PROSPECTIVE LOWER-SECONDARY AND SECONDARY TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING AND THE COLLEGE MATHEMATICS COURSE TOPICS THEY STUDY

by

SHAWN DEE BRODERICK

(Under the Direction of Jeremy Kilpatrick)

ABSTRACT

The need for high quality teachers of school mathematics has never been greater. A call for high quality mathematics teachers in the United States was made recently with the publication of the Common Core State Standards in Mathematics. Evidence from the Teacher Education and Development Study in Mathematics showed the mediocre mathematical knowledge and mathematical pedagogical knowledge of prospective U.S. mathematics teachers. Using quantitative and qualitative methods, I explored the relationship between prospective U.S. lower-secondary and secondary mathematics teachers' knowledge of mathematics and mathematics pedagogy and the program types they were in, courses they took, and topics they studied in courses in college mathematics, school mathematics, and mathematics pedagogy. The results showed that those enrolled in secondary preparation programs had more knowledge of mathematics and mathematics pedagogy than those enrolled in lower-secondary preparation programs. The results also showed that course topics in differential equations; school geometry; school functions, relations, and equations; mathematics instruction; and developing teaching plans for various domains of mathematics and mathematics pedagogy were related to significantly higher knowledge of mathematics and mathematics pedagogy. Recommendations for prospective teacher programs include (a) taking the above-mentioned courses relating to higher mathematical knowledge for teaching, (b) creating higher quality mathematics courses for lower-secondary prospective teachers focusing on special treatments of college mathematics courses designed for their mathematical dispositions and available time in their programs and (c) creating more focused programs that require a wider variety of courses aimed more at the mathematics that prospective teachers will use and less on unrelated abstract collegiate mathematics.

INDEX WORDS: Prospective mathematics teacher education, mathematical knowledge for teaching, Teacher Education and Development Study in Mathematics (TEDS-M), college mathematics, school mathematics, mathematics pedagogy

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A Dissertation Submitted to the Graduate Faculty of The University of Georgia in Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

2013

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August 2013

ACKNOWLEDGEMENTS

I would like to acknowledge the work and support of my wife as I worked on this dissertation. She took care of the many aspects of life here in Athens, GA (e.g., caring for our children) so that I could focus on this study.

I would like to acknowledge the support and guidance of my major professor and for the time he put in helping with planning, reading, and revising my work. I would like to acknowledge the support and guidance of my other committee members and for the time they spent with helping me plan and reviewing my work. I would like to acknowledge the participation of the prospective teachers who made this work possible. Thank you.

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Chapter 1: Rationale and Research Questions

In recent years, the knowledge needed to be successful has changed. It has become more demanding and sophisticated. As such, "teaching mathematics in primary and secondary schools has become more challenging worldwide" (Tatto et al., 2012, p. 17). In order for the United States to remain competitive in this ever-changing world, the National Governors Association and the Council of Chief State School officers organized an initiative to create a common set of standards for mathematics toward which all states could work. They were called the Common Core State Standards in Mathematics (CCSSM).

The framers of the CCSSM called for a change in the mathematics education of U.S. students in Grades K–12 classrooms. They stated that the United States needs to

> establish a shared set of clear educational standards for … mathematics that states can voluntarily adopt … [to] ensure that we maintain America's competitive edge, so that all of our students are well prepared with the skills and knowledge necessary to compete with not only their peers here at home, but with students from around the world. (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010)

After that call was issued, 45 states adopted the proposed standards, and they moved forward in a concerted effort to improve mathematics education in the nation.

The success of any implementation of standards that are internationally competitive, however, depends on the quality of the classroom teacher. The Conference Board of the Mathematical Sciences (CBMS, 2012) stated, "A critical pillar of a strong PreK–12 education is a well-qualified teacher in every classroom" (p. 1). They continued: "International and domestic studies suggest that an important factor in student success is a highly skilled teaching corps" (p. 2). Thus, there is a need for high quality teachers of mathematics.

One international study that surveyed teachers' skills in mathematics was the Mathematics Teaching for the 21st Century (MT21) study (Schmidt et al., 2007; Schmidt, Blömeke, & Tatto, 2011), in which prospective lower-secondary teachers were specifically asked about their mathematical knowledge and beliefs for teaching. MT21 was the precursor of the larger Teacher Education and Development Study in Mathematics (TEDS-M; Tatto, Schwille, Senk, Ingvarson, Peck, & Rowley, 2008; Tatto et al., 2012), which asked prospective elementary, lower-secondary, and secondary teachers about their mathematical knowledge and beliefs for teaching. The aforementioned categories of prospective teachers were called *program groups* by TEDS-M.

In the mathematical knowledge for teaching portions of both of the international surveys, the United States performed from a half of a standard deviation below the international average to a half of a standard deviation above the international average, depending on the program type. Such near-average performance illustrated the need for improving U.S. teachers' mathematical knowledge for teaching.

The results for TEDS-M were reported for overall mathematical knowledge, which was helpful but a little too broad to identify more specific areas of mathematical knowledge that can be improved. I wondered how the prospective teachers performed on the individual domains of mathematical knowledge: (a) number, (b) algebra, (c) geometry, and (d) data. If I could determine the weak areas of these prospective teachers' mathematical knowledge, then teacher educators might improve prospective teachers' performance by focusing on domains of weakness. For example, if the prospective teachers performed poorly in geometry, then a focus on geometry might help improve their mathematical knowledge.

Improving prospective teachers' mathematical knowledge for teaching can come with adjustments in their teacher preparation programs targeted at domains of poor performance. Such adjustments could include examining the college mathematics courses they take and how those courses are set up. Because of the unknown relationship between course taking and specific knowledge domains, I conducted a study of the courses and course topics available to prospective lower-secondary and secondary teachers and how taking, or not taking, such courses and course topics associated with higher or lower mathematical knowledge for teaching. For that study, I asked the following questions:

Research Questions

1. What is the association between program group and the mathematical knowledge for teaching possessed by prospective lower-secondary and secondary teachers in the United States?
2. What is the association between the courses and course topics taken in college mathematics and the mathematical knowledge for teaching possessed by prospective lower-secondary and secondary teachers in the United States?
3. How do prospective lower-secondary and secondary teachers in the United States see the association, if any, between taking courses in college mathematics and the value of those courses as a foundation for their mathematical knowledge for teaching?

Chapter 2: Literature Review and Framework

Because of the importance and influence that prospective mathematics teachers have in the educational system after they graduate and begin teaching, it is vital that they be educated in mathematics and mathematics teaching for the knowledge demands of today's world (Tatto et al., 2012). Many factors go into the education of mathematics teachers, including the type of program they are in (e.g., lower-secondary or secondary) and the course topics they study in that program (e.g., calculus or school geometry). The type of program that prospective teachers are in and the course topics they study will contribute to the knowledge they possess for teaching mathematics when they graduate. It is important to mention that prospective teachers might not have much choice in which program or course topics they study. It is also plausible that program and course topic offerings varied across the universities that participate in international studies.

Early Studies of Mathematical Knowledge for Teaching

For many years, mathematics educators have pondered the knowledge that teachers need to have in order to teach mathematics. Early researchers were concerned with trying to measure that knowledge. Initially, they thought about how many courses teachers took and then looked at student achievement. Ball, Lubienski, and Mewborn (2001) stated, "In the earliest approach to answering questions about the mathematical knowledge that is necessary for teaching, researchers attempted to validate empirically the common maxim that the more mathematical knowledge teachers have, the more mathematical knowledge their students will have" (p. 441). Researchers used the number of college mathematics courses taken by prospective teachers as the indicator of their knowledge level and student achievement scores to indicate the result of that knowledge.

Begle (1979) had taken this approach and found that when prospective teachers took courses past calculus, there were positive main effects regarding students' achievement in only 10% of the cases. Begle also found, however, that in 8% of the cases there were negative main effects in student learning when the teacher had taken courses beyond calculus. Ball et al. (2001) thought it was possible that the effect was due to a disconnection between higher mathematics courses and the mathematics that teachers teach. Ball et al. also observed that as prospective teachers take more advanced mathematics courses, they became accustomed to the lecture format and style of teaching. This type of teaching would not be effective with their future students. Like Begle, Ball et al. concluded that the assumption needed adjustment that the more a teacher knows about his or her subject the more effective he or she will be as a teacher.

Monk (1994) conducted a study similar to Begle's (1979). He surveyed teachers and asked them about the mathematics and science courses they had taken in their undergraduate or graduate programs. Like Begle, Monk found that taking mathematics courses was helpful for the achievement of students of those teachers, but only up to a point. He found that if the prospective teachers took more than five mathematics courses of any kind, the positive effects on student achievement decreased. Monk also found that there were significant positive effects of taking mathematics pedagogy courses. In fact, he found that they contributed more to higher student performance than college mathematics courses did.

Studies like Begle's (1979) and Monk's (1994) supported the existence of a modest positive association between course taking and mathematical knowledge as manifested in student performance. They showed that taking more college mathematics courses past a certain point did not contribute further to higher student performance. These studies, however, did not specify which course topics were the most effective.

Because the relationship between teacher knowledge and student performance is not well established (see, however, Hill, Rowan, & Ball, 2005, for the relationship between teacher knowledge and student achievement in the elementary grades and Baumert et al., 2010, for teacher knowledge and student achievement in the secondary grades), researchers need to investigate prospective teachers' mathematical knowledge from a different angle. Perhaps more work directly measuring mathematical knowledge would be productive. Additionally, because Monk (1994) found that mathematics pedagogy courses contributed more to higher student performance than college mathematics courses did, what might be the relationship of school mathematics courses to performance? Or the preparation program in general? These are important issues that can and should be addressed.

Components of Mathematical Knowledge for Teaching

Much work has been done on identifying the components of the mathematical knowledge that prospective teachers need in order to teach effectively. I call this kind of knowledge *mathematical knowledge for teaching* (MKT). Shulman (1986) described three components of MKT: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge.

Subject matter knowledge is the amount and organization of the teacher's knowledge of mathematical concepts and procedures. In other words, it is mathematical content knowledge, or the mathematics that teachers know and how it is organized in their minds.

Pedagogical content knowledge goes beyond the subject matter into knowledge of content that is specific to teaching. Examples of pedagogical content knowledge in mathematics include the knowledge it takes to present appropriate representations of mathematical concepts to students and a discernment of which mathematical topics are difficult for students to learn and why.

Curricular knowledge includes knowledge of what topics are taught to students in mathematics courses both in the future and in the past, along with what is taught to them in other subjects. Shulman's (1986) ideas have greatly influenced mathematics teacher education. Because teacher educators today know more about the various facets of knowledge of mathematics and mathematics teaching than in the past, they are better equipped to work with prospective teachers and strengthen those areas before the prospective teachers begin teaching.

Ball and colleagues (Ball, 1988, 1990; Ball, Thames, & Phelps, 2008) worked on extending Shulman's ideas of content knowledge for teaching. They divided the mathematical knowledge that teachers need for teaching into *subject matter knowledge* and *pedagogical content knowledge*. Then, through empirical studies, they were able to further investigate each type of knowledge.

Under subject matter knowledge, Ball et al. (2008) characterized *common content knowledge* (CCK), *specialized content knowledge* (SCK), and *horizon content knowledge*.

CCK is the mathematical knowledge and skills that are used by the average adult. Since it is the most prevalent type of mathematical knowledge, prospective teachers need to have a firm awareness of CCK, especially at the grade levels in which they will teach.

SCK is the mathematical knowledge and skills that are used in teaching situations and the knowledge of mathematics beyond what students need to know. This knowledge is a deeper and a more specialized version of the content than what is taught to students. In order to develop this important area of content knowledge for teaching, many preparation programs give prospective teachers a chance to develop their SCK with college courses on school mathematics.

Horizon content knowledge is associated with a part of Shulman's (1986) curricular knowledge in that teachers should know about how mathematical topics are related across the curriculum as they lay foundations for future mathematics learning.

Under pedagogical content knowledge, Ball et al. (2008) characterized *knowledge of content and students* (KCS), *knowledge of content and teaching* (KCT), and *knowledge of content and curriculum*. KCS is a combined knowledge of knowing mathematics and knowing students. This knowledge includes anticipation of what students will think about mathematics, which includes students' difficulties and misconceptions.

KCT combines knowledge of content and knowledge of teaching. For example, teachers draw upon this knowledge when they sequence lessons or choose mathematical examples for a lesson. Ball et al. (2008) used knowledge of content and curriculum much the way Shulman (1986) used curricular knowledge.

Silverman and Thompson (2008) presented another view of MKT. They focused on the knowledge it takes for teacher educators to prompt *key developmental* *understandings* (KDUs) in their prospective teachers. KDUs are advances in conceptual knowledge that promote a "change in the learner's ability to think about and/or perceive particular mathematical relationships" (Simon, 2006, p. 993). The example given by Silverman and Thompson was a teacher educator's tasks designed to help his prospective teachers gain a more conceptual view of area as length times width. He helped them relate length times width to groups of objects, like squares, that cover an entire surface without overlap. In order for the prospective teachers to have an appropriate, more conceptual understanding of area, Simon created a series of tasks that would incrementally advance their knowledge of area to a deeper, more conceptual level. When the prospective teachers completed a series of tasks exploring area in this manner, they experienced an increase in their MKT because of the KDU and could then talk about area in a different, more mathematically correct, and more accessible way.

Usiskin (2001) presented the idea of *teachers' mathematics* as a branch of applied mathematics, which contains the pedagogical content knowledge of teachers among other types of knowledge. He stated that teachers' mathematics includes:

- explanations of new ideas,
- alternate definitions and their consequences,
- why concepts arose and how they have changed over time,
- the wide range of applications of the mathematical ideas being taught,
- alternate ways of approaching problems, including ways with and without calculator and computer technology,
- how problems and proofs can be extended and generalized,
- how ideas studied in school associate with ideas students may encounter in later mathematics study, and
- responses to questions that learners have about what they are learning. (p. 96)

Usiskin continued by stating that this applied mathematics that teachers need to know is only part of the entire field of mathematics and favors certain areas. Some of those areas include number theory, geometry, and foundations of mathematics.

Of all the mathematics courses taught in college (i.e., college mathematics, school mathematics, and mathematics pedagogy, which help increase prospective teachers' MKT), Usiskin (2001) recommended that prospective teachers take college mathematics. Equally important, prospective teachers need to take a "number of mathematics courses that start from the ground up, from the problems faced in the classroom" (p. 97, i.e., school mathematics courses).

Assessments of MKT

The work of Shulman (1986), Ball et al. (2008), and others created a space in which to discuss the various areas of MKT, especially at the elementary level. Once researchers had created that space, measures were designed and implemented to assess these areas of teacher knowledge (see, e.g., Hill, Schilling, & Ball, 2004). The idea was to measure teachers' mathematical knowledge for teaching at any stage of their career, and then find ways to improve that knowledge. Hill and Ball (2004) showed that improvement in mathematical knowledge for teaching was possible in a study with practicing teachers from California. The report of their study showed that, after going through a mathematics professional development summer institute, the teachers' scores increased by an average of roughly 2 to 3 more items correct, out of 23–26 total items, from the pretest to the posttest.

Mathematics Teaching for the 21st Century Study

Using measures of MKT, including items similar to those used in the Hill and Ball (2004) study, the Mathematics Teaching for the 21st Century (MT21; Schmidt et al., 2007) study was created and surveyed the knowledge of prospective middle school teachers from six countries: Bulgaria, Germany, Mexico, South Korea, and the United States. Specifically, the study surveyed and analyzed the prospective teachers' mathematical ability and beliefs about teaching mathematics. Table 1 displays each country's scaled mean scores on the mathematical content knowledge (MCK) items. Table 2 displays each country's scaled mean scores on the mathematical pedagogical content knowledge (MPCK) items. The United States scored below the international mean in all MCK domains—frequently fifth of six. In the MPCK domains, the United States was frequently third of six. Thus, it is crucial that mathematics educators find ways to help prospective teachers in the United States gain more mathematical knowledge for teaching, so they can be effective teachers who are knowledgeable about what students need to know and the best ways to help them get there.

Table 1

Scaled Mean Scores of Six Countries' Prospective Teachers' Mathematical Content Knowledge (MCK) by Domain from the MT21 Study

Country	Number	Algebra	Functions	Geometry	Data
Bulgaria	461	456	477	469	433
Germany	511	476	495	494	497
Mexico	415	452	418	430	453
South Korea	570	586	584	577	567
Taiwan	570	567	582	564	540
United States	451	457	433	459	490

Note. The international mean is 500 with a standard deviation of 100. Adapted from *Teacher Education Matters: A Study of Middle School Mathematics Teacher Preparation in Six Countries*, by W. H. Schmidt, S. Blömeke, and M. T. Tatto, 2011, pp. 131, 138. Copyright 2011 by Teachers College Press.

Table 2

Scaled Mean Scores of Six Countries' Prospective Teachers' Mathematical Pedagogical Content Knowledge (MPCK) by Domain from the MT21 Study

Country	Curriculum	Teaching	Students
Bulgaria	422	400	447
Germany	514	504	518
Mexico	459	438	413
South Korea	486	572	555
Taiwan	540	538	543
United States	507	508	498

Note. The international mean is 500 with a standard deviation of 100. Adapted from *Teacher Education Matters: A Study of Middle School Mathematics Teacher Preparation in Six Countries*, by W. H. Schmidt, S. Blömeke, and M. T. Tatto, 2011, pp. 138. Copyright 2011 by Teachers College Press.

Teacher Education and Development Study in Mathematics

The MT21 study laid additional critical groundwork in the measurement of the mathematical knowledge of prospective teachers and functioned as a precursor to the Teacher Education and Development Study in Mathematics (TEDS-M; Center for Research in Mathematics and Science Education [CRMSE], 2010; Tatto et al., 2008, Tatto et al., 2012). TEDS-M expanded on the MT21 study by investigating prospective teacher education in 17 countries at both the primary and lower-secondary levels (see list of countries in Appendix A). TEDS-M tested two areas of mathematical knowledge: (a) MCK and (b) MPCK. The TEDS-M international report (Tatto et al., 2012) further illustrated the need for teacher educators to strengthen the mathematical knowledge of U.S. prospective teachers, especially those preparing to teach mathematics because of low performance (see Table 3).

At the primary level for the United States, two populations of prospective teachers participated in TEDS-M: primary generalists who were prepared to teach up to Grade 6 and primary specialists. Additionally, there were also two populations of prospective secondary teachers who participated: those prepared to teach up to Grade 10 (lower-secondary) and those prepared to teach up to Grade 11 and beyond (secondary). The MCK scaled mean scores and MPCK scaled mean scores for all four teacher groups are reported in Table 3.

Table 3
Mean Scaled Scores for U.S. Prospective Teachers' Mathematical Knowledge for Teaching from the TEDS-M Study

Program group	MCK	MPCK
Primary (to Grade 6 maximum)	518	544
Primary (mathematics specialists)	520	545
Lower Secondary (to Grade 10 maximum)	468	471
Lower Secondary (to Grade 11 and above)	533	542

Note. Adapted from *Policy, Practice, and Readiness to Teach Primary and Secondary Mathematics in 17 Countries: Findings From the IEA Teacher Education and Development Study in Mathematics (TEDS-M)*, by M. T. Tatto, J. Schwille, S. L. Senk, L. Ingvarson, G. Rowley, R. Peck, … M. Reckase, 2012, pp. 147, 150. Copyright 2012 by the International Association for the Evaluation of Educational Achievement.

At both the primary and lower-secondary levels, U.S. prospective teachers need to improve their MCK and MPCK knowledge. I submit that the relatively low U.S. performance in MT21 and TEDS-M may be linked to relatively low performance by U.S. students in studies, such as the Trends in International Mathematics and Science Study (TIMSS; Mullis, Martin, Foy, & Arora, 2012) and the Programme for International Student Assessment (PISA; Organisation for Economic Cooperation and Development [OECD], 2010).

More importantly, in both MCK and MPCK, the lower-secondary prospective teachers scored the lowest of all U.S. prospective teacher groups. Granted, all prospective secondary teachers were given items in calculus, analysis, linear algebra, and abstract algebra, items that would be difficult for the prospective lower-secondary teachers to complete correctly. So, it seems that program type is associated with differences in MKT.

One important way in which mathematics educators can improve prospective teachers' performance in mathematics and mathematics pedagogy is to make adjustments in their education programs. Teacher educators need to continually consider the effectiveness of specific components of the programs prospective teachers go through. Teacher education programs usually consist of courses, field experiences, and student teaching. In the present study, I investigated the mathematics courses and course topics taken by prospective teachers. Exactly how taking college mathematics courses and course topics is associated with prospective teachers' mathematical knowledge for teaching is still largely unexplored; only few pertinent studies exist (e.g., Begle, 1979; Monk, 1994).

In another such study (Wilson, Cooney, & Stinson, 2005), researchers interviewed practicing high school mathematics teachers and, among other points, discussed the importance of the mathematical knowledge that the teachers had gained from their mathematics courses. The practicing teachers stated that a knowledge of mathematics was necessary in order to teach for understanding, make transitions between topics, provide good examples, sequence lessons, understand student questions, and maintain one's confidence in front of a class of students. One contributor to these useful qualities was college mathematics courses. One teacher in the study asserted:

> I think that a lot of mathematics comes from knowing the upper level math. I can tell my class and I can exude a love of mathematics, even when I am talking about the reflexive property, because I know how it fits into the bigger whole. (p. 91)

Knowledge of higher mathematics can give prospective teachers confidence and help them convey to their students places where the students are headed mathematically.

In contrast, other researchers have stated that college mathematics courses do not help with such practices as teaching for understanding, making transitions, and providing good examples. They claim that college mathematics courses are disconnected from the mathematical knowledge that prospective teachers need in order to teach and interact successfully with their students (Ball, 1990; Ball et al., 2001). For that reason, Ferrini-Mundy and Findell (2001) suggested requiring college mathematics courses that allowed prospective teachers to investigate the school mathematics they would be teaching in a deep way.

In some universities, courses for prospective teachers that allow deep investigations of school mathematics and mathematics pedagogy have been recently implemented. Their impact on and utility for developing prospective teachers' MKT, however, has not been studied. Now that data on course taking and mathematical achievement have been collected on a large scale, through MT21 and TEDS-M, the mathematics courses' relationship to MKT and their utility for the prospective teacher can be addressed.

More specifically, the college mathematics course topics surveyed by MT21 and TEDS-M were separated into three categories on the prospective teacher questionnaires. The categories consisted of (a) college mathematics, (b) school mathematics, and (c) mathematics pedagogy. The course topics surveyed in the MT21 study are listed in Appendix B.

Most of the course topics surveyed in MT21 were from the category of college mathematics. Using the course topic list from MT21, the TEDS-M researchers refined and expanded their list of the three categories of course topics. Such improvements from gathering data for MT21 to gathering data for TEDS-M provided the researchers with the opportunity to collect more detailed data on the types of course topics that are taught in mathematics education programs. The TEDS-M researchers also divided each category of course topics into subcategories based on similar characteristics for later analyses for a technical report. The list of course topics surveyed, by category and subcategory, is given in Appendix C.

The TEDS-M researchers calculated the percent, by country, of those prospective teachers who said they studied the course topics in Appendix C (Tatto et al., 2012). In the United States, the prospective teachers studied 42% of the college mathematics topics, 71% of the school mathematics topics, and 78% of the mathematics pedagogy topics. Although they investigated the variation in the proportion of topics studied among countries, the TEDS-M researchers did not study or report on the utility of individual course topics. More specifically, they did not investigate how studying the course topics was associated with higher or lower mean MKT scores. In the present study, I investigated the associations between individual course taking and course topic studying and knowledge of MCK and MPCK, along with the number, algebra, geometry, and data domains.

Recommendations for Course Taking in Mathematics Teacher Preparation Programs

The Conference Board of the Mathematical Sciences (CBMS) has recommended the courses that prospective mathematics teachers from all levels should take in order to be competent teachers. They have published two sets of recommendations: *The Mathematical Education of Teachers I* (MET I; CBMS, 2001) and *The Mathematical*

Education of Teachers II (MET II; CBMS, 2012). MET II built upon and did not replace MET I.

Recommendations for prospective lower-secondary teachers. MET I (CBMS, 2001) advised that prospective lower-secondary teachers be specialists and have a complete and developed understanding of the mathematics they teach. They needed to be able to build on their students' knowledge of the mathematical concepts from elementary school, and therefore they needed to know elementary mathematics as well.

Prospective lower-secondary teachers should not be left to teach just with the knowledge of middle grades concepts they gained from their middle grades years. Therefore, MET I (CBMS, 2001) recommended that they go through a program that required at least 21 semester hours of mathematics courses. Twelve of those hours should be dedicated to school mathematics. Four 3-hour courses should include a deep look at (a) number and operations; (b) algebra and functions; (c) measurement and geometry; and (d) data analysis, statistics, and probability.

In addition to school mathematics, MET I (CBMS, 2001) recommended that prospective lower-secondary teachers study college mathematics. The recommendation was for them to develop their own mathematical knowledge and make connections between elementary mathematics and middle grades mathematics along with connections between middle grades mathematics and high school mathematics. The background for these college mathematics courses was precalculus or college algebra. Then MET I recommended that prospective middle grades teachers take a calculus course based on concepts and applications and not one typically offered to mathematics majors or engineers.

Further recommended courses included number theory and discrete mathematics because they would help prospective teachers explore many mathematical topics that they would teach. A course in the history of mathematics would give them the historical background for the topics they would teach. Mathematical modeling could show them real-life applications of the topics they would teach. MET I (CBMS, 2001) also recommended that if teachers expected to teach algebra, they should take linear algebra and modern algebra courses. They could also take college geometry if they expected to teach geometry. Such recommendations led to the idea that college courses were a good foundation for teaching corresponding school courses. If prospective teachers took all the above courses, it would total more than 21 hours, which was probably one of the reasons that MET II (CBMS, 2012) upped the minimum recommendation from 21 hours to 24 hours of mathematics courses.

Like MET I, MET II (CBMS, 2012) continued to recommend that prospective lower-secondary teachers take school courses in (a) number and operations, (b) geometry and measurement, (c) algebra and number theory, and (d) statistics and probability. From the first MET report to the second, function concepts were included in the algebra domain, and number theory was added. Also, data analysis was included under the statistics and probability domain.

With regard to college mathematics, many recommended courses from MET I were also in MET II (CBMS, 2012). The writing committee advised that prospective lower-secondary teachers take (a) an introductory statistics course, (b) calculus, (c) number theory, (d) discrete mathematics, (e) history of mathematics, and (f) modeling. A course in statistics was a new recommendation not in MET I.

MET I (CBMS, 2001) did not make any recommendations for mathematics pedagogy or methods courses. MET II (CBMS, 2012), in contrast, recommended that prospective lower-secondary teachers take two courses in methods. To help reduce the number of courses prospective teachers needed to take, MET II stated that these courses could be hybrids of content and methods.

Recommendations for prospective secondary teachers. The MET reports also made recommendations for prospective secondary teachers. MET I (CBMS, 2001) began with four requirements for the mathematical knowledge of all secondary teachers:

- Deep understanding of the fundamental mathematical ideas in grades 9–12 curricula and strong technical skill for application of those ideas.
- Knowledge of the mathematical understandings and skills that students acquire in their elementary and middle school experience, and how they affect learning in high school.
- Knowledge of the mathematics that students are likely to encounter when they leave high school for collegiate study, vocational training or employment.
- Mathematical maturity and attitudes that will enable and encourage continued growth of knowledge in the subject and its teaching. (p. 39)

MET I then elaborated on the five major areas in which prospective secondary teachers need to be proficient: (a) algebra and number theory, (b) geometry and trigonometry, (c) functions and analysis, (d) statistics and probability, and (e) discrete mathematics.

For strengthening algebra and number theory knowledge, MET I recommended courses in calculus and linear algebra. For even further study, courses in abstract algebra and number theory were also suggested. These courses can give secondary teachers insight into why number systems and algebra work as they do. The report, however, stated that many prospective teachers do not have a chance to learn about the connections of college mathematics to school mathematics. They issued a call to mathematicians to assist in helping prospective teachers see those connections.

For geometry and trigonometry knowledge, MET I again recommended calculus and linear algebra. It reported that the specific geometrical aspects of these courses, like graphing and vectors, are useful for prospective teachers. Of course, college geometry courses were emphasized as important. These included axiomatic, coordinate, and non-Euclidean versions. In addition, they specifically stated that modern developments in geometry be included in the coursework.

With regard to functions and analysis knowledge, MET I again recommended calculus and linear algebra, along with possibly other elective courses discussing functions. The report also suggested that the main functional knowledge be developed through a study of high school mathematics from an advanced standpoint, perhaps in a capstone course.

MET I stated that statistics and probability knowledge is best acquired in light of real world situations. Thus, the statistics courses offered by the university that focus on such an approach were recommended by the report.

Discrete mathematics knowledge should also be fostered. MET I included computer science knowledge as well in this area. Prospective teachers should take courses studying mathematical induction. The report stated that prospective teachers usually know how to operate computers, but do not know about its underlying theoretical structures. Such study of computer science was recommended for knowledge of a field in which mathematics has multiple applications.

MET II (CBMS, 2012) provided recommendations for college mathematics courses in light of a short program and a long program. The short program included 33 semester-hours of courses, whereas the long program included 42 semester-hours. Each of those programs was divided into groups of courses taken by a number of undergraduates from other disciplines, courses taken by undergraduate mathematics majors, and courses specific to prospective teachers. The list of courses of each suggested program is included in Appendix D.

At the end of each suggested sequence of courses, MET II allowed nine semester hours of courses specific to prospective teachers. The report recommended that prospective secondary teachers take courses based on (a) high school mathematics from an advanced standpoint, (b) an in-depth investigation of a single topic associated with high school mathematics (e.g., the fundamental theorem of algebra or straight-edge and compass constructions), or (c) mathematics that is useful for the life of the professional teacher (e.g., classical theory of equations or three-dimensional Euclidean geometry).

Prospective Teachers' Perceptions of the Utility of College Mathematics Courses

With regard to prospective teachers' perceptions of the utility of college mathematics courses, the literature is sparse (Clark, Hemenway, St. John, Tolias, & Vakil, 1999). A few articles about calculus classes and one about abstract algebra, however, discuss methods to enhance college mathematics to make the courses more useful to students, which may include prospective teachers.

Olson (1997) pointed out that a main theme of college mathematics courses is proofs. He claimed that students usually cannot understand proofs and are not interested in them. He argued that if traditional methods did not work, then it was important to try alternative methods. One day, he was lecturing on the proof for Newton's Method and through a turn of events found that he had finished with 20 minutes left in the period. He decided to have the students work in groups deriving the method and proving it. He went around to groups and gave pointers. The students soon derived Newton's Method and found a proof for it. Using that experience and subsequent group work sessions, Olson derived and refined his approach to make college courses more meaningful and useful for his students. His approach consisted of the following steps:

1. Choose a proof associated with a memorable picture.
2. Explain the proof completely but quickly, leaving plenty of time for the students to work through it themselves.
3. Emphasize connections between the picture and the algebra.
4. Outline the key steps of the proof and erase everything else.
5. Then form the students into small groups and have them reconstruct the proof, helping them as needed. (p. 124)

He found in subsequent lectures that pictures, a quick introduction to the proof, connecting the proof to the picture, and outlining the proof were optimal for his students. This approach, he found, helped the students get excited as they unraveled the mystery of the mathematics. Such work helped them cultivate a more positive attitude toward proofs. He concluded that his freshman loved calculus proofs because they were able to understand them.

Korey (2002) saw her calculus students as having an appreciation for mathematics, but current mathematics teaching methods had had a negative effect on their perceptions of college mathematics. Thus, they left the course with a more negative attitude about mathematics than they had earlier. Through an NSF-supported project, Learning Mathematics Across the Curriculum, Korey studied students' beliefs about the utility of calculus in special hybrid courses where calculus was studied in context of other subjects like the humanities. She found that students had a renewed interest in mathematics when it was applied to such real-life examples. The students became more competent and showed gains in mathematical reasoning, confidence, and aptitude (see also Wilson et al., 2005). Such characteristics were vital for prospective teachers.

