Donald J. Bindner
On the Space Spanned By the Powers of an Operator and Its Adjoint
(Under the direction of Edward A. Azoff)

In this work, the reflexivity properties of the space spanned by the powers of an operator and its adjoint are investigated. For an operator A, let $\mathcal{T}_A$ be the weak-star closure of $\{A^j, (A^*)^j \mid j \geq 0\}$. Call A star-reflexive if $\mathcal{T}_A$ is reflexive and star-transitive if $\mathcal{T}_A$ is transitive.

The regular unilateral shift is known to be star-transitive, as are the unicellular shifts. We will discover that many non-injective weighted shifts are not star-transitive, but rather star-reflexive. Among the operators found to be star-reflexive are direct sums of non-nilpotent shifts, $0 \oplus T$ for injective shifts T where T^* has a nonzero eigenvalue, and $A \oplus A$ where A is a Jordan matrix.

There is an explicit construction of a star-reflexive weighted shift T similar to the regular unilateral shift S. This contrasts distinctly with the star-transitive nature of S. The similarity that carries S to T is of the simplest type, a diagonal matrix. This example illustrates clearly that the property of being star-reflexive is not invariant under similarity.

The question of similarity and star-reflexivity is also explored for $n \times n$ matrices. It is shown that M_n contains an open dense set of matrices that are both reflexive and star-transitive. It is also shown, for $n \leq 4$, that the only star-reflexive, complex-valued matrices in M_n with n distinct eigenvalues are normal matrices.

Index words: Operator theory, Unilateral shift, Weighted shift, Reflexive, Transitive, Direct sum, Weak-star topology

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Chapter 1

Introduction

One of the fundamental problems in operator theory is to understand the invariant spaces of an arbitrary linear operator on a Hilbert space, that is spaces that are mapped into themselves by the operator. One goal of such a study is to identify when two operators are essentially the same. (Obviously, operators that have different invariant spaces cannot be the same.) This is a manifestation of the classic question shared by all branches of mathematics; how do we differentiate objects from each other?

There can be obstacles to the study of invariant spaces. For example, it is not even known if every linear operator on an infinite dimensional Hilbert space has (nontrivial) invariant spaces. This is known as the invariant subspace problem and forms a central question of operator theory. However, many operators have rich collections of invariant spaces. Among these are the weighted shift, normal, and compact operators.

Generally, the operators of most prominence in the field of operator theory are those that act on infinite dimensional spaces, but there is no lack of interesting finite dimensional questions. Even operators on a four or five-dimensional space can be elusively subtle, as we shall see in Chapter 6.

1.1 Reflexivity

If M is invariant under A (that is $AM \subseteq M$), it is clear that M is invariant under any polynomial in A. Consequently, when investigating invariant spaces, it is natural to look at algebras of operators. It is a time-honored technique to glean information about an operator A from the algebra $\mathcal{A}_A$ it generates. For example, the minimal polynomial of a linear transformation A acting on a finite dimensional vector space tells us the size of the largest Jordan block corresponding to each eigenvalue of A.

The algebraic structure of $\mathcal{A}_A$ does not, however, distinguish between A and $A \oplus A$. For this, we may look at the action of $\mathcal{A}_A$ on the underlying vector space. In particular, the invariant spaces of A determine the rest of its Jordan canonical form.

If $\mathcal{A}$ is an algebra, and $\mathcal{M}$ is the collection of invariant spaces of A, one could ask which operators keep invariant all of the members of $\mathcal{M}$. This set is called the reflexive closure of $\mathcal{A}$, and may be larger than $\mathcal{A}$. If $\mathcal{A}$ exhausts its reflexive closure, it is called reflexive.

One may think of reflexive algebras as algebras having many invariant spaces, or as algebras about which invariant spaces give a lot of information. One simple reflexive algebra is the algebra generated by the identity operator, which leaves every space invariant. The algebra of all bounded operators is also reflexive. Some other straightforward examples can be found in the 2×2 matrices. The algebra $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$ (stars may be chosen freely) is easily seen to be reflexive since $\begin{bmatrix} * \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ * \end{bmatrix}$ are invariant spaces. On the other hand, $\begin{bmatrix} \lambda & * \\ 0 & \lambda \end{bmatrix}$ is not, since its only non-trivial invariant space is $\begin{bmatrix} * \\ 0 \end{bmatrix}$, which is preserved by any upper triangular matrix.

Many of the nicest operators generate reflexive algebras. For example, normal operators generate reflexive algebras, as do large classes of weighted shifts. Of particular note is the regular unilateral shift which generates the reflexive algebra of analytic Toeplitz operators. The operators on finite dimensional spaces that generate reflexive algebras are well classified, and the identification of reflexive algebras in infinite dimensional settings has been widely pursued.

For Hilbert space operators, it is natural to account for the underlying inner product structure by studying the algebra generated by A and A^*. This is highly successful for normal operators. More generally, the study of non-commutative von Neumann algebras sheds light on many non-normal operators; see for example, John Ernest's memoir[12].

Unfortunately, the theory of von Neumann algebras cannot help with some of the most important operators. The best known non-normal operator is the unilateral shift S. Much of this is due to the deep connections between its algebra, $\mathcal{A}_S$, and the function theory developed by A. Beurling. However, the von Neumann algebra generated by S is all of $\mathfrak{B}(\mathcal{H})$ which yields no information about S.

One of the discoveries in the study of reflexivity has been that reflexivity questions can and should be applied to linear spaces that do not happen to be algebras. V. S. Šul′man was the first to adapt the definition of reflexive to apply to arbitrary linear spaces in [31]. The linear space generated by polynomials in A and polynomials in A^* (and closed in the appropriate topology) is generally not an algebra, but it allows us to take into account the adjoint of A in a controlled way, and has the potential for interesting reflexivity properties. We call the space generated by A and A^* the star power span of A, and denote it by $\mathcal{T}_A$. The full space of Toeplitz operators is a space of this type; it is the star power span of the regular unilateral shift. In [9], J. B. Conway and M. Ptak study star power spans of operators which, like the unilateral shift, admit a particularly rich functional calculus.

1.2 Summary of results

The reflexivity properties of linear spaces generated by an operator and its adjoint form the general subject of study in this thesis. Identifying operators A for which $\mathcal{T}_A$ is reflexive is a particular goal, and these operators are referred to as star-reflexive.

Of particular interest are weighted shift operators, which share many properties with the Toeplitz operators. One of the more obvious shared properties is "stripedness", the subject of Chapter 3. The matrix of a Toeplitz operator is easily identified because it remains constant on each subdiagonal (it has constant "stripes"). If A is a weighted shift, the matrix of any member of $\mathcal{T}_A$ is easy to identify; each diagonal is a constant multiple of a power of A or A^*. In fact, the matrix consisting of any single diagonal "stripe" of a member of $\mathcal{T}_A$ always belongs to $\mathcal{T}_A$ as well. The main conclusion of Chapter 3 is that the striped nature of $\mathcal{T}_A$ bleeds over into many spaces that are related to it. Most notably, the reflexive closure of $\mathcal{T}_A$ contains the stripes of all of its members.

Chapter 4 develops some general reflexivity results for direct sums of shifts. Special attention is given to direct sums involving injective shifts T for which T^* has a nonzero eigenvalue. The main result is Theorem 4.8 which characterizes the reflexive closure of the star power span of a direct sum of shifts $T = V \oplus W$ in terms of those of V and W.

Several interesting corollaries follow from Theorem 4.8. First, it gives a general context for some known results, in particular implying that direct sums of injective shifts are reflexive (a result of Hadwin and Nordgren [14]). More generally it implies that any direct sum of (two or more) non-nilpotent shifts is reflexive, and furthermore that the same direct sum is star-reflexive.

Another interesting conclusion from Chapter 4 is that $0 \oplus T$ is star-reflexive whenever T is an injective shift and T^* has a nonzero eigenvalue. This was something of a surprising discovery. The same fact for the regular unilateral shift, an example of a shift of this type, was known only recently (see [3]).

The $0 \oplus T$ discovery of Chapter 4 leads naturally to the analysis taken up in Chapter 5. If S is the unilateral shift, there is another way to think of $0 \oplus S$. The shift $0 \oplus S$ is merely the weighted shift formed by zeroing out the 0th weight of S. Since the regular shift S is not star-reflexive, but $0 \oplus S$ is, a natural question might be to ask where the change in behavior occurs.

Chapter 5 answers this question by looking at unilateral shifts T that are perturbations of the regular shift formed by varying only the 0th weight. The investigation is decidedly analytic. When the 0th weight of T is greater than $\sqrt{1/2}$, it is a consequence of Jensen's inequality that T cannot be star-reflexive. In this case, the star power span of T exhibits fundamentally the same reflexivity properties as occur for the regular shift (i.e. when the 0th weight is one).

When the 0th weight of T is less than $\sqrt{1/2}$, quite the opposite occurs; T is star-reflexive (the same behavior already discovered when the 0th weight is zero). This is quite surprising: even a very concrete matricial view of the star power span of T does not seem indicate such a point of demarcation. For example, the following classes of bounded operators have decidedly different reflexivity properties, despite what would appear on the surface to be only a "cosmetic" difference in their 0th row and column:

$$\begin{bmatrix} \lambda_0 & \frac{1}{4}\lambda_1 & \frac{1}{4}\lambda_2 & \frac{1}{4}\lambda_3 & \cdots \\ \frac{1}{4}\lambda_{-1} & \lambda_0 & \lambda_1 & \lambda_2 & \\ \frac{1}{4}\lambda_{-2} & \lambda_{-1} & \lambda_0 & \lambda_1 & \\ \frac{1}{4}\lambda_{-3} & \lambda_{-2} & \lambda_{-1} & \lambda_0 & \\ \vdots & & & & \ddots \end{bmatrix} \qquad \begin{bmatrix} \lambda_0 & \frac{3}{4}\lambda_1 & \frac{3}{4}\lambda_2 & \frac{3}{4}\lambda_3 & \cdots \\ \frac{3}{4}\lambda_{-1} & \lambda_0 & \lambda_1 & \lambda_2 & \\ \frac{3}{4}\lambda_{-2} & \lambda_{-1} & \lambda_0 & \lambda_1 & \\ \frac{3}{4}\lambda_{-3} & \lambda_{-2} & \lambda_{-1} & \lambda_0 & \\ \vdots & & & & \ddots \end{bmatrix}$$

The bounded operators of the left type form a reflexive class, while the bounded operators of the right type do not. This forms the main result of Chapter 5.

The final chapter compares normality to reflexivity. The intent is to gain some understanding of the way similarity interacts with star power spans. For comparison, note that nearly all of the shifts of Chapter 5 are mutually similar, but they fall into distinctly different classes with respect to star-reflexivity (i.e. with respect to properties of their star power spans). The discussion centers around operators having distinct eigenvalues. Every operator A on $\mathbb{C}^n$ having distinct eigenvalues is similar to a normal operator (which is always both reflexive and star-reflexive). Because basic reflexivity is similarity invariant, A is automatically reflexive. The question of Chapter 6 is whether normality is forced when star-reflexivity is also assumed. The question seems to have a positive answer; we have verified that star-reflexive operators on $\mathbb{C}^n$ with distinct eigenvalues are in fact normal when $n = 2, 3, 4$. For some other classes, the conclusion can be verified for any n, most notably the class of staircase operators with distinct eigenvalues. In a sense, this is a measure of how badly similarity interacts with star-reflexivity. It would seem to indicate that no operator similar to a normal operator can be star-reflexive unless it is normal already. An interesting consequence of the analysis is the discovery of an open dense set of star-transitive operators each of which is similar to a normal operator.

Chapter 2

Basic Principles

2.1 Notation and Terms

A general reference to the topics and terms used throughout this thesis is J. B. Conway's *A Course in Operator Theory*[8], which includes background for the spaces, topologies and operators that will be the subject of study. Further reading about Hardy spaces is available in [15] or [19], and a thorough development of the trace class is in [28]. A basic reference on functional analysis and operator theory is [7]; and the linear algebra topics, including Schur products, are covered well in [16]. A more leisurely discussion of reflexivity and related concepts can be found in [26], although the perspective used in [2] is more parallel to this work.

The following notation will be used throughout:

$\mathcal{H}$ will denote a separable complex (finite or infinite dimensional) Hilbert space. All Hilbert spaces of the same dimension are isometrically isomorphic, so if we wish to specify a finite dimensional space, we shall take it to be $\mathbb{C}^n$. $\mathcal{H}^2$ will be the Hardy space of square integrable analytic functions on the circle, and ℓ^2 will be the space of square summable sequences.

Lower case Arabic letters denote members of $\mathcal{H}$. In particular, f, g are typically used as members of the function space $\mathcal{H}^2$. In this setting, $\hat{f}(j)$ indicates the jth Fourier coefficient of f; that is, f is defined by the series $f(z) = \sum_{j=0}^{\infty} \hat{f}(j) z^j$. Where appropriate (most notably $\mathbb{C}^n$, $\mathcal{H}^2$, and ℓ^2), $\{e_j\}$ denotes the standard orthonormal basis. In particular, $e_j = z^j$ in the Hardy space.

