MATHEMATICAL KNOWLEDGE FOR TEACHING AND COLLABORATIVE COACHING FOR SECONDARY SCHOOL PROFESSIONAL DEVELOPMENT

by

DANIEL JOHANNES BRINK

(Under the Direction of James W. Wilson)

ABSTRACT

This study examined the effectiveness and impact of collaborative coaching, a special type of professional development, on the views and actions of a secondary school mathematics teacher. In particular it examined the following two research questions: 1) In what way did collaborative coaching that specifically focused on the MKT of a secondary school mathematics teacher affect her views on the teaching and learning of mathematics?, and 2) In what way did the participant attend to students' mathematical contributions differently because of her participation in the collaborative coaching? In designing this study, the researcher reviewed literature on teacher knowledge, professional development, and teacher learning. The collaborative coaching focused on the mathematical knowledge for teaching (MKT) of the participant and the researcher. Coaching sessions comprised of both participant and researcher watching video clips from the participant's two Gifted/Honors Algebra II classrooms and working on the mathematics contained in the video clips. The video clips mostly contained students' mathematical contributions from class and the aim of most coaching sessions was to improve, elaborate, discuss, or expand on these mathematical contributions and hence, as a pair, enhance the MKT of both participant and researcher. The participant changed her views on teaching as a teacher-driven endeavor to an exercise with more student involvement. She changed her views on learning as something that was done by passive students taking notes and practicing problems to students actively engaging in mathematical discussions and arguments and learning through this engagement. She changed her attention to students' mathematical contributions by focusing on these contributions for longer periods of time and facilitating interactive student engagement with the contributions rather than merely validating or confirming the contribution herself.

INDEX WORDS: Teacher knowledge, Mathematical knowledge for teaching, Collaborative coaching, Professional development, Secondary school mathematics, Teacher training, Mathematics teacher

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A Dissertation Submitted to the Graduate Faculty of The University of Georgia in Partial

Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

2007

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August 2007

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Aan my liefling, Lise.

En aan ons kinders Mieke, Andre, Louise, Johann…

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CHAPTER ONE

INTRODUCTION AND PROBLEM STATEMENT

This study looked at the effectiveness of a specific type of professional development, called collaborative coaching. By effective professional development I mean (a) professional development that influenced the views of the participant about mathematics teaching by the teacher and mathematics learning by students toward current reform-oriented views, and (b) professional development that had an effect on a particular aspect of teaching practice of the participant with the result of the student becoming more central in the thinking and action of the teacher. The collaborative coaching focused on the mathematical knowledge for teaching (MKT) of a secondary school mathematics teacher. I drew from literature on teacher knowledge, professional development, teacher learning, and collaborative coaching for this study and was also personally driven by experiences as a secondary school mathematics teacher and mathematics department chair.

To motivate the two research questions, I give a brief summary of each body of literature below and elaborate on the phenomenon of coaching generally, and collaborative coaching specifically.

Teacher Knowledge: Researchers have attempted to link the knowledge of mathematics teachers and students' performance for many years. Researchers such as Begle (1972; 1979) and Monk (1994) showed that quantitative measures of teachers' mathematics knowledge, such as the number of undergraduate mathematics courses taken, do not have a positive correlation with student performance. In her book *Knowing and Teaching Elementary Mathematics*, Ma (1999) vividly described the qualitative differences in the profound understanding of fundamental mathematics of Chinese and United States elementary school mathematics teachers and implied these differences accounted for some of the differences in test scores between the two countries. Hill, Rowan, and Ball (2005) found a link between the mathematical knowledge for teaching of elementary school teachers and the performance of students. Similarly Carpenter, Fennema, Peterson, and Carey (1988) showed a correlation between teacher knowledge about problem difficulty and student performance as well as preferred student methods on mathematical tasks and student performance. An equivalent qualitative study for high school teachers has, to my knowledge, not been done. Furthermore, it is limiting to study what mathematics teachers know and to spend a lot of time defining their knowledge. What is missing is a view of, and research on, mathematical knowledge *in the context of teaching* (Ball, Lubienski, & Mewborn, 2001). It is important to know what knowledge teachers have and how this knowledge affects what happens in their classrooms. As Leinhardt, Zaslavsky and Stein (1990) wrote: “Limitations on subject matter knowledge…often reduce the flexibility and creativity of a teacher as well as create a kind of authoritarianism toward the subject and student that permits little or no exploration of ideas” (p. 46).

Professional Development: There is an abundance of literature within the mathematics education research community suggesting that traditional professional development of mathematics teachers has little effect on the teaching and learning of teachers and the learning of students (Garet, Porter, Desimone, Birman, & Yoon, 2001; Halai, 1998; Hawley & Valli, 1999; Kazemi & Franke, 2003; Wilson & Berne, 1999). Ineffective professional development is done by outside experts, in a limited amount of time, at an off-campus location, out of context, without focusing on student learning, and without addressing teacher knowledge and beliefs in a meaningful way. In contrast, research indicated that a serious and sustained focus on mathematical content and teachers' knowledge thereof can have an effect on teacher learning and teaching and student learning (Ball et al., 2001).

Teacher Learning: In order to optimize the learning of participants in collaborative coaching it was imperative that close attention be paid to the teacher learning literature. Two important qualities of professional development that aims to promote teacher change is that it should cause a dilemma for the participants (M. S. Smith, 2000; Wood, Cobb, & Yackel, 1991) and that problem-solving should form a large part of participants' work (Loucks-Horsley, Hewson, Love, & Stiles, 1998). Furthermore, professional development should be relevant to the participants as far as content (working with their own curriculum), venue (working in their own school and classrooms), and students (work of students from their own classes) are concerned (M. S. Smith, 2003; Wood et al., 1991).

The concept of coaching was first explored as an alternative form of professional development by Showers and Joyce (1980; 1982). They called their model *peer coaching* and envisioned pairs of colleagues coaching each other. They defined peer coaching as "providing companionship, helping each other learn to teach the appropriate responses to their students, and figuring out the optimal uses of the model in their courses, and providing each other with ideas and feedback." (Showers & Joyce, 1982, p. 5) All types of coaching incorporated a traditional supervisory model focused on classroom observations and preconference-observation-postconference (Loucks-Horsley et al., 1998). Other important elements of coaching were a strong focus on learning and improvement by participating teachers and an emphasis on interaction and collegiality among participants (Loucks-Horsley et al., 1998).

Showers and Joyce (1982) argued that the professional development designs of their time were effective because they assisted teachers to acquire new skills but were deficient in that they failed to facilitate the transfer of such skills into the classroom. This criticism of professional development was very much in line with current-day literature. The qualities of accountability and feedback that came with coaching made transfer more successful. Other qualities of coaching such as being continuous, deeply embedded in the teacher's classroom and current curriculum, and specific to grade level and content made it adhere to research findings regarding successful professional development. Showers and Joyce (1996) also found that teachers practiced new skills and strategies more regularly and applied them more appropriately in coaching than when they operated alone. Edwards and Newton (1995) found that a specific type of coaching called *cognitive coaching* increased teacher efficacy and deepened their understanding of their own classroom practice. Russo (2004) mentioned the focus on authentic student work and reliance on research as strong attributes of coaching. Yet, despite all the apparent positive qualities of coaching as a form of professional development of teachers, the success of it remained largely anecdotal and there remained a large need for research on the results of coaching on student learning (Russo, 2004).

Apart from Showers and Joyce's (1982) peer coaching, various other forms of coaching developed over the years. *Technical coaching* was used to assist teachers to transfer new skills into their daily practice. This could be done via instructional modeling, joint lesson planning, co-teaching, formal observations and feedback, informal one-on-one contact, and mentoring of new teachers (Poglinco et al., 2003). *Collegial coaching* was applied to increase teachers' professional dialogue, help them reflect on their work, and get them into the classrooms of other teachers (Poglinco et al., 2003). *Cognitive coaching* was designed to enhance teachers' perception of their inner thought processes which would facilitate overt instructional improvements (Costa & Garmston, 1994). While collegial and cognitive coaching aimed at improving existing practices, technical and peer coaching aimed to assist in the acquisition of new skills or the mastery of a new curriculum or style of instruction (Showers & Joyce, 1996). In this study, I used a specific variation of peer coaching, called *collaborative coaching*, because of the focus on “collaborative, peer learning”, “facilitation of learning and not on evaluation of practice”, and “collegial exchange of ideas and problem solving” (Loucks-Horsley et al., 1998, p. 206).

One of the conclusions that could be made from the combination of professional development and teacher knowledge literature is that professional development should focus on teacher knowledge because teacher knowledge (specifically of elementary school mathematics teachers) affects student learning. Another conclusion one could make from the teacher learning literature is that professional development needs to be relevant and perturbing to participants at the same time. With the combination of teacher knowledge, teacher learning, and professional development in mind, I designed the collaborative coaching to focus on the MKT of a high school mathematics teacher and explored its impact on the views and practices of this teacher. I asked the following two research questions:

1. In what way did collaborative coaching that specifically focused on the MKT of a secondary school mathematics teacher affect her views on the teaching and learning of mathematics?

2. In what way did the participant attend to students’ mathematical contributions differently because of her participation in the collaborative coaching?

I would like to define and clarify some of the terms and phrases I used in this study and explain what I meant when I used them.

- *Students' mathematical contributions:* For the purpose of this study I referred to students' mathematical questions, comments, suggestions, and conjectures as students' mathematical contributions.
- *MKT:* I used MKT when referring to the domain of knowledge that a teacher uses in the act of teaching and I conceptualized this domain partitioned in the manner that was done by Ball and her colleagues (2005). This partitioning will be explained in the theoretical framework section in Chapter Two of this study. Furthermore, I refer to my participant (Cindy – a pseudonym) and me "working on our MKT". With this phrase I mean that we examined a particular mathematical idea for the purposes of enhancing our ability to teach it better, to understand related student contributions better, and to understand connections to other parts of mathematics better. After many such coaching sessions I felt we did enhance our MKT, meaning that, through active engagement and problem-solving, we did increase our understanding and our knowledge of the particular mathematical topic.
- *Views on teaching and learning:* When talking about Cindy's views of teaching I will mean the ways she conceptualized teaching done in the best possible manner so that student learning could be optimized. With this way of looking at Cindy's view on teaching I did acknowledge that the realities in which Cindy worked (which I will explain in detail in Chapter Three) affected her view on teaching. I will talk about Cindy's view of learning as the role of the students and their actions in class that optimized the way they learned mathematics.

- *Collaborative coaching:* I used collaborative coaching in this study as a specific form of professional development as defined earlier in this chapter. I must, however, emphasize that I see the collaborative coaching endeavor truly as a combined effort from all the participants. Although one of the participants must take the initiative in the process, all participants should both alternatively be the coaches and the coached.
- *Traditional teaching versus reform-oriented teaching:* For the purposes of this study by the term traditional teaching I will mean that a teacher demonstrated new material to students and they practiced doing it just as the teacher had demonstrated it. The teacher played a very active role in explanations and discussions, acting like the "dispenser of knowledge" (Williams & Baxter, 1996, p. 23), while the students copied notes and practiced problems on their own in class or at home. In this vein, I will use reform-oriented teaching and reform-oriented attention to students and their mathematical contributions as a focus on involving students in actively participating in their own learning. Students' opinions and mathematical contributions are valued and incorporated into lessons because they are seen as ways to help the teacher understand students' thinking. In reform-oriented teaching the teacher takes a backseat in teaching and becomes the "facilitator" of learning (Williams & Baxter, 1996, p. 23) or the "guide-on-the-side" (Nathan & Knuth, 2003, p. 176). The emphasis given to student involvement in mathematics classes by publications such as the three NCTM documents (1989; 1991; 2000) and the British Department for Education and Skills (DfES, 2001) have spoken to the importance of making students' mathematical contributions a central feature in reform-oriented teaching.

- *A teacher's views of teaching and learning:* I see a teacher's views of teaching as his thoughts and ideas about how teaching should be done in order to optimize student learning. I see a teacher's views of learning as his ideas and perceptions of the roles, actions, and activities that students must be engaged in to optimize their learning. A positive change in a teacher's views on teaching and learning would result in the teacher teaching in a less traditional manner and more in a reform-oriented manner. Students will play an increasingly active role in the direction of lessons and in the formulation, validation, and discovery of mathematics in every lesson. Their mathematical thoughts, understanding, misunderstandings, problems, suggestions, and mathematical processes will come to the forefront. This shift in the role of the student will be evident in the way a teacher talks about his or her actions, plans, motivations, and responses before, during, and after a lesson. A teacher affected in a positive manner may put the practical ideas of the abovementioned views into practice. The fact that students would have a greater say in classroom proceedings would be evident. The time spent on, and the depth of dealing with students' mathematical contributions could also be indicative of an increased value placed by a teacher on her students' mathematical contributions. Changes in the views and practice of a teacher "might foreshadow future changes in learning outcomes for students" (M. S. Smith, 2003).

One of the personal drives behind this study was my dissatisfaction with my own teaching. As an experienced high school mathematics teacher, I came to a point in my career where I did not know how to add more value to what I taught my students. I knew the curriculum inside and out and felt pretty confident in my ability to teach it in a manner that made the mathematics understandable to enough students. Although I was also reasonably confident at predicting and dealing with most student misconceptions and typical errors, I had a sense that not all students were benefiting equally from my methods of dealing with their misconceptions and errors. I had also designed a few interesting mathematics lessons along the way to break up the monotony of the rest of the curriculum which I taught in a very traditional manner. I sensed that I still had much to learn and had not yet *arrived* at total competence in mathematics teaching. I felt more of my students could arrive at better and more flexible mathematical understanding as well as a more practical application of the content in their daily lives. In an effort to improve my teaching, I read about teaching techniques and strategies. I also used technology such as PowerPoint and the Internet in my classroom. Yet I still felt more should be done to improve the learning that occurred in my mathematics classroom.

Another personal drive behind this study was dissatisfaction with my professional development efforts. As the head of a high school mathematics department consisting of fourteen teachers, I had similar aspirations for my colleagues' teaching that I had for my own teaching. I wanted my colleagues to improve their teaching just as I wanted to improve my own. I sensed that teaching was not captivating most of them enough and witnessed many of them leave the profession as a result. But just as I did not exactly know what to do to improve my own teaching, I was unsure of what to do to help my colleagues improve their teaching. Furthermore, I did not know how to convey to them the few suggestions I had in an effective manner. I saw the effective professional development of my colleagues as one of the variables that could add value to the job with which I had been entrusted but I did not know how to do this. So I ended up feeling extremely unproductive with the time I allocated toward professional development. Some of the professional development efforts I tried included biannual classroom visits with subsequent debriefing sessions, weekly department meetings with an attempt to focus on research on teaching, and an annual three-day professional development and planning meeting.

This study was designed to meet the abovementioned two professional challenges as well as to begin to help fill the void in research on the effects of coaching on teaching (Russo, 2004). Research on the effectiveness of collaborative coaching would not only benefit people who were interested in professional development but since it would focus on promoting effective teacher actions and healthy teacher views, it would also benefit every classroom teacher. I believe that this type of collaborative coaching would improve the teaching of all participants, including the one who would be the main drive behind it, such as the head of a mathematics department. The design of the study was such that the professional development it might promote would be sustainable by the head of a high school mathematics department as part of his or her regular duties. Details of such practicalities and the differences between this study and a real-life high school mathematics department will be discussed under the *Implications* section in Chapter Five.

CHAPTER TWO

REVIEW OF RELATED LITERATURE

The following review will look at literature on teacher knowledge, professional development, and theories of teacher learning. I believe these categories are essential in designing and implementing effective professional development of high school mathematics teachers. These categories also informed and influenced me greatly in designing the collaborative coaching used in this study.

Teacher Knowledge

Introduction

Many scholars have divided up the teacher knowledge domain. Schwab (1978) divided teacher knowledge into (a) content knowledge, (b) substantive knowledge, (c) syntactic knowledge, and (d) beliefs. Grossman (1990) first divided teacher knowledge into (a) subject-matter knowledge, (b) general pedagogical knowledge, (c) pedagogical content knowledge, and (d) knowledge of context and later (2001) into (a) knowledge of content, (b) knowledge of learners and learning, (c) knowledge of general pedagogy, (d) knowledge of curriculum, (e) knowledge of context, and (f) knowledge of self. Shulman (1986) first divided teacher knowledge into (a) content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge and later (1987) into (a) content knowledge, (b) general pedagogical knowledge, (c) curriculum knowledge, (d) pedagogical content knowledge, (e) knowledge of learners and their characteristics, (f) knowledge of educational context, and (g) knowledge of educational ends. More recently, Ball and various colleagues (2005) divided teacher knowledge into two main categories namely (a) subject matter knowledge and (b) pedagogical content knowledge. They subdivided subject matter knowledge into common content knowledge (CCK), specialized content knowledge (SCK), and knowledge of the mathematics horizon while they subdivided pedagogical content knowledge into knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of curriculum. For reasons specified in the theoretical framework later in this chapter, I preferred the latest division by Ball and her colleagues (2005) and referred to the knowledge teachers have and use in their teaching of mathematics as MKT.

Literature on Types of Teacher Knowledge

The following subsections analyzed various teacher knowledge categories that have been discussed, described, or researched in teacher knowledge literature.

Content Knowledge: The mathematics knowledge that is taught in secondary, undergraduate and graduate mathematics courses is often broadly called content knowledge (Ferrini-Mundy, Floden, McCrory, Burrill, & Sandow, 2003). This body of knowledge includes knowledge of the notation, facts, features, concepts, operations, definitions, axioms, and theorems of mathematics (Kennedy, 1990).

Knowledge of Internal Connections: Teachers possess and should be encouraged to expand their ability to make connections between topics within the curriculum in order to enhance student understanding and interest –internal connections (Ball et al., 2001; Carpenter et al., 1988; Grouws & Schultz, 1996). Silverman (2005) described depth of understanding as a teacher’s ability to connect topics with those of greater conceptual power while breadth of understanding was the ability to connect a topic with topics of less or equivalent conceptual power. It seems instinctive that a teacher must possess the ability to make these deep and broad internal connections before he or she can help students make them. Brophy (1991) discussed the organization of teacher knowledge and said: “Where [teachers’] knowledge is more explicit, better connected, and more integrated, they will tend to teach the subject more dynamically, represent it in more varied ways, and encourage and respond fully to student comments and questions.” (p. 352) One of the participants in a study done by Fernandez (1997) described wanting her students to realize that “every problem is not an individual mountain to conquer” (p. 22) by helping students make connections within mathematics. Ball and Bass (2003) wrote that teaching “involves making connections across mathematical domains, helping students build links and coherence in their knowledge.” (p. 12). Ball (2003) said “teaching requires an awareness and understanding of fundamental mathematical connections” (p. 4) and clarified this notion with an example of multi-digit multiplication that required a connection to place value, geometric representation, and polynomials. Shulman (1986) explained that it was expected of teachers to know which topics were central to a particular section of mathematics and which topics were somewhat peripheral while Schifter (1998) stated that once a topic was introduced, it was never left behind, but was rather revisited again and again.

Knowledge of External Connections: Related to the ability of teachers to make internal connections is their ability to make connections between the curriculum and real-life. Teachers possess and should be encouraged to expand their ability to make connections between the curriculum and real-life applications in order to stimulate student interest in mathematics – external connections (Ball, 2003; Shulman, 1986; Silverman, 2005). Sherin (2002) described a teaching scenario where the concept of slope was introduced by connecting it to the steepness of stairs. In this example, it became very apparent that understanding of the mathematical concept of slope was enhanced when the teachers connected it to a real-life situation that was familiar to students. McNamara (1991) called for a sophisticated understanding of mathematics and its interactions with other subjects by teachers. Although most high school mathematics students would not understand all of the specific mathematics involved, it is important for them to know that mathematics is used to put people on the moon, solve mysterious crimes, cure debilitating diseases, drive powerful computers, optimize economic systems, analyze stock market performance, and unravel the mysteries of DNA (Steen, 1988). It would be a powerful discovery to realize that mathematics is not a human invention without purpose but a way to consistently and systematically quantify activities in nature and to describe characteristics of creation. Where connections between the school curriculum and real-life can be made, every effort should be made to do so as it could only enhance the learning experience.

Knowledge of Different Mathematical Representations: A variety of literature discussed teachers' knowledge of mathematical representations and their ability to move flexibly among different representations. Teachers should possess, and constantly expand their ability to flexibly move among different representations of mathematical problems to broaden student experience and enhance learning (Ball & Bass, 2003; Ball et al., 2001; Grouws & Schultz, 1996; Hill et al., 2005; Hill, Schilling, & Ball, 2004). I refer to mathematical representation as a way "to generate an instance, specimen, example, or image" of the mathematics (Stump, 1997, p. 2). Even (1993) pointed out weaknesses in the ability of preservice teachers to think about the algebraic representations of functions and the nature of the graphical representations of functions and concluded that "powerful content-specific pedagogical preparation based on meaningful and comprehensive subject-matter would enable teachers to teach in the spirit envisioned in the Professional Standards for Teaching Mathematics (National Council of Teachers of Mathematics [NCTM], 1991). Fernandez (1997) pointed out that a variation in representation can be used to clarify student misconceptions, while Ball and Bass (2003) stated that different representations could make mathematical ideas more available to students. Shulman (1986) wrote “since there are no single most powerful forms of representation, the teacher must have at hand a veritable armamentarium of alternative forms of representation” (p. 9). The skill of carefully choosing representations appropriately and mapping between them carefully is becoming evermore important as various technologies such as spreadsheets, graphing tools, and data analysis software enhance teachers’ capabilities to move between varieties of representations of problems comfortably. These different representations afford students with different opportunities to gain mathematical insight and understanding (Ball, 2003; Even, 1990).

Knowledge of Variety in Teaching Options: Literature described teachers’ knowledge of the variety of options that could be used to teach a single topic. Teachers should possess the skills, and constantly expand their skills, to teach any topic with such a variety of methods that it optimizes the chances to resonate with the maximum number of students. Both Shulman (1986) and Ball (2003) emphasized this teacher skill as an essential ingredient in effective teaching and high-quality student learning. Sherin (2002) illustrated how a teacher expanded her knowledge of steepness and commented that “this is precisely the sort of knowledge that could be helpful to her in future attempts to teach the lesson” (p. 142). Teachers often possess a variety of teaching options for any particular topic and make daily choices about which option to use in the classroom. They should be encouraged to think about, and justify, the choices they make. In cases where their choices for certain topics are limited, an expansion of choices would be advantageous.

