\mathbb{P}^n$ is the quotient topology coming from $(\mathbb{R}^{n+1} - \{0\})/\equiv$.

We check that $\equiv$ is an equivalence relation.

1. $\mathbf{x} = 1 \cdot \mathbf{x}$, so $\mathbf{x} \equiv \mathbf{x}$. Thus, $\equiv$ is reflexive.

2. Suppose $\mathbf{x} \equiv \mathbf{y}$. Then there is some $c \in \mathbb{R} - \{0\}$ such that $\mathbf{x} = c\mathbf{y}$. So $(1/c)\mathbf{x} = \mathbf{y}$, with $1/c \in \mathbb{R} - \{0\}$. Hence $\mathbf{y} \equiv \mathbf{x}$ and so $\equiv$ is symmetric.

3. Suppose $\mathbf{x} \equiv \mathbf{y}$ and $\mathbf{y} \equiv \mathbf{z}$. Then there are $c, d \in \mathbb{R} - \{0\}$ such that $\mathbf{x} = c\mathbf{y}$ and $\mathbf{y} = d\mathbf{z}$. So $\mathbf{x} = c(d\mathbf{z}) = (cd)\mathbf{z}$, with $cd \in \mathbb{R} - \{0\}$. Therefore, $\mathbf{x} \equiv \mathbf{z}$ and so $\equiv$ is transitive.

In the case of $n = 1$, we consider the set of all lines that pass through the origin in $\mathbb{R}^2$. Every line that passes through the origin can be identified by another point (a, b) lying on the line. Therefore, we describe any element of $\mathbb{P}^1$ by an equivalence class $[(a, b)]$ or $[a, b]$ for short. Notice that for $a \neq 0$, b/a is the slope of the line in $\mathbb{R}^2$. See Figure 2.1.

Example 2.1.1. Consider the line $\ell \subset \mathbb{R}^2$ given by the equation $4x + 2y = 0$. The slope of this line is $-4/2 = -2$, and so this corresponds to the element $[2, -4] \equiv [1, -2] \in \mathbb{P}^1$.

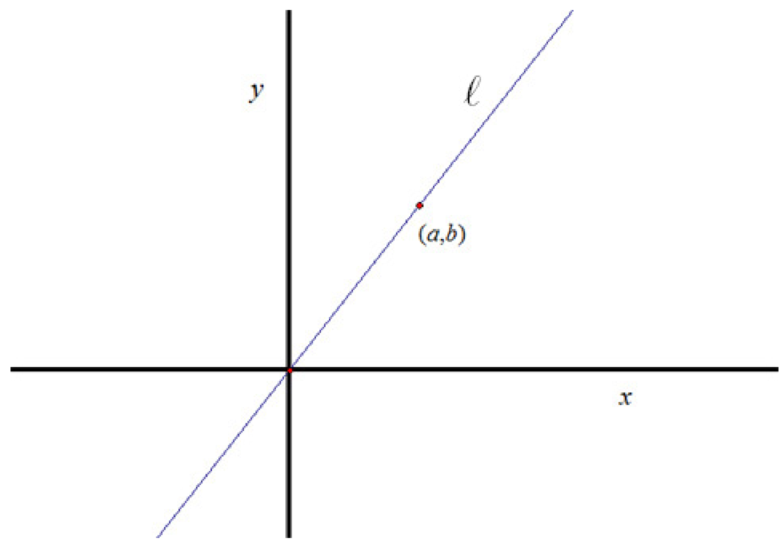


Figure 2.1: Any line passing through the origin is identified by a second point (a, b) lying on the line.

Example 2.1.2. Consider the line $m \subset \mathbb{R}^2$ given by the equation $x = 0$. This line has undefined or infinite slope and $(0, 1)$ is a point on it. Therefore the line m is the element $[0, 1] \in \mathbb{P}^1$. We refer to this as the point at infinity.

The slope of any line through the origin is either a real number or infinite. So we have the one-to-one correspondence between $\mathbb{P}^1$ and $\mathbb{R}^1 \cup \{\infty\}$. We remark here that often it will be convenient to think of $\mathbb{P}^1$ as $\mathbb{R} \cup \{\infty\}$. Now we will briefly discuss the topology of $\mathbb{P}^1$.

For each element $[x_0, x_1] \in \mathbb{P}^1$, consider the unit vector $\mathbf{u} = \dfrac{(x_0, x_1)}{\sqrt{x_0^2 + x_1^2}} \in \mathbb{R}^2$. We have a two-to-one correspondence between points on the unit circle $S^1 = \{(x_0, x_1) : x_0^2 + x_1^2 = 1\}$ and $\mathbb{P}^1$, because vectors $\mathbf{u}$ and $-\mathbf{u}$ represent the same point in $\mathbb{P}^1$. Since each point in the lower semicircle is identified with a point in the upper semicircle, then we only consider the upper semicircle and its endpoints. Note, we still have these endpoints identified with each

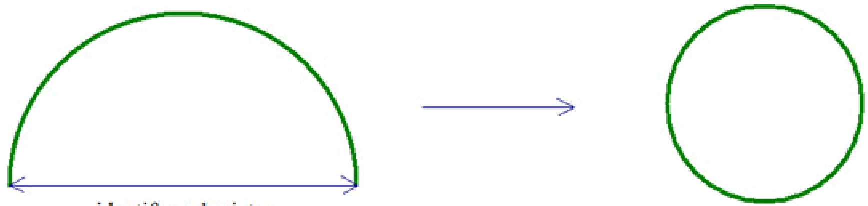


Figure 2.2: $\mathbb{P}^1$ is homeomorphic to the unit circle.

other. If we "glue" these points together, then our semicircle becomes a circle (see Figure 2.2). Therefore $\mathbb{P}^1$ is homeomorphic to the unit circle S^1.

Another way to see that $\mathbb{P}^1$ is homeomorphic to S^1 is by a stereographic projection. This idea is the same as the stereographic projection used to see that the Riemann sphere is the one-point compactification of the complex plane. Consider the unit circle along with the x-axis (a copy of $\mathbb{R}$) in $\mathbb{R}^2$. Let X be the point $(0,1)$. Then any line passing through X that is not horizontal passes through S^1 at P and then meets $\mathbb{R}$ at Q; see Figure 2.3. The mapping $\phi : S^1 - \{X\} \to \mathbb{R}$ which sends P to Q is a homeomorphism. If we extend this by defining $\phi : X \mapsto \infty$, then ϕ is a homeomorphism that maps S^1 to $\mathbb{R} \cup \{\infty\} = \mathbb{P}^1$.

Our next goal is to understand transformations of the projective line to itself, $T : \mathbb{P}^1 \to \mathbb{P}^1$. Since we defined $\mathbb{P}^1$ to be the set of lines through the origin in $\mathbb{R}^2$, we will look to $GL(2,\mathbb{R})$, the group of invertible linear maps from $\mathbb{R}^2$ to itself, to give us motions of $\mathbb{P}^1$. Note that for every $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{P}^1$ and $\lambda \in \mathbb{R} - \{0\}$, $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} \equiv \begin{bmatrix} x \\ y \end{bmatrix}$. Thus, the matrices $A \in \{\lambda \text{ Id} : \lambda \in \mathbb{R} - \{0\}\}$ act as the identity map for points in the projective

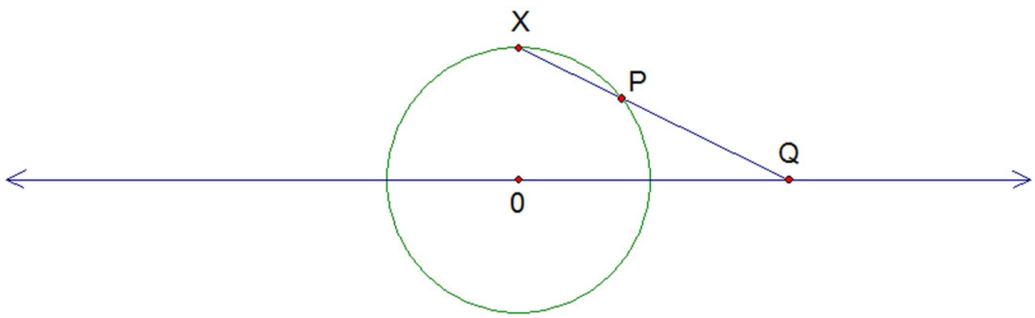


Figure 2.3: Stereographic projection

line. It is easy to check that $\{\lambda \text{ Id} : \lambda \in \mathbb{R} - \{0\}\}$ is the center of $GL(2, \mathbb{R})$. Therefore we can form the quotient group,

$$\text{Proj}(1) = GL(2, \mathbb{R})/\{\lambda \text{ Id} : \lambda \in \mathbb{R} - \{0\}\},$$

which we shall call the **projective group**. If $T \in \text{Proj}(1)$, then we say that T is a **projective transformation**.

Suppose we are given a point $[x_0, x_1] \in \mathbb{P}^1$, with $x_0 \neq 0$. Then $[x_0, x_1] = [1, x_1/x_0]$. By letting $x = \dfrac{x_1}{x_0}$, our point is now $[1, x]$. Suppose $\begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix} \in \text{Proj}(1)$. Then $\begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} \delta + \gamma x \\ \beta + \alpha x \end{bmatrix} \equiv \begin{bmatrix} 1 \\ \frac{\alpha x + \beta}{\gamma x + \delta} \end{bmatrix}$. Therefore, we can write the transformation as the function

$$f(x) = \frac{\alpha x + \beta}{\gamma x + \delta}.$$

Now that we have this explicit formula for T we will go back and consider the case where $x_0 = 0$. On one hand, $\begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \gamma \\ \alpha \end{bmatrix} \equiv \begin{bmatrix} 1 \\ \alpha/\gamma \end{bmatrix}$. On the other, recall that the point $[0, 1]$ represents the point at infinity, which is $\lim_{x \to \infty} [1, x]$. So it makes sense to evaluate

f at the point at infinity by considering $\lim_{x\to\infty} f(x)$. That is, $f(\infty) = \lim_{x\to\infty} \frac{\alpha x+\beta}{\gamma x+\delta} = \frac{\alpha}{\gamma}$. Also, note that $\begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} \gamma \\ -\delta \end{bmatrix} = \begin{bmatrix} 0 \\ \beta\gamma - \alpha\delta \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and so it makes sense to evaluate $f(-\delta/\gamma)$ by considering $\lim_{x\to -\delta/\gamma} \frac{\alpha x+\beta}{\gamma x+\delta} = \infty$.

The rational function $f(x)$ which represents a projective transformation is also known as a **linear fractional transformation** or a **Möbius transformation**.

Lemma 2.1.1. *Let P_1, P_2, P_3 and Q_1, Q_2, Q_3 be two trios of points in $\mathbb{P}^1$. Then there exists a unique projective transformation $T \in \mathrm{Proj}(1)$ such that $T(P_1) = Q_1$, $T(P_2) = Q_2$, and $T(P_3) = Q_3$.*

Proof. Suppose $Q_1 = [\alpha_0, \alpha_1] = [\mathbf{q}_1]$, $Q_2 = [\beta_0, \beta_1] = [\mathbf{q}_2]$, and $Q_3 = [\gamma_0, \gamma_1] = [\mathbf{q}_3]$ are given and let $R_1 = [1, 0] = [\mathbf{r}_1]$, $R_2 = [0, 1] = [\mathbf{r}_2]$, and $R_3 = [1, 1] = [\mathbf{r}_3]$. Since Q_1 and Q_2 are distinct points, then (α_0, α_1) and (β_0, β_1) are linearly independent. So the matrix $\begin{bmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \end{bmatrix}$ is nonsingular. Hence, the system

$$\begin{bmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} \gamma_0 \\ \gamma_1 \end{bmatrix}$$

has the unique (nonzero) solution (λ, μ). Let $\mathcal{T} \in GL(2, \mathbb{R})$ be defined by

$$\mathcal{T} = \begin{bmatrix} \lambda\alpha_0 & \mu\beta_0 \\ \lambda\alpha_1 & \mu\beta_1 \end{bmatrix}.$$

Then we compute $\mathcal{T}\mathbf{r}_i$ for $i = 1, 2, 3$.

$$\mathcal{T}\mathbf{r}_1 = \begin{bmatrix} \lambda\alpha_0 & \mu\beta_0 \\ \lambda\alpha_1 & \mu\beta_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \lambda\mathbf{q}_1, \text{ and } [\lambda\mathbf{q}_1] = [\mathbf{q}_1] = Q_1.$$

$$\mathcal{T}\mathbf{r}_2 = \begin{bmatrix} \lambda\alpha_0 & \mu\beta_0 \\ \lambda\alpha_1 & \mu\beta_1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mu \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \mu\mathbf{q}_2, \text{ and } [\mu\mathbf{q}_2] = [\mathbf{q}_2] = Q_2.$$

$$\mathcal{T}\mathbf{r}_3 = \begin{bmatrix} \lambda\alpha_0 & \mu\beta_0 \\ \lambda\alpha_1 & \mu\beta_1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda\alpha_0 + \mu\beta_0 \\ \lambda\alpha_1 + \mu\beta_1 \end{bmatrix} = \begin{bmatrix} \gamma_0 \\ \gamma_1 \end{bmatrix} = \mathbf{q}_3, \text{ and } [\mathbf{q}_3] = Q_3.$$

Furthermore, we note that $\mathcal{T}$ is unique up to scalar multiples, and so it will correspond to a unique element $T \in \text{Proj}(1)$. We do the same process as above to find the unique element $T_2 \in \text{Proj}(1)$, such that $T_2(R_1) = P_1$, $T_2(R_2) = P_2$, and $T_2(R_3) = P_3$. Then $T = T_1 \circ T_2^{-1}$ is the unique element of $\text{Proj}(1)$ that carries P_i to Q_i, for $i = 1, 2, 3$. □

Remark. Each element of $GL(2, \mathbb{R})$ has 4 parameters and thus 4 degrees of freedom. However, when we consider Proj(1), the parameters are determined up to a scalar multiple. Thus, each element of Proj(1) has 3 degrees of freedom. Note that in the previous proof, each time we specified where a point was sent, we used up one degree of freedom.

Let us continue our investigation by examining the projective plane $\mathbb{P}^2$. Our definition states that $\mathbb{P}^2$ is the set of all lines through the origin in $\mathbb{R}^3$. Just as with $\mathbb{P}^1$, $\mathbf{x} \equiv \mathbf{y}$ if and only if there is some $c \in \mathbb{R} - \{0\}$ such that $\mathbf{y} = c\mathbf{x}$. Thus $\mathbb{P}^2$ is the set of equivalence classes $[(x_0, x_1, x_2)]$ or $[x_0, x_1, x_2]$, where x_0, x_1, and x_2 are not all zero.

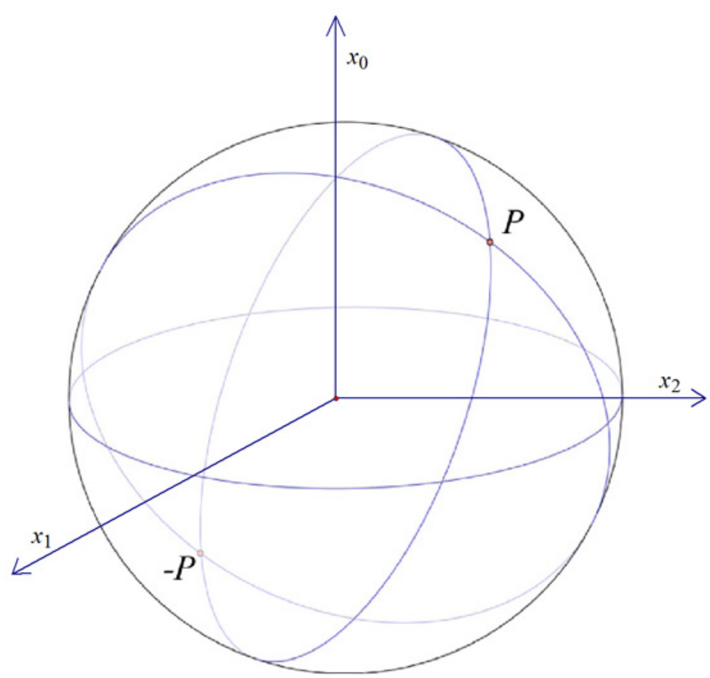


Figure 2.4: Antipodal points of a sphere are identified

Suppose that $[x_0, x_1, x_2] \in \mathbb{P}^2$, such that $x_0 \neq 0$. So, $[x_0, x_1, x_2] = \left[1, \frac{x_1}{x_0}, \frac{x_2}{x_0}\right]$. Letting $\frac{x_1}{x_0} = x$ and $\frac{x_2}{x_0} = y$, we have $[1, x, y]$. These elements are identified by the ordered pairs $(x, y) \in \mathbb{R}^2$. When $x_0 = 0$ then we have $[0, x_1, x_2]$. There is a one-to-one correspondence between these elements and the equivalence classes $[x_1, x_2] \in \mathbb{P}^1$. As before, we think of points with $x_0 = 0$ as being "at infinity," and denote the set of such points by $\mathbb{P}^1_\infty$. Therefore we have a one-to-one correspondence between $\mathbb{P}^2$ and $\mathbb{R}^2 \cup \mathbb{P}^1_\infty$.

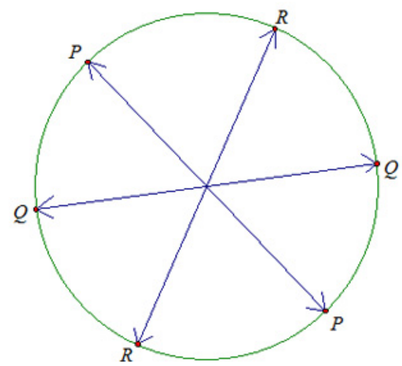


Figure 2.5: $\mathbb{D}$ with antipodal points on the boundary identified.

Now we consider the topology of $\mathbb{P}^2$. Since $\mathbb{P}^2$ is the set of lines through the origin in $\mathbb{R}^3$, then for each element $[x_0, x_1, x_2] \in \mathbb{P}^2$ we consider a unit vector $\mathbf{u} = \frac{(x_0, x_1, x_2)}{\sqrt{x_0^2 + x_1^2 + x_2^2}} \in \mathbb{R}^3$. So then, each point in $\mathbb{P}^2$ is identified by two points $\mathbf{u}$ and $-\mathbf{u}$ on the unit sphere S^2. Hence, we obtain a two-to-one correspondence between S^2 and $\mathbb{P}^2$ by identifying antipodal points of S^2; see Figure 2.4. Notice that every point in the southern hemisphere is identified with a single point in the northern hemisphere. So then we need only to consider the northern hemisphere and the equator. We still have the identification of antipodal points on the equator. We may map each point of the northern hemisphere to points in the interior of the unit disk $\mathbb{D} = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$ by projecting onto the x_1x_2-plane. That is, $(x_0, x_1, x_2) \mapsto (0, x_1, x_2)$. So now we see that $\mathbb{P}^2$ is homeomorphic to a disk with opposite points on the boundary identified; see Figure 2.5.

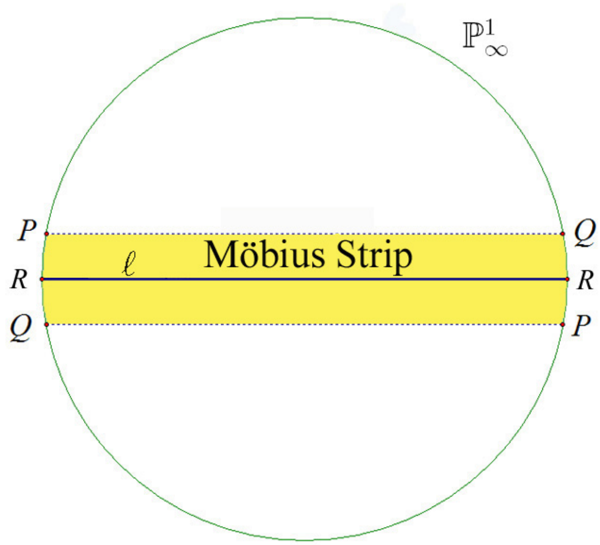


Figure 2.6: $\mathbb{P}^2$ is a Möbius strip with a disk attached to its boundary

Finally, consider a line $\ell \subset \mathbb{D}$ passing through the center and intersecting the boundary at R. A neighborhood of this line forms a Möbius strip, while the rest becomes an open disk because the boundary is identified. Therefore the projective plane is topologically equvalent to a Möbius strip with a disk attached to its boundary (see Figure 2.6).

Definition 2.1.2. Suppose $P = [\mathbf{p}]$ and $Q = [\mathbf{q}]$ are distinct points in $\mathbb{P}^2$, so $\mathbf{p}, \mathbf{q} \in \mathbb{R}^3 - \{0\}$ are linearly independent. Then we define the **projective line** determined by P and Q to be the set of points in $\mathbb{P}^2$ whose corresponding lines in $\mathbb{R}^3$ lie in the plane spanned by $\mathbf{p}$ and $\mathbf{q}$; see Figure 2.7. That is, if $P = [p_0, p_1, p_2]$ and $Q = [q_0, q_1, q_2]$, then

$$\overleftrightarrow{PQ} = \{[s(p_0, p_1, p_2) + t(q_0, q_1, q_2)] : s, t \in \mathbb{R} \text{ not both zero}\}.$$

Remark. Any projective line $\overleftrightarrow{PQ} \subset \mathbb{P}^2$ is a subspace homeomorphic to $\mathbb{P}^1$. This is because we move along $\overleftrightarrow{PQ}$ by allowing $s, t \in \mathbb{R}$ to vary. Thus, any point on $\overleftrightarrow{PQ}$ is given by equivalence classes $[s, t]$.