The inner product of $\mathcal{H}$ will be indicated by angle brackets, as in $\langle e_i, e_j\rangle = 0$ for $i \neq j$. For $f, g \in \mathcal{H}^2$, the standard inner product is $\langle f, g\rangle = \sum_{j=0}^{\infty} \hat{f}(j)\overline{\hat{g}(j)} = \int_{\mathbb{T}} f\overline{g}\, d\mu$ where μ is normalized Lebesgue measure on the circle.

$\mathfrak{B}(\mathcal{H})$ denotes the space of bounded linear operators acting on $\mathcal{H}$. Single operators will be denoted by capital letters, $A, B \ldots$, and sets or linear spaces of operators by script letters $\mathcal{A}, \mathcal{B} \ldots$. The letter S is reserved for the regular shift, which moves a (finite or infinite) basis $Se_j = e_{j+1}$. Similarly, T is a weighted shift: $Te_j = w_j e_{j+1}$, where $\{w_j\}$ is a (finite or infinite) sequence of complex numbers. In each case, when the basis is finite, the last basis vector is sent to zero. Of course, I is the identity operator. $\mathfrak{B}(\mathbb{C}^n)$ is identified with M_n, the space of $n \times n$ matrices with complex entries.

The trace of a positive operator B is $\operatorname{tr} B = \sum_{j=0}^{\infty} \langle Be_j, e_j\rangle$ and may be finite or infinite. The trace norm of any operator B is $||B||_1 = \operatorname{tr}(\sqrt{B^*B})$. The trace class, $\mathfrak{T}$, is the set of operators B with $||B||_1 < \infty$, and forms a complete linear space under the trace norm topology. Every $B \in \mathfrak{T}$, whether positive or not, has a well-defined trace. Moreover, $\mathfrak{B}(\mathcal{H})$ is the dual of the trace class. That is, for every bounded linear functional ϕ defined on the trace class, there is an operator A such that $\phi(B) = \operatorname{tr}(BA)$ for all $B \in \mathfrak{T}$. Whenever at least one of A, B is in the trace class, angle brackets will denote this product trace $\langle A, B\rangle = \operatorname{tr}(BA)$ according to the usual convention.

Dually, each trace class B defines a linear functional ψ on $\mathfrak{B}(\mathcal{H})$ according to the same product trace, $\psi(A) = \langle A, B\rangle$. The *weak-star topology* on $\mathfrak{B}(\mathcal{H})$ is the topology defined by the seminorms $\{\rho_B \mid B \text{ trace class}\}$ where $\rho_B(A) = |\langle A, B\rangle|$. In particular, a net $\{A_j\}$ converges weak-star to A if and only if $\langle A_j, B\rangle \to \langle A, B\rangle$ for every trace class B. The symbol $\operatorname{wk}^* \mathcal{S}$ will be used to indicate the weak-star closure of the set $\mathcal{S}$.

If $\mathcal{S} \subseteq \mathfrak{B}(\mathcal{H})$, write $\mathcal{S}_\perp$ for its *preannihilator*, i.e. $\mathcal{S}_\perp = \{B \in \mathfrak{T} \mid \langle A, B\rangle = 0 \text{ for all } A \in \mathcal{S}\}$. Similarly, the *annihilator* of $\mathcal{T} \subseteq \mathfrak{T}$ is $\mathcal{T}^\perp = \{A \in \mathfrak{B}(\mathcal{H}) \mid \langle A, B\rangle = 0 \text{ for all } B \in \mathcal{T}\}$.

Given vectors $x, y \in \mathcal{H}$, the notation $x \otimes y$ will be used for the rank one operator $u \mapsto \langle u, y\rangle x$. The set of operators of rank one or less is $\mathfrak{F}_1$ and is a total subset of the trace class (that is, linear combinations of rank one members form a dense set in $\mathfrak{T}$). It also forms a weak-star total set in $\mathfrak{B}(\mathcal{H})$. Every member of $\mathfrak{F}_1$ has the form $x \otimes y$ for some x and y, and for $A \in \mathfrak{B}(\mathcal{H})$ we have $\langle A, x \otimes y\rangle = \langle Ax, y\rangle$.

For an operator $A \in \mathfrak{B}(\mathcal{H})$, the (identity-containing) weak-star closed algebra it generates, $\mathrm{wk}^* \operatorname{span}\{A^j \mid j \geq 0\}$, is denoted $\mathcal{A}_A$. Similarly, the *star power span* of A is the weak-star closed span of the powers of A and A^*, and is denoted $\mathcal{T}_A$. Obviously, $\mathcal{T}_A = \mathrm{wk}^*(\mathcal{A}_A + \mathcal{A}_{A^*})$.

2.2 Reflexivity

The following proposition (from [2]) gives three perspectives of reflexivity:

Proposition 2.1. *Suppose $\mathcal{S}$ is a linear subspace of $\mathfrak{B}(\mathcal{H})$, and $A \in \mathfrak{B}(\mathcal{H})$. Then the following are equivalent:*

(1) There is a vector $x \in \mathcal{H}$ which separates A from $\mathcal{S}$ in the sense that $Ax \notin [\mathcal{S}x]^-$ where the bar denotes norm closure (in $\mathcal{H}$).

(2) There is a rank one operator $x \otimes y$ which separates A from $\mathcal{S}$ in the sense that $B \perp x \otimes y$ for every $B \in \mathcal{S}$ but $A \not\perp x \otimes y$.

If $\mathcal{S}$ is an algebra containing the identity operator I, then these conditions are also equivalent to:

(3) There is a subspace of $\mathcal{H}$ invariant for every $B \in \mathcal{S}$ that is not invariant for A.

Historically, property (3) has been the source of the term reflexive closure. The reflexive closure of an (identity-containing) algebra $\mathcal{S}$ is the set of operators that leave invariant all of the invariant spaces of $\mathcal{S}$.

For linear spaces that are not algebras, property (3) cannot be used as the definition of reflexive closure. Property (2) gives a very convenient view of reflexivity, however, and inspires the definition we will use:

Definition 2.2. *If* $\mathcal{S} \subseteq \mathfrak{B}(\mathcal{H})$, *then the* reflexive closure *of* $\mathcal{S}$ *is* $\operatorname{ref}\mathcal{S} = (\mathcal{S}_\perp \cap \mathfrak{F}_1)^\perp$.

In general, $\mathcal{S} \subseteq \mathrm{wk}^* \mathcal{S} \subseteq \operatorname{ref}\mathcal{S}$. A *reflexive space* is a space for which equality holds: $\mathcal{S} = \operatorname{ref}\mathcal{S}$. Reflexive spaces are necessarily weak-star closed.

Opposite to the notion of reflexivity is the property of transitivity. To call $\mathcal{S}$ reflexive is, in effect, to assert that there exist many rank one operators in its pre-annihilator. In contrast, a space $\mathcal{S}$ is called *transitive* when $\mathcal{S}_\perp \cap \mathfrak{F}_1 = \{0\}$, or equivalently when $\operatorname{ref}\mathcal{S} = \mathfrak{B}(\mathcal{H})$. $\mathfrak{B}(\mathcal{H})$ is an example of an obviously transitive space.

It is customary to refer to an individual operator A as reflexive when its weak-star closed algebra $\mathcal{A}_A$ is reflexive. In the same way, call an operator A *star-reflexive* when $\mathcal{T}_A$ is reflexive, and *star-transitive* when $\mathcal{T}_A$ is transitive.

The term star-transitive was introduced by F. Gilfeather and D. R. Larson in [13]. The term star-reflexive is new, but the concept is implicit in [3] and [9].

There is one important sense in which the study of star power spans (whether it be for star-reflexivity or star-transitivity) is more difficult than the study of basic reflexivity. In traditional questions of reflexivity, similarity is an invariant that proves to be very useful. When C, D are invertible operators, $\operatorname{ref}(C\mathcal{S}D) = C\operatorname{ref}(\mathcal{S})D$, and in particular, $\mathcal{S}$ is reflexive if and only if $C\mathcal{S}D$ is reflexive. For an operator A, it follows that $\operatorname{ref}(\mathcal{A}_{C^{-1}AC}) = C^{-1}\operatorname{ref}(\mathcal{A}_A)C$, so A is reflexive exactly when $C^{-1}AC$ is reflexive.

The same is not true for star-reflexivity. The matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ are similar, so their algebras have the same reflexivity properties (both are reflexive, in fact). However, the star power span of A is $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$ (stars may be freely chosen), which is reflexive; and the star power span of B consists of matrices of the form $\begin{bmatrix} a+b+c & a \\ b & c \end{bmatrix}$, which is a transitive space. There are numerous such examples in this thesis. For example nearly all of the shifts encountered in Chapter 5 are mutually similar (and all reflexive), yet they fall into two distinct classes with respect to star-reflexivity.

Despite their different properties with respect to similarity, star-reflexivity and reflexivity are closely related:

Proposition 2.3. *Every star-reflexive operator in M_n is reflexive.*

Proof. Suppose A is star-reflexive, and assume that A is lower triangular without loss of generality. Write $\mathcal{L}$ for the space of lower triangular matrices. We will show that $\mathcal{A}_A = \mathcal{L} \cap \mathcal{T}_A$. This will suffice since $\mathcal{L}$ is reflexive, and the intersection of reflexive spaces is again reflexive.

It is obvious that $\mathcal{A}_A \subseteq \mathcal{L} \cap \mathcal{T}_A$, so assume that B belongs to $\mathcal{L} \cap \mathcal{T}_A$. Then there are polynomials p, q such that $B = p(A) + q(A^*)$. Now $q(A^*) \in \mathcal{L}$ implies that $q(A^*)$ is actually diagonal, and hence $q(A^*)^*$ belongs to $\mathcal{A}_{A^*}$ as well. But that puts $q(A^*)$, hence B, in $\mathcal{A}_A$ as desired. $\square$

2.3 Normal Operators

Normal operators have very nice reflexivity properties. To develop an understanding of them, it is helpful to make use of elementary spaces. Like the definition of reflexive, the notion of elementary involves rank one operators.

Definition 2.4. *If* $\mathcal{S} \subseteq \mathfrak{B}(\mathcal{H})$, *then* $\mathcal{S}$ *is* elementary *if* $\mathcal{S}_\perp + \mathfrak{F}_1 = \mathfrak{T}$.

There are at least two ways to think of elementarity: $\mathcal{S}$ is elementary if every trace class operator can be perturbed by a rank one to yield a member of $\mathcal{S}_\perp$. Another view is that $\mathcal{S}$ is elementary when for every $B \in \mathfrak{T}$ there exists a rank one F inducing the same linear functional on $\mathcal{S}$, i.e. $\langle A, B\rangle = \langle A, F\rangle$ for every $A \in \mathcal{S}$. This second perspective will be very important in the proof of Theorem 4.8.

Lemma 2.5. *Every maximal abelian von Neumann algebra is elementary.*

Proof. It is a consequence of the Spectral theorem that every maximal abelian von Neumann algebra $\mathcal{A}$ is unitarily equivalent to $\mathcal{L}^\infty(\mu)$ acting on $\mathcal{L}^2(\mu)$ (see, for example, Corollary 7.14 of [26]), so there is no loss of generality in assuming $\mathcal{A} = \mathcal{L}^\infty(\mu)$. A general trace class operator is $B = \sum_i f_i \otimes g_i$ where $\sum_i ||f_i||\,||g_i|| < \infty$. By Hölder's inequality, $u = \sum_i f_i \overline{g}_i \in \mathcal{L}^1(\mu)$, so $u = p\overline{q}$ for some $p, q \in \mathcal{L}^2(\mu)$. For any $h \in \mathcal{L}^\infty$, $\langle h, B\rangle = \langle h, p \otimes q\rangle$, so $\mathcal{A}$ is elementary. $\square$

A space $\mathcal{S}$ is *hereditarily reflexive* if each of its weak-star closed subspaces is reflexive. It is well known that this property joins the concepts of elementarity and reflexivity (for a proof, see Proposition 2.10(3) in [2]):

Proposition 2.6. *Let* $\mathcal{S} \subseteq \mathfrak{B}(\mathcal{H})$ *be weak-star closed. Then* $\mathcal{S}$ *is hereditarily reflexive if and only if* $\mathcal{S}$ *is reflexive and elementary.*

Since all von Neumann algebras are reflexive (for example, see Theorem 9.17 of [26]), it follows that every maximal abelian von Neumann algebra is hereditarily reflexive.

Proposition 2.7. *Every normal operator is both reflexive and star-reflexive.*

Proof. Let A be normal, and $\mathcal{S}$ be either $\mathcal{A}_A$ or $\mathcal{T}_A$. Then $\mathcal{S}$ is a weak-star closed subspace of some maximal abelian von Neumann algebra which is hereditarily reflexive by the comment above. Hence $\mathcal{S}$ is reflexive. $\square$

2.4 Finite-dimensional results

Example 2.8. *Let $A \in M_2$. Then*

(1) A is reflexive if and only if it is similar to a normal operator.

(2) If A is normal, then it is star-reflexive; otherwise, it is star-transitive.

Proof. (1) Since similarity does not affect reflexivity of $\mathcal{A}_A$, we may assume that A is in Jordan canonical form. The diagonal case is handled by Proposition 2.7. In the contrary case, we saw in Section 1.1 that all lower-triangular matrices belong to $\mathcal{A}_A$.