Knowledge of Common Student Misconceptions and Errors: A large portion of the teacher knowledge literature has examined teachers’ knowledge of student errors. Teachers should possess, and constantly expand their knowledge of typical student errors. A variety of studies (Ball, 2003; Ball & Bass, 2003; Ball et al., 2001; Carpenter et al., 1988; Hill et al., 2004; Shulman, 1986) referred to knowledge of typical student errors as an essential teacher knowledge skill. According to L. Steffe (personal communication, March 7, 2006) specific attention should be paid to the ways students think and the schema they possess that served as the origin of typical student errors. Teachers with sufficient knowledge in this area would be aware of forthcoming student errors and could act constructively in one of various ways. They could be proactive by treating the cause of the error before it occurs. This could be done by reviewing essential basic tools before introducing a new skill – essentially adjusting the schema of the student. They could also skillfully and naturally allow the error to come to the fore and then expose its inconsistency with the students. They could generate counterexamples or do a simpler example containing the same mathematical principle to highlight the discrepancy (Fernandez, 1997). This knowledge goes hand-in-hand with an awareness of the topics that students find particularly difficult and those topics that take up a central position in a standard high school curriculum (Ball & Bass, 2003). Here the content knowledge of a teacher will play a significant role in identifying these specific topics and organizing learning experiences around them. These errors could be brought about by typical student misconceptions or by typical careless student mistakes. A teacher should be able to distinguish between the two types of errors since they each require a different set of possible responses. Teachers should have knowledge of the conceptual and procedural knowledge students bring to the classroom (Carpenter et al., 1988) and have a repertoire of strategies to deal with these typical student errors.

Knowledge of Students' Mathematical Contributions: Teacher knowledge literature has emphasized knowledge that should allow teachers to attend to students' mathematical contributions in a manner that will enhance learning. This teacher knowledge quality played a central part in the design and execution of this study. Teachers possess and should be encouraged to expand their ability to attend to students’ mathematical contributions in a constructive manner (Ball, 2003; Fernandez, 1997; Hill et al., 2005). The Professional Standards for Teaching Mathematics (NCTM, 1991) emphasized the importance of students’ mathematical contributions and wrote “students are more likely to take risks in proposing their conjectures, strategies, and solutions in an environment in which the teacher respects students’ ideas, whether conventional or standard, whether valid or invalid.” (p. 57). Teachers should therefore encourage classroom discussions over students’ mathematical contributions and allow students to debate and justify statements. This would create the environment advocated by the Professional Standards for Teachers of Mathematics (NCTM, 1991) in which students would feel safe to say what they think and would encourage students to think originally. Teachers need to have a variety of knowledge skills in order to elicit and support such original student thinking. Schifter (1998) remarked that if teachers were to build their instruction around student thinking they would need the skill to interpret students’ mathematical contributions and the skill to challenge students to extend and revise their ideas so that they could become more powerful mathematical thinkers. Ball and Bass (2003) presented an example of non-conventional student strategies to multiply 35 and 25. They made it clear that teachers’ ability to solve mathematics problems did not necessarily equip them to deal constructively with students’ mathematical contributions. They stated further that “learning to size up other methods, determine their adequacy, and compare them, is an essential mathematical skill for teaching” (p. 13). Teachers also need to be able to take students’ answers on written tests and consider the validity of each of those “mathematical contributions” consider presentations of different textbooks for validity and applicability, and change numerical parameters within problems to create high-quality homework and classroom problems (Ball et al., 2001). The mere student question of "why?" calls daily on teachers to warrant claims and justify procedures used in mathematics (Shulman, 1986).

Pedagogical Content Knowledge (PCK): In 1986, Shulman (1986) coined the term pedagogical content knowledge to refer to a specific kind of teacher knowledge. He wrote: "I still speak of content knowledge here, but of the particular form of content knowledge that embodies the aspects of content most germane to its teachablity." (p. 9). According to the *Webster's Third New International Dictionary* (1993), *pedagogy* means the "art" or "science of teaching". PCK therefore means the knowledge to teach content artfully or scientifically (in other words well). It is the knowledge that connects content knowledge with the act of teaching and to the student. PCK is that special blend of content and pedagogy of which a teacher of mathematics can never have enough. Although Marks (1990) described PCK as the quality that should distinguish laypeople and mathematicians from mathematics teachers, the MKT construct by Ball and her colleagues (2005), which I support, classified specialized content knowledge, which fell outside their classification of PCK, should also distinguish teachers. When teaching or planning to teach a mathematics lesson, teachers must harness all their relevant content knowledge while keeping the student and his or her learning firmly at the center of their thinking. They must ensure that their understanding of the content gets transformed into an act that will consider the cognitive processes of their students and will optimize their learning (Sherin, 2002).

Literature on Teacher Knowledge and its Effect on Student Learning

Various scholars have studied the relationship between the components of teacher knowledge and student performance by looking at a range of categories (Begle, 1972, 1979;

Monk, 1994). For the sake of organizing the following section I grouped the literature according to the type of teacher knowledge they involve. I used the MKT construct by Ball and her colleagues (Ball et al., 2005) to differentiate between these studies. I was able to classify certain studies into common content knowledge (CCK) and knowledge of content and students (KCS) specifically but then used the general term MKT for cases where the teacher knowledge quality mentioned in a study did not necessarily pertain to one specific subdivision.

Common Content Knowledge: Some of these teacher knowledge measures were variables such as the number of college-level courses taken, teachers' scores on student tests, and teachers' scores on high school graduation tests. Begle (1972) found a positive main effect between the number of teachers' advanced mathematics courses and student achievement in 10% of students and a negative main effect in 8% of students. Harbison and Hanushek (1992) administered the same fourth grade mathematics test to students and their teachers in Brazil and found the difference in student achievement was linked to teacher scores. Mullens, Murnane, and Willett (1996) used teachers' scores on the Belize National Selection Examination as comparative variable and concluded that students learned more mathematics when their teachers did better on this test.

Knowledge of Content and Students: Carpenter et al. (1988) found a strong relationship between teachers' ability to predict their own students' success and their students' subsequent achievement. In contrast to this finding they did not find a strong correlation between teachers' ability to predict the strategies of their own students and their students' subsequent achievement.

MKT: Both Monk (1994) and Begle (1979) found that mathematics methods courses had a greater impact on student learning than pure mathematics courses. In 1996, the National Commission on Teaching and America's Future (NCTAF) released a report claiming that 90% of the variation in student achievement in reading and mathematics can be attributed to teacher qualifications (National Commission on Teaching and America's Future, 1996). The American Council on Education (1999) also reported: "A thorough grounding in college-level subject matter and professional competence in professional practice are necessary for good teaching….The data are unequivocal: students learn more mathematics when their teachers report having taken more mathematics" (p. 6). In fact, at this point this assertion is taken as fact as demonstrated in Fennema and Franke's chapter in the *Handbook of Research on Mathematics Teaching and Learning* (1992) which opened by declaring: "No one questions the idea that what a teacher knows is one of the most important influences on what is done in classrooms and ultimately on what students learn" (p. 147). Rowan, Chiang and Miller (1997) reported a positive correlation between measures of teachers' use of mathematical knowledge and student performance. Ball and Bass (2003) commented that teachers' agility with content affected the quality of their teaching, while Grouws and Schultz (1996) wrote "providing teachers with certain types of pedagogical content knowledge results in teachers changing their classroom practice in ways that result in increased student learning" (p.444). In her book *Knowing and Teaching Elementary Mathematics* Ma (1999) introduced the concept of profound understanding of fundamental mathematics and indicated the discrepancy between the profound understanding of fundamental mathematics of United States and Chinese teachers. Although not specifically stated, it was implied that this discrepancy was reflected in the discrepancy in comparative test scores between the two countries. Hill et al. (2005) reported that "teachers' content knowledge for teaching mathematics" (p. 396) was a significant predictor for students' gain scores.

A variety of authors reported interesting practical findings when teachers lacked the necessary MKT. Millet and Johnson (1996) described how teachers relied heavily on textbooks and other commercially developed material for their lessons when they experienced a lack of content knowledge. Aubrey (1997) wrote about the difference between Teacher D who had sufficient content knowledge and Teacher C who lacked adequate content knowledge. Teacher D set up explorations and worked toward mathematical relationships while Teacher C was unable to develop effective explanations and questions. Carlsen (1990) investigated the content knowledge of four teachers and found that they talked longer and relied on questions with low cognitive demand when confronted with unfamiliar topics to teach. This finding was confirmed by Brophy (1991) who wrote: “Where [teachers’] knowledge is limited, they will tend to depend on text for content, de-emphasize interactive discourse in favor of seatwork assignments, and in general, portray the subject as a collection of static factual knowledge” (p. 352). Leinhardt, Zaslavsky and Stein (1990) also wrote: “Limitations on subject matter knowledge…often reduce the flexibility and creativity of a teacher as well as create a kind of authoritarianism toward the subject and student that permits little or no exploration of ideas” (p. 46).

Despite the positive results and comments made by some of the studies mentioned above, scholars also offered warnings about linking all types of teacher knowledge to student performance. Begle (1979) stated that the popular notion of teachers’ content knowledge being sufficient to enhance student learning needed drastic modification. Monk (1994) and Begle (1979) reported a threshold of five courses for the impact of mathematics courses on student performance. Since most high school mathematics teachers would have more than five college level mathematics courses, it implied that it would be a futile exercise to attempt to improve student learning only by improving the number of courses teachers took. Rowan et al. (1997) suggested that the effects of teachers on students’ mathematical achievements could be explained not only by teachers’ ability, but also by their motivation and work situation. It could be said that common content knowledge such as the knowledge used in the studies by Begle (1972; 1979) and Monk (1994) might be seen as necessary but not sufficient knowledge for mathematics teachers.

Studies like those done by Begle (1979) and Monk (1994) were indications that it would be incorrect to assume that all types of teacher knowledge have an effect on student learning. Although it is intuitive to want to link a variable like number of college-level mathematics courses of the teacher to student performance, understanding content is not correlated to one's ability to teach it to others (Kennedy, 1990). Furthermore, the knowledge a teacher obtains from taking a mathematics course remains the possession of that teacher. It is uncertain how, and if, this knowledge can cause gains in his or her students' achievement. This is especially true if the mathematics course taken by this teacher was unrelated to school mathematics. It is not possible to assess the quality of the knowledge that was obtained by this course and its applicability by the teacher. Knowledge obtained through rote learning would be different from knowledge obtained from courses that were designed to bring about a comprehensive and articulated understanding of mathematics (Even, 1993; Hill & Ball, 2004). The drive to reform mathematics education by enhancing the quality of student learning requires higher quality mathematics knowledge by the teacher (for example knowledge of the connections between pieces of mathematics rather than isolated knowledge of the individual pieces), not more mathematics knowledge (Schifter, 1998). Typical college level mathematics courses tend to give future teachers more mathematics knowledge but do not enhance their deeper understanding of the mathematics for the sake of teaching it better to students. The opportunity to develop this deep understanding must be provided by high quality professional development (M. S. Smith, 2003) or high quality initial teacher training. An example of initial teacher training that was focused on developing the content knowledge of student teachers was the work of Pat and Alba Thompson on the concept of functions (Thompson & Thompson, 1996).

Although some studies have urged caution in attempting to link teacher knowledge and student achievement, others have recently fueled an enthusiasm shared by many about improving student performance and our ability to influence the variables that do so. Through development of reliable instruments (Ferrini-Mundy & Senk, 2006; Hill et al., 2004) that can measure teacher knowledge in a non-threatening way, we are starting to obtain the ability to assess teacher training and professional development efforts on the terrain where it matters the most – student performance. When feedback comes from studies such as the ones done by Carpenter et al. (1988) and Hill et al. (2005) where researchers linked teacher knowledge to student performance, we get new conviction in our efforts to design professional development courses for in-service teachers that will enhance their knowledge and their students' learning. Since teacher knowledge that could affect student learning is not enhanced by taking more mathematics content courses (Fernandez, 1997), the onus is on mathematics education departments and other professional development providers to provide these learning opportunities for in-service teachers.

Professional Development

The professional development part of this literature review highlights literature which in itself analyzed professional development methods, foci, and content. It was important for the purposes of this study to synthesize criticism against elements of professional development in order to incorporate its good qualities into the design. To conclude this section, I looked at video and its role in the professional development of in-service teachers.

Enough authors have written essays (Hawley & Valli, 1999; Wilson & Berne, 1999) or have done research (Garet et al., 2001; Halai, 1998; Kazemi & Franke, 2003) on professional development of teachers to suggest that traditional in-service programs have limited effect on the teaching and learning of teachers and students. Some of the significant and recurring criticisms against professional development of the recent past were that it was conducted (a) by *outside experts*, (b) in a limited amount of time, (c) without regard to the importance of teacher beliefs, (d) without consideration for modern theories of learning, and most importantly (e) without a clear focus on teacher knowledge. Any professional development exercise must consider these five points to have any hope of affecting the teaching and learning of mathematics.

The outside expert: Traditional professional development has often been conducted by an outsider who acted as the expert who possessed universal answers (Kazemi & Franke, 2003; Lord, 1994; Wilson & Berne, 1999). This "expert" transmitted prepackaged information to a passive audience without regard to its practicality and specific applicability. Professional development of this nature suggested that the keys to effective teaching lie with someone other than the practicing teacher. Furthermore, a bigger criticism against the outside experts was not so much that they were outsiders or that they were experts but that they often used questionable instructional techniques. When prepackaged information was delivered to teachers without engaging them in thinking about mathematics and learning for themselves, it seriously limited the possibility of changing teacher practice and student learning. So outsiders and experts can conduct professional development as long as they use methods of instructional delivery that are in line with reform-oriented practices.

Inadequate time, inappropriate location, and building community: Traditional professional development has sometimes been a sporadic and off-campus activity conducted during the summer vacation. These activities often had no follow-up and teachers were left by themselves until the next summer workshop a year later. Mathematics teachers were also taken away from the familiar setting of their own school building and away from the team of colleagues they worked with every day. This approach neglected the importance of community building during professional development (Lachance & Confrey, 2003). To build up a solid teacher community is an important effort and something that takes a lot of time (Garet et al., 2001; Grossman et al., 2001). Since this study involved collaborative coaching and will ultimately advocate for the use of it for professional development purposes, community building would be an important ingredient in the success of the coaching.

Teacher beliefs: Traditional professional development rarely considered teacher beliefs in the reform of mathematics teaching. This was a major oversight since it is generally viewed as an important variable in teacher change (Thompson, 1984; Wilson & Berne, 1999). Thompson (1992) found that teachers interpreted new information through their old beliefs while Pajares (1992) cited numerous works that claim beliefs “screen[s], redefine[s], distort[s], or reshape[s]” new information (p. 249). The futility of professional development that did not acknowledge the existence of beliefs such as “teaching is telling, knowledge is facts, and learning is recall” (Cohen, 1987) became glaringly obvious. Teachers who hold such beliefs might, for example, want to go back to their own classrooms and teach their students some of the new things they have learned at professional development (i.e. and interesting new way to solve a particular problem) rather than trying to elicit problem solving techniques in their students that they were facilitated to utilize. Ball and Cohen (1999) encouraged professional developers to create “a sufficiently high level of cognitive dissonance to disturb in some fundamental way the equilibrium between teachers’ existing beliefs and practices on the one hand and their experience with subject matter, students’ learning, and teaching on the other” (p. 23). Professional development could be designed to explicitly address teachers’ beliefs, or it could implicitly make attempts to influence teacher beliefs in ways that will result in improved teaching and learning. For example, professional development could explicitly address the importance of students' mathematical contributions in the classroom and address constructive ways to deal with these contributions, and it could implicitly place students' contributions at the center of some of its activities. Both of these choices would address teachers' beliefs and might facilitate change in teachers' actions around the issue of students' mathematical contributions.

Teacher and student learning: The major goal of professional development is the improvement of student learning (Ball & Cohen, 1999; Loucks-Horsley & Matsumoto, 1999; M. S. Smith, 2003). Some of the minor goals of professional development could include invigorating teachers' careers, building professional learning communities, and boosting teachers' morale. To be successful at the major goal of increased student learning in the mathematics classroom, students should be given greater control over their own learning and be held more responsible to pursue their own understanding of the mathematics. In the same manner, teachers at teacher training must be given the opportunity to control their own learning and pursue new knowledge (Little, 1993; Wilson & Berne, 1999). When teachers act as the keepers of knowledge in mathematics classrooms, it affects their teaching strategies and creates a rigid environment that does not encourage learning by students. In the same way that teachers must avoid being the keepers of all mathematics knowledge in their classroom, professional developers must avoid being the keepers of knowledge on effective teaching during teacher training sessions (Simon, 1997).

Teacher Knowledge: Guskey (2003) examined 13 different lists of effective professional development qualities. These lists were all published within the last decade and contained those professional development qualities that had the biggest impact on student learning. From these lists he compiled a new list of 21 characteristics most frequently mentioned. Among the factors mentioned were that professional development should be school-based, allow for sufficient time and other resources, and promote collegiality and collaboration. The ingredient listed by the most professional developers was the enhancement of teachers' knowledge. Guskey (2003) wrote "Helping teachers to understand more deeply the content they teach and the ways students learn that content appears to be a vital dimension of effective professional development" (p. 749). Cohen and Hill (2000) found that when controlling for student characteristics, average mathematics achievement was higher in schools where teachers were involved in extensive professional development that focused on specific content knowledge compared to schools where teachers had not. A focus on content knowledge during professional development also had a positive impact on bringing about change in teacher practice (Garet et al., 2001; Grouws & Schultz, 1996). This change involved a move toward teaching practice such as the practice advocated by the Professional Standards for Teaching Mathematics (NCTM, 1991). Kennedy (1998) examined the content of professional development programs that studied their own impact on student learning. She found that programs focusing on teachers' knowledge had a greater impact on student learning than those focusing on general teaching strategies and behaviors. M.S. Smith (2003) wrote "professional development must provide teachers with the opportunity to improve their understanding of mathematics content and to reflect critically on their learning experiences" (p. 42).

Although the studies which examined various teacher knowledge qualities and student performance that were mentioned above did not necessarily use the particular divisions for mathematical knowledge for teaching in the same manner as Ball and her colleagues (2005), and therefore did not specify the specific teacher knowledge quality they were addressing according to this framework, they all referred to parts of the MKT domain. I also attempted, in the instances that were clear to me, to place some of the literature into the MKT categories in the section *Literature on Teacher Knowledge and its Effects on Student Learning* above. It was clear from this literature that teachers needed strengths in various aspects of the MKT domain to enable their students to develop conceptual understandings about mathematical topics (Borko, 2004; Ma, 1999). Professional development that focused on the enhancement of teachers' MKT seemed to have overwhelming support and had great potential for effectiveness.

The three important foci provided by Sykes (1999) for effective professional development summarized this section well. These points were:

- Professional development should engage teachers simultaneously in learning about subject matter itself, teaching of the subject matter as an intellectual and scholarly endeavor in its own right, and ways that students learn the subject.
- Professional development should ground its content, in part at least, in the content of the student curriculum, in order to establish a direct link between teacher and student learning.
- Professional development should attend to the specificity of teacher learning as it applies to the concepts, ideas, topics, and skills required of students. Teachers should be engage in intellectual work around these building blocks of the student curriculum at relatively specific levels of concentration.

Video: Ever since video equipment became more affordable and accessible it has been utilized successfully in the professional development of mathematics teachers. One of its earliest uses was in micro-teaching (Sherin, 2004) where video was used to model effective teaching by scaling it down in terms of instruction time, class size, or preferred instructional strategies.

Because micro-teaching through video clips relied on a behaviorist view of teaching to train and develop teachers it is no longer a preferred method of professional development. More recently the concept of a video club has emerged as a popular method to employ the characteristics of video as a professional development tool (Sherin, 2003). In these video clubs, participants watched video clips from the classroom of one of the participants and discussed the content of the clips in a variety of ways. Some of the research results from this work included that teacher shifted their primary focus away from the teacher toward students (Sherin & Han, 2004), they developed increasingly complex ways to examine student ideas (Sherin, 2004), and they moved from an evaluative to an interpretive approach in their analysis of video clip content (Sherin & van Es, 2005). Another example where video clips was used to facilitate a form of professional development was where Sherin (2002) examined the ways in which teachers modify their content knowledge and develop new "content knowledge complexes" (p. 149) while teaching a novel six-week section on linear functions and watching video clips of their teaching during video club meetings. The most interesting and relevant research result with video clips for the purposes of this study was Sherin and van Es' (2006) report which found that teachers began focusing more on their own students' mathematical contributions during class as a result of focusing on it more during video club discussions. The question remained for me whether teachers would attend better to their own students' mathematical contributions and whether their views on teaching and learning would change while working on their own MKT during collaborative coaching.

Theory of Teacher Learning

Since the combination of teacher knowledge literature and professional development literature suggested it was a worthwhile endeavor to focus on teacher knowledge, and specifically MKT, during professional development of mathematics teachers, careful consideration must be paid to theories of teacher learning. If the ways in which teachers learn new knowledge and behaviors were not considered, respected, and built into the professional development design, the professional development would have little impact on teacher knowledge.

In order to improve the likelihood that teachers would develop new knowledge they should be brought to a point where they develop a dilemma (or disequilibrium) (M. S. Smith, 2000; Wood et al., 1991) or experience cognitive dissonance (Ball & Cohen, 1999). The dilemma or disequilibrium should occur because of a discrepancy between a teacher's existing knowledge and new situations facilitated by professional development that require different knowledge. This dilemma-induced conflict would serve as a catalyst for productive teacher learning (M. S. Smith, 2000). If professional development could induce disequilibrium within teachers regarding their current mathematics knowledge, their current views on teaching and learning, or their knowledge about their students' mathematical knowledge, the teachers would be more prepared, willing, and open to learning. According to Schifter and Fosnot (1993) and Simon and Schifter (1991), teachers who had the opportunity to examine their current knowledge and beliefs about mathematics and student learning, exhibited changed beliefs which in turn affected their decision making in class, when their current knowledge and beliefs were brought into disequilibrium with new information. The teacher in the study by Wood, Cobb, and Yackel (1991), for example, encountered a dilemma between allowing students the freedom to express their thoughts and providing them with official procedures. In her case, learning took place through the "give answers versus value student thinking" dilemma (M. S. Smith, 2000, p. 354).

She realized that students produced rich and valuable mathematical thoughts and shifted her focus away from her own teaching toward the mathematics of her students.

Teacher learning would be optimized by using relevant curriculum in the setting of the teacher's personal classroom (M. S. Smith, 2003; Wood et al., 1991). This statement corresponded with earlier criticism against professional development that occurred at an off-campus location. Moreover, by situating teacher learning in the daily practice of teachers, they had a greater chance of developing knowledge that was relevant to the practice of teaching (M. S. Smith, 2003). Although it could be beneficial to look at commercially produced classroom videos, for example, it would be best to let teachers examine their own practice by analyzing video of their own teaching (Loucks-Horsley et al., 1998). This would assist in making other variables such as the curriculum, specific blend of students, community, and school administration relevant and applicable to the specific professional development exercise. It would also assist in grounding the discussion of abstract ideas in practice (M. S. Smith, 2003). Furthermore, M.S. Smith (2003) argued that sound teaching practice would not flow from discussing theory but that good theory and general principles would flow from closely examining practice. It is through the reflection that goes with examining authentic classroom practice that teachers would learn and change their practice (M. S. Smith, 2003). When teachers were asked to reflect on their current practice and were simultaneously provided with new experiences, they constructed new knowledge that lead to learning and new behaviors (M. S. Smith, 2000; Wood et al., 1991).