Clark et al. (1999) studied students' perceptions of the utility of abstract algebra. They investigated an implementation of two sections of the course taught by the same instructor. The control group went through a traditional treatment of the course, while the experimental group's course used a constructivist approach to learning, group work, and computer activities. About three-fourths of the students from the experimental group gave an immediate positive reaction to the course when interviewed. In the control group, only about one-third of the students had anything good to say about the course. Most of the students felt negatively about it, mentioning how hard the course was and the low ratio of understanding gained to work invested. Clark et al. asserted that each version was comparable in difficulty and, in the end, students in the experimental course earned better grades than those in the traditional one.

In summary, college mathematics courses can be made more useful to undergraduates, including prospective mathematics teachers, if various pedagogical modifications are made to the traditional format. Such modifications include (a) using groups to grapple with understanding content and using proofs, (b) contextualizing the mathematics, (c) supporting students to construct their knowledge, (d) using activities or tasks, and (e) using technology.

I was not able to locate literature on prospective teachers' perceptions of the utility of their school mathematics and mathematics pedagogy courses, which revealed another gap in the literature. Through the present study, I am adding to the research on the utility of college courses and making recommendations for teacher education programs to strengthen each prospective teacher's MKT.

Chapter 3: Pilot Study

In this chapter, I discuss a pilot study that I conducted in order to better understand the association between the type of teacher preparation program that students were in and their mathematical knowledge for teaching (MKT), and the course topics they took and their MKT. I also conducted interviews with prospective lower-secondary and secondary mathematics teachers who were becoming certified to teach mathematics about their views of the association between their courses and their MKT. At this point I note that the MT21 and the TEDS-M studies surveyed *course topics* (e.g, calculus or school algebra). In my interviews I discussed *courses* (e.g., Calculus 1 or Algebra for Middle School Teachers).

First, I wanted to know how being in a certain kind of certification program to teach mathematics would be associated with the prospective teachers' MKT. For example, how would becoming certified to teach Grades K–8 versus becoming certified to teach Grades 6–12 be associated with prospective teachers' overall MKT and their MKT in certain mathematical domains? Second, I wanted to know how taking specific course topics would be associated with prospective teachers' level of MKT. For example, how would studying calculus be associated with students' MKT and their MKT in certain mathematical domains? Third, I wanted to know how prospective teachers viewed the utility of each course for their MKT. For example, how do prospective teachers think about the utility of studying calculus for their MKT? Because I did not have experience investigating the associations among types of programs, courses and course topics, and

MKT, I conducted a pilot study. The research questions for the pilot study were the same as those for the main study.

Pilot Study Methods

In this section, I describe the process I used to carry out the pilot study and address the research questions.

Program groups, course topics, and measures of MKT. Before I describe the pilot work, I describe how I view the three main variables in this study; namely, (a) prospective teachers' program groups, (b) courses or course topics, and (c) measures of MKT. At the time of the pilot study, the only available data set with appropriate data was that of the Mathematics Teaching in the 21st Century study (MT21). I had access to the data of 381 prospective teachers in the United States. As I went through the data set, I found that the prospective teachers had been asked, in addition to whether they were becoming certified to teach at the middle school, for which other, if any, teaching levels did they intend to become certified? They had the option to choose elementary, high school, both, or neither. Thus, the prospective teachers were in four different program groups: those becoming certified to teach Grades (a) K–8, (b) 6–8, (c) 6–12, and (d) K–12.

The prospective teachers were also asked to check whether or not they had taken certain mathematics course topics from a given list of course topics (Appendix B). There were three categories of mathematics course topics: (a) college mathematics, (b) school mathematics, and (c) mathematics pedagogy.

In the MT21 survey, the prospective teachers were given a subset of 17 mathematics items over five domains: (a) 2 items addressed number, (b) 4 addressed algebra, (c) 3 addressed functions, (d) 4 addressed geometry, and (e) 4 addressed data. Unlike the TEDS-M study, there were no precalculated scaled measures of MKT in the data set I had. Using the given rubric (Schmidt, 2013), I calculated the percent correct the prospective teachers earned on all items. In addition to the overall percent correct, I calculated the percent correct each prospective teacher earned on the items from each of the five domains.

Now I was in a position to make an initial investigation to see how programs and course topics affected MKT overall and for each domain. I chose to use SPSS (IBM SPSS Statistics for Windows, 2010) for the analysis. In SPSS, the data file contained the following variables: (a) each prospective teacher's MKT overall score, (b) his or her MKT score for each of the five domains, (c) the program group to which he or she belonged, and (d) whether or not he or she had taken each course topic from the MT21 survey.

The first question concerned which program group contained the teachers with the highest MKT average and whether that average was significantly different from the average MKT scores of the other program groups. I conducted a one-way analysis of variance (ANOVA) using the prospective teachers' overall MKT scores as the dependent variable and the program group as the factor. Then I ran similar ANOVAs using each domain MKT score as the dependent variable and the program group as the factor.

The second question concerned those course topics taken by the prospective teachers for which there were significantly different average MKT scores between those who had studied the course topic and those who had not. I ran a series of *t* tests, which took the group of those who had studied the course topic, calculated the mean of their percent correct scores, and tested that to see how different it was from the mean score of the group of those who had not studied the course topic. I ran this test for all course topics using the prospective teachers' overall MKT score as the dependent variable. Then, I changed the dependent variable to the each domain's MKT scores and ran a series of *t* tests for the course topics under each domain.

Evolution of the interview protocol. The third question concerned prospective teachers' comments on how they see the association between courses and their MKT. In this section, I discuss my initial attempt at writing the interview protocol and its subsequent iterations based on the pilot interviews with prospective teachers whom I call Nick, Mary, and Jennifer. They were becoming certified to teach Grades 6–12.

I began by formulating some preliminary interview questions that asked about which of the three types of college mathematics courses the prospective teachers had taken and how they thought each had contributed to their ability to teach mathematics and address student ideas. After considering these questions, I decided that I could elicit more specific and deeper answers about courses if I had the prospective teachers solve MKT test items, just as prospective teachers had done in the MT21 study and the TEDS-M study. I decided to ask the prospective teachers to talk about the courses that helped them answer the items, and I modified the interview protocol accordingly.

With the second set of questions, I conducted the first pilot interview with Nick and was able to use the items to successfully guide the conversation to specific courses he had taken and how they had helped him solve some items. I did not know, however, whether Nick had taken more courses than we discussed using the items. So, I began by asking him to list all the college mathematics courses he had taken. At the end of the interview, Nick asked if he could comment on how he thought one particular course was not helpful for his MKT. I realized that it would be good to give the prospective teachers a chance to make general comments on any or all their mathematics courses and added an additional question to that effect at the end of the protocol. After making this modification, I conducted the second pilot interview.

The revised protocol worked well for the second pilot interview with Mary and for most of the third with Jennifer. Jennifer, however, mentioned that even though she had taken a course that helped her solve an item, she said she could have solved it without having any college mathematics courses. I added a question along that line to the interview protocol for each item. The final protocol is given in Table 4.

Table 4
Interview Protocol

Step	Question or action
1	Which mathematics, school mathematics, and mathematics pedagogy courses have you taken in college?
2	Please work through this mathematics item. Please talk about what you are thinking as you respond to the item.
3	[Look for places where the participant should expound his or her thinking and ask him or her to do so when necessary.]
4	[After the participant finishes the item:] Which college mathematics courses helped you to solve this problem? How?
5	Could you have done this item had you not taken any mathematics courses in college?
6	As you worked through the item, did you encounter a topic that your courses should have not covered, but did not? If yes, why? If no, why not?
7	[Repeat Steps 2–6 with several mathematics items from TEDS-M.]
8	In general, which courses have helped your overall mathematical knowledge for teaching and which have not? Why?

Pilot Study Results

Program group and MKT. The results of the first research question are summarized in Table 5. This table shows the results of the ANOVA by program group on overall MKT mean percent correct and each domain's MKT mean percent correct for the MT21 data from the United States.

Table 5

Mean Percent Correct for All MKT Items and Each MKT Domain by Program Group

Program group	*n*	All	Number	Algebra	Functions	Geometry	Data
K–8	169	43	55	36	23	47	52
6–8	104	45	57	40	27	48	53
6–12	94	57	63	51	50	58	61
K–12	14	46	56	35	37	41	54
F		$29.18^{‡}$	3.05^{*}	$8.94^{‡}$	$35.01^{‡}$	$4.12^{†}$	$5.21^{†}$

Note. Data came from the U.S. portion of the MT21 data set.
$^{*}p < .05$, $^{†}p < .01$, $^{‡}p < .001$.

Since all ANOVAs showed significant differences among the groups, I needed to perform some post hoc tests. To see which program groups had significantly higher MKT averages, I performed Tukey HSD post hoc tests on the means for all groups in all domains. Those results are reported in Tables E1–E6 in Appendix E.

The only significant differences involved the Grades 6–12 group. It had significantly higher mean scores than the other groups in every domain except in number, where it was not significantly higher than the Grades 6–8 and K–12 groups; and in algebra, functions, geometry, and data, where it was not significantly higher than the Grades K–12 group.

College mathematics course topics and MKT. Tables F1–F6 in Appendix F contain the results of the series of *t* tests on each course topic and MKT scores by overall performance and by each of the five mathematical domains. Although most course topics did not have a significant association with higher or lower MKT, a few did.

With regard to college mathematics course topics, the overall mean MKT scores of those students who had studied course topics in (a) axiomatic geometry, (b) topology,

(c) linear algebra, (d) abstract algebra, (e) beginning calculus, (f) calculus, (g) multivariate calculus, (h) differential equations, (i) functional analysis, (j) discrete mathematics, (k) mathematical logic, and (l) history of mathematics were significantly higher than the scores of those who had not. For some of those course topics, the differences were highly significant, on the order of $p < .001$. In contrast, the overall mean MKT scores of those students who had studied school mathematics course topics and mathematics pedagogy course topics were not significantly higher than the scores of those who had not. For several of those course topics, those students who had studied the course topics had lower mean MKT scores than those who had not.

Prospective teacher interviews. The pilot interviews that I conducted gave me insight into how prospective teachers view the utility of their college mathematics courses, their school mathematics, and their mathematics pedagogy courses for their MKT.

In the pilot interviews, we discussed their mathematical knowledge for teaching through specific released test items from TEDS-M and ended with a general discussion of each mathematics course they had taken. The test items drew on specific areas of mathematics; areas that international mathematics educators had agreed addressed mathematics topics that prospective teachers needed to know (Tatto et al., 2012). During the pilot interviews, we discussed 14 items, with 2 having two parts. Table G1 in Appendix G contains the categorizations and a brief description of each item.

The prospective teachers frequently commented about how their courses had helped their MKT for solving items and for teaching in general. As noted above, the ANOVA results for the MT21 data showed that the overall mean MKT scores of those students who had taken courses covering the topics in axiomatic geometry, topology, linear algebra, abstract algebra, beginning calculus, calculus, multivariate calculus, differential equations, functional analysis, discrete mathematics, mathematical logic and history of mathematics were higher than the scores for students who had not taken the courses. The three prospective teachers who were interviewed indicated that they had taken courses covering the topics of axiomatic geometry, linear algebra, abstract algebra, beginning calculus, calculus, and multivariate calculus, but not a course covering the topics of topology, differential equations, functional analysis, discrete mathematics, mathematical logic, or history of mathematics, even though the latter courses were available to them. Because the prospective teachers in the pilot interviews did not study some of course topics surveyed in the MT21 survey, I cannot give further insight on how they contributed, or not, to the prospective teachers' MKT.

According to the teachers interviewed, the course covering axiomatic geometry, called Foundations of Geometry I, was somewhat helpful for solving MKT problems and their MKT overall. Nick stated that the course helped him solve Item 610 by helping him reason about the diagonal of a square whose side lengths were 1. He used the Pythagorean Theorem to find that the length of the diagonal would be the square root of 2. He related this calculation to a topic from the same geometry class on the construction of a segment whose length is the square root of 2. Mary also thought that the geometry course was helpful. She said that it was important to know where the geometrical concepts originated when she would teach geometry in school.

The course covering linear algebra, called Introduction to Linear Algebra, significantly helped Nick solve Items 604A and B, which required the prospective teacher to use variables and reason about relative quantities using systems of equations. Because Nick had taken Introduction to Linear Algebra, he was easily able to set up the system and solve for the missing values. He emphasized that his ability to work the problem came from the experience he had gained in Introduction to Linear Algebra. In contrast, Mary had a different view of the utility of Introduction to Linear Algebra. She valued the course for its own sake, but did not believe that it would be helpful for teaching "whatsoever." Jennifer thought that Introduction to Linear Algebra was helpful for Item 814, which involved an operation with matrices.

With regard to the course covering abstract algebra, called Modern Algebra and Geometry I, Nick found his experience from the course to be helpful when solving Item 610. He was able to use his knowledge of irrational numbers to discuss examples and nonexamples of rational and irrational numbers, as required by the item. Mary used her experience from Modern Algebra and Geometry I to solve Item 709, which was about assessing student responses to proving a number theory concept. She stated,

> I'm not saying [this is] the exact proof [that I did in class], but I remember proving something [about] consecutive natural numbers and how the correct way to prove it [is to] not just [prove it for] specific values, but for any natural number.

Mary had responded to the item correctly and easily used the knowledge she had gained in Modern Algebra and Geometry I to do it. Even though she had used her MKT from Modern Algebra and Geometry I in working an item, in general she still did not think the course was valuable to her future teaching, much like her general feelings about Introduction to Linear Algebra.

Jennifer found Modern Algebra and Geometry I very helpful. She thought that it was the most helpful course in assessing the student proofs in working Item 802. She said that she and her classmates had done a similar proof in the course just two weeks before the interview. Item 814 had Jennifer work and reason with matrices and an invented operation. Jennifer said that her knowledge from Modern Algebra and Geometry I helped her work with invented operations. She stated:

> In [Modern Algebra and Geometry I], we talked about properties of some kind of relation, or some kind of set of numbers using a new operation that we define and say we don't know what it does, but we're still going to prove stuff from this operation. So, I feel like that's really beneficial in the understanding of an operation that we don't know what is. … Before [Modern Algebra and Geometry I] I would not have been like, "Oh, it's some random operation that isn't defined that we're just like putting there and we can define it later." I thought that was weird before I saw it in [that] class.

Speaking about the course in general, Jennifer said that Modern Algebra and Geometry I had been a challenging course. She was taking it for a second time. This second attempt had helped her see certain things that she had not seen the first time.

> I talked a lot [in this interview] about things I've learned in abstract algebra that have been beneficial. … I hate that I've taken this class twice. But just in this semester, [I've seen] the benefit in the things that were being taught, and how they relate a lot of [abstract] things because [the material is] not concrete. So you can be like, "Oh, that makes sense in this situation." … Initially when I took it, I [thought] this is horrible. I'm never going to use this. But, now that I know I just can't do it without fully understanding the material. … But it's nice to be able to draw connections, saying this is kind of useful.

Jennifer also mentioned that she used her knowledge of the material, now more fully mastered because of repeating the class, to explain topics in number theory. She said that she could tell her future students about why topics such as the divisibility rules work, instead of just being able to say that they do work.

With regard to the calculus sequence, Calculus 1, Calculus 2, and Multivariable Calculus, each prospective teacher thought that the sequence was helpful. (Note the course topic is *multivariate calculus* and the course is *Multivariable Calculus*.) Nick stated that material from those courses helped him solve Item 705. In it, Nick reasoned about a point on a number line, a line in a plane, and a plane in space. He discussed a little his experience in Multivariable Calculus and how his teacher had presented some three-dimensional graphs to the class, but unfortunately his teacher had not gone over them in much detail. Mary and Jennifer thought that the calculus sequence was helpful to their mathematical knowledge for teaching because of the possibility of teaching precalculus and calculus in high school.

Jennifer added a story of the case of a high school student who needed a teacher that knew the calculus topics, even the topic of multivariate calculus. She said,

> In my high school class, there was a guy who had already taken Calculus AB, [and] already had credit in [Calculus 1]. The [next] semester, he focused on [Calculus 2] stuff. And then our last semester at school, he was doing [Multivariable Calculus] stuff that was through a college textbook so he could potentially test out of it. My teacher had to be familiar with stuff from [Multivariable Calculus], so that she could teach him.

Again, Jennifer thought that someday she might be in that teacher's position and therefore needed to know the material from the calculus sequence. For prospective teachers at lower levels in high school and even in middle school, who do not expect to teach calculus every day, Jennifer thought that taking Multivariable Calculus was still important. She stated,

> I just feel like it's beneficial to have that additional knowledge, especially teaching to a calculus classroom. [You could say] if you take [Multivariable Calculus], these are some of the things that you're going to learn, and these are some of the connections that you're going to be making to that class. … Yeah, I feel like I did that in all my classes at high school, [with teachers saying], "This is something that is going to be useful for you next year in the classes you'll be taking."

Jennifer drew upon the need for prospective teachers to foster horizon content knowledge (Ball et al., 2008), a part of MKT, to make connections to future mathematics classes.

There was one course that the prospective teachers from the pilot interviews had taken and thought was helpful, but that was not fully represented in the MT21 study: Introduction to Higher Mathematics. This course covered mathematical logic, which was included in the MT21 study, but also set theory and the types of proof structures found in higher college mathematics courses. Mary found the course helpful as she solved Item 709, which had the prospective teacher assess sample student work from a number theory proof. About the Introduction to Higher Mathematics course in general, Mary stated,

> I do think, [and] I'm surprised I'm saying this because at the time I didn't think so, but things in this [Introduction to Higher Mathematics] class, besides how to write a proof correctly, I feel like [it helps] you see how to actually prove that something is true and back it up, instead of just making a claim and not understanding why it is the way it is. So, maybe that, knowing strategies, [but] not necessarily a certain proof, but strategies of going about and proving a statement to be true would be helpful in teaching, because then that will help students maybe understand why the things are the way they are.

She thought that her knowledge of how to show the structure of mathematics through proofs would be a valuable asset to her, her teaching, and fostering proving in her students. Jennifer had also taken the course and found it helpful. Item 712 was about an algebraic proof of the quadratic formula. She was able to reason through the item because in one occasion in the course, her teacher had used an algebraic proof of the quadratic formula as an example of a proof. Jennifer also used her knowledge from the course to assess the validity of student proofs in Item 810, which was similar to Item 709. Speaking of the course in general, and despite the knowledge she gained and used from it, Jennifer still debated whether it was useful:

> [Introduction to Higher Mathematics] I thought was useful. Actually, did I? I mean for the proof writing that you have to do in your other classes it was useful. So, if I hadn't taken [Introduction to Higher Mathematics], [Modern Algebra and Geometry I] would definitely not be useful, because I wouldn't learn anything. I would be focused on memorizing the proofs because I wouldn't know how to

> write them. So I feel like [Introduction to Higher Mathematics] is really important, just as a proof-writing class, so that you can take the time to understand the things you're learning in your other classes, and why it's necessary to prove them.

In the end, Jennifer thought that the Introduction to Higher Mathematics class was useful. She decided so after she thought about the degree to which its concepts were used in future higher mathematics classes.

The statistical analysis whose results are in Table F5, showed that students in the MT21 sample who had studied school mathematics course topics had neither higher nor lower overall MKT scores compared with students who had not taken those courses. In the pilot interviews, school mathematics courses were generally beneficial; however, there was some evidence that such courses were not always helpful. For example, Nick thought that his course covering the topics of principles and theory of school arithmetic, called Connections in Secondary Mathematics, was not helpful. He began on his own to discuss the ineffectuality of Connections in Secondary Mathematics, which featured other elementary and middle school mathematics topics such as operations on whole numbers, rationals, and proportional reasoning. On the topic of multiplication, he said:

> In our course right now, we kind of talk about the standard definitions [of operations on whole numbers and fractions] and how will we draw diagrams for [them]. The way I see it, if you try to teach a student in this way, it makes it harder and more confusing even though you're showing what the definition means. I don't feel like it's something a student would get too much out of, even [if applying the problems to] real life. If you know what it means to multiply, [the more conceptual approach is] nice to know, but, it's not [like] you couldn't know how to multiply without actually knowing the natural definition. … But just getting the [conceptual] definition gets confusing, [and] it actually makes multiplication harder for me to understand 'cause I actually [had] a simpler understanding of multiplication before I got to that class. … For me, I just don't see where that course is going, because [the teacher is] saying you probably wouldn't use this to teach a student, but it would be nice to know.

Nick contemplated the possible boundaries of the question of how much and what kind of mathematics a teacher at any level should know, and thought there were some courses that teachers did not need to take. Mary, however, had a somewhat different opinion about the course covering school arithmetic, stating that it was a useful foundational course and that future topics build directly on it. In general, Jennifer had mixed feelings about the course, just as Nick did. She said,

> Talking through fractions is kind of annoying, but really useful. … I mean, I feel like we're just doing the same thing over and over again. But, now that we've done it so many times, even though it's kind of been systematical, I can explain the steps I'm taking while I'm taking them. And like before, all I knew about multiplying and dividing fractions was how to do it. It's like understanding what that looks like in a picture or some physical representation, [which has] been really helpful.

Thus, for Jennifer, even though it was plausible that she did not enjoy her deep look at school arithmetic, she still found it beneficial.

The prospective teachers in the pilot interviews found the most value in their Concepts in Secondary Mathematics course, which covered principles and theory of school algebra and calculus. It also covered precalculus and trigonometry topics, which were not specifically mentioned in MT21, at a deep level. Nick used some of his knowledge gained from the course to solve Item 604, which had him write equations to model a story situation. When asked about this course, Mary said that in general she had gained a lot out of it and that it was "definitely" beneficial. Generally speaking, Jennifer also thought it had a big impact on her. She said,

> [Concepts in Secondary Mathematics] class was insanely wonderful. [We learned] how to explain sine, cosine, and tangent in ways that I didn't even make ever sense of. … So, this was a super, super helpful class.

Jennifer thought that the various ways of discussing the trigonometric functions were helpful for her MKT, and found an inspiring view of trigonometry that she could pass to her students.

In the MT21 analysis, taking the mathematics pedagogy courses was not significantly associated with differences in MKT. The prospective teachers I interviewed, however, saw such courses as making some important contributions to their MKT. Nick was solving Item 704 when he found that an exercise in a mathematics pedagogy course was helping him. He stated,

> In [the] class, we did a little bit with talking about angles and bisectors. We had one problem where it was a fun challenge problem that we were going to do. [The professor] was trying to get us to explore what a mathematical task would be. It took us a while to do it because it had four triangles or something. It was a shape that was cut in a certain way. … So, [in this task] I actually reviewed a lot of stuff from geometry that I hadn't used in a while. … That really refreshed my mind with geometry because I hadn't really taken it since ninth grade.

The professor had used the opportunity to review some mathematics and cover the mathematical pedagogical topic of creating tasks and Nick found it useful for his MKT.

In general, Mary found value in the two mathematics pedagogy courses that she had taken, both focusing on teaching practices in mathematics. Of those courses, she stated,

> With [one of the courses on teaching practices in mathematics], you [learn that you can] get sidetracked with other things while you're teaching, and you kind of think, "Oh, if I teach math then I'm only going to teach how to do a problem." But you have to remember that in order to effectively [teach] you have to understand how a student thinks and what the best way is for you to help him or her in order to actually understand the math. [You can think about how to] question students or what kind of hints [you can] give them and what activities you want them to do. Stuff like that is definitely helpful.

Attention to what you have to know about students in order for them to learn mathematics was beneficial for Mary. Thus, Mary valued the KCS part of her MKT.

Jennifer said that she had learned a lot about questioning in her mathematics pedagogy courses, but not a lot of mathematics. She stated that in one course they learned a lot about how elementary students learn mathematics using activities with counting blocks and activities with things they did with their hands. Of the mathematics that they did do, she said that they learned some mathematics in their readings on student thinking. Those topics were completing the square, multiplication, and factoring.

Mary also took an optional mathematics pedagogy course on teaching mathematics with technology, which was not listed in the MT21 study. It focused on various tools and software to enhance the learning of mathematics. With regard to this course, she stated,

> The [teaching with technology course] helped a lot in seeing math in a different way. A lot of kids might be better at hands-on activities or visuals. A book and a piece of paper and a pencil won't do it for them if they're actually interacting in some kind of applet, some kind of wiki, or whatever that shows you the math. That might be more intriguing for them. So, those [tools] definitely were helpful to them.

In that course, Mary had learned about hands-on tools that enhance mathematics learning for students, another foray into the KCS of MKT. She valued the course and thought that her students would benefit from the knowledge she gained from its strategies.

Pilot Study Discussion

From the pilot study, I learned that my statistical assumptions and methods needed some modification. The knowledge inherent in MKT is a product of all mathematics courses together. Thus, the statistical method I used with the data should consider this conception of MKT. Second, I learned that the pilot interviews went well, and the data I got from the prospective teachers were what I wanted.

Chapter 4: Methods

Participants

For the first two research questions, about the association between the program group and prospective teachers' MKT and the association between studying course topics and prospective teachers' MKT, I performed a secondary analysis of the data from the Teacher Education and Development Study in Mathematics (TEDS-M), which surveyed a sample of prospective teachers from all over the United States. For the third question about how prospective teachers view their college mathematics courses as helpful or not to their MKT, I conducted interviews with a sample of prospective teachers from one university. Below I discuss the participants in each sample.

TEDS-M participants. The TEDS-M study collected data from prospective teachers in 17 countries. There were 608 U.S. lower-secondary and secondary prospective teachers selected for the TEDS-M study from various public universities from around the nation. The TEDS-M researchers, however, were not able to collect a complete set of results from each participant. They had complete data from only 474 of them, which was a completion rate of 78%. The 474 participants were from 46 universities, and each participant was in the last year of his or her teacher preparation program.

There were 121 prospective teachers from a lower-secondary certification program (up to Grade 10) and 353 prospective teachers from a secondary certification program (up to Grade 11 and beyond). I refer to these groups as prospective lower-secondary and prospective secondary teachers, respectively. There were a number of plausible differences within and between groups in the requirements to become certified at those levels (e.g., in courses, field experiences, and student teaching). The only data that I used, however, were the course topics studied by each program group (see Appendix C for a list of the courses surveyed by TEDS-M). Figure 1 shows the percentage of prospective teachers who studied each college mathematics course topic in the survey. Figure 2 shows the percentage of prospective teachers who studied each school mathematics course topic. Figure 3 shows the percentage of prospective teachers who studied each mathematics pedagogy course topic.

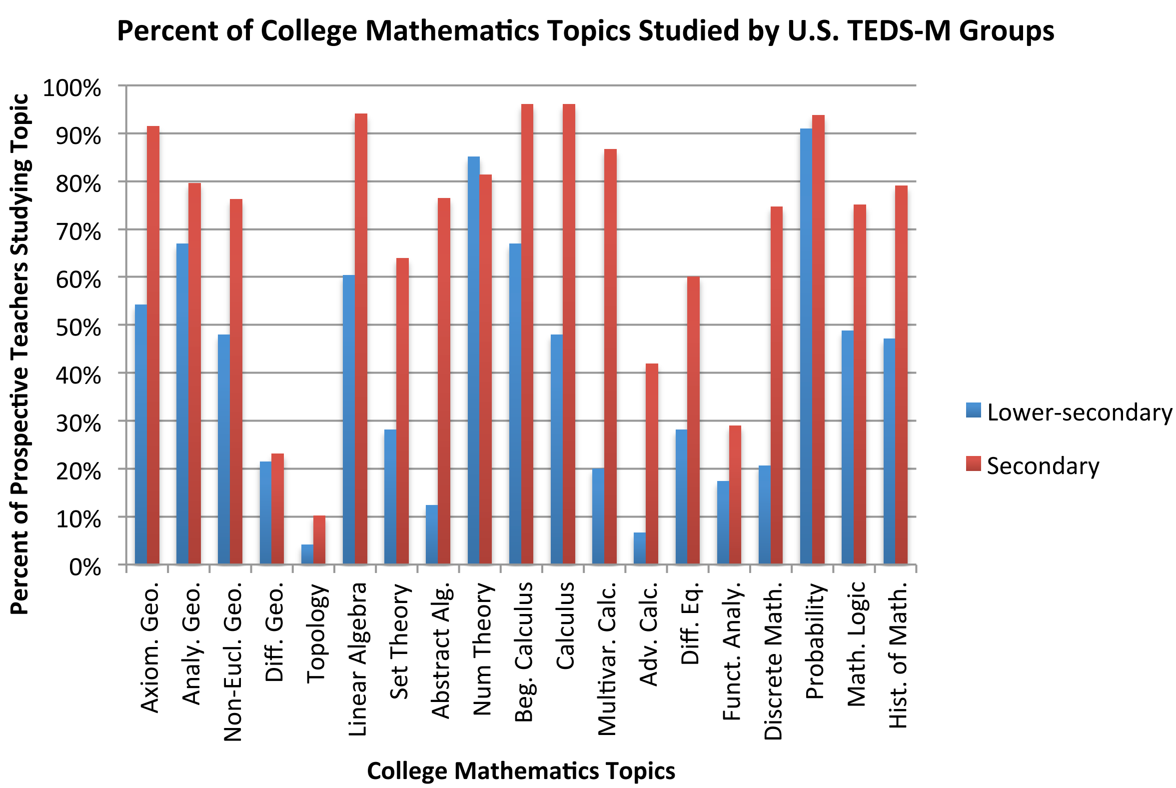

Figure 1. Percent of U.S. prospective teachers who studied college mathematics course topics by program group in TEDS-M. Source: International Association for the Evaluation of Educational Achievement (IEA; 2012).

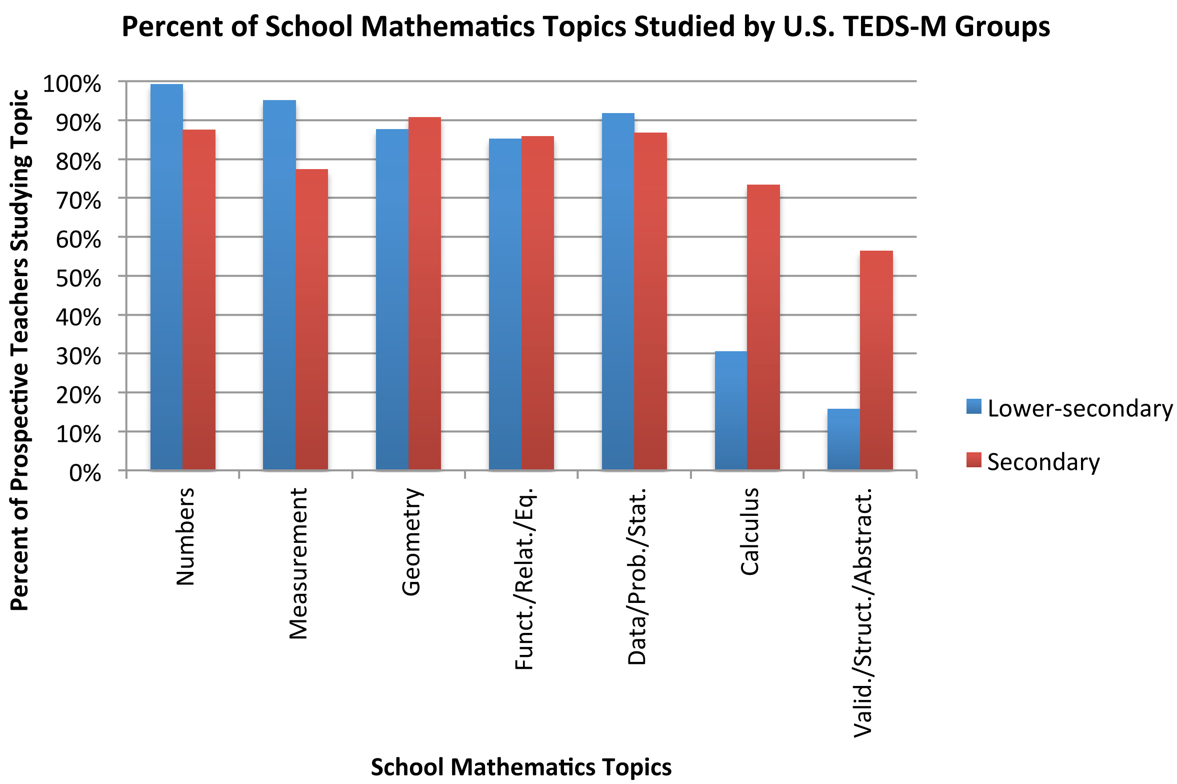

Figure 2. Percent of U.S. prospective teachers who studied school mathematics course topics by program group in TEDS-M. Source: IEA (2012).