(2) The first assertion follows from Proposition 2.7. For the remainder, suppose $A \in M_2$ is not star-transitive. Then we can find nonzero vectors u, v with $u \otimes v \perp \mathcal{T}_A$; that is, $0 = \langle u, v\rangle = \langle Au, v\rangle = \langle u, Av\rangle$. That forces Au, Av to be scalar multiples of u, v respectively, which can only happen when A is normal. $\square$

In a categorical sense, most members of M_2 are diagonalizable and non-normal, i.e., reflexive and star-transitive. In Section 6.3, we will see that this pathology extends to higher dimensions.

The fate of Example 2.8(1) in higher dimensions is a classical result due to J. Deddens and P. Fillmore.

Proposition 2.9. *Suppose T is a nilpotent member of M_n which is in (lower) Jordan Canonical Form. Assume the blocks of T occur in non-increasing order by* *size and write* $m_1 \geq m_2$ *for the sizes of its first two blocks. Then* $\operatorname{ref}\mathcal{A}_T = \mathcal{A}_T + \mathcal{C}$ *where*

$$\mathcal{C} = \{A \in M_n \mid A_{ij} = 0 \text{ for all } i, j \text{ except } 0 \leq j \leq i - m_2 < m_1 - m_2\}$$

In particular, T *is reflexive if and only if* $m_1 = m_2$ *or* $m_1 = m_2 + 1$*.*

This is Theorem 2 of [10]. We will generalize it in Chapter 4. In particular, we will see that for matrices in Jordan form, star-reflexivity is equivalent to reflexivity.

The deepest results of [4] and [9] depend on strengthened versions of Definition 2.4. Proposition 2.11, which ultimately depends on dimension theory for algebraic varieties, will play a crucial role in the proof of Theorem 4.8.

Lemma 2.10. *Every* $(2n-1)$*-dimensional transitive subspace of* M_n *is elementary.*

This is Proposition 4.1(3) of [2]. In view of Example 2.8, we conclude that every member of M_2 is star-elementary.

Proposition 2.11. *Every (weighted) shift of finite order is star-elementary.*

Proof. Suppose T is a weighted shift of order n. Then T can be expressed as an orthogonal direct sum $T = V \oplus W$ where V, W are weighted shifts and W is irreducible of order n. If we consider W as acting on $\mathbb{C}^n$, it can be checked that $\operatorname{ref}\mathcal{A}_W$ contains every lower triangular matrix, so W is star-transitive (this will also follow from Proposition 2.14, since the irreducibility of W implies that it is unicellular). Therefore, W is star-elementary by Lemma 2.10. Since the order of T equals that of W, T is star-elementary as well. □

Lemma 2.12. *Let* $\mathcal{S}$ *be a linear subspace of* M_n*. If* $\dim\mathcal{S} > (n-1)^2$*, then* $\mathcal{S} \cap \mathfrak{F}_1$ *has members other than* 0*.*

Proof. This is Proposition 4.2(1) of [2]. □

Proposition 2.13. *Suppose A in M_n is star-transitive. Then the minimal polynomial of A has degree n.*

Proof. By Lemma 2.12, any subspace of M_n of dimension greater than $(n-1)^2$ has non-trivial intersection with $\mathfrak{F}_1$. Thus if A is to be star-transitive, we must have $\dim(\mathcal{T}_A)_\perp \leq (n-1)^2$, which means $\dim(\mathcal{T}_A) \geq 2n-1$ with in turn implies $\dim(\mathcal{A}_A) = n$. □

2.5 Unicellular operators

In 1957, W. F. Donoghue, Jr.[11] published a paper with an analysis of the weighted shift T sending $e_j \mapsto w_j e_{j+1}$ for the sequence of weights $w_j = 2^{-j}$. He showed that the only invariant spaces of T are the obvious ones, $\text{span}\{e_j \mid j \geq n\}$ for each n. Consequently, T is as far from reflexive as an injective shift can be: its reflexive closure contains every lower triangular bounded operator.

An operator with invariant spaces that are linearly ordered is called *unicellular*, and the Donoghue shift is clearly an operator of this type. The following result of F. Gilfeather and D. R. Larson implies that every unicellular operator is star-transitive.

Proposition 2.14. *[13, Corollary 2.2(iii)] If $A \in \mathfrak{B}(\mathcal{H})$ is unicellular, then $\text{ref}(\mathcal{A}_A) + \text{ref}(\mathcal{A}_{A^*})$ is weak-star dense in $\mathfrak{B}(\mathcal{H})$.*

Their result is actually somewhat stronger: They show that if $\mathcal{A}$ is any nest subalgebra of a von Neumann algebra $\mathfrak{B}$, then $\text{ref}(\mathcal{A}) + \text{ref}(\mathcal{A}^*)$ is weak-star dense in $\mathfrak{B}$.

The class of shifts known to be unicellular has gradually expanded since the shift studied by Donoghue. By the 1970s it was known that if $w_j \downarrow 0$ is in ℓ_p for $p < \infty$, or if $\left(\frac{j+1}{j+2}\right) w_j \downarrow 0$, then T is unicellular [29]. The best known result to date is due to D. V. Yakubovich [34] and states that T is unicellular whenever $w_j \downarrow 0$.

In contrast with the unicellular shifts, D. Sarason proved in [27] that any operator having the same invariant spaces as the regular unilateral shift S (on ℓ^2) must commute with S. The result was based on A. Beurling's classification[5] of the invariant spaces of the shift, and established the reflexivity of the unilateral shift. Gilfeather and Larson[13] were able to show that, like the unicellular shifts, the regular shift is also star-transitive:

Proposition 2.15. *Let $S \in \mathfrak{B}(\mathcal{H}^2)$ be the regular unilateral shift. Then S is reflexive but star-transitive.*

Contrast with Proposition 2.14. For the regular shift S, $\operatorname{ref}(\mathcal{A}_S) + \operatorname{ref}(\mathcal{A}_{S^*})$ is the full space of Toeplitz operators which is much smaller than $\mathfrak{B}(\mathcal{H}^2) = \operatorname{ref} \mathcal{T}_S$.

For all operators, it is clear that $\mathrm{wk}^*\big(\operatorname{ref}(\mathcal{A}_A) + \operatorname{ref}(\mathcal{A}_{A^*})\big) \subseteq \operatorname{ref}(\mathcal{A}_A + \mathcal{A}_{A^*})$, since $\operatorname{ref}(\mathcal{A}_A)$ and $\operatorname{ref}(\mathcal{A}_{A^*})$ are individually contained in the right hand space. When A happens to be star-reflexive, the containment also reverses. It also happens for unicellular operators but not the regular shift, as noted in the previous paragraph.

Definition 2.16. *An operator A is* star-stable *if* $\operatorname{ref}(\mathcal{A}_A) + \operatorname{ref}(\mathcal{A}_{A^*})$ *is weak-star dense in* $\operatorname{ref}(\mathcal{A}_A + \mathcal{A}_{A^*})$.

Review of Example 2.8 shows that $A \in M_2$ is star-stable if and only if A is unicellular or normal.

Proposition 2.17. *The following are equivalent for nilpotent A in M_n:*

(1) A is unicellular.

(2) A is star-transitive.

(3) A^{n-1} is not zero.

(4) A is similar to a shift.

Proof. (1)⇒(2) is Proposition 2.14.

(2)⇒(3) is Proposition 2.13.

(3)⇒(4) Let x be a vector with $A^{n-1}x \neq 0$. Then $\{x, Ax, \ldots, A^{n-1}x\}$ are independent. Let $Be_j = A^j x$ for $0 \leq j < n$. Then $B^{-1}AB$ is a shift.

(4)⇒(1) Clear. □

2.6 Other Reflexivity Facts and Tools

It is a consequence of Proposition 2.9 that $A \oplus A$ is reflexive for any $A \in M_n$. For a shift T with invertible weights, it is also true that $T \oplus T$ is reflexive. This was first shown by A. Lambert[20], who also showed that T is reflexive whenever T^* has a nonzero eigenvalue (a generalization of Sarason's result). D. Hadwin and E. A. Nordgren[14] were later able to improve on Lambert's direct sum result and show that $T_1 \oplus T_2$ is reflexive for any injective weighted shifts. In fact, many direct sums of weighted shifts are (star-)reflexive, and this will follow directly from Theorem 4.8.

Despite the finite dimensional precedent and the weighted shift examples, it turns out that not all operators $A \oplus A$ are reflexive. W. R. Wogen was the first to prove this in a paper of reflexivity counter-examples[33]. Together, Larson and Wogen settled another reflexivity conjecture concerning direct sums by finding a reflexive operator T such that $0 \oplus T$ is not reflexive[21]. This contrasts in a particularly interesting way with Corollary 4.12, where taking a direct sum with 0 makes it <u>more</u> likely for an operator to be star-reflexive.

One novel tool that was used by A. L. Shields[29] is to view weighted shifts as unweighted shifts acting on a weighted space. That is, we should think of T as shifting a non-normalized, orthogonal basis f_j, where $f_j = \left(\prod_{i=0}^{j-1} w_i\right) e_j$. This of course simplifies the nature of T since all of the weights are taken to be one. It is of less use when working simultaneously with T and T^*; all of the “bad stuff” of T is pushed onto T^*, which is no more tractable than with the usual approach. In particular, since star-reflexivity requires an understanding of both T and T^*, Shields' approach is not applied.

It is unfortunate that the lower triangular or upper triangular part of a matrix B need not be a bounded operator (on ℓ_2) even when the operator norm of B is finite. A concrete example of this phenomenon is given by M. D. Choi[6], who proves that $||B|| < \infty$ while $||B_0|| = \infty$ for the Toeplitz matrices

$$B = \begin{bmatrix} 0 & -1 & -\frac{1}{2} & -\frac{1}{3} & \cdots \\ 1 & 0 & -1 & -\frac{1}{2} & \\ \frac{1}{2} & 1 & 0 & -1 & \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 & \\ \vdots & & & & \ddots \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \\ \frac{1}{2} & 1 & 0 & 0 & \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

The use of Cesaro means of the diagonals of a bounded operator to circumvent such difficulties has a long history. In the next chapter, we carry this idea over to preannihilators (in the trace class), thereby allowing us to examine reflexive closures of star power spans of shifts “one diagonal at a time.” This in turn sets up a “local” application of function theory in Chapter 5 which seems quite different from the global functional calculus techniques exploited in [4] and [9].

Chapter 3

Striped Spaces

This chapter explores the relationship between operators and infinite square matrices. One goal is to understand when a given infinite matrix is the matrix of a bounded operator. Another is to give methods for reconstructing an operator from its individual diagonals. Finally, we wish to understand reflexivity properties of linear spaces in terms of their matricial structure.

Recall that up to isometry there is a unique separable infinite-dimensional Hilbert space. For our purposes, there is no useful distinction between $\mathcal{H}^2$ the space of square integrable analytic functions on the circle, and ℓ^2 the space of square summable sequences. There is the obvious "standard isometry" between them given by the Fourier transform, and any operator on $\mathcal{H}^2$ is unitarily equivalent to an operator on ℓ^2 and vice versa. In this sense, the Toeplitz operator $T_z : \mathcal{H}^2 \to \mathcal{H}^2$, which sends a function $f(z)$ to $zf(z)$, is interchangeable with an infinite matrix (acting on ℓ^2 column vectors) that has all 1s on its first subdiagonal. In this way, many interesting operators manifest themselves as infinite matrices.

The results of Sections 3.1 and 3.2 are well-known in the $\mathfrak{B}(\mathcal{H})$ setting; for example, that the boundedness of an infinite matrix corresponds to uniform boundedness of its principal minors. The parallel results for the trace norm are less familiar but are crucial facts when working with our definition of reflexive. For the sake of unity and completeness, a careful treatment of both settings is given. Lemmas 3.20 and 3.21 are also well-known.

3.1 Matrices

When studying any infinite object, it is natural to look for related finite objects. For an infinite square matrix, natural finite objects to study are finite square submatrices. This is analogous to observing the compression of an operator to a suitable finite dimensional space.

For a separable Hilbert space $\mathcal{H}$, let the symbol P_k denote orthogonal projection onto the span of $\{e_0, \dots, e_{k-1}\}$, the first k standard basis vectors.

Lemma 3.1. *(a) For a bounded operator $A \in L(\mathcal{H})$, $P_k A P_k \to A$ in the weak-star topology. (b) For a trace class operator B, $P_k B P_k \to B$ in the trace norm topology.*

Proof. (a) We may assume A is different from 0, or there is nothing to prove. Let C be a rank one operator $C = x \otimes y$. Then

$$\begin{aligned}
\big|\operatorname{tr}\big((A - P_k A P_k)C\big)\big| &= |\operatorname{tr}\big((A - P_k A P_k)x \otimes y\big)| \\
&= |\langle Ax, y\rangle - \langle P_k A P_k x, y\rangle| \\
&= |\langle Ax - AP_k x, y\rangle + \langle AP_k x, y - P_k y\rangle| \\
&\leq |\langle A(x - P_k x), y\rangle| + |\langle AP_k x, y - P_k y\rangle| \\
&\leq ||A||\,||x - P_k x||\,||y|| + ||A||\,||x||\,||y - P_k y||
\end{aligned}$$

which approaches 0 as $k \to \infty$. By linearity, $\operatorname{tr}\big((A - P_k A P_k)C\big) \to 0$ for any finite rank operator C.