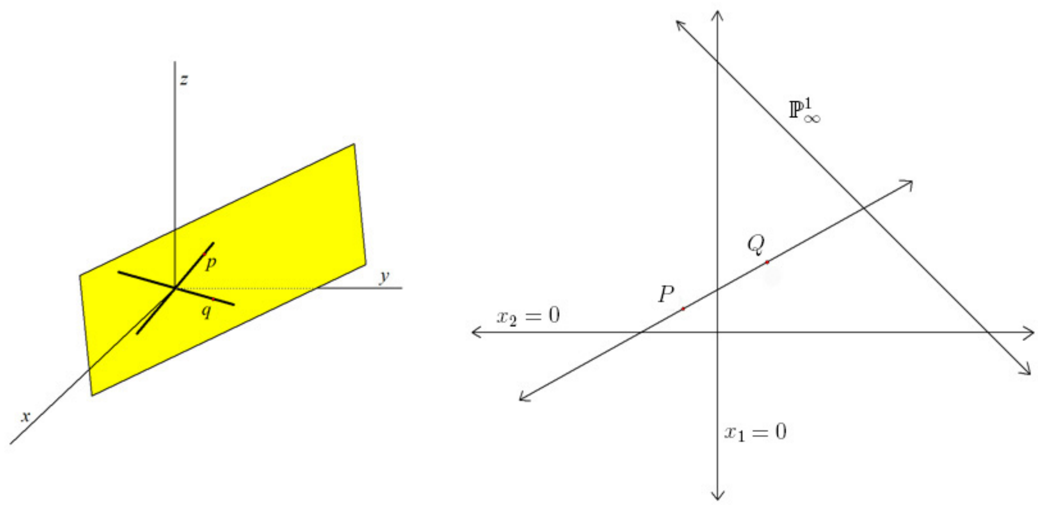


Figure 2.7: Plane in $\mathbb{R}^3$ spanned by $\mathbf{p}$ and $\mathbf{q}$

On the other hand, since any plane through the origin in $\mathbb{R}^3$ can also be given by its normal vector, we have the following lemma.

Lemma 2.1.2. *Every line in* $\mathbb{P}^2$ *is the locus of points* $[x_0, x_1, x_2]$ *satisfying the equation*

$$a_0x_0 + a_1x_1 + a_2x_2 = 0$$

for some $(a_0, a_1, a_2) \neq (0, 0, 0)$.

Proof. Suppose $P = [\mathbf{p}]$ and $Q = [\mathbf{q}]$ are distinct points in $\mathbb{P}^2$. So, $\mathbf{p}$ and $\mathbf{q}$ are linearly independent. Let $\mathbf{a} = (a_0, a_1, a_2) = \mathbf{p} \times \mathbf{q} \neq \mathbf{0}$. Then $\mathbf{a} \cdot \mathbf{p} = 0$ and $\mathbf{a} \cdot \mathbf{q} = 0$. So if $[x_0, x_1, x_2] \in \overleftrightarrow{PQ}$, then $\mathbf{x}$ is some linear combination of $\mathbf{p}$ and $\mathbf{q}$. That is, $\mathbf{x} = \lambda\mathbf{p} + \mu\mathbf{q}$ for some $\lambda, \mu \in \mathbb{R}$ both not zero. Since $\mathbf{x}$ is in the plane spanned by $\mathbf{p}$ and $\mathbf{q}$, it is orthogonal to $\mathbf{a} = \mathbf{p} \times \mathbf{q}$ and therefore it satisfies the homogeneous linear equation $a_0x_0 + a_1x_1 + a_2x_2 = 0$.

Conversely, suppose we are given some $\mathbf{a} = (a_0, a_1, a_2) \neq \mathbf{0}$. Then we can find linearly independent vectors $\mathbf{p} = (p_0, p_1, p_2)$ and $\mathbf{q} = (q_0, q_1, q_2)$ such that $\mathbf{a} \cdot \mathbf{p} = \mathbf{a} \cdot \mathbf{q} = 0$. By letting $P = [\mathbf{p}]$ and $Q = [\mathbf{q}]$, then $\overleftrightarrow{PQ} = \{[\mathbf{x}] : \mathbf{a} \cdot \mathbf{x} = 0\}$. □

Example 2.1.3. Consider points $P = [1, 2, 3]$ and $Q = [1, 0, 1]$. We wish to find $\mathbf{a} = (a_0, a_1, a_2) \neq (0, 0, 0)$, such that the line passing through P and Q satisfies the homogeneous linear equation $a_0x_0 + a_1x_1 + a_2x_2 = 0$.

Let $\mathbf{p} = (1, 2, 3)$ and $\mathbf{q} = (1, 0, 1)$. Then $\mathbf{p} \times \mathbf{q} = (2, 2, -2)$. So take $\mathbf{a} = (1, 1, -1)$. Note that $\mathbf{a} \cdot \mathbf{p} = \mathbf{a} \cdot \mathbf{q} = 0$. Therefore the equation that describes $\overleftrightarrow{PQ}$ is

$$x_0 + x_1 - x_2 = 0.$$

On the other hand, if $\mathbf{a} = (-1, 0, 3)$, then we let $P = [3, 0, 1]$, and $Q = [0, 1, 0] \in \mathbb{P}^2$. Then every point $[\mathbf{x}] \in \overleftrightarrow{PQ}$ satisfies the equation

$$\mathbf{a} \cdot \mathbf{x} = -x_0 + 3x_2 = 0.$$

Remark. If we think of $\mathbb{P}^2$ as $\mathbb{R}^2 \cup \mathbb{P}^1_\infty$, then we can get a Euclidean picture of projective lines. Consider the projective line $\ell = \{[x_0, x_1, x_2] : a_0x_0 + a_1x_1 + a_2x_2 = 0\}$. When $x_0 \neq 0$, then $[x_0, x_1, x_2] \equiv [1, \frac{x_1}{x_0}, \frac{x_2}{x_0}]$. Let $x = x_1/x_0$ and $y = x_2/x_0$. So now we can rewrite the equation of the line

$$\begin{gathered} a_0x_0 + a_1x_1 + a_2x_2 = 0 \\ y = \left(\frac{-a_1}{a_2}\right) x - \frac{a_0}{a_2}. \end{gathered}$$

Therefore, any projective line that is not at infinity can be written and understood in the familiar Euclidean setting. Conversely, each line in $\mathbb{R}^2$ brings about a projective line.

Example 2.1.4. Let $P = [1, 1, 1]$ and $Q = [1, 0, 2]$ be points in $\mathbb{P}^2$. Then the (projective) line $\overleftrightarrow{PQ} = \{[\mathbf{x}] : 2x_0 - x_1 - x_2 = 0\}$. According to the prescription above let us rewrite the equation.

$$2x_0 - x_1 - x_2 = 0$$

$$y = 2 - x.$$

Therefore we can picture the projective line $\overleftrightarrow{PQ}$ as the Euclidean line $y = 2 - x$ with an extra point at infinity.

Theorem 2.1.3. *Two (projective) lines in $\mathbb{P}^2$ have exactly one point of intersection.*

Proof. Let $\ell_1 = \{[\mathbf{x}] : \mathbf{a}\cdot\mathbf{x} = 0\}$ and $\ell_2 = \{[\mathbf{x}] : \mathbf{b}\cdot\mathbf{x} = 0\}$. The vectors $\mathbf{a}$ and $\mathbf{b}$ are unique up to scalar multiplication and they are linearly independent if and only if ℓ_1 and ℓ_2 are distinct. Consider the linear system of equations $\mathbf{a}\cdot\mathbf{x} = \mathbf{b}\cdot\mathbf{x} = 0$. The solution to this system is the subspace of $\mathbb{R}^3$ spanned by $\mathbf{a}\times\mathbf{b}$. Therefore $[\mathbf{x}] = [\mathbf{a}\times\mathbf{b}]$ is the unique point of $\mathbb{P}^2$ such that $[\mathbf{x}] \in \ell_1$ and $[\mathbf{x}] \in \ell_2$. $\square$

Example 2.1.5. Consider the lines

$$\ell_1 = \{[\mathbf{x}] : (2,-1,4)\cdot\mathbf{x} = 0\} \text{ and } \ell_2 = \{[\mathbf{x}] : (5,0,-2)\cdot\mathbf{x} = 0\}.$$

We want to find their point of intersection X.

First note that $\mathbf{a} = (2,-1,4)$ and $\mathbf{b} = (5,0,-2)$ are linearly independent. Hence ℓ_1 and ℓ_2 are distinct. Now we consider the linear system $2x_0 - x_1 + 4x_2 = 5x_0 - 2x_2 = 0$. Solutions to this system are spanned by the vector $\mathbf{x} = \mathbf{a}\times\mathbf{b} = (2,24,5)$. Since this line is given by scalar multiples of $(2,24,5)$, it corresponds to the unique point $X = [2,24,5] \in \mathbb{P}^2$.

Example 2.1.6. Let $P = [1,0,2], Q = [1,3,0], R = [1,-1,-2]$, and $S = [1,1,3] \in \mathbb{P}^2$. Let us find the point $X = \overleftrightarrow{PQ} \cap \overleftrightarrow{RS}$, and show that X lies on the line

$$\ell = \{-50x_0 + 38x_1 + 19x_2 = 0\} \subset \mathbb{P}^2.$$

First, let us find $\mathbf{a} = (a_0, a_1, a_2)$ so that $\overleftrightarrow{PQ}$ is the locus of points $[x_0, x_1, x_2]$ satisfying $a_0x_0 + a_1x_1 + a_2x_2 = 0$. Let $\mathbf{p} = (1,0,2)$ and $\mathbf{q} = (1,3,0)$. Then $\mathbf{a} = \mathbf{p}\times\mathbf{q} = (-6,2,3)$. So,

$$\overleftrightarrow{PQ} = \{[\mathbf{x}] : -6x_0 + 2x_1 + 3x_2 = 0\}.$$

We do the same for $\overleftrightarrow{RS}$. That is, find some $\mathbf{b} = (b_0, b_1, b_2)$, such that $\overleftrightarrow{RS}$ is the locus of points $[x_0, x_1, x_2]$ satisfying $b_0x_0 + b_1x_1 + b_2x_2 = 0$. Let $\mathbf{r} = (1, -1, -2)$ and $\mathbf{s} = (1, 1, 3)$. Then take $\mathbf{b} = \mathbf{r} \times \mathbf{s} = (-1, -5, 2)$. So,

$$\overleftrightarrow{RS} = \{[\mathbf{x}] : -x_0 - 5x_1 + 2x_2 = 0\}.$$

Now to find the point $X = \overleftrightarrow{PQ} \cap \overleftrightarrow{RS}$ we shall consider the linear system

$$-6x_0 + 2x_1 + 3x_2 = -x_0 - 5x_1 + 2x_2 = 0.$$

Solutions to this system are spanned by the vector $\mathbf{a} \times \mathbf{b} = (19, 9, 32)$. Therefore, $X = [19, 9, -32]$ is the unique point of $\mathbb{P}^2$ such that $X = \overleftrightarrow{PQ} \cap \overleftrightarrow{RS}$. To show that X lies on the line $\ell = \{-50x_0 + 38x_1 + 19x_2 = 0\}$, we simply check that the corresponding vector $\mathbf{x} = (19, 9, 32)$ satisfies the homogeneous equation

$$-50x_0 + 38x_1 + 19x_2 = 0.$$

So when $\mathbf{x} = (19, 9, 32)$, we have $-50(19) + 38(9) + 19(32) = -950 + 950 = 0$, as desired.

We will revisit this example in Section 2.4 when we discuss projective duality.

Remark. The lines $\ell_1 : a_0 + a_1x + a_2y = 0$ and $\ell_2 : \tilde{a}_0 + a_1x + a_2y = 0$ are parallel in $\mathbb{R}^2$, and therefore they intersect at the point $[0, -a_2, a_1]$, which is a point at infinity (see Figure 2.8). Moreover, this setup allows us to see that the line at infinity is the set of points at infinity for each direction of line in $\mathbb{R}^2$. This line at infinity is given by $\mathbb{P}^1_\infty = \{x_0 = 0\}$.

Now that we have a feel for $\mathbb{P}^2$, we shall define the group of motions of $\mathbb{P}^2$ as we did for $\mathbb{P}^1$. Since $\mathbb{P}^2$ is defined to be the set of lines that pass through the origin in $\mathbb{R}^3$, we look to $GL(3, \mathbb{R})$ to provide us with transformations from $\mathbb{P}^2$ to itself.

Definition 2.1.3. Let $GL(3, \mathbb{R})$ be the group of invertible 3×3 matrices. Again, the subgroup $\{\lambda \text{ Id} : \lambda \in \mathbb{R} - \{0\}\}$ is the center of $GL(3, \mathbb{R})$ and so we are allowed to form the quotient group

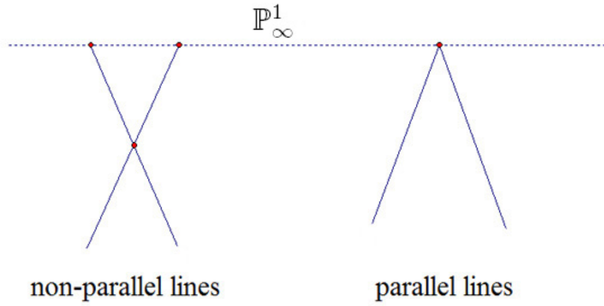


Figure 2.8: Parallel lines intersect at the line at infinity $\mathbb{P}^1_\infty$.

$$\mathrm{Proj}(2) = GL(3, \mathbb{R})/\{\lambda \ \mathrm{Id} : \lambda \in \mathbb{R} - \{0\}\}.$$

We call Proj(2) the group of **projective transformations** on $\mathbb{P}^2$.

Definition 2.1.4. If three or more points of $\mathbb{P}^2$ lie on the same line, then we say that they are **collinear**.

Lemma 2.1.4. *Let $T \in \mathrm{Proj}(2)$ be a projective transformation of $\mathbb{P}^2$, and let X, Y, and Z be collinear points in $\mathbb{P}^2$. Then $T(X), T(Y)$, and $T(Z)$ are collinear.*

Proof. First write $X = [\mathbf{x}]$, $Y = [\mathbf{y}]$, and $Z = [\mathbf{z}]$. Then $\mathbf{x} = (x_0, x_1, x_2)$, $\mathbf{y} = (y_0, y_1, y_2)$, and $\mathbf{z} = (z_0, z_1, z_2)$ are corresponding vectors of $\mathbb{R}^3$. Since X, Y, and Z are collinear, it follows that $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ are linearly dependent, while $\mathbf{x}$, and $\mathbf{y}$ are linearly independent. In other words, there exist $s, t \in \mathbb{R}$ such that $\mathbf{z} = s\mathbf{x} + t\mathbf{y}$. Since $T \in \mathrm{Proj}(2) = GL(3, \mathbb{R})/\{\lambda \mathrm{Id} : \lambda \in \mathbb{R} - \{0\}\}$, we can find some linear map $\mathcal{T} \in GL(3, \mathbb{R})$ which represents T. Then $\mathcal{T}(\mathbf{z}) = \mathcal{T}(s\mathbf{x} + t\mathbf{y}) = s\mathcal{T}(\mathbf{x}) + t\mathcal{T}(\mathbf{y})$. So, the vectors $\mathcal{T}(\mathbf{x}), \mathcal{T}(\mathbf{y})$, and $\mathcal{T}(\mathbf{z})$ which give rise to $T(X), T(Y)$, and $T(Z)$ respectively, are linearly dependent. Hence, $T(X), T(Y)$, and $T(Z)$ are collinear. $\square$

Definition 2.1.5. If three points of $\mathbb{P}^2$ are not collinear, we say they are **in general position**. We say that four points of $\mathbb{P}^2$ are **in general position** if no three of them are collinear.

Theorem 2.1.5. *(The fundamental theorem of projective geometry). Let P_1, P_2, P_3, P_4 and Q_1, Q_2, Q_3, Q_4 be two quartets of points in general position in $\mathbb{P}^2$. Then there exists a unique projective transformation $T \in \mathrm{Proj}(2)$ such that $T(P_1) = Q_1$, $T(P_2) = Q_2$, $T(P_3) = Q_3$, and $T(P_4) = Q_4$.*

Proof. Let $R_1 = [\mathbf{r}_1]$, $R_2 = [\mathbf{r}_2]$, $R_3 = [\mathbf{r}_3]$, and $R_4 = [\mathbf{r}_4]$, where $\mathbf{r}_1 = (1, 0, 0)$, $\mathbf{r}_2 = (0, 1, 0)$, $\mathbf{r}_3 = (0, 0, 1)$, and $\mathbf{r}_4 = (1, 1, 1)$ are representing vectors in $\mathbb{R}^3$. Let $Q_1 = [\mathbf{q}_1]$, $Q_2 = [\mathbf{q}_2]$, $Q_3 = [\mathbf{q}_3]$, and $Q_4 = [\mathbf{q}_4]$, where $\mathbf{q}_1 = (\alpha_0, \alpha_1, \alpha_2)$, $\mathbf{q}_2 = (\beta_0, \beta_1, \beta_2)$, $\mathbf{q}_3 = (\gamma_0, \gamma_1, \gamma_2)$, and $\mathbf{q}_4 = (\delta_0, \delta_1, \delta_2)$ are representing vectors in $\mathbb{R}^3$. Since Q_1, Q_2, and Q_3 are not collinear, $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ are linearly independent. Therefore, the matrix $\begin{bmatrix} \alpha_0 & \beta_0 & \gamma_0 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{bmatrix}$ is invertible. So, the system

$$\begin{bmatrix} \alpha_0 & \beta_0 & \gamma_0 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \\ \nu \end{bmatrix} = \begin{bmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \end{bmatrix}$$

has a unique (nonzero) solution (λ, μ, ν). Let $\mathcal{T} \in GL(3, \mathbb{R})$ be defined by

$$\mathcal{T} = \begin{bmatrix} \lambda\alpha_0 & \mu\beta_0 & \nu\gamma_0 \\ \lambda\alpha_1 & \mu\beta_1 & \nu\gamma_1 \\ \lambda\alpha_2 & \mu\beta_2 & \nu\gamma_2 \end{bmatrix}.$$

Then we compute $\mathcal{T}\mathbf{r}_i$ for $i = 1, 2, 3, 4$.

$$\mathcal{T}\mathbf{r}_1 = \begin{bmatrix} \lambda\alpha_0 & \mu\beta_0 & \nu\gamma_0 \\ \lambda\alpha_1 & \mu\beta_1 & \nu\gamma_1 \\ \lambda\alpha_2 & \mu\beta_2 & \nu\gamma_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \lambda\mathbf{q}_1, \text{ and } [\lambda\mathbf{q}_1] = [\mathbf{q}_1] = Q_1.$$

$$\mathcal{T}\mathbf{r}_2 = \begin{bmatrix} \lambda\alpha_0 & \mu\beta_0 & \nu\gamma_0 \\ \lambda\alpha_1 & \mu\beta_1 & \nu\gamma_1 \\ \lambda\alpha_2 & \mu\beta_2 & \nu\gamma_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mu \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \mu\mathbf{q}_2, \text{ and } [\mu\mathbf{q}_2] = [\mathbf{q}_1] = Q_2.$$

$$\mathcal{T}\mathbf{r}_3 = \begin{bmatrix} \lambda\alpha_0 & \mu\beta_0 & \nu\gamma_0 \\ \lambda\alpha_1 & \mu\beta_1 & \nu\gamma_1 \\ \lambda\alpha_2 & \mu\beta_2 & \nu\gamma_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \nu \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} = \nu\mathbf{q}_3, \text{ and } [\nu\mathbf{q}_3] = [\mathbf{q}_3] = Q_3.$$

$$\mathcal{T}\mathbf{r}_4 = \begin{bmatrix} \lambda\alpha_0 & \mu\beta_0 & \nu\gamma_0 \\ \lambda\alpha_1 & \mu\beta_1 & \nu\gamma_1 \\ \lambda\alpha_2 & \mu\beta_2 & \nu\gamma_2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda\alpha_0 + \mu\beta_0 + \nu\gamma_0 \\ \lambda\alpha_1 + \mu\beta_1 + \nu\gamma_1 \\ \lambda\alpha_2 + \mu\beta_2 + \nu\gamma_2 \end{bmatrix} = \begin{bmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \end{bmatrix} = \mathbf{q}_4, \text{ and } [\mathbf{q}_4] = Q_4.$$

Furthermore, we note that $\mathcal{T}$ is unique up to scalar multiples, and so it will correspond to a unique element $T_1 \in \text{Proj}(2)$. We do the same process as above to find the unique element $T_2 \in \text{Proj}(2)$, such that $T_2(R_1) = P_1$, $T_2(R_2) = P_2$, $T_3(R_3) = P_3$, and $T_4(R_4) = P_4$. Then $T = T_1 \circ T_2^{-1}$ is the unique element of Proj(2) that carries P_i to Q_i, for $i = 1, 2, 3, 4$. $\square$

Remark. Each element of $GL(3, \mathbb{R})$ has 9 parameters and thus 9 degrees of freedom. However, when we consider Proj(2), the parameters are determined up to a scalar multiple. Thus, each element of Proj(2) has 8 degrees of freedom. Note that in the previous proof, each time we specified where a point was sent, we used two degrees of freedom.