Figure 1 shows that there was a large difference between groups in the percentage of students who studied college mathematics course topics. There were more prospective secondary teachers who studied each college mathematics course topics than prospective lower-secondary teachers except for number theory. Additionally, in all course topics—except differential geometry, number theory, and probability—there were many more prospective secondary teachers who studied them than prospective lower-secondary teachers.

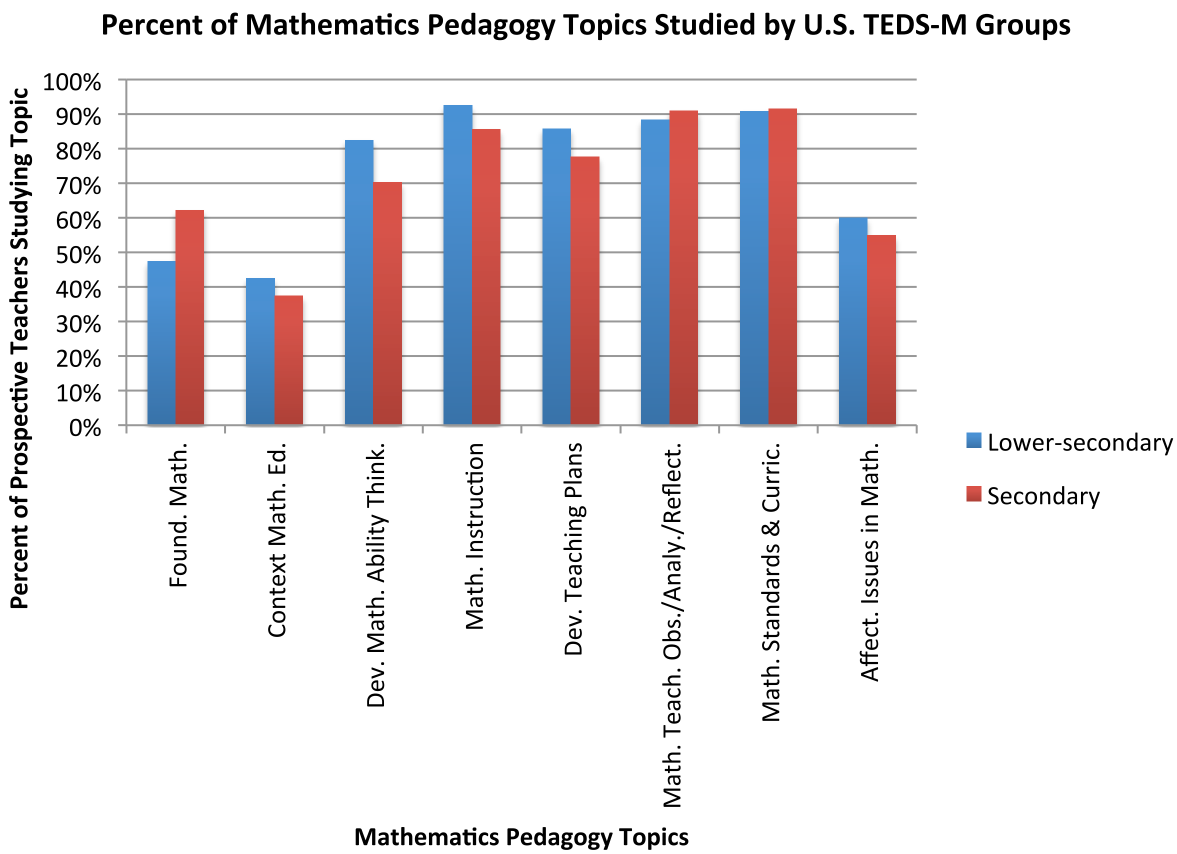

Figure 3. Percent of U.S. prospective teachers who studied mathematics pedagogy course topics by program group in TEDS-M. Source: IEA (2012).

An important idea to remember here, though, is that prospective lower-secondary teachers usually had to specialize in another subject, such as science or social studies, and did not have time to study as many course topics as the prospective secondary teachers. As such, it is noteworthy to see which course topics in college mathematics, out of all the possibilities, were studied by a significant amount of prospective lower-secondary teachers (e.g., geometry, number theory, beginning calculus topics, and probability).

Figure 2 shows that for school numbers and measurement topics, there were more prospective lower-secondary teachers studying those course topics than prospective secondary teachers. For school geometry, functions, and data topics, the percentages of those who studied the course topics were about the same. For the school calculus topic and the school validation, structuring, and abstracting topics, there were relatively more prospective secondary teachers studying those topics. Figure 3 shows that the percent of prospective teachers who studied each mathematics education course topic was about the same across the two groups.

It is to the differences in program requirements reflected in Figures 1 to 3 that I believe that the differences in overall MKT were related. Those becoming certified to teach secondary mathematics studied more (college) mathematics course topics than those becoming certified to teach lower-secondary mathematics.

Interview participants. For the interviews, I selected 10 participants from a large southeastern university in the United States. They were in the last year of their preparation program. Five participants were becoming certified to teach lower-secondary mathematics. This group of participants was selected to resemble the group of participants who were in the lower-secondary group that participated in TEDS-M. The other five were becoming certified to teach secondary mathematics. This group of participants was selected to resemble the group of participants who were in the secondary group that participated in TEDS-M.

With regard to course taking, the programs for each group of prospective teachers interviewed used a cohort model, and so the courses they took tended to be similar within groups. Table H1 in Appendix H contains a list of each of the *courses* taken by the prospective lower-secondary teachers who participated in the interviews and shows how they were associated with the *course topics* from the TEDS-M study. Table H2 contains a similar list for the prospective secondary teachers interviewed.

The prospective lower-secondary teachers interviewed took lower-level mathematics classes than those taken by the prospective secondary teachers interviewed. Even the Calculus 2 course proved too difficult for one prospective lower-secondary teacher. No other prospective lower-secondary teacher took it or any higher mathematics courses.

The prospective secondary teachers took a wider variety of mathematics courses, which were also deeper in content. The school mathematics courses were also deeper in content because they treated secondary mathematics in a deep way. Through these courses, the prospective secondary teachers were better prepared to teach mathematics.

Instruments

I used two types of instruments for the present study. First, since the data that I used to answer the first two research questions came from the TEDS-M study, I discuss how the TEDS-M researchers collected the data that were pertinent to this study. Second, to answer the third research question, I used the interview protocol developed in the pilot study. The protocol for the interviews is in Table 4.

The TEDS-M International Report (Tatto et al., 2012) outlined the instruments used to collect the TEDS-M data. First, from the field tests, the TEDS-M researchers decided that they could not survey the prospective teachers for more than 90 minutes each. Thus, they set up their survey to take 5 minutes to answer background questions, 15 minutes to answer opportunity-to-learn questions, 60 minutes to answer the MKT questions, and 10 minutes to answer the beliefs-about-mathematics- and teaching-questions. Of the MKT questions, about two-thirds addressed MCK, and one-third addressed MPCK. With regard to domain, 30% of the questions focused on number, algebra, and geometry each, with 10% focusing on data. The TEDS-M researchers used a rotated block design in order to ensure domain coverage and have all items answered by at least some of the prospective teachers.

The TEDS-M International Report also released a few MCK and MPCK items to give an indication of the difficulty and type of items that were given to the prospective teachers. Figures 4 and 5 contain items that have, respectively, an MCK and an MPCK application. For the MCK items MFC604A1 and MFC604A2, the content domain was algebra. Figure 6 contains item MFC704, which was an MCK item. Its content domain was geometry. Figure 7 contains item MFC804, an MCK item whose content domain was number.

MFC604A1, A2

The following problems appear in a mathematics textbook for middle school.

1. Peter, David, and James play a game with marbles. They have 198 marbles altogether. Peter has 6 times as many marbles as David, and James has 2 times as many marbles as David. How many marbles does each boy have?
2. Three children Wendy, Joyce and Gabriela have 198 zeds altogether. Wendy has 6 times as much money as Joyce, and 3 times as much as Gabriela. How many zeds does each child have?

(a) Solve each problem.

Solution to Problem 1:

Solution to Problem 2:

Figure 4. Sample MCK algebra item (Tatto et al., 2012, p. 145).

MFC604B

(b) Typically Problem 2 is more difficult than Problem 1 for middle school students. Give one reason that might account for the difference in difficulty level.

Figure 5. Sample MPCK algebra item (Tatto et al., 2012, p. 145).

MFC704

On the figure, *ABCD* is a parallelogram, $\angle BAD = 60°$, segments *AM* and *BM* are angle bisectors of angles *BAD* and *ABC* respectively. If the perimeter of *ABCD* is 6 cm, find the sides of triangle *ABM*.

Write your answers on the lines below.

AB = _______cm

AM = _______cm

BM = _______cm

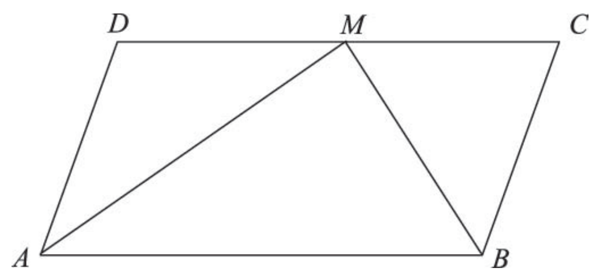

Figure 6. Sample MCK geometry item (Tatto et al., 2012, p. 145).

Procedure

Preparing the TEDS-M data. I first describe how I prepared the data used to answer the first two research questions. The original data set was from TEDS-M 2008 (IEA, 2012).

To indicate the program group for each prospective teacher, the TEDS-M researchers assigned a 5 for lower-secondary prospective teachers or a 6 for secondary prospective teachers.

As part of the TEDS-M survey, each prospective teacher was given a set of mathematics items to measure his or her mathematical knowledge for teaching (MKT). The TEDS-M study used items testing MKT in two areas: MCK and MPCK. The TEDS-M researchers calculated, using item response theory, scaled MCK and MPCK scores for each prospective teacher based on his or her performance on the items.

MFC804

A class has 10 students. If at one time, 2 students are to be chosen, and another time 8 students are to be chosen from the class, which of the following statements is true?

Check one box.

A. There are more ways to choose 2 students than 8 students from the class. ☐
B. There are more ways to choose 8 students than 2 students from the class. ☐
C. The number of ways to choose 2 students equals the number of ways to choose 8 students. ☐
D. It is not possible to determine which selection has more possibilities. ☐

Figure 7. Sample MCK number item (Tatto et al., 2012, p. 146).

In addition to the type of MKT, the items were also categorized by the four mathematical domains they covered: (a) number, (b) algebra, (c) geometry, and (d) data. Because there were not enough items for each mathematical domain, the TEDS-M researchers did not calculate scaled scores of those domains for each prospective teacher. In order to provide some insight to the mean performance on the number items, algebra items, geometry items, and the data items, I used the rubric the TEDS-M researchers provided (IEA, 2012), scored each item as they had, and calculated the mean percent correct for each prospective teacher for each of those four domains.

I was advised by the TEDS-M researchers to be sure to include the prospective teacher weight variable for the U.S. prospective teachers. The TEDS-M researchers calculated a sampling weight for each prospective teacher that adjusted for institutional differences and nonresponse. Nonresponse was a factor that plagued the TEDS-M researchers as they tried to get college students to do mathematics problems (Tatto et al., 2012). By including weights for each prospective teacher, I could make inferences about the entire population using the sample data.

In the data set, I included all the course topic variables (i.e., whether or not a prospective teacher had studied that course topic), which I took from the TEDS-M data file. There were three categories of course topics surveyed by the TEDS-M researchers: (a) college mathematics, (b) school mathematics, and (c) mathematics pedagogy. The prospective teachers were given the options to select from 19 college mathematics course topics, 7 school course topics, and 8 mathematics pedagogy course topics, for a total of 34 course topics.

Methods for exploring the association between program group and MKT. To respond to the first research question, I wanted to investigate the associations between the program groups and the measures of prospective teachers' MCK and MPCK. My goal was to try to understand the large difference between the prospective lower-secondary and secondary teachers on those measures as found in the TEDS-M study. In particular, I wanted to investigate the associations between the program groups and measures of the prospective teachers' knowledge of number, algebra, geometry, and data.

I used a multivariate analysis of variance (MANOVA) because I wanted to investigate how variations in both program groups associated with variations in performance in MKT, represented with multiple dependent variables (i.e., MCK, MPCK, number, algebra, geometry, and data scores), using one statistical model.

The MANOVA was conducted in SPSS using the multivariate general linear model function. The six sets of domain scores were the dependent variables, and the program group variable was the independent variable.

I analyzed the results by investigating how program group associated with higher (or lower) means for MCK, MPCK, number, algebra, geometry, and data. I took note of *F* values to determine the degree to which a group had higher means.

Methods for exploring the association between studying course topics and MKT. For the second research question, I investigated the associations between studying certain college mathematics course topics and the prospective teachers' MCK and MPCK, along with their knowledge of number, algebra, geometry, and data. I used the MANOVA procedure because I wanted to see with one model how studying 34 course topics was associated with scores from the six mathematics domains.

I had learned in the pilot study and in the TEDS-M International Report (Tatto et al., 2012) that the prospective teachers chose to prepare to teach at one of several grade bands. In the pilot study, I found that those individuals who chose to become certified to teach Grades 6–12 had higher MCK and MPCK scores than those who chose to become certified to teach Grades 6–8 (see Table 4). According to the TEDS-M International Report, the prospective secondary teachers in the participating countries had higher MCK and MPCK scores than the prospective lower-secondary teachers (Tatto et al., 2012, p. 147 for MCK and p. 150 for MPCK). To compensate for that selection bias, I used the program group variable as a blocking variable.

Also, it was important to compensate for relative achievement among the prospective teachers. The TEDS-M study data set provided a variable for all prospective teachers based on a survey question that asked them to report the level of grades they usually earned compared with their peers when in high school. The survey question gave options ranging from earning grades always at the top of their year level to earning grades generally below average for their year level on a five-point scale.

The lowest groups, representing those prospective teachers earning average or below average grades compared to their peers, had only 38 and 5 participants, respectively. Therefore, I combined those groups to make a four-point scale to attempt to reduce the amount of noise in the results.

Table 6 shows a breakdown of how the prospective teachers in each level of the self-reported high school grades variable performed. Since the means for each level of the variable mostly decreased according to how they claimed their grades compared in high school, I believe this variable is a viable indicator of relative achievement. Results from analyses specifically using the average or below average group, however, still may be problematic because of the small number of prospective teachers. Means from the portion of the group studying a course topic may be higher than usual compared with those who did not study the course topic because only a very few might have studied the topic.

I performed a MANOVA with blocking variables analysis in SPSS under the general linear model function, using the multivariate option. I used the six measures of MKT as dependent variables and whether each prospective teacher had taken a course addressing a set of 34 mathematics topics as the independent variables.

Table 6
Mean Scores of Each Level of the High School Grades Blocking Variable

Reported high school grades earned	*n*	MCK	MPCK	Number	Algebra	Geometry	Data
Top	132	530	524	58	51	59	53
Near top	199	511	507	51	47	58	47
Above average	99	491	496	46	43	56	43
Average or below average	43	501	510	53	46	53	43

Note. MCK and MPCK scores are scaled means and number, algebra, geometry, and data scores are mean percent correct generated from U.S. TEDS-M data (IEA, 2012).

It was not feasible to run the full factorial MANOVA model and try to investigate all the main effects and interactions of the results. There would be too many. Therefore, I specifically asked SPSS to calculate the main effects for all course topics and the blocking variables for prospective teachers' MCK scores. Because the blocking variables categorized the prospective teachers into groups based on their program groups and perceptions of the grades they earned compared with their peers, I needed to track the interactions between the course topics and those variables.

For example, if an interaction was significant and the overall means were associated with higher MKT scores, then the means for prospective teachers in certain program groups or high school grades levels could be nonsignificantly associated with similar MKT scores or significantly associated with lower MKT scores. Thus, it depended on which group the prospective teachers were in whether or not having studied the course topic was associated with higher MKT scores. In order to home in on the topics that had significant interactions with the blocking variables, I repeatedly ran the MANOVA analysis and removed the insignificant course topic interactions one at a time, until only the significant interactions remained.

Once I had generated the tables of the associations between course topic studying and MKT, I recorded each association's *F* ratio and noted which were significant. Then I noted which course topics had a significant interaction with the blocking variables. Because there were significant interactions, each needed to be investigated at a deeper level in order to see how the prospective teachers in each program group or level of high school grades performed on the specific MKT domain considering whether they had studied the course topic.

For example, if there were a significant interaction between the reported level of high school grades and multivariate calculus course topic for the MCK domain, then I would plot the MCK means of those who studied or did not study multivariate calculus by their claimed level of high school grades. Overall, it might be that the prospective teachers who took the course had a higher mean. With the deeper investigation, the prospective teachers at certain levels might have had higher MCK mean scores, roughly the same MCK mean scores, or lower MCK mean scores.

Methods for interviewing prospective teachers about courses. For the third research question, I believed that it would be beneficial to conduct interviews to expand on and deepen the statistical analyses of the associations between prospective teachers' MKT and having studied a specific course topic. I conducted interviews with a sample of 10 prospective teachers in the last year of their program, giving them the opportunity to discuss the mathematics courses they had taken and the benefits of the topics in those courses for their MKT. Most of the courses covered the 34 topics given to the prospective teachers in the TEDS-M study. A few topics were particular to the specific program at their university.

I requested interviews from 4–6 prospective teachers each from the middle grades and secondary programs at a large southeastern university that had been a participant in both the TEDS-M and the MT21 studies. I had five agree to interview from the lower-secondary program, becoming certified to teach Grades 4–8 mathematics, and five from the secondary program, becoming certified to teach Grades 6–12 mathematics. The interview groups were similar to the Grades 6–8 and 6–12 groups from the MT21 study and the lower-secondary and secondary (Program Groups 5 and 6, respectively) from the TEDS-M study.

In the interviews, I elicited the prospective teachers' thoughts about the utility of their course topics for their MKT in two ways. First, I used some of the same and some similar items given to the TEDS-M and MT21 prospective teachers to elicit discussions of which topics helped them solve the items. I used 14 released items from TEDS-M and 11 unreleased MT21 items that seemed to fit with descriptions of 11 of the unreleased TEDS-M items (see Table G2 in Appendix G). Second, I asked the prospective teachers to comment on the utility of each of their courses in general for their MKT.

As I conducted each interview, I video recorded the prospective teacher's written work as he or she completed the mathematics items. I also used the audio to record our conversations about the survey items and about the general thoughts he or she had about utility of his or her courses and course topics. After each interview, I had the audio transcribed for analysis.

In the analysis phase of the interview data, I read through the transcripts multiple times and coded the dialogue looking for any mention of college mathematics topics and how they contributed to the prospective teachers' MKT. I specifically noted how the prospective teachers discussed (a) each item and how studying certain topics helped them solve the item and (b) the utility of their courses in general for their MKT. I grouped the prospective teachers' comments on the items and which courses and topics were useful by item.

Preparing the results summary. To summarize the results, I first made a table that listed all the mathematics course topics that significantly associated with higher MKT, if taken. It also included the significance levels for each domain.

Second, I made another table that listed all the courses that were cited by the prospective teachers as being influential to their knowledge to solve MT21 and TEDS-M items. I also tracked the frequency with which each course was mentioned. Then I categorized each course into its corresponding course topic as described in the TEDS-M study in order to be able to compare my findings from the interviews with the results of the TEDS-M study.

Chapter 5: Results

Prospective Teachers' Knowledge by Program Group

Table 7 contains the results of the analysis that addressed the first research question on the relationship of the prospective teachers' knowledge by program group. The MANOVA that yielded Table 7 used IRT-scaled individual scores for each prospective teacher's MCK and MPCK and percent correct for the domains of number, algebra, geometry, and data. The scores for MCK and MPCK performance by program group were given in the TEDS-M International Report (Tatto et al., 2012, p. 147). However, the *F* values and performance in the domains of number, algebra, geometry and data were not.

Table 7

Mean Scaled Scores for Mathematical Content and Pedagogical Content Knowledge and Mean Percent Correct for Each Domain by Program Group

Program group	MCK	MPCK	Number	Algebra	Geometry	Data
Lower-secondary[a]	468	471	40	40	50	37
Secondary[b]	553	542	65	55	63	55
F	319.47‡	136.81‡	363.29‡	145.72‡	105.42‡	69.04‡

Note. Domain scores from U.S. data (IEA, 2012) from TEDS-M 2008 study (Tatto et al., 2012).
[a]There were 121 prospective lower-secondary teachers in the study. [b]There were 353 prospective secondary teachers in the study.
‡$p < .001$.

The prospective secondary teachers significantly outperformed the prospective lower-secondary teachers on all measures, as shown by the mean scores and high *F* ratios. Relative to their secondary counterparts, the prospective lower-secondary teachers needed improvement in all domains.

Prospective Teachers' Knowledge by Course Topics Taken

In this section, I report the results of the analysis of the MANOVA with blocking variables in which I used factors of (a) whether students studied each of 34 college mathematics course topics, (b) the program group (blocking variable), (c) the high school grades level (blocking variable), and (d) prospective teacher weights (as a weight factor). Weighted mean scaled scores of MCK and MPCK items, along with mean scores of number, algebra, geometry, and data knowledge were dependent variables. SPSS generated *F* ratios and their significance for all domains. The mean scores, *F* ratios, and their significance level for MCK, MPCK, number, algebra, geometry, and data by course topic are reported in Tables I1–I7 in Appendix I.

The following results show that there were 13 course topics that associated with significantly higher MKT scores as measured by TEDS-M items. I am especially interested in those topics because they associate with higher MKT scores and their true utility can be assessed through interviews. Thirteen of the course topics did not significantly associate with higher or lower MKT scores. I do not say much about these course topics since there is nothing one can conclude about how they associate with MKT scores. The rest of the course topics, or nine of them, were significantly associated with lower MKT scores. The utility of the course topics for prospective teachers' MKT was also addressed through interviews.

I must note that some of the course topics were associated with higher or lower MKT scores in certain domains that did not seem to be related with the course topic. For those topics, perhaps there was some sort of statistical error or fluke. For example, it does not make much sense that studying the course topic of foundations of geometry or axiomatic geometry would be associated with significantly higher data scores (see Table I1).

Table I1 displays the associations between the college mathematics course topics in geometry that were significantly associated with higher or lower MKT scores. The course topics that associated with significant differences in favor of higher MKT scores were the following: (a) foundations of geometry or axiomatic geometry for the data domain, and (b) analytic/coordinate geometry for the geometry domain. The fact that studying foundations of geometry or axiomatic geometry was associated with higher data scores is not intuitively obvious, because data topics are not typically covered in geometry courses. The fact, though, that studying analytic/coordinate geometry was associated with higher geometry scores is encouraging.

Course topics in geometry that were associated with significant differences in favor of lower MKT scores were the following: (a) non-Euclidean geometry for the MPCK, geometry, and data domains; and (b) topology for the MCK, MPCK, number, and algebra domains. It does not seem plausible that studying non-Euclidean geometry could affect a prospective teachers' MPCK since non-Euclidean geometry is a content course and MPCK is pedagogical knowledge. It is not encouraging that those who studied non-Euclidean geometry had significantly lower geometry scores than those who did not. Perhaps it occurred because the geometry items in the TEDS-M survey were from Euclidean geometry. Again, the fact that studying geometry course would be associated with a significant difference in data performance does not make much sense. It is surprising that taking topology would associate with lower scores in nearly all domains. Since so few participants took the course, it is more plausible that the lower scores were somehow unique to that group.

I did not include a table showing the association between studying college mathematics course topics in discrete structures and having higher or lower MKT scores. For none of those course topics did the prospective teachers who had taken them score significantly higher or lower on the MKT items than those who had not taken them. This finding is interesting because one would think that many, if not all, of the course topics in discrete structures would somehow be significantly associated with higher MKT scores. Those course topics are usually important courses in mathematics education programs. Many of the discrete structure course topics in the pilot study were significantly associated with higher MKT scores.

Table I2 displays the association between studying college mathematics course topics in continuity and functions and having higher or lower MKT scores. The course topics that associated with significant differences in favor of higher MKT scores were the following: (a) multivariate calculus for the MCK, number, and data domains; (b) advanced calculus or real analysis or measure theory for the MCK, MPCK, number, algebra, and geometry domains; and (c) differential equations for the algebra domain. It seems surprising that multivariate calculus is the first topic in continuity and functions topics to be significantly associated with higher MKT scores. In the pilot study, the topics of beginning calculus and calculus were also significant. The idea that studying multivariate calculus was associated with higher MCK and number scores is plausible, but that studying it was associated with higher data scores did not seem plausible. It seems plausible that studying advanced calculus or real analysis or measure theory would be associated with higher MCK, number, algebra, or possibly geometry scores, but not too plausible that studying advanced calculus or real analysis or measure theory would be associated with higher MPCK scores, because advanced calculus, real analysis, and measure theory are content courses and not pedagogy courses.

The course topics in continuity and functions that were associated with significant differences in favor of lower MKT scores included theory of real functions, theory of complex functions, and functional analysis for the MPCK and algebra domains. It seems that with some of the more complex topics, like non-Euclidean geometry, associations with lower MKT scores were more plausible. Again, that a content course can be directly associated with lower or higher pedagogical knowledge does not seem plausible. In addition, studying such a complex topic in algebra and having it be associated with a significantly lower algebra score does not seem plausible either.

Table I3 displays the association between studying college mathematics course topics in probability and mathematical logic and having a higher or lower MKT scores. The course topic that was associated with significant differences in favor of a higher MKT score was probability for the algebra domain. Even though it would make more sense for studying probability to be associated with higher data scores, it still seems plausible that studying probability can be associated with higher algebra scores.

Those prospective teachers who studied probability and mathematical logic had significantly lower MKT scores in the following topics: (a) probability for the geometry domain and (b) mathematical logic for the geometry domain. For these associations, I do not think that it is plausible that studying them results in lower geometry knowledge. I do not see much of a connection between the two.

Table I4 displays the association between studying school mathematics course topics and having higher or lower MKT scores. The course topics that were associated with significant differences in favor of higher MKT scores were the following: (a) school geometry for the MCK, geometry, and data domains; (b) school functions, relations, and equations for the MCK, MPCK, algebra, geometry, and data domains; and (c) school calculus for the number domain. It seems fitting that course topics in school mathematics would be significantly associated with higher MKT scores. In those courses, one would study the very topics surveyed in TEDS-M.

The course topics in school mathematics that were associated with significant differences in favor of lower MKT scores were the following: (a) school measurement for the algebra domain; and (b) school data, representations, probability, and statistics for the MPCK and algebra domains. Because the topics of those courses were surveyed in TEDS-M, it is perplexing to see that studying such topics would be associated with lower scores.

Table I5 displays the association between studying mathematics pedagogy course topics in foundations in mathematics pedagogy and having higher or lower MKT scores. The course topics that were associated with significant differences in favor of higher MKT scores were the following: (a) foundations of mathematics for the MCK domain; (b) development of mathematics ability and thinking for the data domain; and (c) affective issues in mathematics for the MPCK, number, and algebra domains. In the case of studying foundations of mathematics pedagogy, to see that studying a topic in pedagogy would be associated with higher scores in content is encouraging. In the pilot study, Nick saw some benefits of covering content in a pedagogy course. Similarly, to see that studying the development of mathematics ability and thinking would be associated with higher data scores is possible, but not plausible. In addition, it is plausible to see that studying affective issues in mathematics would be associated with significantly higher MPCK scores and perhaps the other content domains as well.

The course topic in foundations in mathematics pedagogy that was associated with significant differences in favor of lower MKT scores was the development of mathematics ability and thinking for the number domain. This finding of taking a mathematics pedagogy class and having it be associated with lower number scores is also not very plausible.

Table I7 displays the association between studying instruction in mathematics pedagogy course topics in mathematics teaching and having higher or lower MKT scores. The course topics that were associated with significant differences in favor of higher MKT scores were the following: (a) mathematics instruction for the algebra domain; and (b) developing teaching plans for the MPCK and algebra domains. It seems plausible that the mathematics instruction course could be associated with higher algebra scores if algebra was a focus of examples, but we cannot be certain. Likewise, it is plausible that developing teaching plans was associated with higher MPCK score because both are pedagogy related.

The course topic in instruction in mathematics pedagogy that was associated with significant differences in favor of lower MKT scores was mathematics teaching: observation, analysis, and reflection for the algebra and geometry domains. Again, by taking courses in mathematics (pedagogy), it does not seem plausible that one would score lower on items covering those content areas.

When I used program groups and high school grade levels as blocking variables, there were certain groups or levels that could have had significant interactions with studying course topics. For example, for a single course topic, certain groups or levels could have had higher MKT scores, whereas the other groups or levels could have had lower MKT scores. For that reason, I included the interactions for all courses with the blocking variables in the statistical model.

There were a total of 408 possible interactions in the model (34 courses × 2 blocking variables × 6 domains). Of all possible interactions in the model, 93 were significant. Of those interactions that were significant, 27 involved course topics associating with significantly higher MKT scores, and 66 did not.

For example, the results showed that overall those who studied non-Euclidean geometry scored lower on the geometry items. Also, the interaction of geometry scores blocked by level of high school grades for those who did or did not study non-Euclidean geometry was significant. This interaction meant that even though the overall mean score for geometry was lower for those who studied non-Euclidean geometry, there were some prospective teachers in certain levels of high school grades who studied non-Euclidean geometry and had significantly higher geometry mean scores. More specifically, those who studied non-Euclidean geometry in the near top and above average groups had higher geometry means and the top and average or below average groups had lower means (see Figure J10).

Table J1 in Appendix J contains a list of all 7 course topics that had significant interactions with program group *and* were associated with 5 significantly higher or 2 significantly lower MKT scores. Table J2 contains a list of all 20 course topics that had significant interactions with high school grade levels *and* were associated with 11 significantly higher or 9 lower MKT scores. To investigate these interactions more deeply, I created Figures J1–J27 in Appendix J.

Figure J1 is unique in that it shows a significant interaction between level of high school grades and program group for MPCK. For most of the level of high school grade groups, those enrolled in lower-secondary programs had higher MPCK scores than those in secondary programs, except for those prospective teachers who said that they had average or below average high school grades relative to their peers. This is a curious finding since the MPCK overall average favors the prospective secondary teachers (see Table 6). I am not sure why this interaction worked out this way.

Figures J2–J7 contain interactions for studying course topics and program groups. Figure J2 shows that for geometry scores, of those who studied analytic/coordinate geometry the lower-secondary prospective teachers scored higher on average. For those who did not study analytic/coordinate geometry, the prospective lower-secondary teachers scored lower on average.

Figure J3 shows that for algebra scores, those who studied theory of real functions, theory of complex functions, or functional analysis, the lower-secondary prospective teachers scored lower on average. For those who did not study theory of real functions, theory of complex functions, or functional analysis, the lower-secondary prospective teachers scored higher on average.