When it came to learning new mathematics content or expanding current content, teacher learning happened, just as it did with students, through direct exposure to the content by means of engaging in problem-solving exercises (Loucks-Horsley et al., 1998). Exploring mathematics provided teachers with the opportunity to consider issues of student learning and to construct their own understanding of the mathematics. Furthermore, teachers often made connections that they have never considered before, broadened their understanding of the specific and related topics, considered a variety of solution paths to a certain topic, and grew in their confidence to teach mathematics well (Loucks-Horsley et al., 1998; M. S. Smith, 2003).

Theoretical Framework

Throughout this chapter I have cited and quoted numerous sources proclaiming the importance of teacher knowledge as a variable that holds the key to teacher change and improved student learning. I believe it is important to end this chapter with a vivid description of MKT as the construct for the teacher knowledge domain because it forms the theoretical framework for this study.

The theoretical framework of MKT as conceptualized by Ball and her colleagues (2005) formed the basis for this study and assisted me directly in its design and implementation as well as indirectly in the analysis and interpretation of the collected data. Initially I intended to use the construct proposed by Shulman (1986), and I specifically wanted to focus on the two sections of content knowledge and PCK. I ultimately felt that both these sections were, however, too broad and that they did not distinguish within themselves among different types of important and less important teacher knowledge qualities. For example, from the production-function literature mentioned earlier (Begle, 1972, 1979; Monk, 1994) one got the idea that content knowledge, as a whole, was obsolete as far as its ability to predict student learning. Yet all along, I felt that teacher knowledge qualities such as knowledge of different mathematical representations, knowledge of the content surrounding a specific piece of mathematics, and knowledge of mathematical errors (not typical student errors) were important issues and were either content knowledge or closely related to content knowledge. A practical example related to this issue that most teachers have witnessed is how confused students can become if their parents who work in mathematical fields such as engineering try to explain their mathematics to them. This is a typical example of the teacher (the parent) possessing ample CCK but lacking SCK and knowledge of the mathematical horizon and learning being constricted by it. The construct by Ball and her colleagues (2005) afforded me the opportunity to distinguish between certain less important content knowledge issues (CCK) and important ones that would add value to the teachers' teaching and students' learning (SCK and knowledge of the mathematical horizon).

As far as my choice for the more refined PCK division (Ball et al., 2005) rather than Shulman's (1986) broader division, I found it appealing to have this important knowledge type classified from two perspectives. KCS essentially being PCK from the student's perspective while KCT being PCK from the teacher's perspective. I felt that this distinction was necessary after witnessing the frustrating differences between teachers, who often focused on KCS, and professional developers, who often focused on KCT, during professional development. This distinction enabled me to ensure that the focus of specific collaborative coaching sessions fell on the correct PCK section of teacher knowledge. In cases where I felt inadequate and unsuccessful after our coaching sessions, I often found that it was because I tried to focus on KCT while Cindy focused on KCS.

Ball and her colleagues (2005) conceptualized mathematical knowledge for teaching into two main categories; subject matter knowledge and pedagogical content knowledge. They subdivided subject matter knowledge into common content knowledge (CCK), specialized content knowledge (SCK), and knowledge of the mathematical horizon. They subdivided pedagogical content knowledge into knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of the curriculum.

CCK was defined as the knowledge and mathematical skill expected of any well-educated adult and included practical skills such as recognizing student errors, using mathematical notation correctly, and being able to do assigned student homework. This portion of mathematical knowledge for teaching could be seen as the part that used to broadly be called content knowledge and the type of knowledge that formed the basis for the production-function literature. It was also the part that Shulman called “subject matter knowledge” in his original classification (1986).

Ball and her colleagues (2005) defined SCK as the knowledge and mathematical skill needed by those who teach mathematics to others and included practical skills such as analyzing student errors, evaluating alternative ideas, giving mathematical explanations, and using mathematical representations. Although these qualities describe knowledge needed by a mathematics teacher, the students’ and their influence on teacher knowledge remain on the background for SCK. SCK is applied purely as a specific kind of knowledge about the mathematics of an error, idea, representation, or explanation without regards to student interaction with them.

Knowledge of the mathematical horizon was described as the knowledge and skill that enabled a teacher to make connections between different mathematical topics. Some of the practical implications of such knowledge would be where students display weaknesses from previous sections of mathematics, which Silverman (2005) called breadth of knowledge, or where they show a curiosity regarding the destination that their current mathematics might take them, which Silverman (2005) called depth of understanding.

On the PCK side of the construct, KCS was defined as content knowledge intertwined with knowledge of how students think about, know, and learn particular content and included practical skills such as anticipating student errors and misconceptions, understanding and anticipating students' common computational strategies, and predicting what tasks students might find interesting and manageable. Relative to Shulman's original division (1986) KCS could be seen as a subset of PCK, with PCK itself a subset of MKT. With respect to content knowledge, Hill, Ball, and Schilling (in press) did find, through factor analysis done on items answered on an MKT testing instrument and subsequent interviews with teachers, a distinct difference between KCS and content knowledge. This result confirmed the existence of KCS as an independent part of the MKT construct.

KCT, another subset of PCK, was defined as the knowledge that combined content and teaching and included practical skills such as sequencing content, recognizing pros and cons of different representations, and sizing up students' novel mathematical contributions.

In conclusion, I would like to elaborate on the importance of this theoretical framework to the preparation and execution of this study. First, to ensure that this study added value to the current body of literature on teacher learning and change through professional development, it was essential to place the focus of the collaborative coaching on a construct that was generally valued by the mathematics education community, one that was current, and one that had promise for success in teacher learning and change. Second, instead of being distracted by (sometimes very luring and important) pure content or pure pedagogical issues and knowledge during professional development, MKT directed me toward specific knowledge characteristics such as KCT. Furthermore MKT helped me to ensure that Cindy and I were focusing on the same teacher knowledge category. Third, as an experienced high school mathematics teacher, I had specific things I wanted to learn (teacher knowledge qualities) to improve my own teaching and wanted to focus on these teacher knowledge qualities of my professional development participant. MKT placed these practical teacher knowledge qualities into a framework that assisted me to see them in relation to other important teacher knowledge qualities. In doing so, MKT again enabled me to focus on the ingredients for the collaborative coaching sessions and to see those ingredients in relation to each other and to other irrelevant distracters. MKT as the theoretical framework for this study also had certain limitations. More detail about these limitations will be provided in the *Limitations* section of Chapter Five.

CHAPTER THREE

METHODS AND METHODOLOGY

Participant Selection

I have been involved with Dawson High School where Cindy taught for almost three years. I acted as a supervisor for university undergraduates during their teaching field experience at Dawson High School while Cindy acted as their mentor teacher. During the summer of 2006, I also conducted a one-day professional development event as part of a three-day summer professional development initiative held for all Dawson High School teachers. At the same time as the student teacher supervision, I was also involved in research on professional development that involved the Dawson High School student teachers, mentor teachers and myself. As part of the research and my work to support student teacher development, I held regular cluster meetings with the student teachers and their mentor teachers. During these meetings, we focused on improving the MKT of all teachers involved. Each participant had the task of bringing examples to the meeting of situations, student questions, or mathematics content that could serve as a catalyst for discussions on MKT. During this time two teachers, Cindy Miller and Belinda Daniels, impressed me with their dedication to the meetings and with their commitment in always bringing relevant material to the discussion. Cindy was also particularly excited to share student questions and interesting mathematics content with me during my classroom visits to her student teachers. We often had interesting discussions that focused on mathematics, ways to teach topics better, and students' understanding of specific content. Cindy was always very reflective when our university research demanded that each mentor teacher keep a journal during the student teacher field experience. Brenda was a teacher who changed to teaching from a different previous career and was a very innovative and dedicated teacher. When the time came for me to select a participant, the decision to select Cindy over Brenda was based on her superior teaching experience as well as her willingness to talk about her thoughts on mathematics and mathematics teaching. I also saw in Cindy a participant who would be willing to discuss her practice, open to look critically at her own practice, and open to change some of her thoughts on teaching and learning.

Data Collection Methods and Design of the Study

Data sources for this study include forty 55-minute classroom videos, fifteen one-hour collaborative coaching session videos, one one-hour teacher interview video, field notes from all forty observed lessons, and forty seven pages of hard copies from Cindy's work during the coaching sessions. The forty classroom videos were made up of twenty videos taken during Cindy's fourth period lesson and twenty taken during her sixth period lesson. Cindy taught Gifted/Honors Algebra II during both these lessons. She taught the two classes in the exact same manner, using the exact same preparation and making only minor adjustments from the fourth period to the sixth period. Each class consisted of 22 students. Two sources of video formed the bulk of the data, video from Cindy's lessons and video from our collaborative coaching sessions. For the purposes of Institutional Review Board as well as Grayson County approval, Cindy, all students and their parents signed consent forms. Data collection was conducted from 31 August 2006 until 21 January 2007, interrupted with a month-long break during December 2006. I used one video camera with two cordless microphones strategically placed in the classroom. After slight initial sound problems, the sound produced from these microphones was of extremely high quality. From the back of the classroom I often heard student comments through my headphones that Cindy did not hear from the front of the class. In a few cases, the equipment picked up interesting student comments that Cindy missed and that would have been lost if I did not use such high quality equipment or was just sitting, listening, taking field notes and observing in class.

I videotaped Cindy's two lessons on a Monday or Tuesday and stayed to do a coaching session with her that same day after school. During the videotaping, I would focus on Cindy, her teaching and interactions with the whole class as well as individuals within the class. As soon as something interesting came up during class, I wrote the time and a very brief description of the event in my field notes. When I got home after the day, I immediately digitized the videotapes and made backups on the hard drive of my computer. During the next day or two I carefully watched the video of each lesson and constructed a lesson graph of each lesson. A lesson graph is a time-stamped chronological layout of the events during each lesson, including relevant mathematics, student questions, and small copies of actual pictures taken from the video clip and pasted onto the document. The lesson graphs assisted me to carefully plan the next coaching session which would then typically take place on the Thursday or Friday of the same week. On this day, the schedule of videotaping fourth period, videotaping sixth period, and doing a coaching session after school would repeat. In between each day of data collection, I watched videos of the coaching sessions and wrote lesson graphs for each of them.

I selected video as the medium through which I would conduct the collaborative coaching because of its unique qualities of being flexible in pausing and rewinding clips, and its ability to reproduce what happened in the classroom accurately (Le Fevre, 2004). Another benefit of watching video was that it was found to be a motivator for teachers in professional development (Sherin, 2004). The video clips I included in the coaching sessions, called "video records of practice" (Le Fevre, 2004, p. 238), were mostly focused on students' mathematical contributions. This quality, according to Seago (2004), was one of the essential components of video clips that were used for the professional development of mathematics teachers. I selected them on the basis of their potential to bring forth interesting mathematical discussions (Seago, 2004) and to inspire mathematical conversations from a variety of directions (Le Fevre, 2004). Clips with this quality also afforded me the opportunity to adhere to advice from the teacher learning and professional development literature cited in the previous chapter. Here we saw that professional development should contain authentic content (M. S. Smith, 2003), focus on the MKT of participants (Guskey, 2003) facilitated the pursuit of new knowledge (Little, 1993; Wilson & Berne, 1999), and create a dilemma for the participants (Ball & Cohen, 1999; M. S. Smith, 2000). I also intuitively felt Cindy would benefit from working on mathematics brought forth by students' mathematical contributions as it would assist her in seeing the benefits of carefully attending to these contributions.

Data Analysis

I applied the constant comparative method (Glaser, 1965) as data analysis method for this study. I followed the four steps that form part of this method of analysis namely (1) comparing incidents applicable to each category, (2) integrating categories and their properties, (3) delimiting the theory, and (4) writing the theory.

My first data analysis action happened during data collection when I was videotaping each lesson. I had headphones on during the whole lesson and could clearly hear almost everything said in class. As soon as an event such as a student question, comment or conjecture drew my attention, I wrote down the time and a short description into my field notebook. When time did not permit it, I merely wrote down the time and the abbreviations SQ and SC for student question or student comment and TQ and TC for an interesting teacher question or teacher comment. These cryptic time-stamped field notes helped me immensely later on as I looked at each classroom video.

The next process was to look carefully at each classroom video after it was digitized and saved onto the hard drive of a university computer. While watching each video I kept my field notebook open next to me to make sure that I did not miss any of the events that initially attracted my attention during class. It was surprising to see how many additional events caught my attention during the viewing of each classroom video that I missed during class.

As I watched each video, I carefully constructed its lesson graph. I also made sure that I drew attention to specific events by adding personal comments, suggestions and critical questions on each lesson graph. When the lesson graphs of the most recent data collection were complete, I printed them out and read through each to pick up on common themes that evolved between the two or more lessons. Most of the time, preparation for the next coaching session involved the most recently recorded fourth and sixth period lessons (two to three days earlier) on the same day and therefore covered the same content, teaching, and method of instruction.

In my field notebook, I developed themes and headings as I read through each lesson graph, calling this activity an initial coaching session draft. With access to only the limited amount of data often consisting of only two lessons, and the lesson graphs as analysis tool, I started constructing categories (step one of the four-step process of constant comparison) and placing events in them (Glaser, 1965). I watched each relevant piece of video clip a few times to decide its relevance and theme. As I began to see each event in relation to the rest of the available dataset, I refined each lesson graph by adding comments, reminders, timestamps and references to other similar events. During this activity, I started grouping events together as common themes according to the theoretical framework described in Chapter Two, while other individual events either confirmed their individual importance to the study or convinced me that they were unimportant (step two of the four-step process of constant comparison) (Glaser, 1965). I then drew up a final draft of coaching session content in my field notebook. I time-stamped each entry or group of entries more precisely, clearly marking beginning and ending times for each video clip, in order to create smooth and effective playback during coaching. I then watched each selected clip repeatedly, constructing questions to go with each clip during coaching and writing these down in my field notebook. As a final preparation for the coaching session, I went through the session exactly as I planned it, playing every clip sequentially, and rehearsing every question I wanted to ask Cindy. I carefully made final logistical and administrative notes, in red ink, regarding important things such as the location of the question (before or after the clips) or personal comments I wanted to add during the coaching. So by the time I was ready for a coaching session most of the initial classroom data analysis was done.

Throughout the data collection process, I watched the video of each coaching session and constructed a lesson graph for each. Apart from doing timestamps and writing general descriptions of our work, I also added comments, suggestions, and time stamped references to other sessions and classroom events. Watching the coaching sessions enabled me to bring up previous coaching session events in future coaching sessions which created an important continuum.

At the end of the data collection process, I printed out all classroom and coaching lesson graphs and compiled them into one complete booklet. I carefully read through these, making notes on them and highlighting relevant pieces. Having the hard copies of all lesson graphs at my disposal enabled me to look for incidents that were of specific relevance and which belonged to specific categories. I constantly watched pieces of video, making notes, adjusting lesson graphs, confirming or refuting threads, and adding or removing episodes to and from common threads. At this point the third step of the four-step process of constant comparison process of delimiting theory (Glaser, 1965) started to occur as I began to hone in on less categories and developing reasons for specifically choosing fewer categories. An example of a category that at this stage did not survive the analysis is where I planned to look at incidents when Cindy and I worked on specific MKT activities during our collaborative coaching sessions and to link these to subsequent classroom events that became a positive spin-off from our work. I realized that such a category would only be applicable in a classroom where topics were dealt with for longer and more open-ended problems were part of the curriculum. A longitudinal study of Cindy's classroom and her teaching would have made such a category possible. At this point in the data analysis process, I started developing theory that explained the chosen categories, step four of the four-step constant comparison method (Glaser, 1965). This theory became more explicit as I began thinking about, and writing down, the skeleton for the results of this study. It involved the fact that I started seeing the effects of the collaborative coaching on Cindy's views and actions and attributing the effects specifically to the MKT focus. More details on this theory will be presented towards the end of Chapter Four.

Description of Cindy and Her Environment

Cindy had a total of fifteen years of teaching experience. One of the strengths of her experience as a teacher was that she had taught in a large variety of schools. She had experience in a classroom with a total of 12 struggling students, a small school with a graduation rate of 30%, 8^{th} grade mathematics, science, and P.E. in an alternative school, team-teaching a special education mathematics class, and all the regular mathematics courses ranging from Math Money

Management to Algebra I through AP Statistics. She was also extensively qualified academically with a master’s degree in mathematics education as well as the gifted certification program for her state.

Cindy expressed a preference for teaching juniors as an age group (also the group that she taught while I did my research with her) since they seemed the most focused on their work and on getting good grades to apply to college. She preferred and enjoyed a teaching day that involved teaching a variety of ability levels and content areas. Her strategy to teach students in the lower levels of mathematics was showing students her commitment to them and making mathematics fun for them. As far as teaching gifted students (those that she would be teaching while I did my work with her), Cindy had the following interesting comment during our initial interview:

> I like teaching my gifted kids, but some…I like their questions they have a lot of times, because they seem to think outside the box a lot of times.

Focusing further on Cindy’s initial comments and views on student involvement in class, Cindy had the following to say regarding answering student questions (at 44:10):

> What your job is in that classroom is to almost be like their coach, making sure that…walking around, making sure that they are actively engaged, asking those very…important questions, when they ask you a question, asking a question back to them, to where they are working toward the math.

I also asked Cindy to tell me about the factors that will make her feel like a successful teacher at the end of the year. This was part of her answer:

> If the kids start questioning themselves, if the kids can learn how to question what is going on…if they just start posing some 'what if's?', if everybody start posing 'what if… what if I did that? What would happen?' Because isn't it what learning math is all about?

Since this study focused on the change in Cindy's views and actions that were evident from the manner in which she thought about and attended to students' mathematical contributions, her abovementioned comments were significant. Her actions highlighted in section D of Chapter Four would illustrate that Cindy still had room for improvement to align her actions with these thoughts. As will be illustrated in Section E in Chapter Four it became clear that Cindy did make some progress toward actions that were more in line with her views expressed above.

Cindy's views on teaching and learning and her teaching actions could be described as traditional in the way it was defined in Chapter One. She has developed a typical classroom routine of checking homework, introducing new work without much input from students, doing one or two examples on the board with only low-level questioning, allowing students to practice one or two examples on their own, and starting with the homework if time permitted. Very rarely did Cindy deviate from this pattern to give students opportunities to expose their mathematical thinking through classroom discussions and debates. There was one specific day during our work together that gave me an indication of how Cindy thought about deviations from traditional teaching. On this day, Cindy and I planned some kind of interesting, interactive activity for students around a specific mathematical topic. As I entered the room for the first time on the day this was going to be done, Cindy announced: "We are not going to do that activity today. We need to review for tomorrow's test today and I feel that students have to really learn." Since our planned activity involved the same content as the upcoming test, this statement confirmed that

Cindy preferred traditional teaching and believed that standard review by her was more beneficial to her students than a more reform-oriented activity on the same content.

Cindy maintained a high level of discipline and created an environment in which every student had an opportunity to concentrate and learn at all times. Students often did their homework from the previous day on the board and were then responsible for explaining their work to their peers. This practice, however, seldom led to rich discussions between students, likely because Cindy typically resolved all student questions.

Logistically Cindy taught in four different classrooms during the five months I did research in her class. These changes were brought about by teacher changes and illnesses as well as Dawson High School adding a new building to their school. Furthermore, Cindy and her family moved into a new home during the time of my research. She also became severely ill for at least a week during the time of my research in her classroom. Although these changes and big events did cause some stress in Cindy's life, I did not see them impacting negatively on her teaching (compared to her teaching from the previous two years) and do not believe they affected her participation during our collaborative coaching sessions negatively.

It is very important for the purposes of this study to understand the system in which Cindy taught. Understanding the system will put all the episodes described in Chapter Four as well as my description of Cindy's progress into perspective. Two external variables played a large role in the mathematics that got taught at Dawson High School. First the state required each of its schools to teach a Quality Mathematics Curriculum. This QMC curriculum could rightfully be described as "a mile wide and an inch deep" (Schmidt, Houang, & Cogan, 2002). Grayson County in which Dawson High School was situated required each of its schools to teach a set of MKS's (mathematical knowledge and skills). Since Grayson County prided itself as "a system of world-class schools", their MKS's covered the state-mandated QMC's but went even above and beyond them. So it was very understandable that mathematics teachers in a school such as Dawson High School felt a lot of pressure to teach what they had to teach in the little time they had. When I asked Cindy about reform oriented teaching methods such as groupwork she had the following to say:

> If the state of (name of state) went around and actually redid their curriculum, like they say they are going to, we are going to spend more time and be in depth on things; yes we could actually do more group work. Because then you can investigate things a little bit more. But sometimes to investigate takes such a long time to investigate stuff that…you look at the benefits that you get and the time it has had to take out for you to get that benefit, you have to kind of weigh your results.

This was just one of many instances where Cindy provided me with information that described the pressure she felt as a classroom teacher.

Dawson High School provided a highly organized environment in which to teach mathematics. During every summer the whole subsequent year was precisely set out with every section of work assigned to be done on a specific day of the following year. The exact homework that students would do for each section of mathematics was also planned beforehand. This was then handed to students at the start of each new chapter. The yearly schedule even catered specifically for upcoming public holidays, teacher workdays and other missed teaching days and looked mostly the same every year. This was confirmed by the following comment made by Cindy during coaching on 26 October (at 26:00), "I do not remember feeling this rushed on this day last year."

Within this rigid system, Cindy was also the group leader of the three teachers who taught the Gifted/Honors Algebra II course in which I did my research. She was responsible for weekly meetings with her two colleagues. During these meetings, Cindy and her two colleagues spoke about what sections of the current work to emphasize, what sections of the work to include on tests, what content should be tested next, how students did on the previous test, and who will be writing the next test etc.

All the abovementioned variables resulted in extreme pressure on teachers because of a congested curriculum in this Gifted/Honors Algebra II class. From the first to the last day of the data collection process in Cindy's class, there was not a single day or significant portion of a day that could be spent on unplanned mathematics. The school and county planning required from Cindy to often do two elaborate sections of mathematics in one day. Needless to say, Cindy was under a lot of external pressure to do what was prescribed without room for improvisation on her part. This pressure was superbly summarized by the following comment Cindy made during our 24 October coaching session (at 18:10)

> I hate to say that some days, like today when I did not have a lot of time, I am just…'gosh I just hope Mandy does not ask a ton of questions'…I know that that is the wrong…but here in Grayson County curriculum, I am just hoping today Mandy did not ask a lot of questions.