Example 2.1.7. There is a unique element $T \in \text{Proj}(2)$ such that

$$T([0,0,1]) = [1,2,4] \qquad T([0,2,-2]) = [-3,-5,0]$$
$$T([1,-2,1]) = [2,3,1] \quad T([4,2,3]) = [1,2,1].$$

Note that $[1,2,4]$, $[-3,-5,0]$, $[2,3,1]$, and $[1,2,1]$ are in general position, because no three of the corresponding vectors are linearly dependent. In particular, the matrix $\begin{bmatrix} 1 & -3 & 2 \\ 2 & -5 & 3 \\ 4 & 0 & 1 \end{bmatrix}$ is invertible and the system

$$\begin{bmatrix} 1 & -3 & 2 \\ 2 & -5 & 3 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \mu_1 \\ \nu_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

has the unique solution, $\lambda_1 = 2/5$, $\mu_1 = -3/5$, $\nu_1 = -3/5$. Therefore,

$$T_1 = \begin{bmatrix} \frac{2}{5} \cdot 1 & -\frac{3}{5} \cdot -3 & -\frac{3}{5} \cdot 2 \\ \frac{2}{5} \cdot 2 & -\frac{3}{5} \cdot -5 & -\frac{3}{5} \cdot 3 \\ \frac{2}{5} \cdot 4 & -\frac{3}{5} \cdot 0 & -\frac{3}{5} \cdot 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{9}{5} & -\frac{6}{5} \\ \frac{4}{5} & 3 & -\frac{9}{5} \\ \frac{8}{5} & 0 & -\frac{3}{5} \end{bmatrix}.$$

We will do the same to find T_2. Consider the system of equations

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \lambda_2 \\ \mu_2 \\ \nu_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}.$$

The matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & -2 \\ 1 & -2 & 1 \end{bmatrix}$ is invertible, and we obtain the solution $\lambda_2 = 9$, $\mu_2 = 5$, $\nu_2 = 4$. Therefore,

$$T_2 = \begin{bmatrix} 9 \cdot 0 & 5 \cdot 0 & 4 \cdot 1 \\ 9 \cdot 0 & 5 \cdot 2 & 4 \cdot -2 \\ 9 \cdot 1 & 5 \cdot -2 & 4 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 10 & -8 \\ 9 & -10 & 4 \end{bmatrix}, \text{ and so } T_2^{-1} = \begin{bmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{5} & \frac{1}{10} & 0 \\ \frac{1}{4} & 0 & 0 \end{bmatrix}.$$

Then the composition

$$T = T_1 \circ T_2^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{9}{5} & -\frac{6}{5} \\ \frac{4}{5} & 3 & -\frac{9}{5} \\ \frac{8}{5} & 0 & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{5} & \frac{1}{10} & 0 \\ \frac{1}{4} & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{47}{450} & \frac{101}{458} & \frac{2}{45} \\ \frac{43}{180} & \frac{7}{18} & \frac{4}{45} \\ \frac{1}{36} & \frac{8}{45} & \frac{8}{45} \end{bmatrix}$$

defines a unique element in Proj(2) that gives the desired transformation.

2.2 Perspectivities

Now that we understand points and lines in $\mathbb{P}^2$, we will study the simplest nontrivial correspondence between $\mathbb{P}^1$ and $\mathbb{P}^1$ that occurs in the projective plane. The reader should be aware that in this section, lines are often interpreted to be subspaces of the projective plane. That is, a line ℓ is a copy of $\mathbb{P}^1$ sitting inside $\mathbb{P}^2$.

Definition 2.2.1. Let ℓ be a line in $\mathbb{P}^2$ and let P be a point in $\mathbb{P}^2$ not on ℓ. Then the **projection with center** P is the mapping $\pi : \mathbb{P}^2 - \{P\} \to \ell$ defined by setting $\pi(Q)$ equal to $\overleftrightarrow{PQ} \cap \ell$. See Figure 2.9.

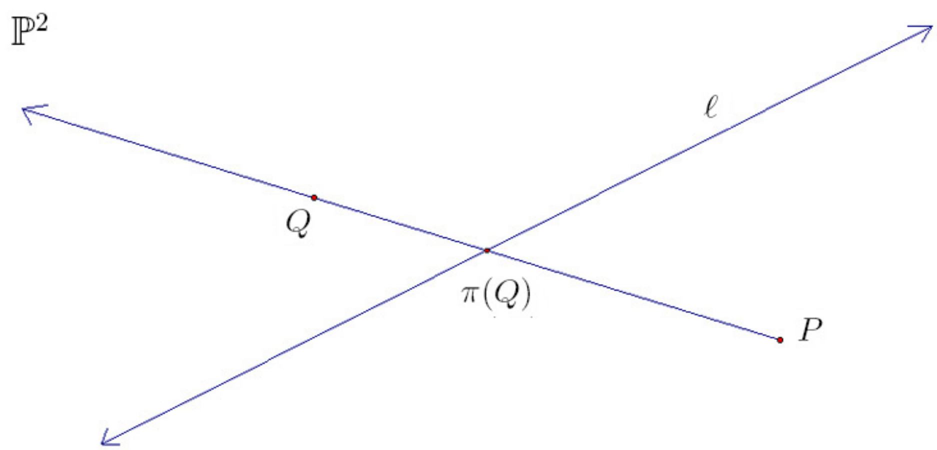


Figure 2.9: The projection π with center P.

Example 2.2.1. Let $P = [1, 0, 0] \in \mathbb{P}^2$ and let $\ell = \{x_0 = 0\} = \mathbb{P}^1_\infty$. Now let $Q = [x_0, x_1, x_2]$ be an arbitrary point not equal to P. By definition, the line

$$\begin{aligned}\overleftrightarrow{PQ} &= \{[s(1, 0, 0) + t(x_0, x_1, x_2)] : s, t \in \mathbb{R} \text{ not both zero}\} \\ &= \{[s + tx_0, tx_1, tx_2] : s, t \in \mathbb{R} \text{ not both zero}\}.\end{aligned}$$

So ℓ intersects $\overleftrightarrow{PQ}$ precisely when $s + tx_0 = 0$. And so $\pi(Q) = \pi([x_0, x_1, x_2]) = [0, tx_1, tx_2] = [0, x_1, x_2]$.

Definition 2.2.2. Suppose ℓ and m are lines in $\mathbb{P}^2$, and let $P \in \mathbb{P}^2$ be a point lying on neither line. Define the mapping $f : \ell \to m$ to be the projection with center P whose domain is restricted to ℓ. That is, for each $Q \in \ell$, let $f(Q)$ be the point of intersection of $\overleftrightarrow{PQ}$ with m. Then we call f a **perspectivity with center** P. See Figure 2.10.

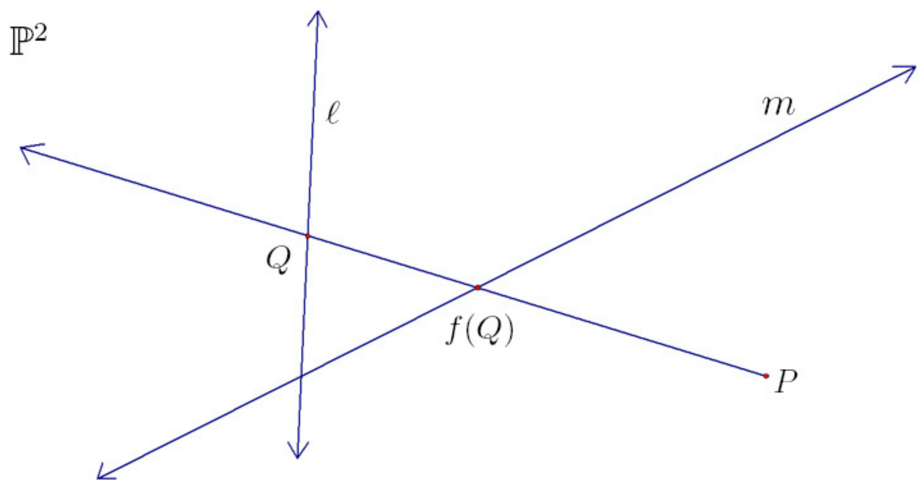


Figure 2.10: The perspectivity $f : \ell \to m$ with center P.

We will show that a perspectivity is a projective transformation $\mathbb{P}^1 \to \mathbb{P}^1$. Without loss of generality, take $\ell = \{x_2 = 0\}$, $m = \{x_1 = 0\}$, and $P = [1, 1, 1]$. Then $\ell \cap m = [1, 0, 0]$. Let $Q \in \ell$ be arbitrary. Then Q is of the form $[1, x, 0]$. We find a homogeneous equation for the line $\overleftrightarrow{PQ}$ by considering the cross product $(1, 1, 1) \times (1, x, 0) = (-x, 1, x - 1)$. We find the point of intersection $\overleftrightarrow{PQ} \cap m = Q'$ by considering $(-x, 1, x - 1) \times (0, 1, 0) = (1 - x, 0, -x)$. Hence $Q' = [x - 1, 0, x] \equiv [1, 0, \frac{x}{x-1}]$. So the perspectivity is given by the linear fractional transformation $f(x) = \dfrac{x}{x-1}$ and is therefore a projective transformation. Thus, every perspectivity is a projective transformation.

Remark. When we first introduced the projective group Proj(1) we defined it so that any $T \in \text{Proj}(1)$ is an automorphism of the same $\mathbb{P}^1$. However, when we consider perspectivites, we are mapping a line $\ell \subset \mathbb{P}^2$ to a second line m in the same picture. Both ℓ and m are copies of $\mathbb{P}^1$ sitting inside the same (projective) plane. Moreover, ℓ and m intersect in exactly one point, which must remain fixed by any perspectivity.

Now that we have a group of motions for $\mathbb{P}^1$ and the space $\mathbb{P}^2$ to use as a setting, we ask the question, "What quantity is left invariant by projective transformations?" The answer is the quantity known as the cross-ratio, which is an element of $\mathbb{R} \cup \{\infty\} = \mathbb{P}^1$ that relates four points $A, B, C, D \in \mathbb{P}^1$.

Definition 2.2.3. Given points $A, B, C, D \in \mathbb{P}^1$ there exists a unique projective transformation $T \in \text{Proj}(1)$ such that $T(A) = 0$, $T(B) = \infty$, and $T(C) = 1$. The **cross-ratio** is defined to be $T(D) \in \mathbb{P}^1$. We denote the cross-ratio by $|A, B; C, D|$.

We derive a formula for the cross-ratio as follows. Let T be given by $f(x) = \dfrac{\alpha x + \beta}{\gamma x + \delta}$. Then $f(A) = \dfrac{\alpha \cdot A + \beta}{\gamma \cdot A + \delta} = 0$ implies that $\alpha \cdot A + \beta = 0$, and so $\beta = -\alpha \cdot A$. Also, we have $f(B) = \dfrac{\alpha \cdot B + \beta}{\gamma \cdot B + \delta} = \infty$ which implies $\gamma \cdot B + \delta = 0$, and so $\delta = -\gamma \cdot B$. Rewriting f we have $f(x) = \dfrac{\alpha(x - A)}{\gamma(x - B)}$. So now $f(C) = \dfrac{\alpha \cdot (C - A)}{\gamma \cdot (C - B)} = 1$, which implies $\dfrac{\alpha}{\gamma} = \dfrac{C - B}{C - A}$. Therefore $f(x) = \dfrac{(C - B)(x - A)}{(C - A)(x - B)}$. Hence, the cross-ratio $f(D) = \dfrac{(C - B)(D - A)}{(C - A)(D - B)}$. Moreover, we see here that where three points are sent uniquely determine a projective transformation.

Remark. If we take one of the points to be infinity (for example, choose $A = \infty$), then the cross-ratio becomes $\displaystyle\lim_{A \to \infty} \frac{(C - B)(D - A)}{(C - A)(D - B)} = \frac{(C - B)}{(D - B)}$.

Theorem 2.2.1. *The cross-ratio is invariant under projective transformations. That is, if* $S \in \text{Proj}(1)$ *and* $A, B, C, D \in \mathbb{P}^1$ *are distinct points then*

$$|A, B; C, D| = |S(A), S(B); S(C), S(D)|.$$

Proof. Let T be the unique element of $\text{Proj}(1)$ such that

$$T : A \mapsto 0,\ B \mapsto \infty,\ C \mapsto 1, \text{ and } D \mapsto |A, B; C, D|.$$

Then we have the following:

$$T \circ S^{-1}(S(A)) \mapsto 0; \quad T \circ S^{-1}(S(B)) \mapsto \infty;$$
$$T \circ S^{-1}(S(C)) \mapsto 1; \quad T \circ S^{-1}(S(D)) \mapsto |A, B; C, D|.$$

Thus $|S(A), S(B); S(C), S(D)| = |A, B; C, D|$. $\square$

Example 2.2.2. Let the line ℓ represent one copy of $\mathbb{P}^1$ and the line m represent a second copy of $\mathbb{P}^1$. Define coordinate systems on ℓ and m and suppose $A = 0$, $B = 1.5$, $C = 0.75$, and $D = 3$ are points on ℓ. Let $\pi : \ell \to m$ be the unique perspectivity with center $P \in \mathbb{P}^2$, such that $\pi(A) = 0, \pi(B) = \infty$, and $\pi(C) = 1$; see Figure 2.11. Then the cross-ratio $|A, B; C, D| = |0, 1.5; 0.75, 3| = \dfrac{(0.75 - 1.5)(3 - 0)}{(0.75 - 0)(3 - 1.5)} = -0.5$. Thus we can conclude that $\pi(D) = -0.5$.

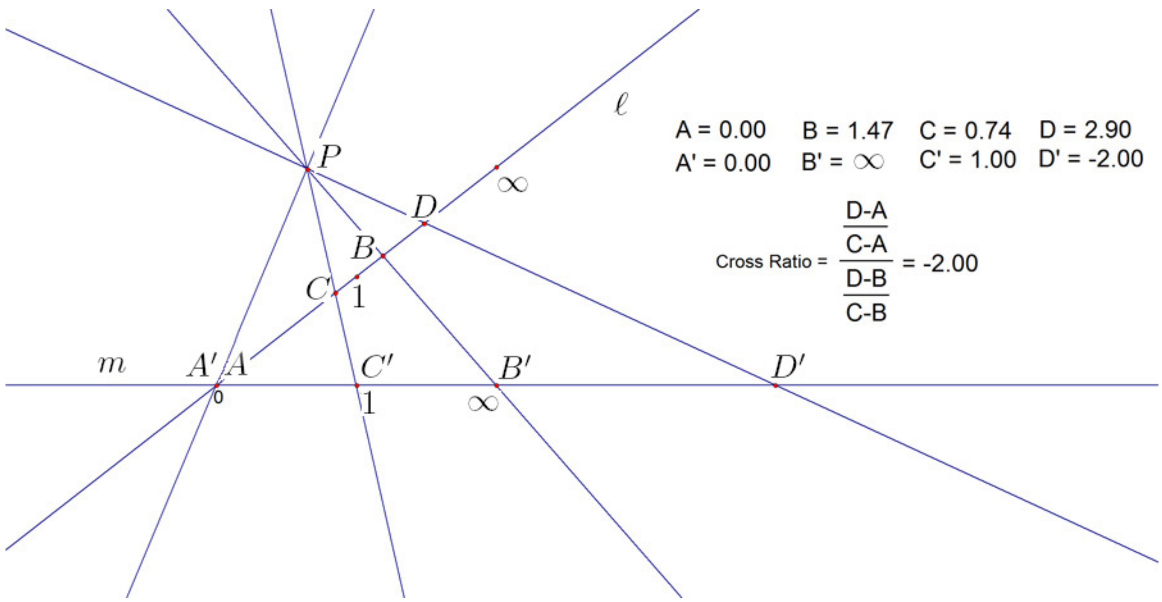


Figure 2.11: Cross-ratio

Example 2.2.3. We can understand the cross-ratio from a more "Euclidean" perspective. Let lines ℓ and m represent copies of $\mathbb{P}^1$ and let π be the perspectivity with center P as in the previous example. This time, however, let us restrict our view to $\mathbb{R}^2$. Now if $\pi : B \mapsto \infty$, the lines $\overleftrightarrow{PB}$ and m are parallel; see Figure 2.12. Then the cross-ratio can be viewed as the following:

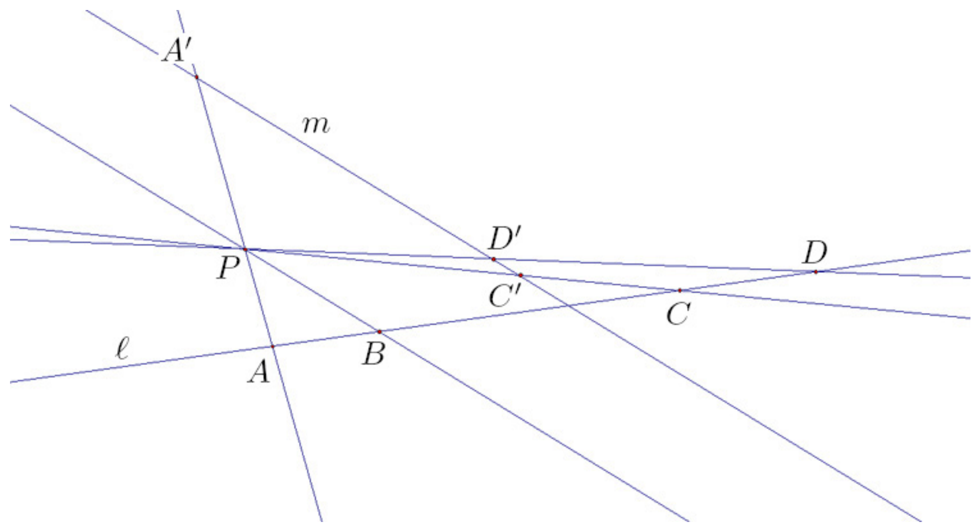


Figure 2.12: Euclidean perspective of the cross-ratio

$$\frac{(CB)}{(DB)}\frac{(DA)}{(CA)} = \frac{(C'B')}{(D'B')}\frac{(D'A')}{(C'A')}.$$

Theorem 2.2.2. *Let $\ell \cong \mathbb{P}^1$ and $n \cong \mathbb{P}^1$ be lines in $\mathbb{P}^2$. For every transformation $T \in \text{Proj}(1)$, there exists a line $m \cong \mathbb{P}^1$ and perspectivities $g : \ell \to m$ and $f : m \to n$, such that T is the composition $f \circ g : \ell \to n$.*

Proof. Recall that any projective transformation is uniquely determined by where it sends three points, say A, B, and C. Let ℓ and n be lines in $\mathbb{P}^2$ and let $A, B, C \in \ell$ and $A'', B'', C'' \in n$. We will show that we can construct perspectivities g and f, so that their composition $f \circ g$ maps A to A'', B to B'', and C to C'', where A, B, C and A'', B'', C'' are arbitrary trios in ℓ and n respectively (Figure 2.13).

Construct any line $m \subset \mathbb{P}^2$ with $A'' \in m$ and construct any point $P \in \overleftrightarrow{AA''}$. Let $g : \ell \to m$ be the perpsectivity with center P. Then, $g : A \mapsto A''$. Next construct lines $\overleftrightarrow{PB}$ and $\overleftrightarrow{PC}$. Let $B' = \overleftrightarrow{PB} \cap m$ and $C' = \overleftrightarrow{PC} \cap m$. So then $g : B \mapsto B'$ and $g : C \mapsto C'$. Now that we have specified where g carries A, B, and C, then it is a unique element of Proj(1).

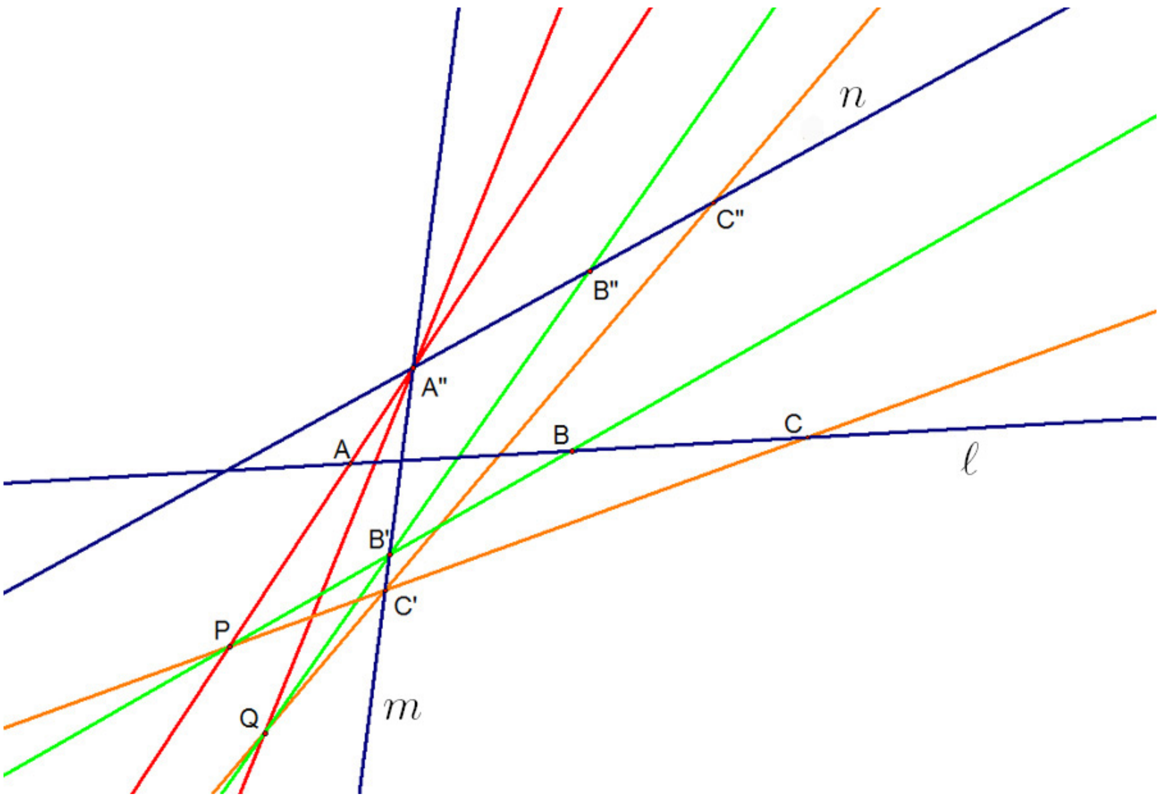


Figure 2.13: Composition of perspectivities

Now we want to construct $f : m \to n$. Construct lines $\overleftrightarrow{B'B''}$ and $\overleftrightarrow{C'C''}$ and let $Q = \overleftrightarrow{B'B''} \cap \overleftrightarrow{C'C''}$. Let $f : m \to n$ be the perspectivity with center Q. Since $m \cap n = A''$, then f will keep A'' fixed. Lines $\overleftrightarrow{B'B''}$ and $\overleftrightarrow{C'C''}$ pass through Q, and so $f : B' \mapsto B''$ and $f : C' \mapsto C''$. Now that we have specified where f maps A'', B', and C', then it is a unique element of Proj(1).