For the rest of the interactions (i.e., Figures J4-J7), those enrolled in lower-secondary programs performed about the same on the MKT domain survey items if they had studied the course topic. If they had not studied the course topic, the lower-secondary students performed much lower.

Figures J8–J27 show details of the interactions between course topics and level of high school grades for the various domains. Figure J11 is unique in that the data scores for all levels of high school grades are lower if non-Euclidean geometry was studied. For all the other significant interactions between courses blocked by level of high school grades for certain domains, there is at least one level that was associated with the opposite outcome than the overall means. That is, if the overall means were associated with higher scores if the topic was studied, there was at least one level of high school grades that was associated with lower scores if the topic was studied, or vice versa. In Figures J8–J27, I have highlighted with a shadow, for the reader, which levels of high school grades went counter to the overall mean.

I found that there were nine situations in which marked differences were seen in MKT performance from the prospective teachers belonging to the average or below average level of high school grades. Table 8 shows the number of prospective teachers from this level who studied and did not study those course. In the table, one can see that there were very few prospective teachers who did not study those course topics compared with the number who did study them, with the exception of non-Euclidean geometry.

Other plausible reasons that could account for the marked differences in MKT scores of those who earned average to below average high school grades include a miscategorization by the prospective teacher of which level of high school grades they should be in. Or, they could have become better students from when they were in high school to the time they participated in the TEDS-M study.

Table 8
Course Topics and Number of Prospective Teachers from the Average or Below Average Level of High School Grades

Table number	Course topic	Number who studied the course topic	Number who did not study the course topic
J8	Foundations of geometry or axiomatic geometry	31	12
J11	Non-Euclidean geometry	25	18
J12	Topology		
J17, J19, J20, J21	School functions, relations, equations	38	5
J22	School data, representations, probability, and statistics	39	4
J26[a]	Mathematics teaching: observation, analysis, and reflection.	37	5

[a]There was one prospective teacher who did not indicate whether or not he or she studied mathematics teaching: observation, analysis, and reflection.

Prospective Teacher Interviews

I included interviews as a part of the study in order to understand the associations between certain college mathematics courses and MKT in a deeper way. I wanted to investigate how prospective teachers agreed (or not) with the associations I found using the TEDS-M data by interviewing a group of prospective lower-secondary teachers and prospective secondary teachers. I also wanted to investigate which courses they thought were most helpful for their MKT and how those courses were helpful. (The association between courses taken by the interviewees and the participants in the TEDS-M study is shown in Appendix H.)

College Mathematics. I discuss the utility of college mathematics first for prospective lower-secondary teachers' MKT and then for prospective secondary teachers' MKT using data from the interviews.

The Utility of College Mathematics Courses for Prospective Lower-Secondary Teachers' MKT. Because of program requirements, the prospective lower-secondary teachers did not take many college mathematics courses (see Table H1 in Appendix H). Most had just taken college Precalculus and Calculus 1. Overall, the prospective lower-secondary teachers did not find these classes too relevant to their MKT or as beneficial as their school mathematics and mathematics pedagogy courses were.

Precalculus. Cora stated that college Precalculus and Calculus 1 were good "foundational classes." Speaking of Precalculus, she stated, "Obviously, it's going to help me with my teaching, because I know where [the middle school curriculum is] headed." In this regard, she found the course helpful for her MKT, or more specifically, her horizon content knowledge. She did not, however, think that the course was as beneficial as her school mathematics courses or mathematics pedagogy courses had been. She said, "But it didn't help me learn how to teach, because it was just not an education class." As with the other prospective lower-secondary teachers, Cora valued courses that directly helped the specialized content knowledge (SCK) of her MCK and her MPCK.

Business Calculus. Ashley took Business Calculus and said that the instructor did not solely focus on business or finance applications of calculus, but calculus concepts for a wide range of real-life examples. She described its utility for her MKT compared to her views of the utility of Calculus 1 for her MKT:

> [In the Business Calculus] class, we did a lot of word problems and real life examples kind of stuff, versus the [Calculus 1] class, where we did a lot of

> numbers and equations. It wasn't bad, but I would say the [Business Calculus course was] really [applicable]. I think understanding that stuff helps to explain other things that are easier than what we did in [Calculus 1], if that makes sense.

Her ability to explain calculus concepts was fostered more in the Business Calculus course than the Calculus 1 course.

Calculus 1. Although three of the five prospective lower-secondary teachers interviewed did not see any benefits to Calculus 1 for their MKT, Melinda found a useful aspect of Calculus 1. She said,

> I'm very glad that I took [Calculus 1], because … it was a hard class. So, it caused me to think more analytically about what I was doing, than I ever have before in a math class. … So, I feel like it prepared me to start [the school and mathematics pedagogy] classes. Because, if I hadn't been made to think more analytically about what I was doing and why I was doing it, then I would have really struggled with these classes.

Calculus 1 was Melinda's first college mathematics course. She found that at the college level, one needs to think more analytically about mathematics than before in order to gain the most from it. Melinda attributed her success in her other courses to that Calculus 1 experience.

College Mathematics in General. Chris stated that he did not see the relevance of higher mathematics courses to teaching middle school mathematics. His thoughts characterized the mentality of most of the prospective lower-secondary teachers:

> I can say that probably Calculus, Linear Algebra and Precal really don't apply to middle school math. I mean they're good, just 'cause it's a higher math for my own knowledge. But, as far as enhancing my teaching of middle school math, those don't apply. They're good, I guess, but I've always said, "Why do you need to take such high math classes to teach not that high of math?"

The fact that most of the prospective lower-secondary teachers that I interviewed thought that college mathematics was far removed from middle school mathematics detracted them from learning higher mathematics and enriching their MKT.

The Utility of College Mathematics Courses for Prospective Secondary Teachers' MKT. The prospective secondary teachers took many more college mathematics courses than the prospective lower-secondary teachers did (see Table G2). Prospective secondary teachers' views of the utility of college mathematics courses were usually different than those of the prospective lower-secondary teachers because they were becoming certified to teach mathematics much closer to college mathematics, even with some overlap. On the whole, prospective secondary teachers found college mathematics courses more helpful for their MKT than the prospective lower-secondary teachers.

Foundations of Geometry I. At the time of the interviews, three of the five prospective secondary teachers were taking Foundations of Geometry I. For that reason, they did not make very many comments about its utility for their MKT. Caitlyn, on the other hand, had already found it helpful for solving a TEDS-M item about the three-dimensional coordinate plane:

> We are working with the *x*, *y*, *z* plane right now … in our geometry class. We are looking at things, like points and lines and $\mathbf{R}^2$ and what they are, and $\mathbf{R}^3$, which is when you add dimension *z*. So, this is very similar to what we are doing in class right now.

Because of her experiences in Foundations of Geometry I, Caitlyn was able to solve this problem and demonstrate that she understood this aspect of her MCK of MKT.

Linear Algebra. With respect to Linear Algebra, Keri commented on its utility for her MKT. She valued the mathematics of matrices and transformations. No one else, however, said much else about Linear Algebra.

Introduction to Higher Mathematics. The Introduction to Higher Mathematics course was a combination of the topics of set theory, mathematical logic, and proof writing. Three of the five prospective secondary teachers expressed thoughts about the utility of Introduction to Higher Mathematics. Overall, they thought that the course helped them progress in their mathematical understanding of proof and its usefulness. Caitlyn said:

> As much as I didn't like [the Introduction to Higher Mathematics] class, it was important to develop a lot of the notation, a lot of the proof vocabulary, a lot of the proof structure. Which, in high school, my proofs were the geometry proofs: [like] side-angle-side. It was very different than this proof. In our classes, we've talked about the importance of developing both informal and formal proof reasoning in high school, which I know I didn't develop. … So, when I took that college course, it was very difficult for me. And I think that if I would have been at least introduced to it before, I wouldn't have been so lost. So, in that way, [Introduction to Higher Mathematics] was necessary for my experience. But, I don't know if it would have been, had I had that in high school.

Caitlyn expressed the importance of developing her ability to prove higher mathematical theorems. She also brought up the fact that her courses in high school could have also made forays into developing this ability to prove. Since her high school courses did not foster this type of thinking, Introduction to Higher Mathematics was useful for her MKT.

Natalie and Michelle discussed a shift in mathematical thinking that occurs when taking Introduction to Higher Mathematics. Natalie stated:

> Yeah, I definitely learned to think differently when I started taking the [Introduction to Higher Mathematics] class. [I learned] a different way about thinking about math, which I guess will probably come in handy when I'm a teacher. … Just to really think deeply about a problem, rather than just how I would have done it in high school.

Not only had Natalie noticed that her thinking changed, but she thought that her new way of thinking would be important to her MKT when she would begin teaching. On a related note, Michelle commented, "It's not the concept so much, but the idea of proving things could be helpful." Michelle focused on the fact that it was important to think deeply about the mathematics by understanding proofs of mathematical concepts, an idea beneficial for one's MKT.

Modern Algebra and Geometry I. This course's main topic was abstract algebra. The prospective secondary teachers interviewed had much to say about the utility of this course for their MKT. Natalie commented that in addition to Introduction to Higher Mathematics, Modern Algebra and Geometry I had helped her make the shift to a deeper thinking of mathematics.

Most of the prospective secondary teachers, however, did not see the course as relevant to the MKT they needed to teach. Keri's thoughts summarized their position well. She said, "[Modern Algebra and Geometry I] was a tough course, and I'm sure I learned some great problem-solving skills, but it was good to get out of there. In [higher] math, all the courses have been difficult, but not relevant, I guess." Keri had touched on the fact that she learned great problem-solving skills, but had missed the mark of the course as a deep look at algebra, which might aid her in teaching algebra. The prospective secondary teachers talked a lot about the course's lack of utility and wanting it to be done quickly.

A possible demonstration of the "great problem-solving skills" mentioned by Keri, which were gained by the prospective secondary teachers, was evident when the prospective secondary teachers were solving TEDS-M items. In particular, there were three items on which the prospective teachers stated they used knowledge gained from the Introduction to Higher Mathematics course and the Modern Algebra and Geometry I course. The first was an item in which the prospective teachers were asked to decide whether certain relations could be considered as equivalence relations. The second and third items included examples of students' proofs of number theory concepts, and the prospective teacher was to assess their validity. The prospective secondary teachers completed the items correctly and easily, whereas the prospective lower-secondary teachers did not complete the items and struggled to reason about them. Therefore, I believe that it is plausible that the Introduction to Higher Mathematics and the Modern Algebra and Geometry I courses were very beneficial for the prospective secondary teachers' MKT. In addition, I would recommend a similar course for prospective lower-secondary teachers, one that would help them gain similar problem-solving skills.

Calculus 1, Calculus 2, and Multivariable Calculus. Michelle captured the reasons that her prospective secondary teacher colleagues found the various calculus courses valuable to take:

> I think [Calculus 1 and Calculus 2] covered AP Calculus AB and BC. So, there's the potential that you're going to be teaching high schoolers the AP Calc AB and BC, so which is what I took in high school my junior year. I took it again in college just to make sure I remembered everything. So, I think you could be put in that situation. So it could be good to have definitely [taken Calculus 1 and Calculus 2].

Michelle viewed Calculus 1 and Calculus 2 as relevant courses for preparing her to teach AP Calculus in high school, and that was the sentiment of three of the other prospective secondary teachers. Only one prospective secondary teacher interviewed, Keri, thought that there should be an additional course on how to teach calculus topics, because she said she had forgotten a lot of calculus at the time of the interview.

There were not many comments on the third semester of calculus topics, but Keri said that she thought Multivariable Calculus was beneficial for her MKT. It was the first college mathematics course she took. From there, she did not study calculus topics for a while and became concerned:

> I almost wish we would have a calculus course later in our curriculum because I know I've forgot a lot of calculus. So, if I go into teaching calculus, which I think I would really enjoy, I will definitely have to relearn a lot. And so, if I took calculus, maybe my last semester, I think it would be beneficial for me to relearn something with calculus.

Keri thought that a calculus course would have been beneficial for her at the end of her program. Instead of just rehashing calculus, I believe that a course on teaching calculus concepts could fulfill her desire to further study calculus and contribute to the MPCK of her MKT at the same time. A similar contribution was accomplished somewhat in the Connections in Secondary Mathematics III course, which covered trigonometry and precalculus teaching.

Elementary Differential Equations. Elementary Differential Equations was valued by two of the secondary teachers, Natalie and Michelle, because of its connections to the real world. However, its applicability to the knowledge to teach middle school was not too apparent to Caitlyn:

> Differential Equations is very much applied math. It's for math and science majors. Those are the only two majors at the university that require this course. I don't think that it would help me in my teaching experience, specifically because I really want to teach middle school, and I think that it's too high for an extension from the calculus that you learn in high school. … It's a lot of harmonic oscillators and spring constants and mixing solutions. Because, you're doing all these operations that seem, you would have no motivation to do them if you didn't have any context. Because you're like, "What the heck am I doing?" So, the way that it becomes a little bit interesting is you would do this if you wanted to find out the composition of the solution in a tank. Or if you wanted to know, you had a mass on a spring, how many times it would bounce before it rested. You have to have the context in order to understand what you're doing. And a lot of the context is applicable for engineering majors and for science majors who will be working with those things in their field. But as a middle or high school math teacher, it wasn't relevant to what I'm going to be doing. It might be cool. I might have my class reference the time where if you want to be a math or science major in college, this might be something you might be doing in your career. But you're not really talking about careers with your high school students very often.

In middle school, Caitlyn could use her knowledge of differential equations to do the very thing that she thought was not relevant. It is important as a mathematics teacher to have knowledge of possible career paths in mathematics. Talking about mathematics in careers is important to motivate students to learn the seemingly inapplicable middle school mathematics (Ellis, 2008).

The knowledge gained from Elementary Differential Equations helped Michelle solve a TEDS-M item. The item had Michelle think about three real-life mathematical situations and determine if they could be modeled by an exponential function. She stated that the differential equations course was the key course that helped her have the knowledge to solve the problem. She also commented on the fact that it was a good refresher for thinking about using functions to model phenomena in the real world.

Statistics. The prospective secondary teachers knew about the importance of their statistics courses from their mathematics pedagogy classes. They saw how statistics was being brought into the high school curriculum and knew that they needed to strengthen their statistical knowledge. Caitlyn's thoughts are representative of those of her peers:

> We just don't have the time to talk about how you can incorporate statistics in every level, now that they're integrating it. So, I think that was the goal of having a separate course, Statistics for Teachers, because it needs its own attention. It's so different than other math, and it's not something that you can just mix in and talk about with all your other stuff in our [mathematics pedagogy] classes.

Caitlyn appreciated having a separate course on statistics education. The Statistics for Teachers course was very influential for her MKT. She continued:

> We did a final project that really incorporated the lesson plan. We had to read a couple of articles about using statistics technology in the classroom, and [that] required us to reach out and look on other experiences in ways we can use statistics in the classroom. I saved all my stuff. I can see myself using that.

The courses that gave the prospective teachers these kinds of materials and opportunities were the ones that they thought helped their MKT the most.

School Mathematics. I discuss the utility of school mathematics courses first for prospective lower-secondary teachers' MKT, and then for prospective secondary teachers' MKT using data from the interviews.

The Utility of School Mathematics Courses for Prospective Lower-Secondary Teachers' MKT. Overall, the prospective lower-secondary teachers thought that the school mathematics courses were the most helpful for gains in their MKT. The TEDS-M items covered many school mathematics topics, but the knowledge gained from the school mathematics courses was only evident at times in the prospective lower-secondary teachers' responses.

Arithmetic and Problem Solving and Arithmetic for Middle School Teachers. The prospective lower-secondary teachers made few comments on the course of Arithmetic for Middle School Teachers. There were not any items solely covering arithmetic in the TEDS-M study. Two of the prospective lower-secondary teachers might have devalued the course because they had taken it so early in their program. Melinda captured their thoughts:

> I would probably say [that taking Arithmetic and Problem Solving] my freshman year … didn't completely help me, just because I wasn't in my major yet, first off. I don't think I understood how it would be implemented into my classroom. … If I probably would have taken that my junior year, it would have helped me more, [rather] than taking it my freshman year.

By her junior year, Melinda thought that she knew better the environment in which she would be teaching and would be better able to focus on learning the mathematics in a way she could use. The awareness of how she would use the knowledge gained in the Arithmetic and Problem Solving course helped her make more focused gains in her MKT.

Algebra for Middle School Teachers. There were four items that prospective lower-secondary teachers attempted using knowledge from the Algebra for Middle School Teachers course. Those items covered (a) a contextualized situation for systems of linear equations, (b) assessing students' work on a number theory proof, (c) modeling real-life mathematical situations with an exponential function, and (d) the probability of an event occurring. This content course was helpful for the prospective lower-secondary teachers to understand what the items required them to do, but none of the prospective lower-secondary teachers was able to complete the items. They lacked the further knowledge that a course in, say, Introduction to Higher Mathematics, would provide for their MKT.

Geometry and Measurement for Middle School Teachers. Overall, the prospective teachers did not have much to say about the Geometry and Measurement for Middle School Teachers course in general, but they did attribute the knowledge they gained from the course as critical for addressing a number of TEDS-M items covering geometry. Cora commented on the positives and negatives of the course:

> [Geometry and Measurement for Middle School Teachers] … was good. It developed a lot [of mathematics], and we happened to use words to explain [the] … math, so that was important. It felt [like] a lot of busy work. It felt like my homework was frustrating at times. [I felt] this was a waste of time. … I think it was important to develop that foundation of really diving into the meaning of things. Like, for those triangles, why [the sum of their angles] makes 180 degrees. It felt primitive, like I said, but it wasn't.

Cora thought it was beneficial for her MKT to look at the content deeply, but at the same time, she found it difficult to focus so much on basic material.

The prospective lower-secondary teachers interviewed mentioned that they had gained a few skills from this course. For example, Chris said:

> [Geometry and Measurement for Middle School Teachers] is more of [problem-solving] methods, like looking at the problem and evaluating what each piece is, and that's kind of what I did here [in this TEDS-M item]. … [What helped the most were the] problem-solving techniques, which is pretty much what we do in [Geometry and Measurement for Middle School Teachers] every day.

He had gained an appreciation for problem-solving techniques. Such techniques and views of problem solving were beneficial for his MKT and for fostering the techniques in his future students. In another example, Ashley used the knowledge she gained from the course of what constitutes a proof on a TEDS-M item about assessing proofs.

School mathematics courses in general. Three of the prospective lower-secondary teachers interviewed made important comments about their school mathematics courses overall. Joan discussed how the prospective lower-secondary teachers thought about the courses as they progressed through the semester:

> [The school mathematics] courses have helped me the most. I was very … resistant to [those] courses in the beginning. … [They] did make me think differently, and [I got] very frustrated and [I] just [wanted] to say, "This is just [how] you do [the mathematics], I don't know the reason behind it." It was hard for me to … explain the math that I was doing. I think learning how to do [the mathematics] and learning why we're doing the math procedure that we're doing has greatly expanded my knowledge.

For her, studying school mathematics in a deeper way and focusing on explaining the topics were challenging as well. Discussing the deeper conceptual meanings of school mathematics that were part of her MKT might help Joan engender such practices in her future classroom.

Like the other prospective lower-secondary teachers interviewed, Chris found that the school mathematics courses were very helpful for refreshing mathematics that he had not seen in years:

> I would say, definitely, the [school] math classes … have been beneficial, just because they are a major refresher. And in those classes we dive deeper into [the mathematics] and really get a better understanding of the … middle school math concepts for our own benefit. So, obviously if I know something better, then I'm going to be able to talk about it more [fluidly] and be able to explain [the answers to the] questions that my students may ask. Without knowing those basic, … background concepts, I would say, then I wouldn't be able to answer those questions.

Chris brought up a couple of good points. He thought that the utility of the courses was in the opportunity they gave him to refresh his school mathematical knowledge, or the mathematics that he would teach. Also, he thought that knowing the mathematics deeply would help him field questions from his students and be able to explain the answers in a more fluid manner.

Cora performed a TEDS-M item about fraction division and stated that she could have completed the problem without any of the mathematics courses offered in her program. She stated:

> I think I could have [done the item] coming out of high school. I don't think I would have thought about it the same way and I don't think I could have explained it. I think right now, I could explain this to kids multiple ways and I could show different representations. I could do this in GSP. I could a lot of stuff with it, more even than what is just shown right here [in the multiple choice options]. Yes, I could have just simply done this coming out of high school, but I can do a lot more because of these classes.

Cora found that the school mathematics and mathematics pedagogy courses helped her deepen her knowledge of fraction division. Her newfound ability to explain the concept in multiple ways using multiple representations expanded her MKT and would be an asset to her in her future teaching.

The Utility of School Mathematics Courses for Prospective Secondary Teachers' MKT. Generally speaking, the prospective secondary teachers interviewed thought their school mathematics courses were beneficial and appreciated the course work they did to strengthen their MKT just as the teachers in the pilot study had done.

Connections in Secondary Mathematics. The Connections in Secondary Mathematics course covered the school arithmetic topics in a deep way, which served as a basis for, and connected such knowledge to, high school mathematics. Because the prospective secondary teachers were more interested in high school topics, they had mixed feelings about how that course had benefitted their MKT.

Keri commented on the lack of utility of her school numbers course: "We spent forever on trying to figure out, what does the different representations of addition—like all the operations of fractions—what does that look like?" Similar to comments made in the pilot study, both Keri and Natalie thought it seemed to require a degree of patience to investigate fraction operations at a deep level.

Other secondary teachers, including Caitlyn, Brian, and Michelle, however, commented favorably on the course's activities and purpose. For example, Caitlyn said:

> The main thing that sticks out to me is [that] we worked a long time on partitioning and dividing fractions and multiplying fractions. We did it all visually with a number line or a double number line or some sort of picture. I don't remember learning fractions that way, and I really liked how [the professor] did it in that class. It gave you the opportunity to actually understand what was going on and not just looking at … dividing a fraction by [a] fraction [by multiplying] by the reciprocal—that being a procedure. This really helped develop the understanding of what we're doing, and why we're doing [it], and why our answers look the way that they do.

For Caitlyn, the course strengthened her MKT by giving her the opportunity to understand those numerical operations.

Concepts in Secondary Mathematics. The Concepts in Secondary Mathematics course, which was soon replaced with Connections in Secondary Mathematics II after my interviews, extensively covered content in functions, relations, and equations, primarily within trigonometry and precalculus. The prospective secondary teachers interviewed enjoyed the course and found it very beneficial for their MKT, just as in the pilot study. Keri's thoughts best expressed their sentiments:

> [Concepts in Secondary Mathematics] was probably the best class that I have taken specifically for teaching. For the content, the content was neat. But really, even though it was a content class, I learned the most about pedagogy in that class than in my other classes. … I have never personally had a teacher that taught that way. I'm not seeing it that way in schools, and so that class was really my only class based off of how they want us to teach. It wasn't something that we paid attention to at the time, because we were so involved in doing the task. But that was the best all-around class that I've taken because I learned both content and pedagogy in it.

Keri was impressed with the content and the structure of the class. It facilitated mathematical discussions with group-worthy tasks. The prospective secondary teachers experienced concrete exemplifications of the theory they were encountering in the teacher preparation program. They integrated the experiences and knowledge into their MKT, and they became a valued part of it.

Connections in Secondary Mathematics III. The Connections in Secondary Mathematics III course deeply covered content in geometry, measurement and data representation, probability, and statistics. With regard to this course, Caitlyn captured her colleagues' points of view, which was that the course was seen as favorable for their MTK:

> We've done a lot with GSP [Geometer's Sketchpad] … It's good working with this technology program because a lot of the classrooms are starting to use more and more technology. Specifically looking at GSP as a way for students to not prove things, but understand how things work for all cases. … So, as a teacher, I

> would like to use that tool because I think it's important for students to be able to do that—to see, explore, investigate. See it for themselves, develop the understanding, and not just be told that this is a truth and you need to accept it. … They accept it because they've explored it on their own. It's not accepting it because it came out of the teacher's mouth; it's accepting it because they know why it's true. They explored it and discovered it on their own.

This course was influential in helping teachers understand what they need to have their students do in order to prove their mathematical ideas in geometry. The prospective secondary teachers now had an insight into what students should learn about geometric concepts and proofs, because they had done it themselves. This new knowledge had expanded their MKT.

Mathematics Pedagogy. I discuss the utility of mathematics pedagogy courses first for prospective lower-secondary teachers' MKT then for prospective secondary teachers' MKT using data from the interviews.

The Utility of Mathematics Pedagogy Courses for Prospective Lower-Secondary Teachers' MKT. For the most part, the prospective lower-secondary teachers found the mathematics pedagogy courses beneficial for their MKT. There were only a few exceptions.

Teaching Number Systems in the Middle School. The prospective lower-secondary teachers had mixed feelings about the Teaching Number Systems in the Middle School course. Cora did not find covering the content or how it was done to be helpful for her MKT.

Melinda and Ashley, on the other hand, found the course to be very helpful for their MKT. Melinda captured this sentiment best:

> So, in [the Teaching Number Systems in the Middle School] class, I started to learn a little more about how to communicate mathematically. We did number lines. We did double number lines, a lot, and we did—we talked about the

> concepts for the first time. … We were made to feel like, in all of these classes … what it feels like to be a middle schooler and to be introduced to these concepts. They're being introduced to me in ways I have never seen before. … I feel like all of our teachers have made progress with us and have made us feel that way. I'm sure it was intentional to be like, "This is the way your kids feel, this is what you need to present them with, because this is what builds conceptual learning," which is what I have learned over the past two years, more so than just teaching to the test.

The most important part of what Melinda shared was the fact that the mathematics pedagogy course helped her learn new material as a middle school student would. In this way, Melinda was able to understand, in some way, how her students would feel when they learned middle school mathematics. This class helped increase her knowledge of content and students (KCS), which is part of the MPCK of MKT.

Teaching Algebra in the Middle School. The prospective lower-secondary teachers interviewed especially appreciated the Teaching Algebra in the Middle School course. Cora characterized the group's sentiment best:

> Then with [Teaching Algebra in the Middle School], this class is really good. It's very challenging, and [the instructor] withholds a lot of information. [He] … sends us in a direction, then he stops, and he makes us struggle, and we leave class kind of like, "Ugh." We have homework, and we're still frustrated and puzzled. Then things connect. And then he kind of leads us again, and it just [like this] again and again. … I talked to a doctoral student, a Ph.D. student, who had this class [another] semester, and she said that it … forever has changed how she thinks about math. So, I see that happening to us.

Cora recognized the importance her teacher placed on the struggle of learning mathematics. When the struggle was past, then she noticed the connections that she was supposed to be making. She saw the process happen over and over. In this way, her MKT increased regarding the benefits and the need to have students struggle with the mathematics.

Teaching Geometry and Measurement in the Middle School. The prospective lower-secondary teachers interviewed thought that the Teaching Geometry and Measurement in the Middle School course helped their MKT. Again, Cora captured their thoughts best:

> [The professor] demonstrated an incredible class where he had great knowledge in understanding of the content, what the class was for, and didn't waste anybody's time. It demanded a lot; this was one of the most challenging classes I had, literally, [and] one of the most difficult classes. He demanded a lot more writing, more … [than] I've ever had before. But, it was very meaningful. I knew that I was learning, and this was going to benefit me. He modeled very good teaching for us, as he structured his class. It was a very great class. … So, I walked out of class with things in hand, things I can implement in my classroom, but then also having witnessed an entire semester of a teacher modeling the class I would want to run.

Cora thought that a mathematics pedagogy course should model good teaching and provide the learner with activities and experience. She also thought that the mathematics pedagogy courses should uses activities that a teacher could take into the classroom and give to students. These components of the course, modeling good teaching and sample activities, helped Cora's MKT.

The Utility of Mathematics Pedagogy Courses for Prospective Secondary Teachers' MKT. The prospective secondary teachers spoke of their mathematics pedagogy classes in only a general way. Michelle characterized the utility of the mathematics pedagogy courses in this way, representing her group:

> [The mathematics pedagogy courses are] really good. First, you have your in class part of it, where you're looking at case studies and reading articles and everything. That's good and gives an idea [of how things could be]. But, then we also get to go into the classroom and see it in practice. The stuff you always read about doesn't really ever play out that way in the classroom. So, those are the most helpful to me because I've always had anxiety of like, "What if I get with students and just don't know." It's definitely just helped me become more comfortable in the classroom with each semester.

Giving secondary teachers' experiences in the classroom before their first year of teaching, or even student teaching, was beneficial for their MKT. From the beginning they were curious to know exactly how they could handle teaching the mathematics. The mathematics pedagogy courses gave secondary teachers such an experience.

Other aspects of mathematics and teaching can be covered in mathematics pedagogy courses. For example, Caitlyn talked about the need to differentiate the mathematics:

> With the [mathematics] pedagogy classes … we didn't do as much content. It was definitely more about differentiation. So, if you have a concept, [what are] the different ways you can present it for students who learn differently? [It] was difficult at first, because it's easy for us to want to teach it the way that we know it, but every student learns in a different way. So, that means that you many have to represent something with numbers with a graph, with a table, with a picture. You may have to do all of those things, or just some of them. Knowing what your students' strengths are helps you decide the way that you're going to present your material.

Knowing the mathematics well enough to differentiate it for one's students, without "dumbing it down," is a vital part of MKT. Mathematics pedagogy courses can be avenues to explore and strengthen this aspect of MKT.

Mathematics pedagogy courses can help develop MKT in another way. They have the ability to strengthen prospective teachers' MCK as well as their MPCK. Through content examples illustrating pedagogical methods, these courses can target secondary teachers' weak content areas and illustrate pedagogy at the same time. For example, Michelle was able to answer a TEDS-M item and cited her present knowledge of the content of the item coming from a review in preparation for learning about observations in mathematics classrooms. She stated that the high school class that prospective teachers in one of the pedagogy courses they were observing was doing a lesson on logarithms. For the mathematics pedagogy course, the professor reviewed logarithms with the prospective teachers, and that review had refreshed Michelle's memory. Without expecting it, Michelle found that her MKT benefitted from the course topic because of the refresher.

Results Summary

Table K1 in Appendix K shows the course topics that were significantly associated with higher MKT through the MANOVA with blocking variables. Table K2 shows the course topics that the prospective teachers specifically stated were helpful in solving the MKT items from the TEDS-M study. The prospective teachers cited specific college courses as helpful for 5 of the 11 number items, 7 of the 15 algebra items, 8 of the 15 geometry items, and 1 of the 6 data items. I also translated the courses that were helpful for the prospective teachers to course topics in order to compare them with the courses from TEDS-M that were associated with higher MKT scores.

There were 5 course topics, out of the 34, that I found to be significantly associated with higher MKT among certain domains and influential in the knowledge prospective teachers used to solve the TEDS-M items. The courses and domains were (a) differential equations for the algebra domain; (b) school geometry for the MCK, geometry, and data domains; (c) school functions, relations, and equations for the MCK, MPCK, geometry, and data domains; (d) mathematics instruction for the algebra domain; and (e) developing teaching plans for the algebra domain.

I believe that both the exploratory nature of this study through its quantitative results and the qualitative results to the research questions have given a detailed picture of how college mathematics courses are associated with prospective teachers' MKT.

Chapter 6: Discussion and Conclusions

Discussion

Program type and MKT. The first research question addressed the association between prospective lower-secondary and secondary teachers and their mathematical knowledge for teaching (MKT), which I have detailed as mathematical content knowledge (MCK), mathematical pedagogical content knowledge (MPCK), and knowledge of number, algebra, geometry, and data.

TEDS-M reported a large difference in both MCK and MPCK between those U.S. prospective teachers enrolled in lower-secondary certification programs and those enrolled in secondary teacher certification programs (Tatto et al., 2012). I looked at the relative weaknesses in performance by U.S. prospective lower-secondary teachers in the domains of number, algebra, geometry, and data. The relative differences between the mean scores for the prospective secondary and lower-secondary teachers were 25% for number, 15% for algebra, 13% for geometry, and 18% for data. Judging by the mean differences, it is plausible that prospective lower-secondary teachers would benefit from assistance in any domain, but it seems that the number domain was the main contributor to the difference between certification groups in their MKT scores.