From what I witnessed during my time at Dawson High School before this research, I believed that Cindy had strong self confidence about her teaching and good self efficacy. The fact that she eagerly agreed to participate in my research seemed to confirm my belief even further. I was therefore very surprised to find that Cindy initially displayed discomfort and nervousness about the content of the clips I showed from her lessons. She seemed anxious that I was going to show her a clip where she made a mathematical error or taught content incorrectly. After a while I felt that I had convinced her that this was not my intention and I could see her relax more during our sessions. Yet, one of our excellent sessions did give her confidence a knock and I sensed her nervousness return until the end of my research. It was the 11 October coaching session concerning Cindy and her class developing a formula for the *y* coordinate of the vertex of a parabola. This episode was described in detail in Section A4 in Chapter Four. Somehow, they developed a formula for all quadratic functions where $a = 1$ and hence obtained the formula $y = \frac{4ac - b^2}{4a^2}$. When they tried to use this formula for quadratic equations with $a \neq 1$ in class, it did not work. This obviously worried and confused Cindy. So when I touched on this issue during our coaching, I believe Cindy saw it as an attempt to "fix her mistake". Although my intensions were totally different from what she believed, the event seemed significant and Cindy could never relax about this issue during future coaching sessions. She often commented right before I played a video clip, "I wonder what I did wrong this time."

All first names, school and county names, as well as official names of state and school curricula used in this study are pseudonyms. In the case of student names, I tried to always makes sure to specifically mention when Justin (in sixth period) and Mandy (in fourth period) were involved since I believe it was generally significant and interesting to see these two students' extensive involvement in classroom discourse as well as their mathematical creativity. I chose to write down my own first name in transcriptions of discussions between Cindy and me because I thought of myself as Cindy's colleague during the coaching, not as an interviewer, teacher, the sole coach (with Cindy as the coached), or the researcher.

CHAPTER FOUR

DATA ANALYSIS AND RESEARCH RESULTS

The research questions for this study were:

1. In what way did collaborative coaching that specifically focused on Cindy's MKT affect her views on the teaching and learning of mathematics?

2. In what ways did Cindy's attention to the students' mathematical contributions change over time as a result of collaborative coaching?

In this chapter I will answer the two research questions of this study by offering data that supported claims that Cindy experienced growth in her views on the teaching and learning of mathematics (research question 1) as well as growth in her attention to students' mathematical contributions (research question 2). The chapter was divided into the following sections:

A. Examples of collaborative coaching sessions between Cindy and me. The data consisted of transcribed collaborative coaching session clips supported by hard copies of all mathematical work.

B. Baseline data consisting of comments by Cindy regarding her views on mathematical teaching and learning. The data consisted of transcribed collaborative coaching session clips.

C. Evidence of growth in Cindy's views on mathematics teaching and learning (Research Question 1). The data consisted of transcribed collaborative coaching session clips.

D. Baseline data where Cindy deals with students' mathematical contributions. The data consisted of transcribed classroom teaching video clips.

E. Evidence of growth in how Cindy attends to students' mathematical contributions (Research Question 2). The data consisted of transcribed classroom video clips.

(Please see Appendix I for a complete summary of the exact dates and times of all episodes described in Sections A-E. The dates, specifically, will help to place the coaching sessions and lessons in relation to each other.)

The rationale for these specific sections was that by reading through Section A one could get a clear picture of the collaborative coaching and the content and features that I believe assisted in bringing about change in Cindy's views and her practice. Then by reading Sections B and C consecutively, one could get a clear picture of a sample of Cindy's old and new views. By reading Sections D and E consecutively, one could get an understanding of her old and new practices. At this point it is important to emphasize that the intervention of only fifteen sessions did not bring about huge change but rather subtle shifts in how Cindy viewed teaching and learning (research question 1). The shifts in Cindy's attention to students' mathematical contributions (research question 2) were less subtle but did not occur as often as one would like. To reiterate, research question 1 was answered by comparing the baseline data presented on some of Cindy's views in Section B with the change reflected in the data reported in Section C. Research question 2 was answered by comparing the baseline data presented in Section D with the evidence of change in the way Cindy attended to students' mathematical contributions presented in the data in Section E. This chapter then concluded by linking the changes to the collaborative coaching sessions generally and specifically to the MKT focus of these coaching sessions.

A. Examples of Collaborative Coaching Sessions

The following six collaborative coaching sessions Cindy and I had all centered around mathematical contributions from students in her two Honors/Gifted Algebra II classes. They described mathematical discussions and work that focused on problematic issues around these students' mathematical contributions and in one instance (Section A6) a focus on an interesting section from the curriculum Cindy taught. The examples represented just a sample of the many interesting mathematical discussions Cindy and I had throughout our fifteen coaching sessions. Some of our discussions did not lead to mathematically satisfactory solutions and certain discussions were certainly more productive than others. I do feel that all of the coaching sessions focused on relevant curriculum in the setting of Cindy's classroom (M. S. Smith, 2003; Wood et al., 1991) and often contained content that caused necessary perturbations in both of us (M. S. Smith, 2000; Wood et al., 1991). These qualities adhered to the teacher learning literature mentioned in Chapter Two and optimized the possibility of having Cindy's view on teaching and learning and her attention to students' mathematical contributions affected by the coaching sessions.

A1. Mandy asked, "Can you solve three variables with only two equations?"

This classroom episode took place during a lesson in which Cindy taught students to use matrices to solve linear equations. She described the process of substituting missing variables with zeros in the coefficient matrix. Mandy's question was:

> So if you can put a zero in for whenever there is not a variable, does that mean you can solve systems of equations with only two equations with three variables if you add another equation that has zeros in for all of them?

As can be seen in the description of this episode in Section D4 below, Cindy seemed unable to mathematically justify or refute this question by Mandy during class. She also attended to Mandy's contribution in a traditional manner by keeping control of the conversation, asking no further questions, and involving no other students in the conversation. To explore the issue further and expand our MKT during the coaching session, I gave Cindy the following two equations each with three variables (see figure 1) $x+y+z=6$ and $x-y-z=-4$ where $x=1; y=2; z=3$. We then explored Mandy's question with the following conversation:

Cindy: I do not know if I will be able to do it.

Danie: No, I want us just to…I have selected values…minus four…and now *x* is one, *y* is two, *z* is three. So they work, those two equations. So now she is saying: 'Okay so put in zeros' and then is thinking now it should be solvable because now you have three equations with three variables.

Cindy: Well is it? (laughs nervously) Must I solve it? If I put zero *x*, zero *y*, zero *z* equal to zero?...Well that…what kind of…but that does not give me any information though.

Danie: Okay, but let us draw up the matrix equation of that then.

Cindy: (writes down the matrix equation of coefficients $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ -4 \\ 0 \end{bmatrix}$– (see figure 1)… We would have to take, we take…and multiply A inverse by B (she writes $\begin{bmatrix} x \\ y \\ z \end{bmatrix}=A^{-1}B$) to get this by itself… Now it depends… are we going to have a determinant that is not workable over here? Is the determinant zero here?

Danie: Yes.

Cindy: And so therefore it cannot be solved.

Danie: That is where it breaks down.

Cindy: (surprised) Oh! That is good!

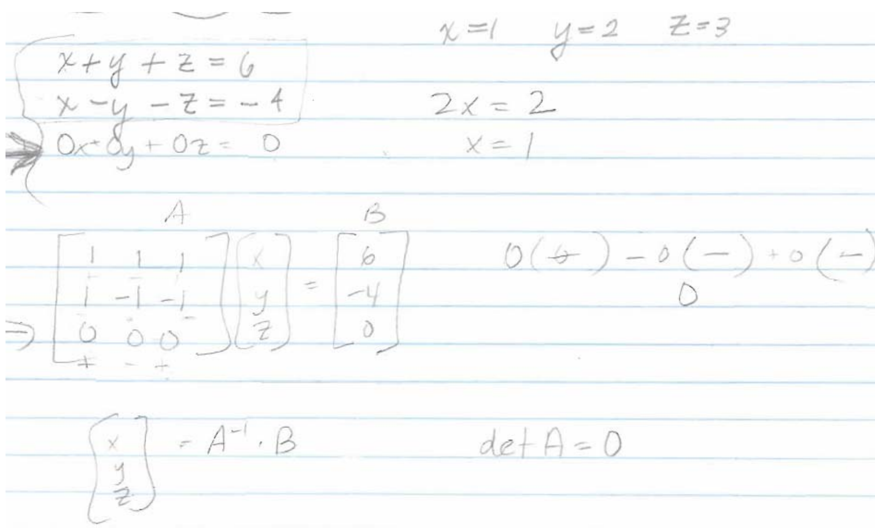

Figure 1.

In this episode we saw that Cindy was able to come up with a mathematically justifiable answer to a student question she had a hard time answering in class. It was, however, clear from the coaching session that she needed certain prompts in order to get to this answer. In cases like this the benefit of colleagues focusing together on a common task through collaborative coaching became clear and provided opportunities for Cindy's growth. With regard to MKT as it was presented as the theoretical framework for this study, I believe Cindy expanded her knowledge of the mathematical horizon during this episode. She was now able to combine prior knowledge on issues such as that three variables usually needed three equations to be solved, determining inverses of matrices, and calculating determinants of matrices and combining this knowledge with her knowledge on the current topic of solving linear equations with matrices.

A2. Duncan asked: "What is the relationship between the *y* intercept of a parabola and the *y* intercepts of its two linear parts?"

During this classroom episode, students were investigating the relationship between the *x* intercepts of two linear equations and the *x* intercept of their product. Surprisingly, Duncan became interested in the relationship between the *y* intercepts of these parts rather than the *x* intercepts and asked Cindy about it. I brought this episode to our coaching session because of the manner in which Cindy struggled to deal with this student question in class (see Section D5 below for a vivid description of Cindy's struggle with Duncan's question). During the discussion that followed Duncan's question in class, Cindy promised to look at the issue of the relationship between the *y* intercepts of the graph of a quadratic function and the graphs of its two linear components. She was not sure at the time what this relationship was and by her own admission did not revisit the issue up to the day of our coaching session five days later. Because of Cindy's struggle with Duncan's mathematical contribution in class, I chose it to be the focus for the discussion during this coaching session. The example we worked with in the following discussion was $y = x - 2$ and $y = x + 3$ and the subsequent quadratic function $y = (x-2)(x+3)$:

Danie: … *x* minus two and $y = x + 3$. I mean in my ten years of teaching I have never thought about that. Just, I mean, it was a brand new question to me. Okay so now, I mean, the matter is now the *y* intercept.

Cindy: Mmm.

Danie: What is the relationship between those two *y* intercepts ($y = x - 2$ and $y = x + 3$) and the *y* intercept of the, of the combination of them. And it is very similar to this now (I point to the sketch on the paper of $y = 2$, $y = 3$ and $y = 6$ (see figure 2)). I mean… won't you, yeah, write the combination…and so what is its *y* intercept?

Cindy: It is going to be negative 6… Because it is when *x* is equal to zero.

Danie: Okay, so can…can you see now what the relationship is of the *y* intercepts of the two linears versus the *y* intercept of the, of the product?

Cindy: It is those two things multiplied together… And the reason why it happened on that every single time is because it was two and one. (Cindy is here referring to the classroom examples in which one of the two linear equations always happened to be one with the result that the quadratic function always shared its *y* intercept with one of the linear functions.) I was wondering about that because I kept noticing it was not one of them, it was just… it was not both of them, it was just the one. And I kept looking at the problem instead of looking down…you know, you are trying to teach and think at the same time… it is like teach them what you got to teach them that day but then I am thinking in the back of my mind going 'you know what, every single one of them was two and one' and I wonder if it was a coincidence or…

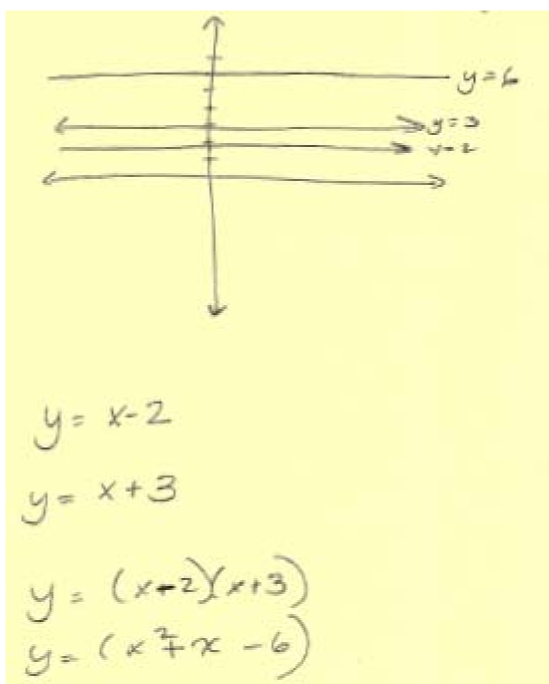

Figure 2.

With regard to the theoretical framework of this study, I believe Cindy expanded her MKT and particularly her knowledge of the mathematical horizon by working on this problem during collaborative coaching. She would now able to combine prior knowledge of algebraic manipulations such as $(x+a)(x+b)=x^2+(a+b)x+ab$ with the current issue of the relationship between the *y* intercepts of two linear equations and the *y* intercept of their product.

A3. Anthony commented: "The vertex is the one place on the graph where a *y*-coordinate has only one *x*-coordinate."

In this classroom episode, Cindy asked her class to make some comments about the vertex of a parabola. One of her students, Anthony, made the following comment:

> It (the vertex) does not have another point that has the same *y* value.

When Anthony made this unique statement in class, Cindy spent more time on it than usual by asking Anthony to repeat his statement and by explaining to all the students what Anthony meant. Furthermore, she confirmed the fact that she thought of the student contribution as unique by making the following statement directly after hearing it again during the coaching session:

> That is pretty good…and it came from him this time and not from Justin over here…and I even told Amy (a colleague) that afternoon, I said 'let me tell you what Anthony did today' and I drew the little picture up and she was like 'oh my gosh!'…so yeah I thought that was pretty neat…it was a very fresh way to look at it. And when we graph, I take it for granted and never say that… I have never had someone say, or me specifically say, 'this one right here is the only point on the parabola that has that *y* value'. There is no other point that will have that same *y* value. It is by itself.

So during the coaching session, I took the opportunity to explore some interesting mathematics contained in this unique statement with Cindy. Our quest was to mathematically prove the statement made by Anthony and my goal was to guide Cindy to not only discover the interesting mathematics contained in the statement but to re-affirm to her that she did have the mathematical ability to mine and explore such statements with her students during the lesson. I asked Cindy to use the literal expression for the *y* value at the vertex, $y = \frac{4ac - b^2}{4a}$, to prove that it produces a unique (and not two different) *x* value. She did so (see figure 3) and concluded:

> $x = \frac{-b}{2a}$, which is the *x* value at the vertex…it's unique…I am missing all this math, I am making all these errors, but when you come down here, it is just $x = \frac{-b}{2a}$ which we know that should be.

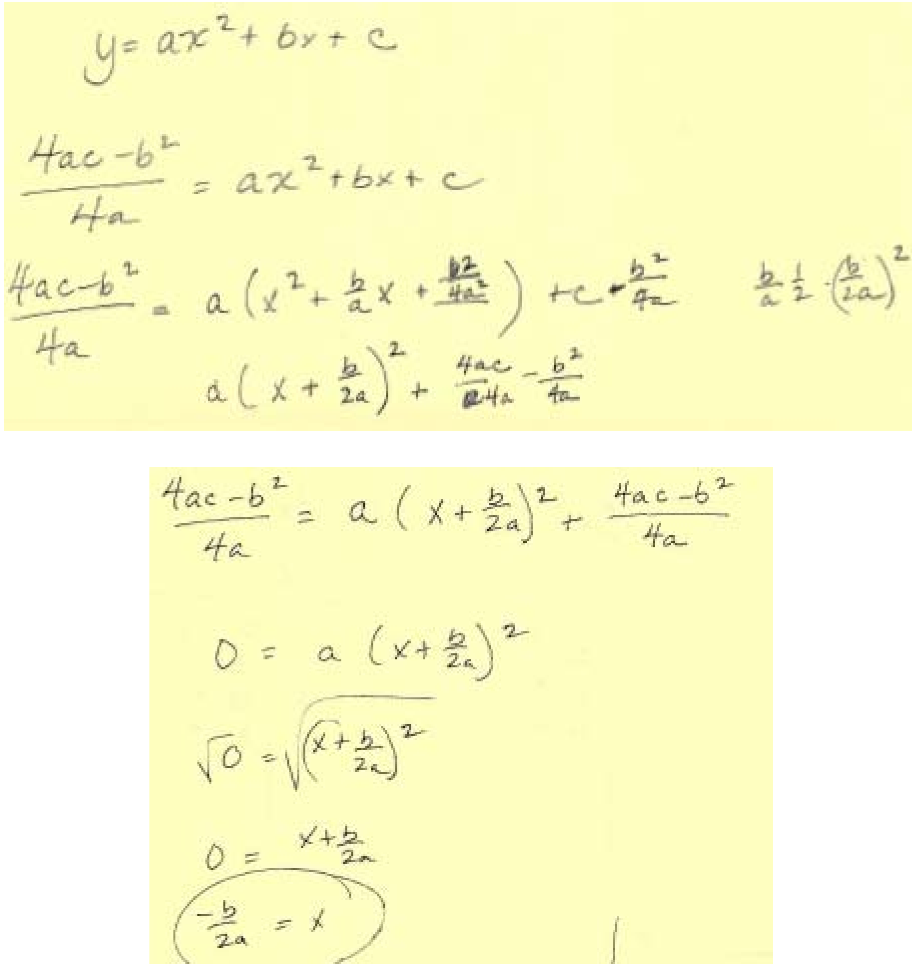

Figure 3.

Cindy continued to exhibit interest in this exercise and good knowledge of the mathematical horizon of this topic by relating our work on the vertex of a quadratic function to local maxima and minima of cubic functions (see figure 4). We discussed the fact that a local minimum of a cubic function was a place where the *y* value uniquely yielded two possible *x* values versus other *y* values that yielded either one or three possible *x* values. At the end of Chapter Four I made the claim that this discussion led to a unique set of actions by Cindy in which she exhibited a willingness to go beyond the boundaries of the school and county curriculum and spend substantial class time on a topic that was not prescribed. This unique action exhibited by Cindy linked changes in Cindy's teaching to the MKT focus of this study (an answer to research question 2), but more on this at the end of this chapter. With regard to the theoretical framework of this study, I believe Cindy acquired new KCT by looking at an alternative way to deal with this novel student contribution and discussing issues relating to it.

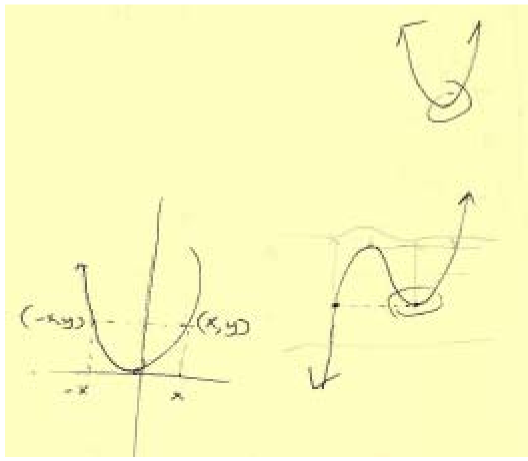

Figure 4.

A4. A discussion about Cindy's incorrect expression for the *y* value at the vertex.

Anthony's very interesting comment (described in A3 above) during Cindy's fourth period lesson about the vertex of a parabola that was unique because of the fact that it was the only point where a *y* value produced only one *x* value was made on 6 October 2006. Cindy and I discussed the mathematics of this matter during our 11 October 2006 coaching session. We showed how Anthony's statement could be proven by substituting $y = \frac{4ac - b^2}{4a}$ into $y = ax^2 + bx + c$, solving for *x*, and showing that it yielded only one answer, namely $x = \frac{-b}{2a}$. On 18 October 2006 Cindy and her students derived a formula for the *y* value at the vertex by completing the square on the standard equation $ax^2 + bx + c = 0$. From the line $\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$ in their work they deduced that the *y* coordinate at the vertex of a parabola was $y = \frac{4ac - b^2}{4a^2}$. The difference between the *y* coordinate we used on 11 October 2006 during coaching and this *y* coordinate (it had an extra *a* in the denominator) stimulated my curiosity and prompted me to bring the issue to the 20 October coaching session. When I introduced the topic, Cindy immediately responded that she and her students attempted to use the expression $y = \frac{4ac - b^2}{4a^2}$ in class and it did not work. She did not know what was wrong and was very adamant that she wanted a solution so that she could fix this mistake the next day in class. We started working on the problem of deciding between the different expressions at 07:55 (see figure 5):

Cindy: Is it because of this number up here? (Cindy circles the coefficient *a* of x^2)

Danie: I think so…I mean, that is what I thought because, whether it is…you see what I made you plug in [during coaching on 11 October 2006], and it worked, was just a 4*a*. And so if it is just…

Cindy: Oh that is it! Because you know what? Our number, our number that we came out with…our number that we…so we can go back into period 4 and clear that up because…We were doing one and we had come out with…oh gosh, let me think of the number. It had a two up here in the front and when we plugged it in up here (Cindy points to the formula) it was…exactly the number, the number down here at the bottom (of the formula) was exactly that number (the coefficient of the x^2) multiplied by the bottom. Like it might come out as the vertex was $\frac{7}{8}$ but we were getting $\frac{7}{16}$.

Danie: So it was wrong by…

Cindy: A factor of *a*.

Danie: Okay, so then, it confirms for me the one you and I used a week ago, which was this one (I point to $y = \frac{4ac - b^2}{4a^2}$) without the square…

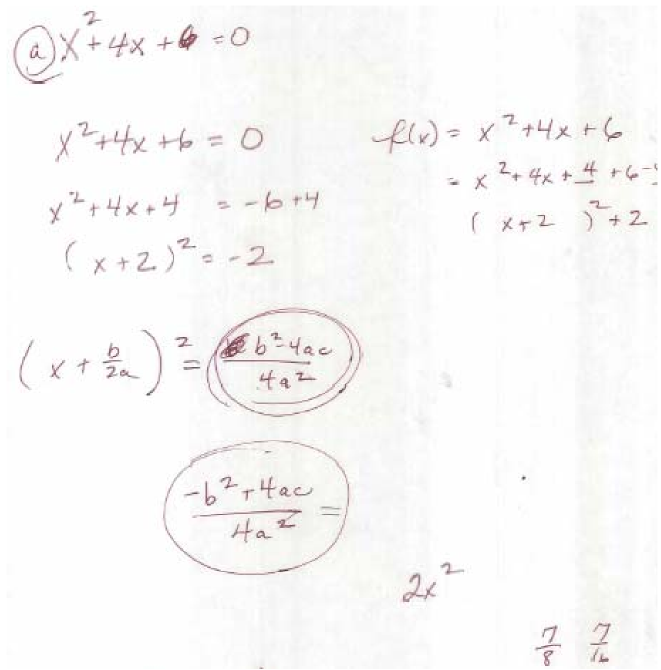

Figure 5.