Let D be an arbitrary trace class operator. Finite rank operators are trace norm dense in the trace class, so we may choose a finite rank C such that $||D - C||_1 < \dfrac{\varepsilon}{3||A||}$. Since C is finite rank, $|\operatorname{tr}\big((A - P_k A P_k)C\big)| < \varepsilon/3$ for sufficiently large k, and

$$\begin{aligned}
\big|\operatorname{tr}\big((A - P_k A P_k)D\big)\big| &= \big|\operatorname{tr}\big(A(D - C)\big) + \operatorname{tr}\big((A - P_k A P_k)C\big) + \operatorname{tr}\big(P_k A P_k (C - D)\big)\big| \\
&\leq ||A||\,||D - C||_1 + |\operatorname{tr}\big((A - P_k A P_k)C\big)| + ||A||\,||C - D||_1 \\
&< \varepsilon.
\end{aligned}$$

Since $\mathrm{tr}(P_kAP_kD) \to \mathrm{tr}(AD)$ for arbitrary trace class D, A is (by definition) the weak-star limit of P_kAP_k.

(b) Let C be a rank one operator $C = x \otimes y$. Then

$$\begin{aligned}
||C - P_kCP_k||_1 &= ||x \otimes y - (P_kx) \otimes (P_ky)||_1 \\
&= ||(x - P_kx) \otimes y + (P_kx) \otimes (y - P_ky)||_1 \\
&\leq ||(x - P_kx) \otimes y||_1 + ||(P_kx) \otimes (y - P_ky)||_1 \\
&= ||x - P_kx||\,||y|| + ||P_kx||\,||y - P_ky|| \\
&\leq ||x - P_kx||\,||y|| + ||x||\,||y - P_ky||
\end{aligned}$$

which approaches 0 as $k \to \infty$. By linearity, $||P_kCP_k - C||_1 \to 0$ for any finite rank operator C.

Let B be an arbitrary trace class operator. Finite rank operators are trace norm dense in the trace class, so we may choose a finite rank C such that $||B - C||_1 < \varepsilon/3$. Since C is finite rank, $||C - P_kCP_k||_1 < \varepsilon/3$ for sufficiently large k, and

$$\begin{aligned}
||B - P_kBP_k||_1 &\leq ||B - C||_1 + ||C - P_kCP_k||_1 + ||P_k(C - B)P_k||_1 \\
&\leq ||B - C||_1 + ||C - P_kCP_k||_1 + ||C - B||_1 \\
&< \varepsilon.
\end{aligned}$$

$\square$

Proposition 3.2. *Let $M = (m_{ij})$ be an infinite matrix. For each k, write T_k for the finite rank operator $T_k := \sum_{i,j=0}^{k-1} m_{ij} e_j \otimes e_i$. Suppose the operator norms of T_k are uniformly bounded.*

(1) The matrix M_k of T_k is obtained by replacing all entries of M except the upper left $k \times k$ corner with zero.

(2) The sequence $\{T_k\}$ converges to a bounded operator T in the weak-star topology.

(3) M is the matrix of T.

(4) If the trace class norms of $\{T_k\}$ are uniformly bounded, then $\lim_{k\to\infty} ||T_k - T||_1 = 0$.

(5) If the matrices $\{M_k\}$ are all positive semi-definite, then $T \geq 0$.

Proof. (1) Clear.

(2) By Alaoglu's Theorem, the unit ball in $L(\mathcal{H})$ is weak-star compact, so the sequence $\{T_k\}$ must have a weak-star limit point T.

Notice that for fixed i, j, the sequence of inner products $\{\langle T_k e_j, e_i\rangle\}$ becomes constant when $k \geq i, j$. Thus $\operatorname{tr}(T_k \cdot e_j \otimes e_i) \to \operatorname{tr}(T \cdot e_j \otimes e_i)$, and $\langle Te_j, e_i\rangle = \langle T_k e_j, e_i\rangle$ whenever $k \geq i, j$. Thus $T_k = P_k T P_k$, and by Lemma 3.1, $T_k \to T$ in the weak-star topology.

(3) As noted above, $\langle Te_j, e_i\rangle = \langle T_k e_j, e_i\rangle = m_{ij}$ for $k \geq i, j$.

(4) We will verify that T is a trace class operator and invoke Lemma 3.1(b). Let $T = W\sqrt{T^*T}$ be the polar decomposition of T so that $||T||_1 = \operatorname{tr}(W^*T) = \displaystyle\sum_{i=0}^{\infty} \langle W^*Te_i, e_i\rangle$. Consider any partial sum $\displaystyle\sum_{i=0}^{N} \langle W^*Te_i, e_i\rangle = \sum_{i=0}^{N} \langle Te_i, We_i\rangle = \operatorname{tr}\big(T \sum_{i=0}^{N}(e_i \otimes e_i)W^*\big)$. Write C for the finite rank operator $\displaystyle\sum_{i=0}^{N}(e_i \otimes e_i)W^*$ and notice that $||C|| \leq ||\sum_{i=0}^{N}(e_i \otimes e_i)||\,||W^*|| \leq 1$. Now by (2), $\operatorname{tr}(TC) = \lim_k \operatorname{tr}(T_k C)$. Since $|\operatorname{tr}(T_k C)| \leq ||T_k C||_1 \leq ||T_k||_1 ||C|| \leq ||T_k||_1$ is uniformly bounded, T is trace class, and Lemma 3.1 applies.

(5) If $\{M_k\}$ are positive semi-definite, then for any f, $\langle T_k f, f\rangle \geq 0$. By (2), $\langle Tf, f\rangle = \lim_k \langle T_k f, f\rangle \geq 0$ so $T \geq 0$. $\square$

As is the usual custom, we will make little distinction between the matrix M and the operator T as they occurred above. We will make references to the matricial properties of T, saying for example that it is "supported on a finite corner" if $T =$

$P_k T P_k$ (since its associated matrix will have only zero entries outside its upper left $k \times k$ corner).

We finish this section by developing some matricial tools that have historically been useful with weighted shifts. Our goal is to ensure that important properties of operators (positivity, boundedness) are preserved under matricial manipulation.

Definition 3.3. *The* Schur *product (also called Hadamard product) of two matrices* $A = (a_{ij})$ *and* $B = (b_{ij})$ *is defined by* $A \circ B = (a_{ij} b_{ij})$.

Theorem 3.4 (Schur Product Theorem). *If* A, B *are positive* $n \times n$ *matrices, then* $A \circ B$ *is also positive.*

Proof. By the Spectral theorem, we may write $A = \displaystyle\sum_{i=1}^{n} a_i x_i \otimes x_i$ and $B = \displaystyle\sum_{i=1}^{n} b_i y_i \otimes y_i$ with $a_i, b_i \geq 0$. So $A \circ B = \displaystyle\sum_{n,m} a_i b_j (x_i \otimes x_i) \circ (y_j \otimes y_j) = \sum_{n,m} a_i b_j h_{ij} \otimes h_{ij}$ where $\hat{h}_{ij}(m) = \hat{x}_i(m) \hat{y}_j(m)$ for each m. Since $A \circ B$ is a sum of positive operators, it must be positive. $\square$

For the next five Lemmas, let K be a positive matrix whose diagonal entries are all bounded by 1.

Lemma 3.5. *If* B *is a positive trace class operator then* $||K \circ B||_1 \leq ||B||_1$.

Proof. Let $K_n = P_n K P_n$. Note that K_n is positive. For positive operators, the trace norm coincides with the trace. By Schur's theorem, we are assured that $K_n \circ B$ is positive, so $||K_n \circ B||_1 = \operatorname{tr}(K_n \circ B) \leq \operatorname{tr} B = ||B||_1$. Since $||P_n(K \circ B)P_n||_1 = ||K_n \circ B||_1 \leq ||B||_1$ for each n, Proposition 3.2 ensures that $||K \circ B||_1 \leq ||B||_1$. $\square$

Lemma 3.6. *If* B *is a self-adjoint trace class operator, then* $||K \circ B||_1 \leq ||B||_1$.

Proof. Using the Spectral Theorem, we may write $B = \sum_{n=1}^{\infty} \lambda_n P_n$, where λ_n are the eigenvalues of B, and P_n are mutually orthogonal projections. Write $C =$

$\sum_{\lambda_n \geq 0} \lambda_n P_n$ and $D = \sum_{\lambda_n < 0} -\lambda_n P_n$. Then C and D are positive, and since all of the eigenvalues are real (i.e. all the terms are accounted for), $B = C - D$. Also $||B||_1 = \sum_{n=1}^{\infty} |\lambda_n| = ||C||_1 + ||D||_1$. Now $||K \circ B||_1 = ||K \circ (C + D)||_1 \leq ||K \circ C||_1 + ||K \circ D||_1 \leq ||C||_1 + ||D||_1 = ||B||_1$. □

Lemma 3.7. *If G is any trace class operator, then $||K \circ G||_1 \leq 2||G||_1$.*

Proof. Define $A = \frac{G+G^*}{2}$ and $B = \frac{G-G^*}{2i}$. Then A and B are self-adjoint and $G = A+iB$. It follows that $||K \circ G||_1 \leq ||K \circ A||_1 + ||K \circ B||_1 \leq ||A||_1 + ||B||_1 \leq 2||G||_1$. □

With some care, the factor of 2 can be eliminated from this proof, so that $||K \circ G||_1 \leq ||G||_1$. However, the proof given is simple and straightforward, and the estimate obtained is sufficient for our purposes.

Lemma 3.8. *If B is any self-adjoint operator, with operator norm $||B||$, then $||K \circ B|| \leq ||B||$.*

Proof. We may assume $||B|| < \infty$, so let m denote the operator norm of B. Then $-mI \leq B \leq mI$. It follows from Schur's theorem that $-K \circ mI \leq K \circ B \leq K \circ mI$, and since $K \circ I \leq I$, we get $-mI \leq K \circ B \leq mI$. For self-adjoint operators, $||B|| = \sup\limits_{||x||=1} |\langle x, Bx \rangle|$, so we have $||K \circ B|| \leq m$. □

Lemma 3.9. *If G is any bounded operator, then $||K \circ G|| \leq 2||G||$.*

Proof. Define $A = \frac{G+G^*}{2}$ and $B = \frac{G-G^*}{2i}$. As before, A and B are self-adjoint, $G = A + iB$, and $||K \circ G|| \leq ||K \circ A|| + ||K \circ B|| \leq ||A|| + ||B|| \leq 2||G||$. □

As with Lemma 3.7, the factor of 2 is not necessary if some care is used in the proof.

We summarize these results in the following Proposition:

Proposition 3.10. *Let K be any positive bounded operator and G be any operator. If G is bounded, then so is $K \circ G$. If G is of trace class, then so is $K \circ G$.*

3.2 Diagonals and Cesaro means

As a matrix, the unilateral shift has nonzero entries only on its first subdiagonal (the same is true for any weighted shift). Any power of the shift or its adjoint is supported in the same way on a single diagonal. For a matrix related somehow to a shift (orthogonal to it, for example), the various diagonals will be of special interest.

Definition 3.11. *Let S denote the unilateral shift. Then the* nth diagonal *of an* operator *A is $(S^*)^n \circ A$ when $n \geq 0$ and $S^{-n} \circ A$ when $n < 0$.*

For convenience, assume the notation $T^{(n)} = T^n$ when $n \geq 0$ and $T^{(n)} = (T^*)^{-n}$ when n is negative. This makes sense for arbitrary operators T, but will be most useful when T is a (weighted) shift. In particular, we may concisely describe the nth diagonal of A as $S^{(-n)} \circ A$.

Corollary 3.12. *If A is bounded (trace class), then every diagonal of A is bounded (trace class).*

Proof. We may write the nth diagonal $S^{(-n)} \circ A$ also as $S^{(-n)}(I \circ (S^{(n)} A))$. Proposition 3.10 guarantees that the operator remains bounded (trace class). □

Definition 3.13. *Let the infinite matrix $C_n = (c_{ij})$ with*

$$c_{ij} = \begin{cases} 1 - \frac{|i-j|}{n+1} & \text{if } |i-j| \leq n \\ 0 & \text{if } |i-j| > n \end{cases}$$

Then the nth Cesaro mean of the diagonals of an operator A is the operator $C_n \circ A$.

It is reasonable to refer to this as a Cesaro mean, since $C_n = \frac{1}{n+1} \sum_{k=0}^{n} T_k$, where $T_k = \frac{1}{2k+1} \sum_{j=-k}^{k} S^{(j)}$

Lemma 3.14. *The matrix C_n is positive.*

Proof. This is proved in [30]. The method is to recognize that the matrix C_n is a Toeplitz matrix, and its associated function is

$$g_n(e^{i\theta}) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ik\theta} = \frac{1}{n+1} \left(\frac{\sin(n+1)\theta/2}{\sin\theta/2}\right)^2 .$$

That is, C_n is the matrix of the Toeplitz operator T_{g_n} on $\mathcal{H}^2$ defined by $T_{g_n} f = P(g_n f)$ for $f \in \mathcal{H}^2$ (where P denotes projection from $\mathcal{L}^2$ onto $\mathcal{H}^2$). Since g_n takes only non-negative values, it is clear that T_{g_n} is a positive operator. Indeed, for any $f \in \mathcal{H}^2$,

$$\langle T_{g_n} f, f\rangle = \langle g_n f, f\rangle = \int_{\mathbb{T}} g_n |f|^2 \, d\mu \geq 0 .$$

As the matrix of a positive operator, C_n is positive. $\square$

Theorem 3.15. *(a) The Cesaro means of the diagonals of a bounded operator A converge to A in the weak-star topology. (b) The Cesaro means of the diagonals of a trace class operator B converge to B in the trace norm topology.*

Proof. (a) Let A, i, j be given. Then for $n \geq i, j$

$$\begin{aligned} \operatorname{tr}\big((A - C_n \circ A) \cdot e_i \otimes e_j\big) &= \langle (A - C_n \circ A) e_i, e_j \rangle \\ &= \frac{|i-j|}{n+1} \langle A e_i, e_j \rangle \end{aligned}$$

which converges to 0 as $n \to \infty$.