As a point of comparison, the findings from the pilot study showed significant differences between those U.S. prospective teachers preparing to teach Grades 6–12 and those preparing to teach Grades 6–8, which were similar to the groups in my study. Those relative differences favored the former group. They were 12% for all domains combined, 6% for number, 11% for algebra, 23% for function, 10% for geometry, and 8% for data, all being statistically significant. In the pilot study, the smallest difference was in number and the largest in function. The differences shown in the MT21 study further confirmed that those becoming certified to teach at the lower-secondary level did not possess the same level of mathematical knowledge as those teaching to certify at the secondary level.

One difference that I noticed between the two studies was that the MT21 researchers did not ask any questions from college mathematics on their test of MCK, whereas the TEDS-M study's MCK survey had some questions from calculus, analysis, linear algebra, and abstract algebra for the prospective lower-secondary and secondary teachers (Tatto et al., 2012, p. 130). Fewer prospective lower-secondary teachers in the present study had studied those topics, and so it is not surprising that they seemed to lack the knowledge necessary to answer the items correctly. Also, if a group of prospective lower-secondary teachers were like the ones I interviewed, they would probably not be able to complete those items, because they had not taken courses that covered those topics. Thus, it is plausible that these difficult questions were a major contributor to the lower scores for the prospective lower-secondary teachers.

Course topic studying and MKT. The second research question addressed the idea that studying certain course topics would be associated with higher, similar, or lower MKT scores. I was most interested in the course topics that were associated with significantly higher MKT scores.

Begle's (1979) study of teachers' course taking and student achievement has some similarities with my statistical findings of prospective teachers' course topic studying and MKT. Begle found that when prospective teachers took courses beyond calculus, only 10% of them showed positive main effects for their students' achievement. He also found that 8% had negative main effects for their students' achievement. Of the course topics in the present study, 21%, or 13, of the topics when studied were associated with significantly higher MKT scores, and 12%, or 9, were associated with lower MKT scores when studied. Compared to Begle's study, I found that studying twice the percent of the course topics were associated with significantly higher MKT scores and one-and-a-half the percent of course topics were associated with significantly lower MKT scores. Thus, the results from the present study were slightly different and more encouraging, since I found that a higher percentage of courses associated with higher MKT scores.

Like Begle's (1979) findings, Monk's (1994) findings were similar to the findings of the present study. Monk found that if prospective teachers took more than five courses, the positive effects on student achievement diminished. I found the same result in the present study. If a prospective teacher were to study the five courses of calculus, multivariate calculus, differential equations, linear algebra, and abstract algebra, some of the additional course topics they might have studied were associated with lower MKT scores (e.g., non-Euclidean geometry). This sequence happened in 127 of the 474 cases.

The MET documents (CBMS, 2001, 2012) recommended courses for prospective lower-secondary and secondary teachers. Of the college mathematics course topics, none recommended by the MET documents coincided with course topics that were significantly associated with higher MKT scores for the prospective lower-secondary teachers in the present study. For the prospective secondary teachers, the course topics of multivariate calculus and advanced calculus, real analysis, and measure theory were recommended by the MET documents and were associated with significantly higher MKT scores.

Of the school course topics recommended by the MET reports, the school number and statistics topics were not associated with higher or lower MKT scores. Thus, I could not determine with the data I have whether or not it would be beneficial to take them. The course topics in school geometry and school functions, relations, and equations would likely be beneficial to take because those topics were associated with higher MKT scores in the present study. In addition, none of the college mathematics course topics were associated with higher MKT scores, but they were not associated with lower MKT scores either. Thus, again I could not determine with the data I have whether or not it would be beneficial to take them.

The MET II document recommended that prospective teachers take courses in mathematics pedagogy. It did not specify which courses, just that they needed to be about teaching and learning. Some ideas could be the mathematics pedagogy courses covering the topics of foundations of mathematics, developing teaching plans, and affective issues in mathematics because in the present study they were associated with significantly higher MPCK scores.

Prospective teacher interviews. Below I discuss the results of the interviews in light of the literature.

College mathematics. Olson (1997) discussed proofs in college courses and said that with more student engagement and practice, the students would become more excited about mathematics and working out its mysteries. The prospective secondary teachers I interviewed had acquired the ability to prove concepts in college mathematics, but they did not find it easy or enjoyable.

The prospective secondary teachers thought that the proofs in the Introduction to Higher Mathematics and Modern Algebra and Geometry I classes were tough and confusing because they had not worked with proofs much before. They also did not know why they had to prove what they did (e.g., the parity of a set of numbers). Some of the prospective secondary teachers, however, saw how the difficulty of abstract algebra had helped them gain some good problem-solving skills.

In the interviews, the need surfaced to be able to do proofs and know that mathematics at the collegiate level was more complex than at the high school level. In order to help prospective teachers with proofs before higher mathematics classes, perhaps it would be prudent to give them more experiences with proofs in the lower college mathematics classes and high school mathematics classes. These experiences were exactly what Olson (1997) recommended. He gave ideas on how to do it and make it relevant for the students.

Clark et al. (1999) discussed students' views of a course on abstract algebra. They ran two sections of the course, one that used technology and one that was traditional. The comments from the students in the traditional section of the course were similar to the comments of the students I interviewed, because each of their sections was traditional. Attempts to conduct the abstract algebra course with technology could be fruitful and help students' motivation and views of the course.

School mathematics. In general, school mathematics courses for the prospective teachers I interviewed began as a bit of mystery. There were only a couple of prospective teachers who discussed the evolution of their thoughts on the utility of school mathematics courses. From those couple of interviews, I gleaned that the prospective lower-secondary and secondary teachers went through a period where they thought they were wasting their time. They thought that they knew the mathematics already and did not need to study it further. These thoughts may have persisted through the entire course. Once they realized they did not know all they needed to about school mathematics, they might have become frustrated because they did not know how to explain the whys of what they did in school mathematics. When they decided to open up to investigating school mathematics in a deep way, it became very fulfilling for them. They gained the ability to explain why they did what they did and to explain the mathematics to others. They were able to do school mathematics more accurately. Thus, the school mathematics courses became some of their favorite courses and those considered most useful for the prospective teacher and his or her MKT.

Additionally, some school mathematics courses promoted more conceptual learning through discussions and group work. The prospective teachers appreciated this difference from their college mathematics courses, which were mostly conducted in lecture format (Olson, 1997). They found it useful to learn mathematics in the manner they were expected to teach it.

Mathematics pedagogy. The utility of mathematics pedagogy courses for their MKT varied among the prospective teachers I interviewed. Some thought that teaching mathematics in a more conceptual way was not possible. Others had the opportunity to implement the theory through field experiences attached to the mathematics pedagogy courses. They thought the ability to see how well the methods in the mathematics pedagogy courses worked was good for their MKT. In addition, the prospective teachers valued the ideas on how to teach the mathematics they learned in the mathematics pedagogy courses, such as how to differentiate the mathematics for their students.

Implications for Practice

With the knowledge gained from this study of how teacher preparation programs and course topics are associated with MKT, mathematics teacher educators and advisors can make more informed decisions about programs and courses for aspiring lower-secondary and secondary teachers.

With regard to implications for program group and their MKT, the TEDS-M research and the present study point to the fact that prospective lower-secondary teachers do not score as well on MKT measures as the prospective secondary teachers (Tatto et al., 2012). As a result, the prospective lower-secondary teachers might need higher quality or more demanding courses. It is implausible that they would be able to take more of the higher mathematics courses common to secondary programs. They do not have the time.

It is also plausible that prospective lower-secondary do not have the disposition. For example, none of the lower-secondary teachers I interviewed had attempted Multivariable Calculus, and several of those who had taken Calculus 1 reported struggling with it. On that note, the Center for Research in Mathematics and Science Education (CRMSE, 2010) stated:

> Increasing the mathematics course-taking requirements by expecting future teachers to be prepared in secondary programs alone might not solve the problem. Such a requirement could have the unintended consequence of creating a shortage of middle school mathematics teachers as many who are interested in middle school might not want to be part of a secondary preparation program. (pp. 2−3)

Implementing a requirement that all prospective lower-secondary teachers go through the secondary certification program might be quite complex and difficult to implement. Perhaps a solution lies in improving the courses they take.

With regard to implications for the course taking of prospective teachers, I believe that improvements can be made. I believe that programs should follow the recommendations of the MET documents (CBMS, 2001, 2012). Some programs might need to require fewer college mathematics courses in order to do so. Because of the value the prospective teachers in the present study placed on the school mathematics courses they took, and the direct association those courses had with the mathematics the prospective teachers would teach, it might be beneficial for teacher preparation programs to offer as wide a variety of those courses as possible. Also, the focus of the TEDS-M items was mainly on lower-secondary and secondary school mathematics. International mathematicians and mathematics educators created these items and thus demonstrated what kind of mathematics they thought was most important for teaching.

A fruitful thought experiment would be the following: What kinds of knowledge do practicing teachers use the most? Is it the knowledge they gain from college mathematics courses, or that from school mathematics courses, or both? If both, what is the optimal combination?

Limitations

There have been some consequences in my study because of the limitations of its design. The TEDS-M data were collected in a specific manner for the purposes and design set forth by the TEDS-M researchers (Tatto et al., 2008). I have taken some of their data for my study, whose purpose was to explore associations between taking courses and prospective teachers' MKT using a MANOVA.

In the MANOVA, I tested the significance of 408 interactions, which clouds the view of what are the real results and what is noise. For example, by using .05 as the significance level, I would find about 20 of those interactions to be significant by chance alone. Thus, one should be cautious about these results. I would like to believe that the more counterintuitive results were a result of false positives, but I have no way to test that.

Another limitation in my use of the TEDS-M data included the fact that I assumed that the course topics meant the same thing to each prospective teacher. For example, a linear algebra course can be taught in several different ways. Thus, it is plausible that those prospective teachers who checked the box that they had studied linear algebra had very different knowledge of what linear algebra is and thus might have answered the MKT items differently.

Similarly, when a prospective teacher selected a course topic like number theory, it is also plausible that a lower-secondary prospective teacher and a secondary prospective teacher had learned the topic differently. It is possible that the lower-secondary teacher thought of number theory as simply the study of factors and multiples. In contrast, the prospective secondary teacher might have taken a course in number theory, and his or her knowledge of number theory would have been much deeper. Yet, in the present study, they both had studied the topic and their experience would have been counted the same. In order to have clearer results, it would have been helpful for the TEDS-M researchers to survey the courses using a finer grain. For example, they could have asked the prospective teacher indicate his or her level of study of each of the mathematical course topics.

Another limitation that has affected the results can be found in some of the figures in Appendix J. In certain figures, the prospective teachers who claimed to have earned average and below average grades in high school show a marked difference in MKT knowledge whether or not they studied certain course topics. One reason for this phenomenon might have been that number of prospective teachers at that level was small, only 43 prospective teachers. For example, Figure J17 shows that there were only 5 prospective teachers from the average or below average level who did not study foundations of geometry or axiomatic geometry. As a result, there were 38 prospective teachers who had studied the topic. The mean scores, then, for those who did or did not study school functions, relations, and equations were derived from a few data points.

A final limitation involved the use of the self-reported variable of level of high school grades to take into account the initial intelligence level of the prospective teachers. More intelligent prospective teachers would do better on the measures of MKT, not necessarily as a product of the courses they took. In order to help those, like myself, with secondary analyses, the TEDS-M researchers could have found a more reliable measure of intelligence.

Future Research

A natural extension of the present study would be to interview practicing teachers and identify which courses they think were the most beneficial for their practice. The idea would be to gain more insight into what should be emphasized most in teacher preparation programs. I believe there would be important differences in the views of practicing teachers depending on the number of years they had been teaching. Newer teachers would plausibly be able to distinguish more of what they had learned in their preparation programs versus from their experience teaching than older teachers would. For that reason, newer teachers would be the most fruitful population to interview. It might also be interesting to investigate the beliefs of more experienced teachers and tap into their recommendations for course topics based on their experiences.

Another extension of this study would involve a deeper look at why prospective lower-secondary teachers scored so low on the TEDS-M items. My hypothesis stems from the inclusion of college items that were too difficult for the prospective teachers at the lower-secondary level. They did not have the experiences that their secondary counterparts had with regard to mathematics courses. It would be important to study their responses at the item level and identify specific areas within the number, algebra, geometry, and data domains with which they had difficulties.

The TEDS-M International Report gave some insight into this phenomenon with their discussion of anchor points (Tatto et al., 2012). The prospective lower-secondary teachers' scaled mean score was near the first anchor point, and so the abilities described for those who scored at the first anchor point would apply. They might have been able to correctly answer those items covering:

> [knowledge of] whole numbers, integers, and rational numbers, and the associated computations; … [evaluation of] algebraic expressions correctly, and [solution of] simple linear and quadratic equations, particularly those that can be solved by substitution or trial and error; … [knowledge of] standard geometric figures in the plane and space; … [identification and application of] simple relations in plane geometry; … [interpretation of and solutions to] more complex problems about numbers, algebra, and geometry if the context or problem type was commonly taught in lower-secondary schools. (pp. 142–144)

Additionally, they had difficulty answering items that involved “describing general patterns, solving multi-step problems with complex linguistic or mathematical relations, and relating equivalent representations of concepts” (Tatto et al., 2012, p. 144). The TEDS-M researchers also stated that the prospective teachers at the first anchor point tended to overgeneralize; could not reason well mathematically; and had difficulty finding faulty arguments and justifying or proving certain conclusions to be true (Tatto et al., 2012). Thus, interventions could be conceived to target areas of weakness in mathematical knowledge and process for the prospective lower-secondary teachers.

Conclusion

To continually respond to the ever-changing knowledge demands each world citizen needs, those in a position to do so should constantly evaluate and reevaluate the effectiveness of mathematics teaching. Quality teachers are required who know mathematics and can effectively teach it to students. Improvements in many areas are needed, not just in program requirements or courses. On the other hand, improvements in programs and courses are possible, as they offer one of the most accessible areas for those who can implement such improvements. I echo the recommendation from the TEDS-M study that “goals for improving mathematics content knowledge and mathematics pedagogy content knowledge among future teachers should be ambitious yet achievable” (Tatto et al., 2012, p. 203). And so, I urge educators to create these ambitious and achievable goals to consistently improve the mathematics education of U.S. teachers and students.

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Appendix A

The following is a list of the 17 countries surveyed in the TEDS-M study (Tatto et al., 2012, p. 17):

- Botswana
- Canada (four provinces)
- Chile
- Chinese Taipei
- Georgia
- Germany
- Malaysia
- Norway
- Oman (lower-secondary teacher education only)
- The Philippines
- Poland
- The Russian Federation
- Singapore
- Spain (primary teacher education only)
- Switzerland (German-speaking cantons)
- Thailand
- United States of America (public institutions only)

Appendix B

The following is a list of college mathematics course topics surveyed in the MT21 study (MT21 data set):

- College mathematics course topics
  - Axiomatic geometry (e.g., Euclidean axioms)
  - Analytic/coordinate geometry (e.g., equations of lines, curves; conic sections; rigid transformations)
  - Non-Euclidean geometry (e.g., projective plane; geometry on a sphere)
  - Differential geometry (e.g., curves and surfaces that can be described by a differential function)
  - Topology
  - Linear algebra (e.g., vector spaces, matrices, dimensions, eigenvalues, eigenvectors)
  - Abstract algebra (e.g., group theory, field theory, ring theory, ideals)
  - Number theory (e.g., divisibility, prime numbers; structuring integers)
  - Beginning calculus topics (e.g., limits, series, sequences)
  - Calculus (e.g., derivatives and integrals)
  - Multivariate calculus (e.g., partial derivatives, multiple integrals)
  - Differential equations (including ordinary differential equations, partial differential equations)
  - Functional analysis (including theory of real functions)
  - Theory of complex functions (e.g., functions with complex variables)
  - Discrete mathematics (e.g., graph theory, game theory, combinatorics; Boolean)
  - Probability
  - Statistics (e.g. distributions, likelihood, statistical inference; linear models; actuarial models)
  - Mathematical logic (e.g., truth tables, symbolic logic, propositional logic, set theory; binary operations)
  - History of mathematics
  - Other mathematics topics (e.g., mathematical optimization, numerical methods; analytical mechanics; modeling)
- School mathematics course topics
  - Principles and theory of school arithmetic
  - Principles and theory of school algebra

    - Principles and theory of school geometry
    - Principles and theory of school probability
    - Principles and theory of school calculus

- Mathematics pedagogy course topics
    - History of school mathematics
    - Mathematics curricula in schools
    - Psychology of mathematics (e.g., cognitive science applied to mathematics; developmental perspectives on the learning & understanding of mathematics, etc.)
    - Methods of teaching mathematics (e.g., using computers/technology; topic-specific methods)
    - Teaching practice in mathematics (e.g., practice teaching peers or students in schools; not part of student teaching)
    - Methods for solving school mathematics problems
    - Assessment in mathematics instruction

Appendix C

The following is a list of college mathematics course topics surveyed in the TEDS-M study (Tatto et al., 2012, pp. 178–179, 181, & 183):

- College mathematics course topics
    - Geometry
        - Foundations of geometry or axiomatic geometry (e.g., Euclidean axioms)
        - Analytic/coordinate geometry (e.g., equations of lines, curves, conic sections, rigid transformations or isometrics)
        - Non-Euclidean geometry (e.g., geometry on a sphere)
        - Differential geometry (e.g., sets that are manifolds, curvature of curves, and surfaces)
        - Topology
    - Discrete structures and logic
        - Linear algebra (e.g., vector spaces, matrices, dimensions, eigenvalues, eigenvectors)
        - Set theory
        - Abstract algebra (e.g., group theory, field theory, ring theory, ideals)
        - Number theory (e.g., divisibility, prime numbers, structuring integers)
        - Discrete mathematics, graph theory, game theory, combinatorics or Boolean algebra
        - Mathematical logic (e.g., truth tables, symbolic logic, propositional logic, set theory, binary operations)
    - Continuity and functions
        - Beginning calculus topics (e.g., limits, series, sequences)
        - Calculus (e.g., derivatives and integrals)
        - Multivariate calculus (e.g., partial derivatives, multiple integrals)
        - Advanced calculus or real analysis or measure theory
        - Differential equations (e.g., ordinary differential equations and partial differential equations)
        - Theory of real functions, theory of complex functions or functional analysis
    - Probability and statistics
        - Probability
        - Theoretical or applied statistics
- School mathematics course topics
    - Numbers, measurement, and geometry

        - Numbers (e.g., whole numbers, fractions, decimals, integer, rational, and real numbers; number concepts; number theory; estimation; ratio and proportionality)
        - Measurement (e.g., measurement units; computations and properties of length, perimeter, area, and volume; estimation and error)
        - Geometry (e.g., 1-D and 2-D coordinate geometry, Euclidean geometry, transformational geometry, congruence and similarity, constructions with straightedge and compass, 3-D geometry, vector geometry)
    - Functions, probability, and calculus
        - Functions, relations, and equations (e.g., algebra, trigonometry, analytic geometry)
        - Data representation, probability, and statistics
        - Calculus (e.g., infinite processes, change, differentiation, integration)
        - Validation, structuring, and abstracting (e.g., Boolean algebra, mathematical induction, logical connectives, sets, groups, fields, linear space, isomorphism, homomorphism)
- Mathematics pedagogy course topics
    - Foundations
        - Foundations of mathematics (e.g., mathematics and philosophy, mathematics epistemology, history of mathematics)
        - Context of mathematics education (e.g., role of mathematics in society, gender/ethnic aspects of mathematics achievement)
        - Development of mathematics ability and thinking (e.g., theories of mathematics ability and thinking; developing mathematical concepts; reasoning, argumentation, and proving; abstracting and generalizing; carrying out procedures and algorithms; application; modeling).
    - Instruction
        - Mathematics instruction (e.g., representation of mathematics content and concepts, teaching methods, analysis of mathematical problems and solutions, problem posing strategies, teacher-pupil interaction)
        - Developing teaching plans (e.g., selection and sequencing the mathematics content, studying and selecting textbooks and instructional materials)
        - Mathematics teaching: observation, analysis and reflection
        - Mathematics standards and curriculum
        - Affective issues in mathematics (e.g., beliefs, attitudes, mathematics anxiety)

Appendix D

The recommended college mathematics courses for prospective secondary teachers (CBMS, 2012, p. 69):

- Short sequence (33 semester-hours).
    - I Courses taken by undergraduates in a variety of majors (15+ semester-hours)
        - Single- and Multi-variable Calculus (9+ semester-hours)
        - Introduction to Linear Algebra (3 semester-hours)
        - Introduction to Statistics (3 semester-hours)
    - II Courses intended for all mathematics majors (9 semester-hours)
        - Introduction to Proofs (3 semester-hours)
        - Abstract Algebra (approach emphasizing rings and polynomials) (3 semester-hours)
        - A third course for all mathematics majors (e.g., Differential Equations) (3 semester-hours)
    - III Courses designed primarily for prospective teachers (9 semester-hours).

- Long sequence (42 semester-hours).
    - I Courses taken by undergraduates in a variety of majors (21 semester-hours)
        - Single- and Multi-variable Calculus (9+ semester-hours)
        - Introduction to Linear Algebra (3 semester-hours)
        - Introduction to Computer Programming (3 semester-hours)
        - Introduction to Statistics I, II (6 semester-hours)
    - II Courses intended for all mathematics majors (12 semester-hours)
        - Introduction to Proofs (3 semester-hours)
        - Advanced Calculus (3 semester-hours)
        - Abstract Algebra (approach emphasizing rings and polynomials) (3 semester-hours)
        - Geometry or Mathematical Modeling (3 semester-hours)
    - III Courses designed primarily for prospective teachers (9 semester-hours).

Appendix E

Table E1

Post Hoc Tests Using Tukey HSD on Mean Differences for All MT21 MKT Items by Program Group

Program group (*a*)	Program group (*b*)	Mean difference (*a* – *b*)
K–8	6–8	–.02
	6–12	–.14‡
	K–12	–.03
6–8	K–8	.02
	6–12	–.12‡
	K–12	–.01
6–12	K–8	.14‡
	6–8	.12‡
	K–12	.11†
K–12	K–8	.03
	6–8	.01
	6–12	–.11†

Note. Data came from the U.S. portion of the MT21 data set (Schmidt, 2013).
†$p < .01$, ‡$p < .001$.

Table E2

Post Hoc Tests Using Tukey HSD on Mean Differences for MT21 MKT Number Items by Program Group

Program group (*a*)	Program group (*b*)	Mean difference (*a* – *b*)
K–8	6–8	–.01
	6–12	–.07*
	K–12	–.00
6–8	K–8	.01
	6–12	–.06
	K–12	.01
6–12	K–8	.07*
	6–8	.06
	K–12	.07
K–12	K–8	.00
	6–8	–.01
	6–12	–.07

Note. Data came from the U.S. portion of the MT21 data set (Schmidt, 2013).
*$p < .05$.

Table E3

Post Hoc Tests Using Tukey HSD on Mean Differences for MT21 MKT Algebra Items by Program Group

Program group (*a*)	Program group (*b*)	Mean difference (*a* – *b*)
K–8	6–8	–.04
	6–12	–.15‡
	K–12	.01
6–8	K–8	.04
	6–12	–.11†
	K–12	.05
6–12	K–8	.15‡
	6–8	.11†
	K–12	.16
K–12	K–8	–.01
	6–8	–.05
	6–12	–.16

Note. Data came from the U.S. portion of the MT21 data set (Schmidt, 2013).
†$p < .01$, ‡$p < .001$.

Table E4

Post Hoc Tests Using Tukey HSD on Mean Differences for MT21 MKT Functions Items by Program Group

Program group (*a*)	Program group (*b*)	Mean difference (*a* – *b*)
K–8	6–8	–.04
	6–12	–.27‡
	K–12	–.14
6–8	K–8	.04
	6–12	–.23‡
	K–12	–.10
6–12	K–8	.27‡
	6–8	.23‡
	K–12	.13
K–12	K–8	.14
	6–8	.10
	6–12	–.13

Note. Data came from the U.S. portion of the MT21 data set (Schmidt, 2013).
‡$p < .001$.

Table E5

Post Hoc Tests Using Tukey HSD on Mean Differences for MT21 MKT Geometry Items by Program Group

Program group (*a*)	Program group (*b*)	Mean difference (*a* – *b*)
K–8	6–8	–.00
	6–12	–.10*
	K–12	.06
6–8	K–8	.00
	6–12	–.10*
	K–12	.06
6–12	K–8	.10*
	6–8	.10*
	K–12	.16
K–12	K–8	–.06
	6–8	–.06
	6–12	–.16

Note. Data came from the U.S. portion of the MT21 data set (Schmidt, 2013).
$^*p < .05$.

Table E6

Post Hoc Tests Using Tukey HSD on Mean Differences for MT21 MKT Data Items by Program Group

Program group (*a*)	Program group (*b*)	Mean difference (*a* – *b*)
K–8	6–8	–.01
	6–12	–.09†
	K–12	–.02
6–8	K–8	.01
	6–12	–.09†
	K–12	–.02
6–12	K–8	.09†
	6–8	.09†
	K–12	.07
K–12	K–8	.02
	6–8	.02
	6–12	–.07

Note. Data came from the U.S. portion of the MT21 data set (Schmidt, 2013).
†$p < .01$.

Appendix F

Table F1

Mean Percent Correct for All MK21 MKT Items and Each MKT Domain by Geometry Course Topic

All			Number			Algebra			Functions			Geometry			Data		
Axiomatic geometry[a]																	
M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t
51	45	23.83‡	61	55	8.79†	45	38	7.71†	38	27	19.46‡	50	49	.06	59	53	9.31†
Analytic/coordinate geometry[b]																	
M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t
48	45	1.89	59	55	3.06	41	41	.00	31	32	.06	51	47	1.83	57	50	8.32†
Topology[c]																	
M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t
52	47	6.88†	65	57	5.87*	48	40	1.15	35	31	1.15	54	49	1.35	58	55	.90

Note. Data came from the U.S. portion of the MT21 data set (Schmidt, 2013). M_S signifies the mean score of those who studied the course topic. M_{NS} signifies the mean score of those who did not study the course topic.

[a]Studied by 141 prospective teachers and not studied by 240. [b]Studied by 273 prospective teachers and not studied by 108. [c]Studied by 37 prospective teachers and not studied by 344.

*$p < .05$, †$p < .01$, ‡$p < .001$.

Table F2

Mean Percent Correct for All MT21 MKT Items and Each MKT Domain by Discrete Structures and Logic Course Topic

Course topic	All			Number			Algebra			Functions			Geometry			Data		
	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t
Linear algebra[a]	49	43	19.04‡	59	54	5.00*	42	37	4.30*	34	24	14.80‡	50	48	.70	58	49	19.26‡
Abstract algebra[b]	54	44	48.94‡	60	57	2.37	50	37	25.28‡	45	26	57.94‡	54	48	5.44*	60	53	11.99†
Discrete mathematics[c]	52	44	37.23‡	60	56	4.59*	46	38	11.46†	40	27	27.15‡	52	48	1.92	60	52	17.51‡
Mathematical logic[d]	50	45	17.17‡	61	55	7.76†	46	37	13.66‡	36	27	13.74‡	51	49	.36	56	54	1.77

Note. Data came from the U.S. portion of the MT21 data set (Schmidt, 2013). M_S signifies the mean score of those who studied the course topic. M_{NS} signifies the mean score of those who did not study the course topic.

[a]Studied by 262 prospective teachers and not studied by 119. [b]Studied by 110 prospective teachers and not studied by 271. [c]Studied by 132 prospective teachers and not studied by 249. [d]Studied by 168 prospective teachers and not studied by 213.

*$p < .05$, †$p < .01$, ‡$p < .001$.

Table F3

Mean Percent Correct for All MT21 MKT Items and Each MKT Domain by Continuity and Functions Course Topic

Course topic	All M_S	All M_{NS}	All t	Number M_S	Number M_{NS}	Number t	Algebra M_S	Algebra M_{NS}	Algebra t	Functions M_S	Functions M_{NS}	Functions t	Geometry M_S	Geometry M_{NS}	Geometry t	Data M_S	Data M_{NS}	Data t
Beginning calculus[a]	50	42	36.94‡	59	54	7.71†	44	34	13.66‡	35	24	19.25‡	51	45	4.45*	58	49	16.82‡
Calculus[b]	50	42	43.98‡	60	53	13.21‡	44	35	13.99‡	37	23	32.01‡	52	46	5.51*	57	51	10.13†
Multivariate calculus[c]	53	44	53.83‡	63	55	17.24‡	48	37	16.74‡	41	26	35.42‡	54	47	5.40*	59	53	10.58†
Differential equations[d]	51	45	25.59‡	60	56	2.70	44	39	5.07*	38	27	21.65‡	52	48	1.39	60	51	20.75‡
Functional analysis[e]	50	46	5.39*	58	47	.17	44	40	3.02	36	29	5.62*	51	49	.37	58	53	5.39*

Note. Data came from the U.S. portion of the MT21 data set (Schmidt, 2013). M_S signifies the mean score of those who studied the course topic. M_{NS} signifies the mean score of those who did not study the course topic.
[a]Studied by 260 prospective teachers and not studied by 121. [b]Studied by 231 prospective teachers and not studied by 150. [c]Studied by 129 prospective teachers and not studied by 252. [d]Studied by 152 prospective teachers and not studied by 229. [e]Studied by 109 prospective teachers and not studied by 272.
*$p < .05$, †$p < .01$, ‡$p < .001$.

Table F4

Mean Percent Correct for All MT21 MKT Items and Each MKT Domain by Probability and History of Mathematics Course Topic

Probability[a]																	
All			Number			Algebra			Functions			Geometry			Data		
M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t
48	45	2.49	57	58	.13	41	40	.30	32	29	.93	50	47	.89	58	49	12.31†
History of mathematics[b]																	
All			Number			Algebra			Functions			Geometry			Data		
M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t
50	46	12.03†	60	57	2.57	43	40	1.86	36	29	8.48†	51	49	.34	60	52	12.60‡

Note. Data came from the U.S. portion of the MT21 data set (Schmidt, 2013). M_S signifies the mean score of those who studied the course topic. M_{NS} signifies the mean score of those who did not study the course topic.
[a]Studied by 289 prospective teachers and not studied by 92. [b]Studied by 124 prospective teachers and not studied by 257.
†$p < .01$, ‡$p < .001$.