I showed Cindy a piece of the classroom video in which they derived their formula. I wanted us to spot the place where the error came in and speculated that it was because of the coefficient of the x^2. As I got to the correct place in the video where Cindy could see her work on the laptop screen she made the following comment before I could play the clip:

Cindy: You know why, because look at that equation right there. When you divide that *a*, when you divide by *a* and now we are coming down here with all these…but see that one has already got a one (Cindy was referring to the classroom example that she used that had a coefficient of one for the x^2), so that one... But I think it would make a difference if it had the two.

Danie: Okay, okay, so that clears it up for me. The only mistake came in, that your numerical comparison to this (the literal equation $ax^2 + bx + c = 0$) was a special case in that it had a one in front of it.

Cindy: Yeah, I think so…

At this point neither of us was totally convinced about our conjecture. A little later on during the coaching session, Cindy worked on the numerical example $2x^2 + 4x + 10 = 0$ (see figure 6). We now saw that the coefficient of the x^2 had an effect on the *y* value at the vertex because the parabola changed when you divided through by it in the process of completing the square. Much later during this coaching session, and after we had been doing other things for at least 20 minutes, Cindy had the following to say:

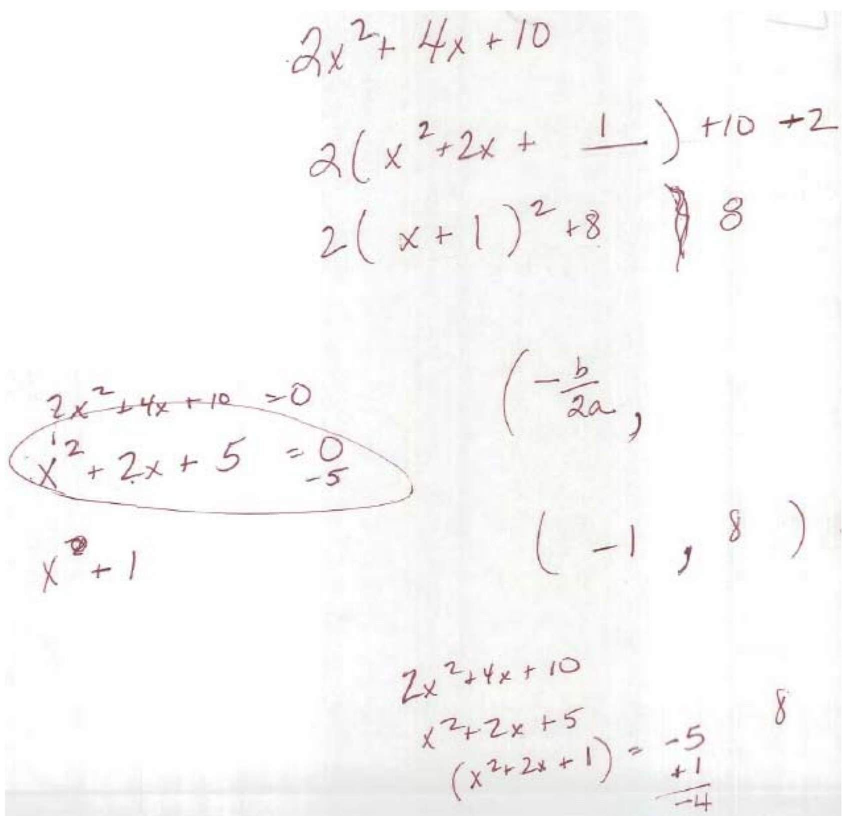

Figure 6.

Cindy: Let us see, with all the stretches and compressions we have been doing…

Danie: …because *a* …

Cindy: …makes it this way (Cindy shows a stretch with her hands).

Danie: Yeah, and it takes the vertex places, I mean it is compression or stretch. So that ties back into that.

So at this point (42:40 of the coaching session) we brought together the *y* coordinate at the vertex of a parabola and the fact that the *a* was responsible for stretches and compressions of the parabola. As far as the theoretical framework for this study goes I believe Cindy expanded her SCK as well as her knowledge of the mathematical horizon surrounding this issue. Her growth in SCK came as a result of inspecting and solving an apparent mathematical error and creating a clear explanation for herself about this error. The tie we made between this error and how it related to the stretching and compressing function of the coefficient of x^2 served to expand her knowledge of the mathematical horizon of this topic.

A5. Cindy algebraically proved that a parabola was symmetric.

During the fourth period lesson on 20 October, Cindy touched on the symmetry quality of a parabola on at least two occasions. On the first occasion (at 23:10), Cindy referred back to Anthony’s statement on 6 October and explained that the vertex was the only point where a *y* value yielded only one *x* value. For every other point on the parabola, every *y* value had two *x* values associated with it and that these values lie symmetrical around the axis of symmetry. After I played this clip to Cindy during our 24 October 2006 coaching session, I asked her whether she was confident that students understood the concept that a parabola was symmetrical or whether she would have liked to add something to this explanation. Her response was as follows:

> I think that they do know that it is symmetrical. I do not know if they fully understand it, but I think that they do understand that the right side is just like the left side.

After this incident, I played a clip within the same lesson (at 25:00) where Cindy again touched on symmetry in discussing the nature of the roots of a parabola, this time explaining that the two *x* intercepts of a parabola lie symmetrical around the axis of symmetry. Duncan immediately added a comment about another parabola that had no *x* intercepts because it had a positive coefficient for x^2 and the vertex lie above the *x* axis. Duncan's participation indicated that he was paying close attention to Cindy's explanation about symmetry. The reason why I played these clips to Cindy was to try to convince her also that Duncan seemed to pay close attention to everything that was going on in class that day. At 27:35 during this same lesson, Duncan asked:

> Is a parabola always going to be like perfectly even and everything? Like it is not going to be like, this side goes up like this [Duncan indicated the shape with his hands] and this side comes over or something?

Here we saw an example of the explanations and discussions during class not being sufficient for one student to understand the concept fully even though this student gave clear evidence of being attentive and involved during the discussion. This prompted me to look at the mathematics involved with Cindy during our coaching session in order to develop alternative methods to explain the concept of symmetry. Without even being prompted about the situation, Cindy immediately spotted the irony in the situation with her clear explanation earlier in the lesson, Duncan's involvement during a subsequent discussion, and his final question. She commented:

But he is talking about symmetry, it is funny. That did happen; I remember that that did happen that day…Maybe these tables, if we did these tables…

At this point in our coaching I wanted to know from Cindy how we could mathematically convince Duncan that a parabola was symmetric. Cindy thought for at least a minute and then came up with an algebraic proof. She showed that you will obtain the same *y* value if you picked two *x* values symmetrical to the axis of symmetry $x = h$ (Cindy chose $x = h + 1$ and $x = h - 1$, substituted these into $y = a(x - h)^2 + k$) and showed that it yielded the same *y* value of $y = a + k$ (see figure 7)). We also discussed the option to do this mathematical proof using the analytical geometry method for obtaining the equation of the loci of all points equidistant to a point and a line. This equation was a quadratic equation and the loci form a parabola, symmetric around the focus. The end result of this discussion was that Cindy had at least three different ways to handle future questions about the symmetry of a parabola. According to this study's theoretical framework, the MKT quality that Cindy enhanced through this episode was KCT.

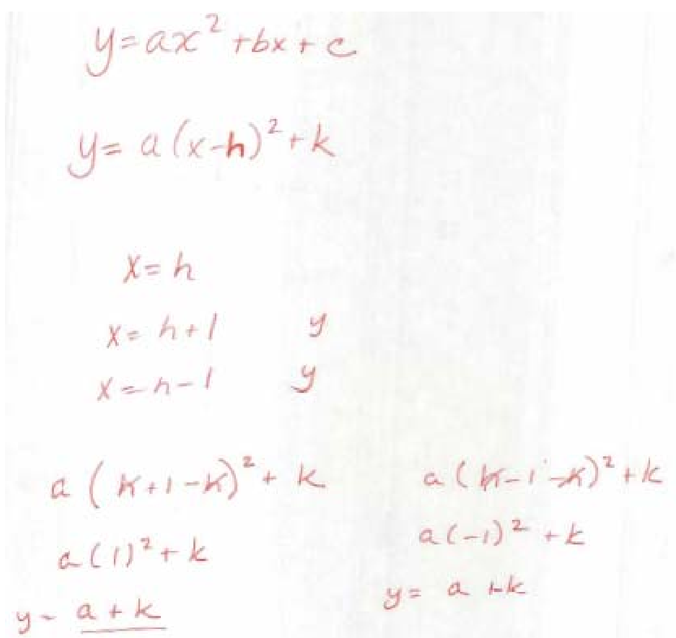

Figure 7.

A6. We looked at the explanation for why $\frac{x}{x+3}=\frac{6}{x-1}$ has two valid solutions but $\frac{x}{x+2}-\frac{8}{x^2-4}=\frac{2}{x-2}$ has only one valid solution.

During Cindy's lesson on the process of solving rational equations on 12 January 2007, I realized that she never focused on the details for why both solutions $x=9$ or $x=-2$ were valid solutions to $\frac{x}{x+3}=\frac{6}{x-1}$ but that $x=6$ was the only valid solution to the equation $\frac{x}{x+2}-\frac{8}{x^2-4}=\frac{2}{x-2}$ since $x=-2$ was a solution that should be discarded. Cindy took great care to draw students' attention to the fact that the process of solving rational equations might produce extraneous solutions while expressions containing no variables in the denominators never will. She meticulously went through the process of checking each solution with students, making sure they knew that $x=-2$ was discarded as a solution for $\frac{x}{x+2}-\frac{8}{x^2-4}=\frac{2}{x-2}$ because it created division by zero. Her focus on the fact that it was discarded as a solution because it created division by zero was confirmed during the following coaching clip:

Danie: When are the two answers that we get both legal answers and when is one an illegal answer that we have to discard? Why does it happen? When does it happen?

Cindy: Well it happens because the denominator cannot be equal to zero.

Because I wanted us to discuss the graphical explanation for this occurrence, I probed further by asking her to discuss the graphical explanation. Cindy took her graphing calculator and drew the left-hand sides and right-hand sides of the equation $\frac{x}{x+2}-\frac{8}{x^2-4}=\frac{2}{x-2}$. She was puzzled about the sketches and the way to interpret them to explain why the solution $x=-2$ should be discarded.

Cindy: Now the thing about it is, right here (Cindy pointed to somewhere on the graphs of her calculator) this one (she pointed to the left-hand side $\frac{x}{x+2}-\frac{8}{x^2-4}$) is going to have, that one is going to have a vertical asymptote at …(she thought for 5 seconds) Let me simplify that left-hand side a little bit. (Cindy obtained a common denominator for the left-hand side and simplified it to $\frac{(x-4)(x+2)}{(x-2)(x+2)}$)…That (Cindy circled the $\frac{x+2}{x+2}$) makes a hole. All right, that one right there has got a hole in the graph, so it is a value that we are not allowed to use. It is a hole literally in the graph. This one (Cindy pointed to the right-hand side $\frac{2}{x-2}$) we are allowed to use that over here, we are allowed to use the negative two over here, but we are not allowed to use it over here, because it is literally a hole in the graph. So what is happening… there is that bold one (Cindy pointed to the graph of $\frac{x}{x+2}-\frac{8}{x^2-4}$), but they are not crossing…but they are not crossing at negative two at all anyway.

Danie: You see but, wait, let me think, you see but, that was my problem, these two (I pointed at both sides of the equation $\frac{x}{x+2}-\frac{8}{x^2-4}=\frac{2}{x-2}$) are not crossing at negative two, and so why do we discard it? But this one, (I pointed at $\frac{x}{x+2}-\frac{8}{x^2-4}$) does not exist at negative two at all…

Cindy: Right, but this one (Cindy pointed at $\frac{x}{x+2}-\frac{8}{x^2-4}$), see what happened to get that common denominator, the least common multiple, you are introducing $x+2$

over here, you are introducing a hole by doing your algebra…So when you try to find the common denominator to look at those two things, you create this extra thing in there, for this one over here (Cindy pointed at $\frac{2}{x-2}$ which she changed to $\frac{2(x+2)}{(x-2)(x+2)}$) that already an extra thing for this one over here (Cindy pointed to $\frac{(x-4)(x+2)}{(x-2)(x+2)}$). It is already a hole for this one over here. And since this one ($\frac{2(x+2)}{(x-2)(x+2)}$) has now created a hole, this one ($\frac{(x-4)(x+2)}{(x-2)(x+2)}$) already had a hole right there, you get that extra value in there, that you think is a value, because you kind of created it by entering this factor ($\frac{x+2}{x+2}$) one in there. It is like you built it up and you put it in there and it is really not there. Does that kind of make sense?

Cindy further speculated about the process of squaring both sides of a radical equation which artificially creates the bottom half of the radical equation graph and yields solutions that would also be discarded. When I asked whether she saw this problem as equivalent to the example of radical equations, she said:

I am wondering, I do not know. I mean I would have to look at some more myself.

Cindy and I both remained unsatisfied regarding the true answer to this mathematical problem for the remainder of this coaching session. Cindy finished the conversation by saying:

But I really wonder if it [the solutions to radical equations] has some kind of connection about this thing right here…because see some of them, you multiply across, a lot of problems, like, you bring in these extra factors of one and nothing is the problem. Maybe

> it is just these freaky ones, when you bring it in and it happens to create… just because of the makeup of the problem, that comes out as a solution, that thing that created a hole.

Regarding the theoretical framework of this study, I believe our work during this episode did improve Cindy's CCK. Although the episode ended with no specific satisfactory solution to our initial problem, Cindy did learn more about the graphs of the equations involved and about the graphical impact of the algebraic process of solving the equation.

The abovementioned six episodes were examples of the many rich and powerful mathematical discussions Cindy and I had during our coaching sessions. On various occasions, I did not have an answer regarding the mathematical topics I picked up from Cindy's class and brought to the coaching sessions. Our work together and the solutions we produced to these problems demonstrated the power that lie in two or more people collaborating on a common problem. It was extremely satisfying to work with Cindy and hear her say things like (18 October 2006 coaching session at 11:00):

> I never thought about that. That is good. You made me think today. You make me think every day.

The six descriptions above were all examples of work that Cindy and I did during our coaching sessions that enhanced elements of her (and my) MKT. Although none of the cases involved mathematical knowledge and skills that Cindy did not possess before the discussion, all six cases were responsible in helping her make connections between parts of her existing knowledge. For example, in A1 above, I have video evidence from the classroom data proving that Cindy knew matrices could not be inverted when their determinants were zero. I also have video evidence that she knew one could mostly not solve three variables with only two equations. I also have video evidence showing her using matrices to solve linear equations. Yet it took some focused work during the collaborative coaching for Cindy to bring all this knowledge together in order to obtain a satisfactory answer to Mandy's question. I believe since Cindy came up with satisfactory solutions to many of the six problems in a straightforward manner it served as inspiration for her to handle future students' mathematical contributions in a way that would be conducive to more mathematical contributions and critical mathematical thinking by her students.

I felt that Cindy acquired more and more mathematical confidence every time we succeeded in solving a problem that I brought from her class. She saw that she was able to deal with these issues and often deal with them in a simple manner. She showed determination to focus on students' mathematical contributions and disappointment regarding cases where she did not. Comments such as "I wish I did not cut him off" testify about this disappointment. Yet I believe every exciting solution to a student's mathematical contribution and every disappointment at not following up on a student's mathematical contribution opened the door for her to be slightly more adventurous within the rigid system of Dawson High School and Grayson County. I also believe that Cindy was convinced, by working on various examples of ingenious student contributions, that students could think about interesting mathematics and that most students' mathematical contributions were worthwhile pursuing in class if time permitted. This belief was confirmed by the increasing number of comments Cindy made regarding the mathematical ingenuity of students such as Justin and Mandy. These were some of the reasons why I believe Cindy changed her views on teaching and learning as well as her attention to students' mathematical contributions.

B. Baseline Data on Cindy's Initial Views on Teaching and Learning

The purpose of Section B was to describe some of Cindy's initial views on teaching and learning. Cindy expressed these views during our initial interview and during the collaborative coaching sessions. To understand how Cindy's views changed over the course of the coaching, Sections C needs to be read in light of the data presented in Section B. I called the data in Section B *baseline data* because, combining what I saw during data collection with my knowledge of Cindy's teaching over the past three years, I feel they accurately portrayed her views on teaching and learning before we started our work together and during the first few weeks of our work together.

B1. Cindy commented on facilitating classroom discussions and her actions and struggles with them.

During our initial interview I asked Cindy whether she ever facilitated discussions around certain mathematical topics and students' mathematical contributions that went on for an extended period of time and involved various students. Her comments were:

> It is hard to get them to do that sometimes. I mean that is your drawback with that. It is because…for the fourth period class, I think they would do that because they bounce…I mean they bounce things off of each other. But that does not happen a lot, I mean you have a lot of kids that are just…they are just waiting for you to say everything. You know and they don't, you know…I do not know, they… a lot of kids just do not talk and just do not…maybe I am not as good at getting them involved but…

I spoke about seeing examples of classrooms where students were constantly and actively involved and where the students realized that classroom discussion was the way teaching happened and learning had to occur. Cindy made the following comments:

> It is hard to be patient, and like…you not be the person talking, and it, I mean, it is hard to be like that, and I guess you just…it is probably me that is not making it happen because I won't let there be silence. Sometimes they are just going to sit there and there might be silence until somebody…you know chooses to chime in and…and have their comment, … but it is hard, it is hard to sit there and be quiet and wait until somebody says, you know, the answer or…I mean, that is hard to do.

Two things stood out from the above comments. Cindy felt pressure to teach in a traditional manner because of student demand. She found it tough, like many teachers do, to refrain from traditional teaching. These two facts combined with evidence from Cindy's day-to-day teaching for more than forty lessons convinced me that she had a very traditional view of teaching and learning. Traditional in the sense that she viewed students as passive receivers of knowledge and therefore placed a lot of emphasis on seatwork such as note taking and practicing example problems, while she, as the teacher, provided knowledge with lots of talking, explaining, and modeling of mathematical concepts. It would be risky to speculate whether Cindy expressed this view due to a pragmatic decision based on students' lack of involvement or whether it was her view on effective teaching more generally.

B2. Cindy explained her plan with students' mathematical contributions.

During this coaching episode I asked Cindy to explain her plan with students' mathematical contributions to me. The following was her reply:

> (Cindy laughed nervously)…I do not know if I have a game plan. I probably should have a game plan. That is probably a problem, that I do not have a game plan. But…I do not know. I try to answer them as best as I can at that moment. Sometimes if there are things that I do not know then I will just have to say: 'I just do not know. We will have to see

> about that' and 'let us think about that together' or… I do that sometimes. 'Let us think…' or I jump to a table or 'what if we did…'. I do not know. But, now if they are wrong, I correct them because they do not keep on going with wrong information. But sometimes I just, I do not know. I do not know if I really have a game plan.

This statement by Cindy shed more light on her views on teaching and learning. From this episode it was clear that at this stage Cindy did not have a clear plan for how to attend to students' mathematical contributions. However, it was significant that throughout her description of her imaginary attention to students' mathematical contributions, Cindy remained the person instructing by teaching in a traditional manner and suggesting all the options. This, together with my experiences of her teaching for numerous lessons, served as confirmation that Cindy had a traditional view on teaching and learning. It was also very significant that Cindy struggled to articulate her plan with students' mathematical contributions beyond trying 'to answer them as best as I can at that moment'. This further confirmed my inference that Cindy had a traditional view of mathematics teaching and learning. This episode exhibited Cindy's views on 14 September and they stood in sharp contrast to her views two months and four months later. Cindy's views on these two later dates were described in C5 below.

B3. Justin commented: "It's like the opposite of a graph."

The classroom incident I played to Cindy involved a classroom discussion around the basics of matrices, in particular the way in which the dimensions of a matrix should be written down (i.e. An $m \times n$ matrix has *m* rows and *n* columns). During this discussion, Justin made the comment: 'It's like the opposite of a graph.' When I played the clip containing Justin's comment to Cindy she responded before I even asked or said anything.

Cindy: I do not know what he was asking, to be quite honest. I do not think I answered his question because I do not even understand… if you listen, did I really even try to answer it? Because I do not even know what he was asking me. "It is kind of like the opposite of a graph."

Danie: Yeah, I was hoping that it links up with this (work done in the coaching session) but then it obviously does not, or it might. I do not know.

Cindy: I do not know. It might be, while we are doing some stuff tomorrow, I will have to… maybe that question will pop back up again. But I really, to tell you the truth, I really do not know… Oh I know what he is saying...

Danie: What is he saying?

Cindy: It is opposite of a graph. I get it now… (Cindy takes a piece of paper and explains it to me (see figure 8)) Which now, I wish I did not cut him off. Because now we can say 'oh yeah it is the opposite, yeah these are up and down and these are left and right, and up and down' you know?

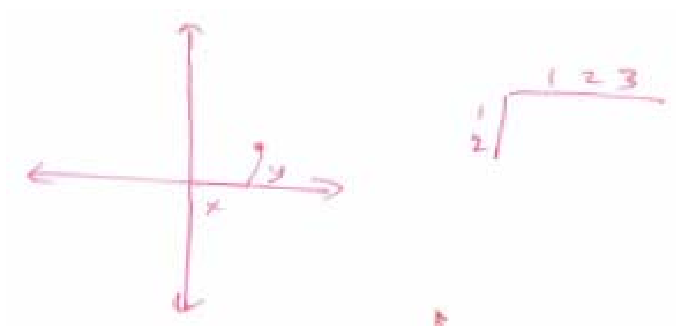

Figure 8.

This episode confirmed my opinion that Cindy held a very traditional view on teaching and learning. It was an exact example of what happened most often in Cindy's class when she did not know how to deal with a student's mathematical contribution. Because she viewed teaching in such a traditional manner no apparent learning took place for anyone in the class when Cindy could not teach it. The lack of learning was evident in the manner in which this episode played out (more will be said on this episode in Section D3).

C. Evidence of Growth in Cindy's Views on Teaching and Learning

This section provided data to answer the first research question by supporting the claim that Cindy's view on teaching and learning did change as a result of our collaborative coaching sessions generally and specifically as a result of the MKT focus of the coaching. It will be important in Section C to pay attention to the focus in Cindy's view on teaching and learning changing away from herself and toward her students. Cindy shifted her view regarding teaching as the responsibility of only the teacher to bringing the students into the equation. She also shifted her view on learning as something passive to a more active endeavor.

C1. Cindy said: "I give it away in class too often."

During this coaching session, I particularly struggled to allow Cindy to express her views without imposing my thoughts and views on her. During this particular incident I was about to say too much regarding an upcoming video clip when I stopped myself with the words: 'I always want to say too much' referring to my role during the coaching sessions. Spontaneously Cindy added:

> Well I give it away a lot of the times in the classroom too. I give away more than I should…I give away, I need to wait, sit back and let them tell me more, I really do. And I see that more and more, here recently, that I really need to sit back and sometimes I repeat what they said real quick. Here recently, I tried to catch myself going 'wait why don't you, repeat that, repeat that and say it to everybody else'… I have tried to let, instead of me just saying it real quick 'o yeah' like I am real excited about it, or stuff, 'Anthony say that again what you just said. Say that to everybody.'