If T is supported on a finite corner, i.e. $T = (t_{ij})$ where $t_{ij} = 0$ when $i \geq k$ or $j \geq k$, then T is a finite combination of operators $e_i \otimes e_j$ and $\lim_{n\to\infty} \operatorname{tr}\big((A - C_n \circ A)T\big) = 0$ by linearity.

Finally, if T is any trace class operator, by Lemma 3.1 we may fix k sufficiently large, with $||T - P_k T P_k||_1 < \dfrac{\varepsilon}{6||A||}$, so that

$$\begin{aligned}\left|\operatorname{tr}\big((C_n \circ A - A)(T - P_k T P_k)\big)\right| &\leq ||(C_n \circ A - A)(T - P_k T P_k)||_1 \\ &\leq ||C_n \circ A - A||\, ||T - P_k T P_k||_1 \\ &\leq 3||A||\, ||T - P_k T P_k||_1 \text{ (by Lemma 3.9)} \\ &< \varepsilon/2.\end{aligned}$$

Notice that this estimate is valid for any choice of n. Since $P_k T P_k$ is supported on a corner, we may choose n sufficiently large so that $\left|\operatorname{tr}\big((A - C_n \circ A)(P_k T P_k)\big)\right| < \varepsilon/2$, and it follows that $\left|\operatorname{tr}\big((A - C_n \circ A)T\big)\right| \leq \left|\operatorname{tr}\big((C_n \circ A - A)(T - P_k T P_k)\big)\right| + \left|\operatorname{tr}\big((A - C_n \circ A)(P_k T P_k)\big)\right| < \varepsilon$.

(b) For a rank one $B = e_i \otimes e_j$, $C_n \circ B = \big(1 - \frac{|i-j|}{n+1}\big)B$ when $n \geq i, j$, so $||B - C_n \circ B||_1 = \frac{|i-j|}{n+1}$ which converges to 0.

If B is supported on a finite corner, that is $B = (b_{ij})$ with $b_{ij} = 0$ when $i \geq k$ or $j \geq k$, then B is a finite linear combination of operators $e_i \otimes e_j$, and by linearity, $||C_n \circ B - B||_1 \to 0$.

Finally, let B be any trace class operator, and let $\varepsilon > 0$. Use Lemma 3.1 to find k sufficiently large so that $||P_k B P_k - B||_1 < \varepsilon/6$. Since $P_k B P_k$ is supported on a finite corner, choose n so that $||C_n \circ (P_k B P_k) - P_k B P_k||_1 < \varepsilon/2$, and conclude that

$$\begin{aligned}||C_n \circ B - B||_1 &\leq ||C_n \circ (B - P_k B P_k)||_1 + ||C_n \circ (P_k B P_k) - P_k B P_k||_1 \\ &\quad + ||P_k B P_k - B||_1 \\ &\leq 2||B - P_k B P_k||_1 + ||C_n \circ (P_k B P_k) - P_k B P_k||_1 \\ &\quad + ||P_k B P_k - B||_1 \text{ (by Lemma 3.7)} \\ &< \varepsilon\end{aligned}$$

□

It is noted in [30] that $A - C_n \circ A$ actually converges to zero in the strong operator topology, however the weak-star convergence result given here will be sufficient for our discussion.

An immediate consequence of this theorem is a concrete description of the weak-star closed algebra generated by a (weighted) shift T. For any bounded operator A, write $A = \sum_{j=-\infty}^{\infty} \alpha_j T^{(j)}$, if $S^{(j)} \circ A = \alpha_j T^{(j)}$ for each j. We call this a *formal power series in* T *and* T^* (actual convergence of the sum in a particular topology is not important). A bounded operator A is in $\mathcal{A}_T$ if and only if it is a formal series $A = \sum_{j=0}^{\infty} \alpha_j T^j$, i.e. when each of the diagonals of A is a scalar multiple of T^n. This is a special case of the following:

Corollary 3.16. *Let* T *be a weighted shift, and* $J \subseteq \mathbb{Z}$*. If* A *is a bounded operator such that* $A = \sum_{j \in J} a_j T^{(j)}$ *is a formal series in* T *and* T^**, then* A *is contained in the weak-star closed span of* $\{T^{(j)} \mid j \in J\}$.

Proof. By Theorem 3.15, A is the weak-star limit of Cesaro means of its diagonals, and each of these means is a polynomial in T and T^*. If any terms in the series for A are missing (i.e. some diagonal of A is zero), then every operator $C_n \circ A$ from the proof of Theorem 3.15 will miss the same terms. □

3.3 Annihilators of Striped Spaces

Definition 3.17. *A set of operators is called* striped *if it contains every diagonal of each of its members.*

Remark: Suppose A and B are bounded operators, one of which belongs to the trace class. If A is supported on its nth diagonal, then $\mathrm{tr}(AB) = 0$ if and only if A is orthogonal to the $(-n)$th diagonal of B.

Proposition 3.18. *(a) If $\mathcal{S}$ is a striped space of operators, then $\mathcal{S}_\perp$ is striped. (b) If $\mathcal{T}$ is a striped space of trace-class operators, then $\mathcal{T}^\perp$ is striped.*

Proof. Let $\mathcal{S}$ be a striped subspace of $L(\mathcal{H})$, and let A be any member of $\mathcal{S}_\perp$. Fix D to be any diagonal of A. By definition, A is orthogonal to every member of $\mathcal{S}$, and in particular to any member supported on a single diagonal. It follows immediately that D is also orthogonal to any member of $\mathcal{S}$ supported on a single diagonal. By Theorem 3.15(a), any $B \in \mathcal{S}$ is a weak-star limit of linear combinations of its diagonals. Since D is orthogonal to every diagonal of B, it follows that D is orthogonal to B, hence $D \in \mathcal{S}_\perp$. The proof of part (b) is a similar application of Theorem 3.15(b). $\square$

Definition 3.19. *Let $F : [a, b] \to L$ be a function into a Banach space L, and P denote a pointed partition of $[a, b]$. Let $||P||$ be the length of the longest interval in P and x_i^* denote the distinguished point in the ith interval.. Define the* Riemann integral of F *to be* $\displaystyle\int_a^b F(x)\,dx = \lim_{||P||\to 0} \sum_{i=1}^{n} F(x_i^*)\Delta x_i$ *when this limit exists.*

Lemma 3.20. *If $F : [a, b] \to L$ is a continuous function taking values in a Banach space L, then $G = \displaystyle\int_a^b F(x)\,dx$ is a well-defined member of L lying in the closed linear span of $\{F(x) \mid x \in [a, b]\}$.*

Proof. Let $\mathcal{R}$ be the collection of Riemann sums of F on the interval $[a, b]$. Let P_1, P_2 denote pointed partitions of $[a, b]$, and let $R_1 = \sum_{i=1}^m F(x_i^*)\Delta x_i$ and $R_2 = \sum_{i=1}^n F(y_i^*)\Delta y_i$ be the associated members of $\mathcal{R}$. We form a relation $\prec$ on $\mathcal{R}$ by setting $R_1 \prec R_2$ if $||P_2|| \le ||P_1||$. Under this relation, $\mathcal{R}$ forms a net. Since L is a complete metric space, we may verify that a limit G exists by checking that $\mathcal{R}$ is a Cauchy net.

Let $\varepsilon > 0$. Since F is uniformly continuous on $[a, b]$, we may choose δ such that $||F(x) - F(y)|| < \dfrac{\varepsilon}{2(b-a)}$ whenever $|x - y| < \delta$.

Let P_1 and P_2 be pointed partitions, with $||P_1||, ||P_2|| < \delta$ for our choice of δ. Let P denote a mutual refinement of $||P_1||$ and $||P_2||$ (i.e. every interval of P is completely contained in some interval of P_1 and in some interval of P_2) with any choice of distinguished points. Let R_1, R_2, and R denote the members of $\mathcal{R}$ associated to P_1, P_2, and P respectively. We will show that $||R_1 - R|| < \varepsilon/2$.

Let x_i^* and z_j^* denote the distingished points in the partitions P_1 and P respectively. Since P is a refinement of P_1, the ith interval of P is a union of subintervals from P_1, say the $j_i, \ldots, j_{i+1} - 1$ intervals. Comparing the ith term of R_1 to the corresponding terms of R,

$$\begin{aligned} ||F(x_i^*)\Delta x_i - \sum_{j=j_i}^{j_{i+1}-1} F(z_j^*)\Delta z_j|| &= ||F(x_i^*) \sum_{j=j_i}^{j_{i+1}-1} \Delta z_j - \sum_{j=j_i}^{j_{i+1}-1} F(x_j^*)\Delta z_j|| \\ &= || \sum_{j=j_i}^{j_{i+1}-1} \big(F(x_i^*) - F(z_j^*)\big)\Delta z_j|| \\ &\le \sum_{j=j_i}^{j_{i+1}-1} \big|\big|F(x_i^*) - F(z_j^*)\big|\big|\Delta z_j \\ &< \frac{\varepsilon}{2(b-a)}\Delta x_i \end{aligned}$$

Thus $||R_1 - R|| < \dfrac{\varepsilon}{2(b-a)} \displaystyle\sum_{i=1}^{m} \Delta x_i = \varepsilon/2$. Similarly, $||R - R_2|| < \varepsilon/2$, so $||R_2 - R_1|| \le ||R_2 - R|| + ||R - R_1|| < \varepsilon$, and the Cauchy condition is verified.

Since $\mathcal{R} \subseteq \operatorname{span}\{F(x) \mid x \in [a,b]\}$, our limit G must land in the closed span of $\{F(x) \mid x \in [a,b]\}$. □

Lemma 3.21. *Continuous linear functionals commute with integration. That is, if X is a continuous linear functional on $L(\mathcal{H})$, then $X\left(\int_a^b F(x)\,dx\right) = \int_a^b X\big(F(x)\big)\,dx$.*

Proof. X sends Riemann sums to Riemann sums, and since it is continuous, it commutes with limits:

$$X\left(\lim_{||P||\to 0} \sum_{i=1}^{n} F(x_i^*)\Delta x_i\right) = \lim_{||P||\to 0} X\left(\sum_{i=1}^{n} F(x_i^*)\Delta x_i\right) = \lim_{||P||\to 0} \sum_{i=1}^{n} X\big(F(x_i^*)\big)\Delta x_i$$

□

The next theorem demonstrates a somewhat amazing property that striped spaces possess. Proposition 3.18 guarantees that the (pre-)annihilator of a striped space is striped. However in $\mathcal{S}_\perp \cap \mathfrak{F}_1$, a set central to questions of reflexivity, often the only diagonal operator is the zero operator, even for a striped space. Yet even in this set, "stripe properties" leak in at the limits.

Theorem 3.22. *If $\mathcal{S} \subseteq L(\mathcal{H}^2)$ is a striped linear space, then the trace-norm closed span of $\mathcal{S}_\perp \cap \mathfrak{F}_1$ is also striped.*

Proof. Write $\mathcal{M}$ for the closure in trace norm of the linear span of members of $\mathcal{S}_\perp \cap \mathfrak{F}_1$. Let $f \otimes g \in \mathcal{M}$. For a complex scalar w of absolute value 1, denote $f_w(z) = f(wz)$.

When $|w| = 1$, $f_w \otimes g_w \in \mathcal{M}$. To check, take $D \in \mathcal{S}$ to be an operator supported on its nth diagonal. For such a D, $Df_w(z) = w^n (Df)(wz)$. Now

$$\begin{aligned} \langle D, f_w \otimes g_w \rangle = \langle Df_w, g_w \rangle &= \int_{\mathbb{T}} Df_w(z) \overline{g(wz)}\, d\mu(z) \\ &= w^n \int_{\mathbb{T}} (Df)(wz) \overline{g(wz)}\, d\mu(z) \\ &= w^{n-1} \int_{\mathbb{T}} (Df)(u) \overline{g(u)}\, d\mu(u) \end{aligned}$$

by simple substitution. But this is $w^{n-1} \langle D, f \otimes g \rangle = 0$, so $f_w \otimes g_w \perp D$.

Now the map $\varphi : \mathbb{T} \to \mathcal{H}^2$ sending $w \mapsto f_w$ is continuous. Indeed, let $f \in \mathcal{H}^2$ and $\varepsilon > 0$ be given. Choose N sufficiently large so that $\sum_{n=N}^{\infty} |\hat{f}(n)|^2 < \frac{\varepsilon^2}{5}$. For each $n < N$, the term $\hat{f}(n) z^n$ is uniformly continuous on $\mathbb{T}$, so there exists some small δ for which the finite sum $\sum_{n=0}^{N-1} |\hat{f}(n)|^2 |w_1^n - w_2^n|^2 < \frac{\varepsilon^2}{5}$ whenever $|w_1 - w_2| < \delta$.