Table F5

Mean Percent Correct for All MT21 MKT Items and Each MKT Domain by School Mathematics Course Topic

Arithmetic[a]																	
All			Number			Algebra			Functions			Geometry			Data		
M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t
45	48	4.50*	55	59	3.24	38	42	2.06	29	32	1.15	46	51	2.32	55	55	.02
Probability[b]																	
All			Number			Algebra			Functions			Geometry			Data		
M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t	M_S	M_{NS}	t
45	48	4.08*	54	49	3.65	39	41	.81	28	32	2.20	46	51	1.96	54	55	.22

Note. Data came from the U.S. portion of the MT21 data set (Schmidt, 2013). M_S signifies the mean score of those who studied the course topic. M_{NS} signifies the mean score of those who did not study the course topic.
[a]Studied by 107 prospective teachers and not studied by 274. [b]Studied by 78 prospective teachers and not studied by 303.
*$p < .05$

Appendix G

Table G1

Released TEDS-M Items Taken by Three Prospective Mathematics Teachers

Item	MCK or MPCK	Domain	Description
604A	MCK	Algebra	Two word problems—solve them
604B	MPCK	Algebra	Two word problems—difficulty level
610	MCK	Number	Represented by irrational numbers
703	MCK	Geometry	Length of ribbon of two gift boxes
704	MCK	Geometry	Triangle in a parallelogram
705	MCK	Geometry	The set of points represented by an equation
709	MPCK	Number	Valid proof
710	MCK	Algebra	Situations that can be modeled by an exponential function
712	MPCK	Algebra	Prove the quadratic formula
802	MCK	Number	Reminder of squared natural number divided by 3
804	MCK	Number	Ways of choosing students
806A	MCK	Data	Graph—student right or wrong
806B	MPCK	Data	Graph—explain student thinking
808	MCK	Geometry	Student work on symmetry
814	MCK	Algebra	Operation with two matrices

Table G2

Unreleased MT21 Items Similar to TEDS-M Items Taken By Prospective Mathematics Teachers

MT21 Item Label	Similar TEDS-M Item Label	MCK or MPCK	Domain	Description
V01	602	MCK	Algebra	True statement about function behavior
J30	603	MPCK	Algebra	Change curriculum
B47	606	MCK	Number	Solution to functional equation
K3	611	MPCK	Number	Relate fraction to real world
H34	613	MCK	Geometry	Perpendicular bisector
A11	701	MPCK	Geometry	Equilateral triangles on a segment – correct drawing
A11B	701B	MCK	Geometry	Equilateral triangles on a segment – true statement
J35	708	MCK	Number	Equivalence relation
I66	715	MPCK	Data	Probability of 1 for the last digit
X22	801	MCK	Number	Smallest set of the solutions for equations
V02	809	MCK	Geometry	Transformations of a flag

Appendix H

Table H1

Courses and Topics Taken by Prospective Lower-Secondary Teachers Interviewed

Courses and topics	Prospective lower-secondary teacher(s)
Foundations of Geometry I	Joan[a]
Linear Algebra	Chris[a]
Precalculus	
School functions, relations, and equations	Chris[a], Cora, Joan[a]
College Algebra	Joan[a]
Introduction to Mathematical Modeling	
School functions, relations, and equations	Cora
Business Calculus	
Calculus	
Multivariate calculus	Ashley
Calculus I	
Beginning calculus topics	
Calculus	Ashley, Chris[a], Cora, Melinda, Joan[b]
Calculus 2	Chris[c]
Arithmetic and Problem Solving	
School numbers	Cora, Melinda
Arithmetic for Middle School Teachers	
School numbers	Ashley, Chris, Joan

(continued)

Table H1 (continued)

Courses and topics	Prospective lower-secondary teacher(s)
Algebra for Middle School Teachers	
School functions, relations, and equations	Ashley[e], Chris[e], Cora[e], Melinda[e], Joan[e]
Geometry and Measurement for Middle School Teachers	
School geometry	
School measurement	Ashley, Chris, Cora, Melinda, Joan
Teaching Number Systems in the Middle School	
School numbers	
Context of mathematics education	
Development of mathematics ability and thinking	
Mathematics instruction	
Mathematics standards and curriculum	
Mathematics teaching and learning with technology[d]	Ashley, Chris, Cora, Melinda, Joan
Teaching Geometry and Measurement in the Middle School	
School geometry	
School measurement	
Development of mathematics ability and thinking	
Mathematics instruction	
Mathematics standards and curriculum	
Mathematics teaching and learning with technology[d]	Ashley, Chris, Cora, Melinda, Joan
Teaching Algebra in the Middle School	
School functions, relations, and equations	
Development of mathematics ability and thinking	
Mathematics instruction	
Mathematics standards and curriculum	
Mathematics teaching and learning with technology[d]	Ashley[e], Chris[e], Cora[e], Melinda[e], Joan[e]

[a]Course not taken at university. [b]Course dropped and retaken. [c]Course dropped. [d]Course topic not included in TEDS-M.
[e]Courses in progress at time of interview.

Table H2

Courses and Topics Taken by Prospective Secondary Teachers Interviewed

Courses and topics	Prospective secondary teacher(s)
Foundations of Geometry I	
Foundations of geometry or axiomatic geometry	
Non-Euclidean geometry	Caitlyn[a], Natalie[a], Keri, Michelle[a]
Differential Geometry	
Differential geometry	Brian[a]
Introduction to Linear Algebra	
Linear algebra	Caitlyn, Natalie, Keri, Brian, Michelle
Modern Algebra and Geometry I	
Abstract algebra	Caitlyn, Natalie, Keri, Brian, Michelle
Number Theory	
Number theory	Brian[a]
Calculus 1	
Beginning calculus topics	
Calculus	Caitlyn, Natalie, Brian, Michelle
Calculus 2	
Beginning calculus topics	
Calculus	
Multivariate calculus	Caitlyn, Natalie, Brian, Michelle
Sequences and Series	
Beginning calculus topics	Natalie, Brian, Michelle
Multivariable Calculus	
Multivariate calculus	Caitlyn, Natalie, Keri, Brian, Michelle
Elementary Differential Equations	
Differential equations	Caitlyn[a], Natalie[a], Michelle
Introductory Statistics	
Theoretical or applied statistics	Caitlyn, Natalie, Michelle

(continued)

Table H2 (continued)

Courses and topics	Prospective secondary teacher(s)
Statistical Methods	
Theoretical or applied statistics	Caitlyn, Natalie, Keri, Michelle
Probability and Statistics for Secondary Teachers	
Theoretical or applied statistics	
Probability	Caitlyn, Natalie, Keri
Probability	
Probability	Brian
Introduction to Higher Mathematics	
Mathematical logic	
Set theory	
Writing college level proofs[b]	Caitlyn, Natalie, Keri, Brian, Michelle
Connections in Secondary Mathematics	
School numbers	
School data representation, probability, and statistics	
Mathematics standards and curriculum	
Development of mathematics ability and thinking	Caitlyn, Natalie, Keri, Brian, Michelle
Concepts in Secondary Mathematics	
School functions, relations, and equations	
Mathematics teaching and learning with technology[b]	Caitlyn, Natalie, Keri, Brian, Michelle
Connections in Secondary Mathematics III	
Beginning calculus topics	
School geometry	
School measurement	
School data representation, probability, and statistics	
Development of mathematics ability and thinking	
Mathematics instruction	Caitlyn[a], Natalie[a], Keri[a], Brian[a], Michelle[a]

(continued)

Table H2 (continued)

Courses and topics	Prospective secondary teacher(s)
Pedagogy 1 – Focus on Student Thinking	
Development of mathematics ability and thinking	
Mathematics instruction	Caitlyn, Natalie, Keri, Brian, Michelle
School Practicum in Secondary Mathematics	
Mathematics instruction	
Mathematics teaching: observation, analysis, and reflection	
Affective issues in mathematics	Caitlyn, Natalie, Keri, Brian, Michelle
Teaching and Learning Secondary School Mathematics	
Mathematics instruction	
Mathematics teaching: observation, analysis, and reflection	
Affective issues in mathematics	
Developing teaching plans	Caitlyn[a], Natalie[a], Keri[a], Brian[a], Michelle[a]

[a]Courses in progress at time of interview. [b]Course topic not included in TEDS-M.

Appendix I

Table I1

Results of MANOVA Comparing the Mean Scores for Mathematical Knowledge for Teaching by Geometry Course Topics Associating with Significantly Higher or Lower Domain Scores

Foundations of geometry or axiomatic geometry[a]																	
MCK			MPCK			Number			Algebra			Geometry			Data[b]		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
497	489	.58	473	447	3.31	46	47	.19	42	37	3.18	59	57	.43	50	36	8.80†
Analytic/coordinate geometry[c]																	
MCK			MPCK			Number			Algebra			Geometry[d]			Data		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
500	486	3.84	459	461	.04	46	47	.48	39	40	.28	61	54	14.00‡	46	40	3.22
Non-Euclidean geometry[e]																	
MCK			MPCK[b]			Number			Algebra			Geometry[b]			Data[b]		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
483	503	3.61	441	479	6.26*	45	49	1.26	39	41	.28	54	62	7.89†	33	54	16.90‡
Topology[f]																	
MCK[b]			MPCK[b]			Number			Algebra			Geometry			Data		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
481	505	5.68*	441	479	7.45†	44	50	3.90*	36	43	5.27*	56	60	2.41	44	42	.14

Note. Data came from the U.S. portion of the TEDS-M data set (IEA, 2012). M_S signifies the mean score of those who studied the course topic. M_{NS} signifies the mean score of those who did not study the course topic.
[a]Studied by 375 prospective teachers and not studied by 83. [b]Study or not study of topic had interaction with domain by level of high school grades (see Appendix J for breakdown). [c]Studied by 351 prospective teachers and not studied by 107. [b]Study or not study of topic had interaction with domain by program group (see Appendix J for breakdown). [e]Studied by 317 prospective teachers and not studied by 141. [f]Studied by 40 prospective teachers and not studied by 418.
*$p < .05$, †$p < .01$, ‡$p < .001$.

Table I2

Results of MANOVA Comparing the Mean Scores for Mathematical Knowledge for Teaching by Continuity and Functions Course Topics Associating with Significantly Higher or Lower Domain Scores

Multivariate calculus[a]																	
MCK			MPCK			Number			Algebra			Geometry			Data		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
505	481	7.01†	469	451	1.94	53	40	24.23‡	39	40	.35	58	57	.18	48	39	4.54*
Advanced calculus or real analysis or measure theory[b]																	
MCK			MPCK[c]			Number			Algebra[c]			Geometry			Data		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
506	480	9.37†	477	442	8.62†	50	44	5.66*	43	36	8.17†	61	55	7.25†	47	40	2.67
Differential equations[d]																	
MCK			MPCK			Number			Algebra[c]			Geometry			Data		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
493	493	.01	467	453	1.39	45	49	3.15	43	36	8.88†	56	59	1.88	46	41	1.48
Theory of real functions, theory of complex functions, or functional analysis[e]																	
MCK			MPCK			Number			Algebra[f]			Geometry			Data		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
488	498	1.08	444	475	5.99*	49	45	1.94	37	42	4.30*	56	60	1.98	40	42	2.99

Note. Data came from the U.S. portion of the TEDS-M data set (IEA, 2012). M_S signifies the mean score of those who studied the course topic. M_{NS} signifies the mean score of those who did not study the course topic.
[a]Studied by 318 prospective teachers and not studied by 140. [b]Studied by 149 prospective teachers and not studied by 309. [c]Study or not study of topic had interaction with domain by level of high school grades (see Appendix J for breakdown). [d]Studied by 236 prospective teachers and not studied by 222. [e]Studied by 117 prospective teachers and not studied by 341. [f]Study or not study of topic had interaction with domain by program group (see Appendix J for breakdown).
*$p < .05$, †$p < .01$, ‡$p < .001$.

Table I3

Results of MANOVA Comparing the Mean Scores for Mathematical Knowledge for Teaching by Probability and Mathematical Logic Course Topics Associating with Significantly Higher or Lower Domain Scores

Probability[a]																	
MCK			MPCK			Number			Algebra			Geometry			Data		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
497	489	.57	467	453	.93	48	46	.39	43	36	4.93*	55	61	4.58*	42	44	.14
Mathematical logic[b]																	
MCK			MPCK			Number			Algebra			Geometry			Data		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
491	495	.35	462	458	.21	48	46	1.47	40	39	.53	56	60	4.15*	42	44	.52

Note. Data came from the U.S. portion of the TEDS-M data set (IEA, 2012). M_S signifies the mean score of those who studied the course topic. M_{NS} signifies the mean score of those who did not study the course topic.
[a]Studied by 426 prospective teachers and not studied by 32. [b]Studied by 324 prospective teachers and not studied by 134.
*$p < .05$.

Table I4

Results of MANOVA Comparing the Mean Scores for Mathematical Knowledge for Teaching by School Mathematics Course Topics Associating with Significantly Higher or Lower Domain Scores

School measurement[a]																	
MCK			MPCK			Number			Algebra			Geometry			Data		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
488	498	1.36	450	470	2.34	46	47	.15	37	42	3.92*	56	60	3.09	43	43	.01
School geometry[b]																	
MCK[c]			MPCK			Number			Algebra			Geometry			Data[c]		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
505	481	4.45*	474	446	3.28	49	44	2.33	41	38	1.09	62	53	9.59†	50	37	5.89*
School functions, relations, and equations[d]																	
MCK[e]			MPCK[e]			Number			Algebra[e]			Geometry[e]			Data[e]		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
516	471	7.90†	504	415	16.13‡	50	44	2.13	46	33	7.92†	65	51	10.04†	53	33	7.06†
School data representations, probability, and statistics[f]																	
MCK			MPCK[e]			Number			Algebra			Geometry			Data		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
481	505	1.62	427	493	6.00*	46	47	.04	33	46	4.96*	56	60	.57	35	51	3.22
School calculus[g]																	
MCK			MPCK			Number[c]			Algebra			Geometry			Data		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
496	490	.50	459	461	.06	50	44	8.28†	39	40	.23	58	58	.10	45	42	.71

Note. Data came from the U.S. portion of the TEDS-M data set (IEA, 2012). M_S signifies the mean score of those who studied the course topic. M_{NS} signifies the mean score of those who did not study the course topic.
[a]Studied by 377 prospective teachers and not studied by 81. [b]Studied by 411 prospective teachers and not studied by 47. [c]Study or not study of topic had interaction with domain by program group (see Appendix J for breakdown). [d]Studied by 392 prospective teachers and not studied by 66. [e]Study or not study of topic had interaction with domain by level of high school grades (see Appendix J for breakdown). [f]Studied by 404 prospective teachers and not studied by 54. [g]Studied by 284 prospective teachers and not studied by 174.
*$p < .05$, †$p < .01$.

Table I5

Results of MANOVA Comparing the Mean Scores for Mathematical Knowledge for Teaching by Course Topics on Foundations in Mathematics Pedagogy Associating with Significantly Higher or Lower Domain Scores

Foundations of mathematics[a]																	
MCK[b]			MPCK			Number			Algebra			Geometry			Data		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
501	485	6.12*	465	455	1.15	49	45	3.42	41	38	3.12	59	57	1.54	44	43	.22
Development of mathematics ability and thinking[c]																	
MCK			MPCK			Number[b]			Algebra			Geometry			Data[d]		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
496	490	.72	455	465	.83	45	49	3.89*	40	39	.29	59	57	1.18	48	37	6.14*

Note. Data came from the U.S. portion of the TEDS-M data set (IEA, 2012). M_S signifies the mean score of those who studied the course topic. M_{NS} signifies the mean score of those who did not study the course topic.

[a]Studied by 265 prospective teachers and not studied by 193. [b]Study or not study of topic had interaction with domain by level of high school grades (see Appendix J for breakdown). [c]Studied by 335 prospective teachers and not studied by 123. [d]Study or not study of topic had interaction with domain by program group (see Appendix J for breakdown).

*$p < .05$.

Table I6

Results of MANOVA Comparing the Mean Scores for Mathematical Knowledge for Teaching by Course Topics on Instruction in Mathematics Pedagogy Associating with Significantly Higher or Lower Domain Scores

Mathematics instruction[a]																	
MCK			MPCK			Number			Algebra			Geometry			Data		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
490	496	.38	463	457	.18	45	49	2.00	43	36	4.55*	57	59	.61	45	41	.79
Developing teaching plans[b]																	
MCK			MPCK			Number			Algebra			Geometry			Data		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
498	488	1.50	473	447	4.69*	48	46	1.00	43	36	8.33†	57	59	1.11	45	42	.40
Mathematics teaching: observation, analysis and reflection[c]																	
MCK			MPCK			Number			Algebra[d]			Geometry[d]			Data		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
477	509	3.73	445	475	1.61	46	48	.17	35	45	4.26*	52	64	6.76†	38	49	1.89
Affective issues in mathematics[e]																	
MCK			MPCK			Number			Algebra[d]			Geometry			Data		
M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F	M_S	M_{NS}	F
499	487	2.16	477	443	10.08†	49	45	4.08*	42	37	4.53*	59	60	1.66	41	46	1.63

Note. Data came from the U.S. portion of the TEDS-M data set (IEA, 2012). M_S signifies the mean score of those who studied the course topic. M_{NS} signifies the mean score of those who did not study the course topic.

[a]Studied by 401 prospective teachers and not studied by 57. [b]Studied by 366 prospective teachers and not studied by 92. [c]Studied by 415 prospective teachers and not studied by 43. [d]Study or not study of topic had interaction with domain by level of high school grades (see Appendix J for breakdown). [e]Studied by 258 prospective teachers and not studied by 200.

*$p < .05$, †$p < .01$.

Appendix J

Table J1

Significant Interactions of Program Group with Course Topics by Domain

Interaction	Domain	*F* value
Program group × high school grades level	MPCK	2.66*
Program group × analytic/coordinate geometry	Geometry	10.70†
Program group × theory of real functions	Algebra	14.76‡
Program group × school geometry	MCK	4.36*
	Data	6.23*
Program group × school calculus	Number	11.50‡
Program group × development of mathematics ability and thinking	Data	9.52†

Note. Data for table from U.S. TEDS-M data (IEA, 2012).
**p* < .05, †*p* < .01, ‡*p* < .001.

Table J2

Significant Interactions of Level of High School Grades with Course Topics by Domain

Interaction	Domain	*F* value
High school grades level × foundations of geometry	Data	3.65*
High school grades level × non-Euclidean geometry	MPCK	5.01†
	Geometry	4.35†
	Data	3.91†
High school grades level × topology	MCK	2.94*
	MPCK	3.38*
High school grades level × advanced calculus or real analysis	MPCK	4.38†
	Algebra	2.73*
High school grades level × differential equations	Algebra	3.70*
High school grades level × school functions, relations, and equations	MCK	6.13‡
	MPCK	8.02‡
	Algebra	4.27†
	Geometry	4.32†
	Data	4.13†
High school grades level × school data representations	MPCK	4.95†
High school grades level × foundations of mathematics	MCK	6.58‡
High school grades level × development of mathematics ability and thinking	Number	2.94*
High school grades level × mathematics teaching: observation, analysis and reflection	Algebra	4.19†
	Geometry	4.87†
High school grades level × affective issues in mathematics	Algebra	3.36*

Note. Data for table from U.S. TEDS-M data (IEA, 2012).
*$p < .05$, †$p < .01$, ‡$p < .001$.

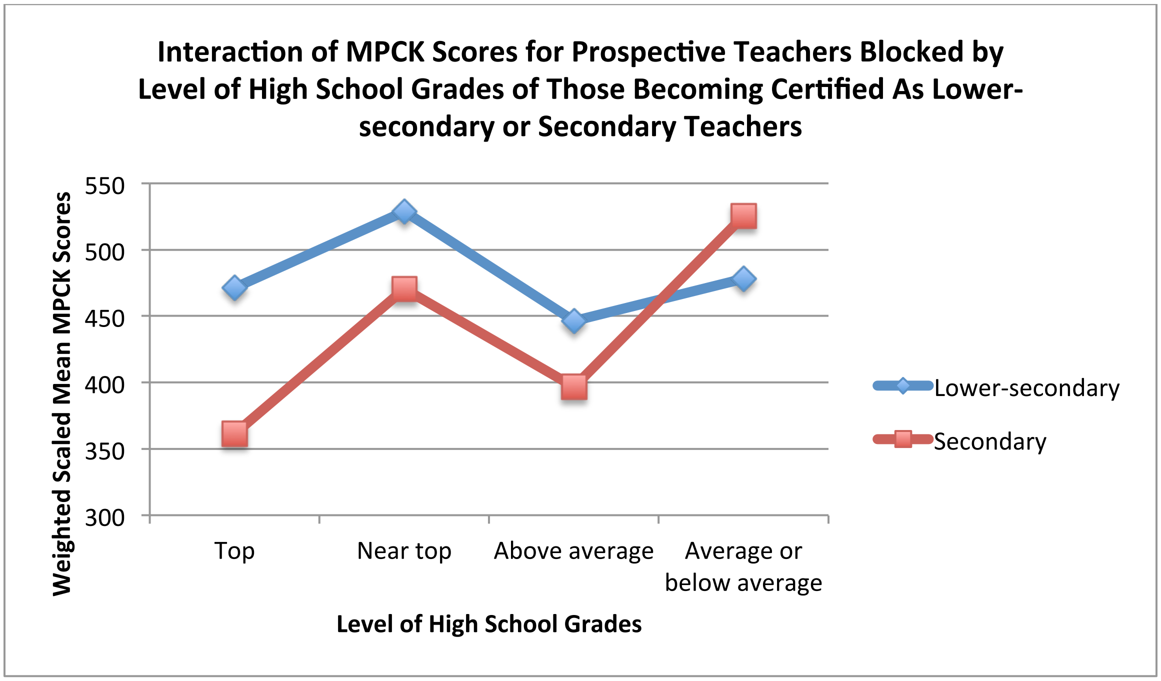

Figure J1. Decomposing the interaction of MPCK scores blocked by level of high school grades of those becoming certified as lower-secondary or secondary teachers. Data for figure from U.S. TEDS-M data (IEA, 2012).

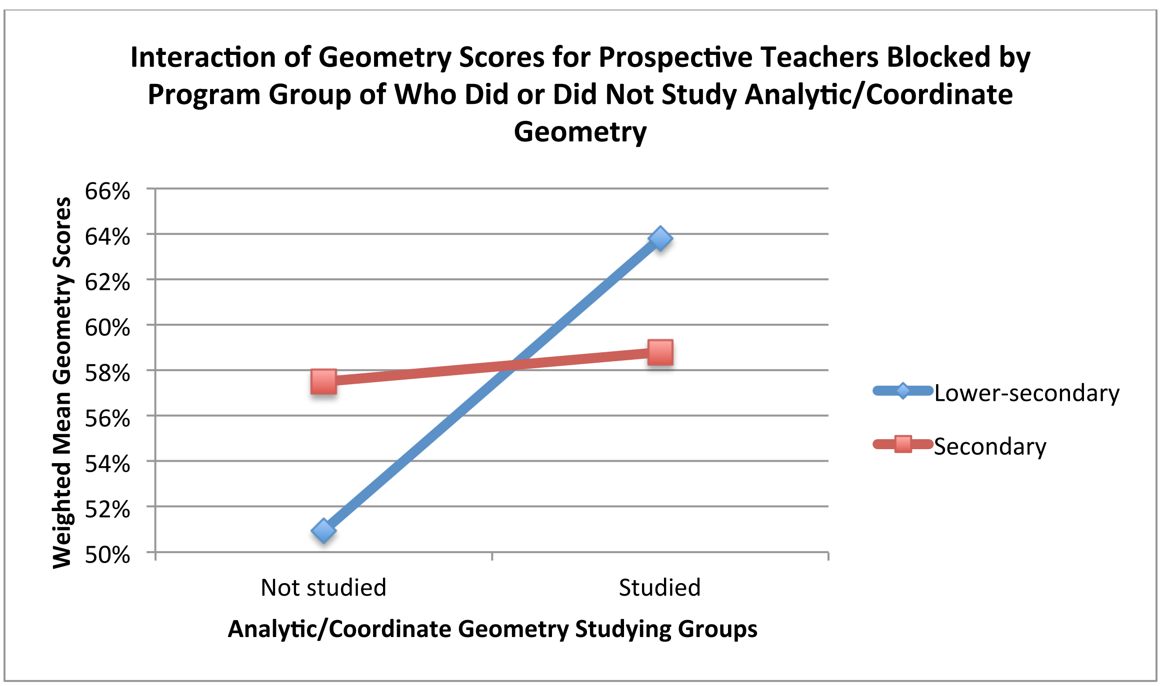

Figure J2. Decomposing the interaction of geometry scores blocked by program group of those who did or did not study analytic/coordinate geometry. Data for figure from U.S. TEDS-M data (IEA, 2012).

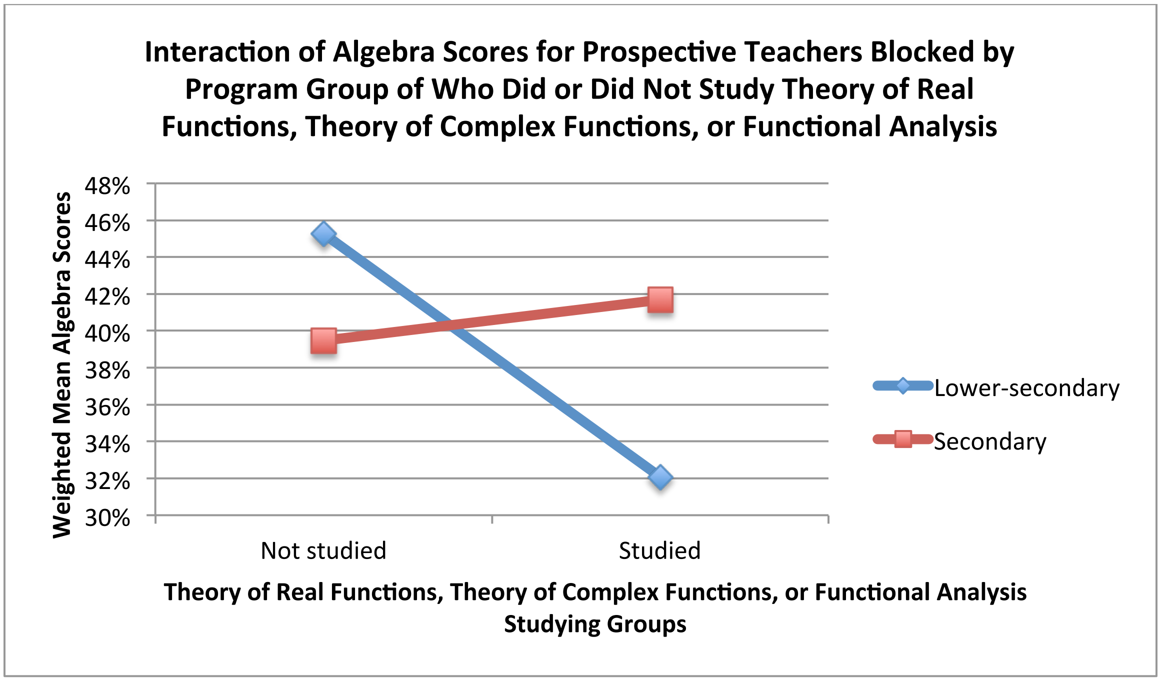

Figure J3. Decomposing the interaction of algebra scores blocked by program group of those who did or did not study theory of real functions, theory of complex functions, or functional analysis. Data for figure from U.S. TEDS-M data (IEA, 2012).

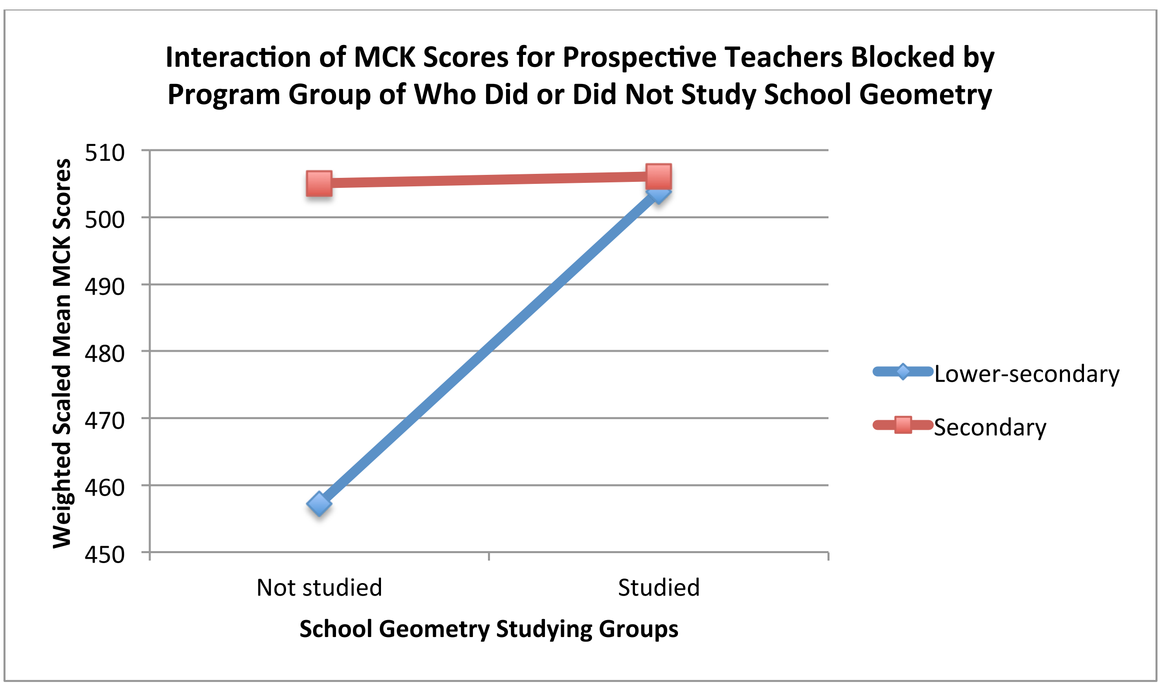

Figure J4. Decomposing the interaction of MCK scores blocked by program group of those who did or did not study school geometry. Data for figure from U.S. TEDS-M data (IEA, 2012).

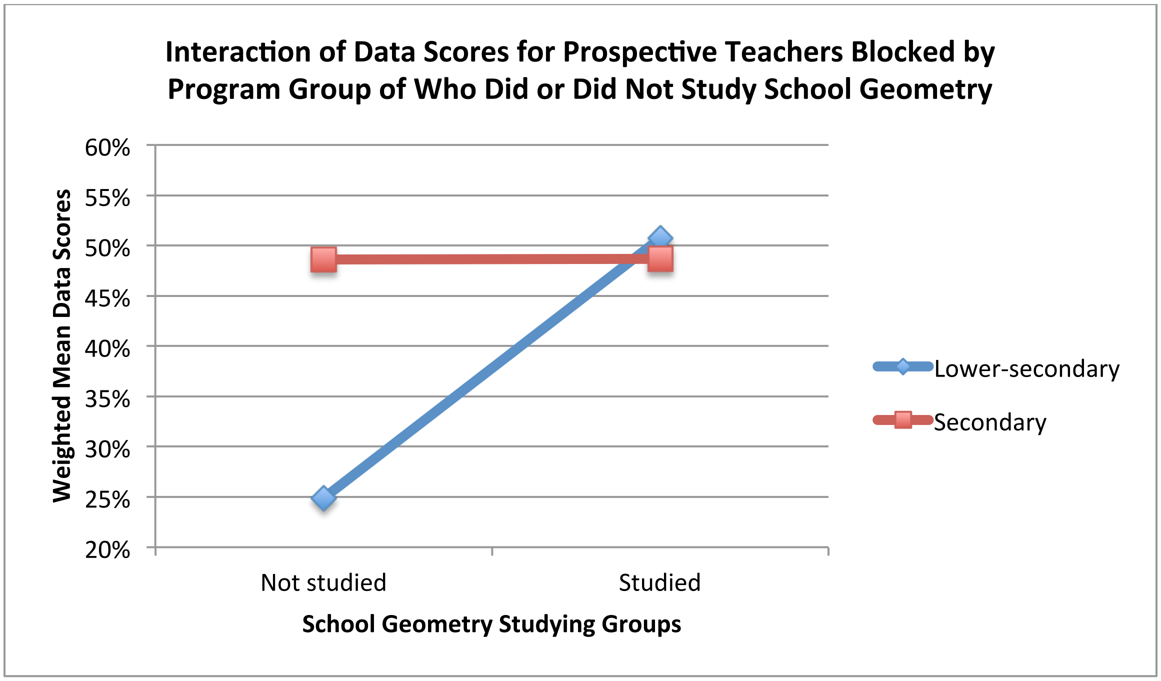

Figure J5. Decomposing the interaction of data scores blocked by program group of those who did or did not study school geometry. Data for figure from U.S. TEDS-M data (IEA, 2012).