Danie: Okay, but it takes willpower to change such a habit isn't it?

Cindy: It is, and I do not do it all the time, but I have tried to, I have…I do not know if you have noticed, but I have…I have really been trying to not like say so much and let them say a little bit more.

Although the practical move toward reform-oriented teaching had not yet happened this early in our work together, Cindy at least expressed a desire to hold back on her teaching and to allow students to be more active. The fact that she acknowledged the difficulty in limiting her traditional teaching responsibilities indicated that a shift in her view on teaching and learning started to take place. Previously, I do not believe that Cindy would have found fault with telling students everything and I witnessed numerous teaching episodes that supported this inference. At least now there was a hint of tension between her old ways of teaching and some new views on teaching and learning where students could become more active in the learning process.

C2. Cindy commented: "I was proud of myself."

During this coaching session, Cindy and I looked at a piece of classroom video wherein students were making a conjecture about the slope of two linear equations (i.e. $y=-2x+1$ and $y=x+2$) and the resulting *a* value (the coefficient of x^2) of the product, a quadratic equation ($y=-2x^2-3x+2$). The resulting *a* value then determined the direction of the parabola. The students correctly conjectured that if the slopes of the two lines were opposite like in our example, then the parabola opened downwards while it opened upwards when the slopes were the same (i.e. two positive or two negative slopes for the linear factors both caused a positive *a* for the quadratic equation). Cindy's comments (at 45:04) about her involvement in this discussion were significant:

> But they came up with it. Like I did not tell them that… we lead up to that but I did not tell them ‘okay this is…’. They came up with that. Yes… Which I was proud of myself that I did not say ‘hey look…’. See that is what, that is what I was talking about earlier. It is like I am trying to not tell them everything… to get them to see more the patterns… lead them there. Or give them some information, maybe they will start seeing some of the patterns. And I think that is why they are seeing more. Like they are seeing even more patterns, they are looking for more patterns than I am looking for now.

On various previous occasions, Cindy attributed the success of students to come up with novel conjectures to her own interventions. She mentioned her own actions such as ‘I made them look at more examples’ or ‘I am looking at the patterning that is going on’. It is therefore noteworthy that on this occasion Cindy attributed the success of students to come up with a novel conjecture to the students themselves. This was testimony to the fact that Cindy’s views on teaching and learning were beginning to change. It was evident how she now started to believe that teaching and learning were not just controlled by the teacher but might be accomplished by the students individually or cooperatively.

C3. Cindy said: “I just hope Mandy does not ask a ton of questions.”

Before the 24 October coaching session, Cindy and I were informally discussing the demands of the Grayson County curriculum. During this unrecorded discussion Cindy expressed frustration and disappointment at not being able to allow students more time to conjecture and debate interesting mathematical topics. Cindy again voiced her frustration during the same 24 October 2006 coaching session. At 18:10 she said:

> I hate to say that some days, like today when I did not have a lot of time, I am just…’gosh I just hope Mandy does not ask a ton of questions’…I know that that is the

> wrong…but here in Grayson County curriculum, I am just hoping today Mandy did not ask a lot of questions.

To deal with students' mathematical contributions in a traditional manner where the teacher just gives a mathematically correct answer, explanation, or justification does not take up a lot of time. Such teacher actions were ideal in a school system such as the Grayson County school system with its extremely full curriculum. Cindy's concern about the number of questions Mandy would ask on that specific day was an indication that she started to think less traditional about dealing with students' mathematical contributions. I believe that Cindy was beginning to see the value of student involvement in the teaching process and because such teaching demanded more time than mere *chalking and talking* felt this expressed pressure.

C4: Cindy's new progress report comments.

By the coaching session on 24 October Cindy had two opportunities to communicate to parents concerning their child's progress in class. This communication happened through a progress report after six weeks into the semester and one after twelve weeks. Because I did not start working with Cindy until after four weeks into the semester we worked together for only two weeks by the time the first progress report was sent out but for 8 weeks by the time the second progress report was sent out. Cindy made the following remarks regarding the comments she used on these progress reports:

> I was amazed when I did their progress reports and I did their comments, about how many different kids, here recently that I could not put it on before, but yet, one of the comments was 'seeks additional information' and…which was a lot of kids now, and actually here recently 'has enthusiasm toward learning', which was because I think they ask tons of questions now. I mean they seem to ask a lot of questions. I was looking at the

comments I made last time, because they are still there, in the computer, I can look at the new comments that I am making, and I did not make that comment that much the first six weeks, but the second six weeks, it just seems like I was able to, for several kids, put that comment on there.

Generic computer generated progress report comments take a lot of thought. It is often very difficult and requires a lot of thinking to choose a comment that accurately reflects an individual student's progress. The fact that Cindy started using these two comments regularly during the time period that we were working together indicated that students and their mathematical thoughts and actions had moved more towards the forefront of Cindy's thinking. The two attributes Cindy mentioned are such important student qualities and noticing them in her students was indicative of a change in Cindy's views on teaching and learning.

C5. Cindy commented about her goals with students' mathematical contributions.

Subsequent to the coaching episode described in B2 above concerning Cindy's plans with students' mathematical contributions during the 14 September coaching sessions, I asked her some equivalent questions during the 14 November and 19 January coaching sessions. Cindy's responses were significantly different than on 14 September in a variety of ways. Below are Cindy's responses regarding:

Her goals when students asked her questions in class:

My goal is by the time we are finished, is for them to understand or have their question answered. But sometimes it is not, sometimes I am quick to answer it, but sometimes I am not so quick to answer it, and then sometimes I'll let other people answer it or…

How she viewed the other students in that situation:

Well hopefully, if they are paying attention, they are going to help answer that question too. And that has happened in here, like sometimes some student will ask a question and other people will end up answering their question and it will not necessarily be me. Which is actually probably the goal, most of the time, for me not to be the person that answers the question. Sometimes even they will go ‘Where did that two come from?’ and then one kid across the room is telling them where it came from, I mean explain it to them where it came from. And that is actually better if somebody else tells where…and sometimes I even do pose that question: ‘Where did the two come from? Anybody remember where the two comes from?’ I do not do it that often but that is probably the better thing to do ‘Where did the two come from?’ and then have somebody else voice in where the two came from.

Again her goals with students’ mathematical contributions:

I want them to make conjectures. I want them to be looking ahead at the math and make some conjectures…for them to see what is going on with the math and me not to tell them everything that is going on with the math. I think it is becoming more habit…I mean, I am trying to get out of the habit of just telling them stuff all the time…I try to, if it is a situation where I feel like we can discuss it that way then that is what I try to do. I guess I am trying to balance, there is a balance point…I know this is gifted and honors, but I still, in algebra, I still try to not tell them everything…I try to do that but I think it has been more this year than it has been in the past years. I am more conscious of it, or maybe, like I said, maybe it is just more a habit.

In contrast with the episode described in B2 Cindy was very articulate about her thoughts and certain about her plans during these two episodes. Her certainty in itself communicated a change in her views on teaching and learning, but more so did her actual comments. From these comments it was evident that Cindy now saw students as an integral part in the teaching and learning of mathematics and that they had an active role to play in the mathematics classroom. Although she admitted to struggling between being aware of the right thing to do and actually doing it, Cindy demonstrated evidence of taking the first steps in teacher change by changing her views on teaching and learning.

C6. Cindy commented about the benefits of our coaching sessions.

I initiated this coaching episode by reminding Cindy that a while ago she expressed the opinion that she was benefiting from our collaborative coaching sessions. I asked about her focus on students and their mathematical contributions and whether she had changed this focus since our coaching sessions began. Cindy responded in the following manner:

> Yes, I do look at students and their input a lot differently. Like I let them talk a lot more now. I mean, and even like right now, we are back in the room, especially fourth period, so if you have noticed, the kids are going to the board a lot more. Because, especially that fourth period class, which is I think, bright, I think they are overall a little bit brighter sometimes, all of them together. But I am letting them talk a lot more and explain what they have to explain, so instead of me doing all of that. I let them talk a lot more.

At 46:10, I essentially repeated the question regarding students and their mathematical contributions to Cindy. She responded in the following manner:

> Yes, and I am not so quick to tell them that they are wrong when they are. Because they need to figure out that they are wrong. I am not so quick to say ‘no you cannot do that’. Sometimes I do say ‘no you cannot do that’.

The response from Cindy during this episode was strikingly different from the response she gave in B1. She came across very confident and purposeful and provided evidence that her views on how to best teach mathematics and how students best learn mathematics had started to change. She spoke specifically about students and the active role they now started to play in her class and how students moved toward the forefront (doing mathematics on the board, talking more) when it came to doing mathematics and herself moving more to the background (not being so quick to tell them that they were wrong).

To conclude the description of Cindy's views on teaching and learning and provide an answer to the first research question of this study, I would like to summarize the change that occurred in Cindy's views. Cindy's views on teaching and learning as described by her comments on these views and some of her actions portraying these views did show a shift during the course of the fifteen collaborative coaching sessions. In particular, Cindy's views moved away from a view of teaching as done by only the teacher from the front of the classroom and of learning as done by students passively receiving instruction. She now viewed teaching more as an interactive process orchestrated by the teacher and involving students while she viewed learning as a process that took place in the act of being involved in discussions and explanations.

I have to emphasize that I do not believe that Cindy's views changed drastically because of our work. For a drastic change in views to occur, I believe collaborative coaching such as this would have to continue for much longer and the practical rewards of subtle shifts in her views would have to be far greater than they were during this short period of time. Cindy's rigid environment also played a role in setting the stage for only certain types of teaching to be practical and possible which would play a role on teachers' views on teaching and learning in the long run.

D. Baseline Classroom Data Where Cindy Attended To Actual Students' Mathematical Contributions

At the beginning of my work with Cindy, she was a very traditional teacher by the same definition used and defined in Chapter One and throughout this study. She rarely deviated from the teaching format where she explained new mathematics on the board and students practiced equivalent examples individually in class or at home. The attribute I was most concerned about during our work, and the subject of my second research question, was how Cindy attended to students' mathematical contributions. Cindy could be described as a traditional teacher in this part of her teaching also. Very rarely did she involve anyone else in attending to contributions from students in her class. After careful data analysis and getting to know Cindy much better over the five months we worked together, I could give three reasons for the way Cindy attended to students' mathematical contributions.

First, as I described in detail in Chapter Three, Cindy was a veteran high school mathematics teacher who had taught mathematics to a wide spectrum of high school mathematics students. In the interview at the start of the research, she also expressed enjoyment in teaching a variety of students every day. She was therefore used to helping her students do as well as possible irrespective of their mathematical inclination and talent. Due to her extensive experience, she was also comfortable with teaching a variety of mathematics from Math Money Management to Algebra I through AP statistics. These factors contributed in making Cindy a confident high school mathematics teacher who felt that she could explain mathematics to her students effectively. As a result of her confidence, Cindy was used to dealing with all students' mathematical contributions herself.

A second factor that I described in detail in Chapter Three was the environment in which Cindy worked. Due to state, county and school demands Cindy and her colleagues felt a lot of pressure to teach a large amount of prescribed content in the academic year. Because of this external pressure Cindy did not feel she had the luxury to deviate from her very rigid planning. She often expressed the opinion that things such as groupwork took too long and that she just could not afford to teach that way.

Third, on certain occasions Cindy simply lacked the CCK and the SCK to support discussion around interesting student contributions, as do many teachers. The teaching actions to which Cindy reverted back, such as to talk more and to rely on passive seatwork by students, resembled those described by Aubrey (1997), Brophy (1991), and Millet and Johnson (1996) as actions typical of someone who lacked essential knowledge.

In the beginning of my data collection in Cindy's classroom, she rarely used students' mathematical contributions as an opportunity for learning by the student who asked the question (by quickly supplying answers to a student's contributions, I believe that the student received an answer but rarely learned) or by any other student in the class. In almost no initial instances in the classroom video data did she use a student's mathematical contribution as a springboard for a discussion in class. Her automatic initial reaction to a question, comment or conjecture was to immediately answer, validate or refute it. An example that illustrated this trait occurred on 20 September 2006 during Cindy's sixth period lesson (at 06:40). Cindy was explaining to students how to sketch the line $y = -\frac{3}{4}x + 2$ by using the slope. She started at the point $(0;2)$ and because the scales on the axes on the blackboard were done in halves (and not ones), Cindy moved three halves (and not ones) down and four halves to the right. This unexpected choice by

Cindy invited a chorus of student voices all disagreeing with her decision. Some of the comments were:

Paul: No.

Peter: Mmm.

Joseph: No because, see everything is two.

Philip: You have halves.

Amongst the many voices one could clearly hear one student confidently say: “It does not matter” indicating that he knew what Cindy did and why she did it. So Cindy here had students who participated in the debate and who had arguments for both possibilities, yet she did not use this situation to allow students to justify and rethink their understanding. Even in cases when a teacher did not have proponents for both sides of the argument, the fact that many students voiced their dissatisfaction in her action should have been enough for Cindy to attend to their contributions better. She did not, and went on to explained herself:

Cindy: Yeah, but I still did three blocks. If I did the three, if I did down three, one, two, three (Cindy went down),…one, two, three, four (Cindy went to the right). Because slope is a ratio right?

Joseph: Oh yeah. Yeah.

Cindy: So since I have got everything blocked off, like two blocks is equal to one, I still can just go… Does everybody understand that it is still going to be okay? To go down three and over four, because slope is just a ratio… it’s a change in y value divided by a change in x value.

This type of (lack of) attention to students’ mathematical contributions in class stood in sharp contrast to the aims and goals prescribed by a variety of publications. The NCTM

Professional Standards for Teaching Mathematics (NCTM, 1991) encouraged teachers to listen carefully to students' ideas, ask them to clarify and justify their statements, ask them to convince themselves and other students of the validity of their statements, and pursue some of these statements as content for further discussion. The NCTM *Principles and Standards for School Mathematics* (NCTM, 2000) stated that "teachers' actions are what encourage students to think, question, solve problems, and discuss their ideas, strategies, and solutions" (p. 18) while the NCTM *Professional Standards for Teaching Mathematics* stated that "opportunities to explain, conjecture, and defend one's ideas orally and in writing can stimulate deeper understanding of concepts and principles…In the process of discussing mathematical concepts and symbols, students become aware of the connections between them" (1991, p. 78). The NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989) also encouraged the idea that students must develop the skill to "reflect upon and clarify their thinking about mathematical ideas and relationships" and to "make and test conjectures" (p. 140). The British Department for Education and Skills (2001) encouraged interactive teaching where "asking for explanations, giving time for pupils to think before inviting an answer, listening carefully to pupils' responses and responding constructively…, challenging their assumptions and making them think…, asking pupils to suggest a line of inquiry…, discussing pupils' justifications of the methods or resources they have chosen" (p. 27). The five episodes described below should serve as baseline data for the manner in which Cindy dealt with students' mathematical contributions and give an understanding for exactly how she handled these contributions. Although her growth in this area was not extensive, I believe that the growth reported in Section E below will be clearer when seen against the backdrop of her initial actions in Section D.

D1. Mandy asked: "What are functions used for in real life?"

This was an introductory lesson on function.

Mandy: What's the point of functions in everyday life? How are they used?

Cindy: Well, certain everyday situations are just relations and certain everyday situations are functions, and… really just today, we dealing with what is a function? We will have situations, there are business situations that are functions. There are actually business situations that are not functions initially, but we restrict back the domain so that we end up, a lot of times we will maybe restrict our domain somehow so that we do end up with a function. Some business situations are not functions initially, but we chop off part of it, we will talk about it a little later this chapter, which is called piecewise function, so that we get little pieces of it that are functions. But a lot of times, in situations, we like to input one value, which is your *x* value, and know that the output is only going to be one *y* value. Okay, because it is very, like especially in business situations, you do not want to put in: I want to make 500 products and be getting out two different answers for cost. You probably would not want to do that. Okay, you know, I am wondering what is the cost for these 500 products. Well then you input that one value and you get two different ones. So you might not want to do that. But you might… things set a little differently. A lot of times things are not functions and then you need to try to manipulate them with restricting domain so that we end up with functions. Well actually restricting ranges a lot of times, excuse me, so that we end up with functions. We chop off pieces, we put things together, but in everyday life mostly we would like to have functions. We like when we put an *x* in we get one *y* back out. When we have an input we have one output. That is what we like to have. Does it not make it easier if you have an input and you have one output? I do not think I answered that very well, but...

Like many other students' mathematical contributions in Cindy's class, this one would have surprised most teachers and would have caused them to think really hard to optimize its use for student learning. Not only was it hard because of the choices available as answers to the question but also because it would have been difficult to orchestrate a classroom discussion centering on the question. Since this lesson was about the introduction of functions, most students would likely not have been able to contribute much toward answering this question. Yet, the fact that Cindy ended her answer to this very interesting student contribution in the manner she did confirmed that she was unsure about the quality of her explanation. It also confirmed that this episode was an example of the fact that Cindy still had room for improvement in dealing with students' mathematical contributions. Even after 15 years of teaching she admitted during the subsequent coaching session that she was caught off guard by this question. She also did not draw on the relevant MKT to answer the question or did not posses the relevant MKT to answer it effectively. What was even more important was that Cindy showed in this instance, and various other early instances, no inclination to obtain any student involvement in the exercise of dealing with students' mathematical contributions. She did choose to spend time on the contribution, but elected to use the time to try to construct an answer on her own.

D2. Tanya asked: "Will it ever be a horizontal line?"

During this classroom episode Cindy had $f(x)=|x|$ on the board and explained the effects of a change to the equation on the graph of the function. She explained that the *a* in $f(x)=|ax|$ determined how "slim or wide" the v of the graph would be, specifically that if *a* got closer and closer to zero, the graphs went "wider and wider".

Tanya: Will it ever be a horizontal line?

Cindy: …but it would not be…it will still be…I can get it really flat, but I do not think that… That is a good question. I do not think I will be able to flatten it out all the way.

Andy: That would just be *x*.

Cindy: Unless I…

Andy: That would just be *y* equals…

Cindy: Well the only way that I could do that is if I maybe multiply by zero on the outside. But then I do not really have absolute value any more. That is what I am thinking. Okay? Yeah, but that is the only way I think it can flatten out to be a line but that would be y equals zero. But yeah, if your *a* is equal to zero. Uhm, good question though.

From the transcript above we could see that Cindy clearly did not have a good answer ready for this student question. She also did not use most of the practices suggested earlier by the NCTM (1989; 1991; 2000) and DfES (2001) documents to try to involve students in this discussion. It was also clear from this example that Andy had thoughts he wanted to contribute, yet at no stage did Cindy call on him, involve him, acknowledge his contribution, or included his comments in her own explanation. Learning surrounding this very interesting student contribution was therefore reduced to only passively listening to the attempted explanation given by Cindy.

D3. Justin commented: "It is like the opposite of a graph."

Cindy was busy introducing the notation for writing down the dimensions of the matrix $B = \begin{bmatrix} 2 & 5 & -3 \\ 7 & -1 & 0 \end{bmatrix}$ to her students. Students were busy discussing ways to memorize the fact that the dimensions were written down in the order of rows by columns.

Justin: It's like the opposite of a graph.

Tommy: What are you talking about? You confuse me so much.

Justin: *x* and *y*…Which of course, they could not make them the same.

Although a variety of students and Cindy added comments at this point, none of their remarks concerned Justin's statement. For a minute or two it was as if the class had a discussion about Justin's contribution but the actual discussion had nothing to do with his statement. Confirmation of the fact that Cindy did not think she handled Justin's mathematical contribution well was found in the following comment she made during our 10.03.2006 coaching session (at 29:50):

> I do not think I answered his question because I do not even understand… if you listen, did I really even try to answer it? Because I do not even know what he was asking me.

After admitting that she did not know what Justin meant during the coaching session, it took Cindy 25 seconds to understand what he meant and to start explaining it to me (see figure 8 in Section B3). Justin merely commented that the Cartesian coordinates of a point were expressed by first the *x* coordinate (horizontal dimension) and then the *y* coordinate (vertical dimension) while the dimensions of a matrix was expressed as rows (vertical dimension) by columns (horizontal dimension). The fact that she was able to understand Justin's comment so quickly confirmed that Cindy did not pay much attention to it during class. Not only would it have been beneficial to focus the attention of the other students on a discussion about Justin's statement, but it may have given Cindy the time to gather her own thoughts in order to understand what the student meant. Also, according to the NCTM *Curriculum and Evaluations Standards for School Mathematics* (1989) Justin should have been given the opportunity to "reflect upon and clarify" (p. 140) his mathematical thinking .

D4. Mandy asked: "Can you solve three variables with only two equations?"

Cindy was busy going over the process where students used matrices to solve linear equations. She taught the principle at the hand of a textbook example and specifically explained that she selected the example because it had some variables missing (not all three equations had three variables each). She wanted students to see what to do when that happened. She emphasized that where there were variables missing students should include a zero in that position. The textbook example was

$$\begin{aligned}&-2x + y + 6z = 18\\&5x + 8z = -16\\&3x + 2y - 10z = -3\end{aligned}$$

Mandy: So if you can put a zero in for whenever there is not like a variable, does that mean you can solve systems of equations with only two equations with three variables if you add another equation that has zeros in for all of them?

Cindy: No, probably not. It kind of depends on what the equations look like, but maybe not. And that really depends on how, what variables the equations… What, you have two var… you have two equations but three variables?

Mandy: Yeah.

Cindy: It depends on what they are or how it looks.

Mandy: Oh.

Cindy: You might not be able to do that.

In class Cindy only dealt with this question for 35 seconds. She involved no students in answering the question. All of this was confirming evidence that Cindy did not deal with this student's mathematical contributions in a manner that could enhance student learning. Although Cindy did not know the answer to Mandy's question in class, I did see evidence later in the study that she successfully managed to attend to students' mathematical contributions constructively even when she did not know the mathematical answer to their contribution.

D5. Duncan asked: "What is the relationship between the *y* intercept of a parabola and the *y* intercepts of its two linear parts?"

This classroom episode is the same as the one described in A2 above. Cindy was taking students through a discovery exercise where they sketched two linear functions $y = 2x - 2$ and $y = 2x + 1$ and the quadratic equation formed by the product $y = (2x - 2)(2x + 1)$. The goal was for students to see that the *x*-intercepts of the quadratic equation corresponded with the two *x*-intercepts of the linear equations. The goal was *not* to look at the relationship between the respective *y* intercepts but a student brought this up.

Duncan: It is also on the y axis, isn't it?

Peter: Maybe that is a coincidence.

Cindy: Now uhm…that right there (Cindy points to the *y*-intercept of $y = 2x + 1$**)** is crossing at the *y* axis at zero, one, the line is, but if you notice, the parabola is…Oh down here (Cindy points to the *y*-intercept of $y = 2x - 2$ which is also the *y*-intercept of the parabola)?