Therefore,

$$\begin{aligned} ||f_{w_1} - f_{w_2}||^2 &= \sum_{n=0}^{\infty} |\hat{f}(n)|^2 |w_1^n - w_2^n|^2 \\ &= \sum_{n=0}^{N-1} |\hat{f}(n)|^2 |w_1^n - w_2^n|^2 + \sum_{n=N}^{\infty} |\hat{f}(n)|^2 |w_1^n - w_2^n|^2 \\ &< \frac{\varepsilon^2}{5} + \frac{\varepsilon^2}{5} \cdot 4 \\ &= \varepsilon^2 \end{aligned}$$

whenever $|w_1 - w_2| < \delta$. So $\varphi(w_2) \to \varphi(w_1)$ as $w_2 \to w_1$.

In addition, the maps $f \mapsto f \otimes g$ and $g \mapsto f \otimes g$ are both continuous, so the map $w \mapsto f_w \otimes g_w$ is also continuous. Define $F(\theta) = f_{e^{i\theta}} \otimes g_{e^{i\theta}}$. Then F is a continuous mapping from $[0, 2\pi]$ to $L(\mathcal{H})$. Since $F(\theta) \in \mathcal{M}$ for each θ, we know $\int F(\theta)\, d\theta$ is a member of $\mathcal{M}$, and Lemma 3.21 assures us that this integral may be computed coordinate wise. However,

$$f_w \otimes g_w = \begin{bmatrix} f_0\overline{g_0} & f_0\overline{g_1 w} & f_0\overline{g_2 w^2} & \cdots \\ f_1\overline{g_0}w & f_1\overline{g_1} & f_1\overline{g_2 w} & \cdots \\ f_2\overline{g_0}w^2 & f_2\overline{g_1}w & f_2\overline{g_2} & \cdots \\ \vdots & & & \ddots \end{bmatrix}$$

where $w = e^{i\theta}$. Integrating this matrix coordinate wise with respect to θ on the interval $[0, 2\pi]$ has the effect of preserving the main diagonal and making the rest of the matrix 0. So we conclude that $\mathcal{M}$ contains the 0th stripe of $f \otimes g$, and hence of each of its operators. By multiplying first by appropriate scalars w^k or $\overline{w}^k$ and then integrating, we see that in fact $\mathcal{M}$ is striped. $\square$

3.4 Reflexivity of Striped Spaces

Corollary 3.23. *If $\mathcal{S} \subseteq L(\mathcal{H}^2)$ is a striped linear space, then* $\operatorname{ref}(\mathcal{S})$ *is striped.*

Proof. Let M be as in the previous theorem. Then $\operatorname{ref}(\mathcal{S}) = M^{\perp}$ and is striped by Proposition 3.18. $\square$

Observe that a striped subspace of a reflexive space need not be reflexive. For example M_2 is reflexive, while the algebra $\left\{ \begin{bmatrix} \lambda & 0 \\ \mu & \lambda \end{bmatrix} \right\}$ is a striped but non-reflexive subspace.

Definition 3.24. *A striped linear space of operators* $\mathcal{S}$ *is* unistriped *if for each* j, $S^{(j)} \circ \mathcal{S}$ *is at most one-dimensional.*

In any striped space $\mathcal{S}$, an operator A is the formal sum of its diagonals: $A = \sum_{j \in \mathbb{Z}} A_j$, where $A_j = S^{(-j)} \circ A$ are members of $\mathcal{S}$. In a unistriped space $\mathcal{S}$, this representation is even more compelling because it is possible to choose a distingished member of each one-dimensional space $S^{(-j)} \circ \mathcal{S}$. The terms of the sum then become scalar multiples of the distinguished representatives, forming a kind of formal Laurent series for A.

Proposition 3.25. *Every weak-star closed, striped subspace of a reflexive, unistriped space is reflexive.*

Proof. Let $\mathcal{T}$ be a reflexive unistriped space, and let $\mathcal{S}$ be a weak-star closed, striped subspace. Then $\operatorname{ref} \mathcal{S} \subseteq \operatorname{ref} \mathcal{T} = \mathcal{T}$. Let $J = \{j \mid S^{(-j)} \circ \mathcal{S} = \langle 0 \rangle\}$ and $A \in \operatorname{ref} \mathcal{S}$. Since $A \in T$, write $A = \sum_j A_j$ where $A_j = S^{(-j)} \circ A \in \mathcal{T}$. It is clear that $A_j = 0$ for $n \in N$, so $A = \sum_{j \notin J} A_j$. Because $S^{(-j)} \circ \mathcal{T}$ has dimension 1 for $j \notin J$, it must be that $A_j \in \mathcal{S}$ when $j \notin J$. Therefore, $A \in \mathcal{S}$ by Theorem 3.15. $\square$

Notice that this is similar to being hereditarily reflexive[2]. Any space that is both reflexive and elementary is hereditarily reflexive; every weak-star closed subspace is reflexive. However, a unistriped reflexive space need not be elementary. For example take $\mathcal{A} = \{(a_{ij}) \in M_2 \mid a_{22} = 0\}$. Then $\mathcal{A}$ is reflexive and it is clearly unistriped, but is has a (non-striped) non-reflexive subspace $\mathcal{B} = \{(b_{ij}) \in \mathcal{A} \mid b_{12} = b_{21}\}$.

Corollary 3.26. *Every star-reflexive weighted shift is reflexive.*

Proof. The algebra and star power span of a weighted shift T are always unistriped spaces by Corollary 3.16. Since $\mathcal{A}_T$ is weak-star closed, it is reflexive whenever $\mathfrak{T}_T$ is, by Proposition 3.25. $\square$

Chapter 4

Reflexivity of Direct Sums of Shifts

In this chapter, we investigate the reflexivity properties of direct sums of weighted shift operators. We highlight a well-studied class of operators, invertibly weighted shifts on $\mathcal{H}^2$ whose adjoint has a nonzero eigenvalue. In this category is the most well-studied shift, the regular unilateral shift.

The weak-star closed algebra, $\mathcal{A}_S$, generated by the unilateral shift is the class of analytic Toeplitz operators, a reflexive algebra. The space $\mathcal{T}_S$ is the full space of Toeplitz operators, a transitive space. Interestingly, it was shown in [3] that every weak-star closed subspace of $\mathcal{T}_S$ is either transitive or reflexive. It is of particular note that any (weak-star closed) subspace of $\mathcal{T}_S$ having zeros on an entire diagonal is reflexive.

We first develop injective shifts T for which T^* has a nonzero eigenvalue. All such shifts are known to be reflexive[20], but their star-reflexivity properties are harder to categorize. However, the techniques of Chapter 3 apply to $\mathcal{T}_T$ which is a unistriped space. An especially interesting fact is that omitting diagonals from a unistriped space makes it more likely to be reflexive. Theorem 4.4 shows that for many shifts T, omitting a single diagonal from the star power span of T will yield a reflexive space.

We end with direct sums, giving a general reflexivity result for the sum $V \oplus W$ where V and W are shifts. This, together with the result of Section 4.2, tells us that many non-injective shifts are star-reflexive. It also affords us with a new proof of a classical reflexivity result for $n \times n$ matrices as well as new star-reflexivity results in the same setting.

4.1 Weight Sequences

Let T denote a unilateral weighted shift with invertible weights $\{w_j\}_{j=0}^{\infty}$; that is, T is the linear function sending $T(z^j) = w_j z^{j+1}$ for $j \geq 0$. In general, the weights may be complex numbers $w_j = r_j e^{i\theta_j}$; however there will be no loss of generality in assuming they are positive since such T is unitarily equivalent to a shift with weight sequence r_j. The unitary U that implements the equivalence is defined by $U(z^j) = \left(e^{i\sum_{k=0}^{j-1}\theta_k}\right) z^j$.

It should be no surprise that the weight sequence plays an important role in the study of a shift T. So we begin with some basic facts about $\{w_j\}$.

Lemma 4.1. *Let $\{w_j \mid j \geq 0\}$ be a sequence of positive real numbers and $k \geq 0$. Then $\liminf\limits_{j\to\infty}[w_0 \cdots w_{j-1}]^{1/j} = \liminf\limits_{j\to\infty}[w_{k+1} \cdots w_{k+j}]^{1/j}$.*

Proof. It is sufficient to prove this for k=0. Write $a_j = [w_0 \cdots w_{j-1}]^{1/j}$ and $b_j = [w_1 \cdots w_j]^{1/j}$. Then $a_{j+1} = [w_0]^{\frac{1}{j+1}}[b_j]^{1+\frac{1}{j}}$. The exponents converge to 0 and 1 respectively, so we can see that $\{a_j\}$ and $\{b_j\}$ have the same subsequential limits. The smallest such limit is the common $\liminf$ of the sequences $\{a_j\}$ and $\{b_j\}$. □

Definition 4.2. *If T is a weighted shift with positive weight sequence $\{w_j\}$, write $\rho(T)$ for $\liminf_{j\to\infty}[w_0 \cdots w_{j-1}]^{1/j}$.*

As the next Lemma will show, the criterion that T^* have a nonzero eigenvalue amounts to $\rho(T) > 0$.

Lemma 4.3. *Let T be a weighted shift, mapping $Te_j = w_j e_{j+1}$ with positive weights w_j.*

(1) If λ is an eigenvalue of T^*, then $|\lambda| \leq \rho(T)$.

(2) If $|\lambda| < \rho(T)$, then λ is an eigenvalue of T^*.

Proof. The function $f = \sum_{j=0}^{\infty} a_j z^j$ is an eigenvector of T^* with eigenvalue λ if and only if

$$\begin{aligned} \lambda a_0 &= w_1 a_1 \\ \lambda a_1 &= w_2 a_2 \\ \lambda a_2 &= w_3 a_3 \\ &\vdots \end{aligned}$$

Without loss of generality, assume $a_0 = 1$ and solve for each coefficient in turn. Then $a_j = \dfrac{\lambda^j}{w_1 \cdots w_j}$. These are the coefficients of a function in $\mathcal{H}^2$ if and only if $\{a_j\}$ is square summable. This is guaranteed when $|\lambda| < \rho(T)$ and impossible when $|\lambda| > \rho(T)$. $\square$

4.2 Linear spaces generated by T and T^*

As the next Theorem and Corollary show, removing diagonals from many striped spaces makes them more likely to reflexive. Here, removing just one diagonal from the star power span of T is enough to guarantee reflexivity.

Theorem 4.4. *Let T be an injective shift such that T^* has a nonzero eigenvalue. Then the weak-star closed span of $\{T^{(n)} \mid n \neq 0\}$ is reflexive.*

Proof. Let $\{w_j \mid j \geq 0\}$ denote the weights of the operator T. Take each weight to be positive without loss of generality.

Write $\mathcal{B}$ for the weak-star closure of span$\{T^{(n)} \mid n \neq 0\}$. We will show that $(\mathcal{B}_\perp \cap \mathcal{F}_1)^\perp \subseteq \mathcal{B}$. Write M for the closed linear span of the rank one members of $\mathcal{B}_\perp$ and let $B \in M^\perp$. We have to show that $B \in \mathcal{B}$.

A matricial view will help keep the proof clear. The shift T can be expressed as a matrix

$$T = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ w_0 & 0 & 0 & \\ 0 & w_1 & 0 & \\ 0 & 0 & w_2 & \\ \vdots & & & \ddots \end{bmatrix}$$

We wish to learn the nature of B as a matrix $B = (b_{ij})$.

The rank one $z^j \otimes z^j$ is obviously in M for each j, so B contains zeros on its main diagonal.

For any complex $|a| < \rho(T)$ and any integer $k \geq 0$, form the rank one operator $f \otimes g = z^k(z - a/w_k) \otimes z^{k+1} \sum_{j=0}^{\infty} \frac{(\overline{a}z)^j}{w_{k+1} \cdots w_{k+j}}$. Matricially, this is a relatively simple operator. For example, when $k = 0$ the matrix of $f \otimes g$ is

$$F = \begin{bmatrix} 0 & -\frac{a}{w_0} & -\frac{a^2}{w_0 w_1} & -\frac{a^3}{w_0 w_1 w_2} & \cdots \\ 0 & 1 & \frac{a}{w_1} & \frac{a^2}{w_1 w_2} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & & & \ddots & \end{bmatrix}$$

For $n \geq 1$ we have $T^n \perp f \otimes g$ and $(T^*)^n \perp f \otimes g$, so $f \otimes g \in M$ (and since M is self-adjoint, $g \otimes f \in M$ also). This is particularly concrete for T and F where it is straightforward to check that $\operatorname{tr}(TF) = 0$, $\operatorname{tr}(T^2 F) = 0$, and so on. It is also clear that $\operatorname{tr}((T^*)^n F) = 0$ for $n \geq 1$.