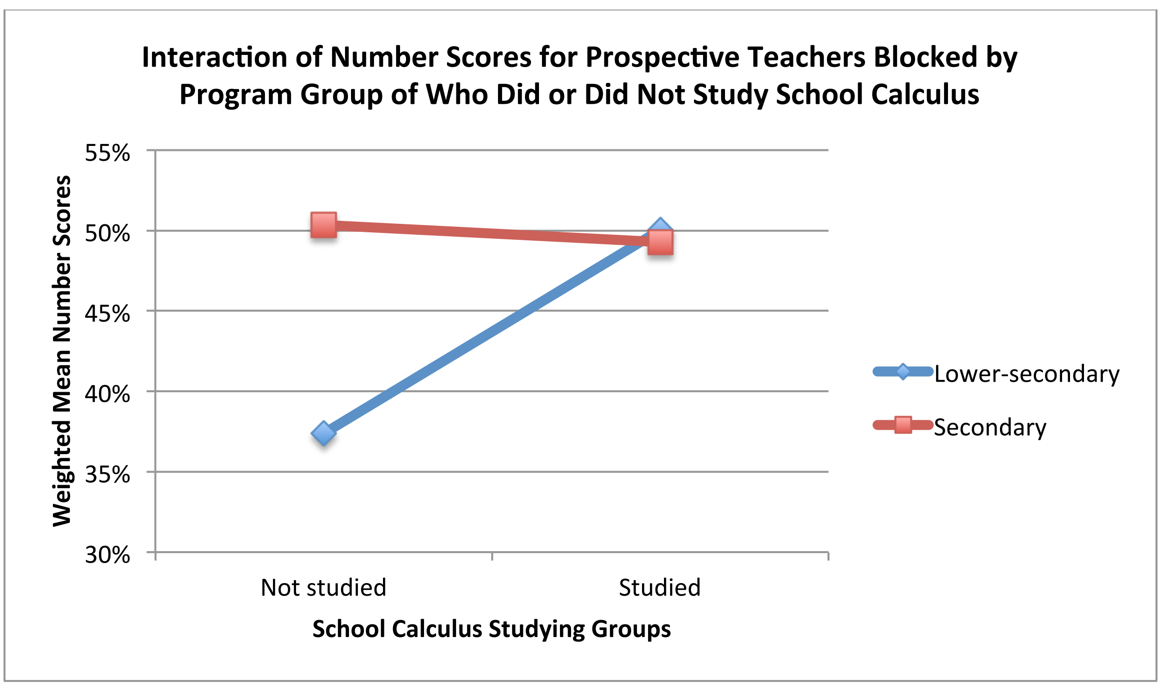

Figure J6. Decomposing the interaction of number scores blocked by program group of those who did or did not study school calculus. Data for figure from U.S. TEDS-M data (IEA, 2012).

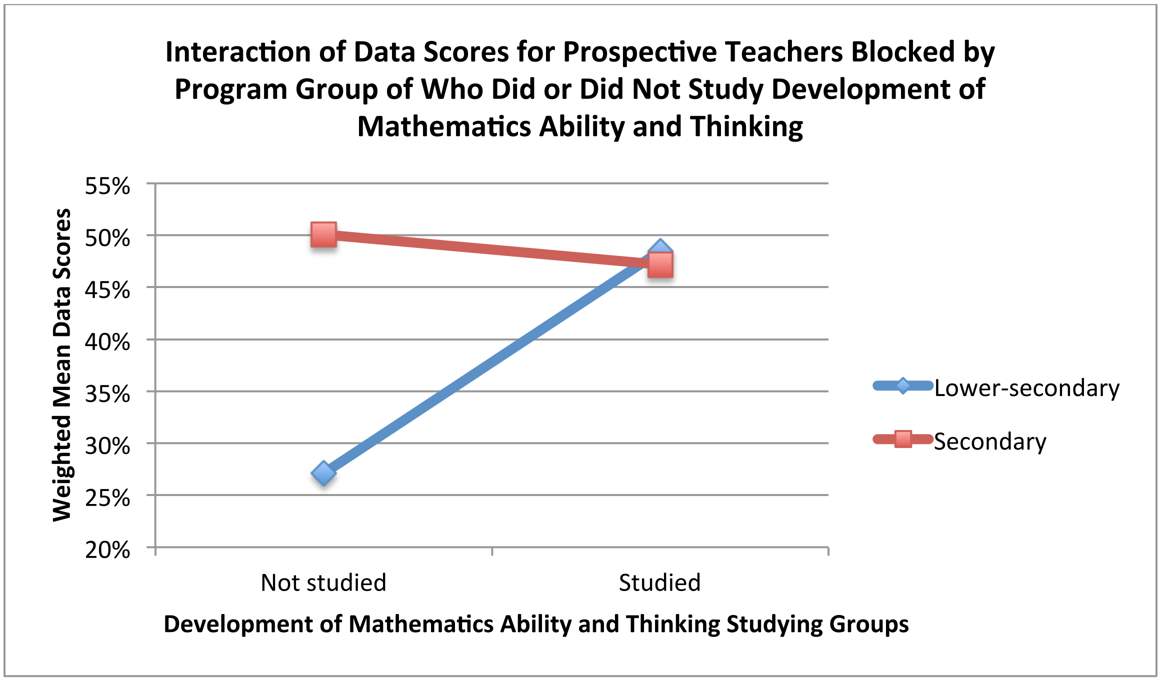

Figure J7. Decomposing the interaction of data scores blocked by program group of those who did or did not study development of mathematics ability and thinking. Data for figure from U.S. TEDS-M data (IEA, 2012).

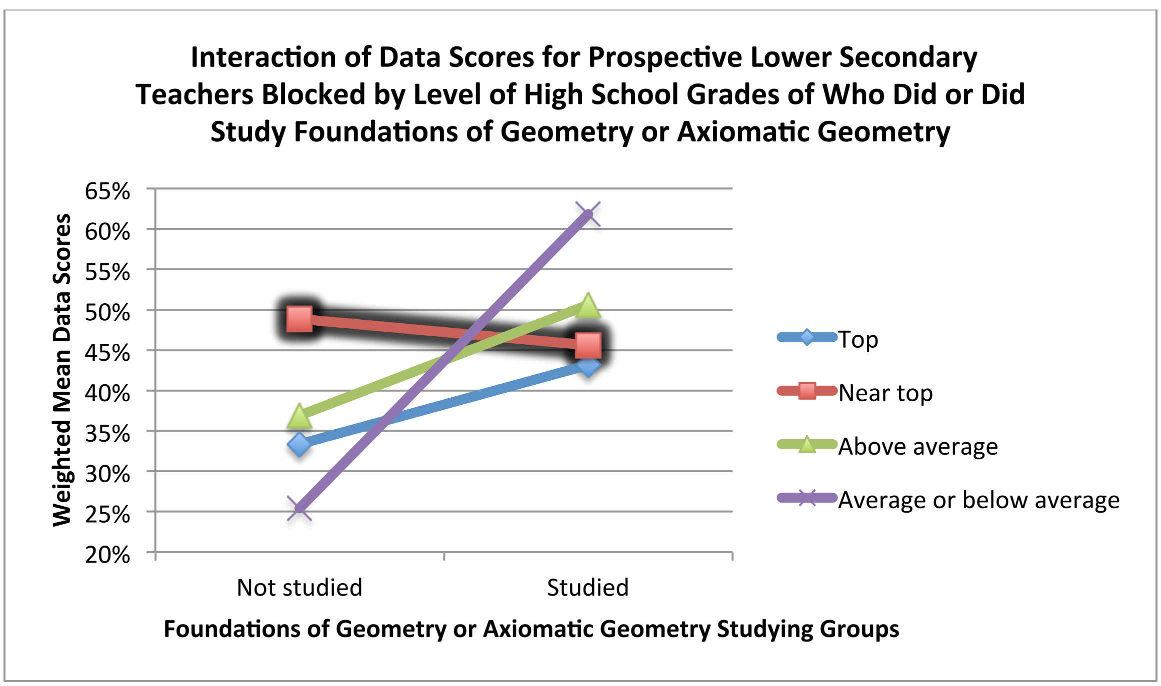

Figure J8. Decomposing the interaction of data scores blocked by level of high school grades of those who did or did not study foundations of geometry or axiomatic geometry. Data for figure from U.S. TEDS-M data (IEA, 2012).

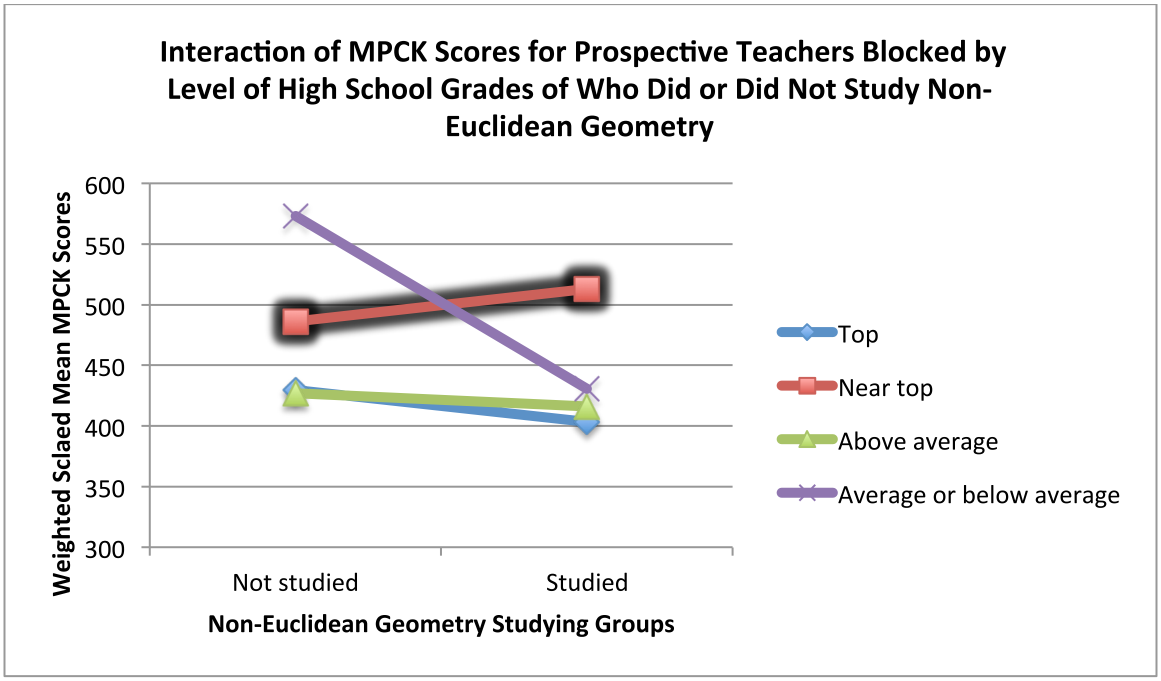

Figure J9. Decomposing the interaction of MPCK scores blocked by level of high school grades of those who did or did not study non-Euclidean geometry. Data for figure from U.S. TEDS-M data (IEA, 2012).

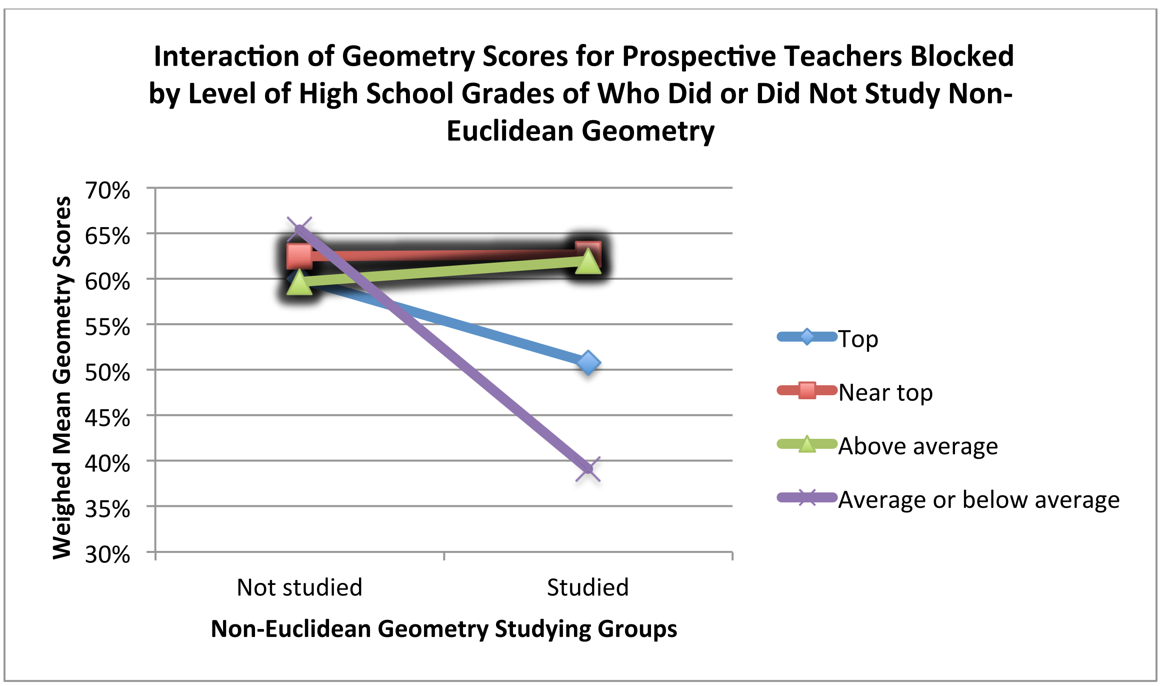

Figure J10. Decomposing the interaction of geometry scores blocked by level of high school grades of those who did or did not study non-Euclidean geometry. Data for figure from U.S. TEDS-M data (IEA, 2012).

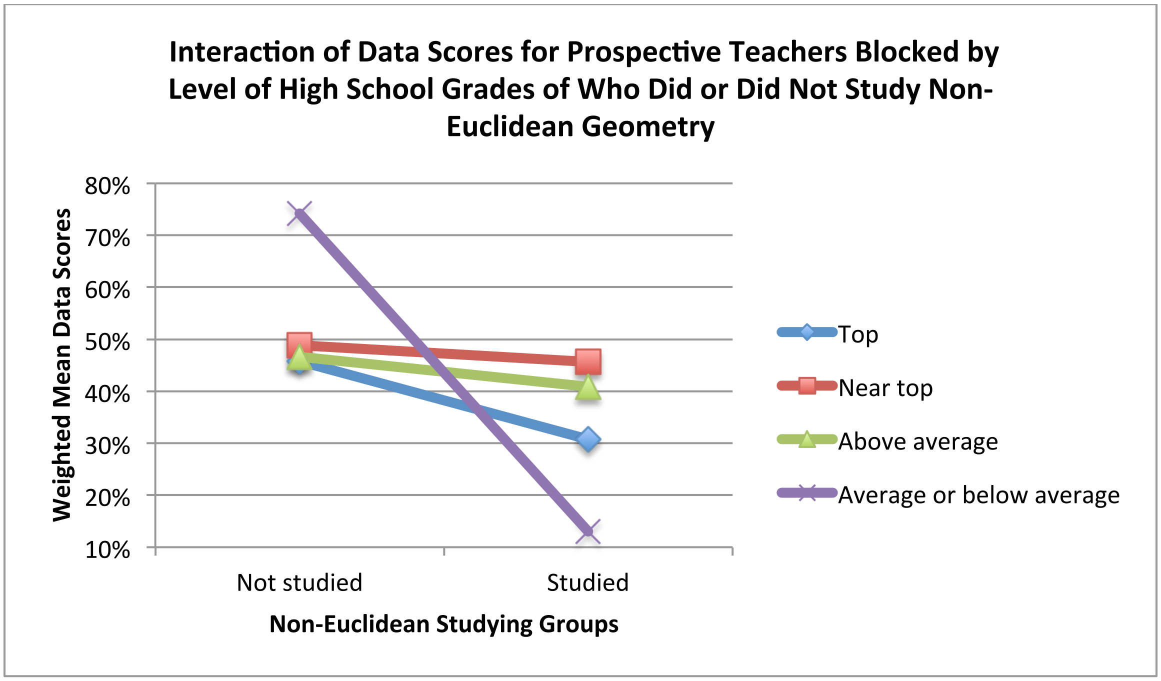

Figure J11. Decomposing the interaction of data scores blocked by level of high school grades of those who did or did not study non-Euclidean geometry. Data for figure from U.S. TEDS-M data.

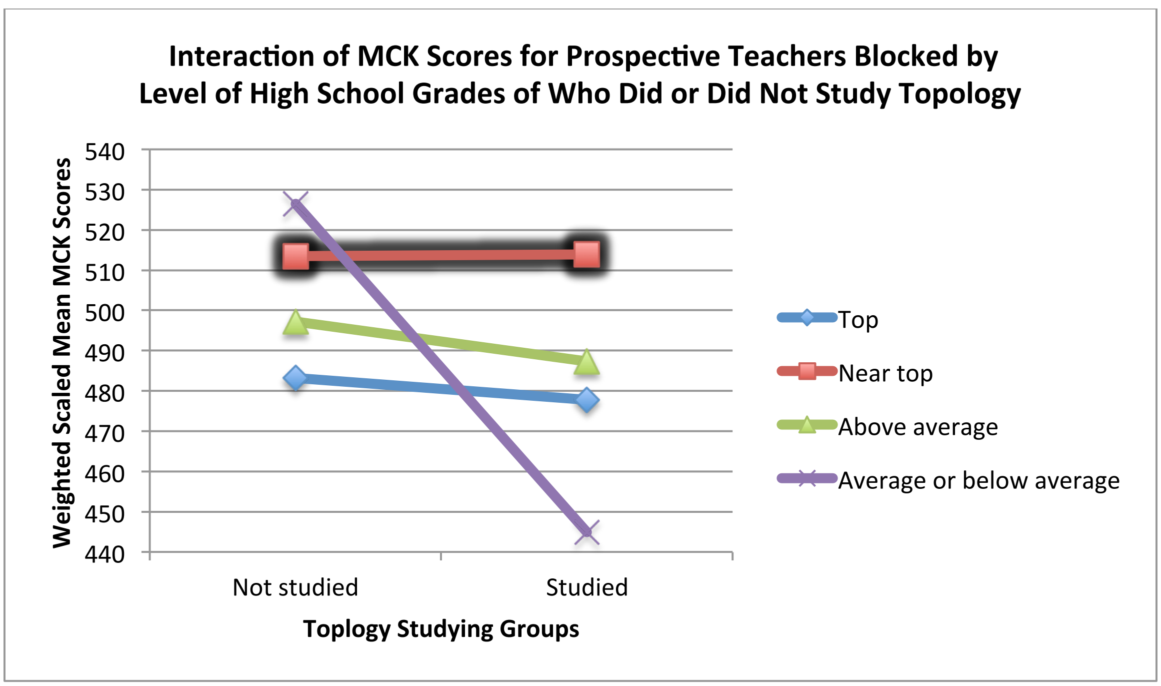

Figure J12. Decomposing the interaction of MCK scores blocked by level of high school grades of those who did or did not study topology. Data for figure from U.S. TEDS-M data (IEA, 2012).

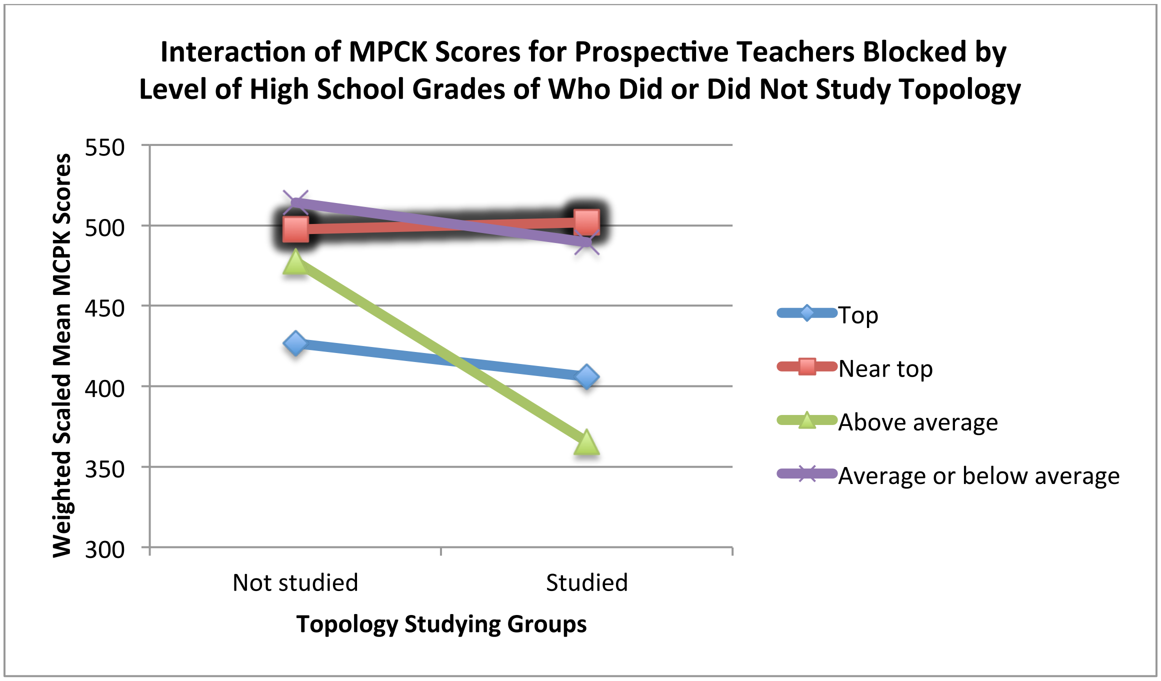

Figure J13. Decomposing the interaction of MPCK scores blocked by level of high school grades of those who did or did not study topology. Data for figure from U.S. TEDS-M data (IEA, 2012).

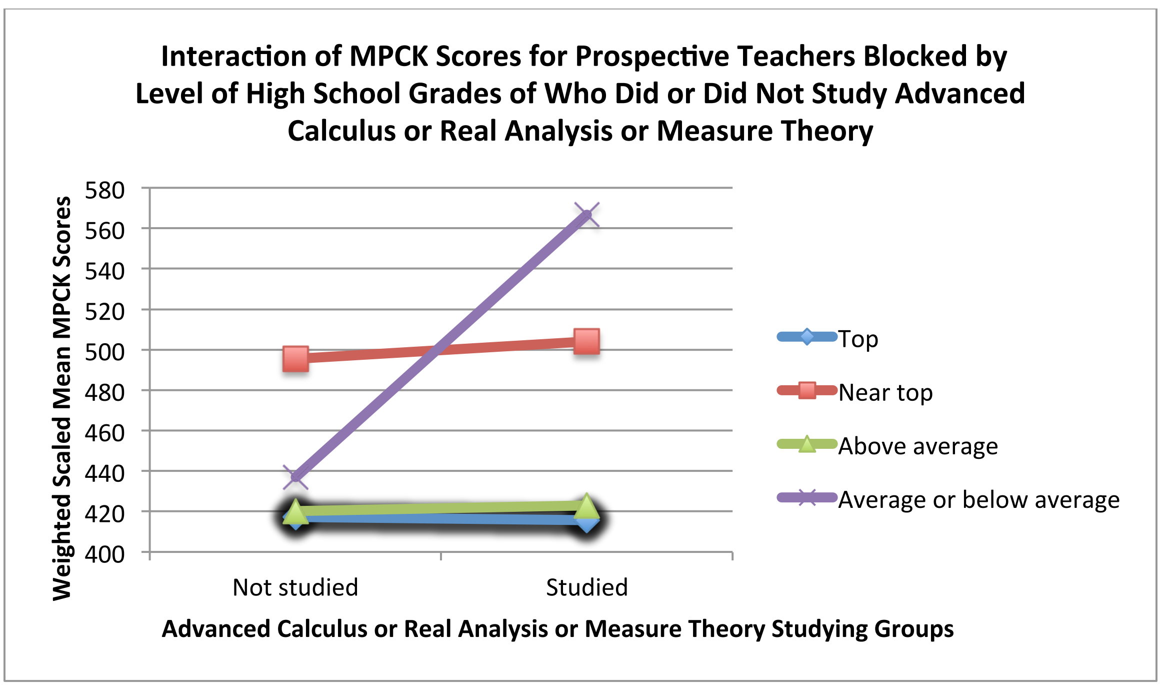

Figure J14. Decomposing the interaction of MPCK scores blocked by level of high school grades of those who did or did not study advanced calculus or real analysis or measure theory. Data for figure from U.S. TEDS-M data (IEA, 2012).

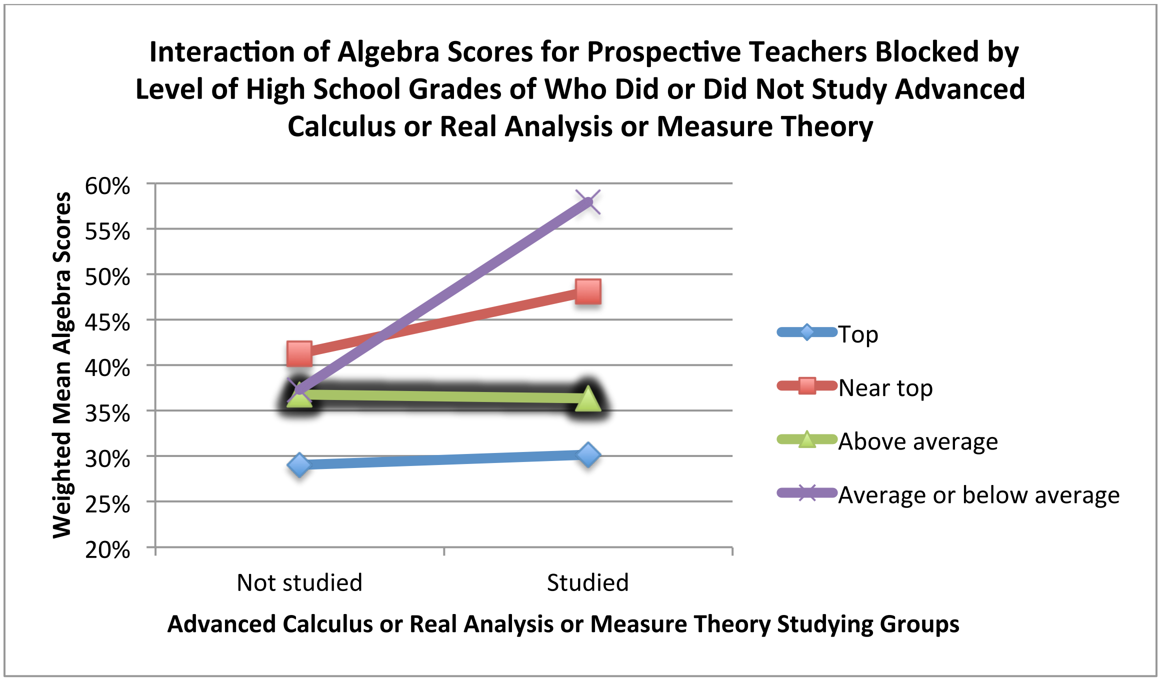

Figure J15. Decomposing the interaction of algebra scores blocked by level of high school grades of those who did or did not study advanced calculus or real analysis or measure theory. Data for figure from U.S. TEDS-M data (IEA, 2012).

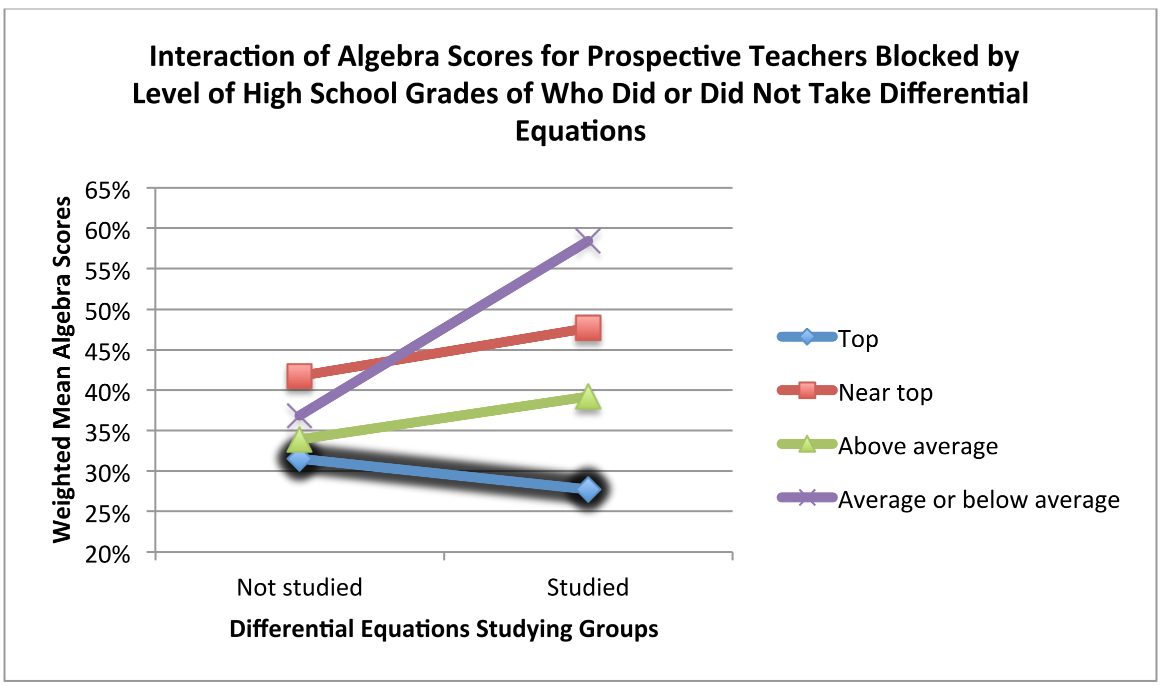

Figure J16. Decomposing the interaction of algebra scores blocked by level of high school grades of those who did or did not study differential equations. Data for figure from U.S. TEDS-M data (IEA, 2012).

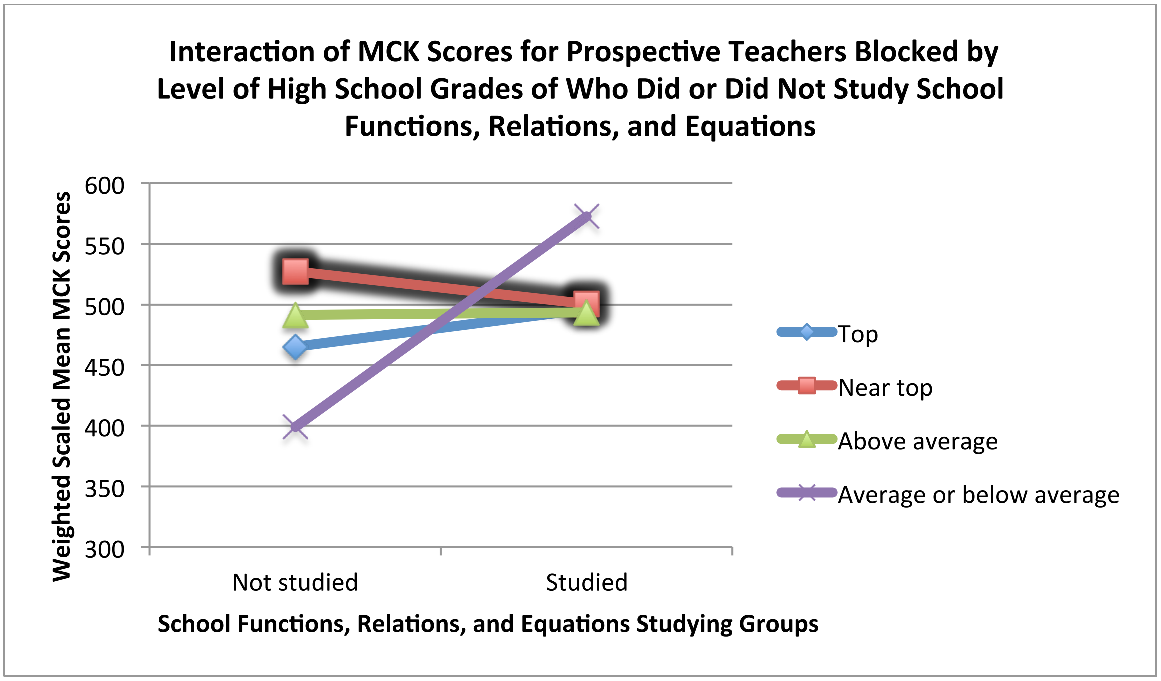

Figure J17. Decomposing the interaction of MCK scores blocked by level of high school grades of those who did or did not study school functions, relations, and equations. Data for figure from U.S. TEDS-M data (IEA, 2012).