Duncan: Yes.

Cindy: That just happens to be a coincidence… well. We'll…we'll look at that. I am not quite sure. I have never really looked at that.

Joseph: Well, three of these, three of these do cross on the line.

Cindy: Do they? At the same place?

Joseph: Yes. I think.

Cindy: Well we are going to have to look and see if more of those… because I have not really looked at that aspect. I have just been looking at that (Cindy points to the one *x*-intercept) and that (Cindy points to the other *x*-intercept) and not really looked down there at that value. That might be something else that they did not ask us to discover down here but yet is something that we can look at. We will just have to do more of them to test it out. Okay?

By the time we looked at this discussion again in the following coaching session, Cindy started off by saying: "Which I have not really thought about that." It was also evident from the lesson video that Cindy did not consider involving any students in the quest for an answer to this evolving conjecture. Joseph also seemed to be attentive during Cindy's short conversation with Duncan and aware of the conjecture in question, yet Cindy did not involve him at all.

E. Evidence of Growth in How Cindy Attended to Students' Mathematical Contributions

This section focused on providing the answer to the second research question of this study. The data presented below will provide the evidence to support the claim that Cindy did begin to change the way she attended to students' mathematical contributions. The chapter will end with a discussion of reasons why this change happened as a result of our work during the collaborative coaching sessions. Change in the way Cindy attended to students' mathematical contributions started to occur by the fifth collaborative coaching session. It is also important to note that change in Cindy's actions did not happen in a linear fashion, always just steadily improving. There were often times, even in lessons toward the end of the study, where Cindy demonstrated how tough it was to get rid of old habits and where she attended to students' mathematical contributions in a very traditional manner.

E1. Cassandra said: "Subtract $\frac{b^2}{4a^2}$ from the right-hand side."

In this classroom episode, Cindy was busy solving for *x* in $ax^2 + bx + c = 0$ by completing the square. When she got down to the step where the square of half of $\frac{b}{a}$ should be added to both sides of the equation, Cassandra wanted to subtract this expression from the right-hand side of the equation in stead of adding it. The following transcript illustrates the subsequent classroom discussion:

Cindy: Now, since I added it ($\frac{b^2}{4a^2}$) here, what have I got to do to the other side?

Cassandra: Add it.

Tommy: Add it to the other side.

Duncan: Subtract it from the other side right?

Cassandra: Add it. Subtract it.

Tommy: No it is adding, because we are adding it to both sides.

Cindy: Okay, well, let us get this cleared up.

Tommy: No it is adding because we are adding it to both sides.

Cassandra: No. Sweetie we are subtracting it.

Various students argue about it for 15 seconds, Cindy says nothing.

Cindy: Okay, I am going to wait until you all settle this.

Cassandra: It is subtracting.

Tommy: It is adding.

Bethany: I think we should do whatever she tells us to do.

Various students get an opportunity to voice their opinions.

Cassandra: You have to act like it was not even there…

Tommy: Yes so you have to add it…

Cassandra: No I remember it was adding and subtracting…

Philip: Let her (Cindy) explain. If you guys will just be quiet she will explain it to you.

Cindy: No, no actually I am letting you all have a little argument about this.

Various students again argue their viewpoints.

Cindy: Okay, Duncan…

Duncan: It is add, because in the other one you have the f of *x* on the other side and we did not want to mess with that, so it was all on one side. So you could not add something to that one side that was not already there. Now you have two sides, now we can add it to the other side.

Various students, including Tommy, Cassandra, and Duncan continue to heatedly debate this issue for another 30 seconds.

It was significant that the answer to the student question in this episode did not come from Cindy. Although she previously demonstrated a tendency to get involved in such mathematical conversations very quickly, on this occasion Cindy stood back in silence, and even encouraged this debate to continue on three different occasions. Duncan initially proposed subtracting $\frac{b^2}{4a^2}$ from the right-hand side but ended this episode explaining to the whole class why it should be added. I believe that the reason why students like Duncan and Cassandra came to a satisfactory conclusion to their problem was due to the fact that Cindy handled Duncan's mathematical contribution in a manner that was conducive to beneficial and constructive learning by all students. Because of the time Cindy allowed this debate to continue, Duncan had the opportunity to learn from his peers and correct his own mistake. I believe he had a richer mathematical experience during this event than in a case where Cindy immediately supplied him with a correct answer without peer input. I say this because he had the opportunity to listen to, and consider various options for a solution rather than being told what the correct solution was. Cindy seemed to agree that this was an episode in which she did well by making the following comment during our 20 October coaching session:

> You know they argued for a long time, but I think it was beneficial for them to argue about it, and work it out them…like talk among each other and try to figure it out instead of me just telling them: 'No this is what you do'…I do not think that it was bad that they argued with each other.

E2. Tanya asked: "What do you do to the other side?"

During this classroom episode, Cindy was busy changing the function $f(x) = x^2 + 10x + 8$ into vertex form by completing the square. Tanya's comments came, during the first step, directly after Cindy added 25 to the function.

Tanya: So then, what do you do to the…you know, what do you do to the other side?

Cindy: Well I do not have another side.

Tanya: I know, so you are just adding 25 out of nowhere?

Cindy: Out of nowhere, yeah…Am I allowed to do that?

Tanya: No.

Cindy: No I am not, am I?

Tanya: No.

Cindy kept quiet for a few seconds. Various students made comments about what to do at this stage.

Tanya: I do not understand…I do not understand how you just add…

Cindy: Tanya, ask your question again.

Tanya: I do not understand how you just add…I know why you add because you want to complete the square, but I want to know how you make that which you just wrote ($f(x) = x^2 + 10x + 25 + 8$) equal to that…the original equation.

Cindy: Listen to Tanya's question. Her question…she has a very valid comment right now. I just added 25 out of nowhere. Is this equation (Cindy points at $f(x) = x^2 + 10x + 25 + 8$) equal to this equation (Cindy points at $f(x) = x^2 + 10x + 8$) any longer?

Students: No it is not.

Cindy: how can we compensate here for the fact that I just added a 25 over here?

Tanya: Subtract 25.

Although Tanya had a problem that could have been easily solved by merely giving a short, correct answer to her problem, Cindy exercised a large amount of restraint during the two minutes of classroom debate. The manner in which Cindy stepped back from her traditional role as teacher contrasted greatly with the episodes mentioned in Section D where she applied a traditional approach. The end result was excellent participation and reasoning by a variety of students and a satisfactory outcome for Tanya.

E3. Tommy said: "If it is negative it is a compression."

Cindy was busy revising the vertex form of a quadratic function and the effects on the graph of the various parts. After discussing the effect of the *k* in $f(x) = a(x-h)^2 + k$, Cindy asked Tommy to comment on the effect of *a*.

Cindy: Now Tommy, what did this number up here (Cindy points to the *a*) do?

Tommy: Compression.

Cindy: Yes, and how did we know whether it was compression or a stretch?

Tommy: By the negative or positive sign. If it is negative it is a compression, if it is positive…

Various students contribute their opinions after this error by Tommy. Cindy stood back and allowed them to debate for a while.

Cindy: All right, wait, wait, TR you back there are disagreeing with him.

TR: Never mind.

Cindy: You are saying 'no'.

TR: Never mind.

Cindy: No go ahead.

TR: I do not know.

Cindy: He said that if it was positive or negative, that told us whether it was a compression or a stretch.

TR: Yes that is probably right.

Cindy: All right, Susan…

Susan: If it is less than zero? Less than one, it is a compression…and if it is more than one, it is a stretch.

Various students add their comments at this stage.

Cindy: So anything less than one, any value that is less than one (she wrote $a < 1$ on the board) is going to be...

Susan: …a compression…

Cindy: You mean even like negative four?

Various students commented. After these comments, many students assisted in constructing the correct answer $-1 < a < 1$ that Cindy wrote on the board.

Again Cindy demonstrated excellent restraint and the result was the construction of a correct answer to the problem by a variety of students, reaching consensus while being very active and vocal during a lively debate. None of this would have been possible if Cindy merely supplied the correct answer to the student's mathematical contributions. In this case also, Cindy's altered attention to a student's mathematical contribution was evident and it lead to learning by many students.

As a teacher, it is easier to act in a manner that is conducive to student learning when you know and understand the mathematics and the student thinking involved. When student thoughts are not so clear and their statements or misunderstandings hard to understand at first, it becomes easier to dismiss them and handle them with less care and consideration (Aubrey, 1997). It was therefore encouraging to even see Cindy attend to a student's mathematical contribution she did not understand at first in a manner that was more in line with suggestions from documents such as those by the NCTM (1989; 1991; 2000) and DfES (2001). On 6 October during the sixth period (at 22:20), Justin constructed a conjecture regarding two linear equations and the quadratic that is formed by the product (even though Cindy and I looked at this conjecture again during coaching, neither of us understood it or could reproduce it in our own words). On 24

October during the sixth period, Josh constructed a conjecture about the differences in the *y* values obtained from the tables of $f(x) = x^2$, $f(x) = 2x^2$, and $f(x) = 3x^2$. Although Cindy admitted in both cases that she did not understand the student conjectures, she gave the two students ample time to explain their conjecture and to other students to think about them and make comments.

E4. Josh asked: "Are you always going to shade, no matter the equation?"

During this classroom episode, Cindy explained to students how to select (or shade) the set of solutions to quadratic inequalities. She demonstrated the number line method where values on a number line were selected from the three regions formed by the zeros and substituted into the quadratic equation to determine whether the region yielded a positive or a negative outcome (*y* value). She then proceeded to complement this method by showing students how to use the graph of the quadratic equation to see which region yielded positive and negative *y* values. She urged students to shade the region that they selected as the set of solutions and gave them some to do for homework. At 50:54, as the students were getting ready to leave the class, Josh asked the following question:

Are you always going to shade, no matter the equation?

The result of this question was that Cindy made all the students sit back down while she asked Josh to repeat his question. Cindy then proceeded to assign the students the homework task of finding an example where one would always shade and an example where one would never shade. Although I did not videotape during the next day, Cindy reported back that most students found examples of both cases she requested. This episode testified to the fact that Cindy became more willing, within the rigid system where homework was planned and predetermined months in advance, to be lead by students' mathematical contributions and to alter her homework plans.

E5. Tommy commented: "If you have a numerator over a denominator that equals a numerator over the same denominator…"

During the following classroom episode, Cindy was going through a student's solution of the equation $\frac{x}{2}+\frac{5}{3}=\frac{7}{6}$ on the board. The method that Cindy and most of the students preferred was to give the equation a common denominator on both sides of the equal sign, and then to drop the denominators throughout and be left with only the numerators that were equal. As Cindy was about to explain this strategy to her students, Tommy sensed the explanation was approaching, he interrupted Cindy, and the following discussion occurred:

Tommy: If you have something over six equals something over six, can't you just do that something equals the something?

Cindy: Thank you Tommy, yes…Why don't you say that again, but maybe not the something.

Tommy: All right, if you have a numerator over a denominator that equals a numerator over the same denominator, you can just…the one numerator equals the other numerator.

Amy: It is like the denominators are equal.

Various students add their comments at this stage.

Amy: It is as if you have ones, if you are going over ones. It would be the same thing.

Tommy: If the denominators are the same on both, then the top equal.

Cindy: Yes, is that…do you agree with that.

Amy: Yes.

Various other students now voiced their agreement.

This episode contained evidence of some of the growth Cindy experienced during our work together. It was an instance that demonstrated Cindy's newfound ability to step aside and allow students to debate and argue over simple mathematical issues and these debates leading to clarifications by students for other students. Although Cindy fully intended to explain this concept to her students, at the end of this discussion she was satisfied that it was sufficiently dealt with during the discussion and did not return to it again. This was an example of the favorable situation where students' mathematical contribution dictated the direction of part of a lesson (NCTM, 2000; J. P. Smith, 1996). Episodes such as this one were very rare in the beginning of this research when Cindy demonstrated a tendency to dictate the course and content of such small discussions.

E6. Chris stated: "Switch the $\frac{8}{x^2-4}$ with the $\frac{2}{x-2}$."

The classroom episode occurred when Cindy and her students were getting ready to solve $\frac{x}{x+2}-\frac{8}{x^2-4}=\frac{2}{x-2}$ on the board. Cindy asked the class to choose whether they wanted to do the problem by finding the lowest common denominator or whether they wanted to multiply through by the lowest common denominator.

Chris: Would it not be easier to start out by like switching the place of the x…

Cindy: Wait, wait, wait, let me hear this…

Chris: Would it not be easier to start out by switching the place of the eight over whatever with the two over whatever because that…

Tommy: You cannot do that until you have the common denominator.

Chris: You can switch them at any time, it is just adding them, add both to one side and then subtract both from one side.

Cindy: Oh you want to move the $\frac{8}{x^2-4}$ over to the other side as a positive $\frac{8}{x^2-4}$, and then you want to move the other one…what is that going to do?

Amy: Well yeah, then it is going to…

Tommy: No but that will be difference of squares.

Amy: But what is that going to do for you though?

Chris: I do not know. It seems like it would work out easier.

Various students give their opinion.

Tommy: Well if you did that, if you did that, the common denominator will be the difference of two squares…of x^2-4.

Amy: Actually I see what he is saying now. That makes sense because if you move that one to the other side, all you have to do is…

Allison: From there all you would need to do is multiply that one by $x-2$ and that one by $x+2$.

Amy: That is true, it is the same thing.

Tommy: But what I am saying is, what I am saying is if you do what Chris does, the common denominator of both sides is going to be the, is going to be the difference of two squares of x^2-4.

Amy: It already is right now.

Cindy: It already is. Is that not what you were saying Jessica?

Jessica: Yes, all you …

Cindy: The common denominator is going to be the common denominator no matter which side you have all the stuff move to.

Amy: So there is really no point in moving them.

Chris: Right, it is just…yeah but if you have some other big problem it is going to be a lot easier to have all the factors…

For more than two minutes Cindy stepped back without intervening, allowing students to take part in a very interesting mathematical discussion. This happened despite the fact that Cindy knew, and confirmed during the coaching session that she knew, Chris's proposal was trivial. This was another indication that Cindy started to change the way she attended to students' mathematical contributions. During the subsequent coaching session discussion on this issue, Cindy explained that she even thought about a typical mistake that students were likely to make if they followed Chris's method. She mentioned that they might have seen an expression like $\frac{x}{x+2}-\frac{2}{x-2}=\frac{8}{x^2-4}$ as one where the lowest common denominator was already determined by reasoning "the left-hand side contains both a $x+2$ and a $x-2$ and the right-hand side contains both a $x+2$ and a $x-2$". This was an occasion where Cindy exhibited excellent KCS and utilized it in a manner that assisted with student learning. The fact that Cindy had this specific concern about the mathematical issue at hand but did not voice it was further evidence of change in the way she attended to this mathematical contribution. It was very satisfactory to see students like Amy come full circle with Chris's argument without any intervention from Cindy. Other students like Tommy were also learning actively by having to defend their arguments. An episode such as this did a lot to remove Cindy as the final authority that had to validate all mathematics that went on in the classroom and commanded students to take more responsibility for their own learning.

To conclude the two sections on Cindy's attention to students' mathematical contributions and provide the answer to the second research question, I would like to now summarize the change observed between the episodes in Section D and Section E. Over the course of the collaborative coaching Cindy attended differently to students' mathematical contributions. In the beginning Cindy tried to answer all questions by herself and did not give other students an opportunity to voice their opinions. Later on Cindy became much more open to input from other students. In fact she actively invited student participation (TR in Section E3) and actively took a back seat while encouraging students to argue their cases (Section E1). She allowed students' mathematical contributions to dictate the sequence of some of her lessons (Tommy on Section E5) and some of her homework assignments (Josh in Section E4). She even allowed lengthy discussions about issues that had little mathematical significance (Chris in Section E7). Cindy's actions were now more in line with "a discourse which values asking questions and sense-making, pursuing mathematical hunches and developing arguments, and thinking critically about the uses and consequences of uses of mathematics in society" (Nicol, 1999, p. 46). Cindy exhibited a shift in what actions she valued as described by M.S. Smith (2000) as "give answers versus value student thinking" (p. 354).

To put the change in Cindy's attention to student contributions in perspective, there were still many times that Cindy reverted back to the traditional way of attending to students' mathematical contributions. In fact, such instances were still in the majority when our work together ended. A larger and permanent shift in Cindy's attention where Cindy would exhibit reform-oriented attention on a more regular basis would only be brought about by a longer and more sustained collaborative coaching effort.

Contributing Cindy's Change to the Collaborative Coaching Sessions and its Focus on MKT

A variety of factors served as a basis for me to infer that the change witnessed in Cindy's views on teaching and learning and her altered attention to students' mathematical contributions could be attributed to our work in the collaborative coaching sessions generally. More specifically, I believe that at least part of the reason for Cindy's change could be attributed to the explicit focus on MKT during the collaborative coaching sessions. I would like to offer the following four arguments to authenticate this claim.

First, during the final part of the 11 October coaching session (at 51:00 to 55:00) Cindy spoke to me about her changed actions in class. She mentioned the fact that she focused more on students' mathematical contributions and when I asked her for the reasons for this change, she replied:

> I think it is because we are working together. I mean, honestly, I do not mean to just say that…

So here Cindy unambiguously attributed her growth to our work together. Furthermore, she elaborated by immediately adding:

> I am really trying to pay attention to what they are asking, so that I can really get to the root of what they are thinking…because what they are asking is kind of letting me know what they are thinking…It is like I am trying to listen to them a little bit more so that, maybe I can give them examples like 'okay this is what you think, but let me give you something that is close to what you are thinking…' So I am trying to be a little bit more conscious about what they are asking so maybe that leads me to think about what they are thinking.

While the first part of Cindy's statement linked the growth in her attention to students' mathematical contributions to our collaborative coaching in general, this part of the statement linked her growth specifically to the MKT focus. Up to this point in our coaching Cindy seemed specifically unsure about the mathematical component of students' contributions in a number of instances (i.e. Justin's comment: 'It's like the opposite of a graph' in Section B3 above). Our subsequent work on this mathematics during the coaching sessions gave Cindy clarity about the students' contributions in all but one instance during all fifteen coaching sessions. In other words, our MKT work during coaching cleared up Cindy's confusion about certain students' mathematical contributions and made her more attentive toward these contributions. Her improved attention was highlighted by "so that I can really get to the root of what they are thinking" (a KCS quality) and "maybe I can give them examples like…" (a KCT quality). For Cindy to juxtapose the statement that she contributed her growth to the coaching sessions and the statement about student thinking mentioned in the quotes above after five coaching sessions that focused mainly on MKT issues, and hardly ever on the general issue of student thinking, was an acknowledgement to the contribution of our focus on MKT to her altered views and actions. Because Cindy continued with the abovementioned conversation by mentioning certain MKT qualities that she would draw on to elicit better student thinking (see Section C2 above), I believe this episode was more an example of how our work and focus on MKT impacted on Cindy's altered actions than her altered views.

Second, during the collaborative coaching sessions, I used classroom situations from Cindy's own classes in which students made interesting mathematical contributions. The coaching content was therefore firmly situated in Cindy's own classroom. I showed these contributions to Cindy and together we worked on the mathematics involved in these questions.

Like the research on teacher learning suggested, most of our coaching sessions that focused on MKT, and especially those listed in Section A above, caused disequilibrium (M. S. Smith, 2000) and a high level of cognitive dissonance in Cindy (Ball & Cohen, 1999)**.** This was brought about by Cindy seeing mathematical topics she had taught for many years in fresh ways (why three variables could not be solved with only two equations by making use of the coefficient matrix), by discovering connections between topics for the first time (i.e. the relationship between the *y* intercept of two linear functions and the *y* intercept of the product), and by struggling with topics she thought she had under control (why certain rational equations yield extraneous solutions and other do not). They contained authentic problems from Cindy's class which excited her and, as in the example where Cindy excitedly spoke to her colleague about Anthony's contribution in Section A3 above, she showed that they caused her to realize that students could produce valuable and interesting mathematical contributions if given the opportunity. I also believe Cindy saw the potential contained in these mathematical contributions to be used as a springboard for rich mathematical classroom discussions. After all, they were responsible to teach her some things which excited her. My beliefs were substantiated by Cindy's energized comment during our second-to-last coaching session on 16 January 2007. She said: "You made me think today. You make me think every day."

A third reason why I feel that Cindy's growth could be attributed to our work in general and to its MKT focus specifically had to do with the video clips I used during coaching. In selecting video clips to use during coaching, I mostly focused on episodes that contained mathematics and Cindy's attention to students' mathematical contributions which I felt could be expanded, cleared up, altered, or improved. They were therefore episodes that contained the implicit message that Cindy's attention to these mathematical issues was limited. Most of our work during the coaching sessions also provided proof that this implicit message was indeed true since collaboratively we constructed expansions, clarifications, alterations and improvements. So I believe that this implicit message through the focus on MKT affected Cindy to change her attention to students' mathematical contributions. My belief was substantiated by the increasing number of times Cindy commented about the ingenuity of various students in her class. In other words, the disequilibrium brought about by the focus on MKT expansions of Cindy's work brought about the change in her practice.

Finally, on 6 October 2006 Anthony made his interesting comment about the unique feature of the vertex of a parabola (Section A3). On 11 October 2006 Cindy and I discussed this comment, worked on the mathematics of it and expanded our KCT by developing an algebraic proof for Anthony's conjecture. On 18 October Cindy exhibited a unique teaching action where she deviated from the prescribed lesson for the day and developed, with a relatively substantial amount of help from the class, an expression for the *y* coordinate of the vertex of a parabola ($y = \frac{4ac - b^2}{4a^2}$). The method she employed involved the same mathematical manipulations we used during the 11 October coaching session to prove Anthony's statement. This happened despite the fact that Cindy advanced through the 11 October episode tentatively while making, by her own admission, quite a few mathematical errors. On 19 October, a day on which I was not collecting data, Cindy tested $y = \frac{4ac - b^2}{4a^2}$ in class with her students (and found a mistake in it). On 20 October Cindy and I discussed $y = \frac{4ac - b^2}{4a^2}$ during coaching and discovered the unique error in it. Again we employed mathematical manipulations similar to the ones used on 11 October. During the 20 October coaching session Cindy agreed that she would facilitate a guided discovery exercise to help students discover the error and build a new, correct expression for the *y* coordinate of the vertex of a parabola. Furthermore, during the subsequent test on the chapter on parabolas, almost 20% of the students used this developed expression for the *y* coordinate of the vertex. This formula was not explained, developed, or used in the students' textbook. When I prompted Cindy about the reasons why she developed this unique expression, even when it was not in the textbook nor prescribed by neither the Dawson High School nor Grayson County plan, she could not explain why. After playing the video of the 18 October lesson to her again during the 20 October coaching session, she could still not say why she did it. I believe that the MKT focus of our collaborative coaching session on 11 October lead to Cindy's interests, her unique teaching actions exhibited on 18 October, and the use of this expression by a fifth of her students during the test. This chain of episodes provided evidence that specifically the MKT focus of the coaching sessions affected Cindy's actions.