Now, we merely observe that the projective transformation $f \circ g : \ell \to n$ carries A to A'', B to B'', and C to C'', as desired. □

Remark. Note that if $A = A''$, then we can achieve the tranformation with a single perspectivity with center $\overleftrightarrow{BB''} \cap \overleftrightarrow{CC''}$.

2.3 The theorems of Desargues and Pappus

We now turn to some very important and beautiful results of projective geometry discovered by Desargues and Pappus. We remark here that there are special cases of these theorems can be proved using facts from Euclidean geometry. We shall provide the statements and proofs of these special cases before each theorem as a motivating example.

Theorem 2.3.1. *(Special case of Desargues). Let $\triangle PQR$ and $\triangle P'Q'R'$ be triangles in the projective plane. Suppose the three lines $\overleftrightarrow{PP'}$, $\overleftrightarrow{QQ'}$, and $\overleftrightarrow{RR'}$ are concurrent. If $\overleftrightarrow{PQ} \parallel \overleftrightarrow{P'Q'}$ and $\overleftrightarrow{QR} \parallel \overleftrightarrow{Q'R'}$, then $\overleftrightarrow{PR} \parallel \overleftrightarrow{P'R'}$. See Figure 2.14.*

Proof. Let $X = \overleftrightarrow{PP'} \cap \overleftrightarrow{QQ'} \cap \overleftrightarrow{RR'}$. Since $\overleftrightarrow{PQ} \parallel \overleftrightarrow{P'Q'}$ then $\triangle XQR \sim \triangle XP'R'$. Thus, $\dfrac{XP}{XP'} = \dfrac{XR}{XR'}$. Similarly, since $\overleftrightarrow{QR} \parallel \overleftrightarrow{Q'R'}$ then $\triangle XQR \sim \triangle XQ'R'$. So, $\dfrac{XQ}{XQ'} = \dfrac{XR}{XR'}$. Therefore, $\dfrac{XP}{XP'} = \dfrac{XR}{XR'}$ and thus $\triangle XPR \sim \triangle XP'R'$. We conclude that $\overleftrightarrow{PR} \parallel \overleftrightarrow{P'R'}$. □

Now we proceed with Desargues' theorem.

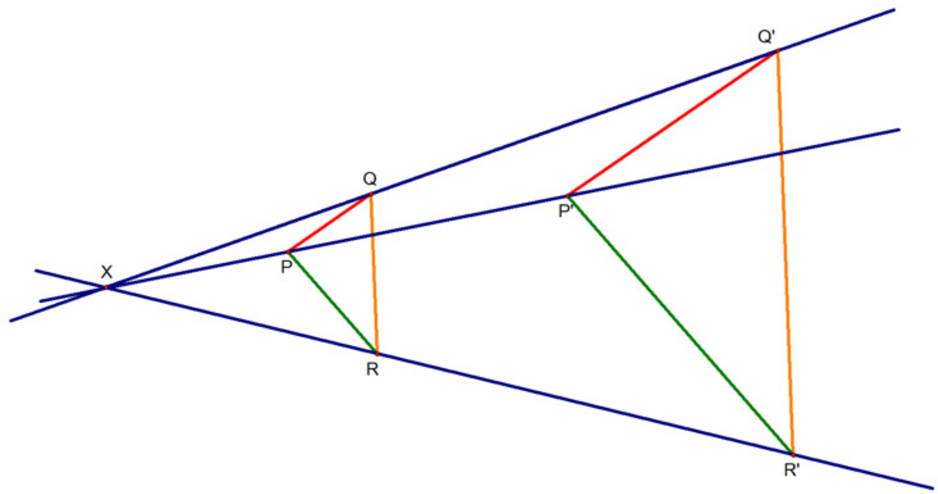


Figure 2.14: Special case of Desargues' theorem

Theorem 2.3.2. *(Desargues). Let $\triangle PQR$ and $\triangle P'Q'R'$ be triangles in the projective plane. Suppose the three lines $\overleftrightarrow{PP'}$, $\overleftrightarrow{QQ'}$, and $\overleftrightarrow{RR'}$ are concurrent. Then the three points $A = \overleftrightarrow{PQ} \cap \overleftrightarrow{P'Q'}$, $B = \overleftrightarrow{QR} \cap \overleftrightarrow{Q'R'}$, and $C = \overleftrightarrow{PR} \cap \overleftrightarrow{P'R'}$ are collinear. (See Figure 2.15).*

Note that in the special case of the theorem we just proved, the points A, B, and C were on the line at infinity.

Proof. Without loss of generality, assume $P = [1, 0, 0]$, $Q = [0, 1, 0]$, $R = [0, 0, 1]$, and $X = \overleftrightarrow{PP'} \cap \overleftrightarrow{QQ'} \cap \overleftrightarrow{RR'} = [1, 1, 1]$. Then since P' lies on the line $\overleftrightarrow{XP}$, Q' lies on the line $\overleftrightarrow{XQ}$, and R' lies on the line $\overleftrightarrow{XR}$, there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that $P' = [\alpha, 1, 1]$, $Q' = [1, \beta, 1]$, and $R' = [1, 1, \gamma]$. Now our goal is to determine coordinates for points A, B, and C.

We defined $A = \overleftrightarrow{PQ} \cap \overleftrightarrow{P'Q'}$, so we find equations for both $\overleftrightarrow{PQ}$ and $\overleftrightarrow{P'Q'}$ and find their intersection. Let $\mathbf{p} = (1, 0, 0)$, $\mathbf{q} = (0, 1, 0)$, $\mathbf{p}' = (\alpha, 1, 1)$, and $\mathbf{q}' = (1, \beta, 1)$ be vectors in $\mathbb{R}^3$ corresponding to P, Q, P', and Q'. Then $\mathbf{p} \times \mathbf{q} = (0, 0, 1)$ and so $x_2 = 0$ is the equation for $\overleftrightarrow{PQ}$. Similarly, $\mathbf{p}' \times \mathbf{q}' = (1-\beta, 1-\alpha, \alpha\beta - 1)$, and so $(1-\beta)x_0 + (1-\alpha)x_1 + (\alpha\beta - 1)x_2 = 0$

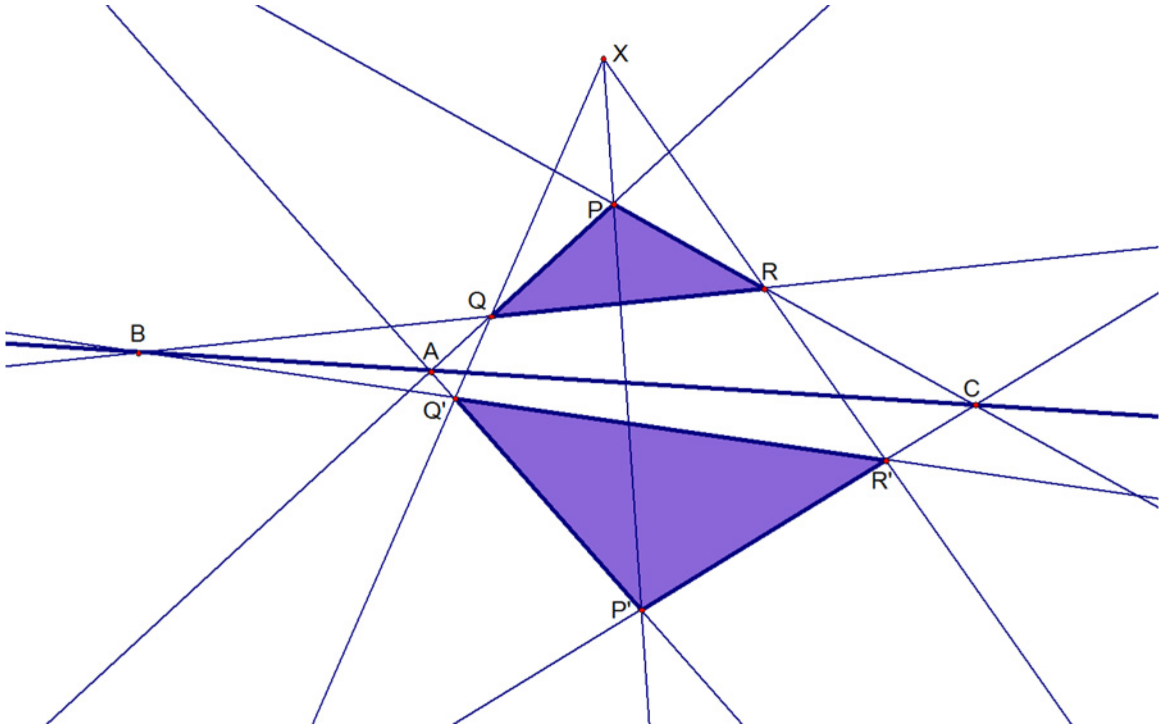


Figure 2.15: Desargues' theorem

is the equation for $\overleftrightarrow{P'Q'}$. We find coordinates for A by taking the cross product $(0,0,1)\times(1-\beta,1-\alpha,\alpha\beta-1)=(\alpha-1,1-\beta,0)$.

We defined $B=\overleftrightarrow{QR}\cap\overleftrightarrow{Q'R'}$, so we find equations for both $\overleftrightarrow{QR}$ and $\overleftrightarrow{Q'R'}$ and find their intersection. Let $\mathbf{r}=(0,0,1),\mathbf{r}'=(1,1,\gamma)\in\mathbb{R}^3$. Then $\mathbf{q}\times\mathbf{r}=(1,0,0)$ and so $x_0=0$ is the equation for $\overleftrightarrow{QR}$. Similarly, $\mathbf{q}'\times\mathbf{r}'=(\beta\gamma-1,1-\gamma,1-\beta)$, and so $(\beta\gamma-1)x_0+(1-\gamma)x_1+(1-\beta)x_2=0$ is the equation for $\overleftrightarrow{Q'R'}$. We find coordinates for B by taking the cross product $(1,0,0)\times(\beta\gamma-1,1-\gamma,1-\beta)=(0,\beta-1,1-\gamma)$.

We defined $C=\overleftrightarrow{PR}\cap\overleftrightarrow{P'R'}$, so we find equations for both $\overleftrightarrow{PR}$ and $\overleftrightarrow{P'R'}$ and find their intersection. $\mathbf{p}\times\mathbf{r}=(0,-1,0)$, and so $x_1=0$ is the equation for $\overleftrightarrow{PR}$. Similarly, $\mathbf{p}'\times\mathbf{r}'=(\gamma-1,1-\alpha\gamma,\alpha-1)$, and so $(\gamma-1)x_0+(1-\alpha\gamma)x_1+(\alpha-1)x_2=0$ is the equation for

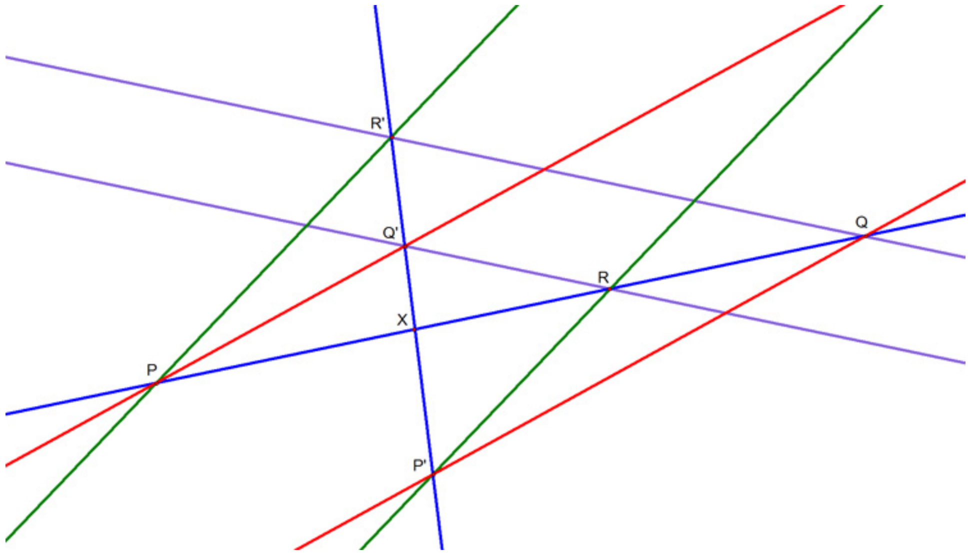


Figure 2.16: Special case of Pappus' theorem

$\overleftrightarrow{P'R'}$. We find coordinates for C by taking the cross product $(0,-1,0)\times(\gamma-1, 1-\alpha\gamma, \alpha-1) = (1-\alpha, 0, \gamma-1)$.

Now, the sum of the homogeneous coordinate vectors of points A, B, and C is 0 and so we conclude that A, B, and C are collinear. $\square$

As we did with Desargues, we will start with a special case of Pappus' theorem that we prove using only facts from Euclidean geometry.

Theorem 2.3.3. *(Special case of Pappus). Let P, Q, R and P', Q', R' be triples of collinear points, and lines $\overleftrightarrow{PQR} \not\parallel \overleftrightarrow{P'Q'R'}$. If $\overleftrightarrow{PQ'} \parallel \overleftrightarrow{P'Q}$ and $\overleftrightarrow{PR'} \parallel \overleftrightarrow{P'R}$, then $\overleftrightarrow{QR'} \parallel \overleftrightarrow{Q'R}$. See Figure 2.16.*

Proof. Let $X = \overleftrightarrow{PQR} \cap \overleftrightarrow{P'Q'R'}$. Then since $\overleftrightarrow{PQ'} \parallel \overleftrightarrow{P'Q}$ then we have $\triangle PXQ' \sim \triangle QXP'$. Thus $\dfrac{PX}{Q'X} = \dfrac{QX}{P'X}$. Similarly, since $\overleftrightarrow{PR'} \parallel \overleftrightarrow{P'R}$, then $\triangle PXR' \sim \triangle RXP'$ and so $\dfrac{PX}{R'X} = \dfrac{RX}{P'X}$. Putting this together, we have

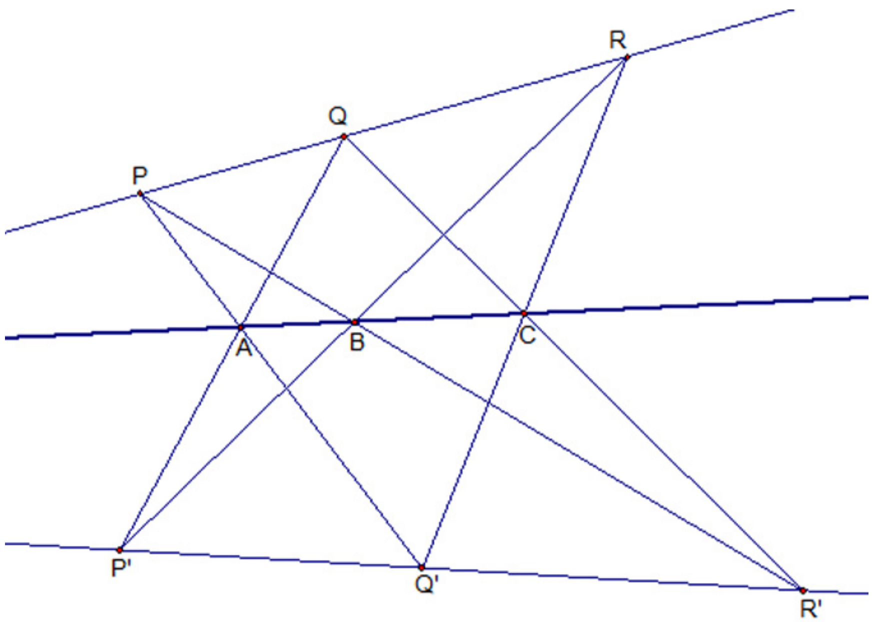


Figure 2.17: Pappus' theorem

$$\frac{\frac{RX}{P'X}}{\frac{QX}{P'X}} = \frac{\frac{PX}{R'X}}{\frac{PX}{Q'X}} \Longrightarrow \frac{RX}{QX} = \frac{Q'X}{R'X}.$$

Therefore $\triangle RXQ' \sim \triangle QXR'$, which means $\overleftrightarrow{QR'} \parallel \overleftrightarrow{Q'R}$. $\square$

Theorem 2.3.4. *(Pappus). Let P, Q, R and P', Q', R' be triples of collinear points. Let $\overleftrightarrow{PQ'} \cap \overleftrightarrow{P'Q} = A$, $\overleftrightarrow{PR'} \cap \overleftrightarrow{P'R} = B$, and $\overleftrightarrow{QR'} \cap \overleftrightarrow{Q'R} = C$. Then $A, B,$ and C are collinear, as well. See Figure 2.17.*

Again, we remark that the special case of Pappus' theorem we started with is when $A, B,$ and C lie on the line at infinity.

Proof. Assume, without loss of generality, that $P = [1,0,0], P' = [0,1,0], Q = [0,0,1]$, and $Q' = [1,1,1]$. It is easy to check that $\overleftrightarrow{PQ} = \{[\mathbf{x}] : x_1 = 0\}$, and $\overleftrightarrow{P'Q'} = \{[\mathbf{x}] : x_0 - x_2 = 0\}$. Therefore R is of the form $[1,0,\alpha]$ for some nonzero α, and R' is of the form $[\beta,1,\beta]$ for some nonzero β or $R' = [1,0,1]$. If $R' = [1,0,1]$, then it is easy to check that A, B, and C are collinear. So we will assume the former and write down equations for the lines $\overleftrightarrow{PQ'}$, $\overleftrightarrow{P'Q}$, $\overleftrightarrow{PR'}$, $\overleftrightarrow{P'R}$, $\overleftrightarrow{QR'}$, and $\overleftrightarrow{Q'R}$:

$$\begin{aligned}
\overleftrightarrow{PQ'} &= \{\mathbf{x} : -x_1 + x_2 = 0\}, & \overleftrightarrow{P'Q} &= \{\mathbf{x} : x_0 = 0\} \\
\overleftrightarrow{PR'} &= \{\mathbf{x} : -\beta x_1 + x_2 = 0\}, & \overleftrightarrow{P'R} &= \{\mathbf{x} : \alpha x_0 - x_2 = 0\} \\
\overleftrightarrow{QR'} &= \{\mathbf{x} : -x_0 + \beta x_1 = 0\}, & \overleftrightarrow{Q'R} &= \{\mathbf{x} : \alpha x_0 + (1-\alpha)x_1 - x_2 = 0\}
\end{aligned}$$

Now we can proceed to compute the coordinates for points A, B, and C.

$$\begin{aligned}
A &= \overleftrightarrow{PQ'} \cap \overleftrightarrow{P'Q} = [0,1,1] \\
B &= \overleftrightarrow{PR'} \cap \overleftrightarrow{P'R} = [\beta,\alpha,\alpha\beta] \\
C &= \overleftrightarrow{QR'} \cap \overleftrightarrow{Q'R} = [\beta,1,\alpha\beta-\alpha+1]
\end{aligned}$$

Now consider the following linear combination of the representing vectors in $\mathbb{R}^3$:

$$(\alpha-1)(0,1,1) - (\beta,\alpha,\alpha\beta) + (\beta,1,\alpha\beta-\alpha+1) = 0.$$

Hence, the points A, B, and C are collinear. $\square$

Remark. Note that the following converse is logically equivalent to the original statement of Pappus' theorem.

Let P,Q,R and P',Q',R' be triples of points. Let $\overleftrightarrow{PQ'} \cap \overleftrightarrow{P'Q} = A$, $\overleftrightarrow{PR'} \cap \overleftrightarrow{P'R} = B$, and $\overleftrightarrow{QR'} \cap \overleftrightarrow{Q'R} = C$. Then if P,Q,R are collinear and A,B,C are collinear, it follows that P',Q', and R' are collinear as well.

If we swap the labels $P' \longleftrightarrow A, Q' \longleftrightarrow B$, and $R' \longleftrightarrow C$, then we have exactly the original statement of Pappus' theorem.

2.4 Projective duality

We now bring attention to an interesting twist to the projective plane. Recall that we defined points in $\mathbb{P}^2$ as equivalence classes of nonzero vectors in $\mathbb{R}^3$. Let $\mathbb{R}^{3*}$ denote the set of all linear functionals $f : \mathbb{R}^3 \to \mathbb{R}$. Then $f \in \mathbb{R}^{3*}$ is of the form

$$f(x) = a_0x_0 + a_1x_1 + a_2x_2$$

for some $\mathbf{a} = (a_0, a_1, a_2) \in \mathbb{R}^3$. We call $\mathbb{R}^{3*}$ the dual space of $\mathbb{R}^3$. Note that for every $\mathbf{a} \in \mathbb{R}^3$ there is an associated linear functional $f \in \mathbb{R}^{3*}$ given by $f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$, and so there is a one-to-one correspondence between $\mathbb{R}^3$ and $\mathbb{R}^{3*}$. In fact, $\mathbb{R}^3$ is isomorphic to $\mathbb{R}^{3*}$.