By Theorem 3.22, M contains the diagonal stripes of $f \otimes g$, that is the rank two operators $z^{k+1} \otimes \frac{z^{k+j+1}}{w_{k+1} \cdots w_{k+j}} - z^k \otimes \frac{z^{k+j}}{w_k \cdots w_{k+j-1}}$ for each $k \geq 0$, and $j \geq 1$. For

F this means that M contains all of its "stripes", and it follows that

$$\operatorname{tr} B \begin{bmatrix} 0 & 0 & \cdots & -\frac{a^j}{w_0\cdots w_{j-1}} & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & \frac{a^j}{w_1\cdots w_j} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & & & & & & \ddots \end{bmatrix} = 0$$

In particular we learn that $w_0 b_{1,j+1} = w_j b_{0,j}$ for $j \geq 1$. Similarly, using stripes of F^*, it must be that $w_0 b_{j+1,1} = w_j b_{j,0}$.

In general, since $B \perp z^{k+1} \otimes \frac{z^{k+j+1}}{w_{k+1}\cdots w_{k+j}} - z^k \otimes \frac{z^{k+j}}{w_k \cdots w_{k+j-1}}$ and its adjoint for each $k \geq 0$, it is forced that $w_k b_{k+1,j+1} = w_{k+j} b_{k,j}$ and $w_k b_{j+1,k+1} = w_{k+j} b_{j,k}$ for $j > k \geq 0$. On any diagonal of B, fixing one entry forces the values of all the other entries on that diagonal, and it must be that the diagonal is a scalar multiple of T^n (if below the main diagonal) or $(T^*)^n$ (if above). Thus B is a formal series in T and T^*, and by Corollary 3.16, $B \in \mathcal{B}$. □

Corollary 4.5. *Let T be an injective weighted shift such that T^* has a nonzero eigenvalue. If $0 \notin J \subset \mathbb{Z}$ then the weak-star closed span of $\{T^{(n)} \mid j \in J\}$ is reflexive.*

Proof. The unistriped space $\text{wk}^* \operatorname{span}\{T^{(n)} \mid j \neq 0\}$ is reflexive by Theorem 4.4, so Proposition 3.25 applies. □

4.3 Direct Sums of Shifts

Definition 4.6. *For a shift T, let the* order *of T be* $\inf_{n\geq 1}\{n \mid T^n = 0\}$.

An injective shift on $\mathcal{H}^2$ has infinite order, and a non-injective shift may have infinite or finite order. A shift on $\mathbb{C}^n$ will necessarily have order at most n.

Lemma 4.7. *Let W be a weighted shift of order $n < \infty$. Let (d_j) be any complex numbers for $-n < j < n$. Then there exists a rank one matrix F such that $\operatorname{tr}(W^{(j)}F) = d_j$ for each j.*

Proof. Without loss of generality, assume $w_0, \dots, w_{n-1}$ are positive. Define $\tilde{F} = (\tilde{f}_{ij})$ by setting $\tilde{f}_{0j} = \frac{d_j}{\prod_{i<j} w_i}$, $\tilde{f}_{j0} = \frac{d_{-j}}{\prod_{i<j} w_i}$ and let all other entries be zero. Then $\operatorname{tr}(W^{(j)}\tilde{F}) = d_j$ for each j but $\tilde{F}$ is not rank one. Apply Proposition 2.11 to find a rank one F having $\operatorname{tr}(W^{(j)}F) = \operatorname{tr}(W^{(j)}\tilde{F})$. $\square$

The next theorem describes the reflexive closure of $\mathcal{A}_T$ and $\mathcal{T}_T$ where T is a direct sum of weighted shifts $V \oplus W$, and should be compared to Proposition 2.9. In particular, when V is the zero operator (on $\mathbb{C}^n$ or $\mathcal{H}^2$), it shows that T is star-reflexive exactly when $\operatorname{span}\{T^j, (T^*)^j \mid j \geq 1\}$ is reflexive (compare with Theorem 4.4).

Theorem 4.8. *Let $T = V \oplus W$ where V and W are weighted shifts (either on $\mathbb{C}^n$ or $\mathcal{H}^2$). Set $k_1 = \operatorname{order}(V)$, $k_2 = \operatorname{order}(W)$ and assume $k_1 \leq k_2 \leq \infty$. Let*

- $\mathcal{A} = \operatorname{wk}^* \operatorname{span}\{T^j \mid j \geq 0\}$
- $\mathcal{B} = \operatorname{wk}^* \operatorname{span}\{T^j, (T^*)^j \mid j \geq 0\}$
- $\mathcal{C} = \operatorname{wk}^* \operatorname{span}\{T^j \mid j \geq k_1\}$
- $\mathcal{D} = \operatorname{wk}^* \operatorname{span}\{T^j, (T^*)^j \mid j \geq k_1\}$

Then (a) $\operatorname{ref} \mathcal{A} = \operatorname{wk}^* \operatorname{span}\{T^j \mid j < k_1\} + \operatorname{ref} \mathcal{C}$ *and (b)* $\operatorname{ref} \mathcal{B} = \operatorname{wk}^* \operatorname{span}\{T^{(j)} \mid -k_1 < j < k_1\} + \operatorname{ref} \mathcal{D}$.

In particular, T is reflexive if and only if $\mathcal{C}$ is reflexive, and T is star-reflexive if and only if $\mathcal{D}$ is reflexive.

Proof. We prove (b), and note that (a) is similar. Let $e_0, e_1, \dots$ refer to the basis shifted by V (which may be finite or infinite), and $f_0, f_1, \dots$ refer to the basis shifted by W. Assume the weights of V and W are non-negative real.

By Corollary 3.23, we know that $\operatorname{ref}\mathcal{B}$ is striped. So let $B \in \operatorname{ref}\mathcal{B}$ be any operator supported on its nth diagonal and write $B = (b_{ij})$. We must show that $B \in \mathrm{wk}^* \operatorname{span}\{T^{(n)} \mid -k_1 < n < k_1\} + \operatorname{ref}\mathcal{D}$.

First take $|n| < k_1$, and for simplicity assume $n \geq 0$ (negative n is similar). Consider V and W as matrices, and write $V^{(n)} = (v_{ij}), W^{(n)} = (w_{ij})$. Let i, j be given (where i, j are small enough that $i + n, j + n$ refer to a valid columns of V, W respectively if acting on a finite dimensional space), and form the rank one operator $(v_{i,i+n} f_{j+n} - w_{j,j+n} e_{i+n}) \otimes (f_j + e_i)$. This rank one is orthogonal to $T^{(j)}$ for all j, and therefore its diagonals must be orthogonal to B by Theorem 3.22. In particular, the rank two operator $v_{i,i+n} f_{j+n} \otimes f_j - w_{j,j+n} e_{i+n} \otimes e_i$ is orthogonal to B. However this rank two has the effect of "comparing" the $w_{j,j+n}$ entry of $W^{(n)}$ to the $v_{i,i+n}$ entry of $V^{(n)}$. If either of $w_{j,j+n}$ or $v_{i,i+n}$ is nonzero, then the corresponding b_{ij}s must be in the same ratio. Since $n < k_1$, $V^{(n)}$ and $W^{(n)}$ must both be nonzero somewhere, so letting i and j vary, it follows that B is a scalar multiple of $T^{(n)}$.

If $k_1 = \infty$, there is nothing left to show. Otherwise, assume B is supported on the nth diagonal for $|n| \geq k_1$. We will show that B is orthogonal to every rank one that is orthogonal to $\mathcal{D}$.

Clearly B is a block matrix $B = \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix}$ with B_{00}, B_{01}, B_{10} all zero.

Let $A \in \mathcal{D}_\perp \cap \mathfrak{F}_1$. Then A is a block matrix $A = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix}$ where $A_{11} \perp W^{(n)}$ for $|n| \geq k_1$.

Now B is orthogonal to A if and only if it is orthogonal to any block matrix $\begin{bmatrix} * & * \\ * & A_{11} \end{bmatrix}$. Form a new block matrix $\hat{A}$, setting $\hat{A}_{11} = A_{11}$. For $|j| < k_1$ define $d_j = -\operatorname{tr}(W^{(j)} A_{11})$. By Lemma 4.7, there exists a rank one $\hat{A}_{00}$ with $\operatorname{tr}(V^{(j)} \hat{A}_{00}) = d_j$

for $|j| < k_1$, and $\hat{A}_{01}, \hat{A}_{10}$ may be chosen so that $\hat{A}$ remains rank one. By construction $\mathcal{B} \perp \hat{A}$, so $B \perp \hat{A}$. Therefore $B \perp A$ and must be a member of ref $\mathcal{D}$. □

The preceding result is equally valid when V, and W are themselves direct sums of shifts.

Corollary 4.9. *A direct sum of any shift with a (star-)reflexive shift of equal or greater order is (star-)reflexive. In particular, a direct sum of (star-)reflexive shifts is again (star-)reflexive.*

Proof. Let $T = V \oplus W$ as in Theorem 4.8, where W is (star-)reflexive. The sets $\mathcal{C}$ and $\mathcal{D}$ are striped subspaces of spaces $0 \oplus \mathcal{A}_W$ and $0 \oplus \mathcal{T}_W$, respectively. When $\mathcal{A}_W$ and $\mathcal{T}_W$ are reflexive spaces, so are $0 \oplus \mathcal{A}_W$ and $0 \oplus \mathcal{T}_W$, and Proposition 3.25 applies. □

Corollary 4.10. *A direct sum of shifts which contains shifts of arbitrarily large finite order is (star-)reflexive.*

Proof. Let T be the (infinite) direct sum of shifts. Write T as $V \oplus W$ where W and V are direct sums involving shifts of arbitrary finite order, so that order $W =$ order $V = \infty$. Then by Theorem 4.8, T is (star-)reflexive. □

Corollary 4.11. *A direct sum of shifts containing two or more shifts of infinite order is (star-)reflexive.*

Note that neither of the infinite order shifts needs to be injective. For comparison, Lambert[20] showed that $T \oplus T$ is reflexive for an injective weighted shift $T \in \mathfrak{B}(\mathcal{H}^2)$. Hadwin and Nordgren[14] improved this to show that any direct sum of at least two injective shifts is reflexive.

Corollary 4.12. *Let $T = V \oplus W$ where V is a shift of order $k \leq \infty$ (either on $\mathbb{C}^n$ or $\mathcal{H}^2$), and W is an injective shift on $\mathcal{H}^2$ such that W^* has a nonzero eigenvalue. Then T is (star-)reflexive.*

In particular $0 \oplus W$ is (star-)reflexive.

Proof. When $k = \infty$ there is nothing to show. For $k < \infty$, the set $\mathcal{D}$ occurring in Theorem 4.8 is reflexive by Corollary 4.5. □

When W is the regular unilateral shift, this is a special case of a recent result by E. A. Azoff and M. Ptak [3]. They show that every intransitive, weak-star closed subspace of Toeplitz operators (in this case $\mathcal{D}$) is reflexive.

Finally, we get this generalization of a result due originally to J. A. Deddens and P. A. Fillmore[10].

Corollary 4.13. *Suppose T is a nilpotent member of M_n which is in (lower) Jordan Canonical Form. Assume the blocks of T occur in non-increasing order by size and write $m_1 \geq m_2$ for the sizes of its first two blocks. Then $\operatorname{ref} \mathcal{A}_T = \mathcal{A}_T + \tilde{\mathcal{C}}$ and $\operatorname{ref} \mathcal{T}_T = \mathcal{T}_T + \tilde{\mathcal{D}}$ where*

$$\tilde{\mathcal{C}} = \{A \in M_n \mid A_{ij} = 0 \text{ for all } i, j \text{ except } 0 \leq j \leq i - m_2 < m_1 - m_2\}$$
$$\tilde{\mathcal{D}} = \tilde{\mathcal{C}} + \tilde{\mathcal{C}}^*$$

In particular, T is star-stable.

Proof. Write W for the largest block of T, and V for the rest, so that T is unitarily equivalent to $V \oplus W$. Then this unitary equivalence carries $\tilde{\mathcal{C}}$ to the reflexive closure of the set $\mathcal{C}$, and $\tilde{\mathcal{D}}$ to the reflexive closure of the set $\mathcal{D}$, as they occur in Theorem 4.8. □

Chapter 5

Perturbations of the Regular Shift

In Corollary 4.12, we learned that if W is an injective shift on $\mathcal{H}^2$ and W^* has a nonzero eigenvalue, then $0 \oplus W$ is a star-reflexive operator. The regular shift S has the property that S^* has nonzero eigenvalues, making $0 \oplus S$ the simplest operator for which the Corollary applies.

Now, $0 \oplus S$ is unitarily equivalent to the shift $T = S - z \otimes 1$, the shift where the 0th weight is zero and the remaining weights are one. Matricially, T and S differ in only one coordinate:

$$S = \begin{bmatrix} 0 & & & \cdots \\ 1 & 0 & & \\ 0 & 1 & 0 & \\ \vdots & & & \ddots \end{bmatrix} \qquad T = \begin{bmatrix} 0 & & & \cdots \\ 0 & 0 & & \\ 0 & 1 & 0 & \\ \vdots & & & \ddots \end{bmatrix}$$

Yet a distinct dichotomy is established: S is star-transitive, while T is star-reflexive.

The goal of this chapter is to understand what happens in general when the 0th weight of the unilateral shift is perturbed. These shifts form a one parameter family for which the star-reflexivity properties can be determined. The surprise is that the value $\sqrt{1/2}$ is a "magic value". When the 0th weight of T has magnitude larger than $\sqrt{1/2}$, T is star-transitive; when the magnitude is smaller, T is star-reflexive.