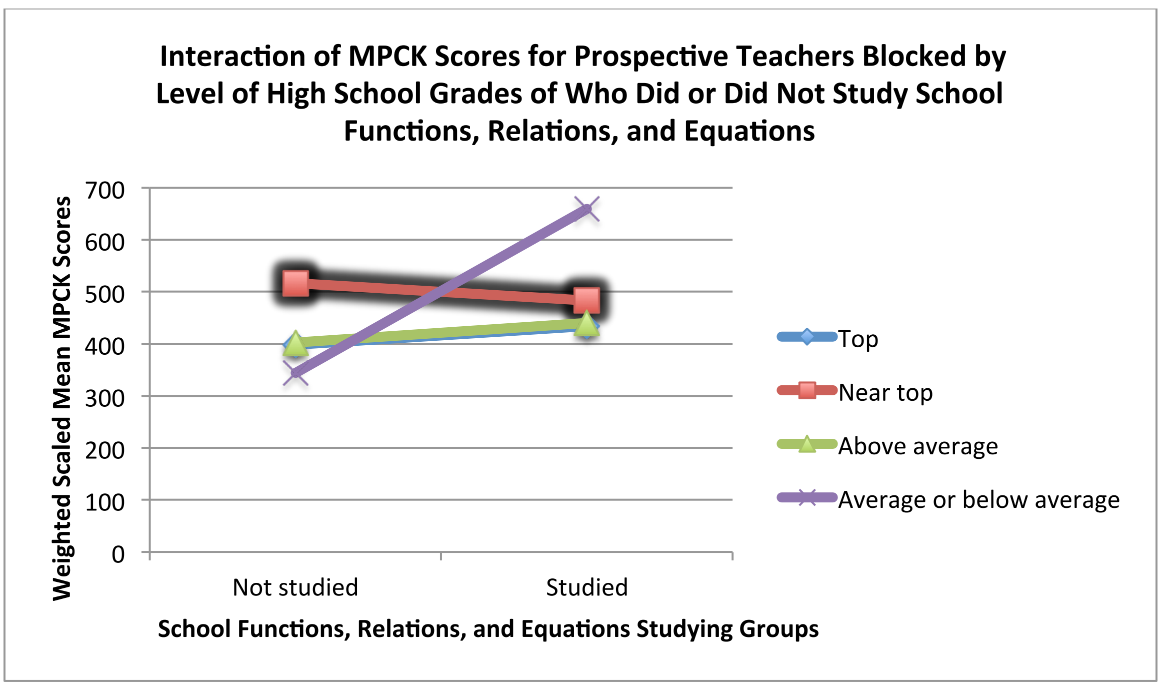

Figure J18. Decomposing the interaction of MPCK scores blocked by level of high school grades of those who did or did not study school functions, relations, and equations. Data for figure from U.S. TEDS-M data (IEA, 2012).

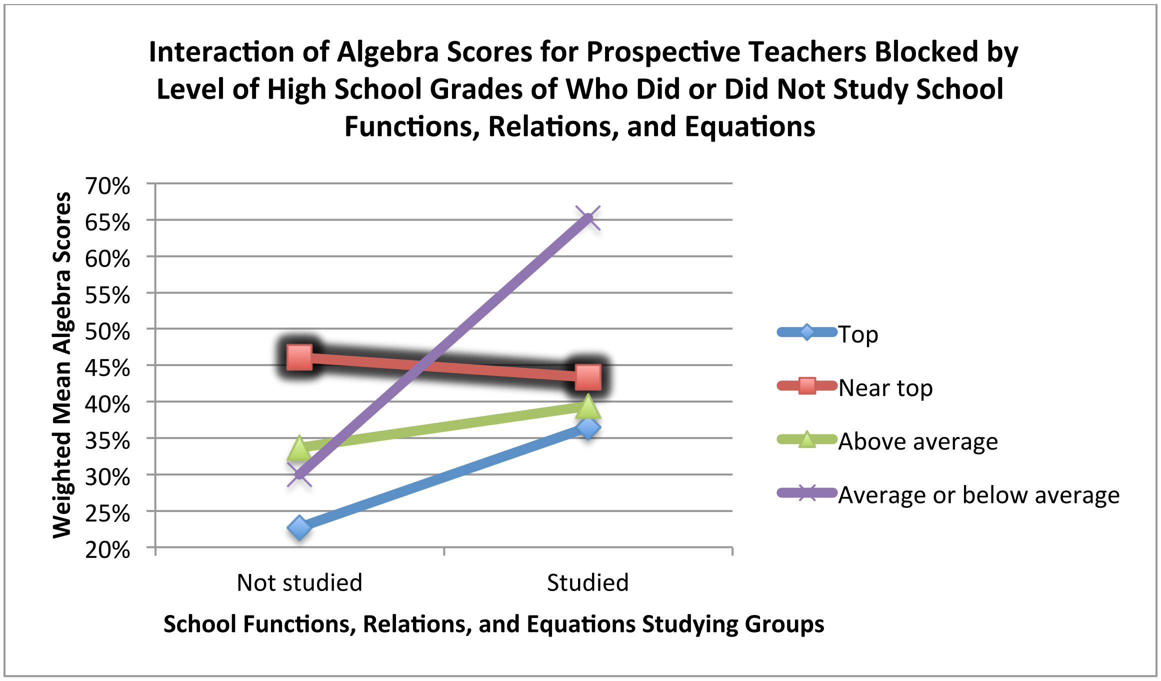

Figure J19. Decomposing the interaction of algebra scores blocked by level of high school grades of those who did or did not study school functions, relations, and equations. Data for figure from U.S. TEDS-M data (IEA, 2012).

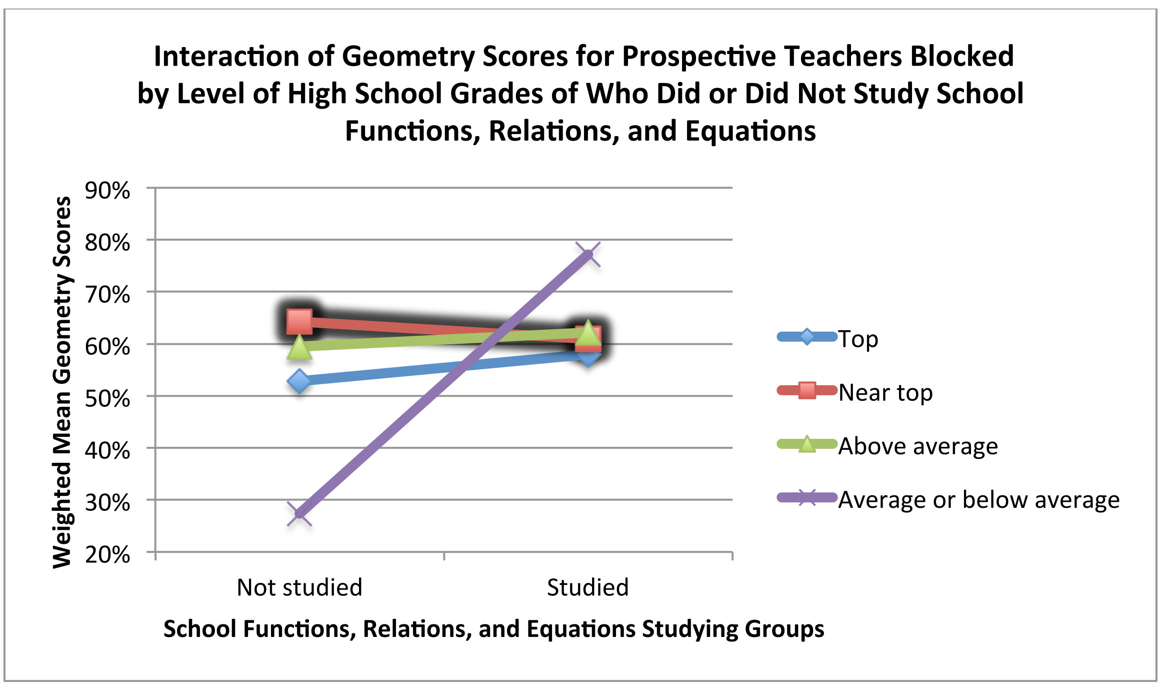

Figure J20. Decomposing the interaction of geometry scores blocked by level of high school grades of those who did or did not study school functions, relations, and equations. Data for figure from U.S. TEDS-M data (IEA, 2012).

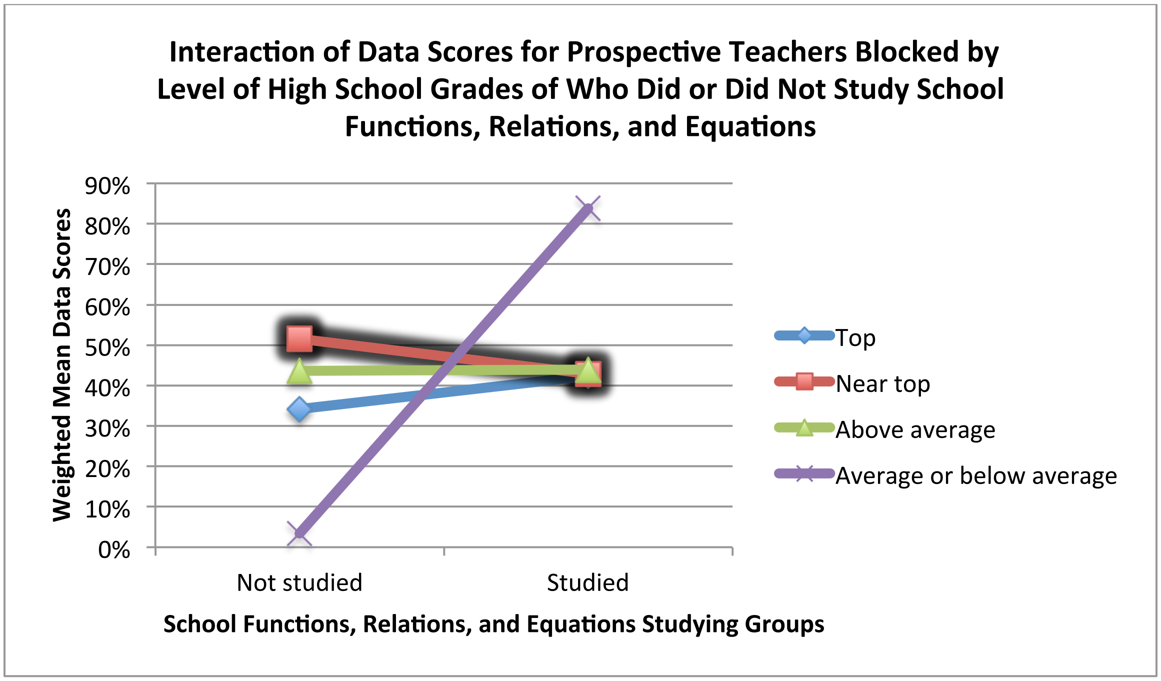

Figure J21. Decomposing the interaction of data scores blocked by level of high school grades of those who did or did not study school functions, relations, and equations. Data for figure from U.S. TEDS-M data (IEA, 2012).

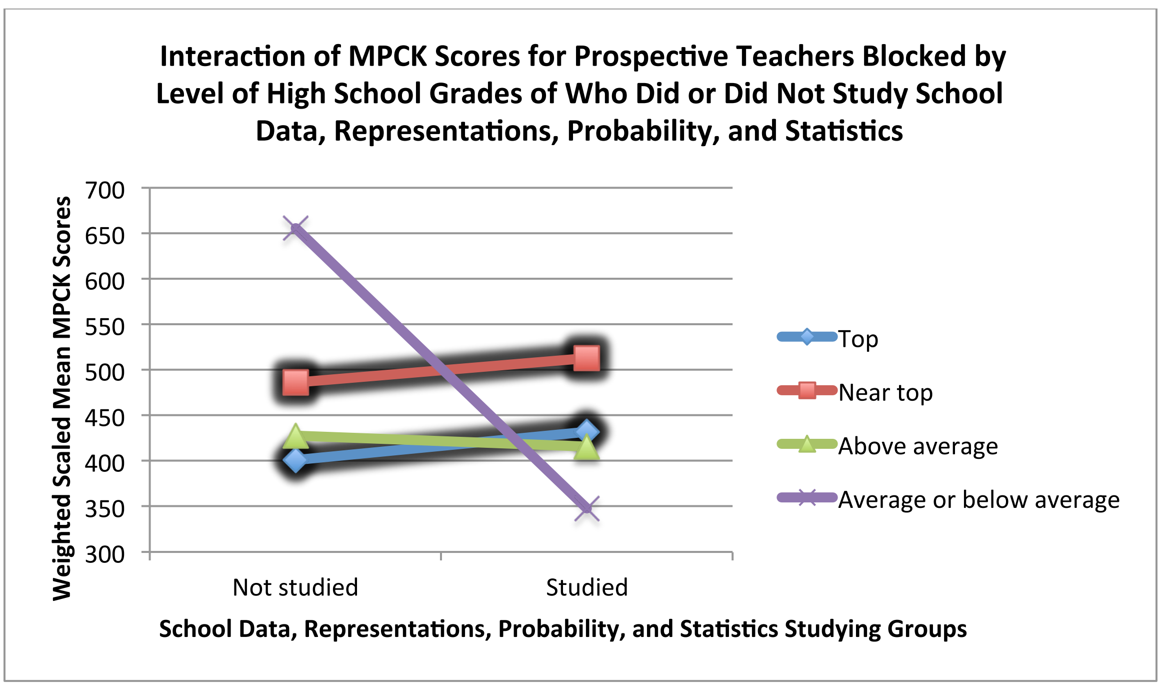

Figure J22. Decomposing the interaction of MPCK scores blocked by level of high school grades of those who did or did not study school data, representations, probability, and statistics. Data for figure from U.S. TEDS-M data (IEA, 2012).

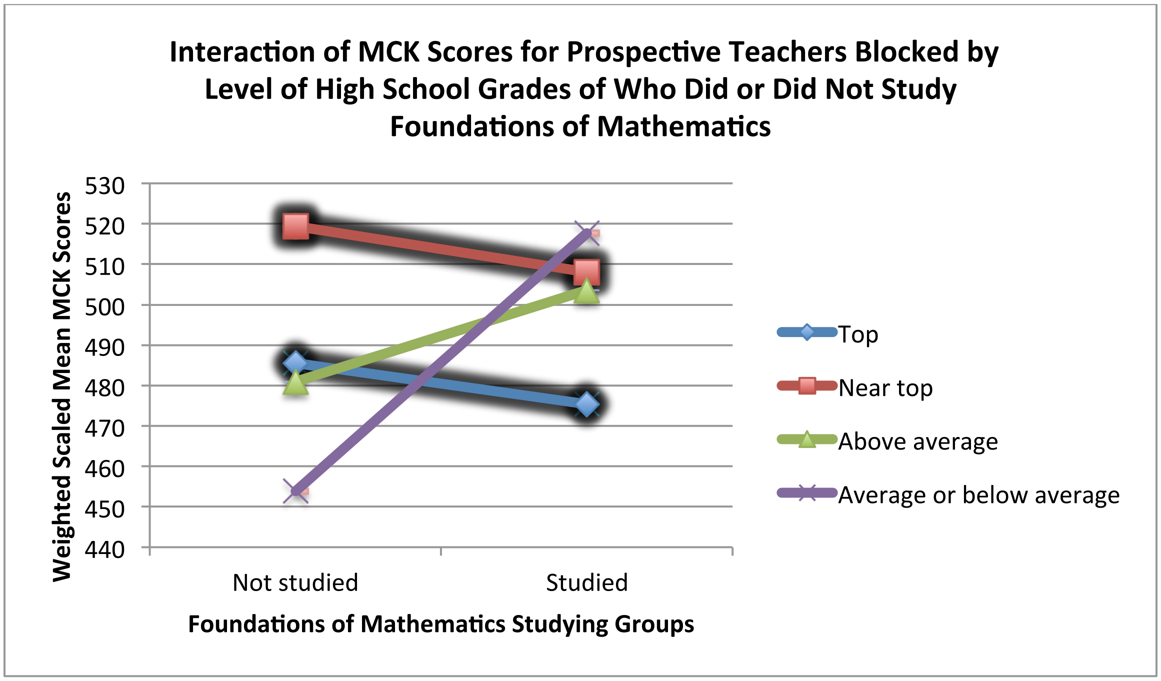

Figure J23. Decomposing the interaction of MCK scores blocked by level of high school grades of those who did or did not study foundations of mathematics. Data for figure from U.S. TEDS-M data (IEA, 2012).

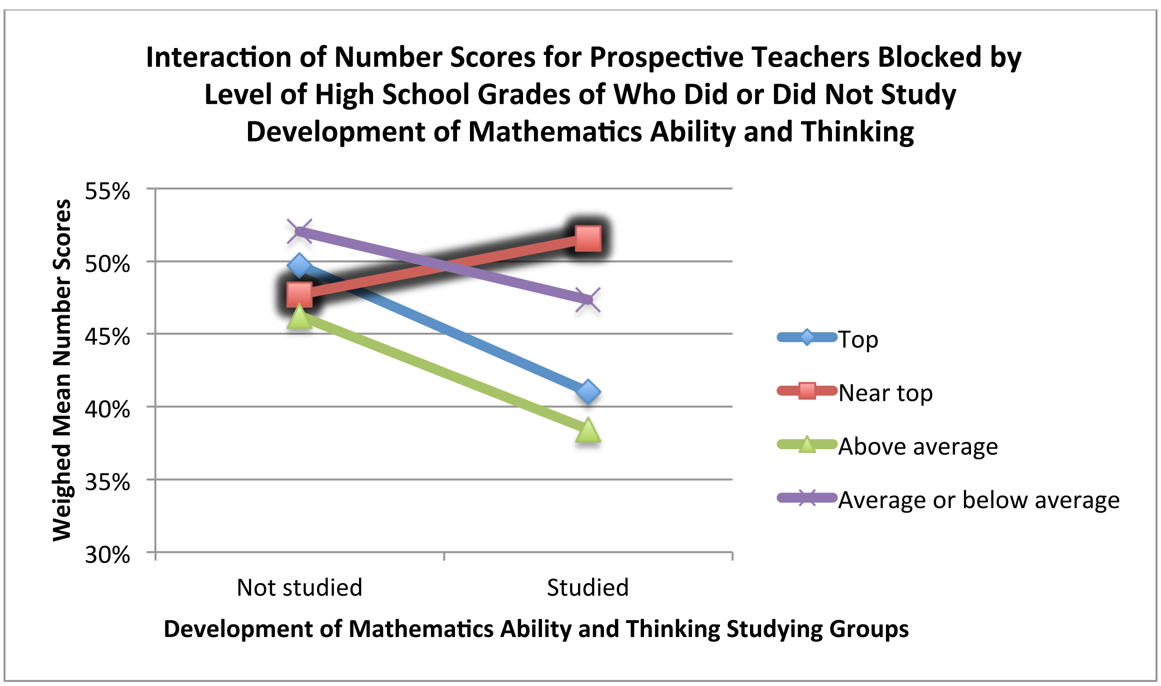

Figure J24. Decomposing the interaction of number scores blocked by level of high school grades of those who did or did not study development of mathematics ability and thinking. Data for figure from U.S. TEDS-M data (IEA, 2012).

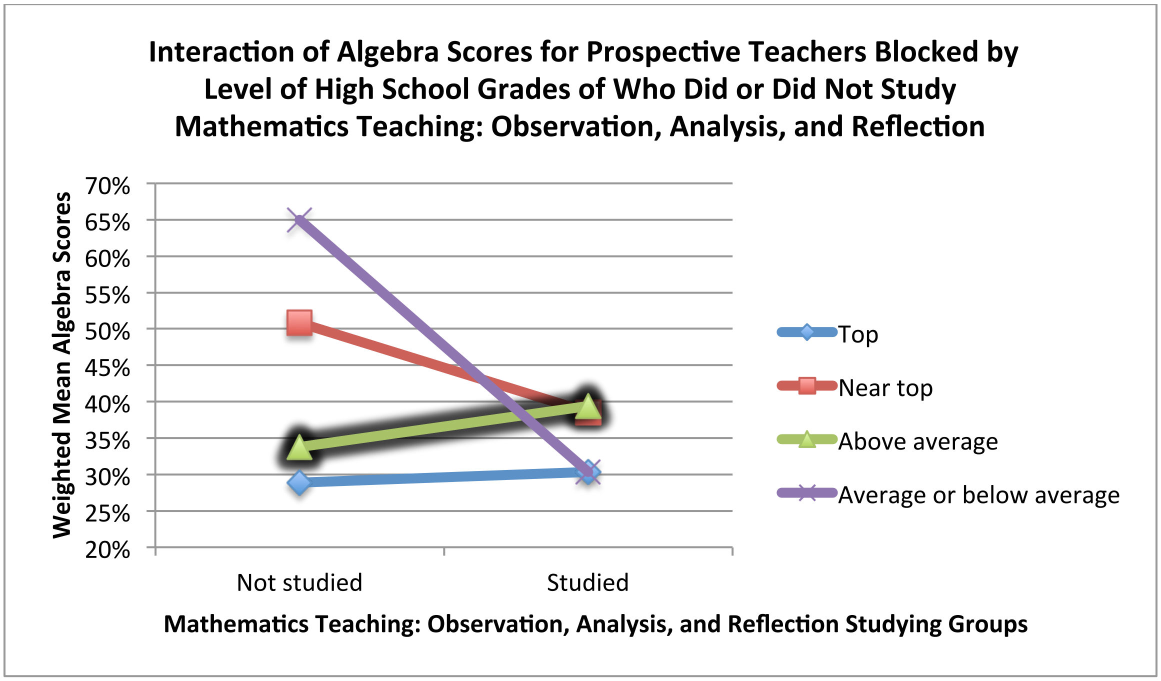

Figure J25. Decomposing the interaction of algebra scores blocked by level of high school grades of those who did or did not study mathematics teaching: observation, analysis, and reflection. Data for figure from U.S. TEDS-M data (IEA, 2012).

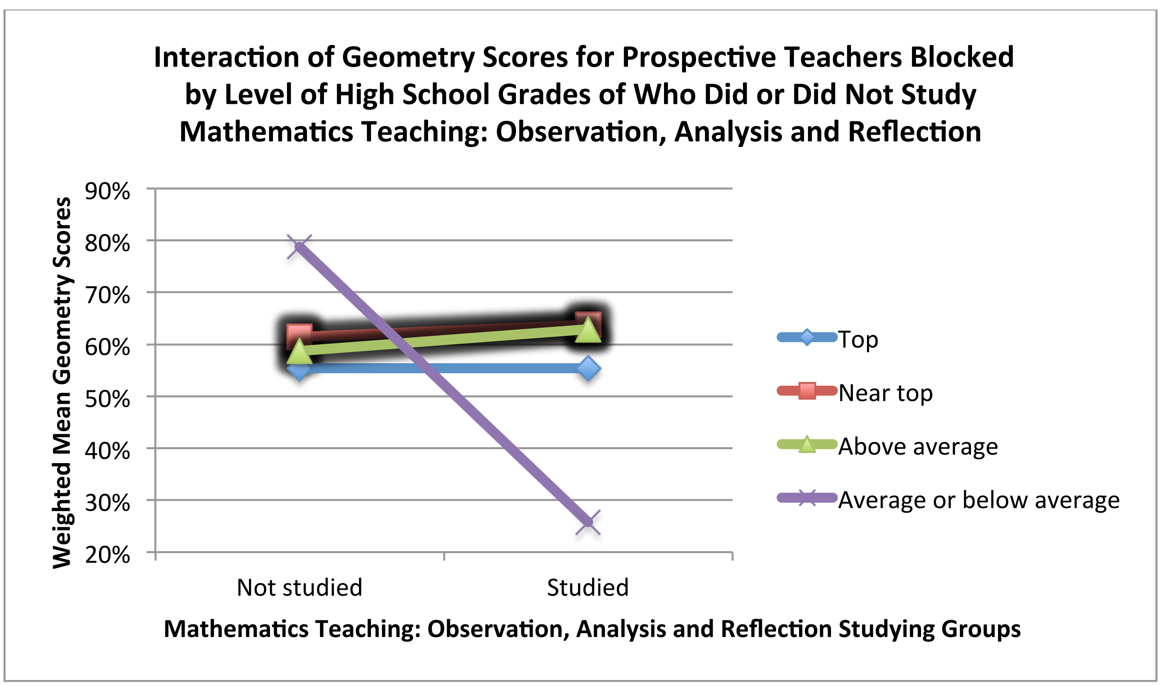

Figure J26. Decomposing the interaction of geometry scores blocked by level of high school grades of those who did or did not study mathematics teaching: observation, analysis, and reflection. Data for figure from U.S. TEDS-M data (IEA, 2012).

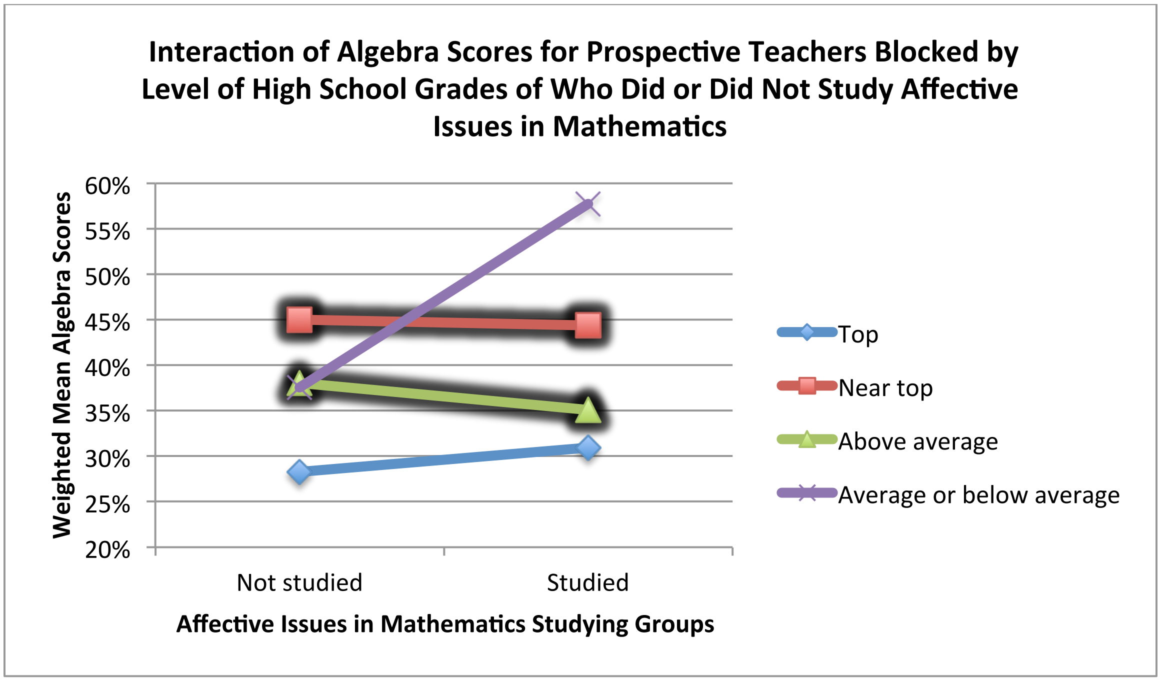

Figure J27. Decomposing the interaction of algebra scores blocked by level of high school grades of those who did or did not study affective issues in mathematics. Data for figure from U.S. TEDS-M data (IEA, 2012).

Appendix K

Table K1

P-values of Course Topics That Were Associated With Higher MKT Scores When Studied by Prospective Teachers

Foundations of geometry or axiomatic geometry					
MCK	MPCK	Number	Algebra	Geometry	Data
—	—	—	—	—	.003
Analytic/coordinate geometry					
MCK	MPCK	Number	Algebra	Geometry	Data
—	—	—	—	.000	—
Multivariate calculus					
MCK	MPCK	Number	Algebra	Geometry	Data
.009	—	.000	—	—	.034
Advanced calculus or real analysis or measure theory					
MCK	MPCK	Number	Algebra	Geometry	Data
.002	.004	.018	.005	.007	—
Differential equations					
MCK	MPCK	Number	Algebra	Geometry	Data
—	—	—	.003	—	—
Probability					
MCK	MPCK	Number	Algebra	Geometry	Data
—	—	—	.027	—	—
School geometry					
MCK	MPCK	Number	Algebra	Geometry	Data
.036	—	—	—	.002	.016
School functions, relations, and equations					
MCK	MPCK	Number	Algebra	Geometry	Data
.005	.000	—	.005	.002	.008
School calculus					
MCK	MPCK	Number	Algebra	Geometry	Data
—	—	.004	—	—	—
Foundations of mathematics					
MCK	MPCK	Number	Algebra	Geometry	Data
.014	—	—	—	—	—
Development of mathematics ability and thinking					
MCK	MPCK	Number	Algebra	Geometry	Data
—	—	—	—	—	.014

(continued)

Table K1 (continued)

Mathematics instruction					
MCK	MPCK	Number	Algebra	Geometry	Data
—	—	—	.034	—	—
Developing teaching plans					
MCK	MPCK	Number	Algebra	Geometry	Data
—	.031	—	.004	—	—
Affective issues in mathematics					
MCK	MPCK	Number	Algebra	Geometry	Data
—	.002	.044	.034	—	—

Note. P-values derived from U.S. data (IEA, 2012) from TEDS-M 2008 study (Tatto et al., 2012).

Table K2
Course Topics That Helped Prospective Teachers Solve MKT Items

Foundations of geometry or axiomatic geometry					
MCK	MPCK	Number	Algebra	Geometry	Data
2	—	—	—	2	—
Abstract algebra					
MCK	MPCK	Number	Algebra	Geometry	Data
1	1	1	1	—	—
Precalculus (college)					
MCK	MPCK	Number	Algebra	Geometry	Data
1	—	—	1	—	—
Beginning calculus					
MCK	MPCK	Number	Algebra	Geometry	Data
2	—	—	2	—	—
Differential equations					
MCK	MPCK	Number	Algebra	Geometry	Data
1	—	—	1	—	—
Statistics for teachers					
MCK	MPCK	Number	Algebra	Geometry	Data
—	1	—	—	—	1
School numbers					
MCK	MPCK	Number	Algebra	Geometry	Data
1	2	2	1	—	—
School algebra					
MCK	MPCK	Number	Algebra	Geometry	Data
3	1	1	3	—	—
School geometry					
MCK	MPCK	Number	Algebra	Geometry	Data
6	2	1	—	6	1
School functions, relations, and equations					
MCK	MPCK	Number	Algebra	Geometry	Data
3	1	2	—	2	—

(continued)

Table K2 (continued)

Mathematics pedagogy (secondary)					
MCK	MPCK	Number	Algebra	Geometry	Data
1	2	—	3	—	—
Mathematics instruction in geometry					
MCK	MPCK	Number	Algebra	Geometry	Data
5	1	1	1	3	—
Mathematics instruction in algebra					
MCK	MPCK	Number	Algebra	Geometry	Data
3	1	1	3	—	—

Note. Counts under the MCK and MPCK categories are totals of the counts by number, algebra, geometry, and data domains.