Because $\mathbb{R}^{3*}$ is a vector space, we can consider the set of lines through the origin in $\mathbb{R}^{3*}$ just as we did for $\mathbb{R}^3$. So then we have a second copy of the projective plane called the **dual projective plane**, denoted $\mathbb{P}^{2*}$. The isomorphism $\mathbb{R}^3 \to \mathbb{R}^{3*}$ induces a one-to-one correspondence between $\mathbb{P}^2$ and $\mathbb{P}^{2*}$. For every point $[\pi_0, \pi_1, \pi_2] \in \mathbb{P}^{2*}$ there is a corresponding projective line $\ell_\Pi \subset \mathbb{P}^2$ given by $\ell_\Pi = \{[\mathbf{x}] : \pi_0x_0 + \pi_1x_1 + \pi_2x_2\}$. Note that this mapping is well-defined.

Definition 2.4.1. Let P be a point in $\mathbb{P}^2$. Then the set of all lines in $\mathbb{P}^2$ that pass through P is called the **pencil of lines passing through** P, denoted λ_P.

Suppose $P = [p_0, p_1, p_2] \in \mathbb{P}^2$ and $\Pi = [\pi_0, \pi_1, \pi_2] \in \mathbb{P}^{2*}$. Then Π corresponds to the line $\ell_\Pi = \{[\mathbf{x}] : \pi_0x_0 + \pi_1x_1 + \pi_2x_2 = 0\} \subset \mathbb{P}^2$. Note that P lies on ℓ_Π if and only if $\pi_0p_0 + \pi_1p_1 + \pi_2p_2 = 0$. Now, keeping $\mathbf{p}$ fixed, $\pi_0p_0 + \pi_1p_1 + \pi_2p_2 = 0$ gives us a homogeneous linear equation in $\boldsymbol{\pi}$, and thus defines a line in $\mathbb{P}^{2*}$.

Consider the lines $\ell_\Pi = \{[\mathbf{x}] : \pi_0x_0 + \pi_1x_1 + \pi_2x_2 = 0\} \subset \mathbb{P}^2$ and $\lambda_P = \{[\boldsymbol{\xi}] : p_0\xi_0 + p_1\xi_1 + p_2\xi_2 = 0\} \subset \mathbb{P}^{2*}$. Each point $[\boldsymbol{\xi}] \in \lambda_P$ corresponds to a line in $\mathbb{P}^2$ satisfying $p_0\xi_0 + p_1\xi_1 + p_2\xi_2 = 0$, which passes through the point P. Hence the line $\lambda_P \subset \mathbb{P}^{2*}$ is the pencil of lines through the point $P \in \mathbb{P}^2$. The line ℓ_Π is a member of this pencil if and only if $P \in \ell_\Pi$, if and only if $\pi_0p_0 + \pi_1p_1 + \pi_2p_2 = 0$. See Figure 2.18.

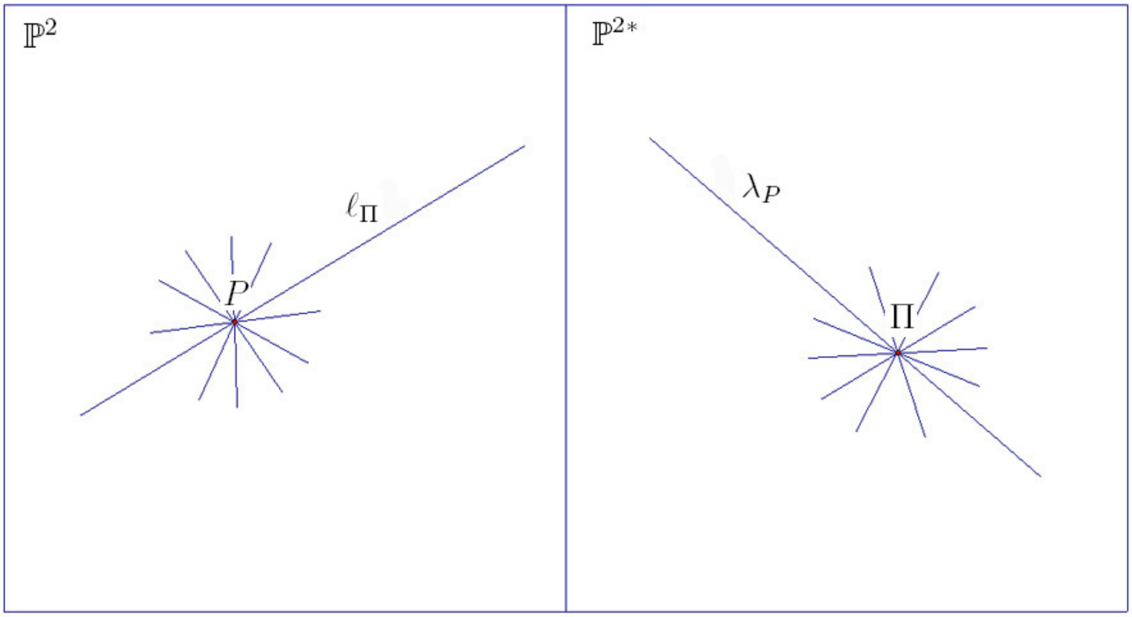


Figure 2.18: Dually, points correspond to lines and a line is a pencil of lines through a point.

Definition 2.4.2. We define **projective duality** to be the correspondence

$$\mathbb{P}^2 \longleftrightarrow \{ \text{ lines in } \mathbb{P}^{2*}\}$$

$$\text{point } P \leftrightsquigarrow \lambda_P = \text{the pencil of lines through } P.$$

We will now make some observations, which describe some of the interplay between $\mathbb{P}^2$ and $\mathbb{P}^{2*}$. Suppose $\Pi = [\boldsymbol{\pi}], \Gamma = [\boldsymbol{\gamma}] \in \mathbb{P}^{2*}$ correspond to lines $\ell_\Pi, \ell_\Gamma \subset \mathbb{P}^2$. Let $\mu \subset \mathbb{P}^{2*}$ be the line that joins points Π and Γ. By projective duality, the line μ will correspond to a pencil of lines passing through some point $P = [\mathbf{p}] \in \mathbb{P}^2$. Therefore $P \in \ell_\Pi$ and $P \in \ell_\Gamma$ and since ℓ_Π and ℓ_Γ intersect in exactly one point, it follows that $\ell_\Pi \cap \ell_\Gamma = P$. See Figure 2.19.

Now let $\Pi, \Gamma, \Sigma \in \mathbb{P}^{2*}$ correspond to lines $\ell_\Pi, \ell_\Gamma, \ell_\Sigma \subset \mathbb{P}^2$. As before, take μ to be the line in $\mathbb{P}^{2*}$ that joins Π and Γ. From our previous observation, the line μ corresponds to the pencil of lines through the point $P = \ell_\Pi \cap \ell_\Gamma$. Now if Σ also lies on μ then the line ℓ_Σ must

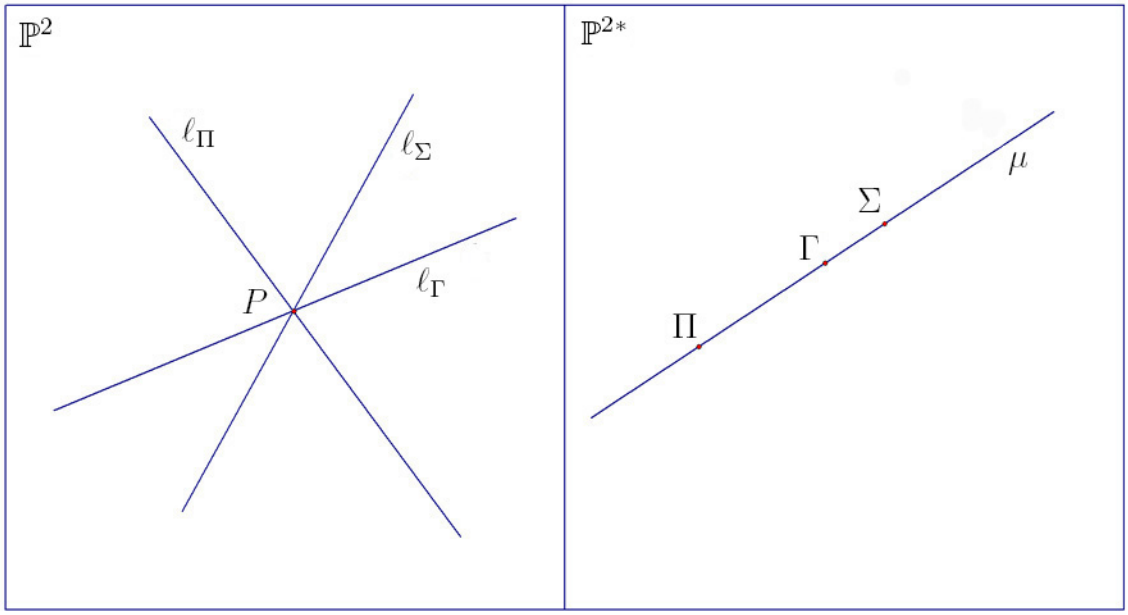


Figure 2.19: Concurrent lines translate dually to collinear points.

pass through P, which implies that ℓ_Π, ℓ_Γ, and ℓ_Σ are concurrent. Conversely, if we assume that the line ℓ_Σ passes through P, then Σ must lie on μ in $\mathbb{P}^{2*}$. Therefore, we conclude that points Π, Γ, and Σ in $\mathbb{P}^{2*}$ are collinear if and only if their corresponding lines ℓ_Π, ℓ_Γ, and ℓ_Σ are concurrent in $\mathbb{P}^2$. See Figure 2.19. Since $\mathbb{P}^{2*}$ can be thought of as a second copy of $\mathbb{P}^2$, then dually, we can also make the statement that points P, Q, and R in $\mathbb{P}^2$ are collinear if and only if their corresponding lines λ_P, λ_Q, and λ_R in $\mathbb{P}^{2*}$ are concurrent.

Remark. We note here that every theorem that is valid in $\mathbb{P}^2$ has a dual theorem valid in $\mathbb{P}^{2*}$. We obtain the dual statement by making the following changes:

$$\text{point} \longleftrightarrow \text{line},$$

$$\text{collinear} \longleftrightarrow \text{concurrent},$$

$$\text{join} \longleftrightarrow \text{intersection}.$$

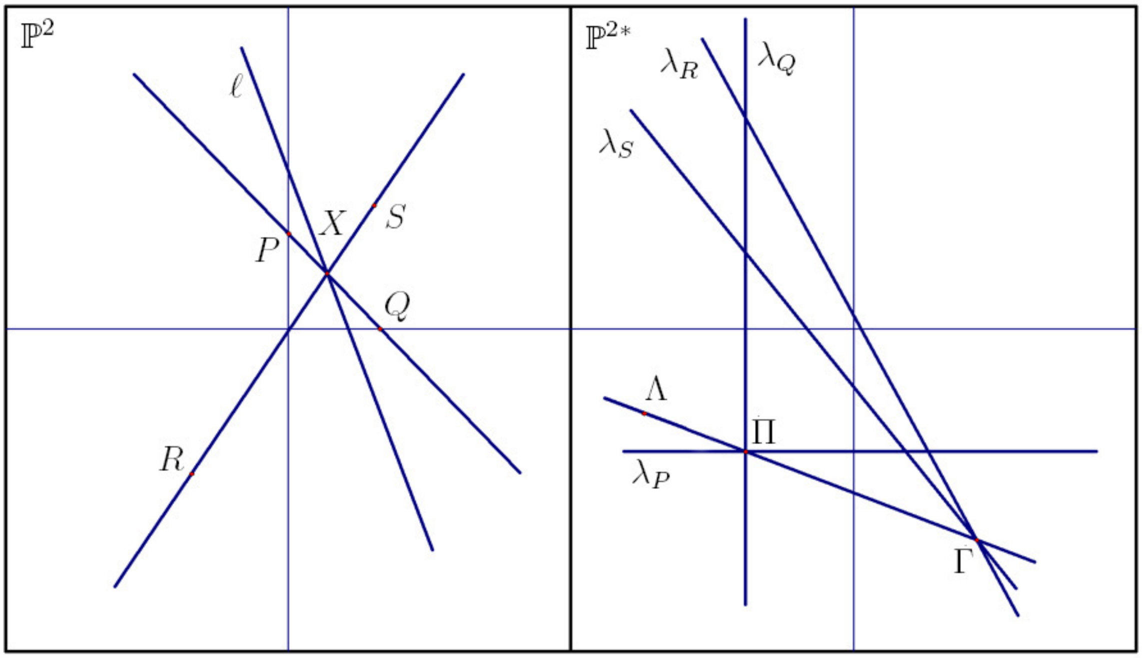


Figure 2.20: Example from page 13 with dual.

Example 2.4.1. Recall Example 2.1.6 on page 13. Let us understand this same example in the dual setting (see Figure 2.20). First we started with points $P = [1, 0, 2]$, $Q = [1, 3, 0]$, $R = [1, -1, -2]$, and $S = [1, 1, 3]$. Under projective duality these points correspond to the following lines in $\mathbb{P}^{2*}$:

$$P = [1, 0, 2] \leftrightsquigarrow \lambda_P = \{[\boldsymbol{\xi}] : \xi_0 + 2\xi_2 = 0\}$$

$$Q = [1, 3, 0] \leftrightsquigarrow \lambda_Q = \{[\boldsymbol{\xi}] : \xi_0 + 3\xi_1 = 0\}$$

$$R = [1, -1, -2] \leftrightsquigarrow \lambda_R = \{[\boldsymbol{\xi}] : \xi_0 - \xi_1 - 2\xi_2 = 0\}$$

$$S = [1, 1, 3] \leftrightsquigarrow \lambda_S = \{[\boldsymbol{\xi}] : \xi_0 + \xi_1 + 3\xi_2 = 0\}$$

Then in $\mathbb{P}^2$ we found lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$. Dually, these lines are pencils of lines through the points Π and Γ respectively. That is,

$$\overleftrightarrow{PQ} = \{[\mathbf{x}] : -6x_0 + 2x_1 + 3x_2 = 0\} \leftrightsquigarrow \text{pencil of lines through } \Pi = [-6, 2, 3]$$
$$\overleftrightarrow{RS} = \{[\mathbf{x}] : -x_0 - 5x_1 + 2x_2 = 0\} \leftrightsquigarrow \text{pencil of lines through } \Gamma = [-1, -5, 2].$$

Next, we found the intersection $\overleftrightarrow{PQ} \cap \overleftrightarrow{RS} = X \in \mathbb{P}^2$. The dual statement here would be to find the line $\overleftrightarrow{\Pi\Gamma}$. In other words,

$$\overleftrightarrow{PQ} \cap \overleftrightarrow{RS} = X = [19, 9, -32] \leftrightsquigarrow \overleftrightarrow{\Pi\Gamma} = \{[\boldsymbol{\xi}] : 19\xi_0 + 9\xi_1 - 32\xi_2 = 0\}.$$

Lastly, we checked that the point $X \in \mathbb{P}^2$ was contained in the line ℓ. Under duality this translates to checking to see that the line $\overleftrightarrow{\Pi\Gamma}$ contains the point $\Lambda \in \mathbb{P}^{2*}$, where $\Lambda \leftrightsquigarrow \ell$. That is,

$$X \in \ell = \{[\mathbf{x}] : -50x_0 + 38x_1 + 19x_2 = 0\} \leftrightsquigarrow \overleftrightarrow{\Pi\Gamma} \ni \Lambda = [-50, 38, 19].$$

We close this section by going back to Desargues' and Pappus' theorems and considering them in the dual setting.

Theorem 2.4.1. *(Desargues' dual). Let $\triangle\Pi\Gamma\Xi$ and $\triangle\Pi'\Gamma'\Xi'$ be triangles in $\mathbb{P}^{2*}$. Suppose the three points $\Lambda = \overleftrightarrow{\Pi\Gamma} \cap \overleftrightarrow{\Pi'\Gamma'}$, $\Sigma = \overleftrightarrow{\Gamma\Xi} \cap \overleftrightarrow{\Gamma'\Xi'}$, and $\Omega = \overleftrightarrow{\Pi\Xi} \cap \overleftrightarrow{\Pi'\Xi'}$ are collinear. Then the three lines $\overleftrightarrow{\Pi\Pi'}$, $\overleftrightarrow{\Gamma\Gamma'}$, and $\overleftrightarrow{\Xi\Xi'}$ are concurrent.*

Before we prove Desargues' dual theorem, we remark that the dual statement is the converse of the orginal statement of Desargues' theorem. Also, the picture is the same as in Desargues' theorem (Figure 2.15) if we relabel appropriately.

Proof. Let A, A', B, B', C, and C' be points in $\mathbb{P}^2$ such that under projective duality, $A \longleftrightarrow \overleftrightarrow{\Pi\Gamma}$, $A' \longleftrightarrow \overleftrightarrow{\Pi'\Gamma'}$, $B \longleftrightarrow \overleftrightarrow{\Gamma\Xi}$, $B' \longleftrightarrow \overleftrightarrow{\Gamma'\Xi'}$, $C \longleftrightarrow \overleftrightarrow{\Pi\Xi}$, and $C' \longleftrightarrow \overleftrightarrow{\Pi'\Xi'}$. Then under projective duality $\Lambda \longleftrightarrow \overleftrightarrow{AA'}$, $\Sigma \longleftrightarrow \overleftrightarrow{BB'}$, and $\Omega \longleftrightarrow \overleftrightarrow{CC'}$. Now since we are assuming that Λ, Σ, and Ω are collinear in $\mathbb{P}^{2*}$, then it follows that $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are concurrent in $\mathbb{P}^2$. Let $P = \overleftrightarrow{AC} \cap \overleftrightarrow{A'C'}$, $Q = \overleftrightarrow{AB} \cap \overleftrightarrow{A'B'}$, and $R = \overleftrightarrow{BC} \cap \overleftrightarrow{B'C'}$. Then by Desargues' theorem, P, Q, and R are collinear in $\mathbb{P}^2$. Under projective duality, $P \longleftrightarrow \overleftrightarrow{\Pi\Pi'}$, $Q \longleftrightarrow \overleftrightarrow{\Gamma\Gamma'}$,

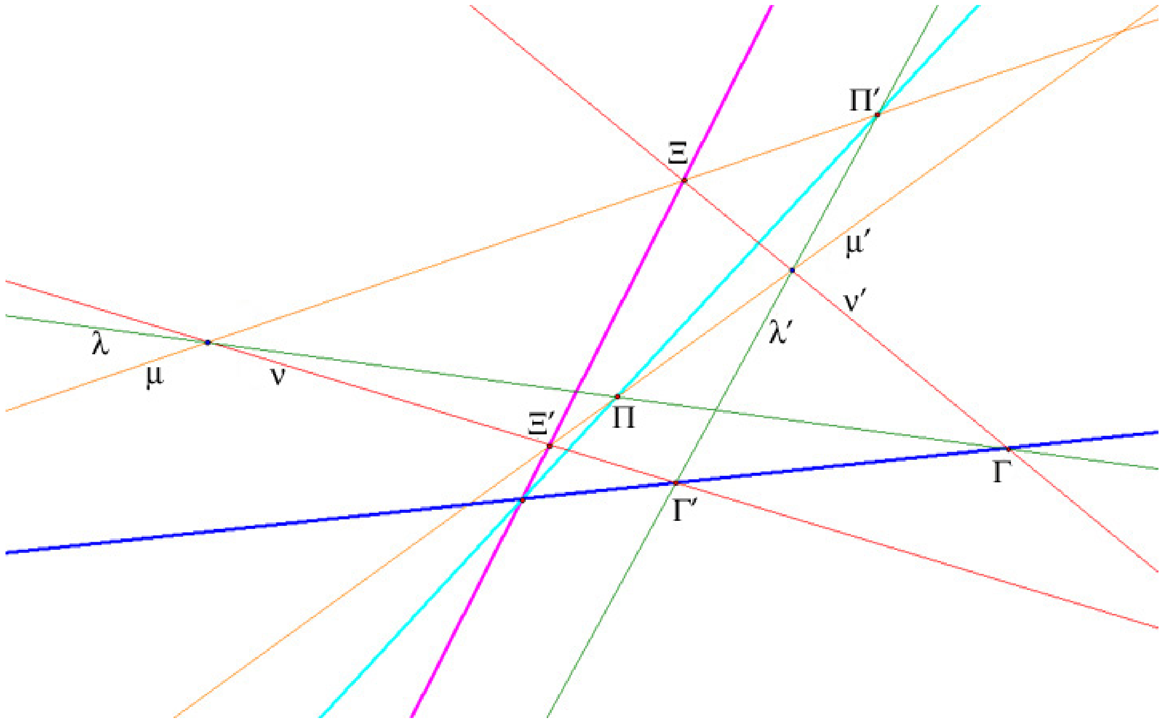


Figure 2.21: Pappus' dual theorem

and $R \longleftrightarrow \overleftrightarrow{\Xi\Xi'}$. Therefore it follows from our observations above that $\overleftrightarrow{\Pi\Pi'}$, $\overleftrightarrow{\Gamma\Gamma'}$, and $\overleftrightarrow{\Xi\Xi'}$ are concurrent. $\square$

Theorem 2.4.2. *(Pappus' dual). Let λ, μ, ν and λ', μ', ν' be triples of concurrent lines in $\mathbb{P}^{2*}$. Let $\lambda \cap \mu' = \Pi$, $\lambda' \cap \mu = \Pi'$, $\lambda \cap \nu' = \Gamma$, $\lambda' \cap \nu = \Gamma'$, $\mu \cap \nu' = \Xi$, and $\mu' \cap \nu = \Xi'$. Then the lines $\overleftrightarrow{\Pi\Pi'}$, $\overleftrightarrow{\Gamma\Gamma'}$, and $\overleftrightarrow{\Xi\Xi'}$ are concurrent as well (see Figure 2.21).*

Proof. Let P, Q, R, P', Q', R' be points in $\mathbb{P}^2$ corresponding to lines $\lambda, \mu, \nu, \lambda', \mu', \nu'$ respectively. Since λ, μ, and ν are concurrent in $\mathbb{P}^{2*}$, then under projective duality P, Q, and R are collinear in $\mathbb{P}^2$. A similar result is true for the primed versions. Considering projective duality again, $\Pi \longleftrightarrow \overleftrightarrow{PQ'}$, $\Pi' \longleftrightarrow \overleftrightarrow{P'Q}$, $\Gamma \longleftrightarrow \overleftrightarrow{PR'}$, $\Gamma' \longleftrightarrow \overleftrightarrow{P'R}$, $\Xi \longleftrightarrow \overleftrightarrow{Q'R}$, and $\Xi' \longleftrightarrow \overleftrightarrow{QR'}$.