In research that involved people as participants, there would always be the possibility of participants saying and doing what they know the researcher wanted to hear or see. I do not believe that the data I presented here was affected by this reality in any significant way and believe that the following explanation would clearly state this point. On the last day of our work together I asked Cindy what she told colleagues who asked what I was studying. To my surprise Cindy gave me an answer that convinced me she did not know what I was studying, whether I was focusing on her or her students, or what I was looking for in her or her students. She said:

I do not know. I just tell them that you are working on your dissertation.

Even with more prompts, Cindy could not articulate what it was that I was looking for in her class. I quickly ended this line of inquiry as I could see that she was embarrassed about this and felt that she had to know what I was researching but did not know. Furthermore, at the beginning of my study I gave Cindy a copy of my prospectus. This document clearly explained my research and the focus of my study. When I helped Cindy move from one class to another I found this document buried under a pile of other administrative work. Although I did not want to embarrass Cindy by asking whether she had read it, I could see from the state of the document and where I found it that she likely had not read it. This fact further helped confirm my belief that Cindy did not know what my focus was in her classroom. Therefore, I do not believe that Cindy ever said what I wanted to hear or demonstrated teaching that she thought I wanted to see. After my initial disappointment with Cindy’s answer to her colleagues about my research interests I quickly realized that this was a strength of this study and not a weakness. This episode convinced me that the views Cindy expressed and the teaching actions she performed were as authentic as could be expected with another person in the room.

CHAPTER FIVE

SUMMARY AND CONCLUSIONS

Summary

I conducted the study reported here in the quest for answers to the following two research questions:

1. In what way did collaborative coaching that specifically focused on the MKT of a secondary school mathematics teacher affect her views on the teaching and learning of mathematics?
2. In what way did the participant attend to students' mathematical contributions differently because of her participation in the collaborative coaching?

During the process of planning, designing, and implementing the study I reviewed and analyzed literature on teacher knowledge, professional development, and teacher learning. I considered two types of teacher knowledge literature namely 1) literature that described the various types of teacher knowledge and 2) literature that considered the effects of the various types of teacher knowledge on student learning. The literature on the different types of teacher knowledge assisted me to design a study that would focus on important teacher knowledge qualities such as the mathematical knowledge to deal with students' mathematical contributions in a manner that would enhance student learning (Ball, 2003; Fernandez, 1997; Hill et al., 2005; NCTM, 1991). What teachers know and the knowledge they were able to use in their teaching were important variables to consider as it had an effect on student learning (Aubrey, 1997; Ball & Bass, 2003; Carlsen, 1990; Carpenter et al., 1988; Fennema & Franke, 1992; Grouws &

Schultz, 1996; Hill et al., 2005; Ma, 1999; Millet & Johnson, 1996; Rowan et al., 1997). It was important for this study, designed to influence a high school teacher's views and actions through work on her MKT, to be aware of studies reporting on the impact of teacher knowledge in general, and MKT specifically, on student performance. Some of the important ones were the elementary school studies done by Hill et al. (2005) and Carpenter et al. (1988). Both these studies empirically linked types of teacher knowledge to types of student performance.

Because this study looked at the effects of professional development on the views and actions of a teacher, professional development literature influenced me in the design and implementation choices. This literature recommended focusing on parts of the participant's mathematical knowledge for teaching (Cohen & Hill, 2000; Garet et al., 2001; Grouws & Schultz, 1996; Guskey, 2003), doing the professional development over an extended period of time (Garet et al., 2001; Grossman et al., 2001), and to take into consideration ways that students and teachers learn effectively (Little, 1993; Wilson & Berne, 1999).

Furthermore, teacher learning literature emphasized the importance of relevant content for professional development (M. S. Smith, 2003; Wood et al., 1991), creating dilemma-induced conflict in the participant (M. S. Smith, 2000; Wood et al., 1991), and involving her in problem-solving during the professional development (Loucks-Horsley et al., 1998).

Because of my interest to design professional development that would be sustainable by the chair of a mathematics department in a high school, I chose collaborative coaching from a variety of coaching formats as the specific form of professional development (Costa & Garmston, 1994; Edwards & Newton, 1995; Loucks-Horsley et al., 1998; Poglinco et al., 2003; Russo, 2004; Showers & Joyce, 1982, 1996).

The MKT construct developed by Ball and her colleagues (2005) formed the theoretical framework of this study. Although Shulman's content knowledge and PCK categories (1987) initially offered an appealing lens through which to consider teacher knowledge, I found both categories too broad in that they did not discriminate between important and less important teacher knowledge qualities. The MKT construct that subdivided both content knowledge and PCK into further categories afforded me the opportunity to focus on more specific teacher knowledge qualities such as SCK, KCS, KCT, and knowledge of the mathematical horizon.

I selected Cindy Miller as the participant of my study. Cindy was an experienced high school mathematics teacher who worked in a rigid school system as it did not allow for any deviation from the daily, monthly and yearly planning. Cindy taught this curriculum in a traditional manner. She started out with a rigid view on teaching which placed the teacher in total control of all teaching. Cindy's view on learning was also very traditional as she saw students in a passive role in the classroom, practicing problems, copying notes, and relying on her explanations to make sense of all mathematics. Cindy attended to students' mathematical contributions in a traditional manner. This involved answering all questions and validating all comments and conjectures without involving other students in the process.

During the collaborative coaching sessions Cindy and I focused on interesting student contributions from her classes. We watched video clips where students made mathematical contributions and then discussed the mathematical (SCK and knowledge of the mathematical horizon) and teaching implications (KCT and KCS) of these contributions. During my questioning I always focused on the MKT components involved in the episodes and stayed clear of pedagogical discussions as far as possible. Some of the topics for which we enhanced our MKT through collaborative coaching were: using matrices to prove that three variables could (mostly) not be solved by two equations, developing three ways to algebraically prove that a parabola was symmetric, examining the relationship between the *y* intercept of two linear equations and the *y* intercept of the product, and discussing unique ways to look at the vertex of a parabola.

Results

Cindy showed change in her views on teaching and learning and demonstrated change in her attention to students' mathematical contributions. The answer to my first research question came when she started to view teaching as something that was not just done by a teacher talking but that it should involve students. She started to view learning as something that could be enhanced by variables such as whole class discussions with peers and students answering each others' questions. The answer to the second research question was that Cindy started to attend to students' mathematical contributions in a more reform-oriented manner as advocated by the three NCTM documents (1989; 1991; 2000) and the British Department for Education and Skills document (2001). I now witnessed various episodes where Cindy involved students in discussions about a student contribution by encouraging them to clarify, justify, and argue their points. She deliberately stood back and allowed students to contribute to the teaching and learning process in her classes by making and testing conjectures themselves. Although this change did not occur constantly, the fact that it did occur on occasion was a significant change to a teacher who had strong traditional views and actions and who taught in a rigid, traditional system.

Linking the Research Results to the MKT-Focused Collaborative Coaching

I attributed the changes in Cindy's views and her actions to our collaborative coaching and specifically to the MKT focus of our work. (For a detailed description of this section, see the end of Chapter Four.) First, Cindy herself explicitly attributed her change to our work together after the first five sessions. Specifically, she mentioned MKT qualities such as “so that I can get to the root of what they are thinking” (a KCS quality) and “maybe I can give them examples…” (a KCT quality) in her explanation for why she attributed her growth to our work together. Second, the combination of disequilibrium (brought about by working on students’ mathematical contributions) and excitement (brought about by managing to elaborate on these contributions during coaching) motivated Cindy and convinced her to think about students’ and their learning differently and to attend to their contributions with more care. Third, our work on students’ mathematical contributions was prompted by classroom video clips. The inherent message from these clips was that the classroom episodes were mathematically limited. Our work during coaching confirmed this message as we expanded the related mathematics in all but one coaching session. Seeing numerous classroom episodes mathematically expanded caused Cindy to experience a dilemma in the way she viewed teaching and learning and the way she attended to students’ mathematical contributions. I believe this dilemma, brought about by our MKT-focused work, caused students to move into an active role in Cindy’s views on teaching and learning as well as in the way she attended to their contributions. Finally, our work during one particular coaching session on 11 October set off a series of unique actions by Cindy that I believe could logically only be attributed to the MKT focus of our coaching sessions. I believe linking Cindy’s progress to the MKT focus of our collaborative coaching could be backed up with what Brophy (1991) wrote: “Where [teachers’] knowledge is more explicit, better connected, and more integrated, they will tend to teach the subject more dynamically, represent it in more varied ways, and encourage and respond fully to student comments and questions” (p. 352). In relation to this quote from Brophy, I believe Cindy was at a point in her career that sustained collaborative coaching that focused on MKT would enable her to teach in such a manner. Without sustained collaborative coaching, I doubt if Cindy would maintain all aspects of the progress she made throughout the coaching.

Implications

This study showed that collaborative coaching could be used in a one-on-one setting as a successful format for professional development. The effects of collaborative coaching should no longer be called purely anecdotal (Russo, 2004). This research supports the claim that coaching could be used as an effective tool for professional development.

The collaborative coaching was responsible for numerous advances in the MKT of both Cindy and me. These advances in teachers' MKT have proven difficult to me to achieve through other professional development formats I have used. This study showed that collaborative coaching was an effective way to focus on the MKT of a teacher and to have an effect on it. This conclusion is an important addition to the secondary school teacher knowledge literature.

Considering that Cindy and I were both experienced high school mathematics teachers with extensive graduate school and mathematics qualifications, I believe this study provides evidence that collaborative coaching focused on the MKT of participants might contribute to an increase in the MKT of other teachers, including elementary and middle school teachers who were not specifically trained as mathematics teachers. I suspect that the increase for elementary and middle school teachers might come more in CCK than it did for the two of us. When the result of teachers' MKT on student performance by Hill et al. (2005) is considered, it would be a worthwhile exercise to invest in professional development advocated by this research.

This study originated out of my desire to find answers for my own teaching dilemmas and my professional development dilemmas as the head of a high school mathematics department. So it was not only important for me to find answers to the question of effective professional development but also an answer to the dilemma of practical and applicable professional development as part of the job description of the head of a mathematics department. This does not exclude other department members to take the active role in the coaching, sharing or rotating this active role, or pairing up with one other colleague for the collaborative coaching. All of these variables would be determined by the unique dynamics within the department. I believe that the professional development Cindy and I did through our collaborative coaching sessions would be viable in a high school setting without changing the elements that made it effective. For this type of collaborative coaching to be duplicated effectively by the head of a mathematics department or any other department member, the following modifications will have to be done:

- Participation will change from two to potentially fifteen or more teachers. This will give the group a different dynamic but will not affect the MKT focus. If anything, more people will be able to contribute which could have a positive effect on other participants' MKT. On the other hand, the social dynamic of a bigger group would be different and should present different challenges. A teacher who might be willing to open up her practice to one person might feel more inhibited when her practice is opened up to a group of colleagues.
- Video clips will have to come from the classrooms of all participants. This should not affect the professional development negatively nor should it change it. Video clips are used to authenticate the problems that are used in the professional development (M. S. Smith, 2003).

- Collaborative coaching will possibly not be done twice per week. If mathematics departments have a weekly meeting, the sessions could be done once per week and sustained year-round. In that regard the real high school mathematics department would be able to do better than this study since it would be able to adhere to literature that advocate for a sustained professional development effort (Ball et al., 2001).

Limitations

The theoretical framework of a study should be the tool used to design the study, practically implement the study, analyze the data, report the research results, and interpret the research results to answer the research questions and reach conclusions. With these criteria for a theoretical framework in mind, MKT had certain limitations as the theoretical framework for this study. The first limitation pertained to the data analysis and the results and conclusion stages of this study. These stages of this study focused on the views and actions of the participant as they pertained to the two research questions. In order to answer these research questions, I paid attention to traditional teacher views and actions and compared them to reform-oriented views and actions. Teacher knowledge categories such as the ones in the MKT construct did not assist me with this process nor did it afford me the opportunity to talk about the changes I witnessed in Cindy’s views and actions in terms of the MKT categories. I believe that this framework would serve better in assisting with an analysis of the data and answering research questions in a study that employed the same coaching strategy but that looked at the effects longitudinally. MKT could, in this regard, be used to look at a research question such as: “In what way does growth in the MKT of a teacher over two or three years impact her views on teaching and learning and her teaching actions?” In such study mathematics content would be discussed and taught more than once. MKT improvements could then be discussed in terms of this framework and it could then be used to discuss teacher change. For example, in year two of a study on the effects of collaborative coaching which focused on MKT, witnessed changes in the way a participant attended to students' mathematical contributions could possibly be contributed more directly to a growth in MKT as a result from work in the previous year rather than a general MKT focus during coaching. As we witnessed in a variety of cases with Cindy, lack of appropriate MKT handicapped her in certain episodes. Although lack of MKT is not an excuse to brush off students' mathematical contributions, improved MKT could serve a teacher well when attending to complicated students' mathematical contributions. In this study, which gave a snapshot of Cindy's MKT, the MKT construct as a theoretical framework was not particularly useful to analyze growth in her views and actions.

The MKT construct is currently used in practice as a static measurement of what teachers know (Hill et al., 2005) or a way to classify various teacher knowledge qualities under specific categories (Ball et al., 2005). It therefore did not assist me to talk about the changes that I believe occurred in Cindy's MKT. Although changes in Cindy's MKT were not a focus of this study or part of any research question, I believe that changes in Cindy's MKT did occur in a meaningful way. Cindy, through our coaching sessions, did expand certain MKT qualities such as her ability to make mathematical connections between certain parts of the curriculum (KCT), but the MKT construct did not assist me to link these coaching sessions to the growth I witnessed in her views on teaching and learning and her attention to students' mathematical contributions. To make the link between our coaching and changes in Cindy's views and actions (described in the *Results* section above) I rather used (non MKT) comments from Cindy and her answers to (non MKT) questions during the collaborative coaching sessions.

It became evident during this study that, although I did not have a bias towards any specific section of the MKT construct, not all of the MKT categories were equally useful during the collaborative coaching. In particular, I did not find CCK to be a particularly helpful category and still had trouble practically distinguishing between SCK and KCS during our work. These problems might suggest rethinking the MKT categories altogether or putting in a greater effort to define each category better and to give more practical day-to-day teacher knowledge qualities as examples of each category.

The study made no claims about what specific advantages would flow from collaborative coaching that focused on MKT to larger groups of participants. The unique dynamic of this study made for certain types of unique interactions that would be difficult to duplicate in a larger group. Among other things, the fact that all video clips came from Cindy's class and was therefore of specific personal interest to her would be different in a larger group. The logistical issues around the meetings of a pair of people versus that of a larger group would also play a role in how this study could be duplicated and generalized. Although I believe that collaborative coaching would be practically applicable within a high school mathematics department and a useful professional development effort, the extent and specific characteristics of the impact would be hard to predict or equate with the results from this case study.

Cindy taught in a specific setting that placed a specific set of conditions on her teaching. I believe that the results of this study should be viewed in light of these unique constraints and that different constraints would impact differently on the results. If anything, I think that possible results might just be more favorable than what was achieved within this rigid system. The fact that Cindy altered the homework she assigned to students once (Section E4), skipped the explanation of a specific concept because of students' discussion thereof (Section E5), or allowed students to discuss mathematics interesting to them for a few minutes on a number of occasions (Sections E1, E2, E3, and E6) was huge when seen against the backdrop of the constraining system within Dawson High School.

As participant, Cindy with her traditional views and teaching methods brought a unique set of variables to the collaborative coaching. These variables played a unique role which will be different with larger groups as well as with individuals with different views and teaching methods. Although I believe traditional teaching and a traditional view of teaching and learning is prevalent among secondary school teachers, the collaborative coaching might yield a different set of results when participants were more reform-oriented from the outset.

Future Research

Because this study was done with only one participant, the first line of inquiry calls for examining the effects of involving more participants in this type of collaborative coaching. Although, due to the results from this study, the success of coaching as a form of professional development would no longer be described as purely anecdotal (Russo, 2004), the results of coaching among larger groups would still require further research. Such research would assist in making better assumptions and recommendations about the practical applicability of this type of collaborative coaching. When looking at the work of numerous authors who described effective professional development (Franke & Kazemi, 2001; Garet et al., 2001; Halai, 1998; Hawley & Valli, 1999; Wilson & Berne, 1999), we can see that collaborative coaching fits the profile of effective professional development well. Therefore, it is intuitive to think that collaborative coaching focused on the MKT of a larger group would be an effective tool for teacher change. The field of professional development needs research to confirm this intuition.

A second line of inquiry that would extend this work would be to do a longitudinal study to track the permanency of these changes in a teacher. A longer period of coaching would be more beneficial (Garet et al., 2001; Grossman et al., 2001) and it would be valuable to research the type of changes in teachers' views and actions over a longer period of time. Moreover, this study did not claim that the changes in Cindy's views and her actions represented the majority of her teaching. On the contrary, Cindy's attention to students' mathematical contributions in a reform-oriented manner was still by far in the minority. Research conducted over a longer period of time will tell whether this ratio of reform-oriented versus traditional teaching would remain or change.

Another line of future inquiry that would be an advantageous extension of this work would be to see whether the advances made toward reform-oriented views and teaching would affect student learning. It seems significant if a traditional teacher changed his views on teaching and learning from pure teacher-driven instruction toward student discussion and involvement. Whether this change would result in improved student learning or student achievement would need research confirmation.

The specifics and extent of Cindy's MKT growth was not a focus of this study. I believe that it would be a worthwhile endeavor to follow up a study such as this with an in-depth look at the specific nature of Cindy's MKT change and the extent thereof. Although we worked on only a selected section of the curriculum, I suspect that Cindy gained knowledge that would impact the teaching of other related content also. Such a suspicion would only be validated through further research. Furthermore, the impact of Cindy's MKT growth on her teaching and her students' learning could be another very important line of inquiry. The question remains whether a secondary school teacher’s MKT has such a strong impact on students’ achievement as the MKT of elementary school teachers (Hill et al., 2005).

Final Thoughts

A researcher involved in any research often cannot avoid being affected by some aspect of the research. In this regard I was also affected by my own research in a major way. First, I was affected by my own research by being present in another teacher’s classroom for almost five months. Since most teachers live very isolated professional lives from each other, this experience was a huge privilege for me. I learned an enormous amount by just being present when Cindy taught. Her unique social skills, administrative processes, grading methods and practices, and procedures of discipline and classroom management taught me a number of valuable lessons.

I expanded my MKT through our work together and saw this as one of the values of discussing mathematical issues with a colleague. On many occasions I was surprised to learn something new about mathematical content that I have been teaching for years, some of which so simple, I would almost be ashamed to elaborate on them. Yet, I do not believe my experiences would be unique, especially since I saw Cindy also learning new things or seeing old things in new ways. Mathematics is such a vast, interconnected field of ideas and concepts that no one, especially not any mathematics teacher, dares to approach it with any form of academic arrogance.

The mathematical content that is taught in a mathematics class is an important variable but is by far not the most important variable in the mathematics classroom. I witnessed in Cindy’s class content that was very new to me as a foreigner. I was also aware of other content that was missing from Cindy’s class that was a regular part of the curriculum I taught. Yet, I do not believe these seemingly major differences in curriculum would benefit or handicap the mathematical learning, understanding, or performance of any of the students involved. This has convinced me of the insignificance of curricular debates such as what content should be included into a curriculum or the sequencing of content within a curriculum. What matters much more is how we teach mathematics, not what mathematics we teach.

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APPENDIX A

Exact Dates and Times of Events Mentioned in Chapter Four

A1. 10.06.2006 Coaching at 30:04: Mandy asked, "Can you solve three variables with only two equations?" (Lesson on 10.03.2006 4th at 30:30)

A2. 10.11.2006 Coaching at 04:00: Duncan asked: "What is the relationship between the *y* intercept of a parabola and the *y* intercepts of its two linear parts?" (Lesson on 10.6.2006 4th @ 02:35)

A3. 10.11.2006 Coaching at 12:20: Anthony commented: "The vertex is the one place on the graph where a *y*-coordinate has only one *x*-coordinate." (Lesson on 10.6.2006 6th @ 40:00)

A4. 10.20.2006 Coaching at 07:55 up to 18:00: The discussion about Cindy's incorrect expression for the *y* value at the vertex.

A5. 10.24.2006 Coaching at 04:20: Cindy algebraically proved that a parabola was symmetric.

A6. 01.16.2007 Coaching at 22:40: We looked at the explanation for why $\frac{x}{x+3}=\frac{6}{x-1}$ has two valid solutions but $\frac{x}{x+2}-\frac{8}{x^2-4}=\frac{2}{x-2}$ has only one valid solution.

B1. Interview at 16:40 and 18:15: Cindy commented on facilitating classroom discussions and her actions and struggles with them.

B2. 09.14.2006 Coaching at 01:00:40: Cindy explained her plan with students' mathematical contributions.

B3. 10.03.2006 Coaching at 29:20 and 33:40: Justin commented: "It's like the opposite of a graph."

C1. 10.11.2006 Coaching at 32:30: Cindy said, "I give it away in class too often."

C2. 10.11.2006 Coaching at 45:18: Cindy commented: "I was proud of myself."

C3. 11.24.2006 Coaching at 18:10: Cindy said: 'I just hope Mandy does not ask a ton of questions'

C4: 11.14.2006 Coaching at 11:00: Cindy's new progress report comments.

C5. 11.14.2006 Coaching at 01:00 and 1.19.2007 Coaching at 03:00 – 08:00: Cindy commented about her goals with students' mathematical contributions.

C6. 1.16.2007 Coaching at 41:50: Cindy made comments about the benefits of our coaching sessions.

D1. 08.29.2006 4th at 36:38 – 38:56: Mandy asked: "What are functions used for in real life?"

D2. 09.08.2006 4th at 42:15 – 42:55: Tanya asked: "Will it ever be a horizontal line?"

D3. 09.27.2006 6th at 24:05: Justin commented: "It is like the opposite of a graph."

D4. 10.03.2006 4th at 30:30: Mandy asked: "Can you solve three variables with only two equations?"

D5. 10.06.2006 4th at 27:55: Duncan asked: "What is the relationship between the *y* intercept of a parabola and the *y* intercepts of its two linear parts?"

E1. 10.18.2006 4th at 06:20 – 08:00: Cassandra said: "Subtract $\frac{b^2}{4a^2}$ from the right-hand side."

E2. 10.13.2006 6th at 18:55: Tanya asked: "What do you do to the other side?"

E3. 10.13.2006 6th at 12:10: Tommy said: "If it is negative it is a compression."

E4. 10.24.2006 4th at 50:54: Josh asked: "Are you always going to shade, no matter the equation?"

E5. 01.12.2007 6th at 34:00: Tommy commented: “If you have a numerator over a denominator that equals a numerator over the same denominator…”

E6. 01.12.2007 6th at 48:50: Chris stated: “Switch the $\frac{8}{x^2-4}$ with the $\frac{2}{x-2}$.”