Let $\overleftrightarrow{PQ'} \cap \overleftrightarrow{P'Q} = A$, $\overleftrightarrow{PR'} \cap \overleftrightarrow{P'R} = B$, and $\overleftrightarrow{QR'} \cap \overleftrightarrow{Q'R} = C$. By Pappus' theorem, the points $A, B, C \in \mathbb{P}^2$ are collinear. Also, under projective duality we have $A \longleftrightarrow \overleftrightarrow{\Pi\Pi'}$, $B \longleftrightarrow \overleftrightarrow{\Gamma\Gamma'}$, and $C \longleftrightarrow \overleftrightarrow{\Xi\Xi'}$. Therefore, $\overleftrightarrow{\Pi\Pi'}$, $\overleftrightarrow{\Gamma\Gamma'}$, and $\overleftrightarrow{\Xi\Xi'}$ are concurrent in $\mathbb{P}^{2*}$. $\square$

Chapter 3

Conics in the projective plane

We now direct the focus of this paper to the investigation of conics in the projective plane. We have studied lines as being the solution set to a homogeneous linear equation, and so it is natural to progress to the solutions of quadratic equations. Our first goal is to show a one-to-one correspondence between elements of Proj(1) and conics. Before we depart, let us make the following definition.

Definition 3.0.3. A **conic** is the set of points in $\mathbb{P}^2$ satisfying a homogeneous quadratic equation in $\mathbf{x} = (x_0, x_1, x_2)$. That is, a conic is the set

$$\mathcal{C} = \{ax_1^2 + bx_1x_2 + cx_2^2 + dx_0x_1 + ex_0x_2 + fx_0^2 = 0\} \subset \mathbb{P}^2,$$

where $(a, b, c, d, e, f) \neq \mathbf{0}$. Furthermore, a conic is said to be **nondegenerate** if it is nonempty and is neither a point nor a pair of lines. We shall see the algebraic characterization of nondegenerate conics in section 3.3.1 when we study quadratic forms.

Remark. If we only consider those points of $\mathbb{P}^2$ that are not in $\mathbb{P}^1_\infty$, then $x_0 \neq 0$ and so we can let $[x_0, x_1, x_2] = [1, x, y]$. Then our conic becomes

$$\mathcal{C} = \{ax^2 + bxy + cy^2 + dx + ey + f = 0\},$$

which is the familiar equation of a conic (section) in $\mathbb{R}^2$. Recall that we can classify nondegenerate conics by the discriminant. That is, if $b^2 - 4ac < 0$ then the equation represents an ellipse. Moreover, if it is also the case that $a = c$ and $b = 0$, then the equation represents a circle. If $b^2 - 4ac = 0$ then we have a parabola, and if $b^2 - 4ac > 0$, then the conic is a hyperbola.

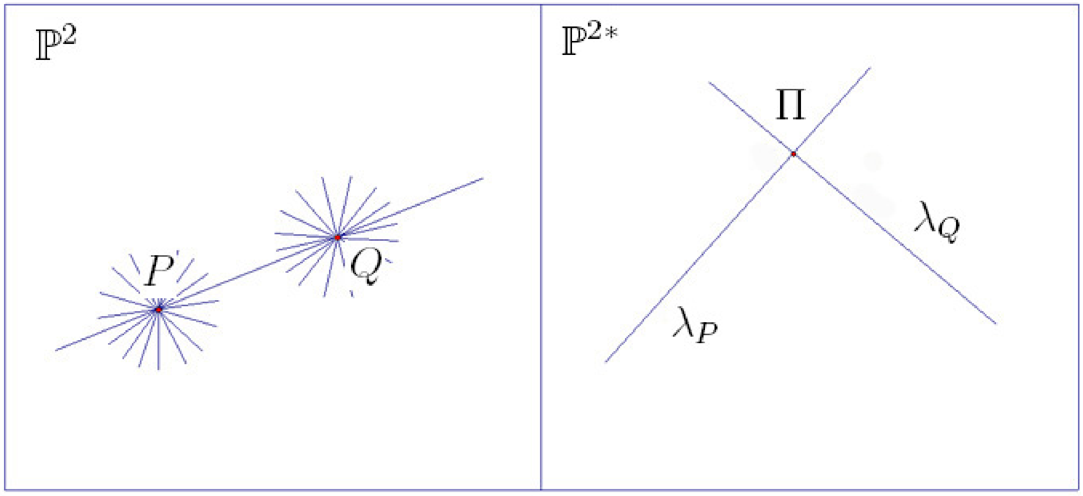


Figure 3.1: The pencils λ_P and λ_Q in $\mathbb{P}^2$ and in $\mathbb{P}^{2*}$

3.1 Projective generation of the conic

Let P and Q be points in $\mathbb{P}^2$ and let λ_P and λ_Q be their respective pencils of lines. Under projective duality, λ_P and λ_Q are lines in $\mathbb{P}^{2*}$. So there are vectors $\boldsymbol{\zeta}, \boldsymbol{\eta}, \boldsymbol{\rho}, \boldsymbol{\sigma} \in \mathbb{R}^{3*}$ such that

$$\lambda_P = \{[s\boldsymbol{\zeta} + t\boldsymbol{\eta}] : s, t \in \mathbb{R} \text{ not both zero}\} \text{ and}$$
$$\lambda_Q = \{[s\boldsymbol{\rho} + t\boldsymbol{\sigma}] : s, t \in \mathbb{R} \text{ not both zero}\}.$$

Let $\overleftrightarrow{PQ} \subset \mathbb{P}^2$ correspond to the point $\Pi \in \mathbb{P}^{2*}$. Since $\overleftrightarrow{PQ}$ is common to both pencils, it follows that lines λ_P and λ_Q intersect at Π. Therefore, without loss of generality, take $\boldsymbol{\rho} = \boldsymbol{\eta} = \boldsymbol{\pi}$. See Figure 3.1.

If we fix $[s, t] \in \mathbb{P}^1$ then we have fixed two points in $\mathbb{P}^{2*}$, one on λ_P and the other on λ_Q. Back in $\mathbb{P}^2$, these points correspond to lines

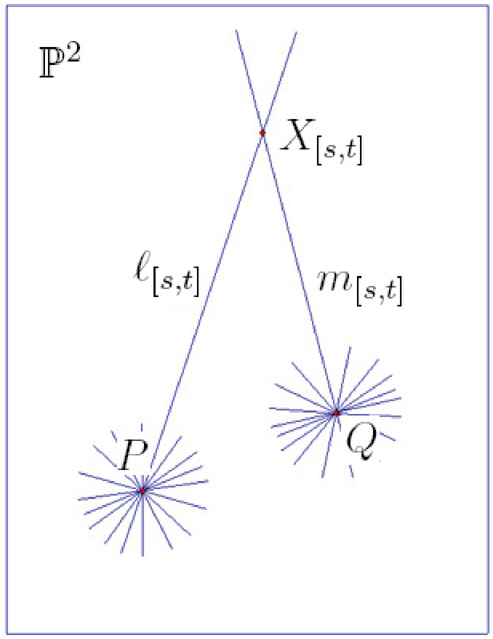


Figure 3.2: The intersection point $X_{[s,t]}$ of $\ell_{[s,t]}$ and $m_{[s,t]}$.

$$\ell_{[s,t]} = \{[\mathbf{x}] : (s\boldsymbol{\zeta} + t\boldsymbol{\pi}) \cdot \mathbf{x} = 0\}, \text{ which passes through } P, \text{ and}$$

$$m_{[s,t]} = \{[\mathbf{x}] : (s\boldsymbol{\pi} + t\boldsymbol{\sigma}) \cdot \mathbf{x} = 0\}, \text{ which passes through } Q.$$

Let $X_{[s,t]} = \ell_{[s,t]} \cap m_{[s,t]}$. We want to understand the locus of points $X_{[s,t]}$ as $[s,t]$ varies through $\mathbb{P}^1$. See Figure 3.2.

Since $X_{[s,t]} = \ell_{[s,t]} \cap m_{[s,t]}$, its coordinates (up to scalar multiples) give the nontrivial solution to the system

$$(s\boldsymbol{\zeta} + t\boldsymbol{\pi}) \cdot \mathbf{x} = 0$$
$$(s\boldsymbol{\pi} + t\boldsymbol{\sigma}) \cdot \mathbf{x} = 0$$

Consider this same system as a system of equations in (s,t). That is,

$$s(\boldsymbol{\zeta} \cdot \mathbf{x}) + t(\boldsymbol{\pi} \cdot \mathbf{x}) = 0$$
$$s(\boldsymbol{\pi} \cdot \mathbf{x}) + t(\boldsymbol{\sigma} \cdot \mathbf{x}) = 0$$

This system has a nontrivial solution if and only if

$$\det\left(\begin{bmatrix} \boldsymbol{\zeta} \cdot \mathbf{x} & \boldsymbol{\pi} \cdot \mathbf{x} \\ \boldsymbol{\pi} \cdot \mathbf{x} & \boldsymbol{\sigma} \cdot \mathbf{x} \end{bmatrix}\right) = 0.$$

If $\boldsymbol{\zeta} = (\zeta_0, \zeta_1, \zeta_2)$, $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2)$, and $\boldsymbol{\sigma} = (\sigma_0, \sigma_1, \sigma_2)$, then the previous equation becomes

$$(\zeta_1\sigma_1 - \pi_1^2)x_1^2 + (\zeta_1\sigma_2 + \zeta_2\sigma_1 - 2\pi_1\pi_2)x_1x_2 + (\zeta_2\sigma_2 - \pi_2^2)x_2^2 + (\zeta_0\sigma_1 + \zeta_1\sigma_0 - 2\pi_0\pi_1)x_0x_1 +$$
$$(\zeta_0\sigma_2 + \zeta_2\sigma_0 - 2\pi_0\pi_2)x_0x_2 + (\zeta_0\sigma_0 - \pi_0^2)x_0^2 = 0,$$

which is by definition a conic.

Theorem 3.1.1. *Every projective transformation ϕ brings about a conic in $\mathbb{P}^2$, and conversely, each conic specifies a projective transformation.*

Proof. Let $\phi : \lambda_P \to \lambda_Q$ be given by $\phi(t) = \dfrac{\alpha t + \beta}{\gamma t + \delta}$. For convenience, let $P = [1, 0, 0]$ and $Q = [1, 1, 0]$. An arbitrary line $\ell \in \lambda_P$ is of the form

$$\ell = \{[\mathbf{x}] : -tx_1 + x_2 = 0\}, \text{ where } t \in \mathbb{P}^1.$$

Also, an arbitrary line $m \in \lambda_Q$ is of the form

$$m = \{[\mathbf{x}] : t'x_0 - t'x_1 + x_2 = 0\}, \text{ where } t' \in \mathbb{P}^1.$$

Now set $t' = \dfrac{\alpha t + \beta}{\gamma t + \delta}$. Then we obtain the equation

$$t' = \frac{\alpha\left(\dfrac{x_2}{x_1}\right) + \beta}{\gamma\left(\dfrac{x_2}{x_1}\right) + \delta}.$$

Plugging this back into the equation for m, we obtain

$$\frac{\alpha\left(\dfrac{x_2}{x_1}\right)+\beta}{\gamma\left(\dfrac{x_2}{x_1}\right)+\delta}x_0 - \frac{\alpha\left(\dfrac{x_2}{x_1}\right)+\beta}{\gamma\left(\dfrac{x_2}{x_1}\right)+\delta}x_1 + x_2 = 0.$$
$$\alpha x_0x_2 + \beta x_0x_1 - \alpha x_2x_1 - \beta x_1^2 + \gamma x_2^2 + \delta x_1x_2 = 0.$$

Remark. Note that we can determine which type of conic (ellipse, parabola, or hyperbola) in $\mathbb{R}^2$ that this projective transformation gives. If we assume $x_0 \neq 0$ and let $x = \dfrac{x_1}{x_0}$ and $y = \dfrac{x_2}{x_0}$, then the last equation becomes

$$-\beta x^2 + (\delta - \alpha)xy + \gamma y^2 + \beta x + \alpha y = 0.$$

So we can classify the conic based on the sign of the discriminant, $(\delta - \alpha)^2 + 4\beta\gamma$.

To prove the converse we let $\mathcal{C}$ be a conic given by the equation

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_0x_1 + ex_0x_2 + fx_0^2 = 0.$$

Choose $P, Q \in \mathcal{C}$. By Theorem 2.1.5, we may take $P = [1, 0, 0]$ and $Q = [0, 1, 0]$. Therefore $a = f = 0$ and so our conic reduces to

$$\mathcal{C} = \{bx_1x_2 + cx_2^2 + dx_0x_1 + ex_0x_2 = 0\}.$$

Suppose ℓ_R is a member of the pencil λ_P of lines through P which also contains the point $R = [r_0, r_1, r_2]$. Then

$$\ell_R = \{[\mathbf{x}] : -r_2x_1 + r_1x_2 = 0\}.$$

If we let $r_2/r_1 = t \in \mathbb{R} \cup \{\infty\}$ then we deduce that lines ℓ passing through P are of the form $x_2 = tx_1$.

Suppose that m_R is a member of the pencil λ_Q of lines through Q which also contains the point $R = [r_0, r_1, r_2]$. Then

$$m_R = \{[\mathbf{x}] : r_2x_0 - r_0x_2 = 0\}.$$

If we let $r_0/r_2 = t' \in \mathbb{R} \cup \{\infty\}$ then we deduce that lines m passing through Q are of the form $x_0 = t'x_2$.

Now taking the point $[t'tx_1, x_1, tx_1]$ and plugging it into the equation for $\mathcal{C}$ gives

$$bt + ct^2 + dt't + et't^2 = 0, \text{ which implies}$$
$$t' = \phi(t) = \frac{-ct - b}{et + d}.$$

Therefore the conic $\mathcal{C}$ is specified by the projective transformation ϕ. $\square$

Corollary 3.1.2. *If the projective transformation* $\phi \in \mathrm{Proj}(1)$ *is a perspectivity, then the conic it specifies is degenerate (i.e., a pair of lines).*

Proof. Without loss of generality, we can take ϕ to be the perspectivity given by $\phi = \dfrac{t}{t-1}$. Then the resulting conic is $\mathcal{C} = \{[\mathbf{x}] : x_2(x_2 - 2x_1 + x_0) = 0\}$, which is a pair of lines. $\square$

Example 3.1.1. Suppose $P = (-1, 0)$ and $Q = (1, 0)$ are points in $\mathbb{R}^2$. We wish to find the mapping f that sends lines through P to lines through Q by $f : \ell \mapsto m$ only if $\ell \perp m$. It is easy to check that

$$\{\ell : \ell = \{y = tx + t\}, t \in \mathbb{R} \cup \{\infty\}\}$$

gives the set of lines which pass through P, and

$$\{m : m = \{y = t'x - t'\}, t' \in \mathbb{R} \cup \{\infty\}\}$$

gives the set of lines which pass through Q.

Since we want lines ℓ and m to be perpendicular then we set $t' = -1/t$. Then the equation for m is given by

$$y = tx + t \Longrightarrow t = \frac{1-x}{y}.$$

Substituting this back into the equation for ℓ gives

$$y = \left(\frac{1-x}{y}\right)x + \frac{1-x}{y} \Longrightarrow x^2 + y^2 = 1,$$

which is the equation of the unit circle centered at the origin. In Section 3.3 we will learn how to embed the circle and other conics in $\mathbb{P}^2$.

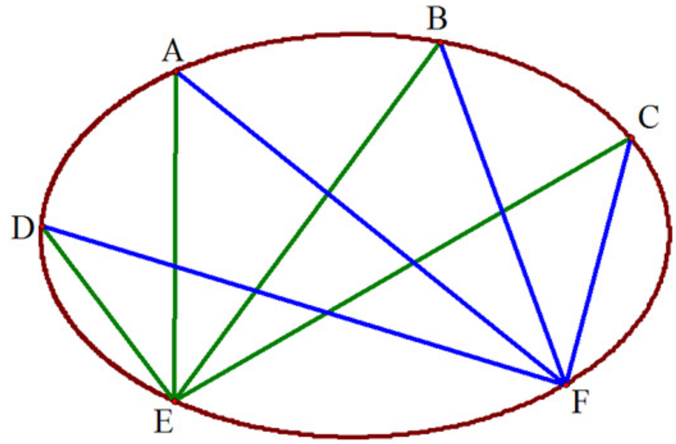


Figure 3.3: Conic interpretation of cross-ratio

3.2 The theorems of Pascal and Brianchon

In this section we will state and prove Pascal's theorem, which is a classical and essential when discussing conics in projective geometry. Then we state Brianchon's theorem, the dual of Pascal. We shall start with the following definition.

Definition 3.2.1. We say that *five* points in the plane are **in general position** if no four are collinear.

Lemma 3.2.1. *Suppose A, B, C, D, E, F are distinct points on a nondegenerate conic $\mathcal{C}$. Then*

$$|\overleftrightarrow{EA}, \overleftrightarrow{EB}; \overleftrightarrow{EC}, \overleftrightarrow{ED}| = |\overleftrightarrow{FA}, \overleftrightarrow{FB}; \overleftrightarrow{FC}, \overleftrightarrow{FD}|,$$

where we are computing the cross-ratio in λ_E, $\lambda_F \cong \mathbb{P}^1$.

Proof. Let $T : \lambda_E \to \lambda_F$ be the unique element of $\mathrm{Proj}(1)$ so that

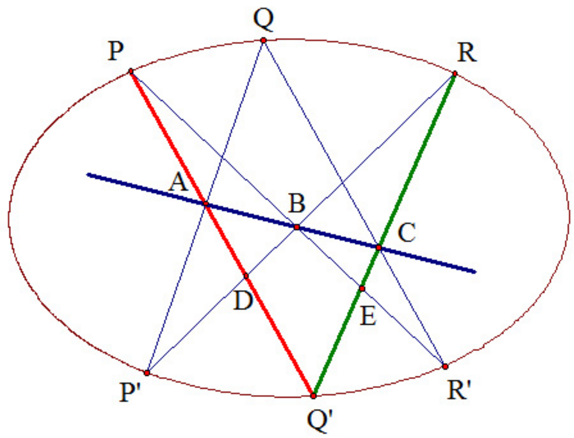


Figure 3.4: Pascal's Theorem

$$T(\overleftrightarrow{EA}) = \overleftrightarrow{FA},\ T(\overleftrightarrow{EB}) = \overleftrightarrow{FB}, \text{ and } T(\overleftrightarrow{EC}) = \overleftrightarrow{FC}.$$

Then for any point $D \in \mathcal{C}$, $T(\overleftrightarrow{ED}) = \overleftrightarrow{FD}$. Therefore the cross-ratios are equal. $\square$

From this lemma we deduce that fixing five points A, B, C, E, F in general position, we can obtain a unique transformation $T : \lambda_E \to \lambda_F$ and thus determine a conic section as the locus of points of intersection of the members of λ_E with their respective images in λ_F.

Theorem 3.2.2. *(Pascal). Let P, Q, R, P', Q', and R' be vertices of a hexagon that is inscribed in a conic. Then the three points $A = \overleftrightarrow{PQ'} \cap \overleftrightarrow{QP'}$, $B = \overleftrightarrow{PR'} \cap \overleftrightarrow{RP'}$, and $C = \overleftrightarrow{QR'} \cap \overleftrightarrow{RQ'}$ are collinear.*

Proof. Let $D = \overleftrightarrow{PQ'} \cap \overleftrightarrow{RP'}$ and $E = \overleftrightarrow{PR'} \cap \overleftrightarrow{RQ'}$. The projection $\pi_{P'} : \mathbb{P}^2 - \{P'\} \to \overleftrightarrow{PQ'}$ gives the following:

$$\pi_{P'}(P) = P,\ \pi_{P'}(Q) = A,\ \pi_{P'}(R) = D,\ \pi_{P'}(Q') = Q'.$$

Therefore $|\overleftrightarrow{P'P}, \overleftrightarrow{P'Q}; \overleftrightarrow{P'R}, \overleftrightarrow{P'Q'}| = |P, A, D, Q'|$. Similarly, the projection $\pi_{R'} : \mathbb{P}^2 - \{R'\} \to \overleftrightarrow{RQ'}$ gives the following:

$$\pi_{R'}(P) = E,\ \pi_{R'}(Q) = C,\ \pi_{R'}(R) = R,\ \pi_{R'}(Q') = Q'.$$

Therefore $|\overleftrightarrow{R'P}, \overleftrightarrow{R'Q}; \overleftrightarrow{R'R}, \overleftrightarrow{R'Q'}| = |E, C; R, Q'|$. Let $T : \lambda_{P'} \to \lambda_{R'}$ be such that

$$T : \overleftrightarrow{P'P} \mapsto \overleftrightarrow{R'P},\ \overleftrightarrow{P'Q} \mapsto \overleftrightarrow{R'Q},\ \overleftrightarrow{P'R} \mapsto \overleftrightarrow{R'R},\ \overleftrightarrow{P'Q'} \mapsto \overleftrightarrow{R'Q'}.$$

By the previous theorem, $|\overleftrightarrow{P'P}, \overleftrightarrow{P'Q}; \overleftrightarrow{P'R}, \overleftrightarrow{P'Q'}| = |\overleftrightarrow{R'P}, \overleftrightarrow{R'Q}; \overleftrightarrow{R'R}, \overleftrightarrow{R'Q'}|$, and thus $|P, A; D, Q'| = |E, C; R, Q'|$. The perspectivity $\pi_B : \overleftrightarrow{PQ'} \to \overleftrightarrow{RQ'}$ with center B sends P to E, D to R, and Q' to Q', and so it must also send A to C. Therefore (computing the cross-ratio as elements in $\lambda_P \cong \mathbb{P}^1$), $|\overleftrightarrow{BP}, \overleftrightarrow{BA}; \overleftrightarrow{BD}, \overleftrightarrow{BQ'}| = |\overleftrightarrow{BE}, \overleftrightarrow{BC}; \overleftrightarrow{BR}, \overleftrightarrow{BQ'}|$. Since $\overleftrightarrow{BD} = \overleftrightarrow{BR}$ and $\overleftrightarrow{BP} = \overleftrightarrow{BE}$, it must be the case that $\overleftrightarrow{BA} = \overleftrightarrow{BC}$. Therefore, A, B, and C are collinear. $\square$

Remark. Notice that if our conic happens to be the degenerate case of a pair of lines, then Pascal's theorem reduces to Pappus' theorem. The dual of Pascal's theorem is also known as Brianchon's theorem, as he discovered it nearly 170 years after Pascal.

Theorem 3.2.3. *(Brianchon). If a hexagon is circumscribed about a conic, then its three diagonals are concurrent; see Figure 3.5.*

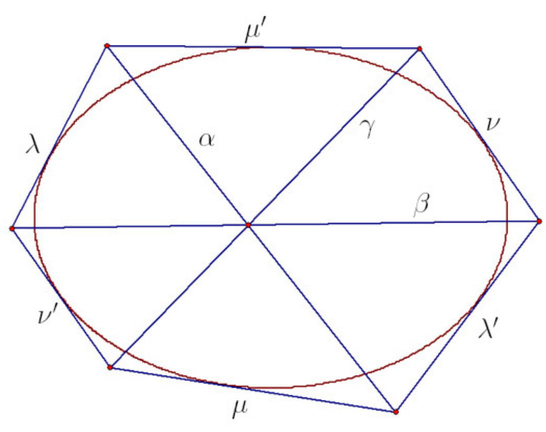


Figure 3.5: Brianchon's Theorem

3.3 Quadrics and embedding $\mathbb{R}^2$ conics in $\mathbb{P}^2$

In this section we will study quadratic forms and use them to prove that all nondegenerate conics are projectively equivalent. Then we will look at the ellipse, parabola, and hyperbola separately and interpret each in $\mathbb{P}^2$. It will be important to keep in mind that the line at infinity is given by $\mathbb{P}^1_\infty = \{x_0 = 0\}$.

3.3.1 Quadratic forms

Definition 3.3.1. Let A be an $n \times n$ symmetric matrix. The associated **quadratic form** $Q : \mathbb{R}^n \to \mathbb{R}$ is defined by $Q(\mathbf{x}) = A\mathbf{x} \cdot \mathbf{x}$.

Lemma 3.3.1. *Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form. Then there is an orthonormal basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ for $\mathbb{R}^n$ and real numbers $\lambda_1, \ldots, \lambda_n$ such that, writing $\mathbf{x} = \sum_{i=1}^{n} y_i \mathbf{v}_i$, we have $Q(\mathbf{x}) = \tilde{Q}(\mathbf{y}) = \sum_{i=1}^{n} \lambda_i y_i^2$.*

Proof. By definition, $Q(\mathbf{x}) = A\mathbf{x} \cdot \mathbf{x}$ for some symmetric matrix A. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the symmetric linear map that is given by A. By the Spectral theorem, there are real numbers $\lambda_1, \ldots, \lambda_n$ and an orthonormal basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with the property that $T\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for each $i = 1, \ldots, n$. Then we have

$$Q(\mathbf{v}_i) = T(\mathbf{v}_i) \cdot \mathbf{v}_i = (\lambda_i \mathbf{v}_i) \cdot \mathbf{v}_i = \lambda \|\mathbf{v}_i\|^2 = \lambda_i(1) = \lambda_i.$$

Thus,

$$\begin{aligned} Q(\mathbf{x}) = Q\left(\sum_{i=1}^{n} y_i \mathbf{v}_i\right) &= T\left(\sum_{i=1}^{n} y_i \mathbf{v}_i\right) \cdot \left(\sum_{j=1}^{n} y_j \mathbf{v}_j\right) \\ &= \left(\sum_{i=1}^{n} \lambda_i y_i \mathbf{v}_i\right) \cdot \left(\sum_{i=j}^{n} y_j \mathbf{v}_j\right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i y_i y_j (\mathbf{v}_i \cdot \mathbf{v}_j). \end{aligned}$$

Note that whenever $i \neq j$, $\mathbf{v}_i \cdot \mathbf{v}_j = 0$, and when $i = j$, $\mathbf{v}_i \cdot \mathbf{v}_j = 1$. So then we have

$$Q(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i y_i^2 = \tilde{Q}(\mathbf{y}).$$

□

Example 3.3.1. Let $Q(x_1, x_2) = 6x_1x_2 + 8x_2^2$. The corresponding symmetric matrix A is

$$A = \begin{bmatrix} 0 & 3 \\ 3 & 8 \end{bmatrix}.$$

The characteristic polynomial of A is $p(t) = t^2 - 8t - 9$ and so the eigenvalues are $\lambda_1 = 9$, $\lambda_2 = -1$. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$. We wish to normalize them in order to obtain an orthonormal basis $\mathbf{v}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Therefore,

$$A = P\tilde{A}P^{-1}, \text{ where } P = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \text{ and } \tilde{A} = \begin{bmatrix} 9 & 0 \\ 0 & -1 \end{bmatrix}.$$

Making the substitution $\mathbf{y} = P^{-1}\mathbf{x} = P^T\mathbf{x}$, we obtain

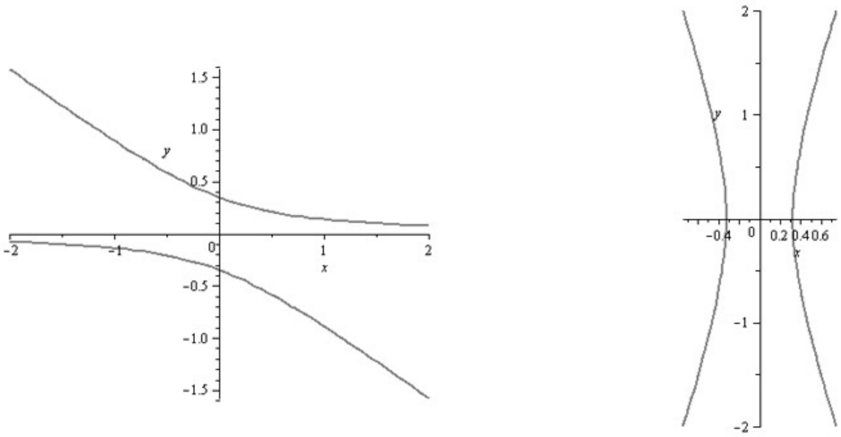
$$Q(\mathbf{x}) = A\mathbf{x} \cdot \mathbf{x} = \mathbf{x}^T A\mathbf{x} = \mathbf{x}^T (P\tilde{A}P^T)\mathbf{x} = (P^T\mathbf{x})^T \tilde{A}(P^T\mathbf{x}) = \mathbf{y}^T \tilde{A}\mathbf{y} = 9y_1^2 - y_2^2.$$

So in the y-coordinates, the quadratic form is given by

$$\tilde{Q}(\mathbf{y}) = 9y_1^2 - y_2^2,$$

We wish to identify the conic section $Q(\mathbf{x}) = 1$. If we use the y-coordinate form, we have $9y_1^2 - y_2^2 = 1$. Now the conic is much easier to understand and we see that this is a hyperbola with asymptotes $y_2 = \pm\frac{1}{3}y_1$. See Figure 3.6.

Consider the symmetric matrix $C = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$. The associated quadratic form Q is given by



$6x_1x_2 + 8x_2^2 = 1$ $\qquad$ $9y_1^2 - y_2^2 = 1$

Figure 3.6: Compare $6x_1x_2 + 8x_2^2 = 1$ with $9y_1^2 - y_2^2 = 1$.

$$Q(\mathbf{x}) = C\mathbf{x} \cdot \mathbf{x} = ax_0^2 + bx_0x_1 + cx_1^2 + dx_0x_2 + ex_1x_2 + fx_2^2.$$

Thus, any 3×3 symmetric matrix determines a conic. Note that the matrix C has six independent parameters. However, these parameters are determined up to scalars and so a conic has five degrees of freedom. This checks with the observation we made earlier that five points in general position determine a conic.

Lemma 3.3.2. *Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form. Then there is an orthogonal basis $\mathbf{v}'_1, \dots, \mathbf{v}'_n$ so that, writing $\mathbf{x} = \sum_{i=1}^{n} z_i \mathbf{v}'_i$, $Q(\mathbf{x}) = \sum_{i=1}^{n} \epsilon_i z_i^2$, where $\epsilon_i = -1, 0,$ or 1.*

Proof. As before, write $Q(\mathbf{x}) = T(\mathbf{x}) \cdot \mathbf{x}$ for some symmetric linear map T. Then by the spectral theorem, we can find an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$. Suppose we reorder the eigenvalues so that $\lambda_1, \dots, \lambda_k > 0$, $\lambda_{k+1}, \dots, \lambda_m < 0$, and $\lambda_{m+1}, \dots, \lambda_n = 0$. Let $\mathbf{v}'_j = \dfrac{1}{\sqrt{|\lambda_j|}}\mathbf{v}_j$ for every $j = 1, \dots, m$ and $\mathbf{v}'_j = \mathbf{v}_j$ for $j = m+1, \dots, n$. As in Lemma 3.3.1, we can obtain the quadratic form $\tilde{Q}(\mathbf{y}) = \sum_{i=1}^{n} \lambda_i y_i^2$ by writing $\mathbf{x} = \sum_{i=1}^{n} y_i \mathbf{v}_i$. By setting $\mathbf{x} = \sum_{i=1}^{n} z_i \mathbf{v}_i'$, we have $z_i = \sqrt{|\lambda_i|} y_i$ for each $i = 1, \ldots, m$ and $z_i = y_i$ for $i = m+1, \ldots, n$. Thus,

$$Q(\mathbf{x}) = \tilde{Q}(\mathbf{y}) = \sum_{i=1}^{n} \lambda_i y_i^2 = \sum_{i=1}^{m} \lambda_i y_i^2 = \sum_{i=1}^{m} \frac{\lambda_i}{|\lambda_i|} z_i^2 = \sum_{i=1}^{k} z_i^2 - \sum_{i=k+1}^{m} z_i^2.$$

□

We will now consider the case when $n = 3$ and return to projective conics. In the following proposition we use Lemmas 3.3.1 and 3.3.2 to algebraically characterize degenerate and nondegenerate conics.

Proposition 3.3.3. *Suppose $\mathcal{C}$ is a conic given by the 3×3 symmetric matrix C. Let $Q(\mathbf{x}) = C\mathbf{x} \cdot \mathbf{x}$ be the associated quadratic form. Writing $Q(\mathbf{x}) = \sum_{i=1}^{n} \epsilon_i z_i^2$ as in Lemma 3.3.2, the conic $\mathcal{C}$ is nondegenerate if and only if the ϵ_i's are neither all 1 nor all -1, and no $\epsilon_i = 0$.*

Proof. Consider the following cases:

1. Suppose $\epsilon_i = 1$ for every i or $\epsilon_i = -1$ for every i. If either of these cases happen we obtain the equation $x_0^2 + x_1^2 + x_2^2 = 0$. There is no solution in $\mathbb{P}^2$ to this equation and so we say that this conic is empty, which is a degenerate conic.

2. Suppose $\epsilon_i = 0$ and ϵ_j and ϵ_k are either both 1 or both -1. In each of these cases we obtain the equation $x_j^2 + x_k^2 = 0$. This equation is satisfied only when $x_j = x_k = 0$. Therefore the conic is a single point, which is also a degenerate conic.

3. Suppose $\epsilon_i = -1$, $\epsilon_j = 0$, and $\epsilon_k = 1$. Then we have the equation $x_i^2 - x_k^2 = 0$. Since the left hand side is a difference of squares, we factor it to obtain $(x_i - x_k)(x_i + x_k) = 0$. This equation gives two lines $x_i = x_k$ and $x_i = -x_j$. As we stated before, a pair of lines is a degenerate conic.

4. Suppose two of the $\epsilon_i = \epsilon_j = 0$ and $\epsilon_k = \pm 1$. In this case, we have the equation $x_k^2 = 0$. Thus the conic is a single line that we count twice. This is the final case of a degenerate conic.

5. Suppose $\epsilon_i = \epsilon_j = 1$ and $\epsilon_k = -1$ or vice versa. In this final case we have the equation $x_i^2 + x_j^2 - x_k^2 = 0$. The conic presented here is nondegenerate.

In summary, in order for $\mathcal{C}$ to be nondegenerate, then $\epsilon_i \neq 0$ for any i and $\epsilon_i \neq \epsilon_j$ for some i and j. $\square$

Theorem 3.3.4. *All nondegenerate conics in $\mathbb{P}^2$ are projectively equivalent.*

Proof. Let C be a 3×3 symmetric matrix that defines the conic $\mathcal{C}$ by the equation

$$Q(\mathbf{x}) = \mathbf{x}^T C \mathbf{x} = 0.$$

Apply the projective transformation $A \in \text{Proj}(1)$ to our conic. Then the transformed equation is

$$(A\mathbf{x})^T C (A\mathbf{x}) = \mathbf{x}^T (A^T C A)\mathbf{x} = \mathbf{x}^T C' \mathbf{x} = 0 \text{ (where } C' = A^T C A),$$

which is another conic $\mathcal{C}'$. Note that $\mathcal{C}'$ is a nondegenerate conic, since $\det C \neq 0$ if and only if $\det C' \neq 0$ and the eigenvalues of C are neither all 1 nor all -1 if and only if the eigenvalues of C' are neither all 1 nor all -1.

Suppose $\mathcal{C}$ is a nondegenerate conic. Then by Proposition 3.3.3 we can transform $\mathcal{C}$ into one of the following:

$$x_0^2 + x_1^2 - x_2^2 = 0 \quad (1)$$
$$x_0^2 - x_1^2 - x_2^2 = 0 \quad (2)$$

It remains to show that these conics are projectively equivalent to each other. Setting $x_0 = y_2$, $x_1 = y_1$, and $x_2 = y_0$ changes equation (1) into $y_0^2 + y_1^2 - y_2^2 = 0$, which is equation (2). This change of variable is given by

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}.$$

Note that the 3×3 matrix is invertible and so we have projective equivalence among all conics. □

Lemma 3.3.5. *Let $\mathcal{C}$ be a nondegenerate conic. Then any line $\ell \subset \mathbb{P}^2$ meets $\mathcal{C}$ in at most two places.*

Proof. Without loss of generality, let $\ell = \{x_2 = 0\}$. Then $\mathcal{C} \cap \ell = \{[x_0, x_1, 0] \in \mathbb{P}^2 : ax_1^2 + dx_0x_1 + fx_0^2 = 0\}$. Either $[0,1,0] \in \mathcal{C} \cap \ell$ or $[0,1,0] \notin \mathcal{C} \cap \ell$. If it is the case that $[0,1,0] \in \mathcal{C} \cap \ell$, then $a = 0$ and $[d, -f, 0]$ is the only other possible point of intersection. Note that we cannot have $d = f = 0$ as well, because that would mean that $\ell \subset \mathcal{C}$, which contradicts the fact that $\mathcal{C}$ is nondegenerate.

If we have the other case, where $[0,1,0] \notin \mathcal{C} \cap \ell$, then any point of intersection is of the form $[1, t, 0]$ for $t \in \mathbb{R}$. Substituting this back into our equation for $\mathcal{C}$ yields $at^2 + dt + f = 0$. This is just a quadratic equation, which has at most two real roots. □

Remark. When $\mathcal{C} \cap \ell$ is a single point P, then ℓ is the tangent line to $\mathcal{C}$ at P. The only time $at^2 + dt + f = 0$ has exactly one solution is when $d^2 - 4af = 0$ and so there is only one instance when ℓ intersects $\mathcal{C}$ in exactly one point, namely when $\ell \cap \mathcal{C} = [1, -d/2a, 0]$. Therefore ℓ must lie tangent to $\mathcal{C}$ at the point P.

3.3.2 Embedding the ellipse

The ellipse $\mathcal{E} \subset \mathbb{R}^2$ is given by the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, for $a, b \in \mathbb{R} - \{0\}$. To understand this with projective coordinates we let $x = \dfrac{x_1}{x_0}$ and $y = \dfrac{x_2}{x_0}$. Then we obtain the equation

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = x_0^2.$$

Since there is no (real) solution to this equation when $x_0 = 0$, then we conclude that the intersection $\mathcal{E} \cap \mathbb{P}^1_\infty = \emptyset$ (see Figure 3.7).

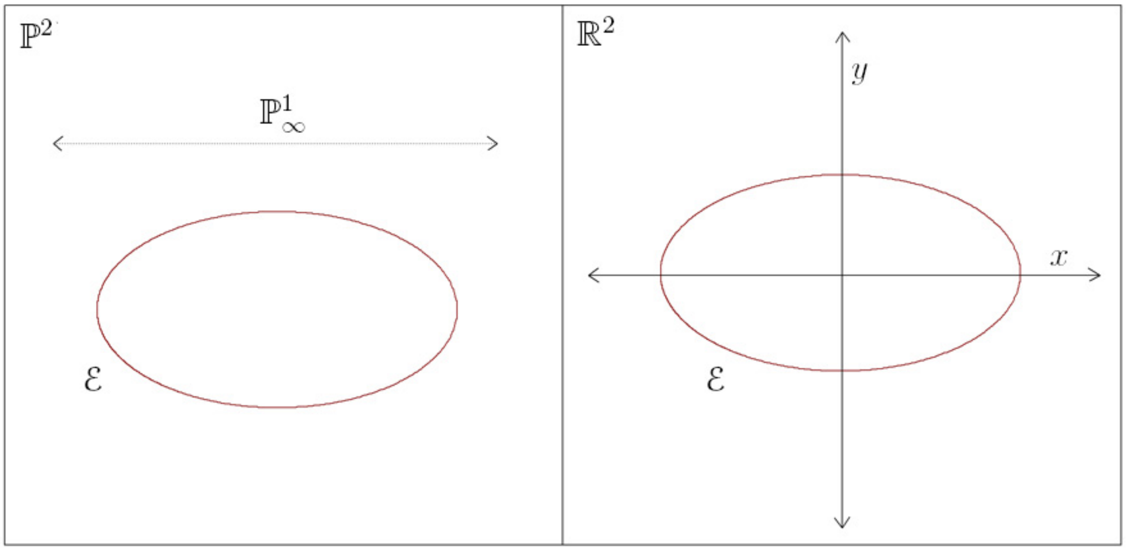


Figure 3.7: The intersection of $\mathcal{E}$ and $\mathbb{P}^1_\infty$ is empty

3.3.3 Embedding the parabola

The parabola $\mathcal{P} \subset \mathbb{R}^2$ is given by the equation $x^2 = 4ay$, for $a \in \mathbb{R} - \{0\}$. As we did above, we let $x = \dfrac{x_1}{x_0}$ and $y = \dfrac{x_2}{x_0}$. Then we obtain the equation

$$x_1^2 = 4ax_2x_0.$$

When we set $x_0 = 0$, then the equation is satisfied when $x_1 = 0$. Hence $\mathcal{P} \cap \mathbb{P}^1_\infty = [0, 0, 1]$ (see Figure 3.8).

3.3.4 Embedding the hyperbola

The hyperbola $\mathcal{H} \subset \mathbb{R}^2$ is given by $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, for $a, b \in \mathbb{R} - \{0\}$. We make our substitution once more and obtain

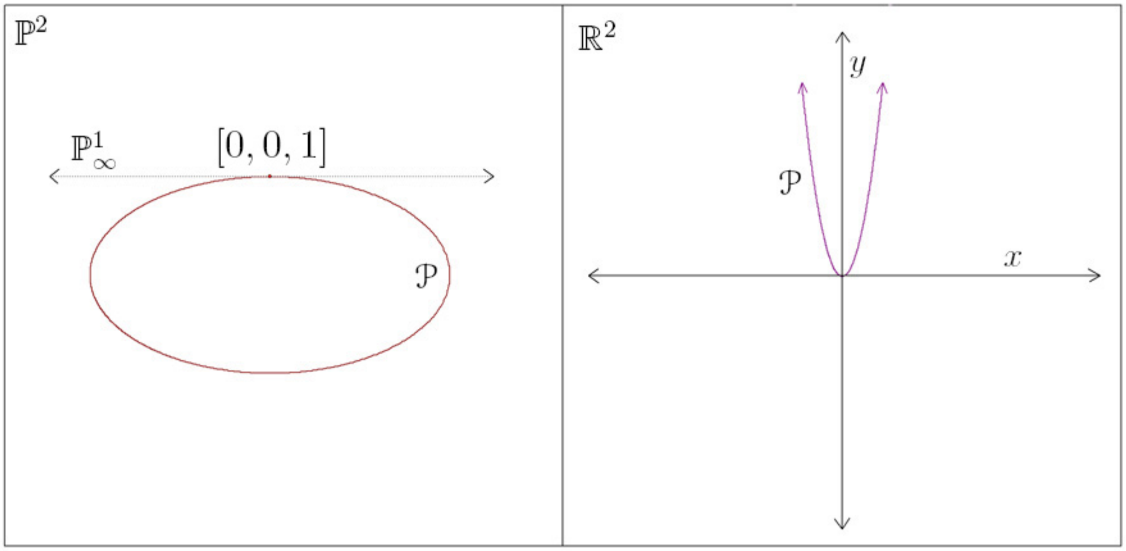


Figure 3.8: The intersection of $\mathcal{P}$ and $\mathbb{P}^1_\infty$ is $[0, 0, 1]$

$$\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = x_0^2.$$

If we set $x_0 = 0$, then the equation is satisfied when $x_1 = \pm a$ and $x_2 = \pm b$. Therefore we have two points of intersection. Namely $\mathcal{H} \cap \mathbb{P}^1_\infty = \{[0, a, b], [0, -a, b]\}$ (see Figure 3.9).

3.4 The dual conic $\mathcal{C}^*$

In this section we extend projective duality to our study of conics.

Lemma 3.4.1. *Let $\mathcal{C}$ be a nondegenerate conic given by the equation $\mathbf{x}^T C\mathbf{x} = 0$, where C is a 3×3 invertible, symmetric matrix. Then the tangent line of $\mathcal{C}$ at the point $P = [\mathbf{p}]$ is $(C\mathbf{p}) \cdot \mathbf{x} = 0$.*

Proof. The line $\ell = \{(C\mathbf{p}) \cdot \mathbf{x} = 0\}$ passes through point P since $\mathbf{p}^T C\mathbf{p} = 0$ (P lies on $\mathcal{C}$). Suppose $Q = [\mathbf{q}]$ is some other point that lies on $\mathcal{C}$, then $\mathbf{q}^T C\mathbf{q} = 0$. Now if Q also lies on

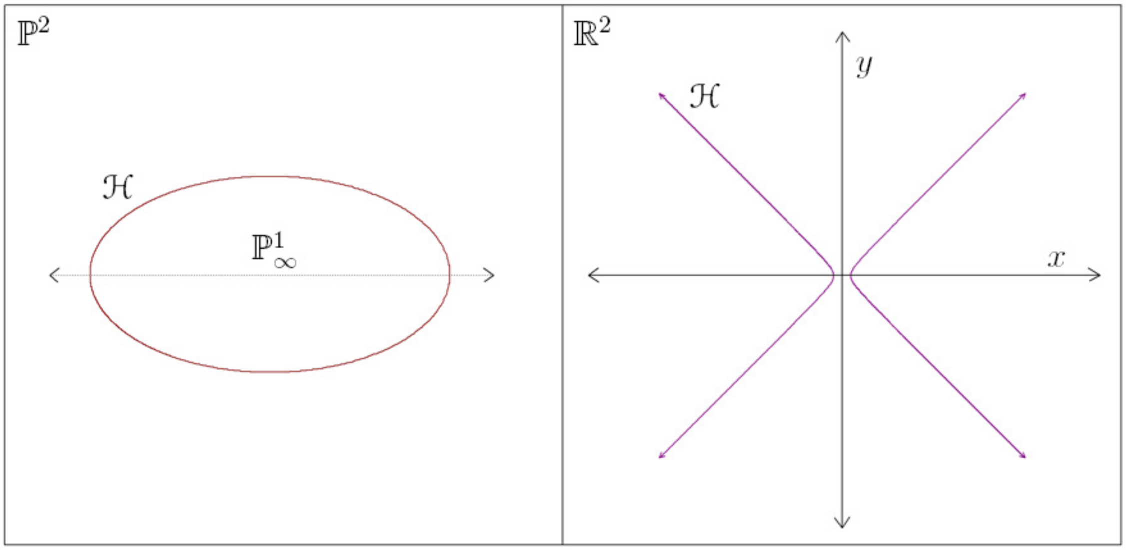


Figure 3.9: The intersection of $\mathcal{H}$ and $\mathbb{P}^1_\infty$ is $\{[0, a, b], [0, -a, b]\}$

ℓ then $\mathbf{p}^T C\mathbf{q} = 0$. This means that $(\mathbf{p} + t\mathbf{q})^T C(\mathbf{p} + t\mathbf{q}) = 0$ for all t. Thus $\ell \subset \mathcal{C}$, which contradicts the fact that $\mathcal{C}$ is nondegenerate. Therefore ℓ meets $\mathcal{C}$ only at P, so it is tangent to $\mathcal{C}$ at P. □

We also give an alternate proof, which uses calculus to find the tangent line.

Proof. Suppose that $\mathcal{C} = \{[\mathbf{x}] : ax_1^2 + bx_1x_2 + cx_2^2 + dx_0x_1 + ex_0x_2 + fx_0^2 = 0\}$. Then the invertible, symmetric matrix $C = \begin{bmatrix} f & d/2 & e/2 \\ d/2 & a & b/2 \\ e/2 & b/2 & c \end{bmatrix}$. Now suppose S is the level surface given by $F(x_0, x_1, x_2) = ax_1^2+bx_1x_2+cx_2^2+dx_0x_1+ex_0x_2+fx_0^2 = 0$ in $\mathbb{R}^3$. Since $P = [p_0, p_1, p_2]$ satisfies the equation of the conic, then $\mathbf{p} = (p_0, p_1, p_2)$ is a point on the surface S. Hence, the normal vector of the tangent plane to S at (p_0, p_1, p_2) is

$$\nabla F(p_0, p_1, p_2) = (dp_1 + ep_2 + 2fp_0, 2ap_1 + bp_2 + dp_0, bp_1 + 2cp_2 + ep_0).$$

Therefore, the tangent line ℓ_P to $\mathcal{C}$ at P is given by the equation

$$\nabla F(p_0, p_1, p_2) \cdot x = 0.$$

However, $C\mathbf{p} \equiv \nabla F(p_0, p_1, p_2)$, and so the tangent line of $\mathcal{C}$ at P is $(C\mathbf{p}) \cdot \mathbf{x} = 0$. $\square$

Definition 3.4.1. Let $\mathcal{C} \subset \mathbb{P}^2$ be a nondegenerate conic. Then the **dual conic** is defined to be $\mathcal{C}^* = \{\ell_P \in \mathbb{P}^{2*} : \ell_P \text{ is the tangent line to } \mathcal{C} \text{ at } P\}$. See Figure 3.10.

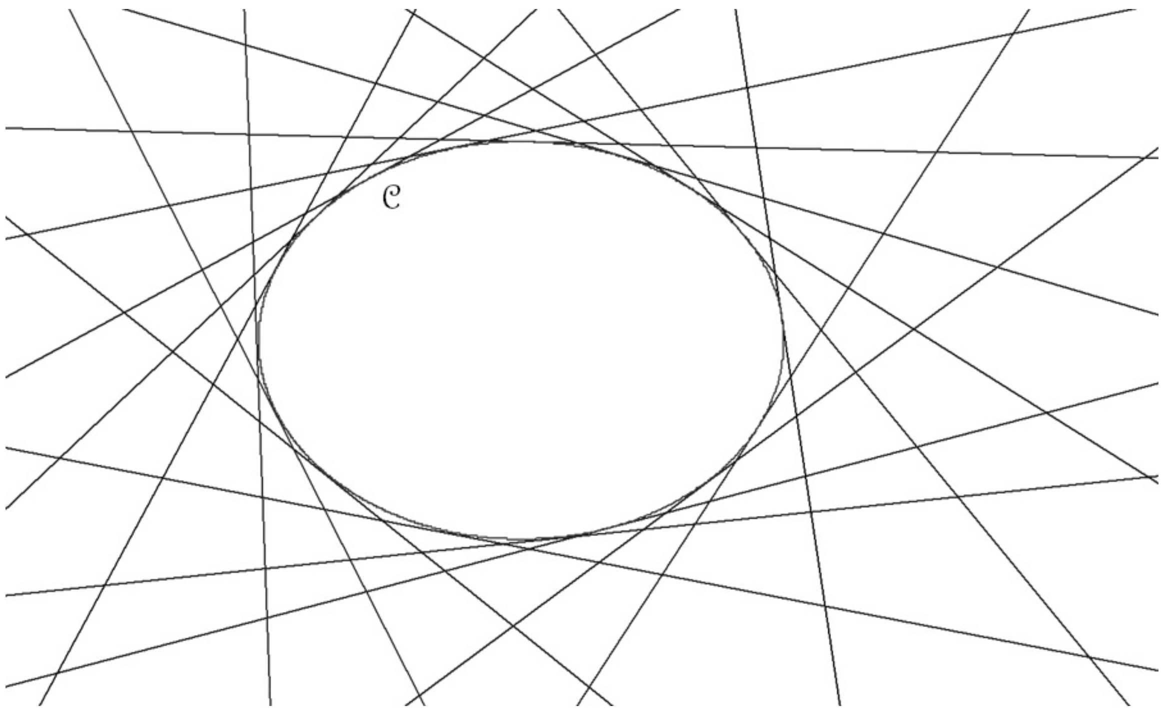


Figure 3.10: The dual conic $\mathcal{C}^*$.

Theorem 3.4.2. *The dual conic $\mathcal{C}^* \subset \mathbb{P}^2$ of the nondegenerate conic $\mathcal{C} \subset \mathbb{P}^2$ is another nondegenerate conic.*

Proof. Suppose $\mathcal{C}$ is given by the equation $\mathbf{x}^T C\mathbf{x} = 0$, where C is a 3×3 invertible, symmetric matrix and suppose that $P = [\mathbf{p}]$ is an arbitrary point lying on $\mathcal{C}$: $\mathbf{p}^T C\mathbf{p} = 0$. By Lemma 3.4.1, the equation of the tangent line of $\mathcal{C}$ at P is $\ell = \{(C\mathbf{p}) \cdot \mathbf{x} = 0\}$. If we let $\boldsymbol{\xi} = C\mathbf{p}$, then we obtain $C^{-1}\boldsymbol{\xi} = \mathbf{p}$. Substituting this back in for $\mathbf{p}$ yields

$$\mathbf{p}^T C\mathbf{p} = (C^{-1}\boldsymbol{\xi})^T C(C^{-1}\boldsymbol{\xi}) = \boldsymbol{\xi}^T(C^{-1})^T\boldsymbol{\xi} = \boldsymbol{\xi}^T C^{-1}\boldsymbol{\xi} = 0,$$

which is a conic. □

Remark. From the previous proof, we see that when a conic $\mathcal{C}$ is defined by a 3×3 invertible, symmetric matrix C, then the dual conic $\mathcal{C}^*$ is defined by the inverse matrix C^{-1}.

Definition 3.4.2. Let $\{C_t : t \in \mathbb{P}^1\}$ be a family of curves in $\mathbb{P}^2$. We say that the curve C is the **envelope** of the family of curves if each member of the family is tangent to C at some point (depending on t).

Proposition 3.4.3. *Suppose the projective transformation $T : \lambda_P \to \lambda_Q$ determines the conic $\mathcal{C}$ by considering the locus of points of intersection of the members in the pencil λ_P with their respective images in the pencil λ_Q. In $\mathbb{P}^{2*}$, λ_P and λ_Q are lines and so the projective transformation T maps points in λ_P to points in λ_Q. The conic $\mathcal{C}$, which is a set of points in $\mathbb{P}^2$, is a family of lines in $\mathbb{P}^{2*}$, where each line in the family is the join of a point $\Pi \in \lambda_P$ and a point $\Gamma \in \lambda_Q$ and $T : \Pi \mapsto \Gamma$. The envelope of this family of lines is the dual conic $\mathcal{C}^*$.*

Proof. Suppose that ℓ and m are members of the pencils λ_P and λ_Q respectively and that $T : \ell \mapsto m$. Then their intersection $\ell \cap m = X$ is a point lying on the conic $\mathcal{C}$. Let t_X be the tangent line to $\mathcal{C}$ at X. Then under projective duality, X corresponds to the line λ_X joining Π and Γ, and t_X corresponds to the point Σ lying on λ_X. Since t_X is tangent to $\mathcal{C}$ at X, then $\Sigma \in \mathcal{C}^*$. It remains to show that λ_X is tangent to $\mathcal{C}^*$. Suppose that λ_X intersects $\mathcal{C}^*$ in a second point $\Sigma' \in \mathbb{P}^{2*}$. In $\mathbb{P}^2$, Σ' corresponds to a line that passes through X and is tangent to $\mathcal{C}$. Since $X \in \mathcal{C}$ then Σ' corresponds to the tangent line t_X. Hence $\Sigma' = \Sigma$ and so λ_X is tangent to $\mathcal{C}^*$. See Figure 3.11. □

The picture that we see of the dual conic $\mathcal{C}^*$ in $\mathbb{P}^{2*}$ is the envelope of some family of lines, and the family of lines itself is the dual picture of the original conic $\mathcal{C}$. The lines that make up the family are established by a projective transformation $T : \lambda_P \to \lambda_Q$, where $\overleftrightarrow{\Pi\Gamma}$ is a member of the family if and only if $T : \Pi \mapsto \Gamma$. See Figure 3.12.

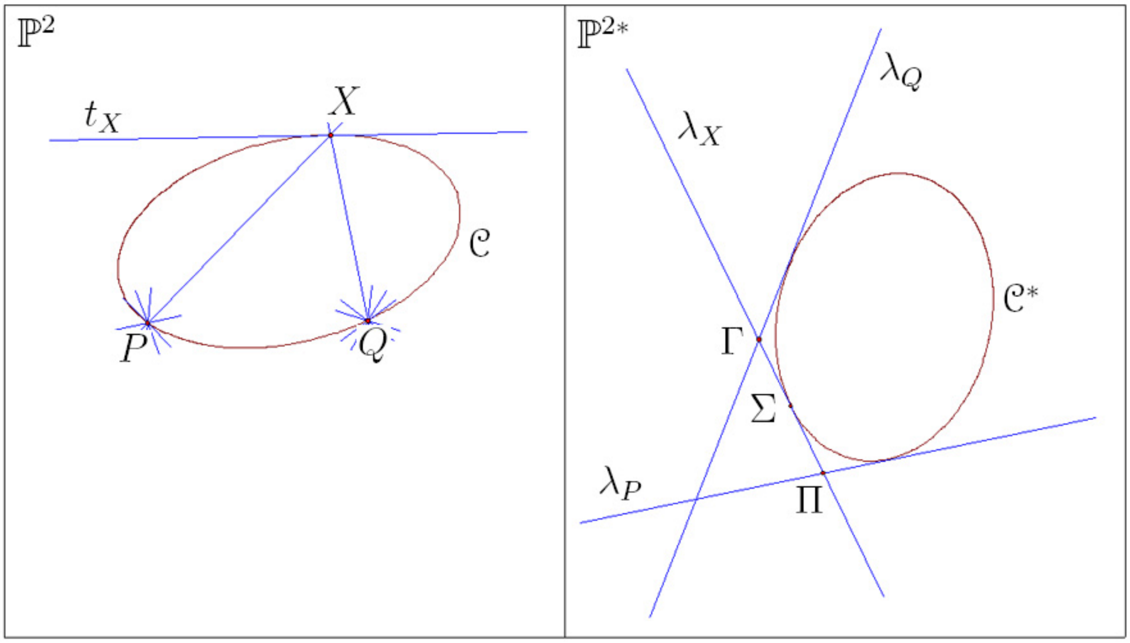


Figure 3.11: Comparing the conic $\mathcal{C}$ with its dual $\mathcal{C}^*$.

3.5 Classification of conics

In this last section we close with a question that remains to be answered. Suppose λ, μ, and ν are projective lines in $\mathbb{P}^{2*}$ and let $T : \lambda \to \nu$ be a transformation written as a composition of perspectivities $f : \mu \to \nu$ with center Π and $g : \lambda \to \mu$ with center Γ, as in Theorem 2.2.2. By translating this setup to $\mathbb{P}^2$, we interpret T as a projective transformation from a pencil of lines passing through some point A to another pencil of lines passing through some point C by first going through the auxiliary pencil of lines through B. The points of perspectivity Π and Γ in $\mathbb{P}^{2*}$ become lines (of projection) ℓ_Π and ℓ_Γ in $\mathbb{P}^2$. We wish to classify the conic generated by $T = f \circ g$ based on the geometry of the perspectivities. See Figure 3.13.

We now provide the answer to some particular cases. Recall that a single perspectivity results in a degenerate conic. So if the first line of projection ℓ_Π is a member of the pencil of

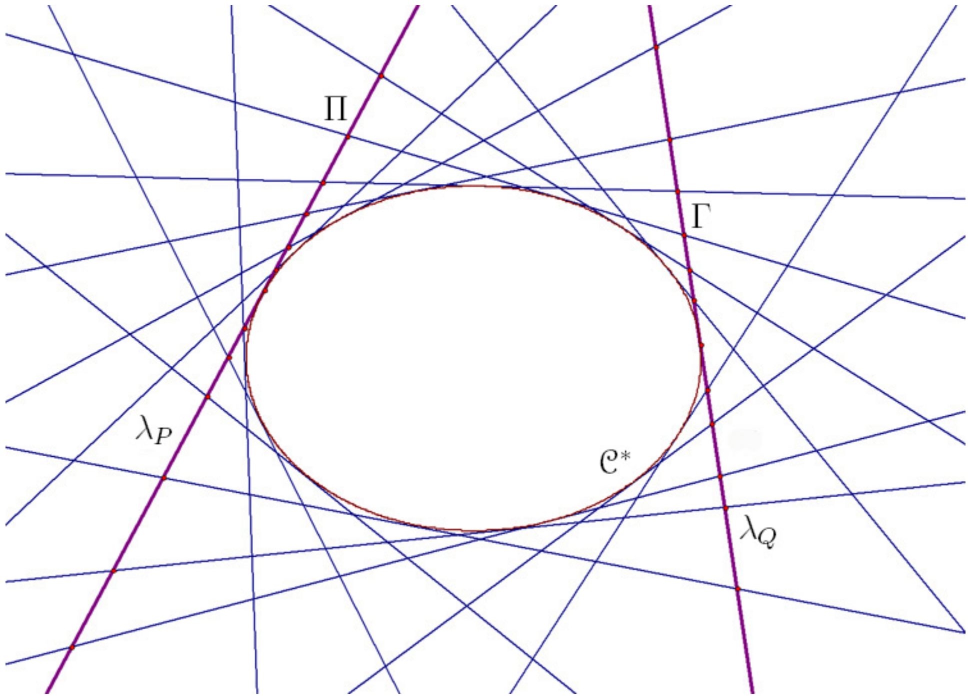


Figure 3.12: The dual conic $\mathcal{C}^*$

points through A or the pencil of points through B, then T is reduced to a single perspectivity. Therefore, T generates a pair of lines, which is a degenerate conic. We get the same result if the second line of projection ℓ_Γ is a member of the pencil of lines through B or the pencil of points through C.

Consider the case where $A \notin \mathbb{P}^1_\infty$ and $C \in \mathbb{P}^1_\infty$. Then the line at infinity $\mathbb{P}^1_\infty$ is a member of the pencil of points through C. Suppose m is a member of the pencil of points through A and $T : m \mapsto \mathbb{P}^1_\infty$. Then either $m \cap \mathbb{P}^1_\infty = C$ or $m \cap \mathbb{P}^1_\infty \neq C$. If m intersects the line at infinity at the point C then the line at infinity is tangent to the conic generated by the transformation T and is therefore a parabola. On the other hand if m intersects the line at infinity at some other point other than C, then the conic will cross $\mathbb{P}^1_\infty$ in two places, resulting in a hyperbola.

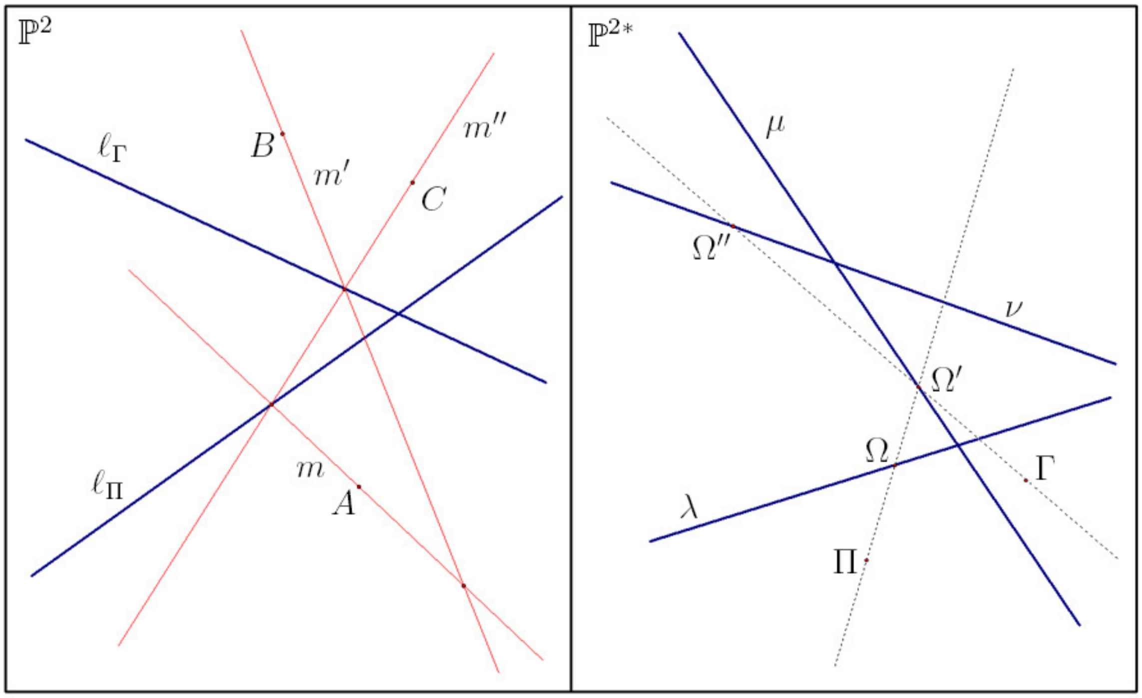


Figure 3.13: The dual of the composition of perspectivities.

The last obvious case is when both A and C lie on the line at infinity. In this case, the conic generated by T clearly intersects the line at infinity at points A and C. So in this case the conic will by a hyperbola.

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