This should be compared with a result of J. B. Conway and M. Ptak. In [9], they show that for a weighted shift T satisfying the properties

1. $||T|| = \lim_j ||T^j||^{1/j}$

2. $\lim_j T^j x = 0 = \lim_j (T^*)^j x$ for all $x \in \mathcal{H}^2$.

the space $\mathcal{T}_T$ is hyperreflexive (a property stronger than reflexivity), so T is star-reflexive. This complements Theorem 5.6, which demonstrates star-reflexivity for a class of shifts that satisfies Property 1 but not 2. It is also notable that many of the star-transitive shifts of Corollary 5.3 also satisfy only Property 1.

We begin by characterizing the members of $(\mathcal{T}_T)_\perp \cap \mathfrak{F}_1$.

Theorem 5.1. *Let $T = S + (c-1)z \otimes 1$, for $c > 0$. In order for $f \otimes g$ to be orthogonal to $\mathcal{T}_T$, it is necessary and sufficient that*

$$\big(f(z) + (c-1)\hat{f}(0)\big)\big(\overline{g(z) + (c-1)\hat{g}(0)}\big) = (c^2-1)\hat{f}(0)\overline{\hat{g}(0)} \tag{5.1}$$

for almost all z of absolute value one.

Proof. There are three cases to consider: (1) both of $\hat{f}(0), \hat{g}(0)$ are zero, (2) exactly one of $\hat{f}(0), \hat{g}(0)$ is zero, or (3) both of $\hat{f}(0), \hat{g}(0)$ are nonzero.

(1) Assume $\hat{f}(0) = \hat{g}(0) = 0$. Then Equation (5.1) is equivalent to $f\overline{g} \equiv 0$.

For $n \geq 0$, $T^n \perp f \otimes g$ if and only if

$$0 = \langle T^n, f \otimes g\rangle = \langle T^n f, g\rangle = \langle S^n f, g\rangle = \int_{\mathbb{T}} z^n f \overline{g}\, d\mu\,.$$

Similarly, for $n \geq 0$, $(T^*)^n \perp f \otimes g$ if and only if

$$0 = \langle (T^*)^n, f \otimes g\rangle = \langle f, T^n g\rangle = \langle f, S^n g\rangle = \int_{\mathbb{T}} \overline{z}^n f \overline{g}\, d\mu\,.$$

The functions $\{z^n\}_{n\in\mathbb{Z}}$ are weak-star dense in $\mathcal{L}^\infty(\mathbb{T})$, so it follows that $f \otimes g \perp \mathcal{T}_T$ if and only if $f\overline{g} \equiv 0$.

(2) Assume $\hat{f}(0) = 0$ and $\hat{g}(0) \neq 0$. Then Equation (5.1) is equivalent to $f\big(\overline{g + (c-1)\hat{g}(0)}\big) \equiv 0$.

For $n \geq 0$, $T^n \perp f \otimes g$ if and only if

$$0 = \langle T^n f, g\rangle = \langle S^n f, g\rangle = \langle S^n f, g + (c-1)\hat{g}(0)\rangle = \int_{\mathbb{T}} z^n f\big(\overline{g + (c-1)\hat{g}(0)}\big)\, d\mu,$$

and for $n \geq 1$, $(T^*)^n \perp f \otimes g$ if and only if

$$0 = \langle f, T^n g \rangle = \big\langle f, S^n\big(g + (c-1)\hat{g}(0)\big)\big\rangle = \int_{\mathbb{T}} \overline{z}^n f\,\overline{\big(g + (c-1)\hat{g}(0)\big)}\, d\mu\,.$$

So $f \otimes g \perp \mathcal{T}_T$ if and only if $f\,\overline{\big(g + (c-1)\hat{g}(0)\big)} \equiv 0$.

(3) Assume $\hat{f}(0) \neq 0$ and $\hat{g}(0) \neq 0$.

For $n \geq 1$, $T^n \perp f \otimes g$ if and only if

$$\begin{aligned} 0 &= \langle T^n f, g \rangle \\ &= \big\langle S^n\big(f + c - 1\big), g \big\rangle \\ &= \Big\langle S^n\big(f + (c-1)\hat{f}(0)\big), g + (c-1)\hat{g}(0) \Big\rangle \\ &= \int_{\mathbb{T}} z^n \big(f + (c-1)\hat{f}(0)\big)\overline{\big(g + (c-1)\hat{g}(0)\big)}\, d\mu \end{aligned}$$

and for $n \geq 1$, $(T^*)^n \perp f \otimes g$ if and only if

$$\begin{aligned} 0 &= \langle f, T^n g \rangle \\ &= \big\langle f, S^n\big(g + (c-1)\hat{g}(0)\big)\big\rangle \\ &= \Big\langle f + (c-1)\hat{f}(0), S^n\big(g + (c-1)\hat{g}(0)\big)\Big\rangle \\ &= \int_{\mathbb{T}} \overline{z}^n \big(f + (c-1)\hat{f}(0)\big)\overline{\big(g + (c-1)\hat{g}(0)\big)}\, d\mu \end{aligned}$$

It follows that $f \otimes g \perp \operatorname{span}\{T^n, (T^*)^n \mid n \geq 1\}$ if and only if $\big(f + (c-1)\hat{f}(0)\big)\overline{\big(g + (c-1)\hat{g}(0)\big)} = \alpha$ for some constant α, and expanding the product reveals that $f\overline{g} = \alpha - \overline{\hat{g}(0)}(c-1)f - \hat{f}(0)(c-1)\overline{g} - (c-1)^2\hat{f}(0)\overline{\hat{g}(0)}$. Integrating shows that $\alpha = (c^2-1)\hat{f}(0)\overline{\hat{g}(0)}$, when $f \otimes g$ is also orthogonal to $I = T^0$. $\square$

The next Corollary proves half of our dichotomy. It makes use of Jensen's inequality (a proof can be found on pp. 51–52 of [15]):

Lemma 5.2 (Jensen's Inequality). *If f is a function in $\mathcal{H}^1$, then*

$$\log|\hat{f}(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{it})|\, dt$$

Corollary 5.3. *Let $T = S + (c-1)z \otimes 1$ be a perturbation of the shift. If $c > 1/\sqrt{2}$, then $\mathcal{T}_T$ is transitive.*

Proof. We may assume $c \neq 1$, or $T = S$ (for which the result is well known [13]). Assume $f \otimes g \perp \mathcal{T}_T$.

If $\hat{f}(0) = 0$ and $\hat{g}(0) = 0$, then by Theorem 5.1, $f\overline{g} \equiv 0$. Since f, g are members of $\mathcal{H}^2$ it follows that one of f, g is zero a.e., so $f \otimes g = 0$.

If $\hat{f}(0) = 0$ and $\hat{g}(0) \neq 0$, then by Theorem 5.1, $f\overline{\big(g + (c-1)\hat{g}(0)\big)} \equiv 0$. For $c \neq 0$, $g + (c-1)\hat{g}(0)$ must be nonzero because it has a nonzero constant term. It follows that f is zero a.e., whence $f \otimes g = 0$.

If f and g have nonzero constant terms, then by scaling we may assume $\hat{f}(0) = \hat{g}(0) = 1$. Write $F(z) = f(z) + (c-1)$, and $G(z) = g(z) + (c-1)$. By Theorem 5.1, $F\overline{G} \equiv c^2 - 1$. Lemma 5.2 gives

$$\int_{\mathbb{T}} \ln |F(z)|\, d\mu \geq \ln |F(0)| = \ln(1 + c - 1) = \ln c$$

and

$$\int_{\mathbb{T}} \ln |G(z)|\, d\mu \geq \ln |G(0)| = \ln(1 + c - 1) = \ln c$$

and it must follow that

$$\ln c^2 \leq \int_{\mathbb{T}} \ln |F\overline{G}|\, d\mu = \int_{\mathbb{T}} \ln |c^2 - 1|\, d\mu = \ln |c^2 - 1|$$

This inequality may be satisfied only when $c^2 \leq 1 - c^2$, i.e. when $c \leq 1/\sqrt{2}$, which finishes the proof. $\square$

In Corollary 5.3, the matter of finding $f \otimes g$ in $(\mathcal{T}_T)_\perp$ is translated into the task of finding $\mathcal{H}^2$ functions F, G with $F(0) = G(0) = c$ and $F(z)\overline{G(z)} = c^2 - 1$ for almost every z of absolute value one. Jensen's inequality disallows this when $|c| > 1/\sqrt{2}$, but the situation is very different for $|c| < 1\sqrt{2}$.

Proposition 5.4. *For $0 < c < 1/\sqrt{2}$, the general solution to the equations*

- $F(0) = G(0) = c$
- $F(z)\overline{G(z)} = c^2 - 1$ *for all* $|z| = 1$

with rational $F, G \in \mathcal{H}^2$ *is*

$$F(z) = c\prod_{j=1}^{n}\left(\frac{1}{a_j}\frac{a_j - z}{1 - b_j z}\right)$$
$$G(z) = c\prod_{j=1}^{n}\left(\frac{1}{\overline{b_j}}\frac{\overline{b_j} - z}{1 - \overline{a_j} z}\right)$$

where $0 < |a_j|, |b_j| < 1$ *and* $\prod_{j=1}^{n} a_j b_j = \frac{c^2}{c^2-1}$.

Proof. A direct computation show that the displayed F, G satisfy the desired conditions. For the converse, suppose F, G are rational functions solving the equations. Set $F(z) = p(z)/q(z)$, and write $p(z) = c_1 \prod_{i=1}^{m}(a_i - z)$ and $q(z) = \prod_{j=1}^{n}(1 - b_j z)$. Since $F \in \mathcal{H}^2$, no zero of q may lie inside the closed unit disk, so $|b_j| < 1$ for all j. Also $c = F(0) = c_1 \prod_{i=1}^{m} a_i$, or equivalently $c_1 = c\prod_{i=1}^{m}\frac{1}{a_i}$. So $F(z) = \dfrac{c\prod_{i=1}^{m}\left(\frac{1}{a_i}(a_i - z)\right)}{\prod_{j=1}^{n}(1 - b_j z)}$.

Since $F\overline{G} \equiv c^2 - 1$, $\overline{G(z)} = \dfrac{(c^2-1)\prod_{j=1}^{n}(1 - b_j z)}{c\prod_{i=1}^{m}\left(\frac{1}{a_i}(a_i - z)\right)}$ for $|z| = 1$, so $G(z) = z^{m-n}\dfrac{(c^2-1)}{c}\dfrac{\prod_{j=1}^{n}(\overline{b_j} - z)}{\prod_{i=1}^{m}\left(\frac{1}{\overline{a_i}}(1 - z\overline{a_i})\right)}$. This implies that $n = m$ and requires that $|a_i| < 1$ for all i. Finally, $G(0) = c$ forces $\dfrac{c^2-1}{c}\prod_{j=1}^{n}\overline{a_j}\overline{b_j} = c$, which is equivalent to the stated condition $\prod_{j=1}^{n} a_j b_j = \frac{c^2}{c^2-1}$ since c is real. □

The necessity of $c < 1/\sqrt{2}$ is apparent for this construction, since it is equivalent to $\left|\frac{c^2}{c^2-1}\right| < 1$ and is required if we are to choose a_j, b_j with modulus less than one. Therefore, there are no rational solutions when $c = 1/\sqrt{2}$.

The simplest example of a rank one operator $f \otimes g$ created from the functions in Proposition 5.4 corresponds to:

$$f(z) = 1 - c + \frac{c}{a}\frac{a - z}{1 - bz}$$
$$g(z) = 1 - c + \frac{c}{\overline{b}}\frac{\overline{b} - z}{1 - \overline{a}z}$$

The symmetry in the definitions of f and g is reflected in the matricial representation of $f \otimes g$. Expand each

$$f(z) = 1 + \frac{1}{c}\sum_{j=1}^{\infty}(bz)^j \tag{5.2}$$
$$g(z) = 1 + \frac{1}{c}\sum_{j=1}^{\infty}(\overline{a}z)^j$$

Then $f \otimes g$ is represented by the matrix

$$f \otimes g = \frac{1}{c^2}\begin{bmatrix} c^2 & ca & ca^2 & ca^3 & \cdots \\ cb & ba & ba^2 & ba^3 & \\ cb^2 & b^2a & b^2a^2 & b^2a^3 & \\ cb^3 & b^3a & b^3a^2 & b^3a^3 & \\ \vdots & & & & \ddots \end{bmatrix} = \frac{1}{c^2}\begin{bmatrix} c^2 & ca & ca^2 & ca^3 & \cdots \\ cb & \xi^1 & a\xi^1 & a^2\xi^1 & \\ cb^2 & b\xi^1 & \xi^2 & a\xi^2 & \\ cb^3 & b^2\xi^1 & b\xi^2 & \xi^3 & \\ \vdots & & & & \ddots \end{bmatrix} \tag{5.3}$$

where $\xi = ab = \frac{c^2}{c^2-1}$.

The existence of (any) rank one operators in the preannihilator, makes it apparent that the space $\mathfrak{T}_T$ is not transitive when $c < 1/\sqrt{2}$. Unfortunately, the operators of the type given in Equation (5.2) do not form a total subset of $(\mathfrak{T}_T)_\perp$, the condition necessary to show that T is star-reflexive. For example, many diagonal operators are orthogonal to these rank ones, which all share the same main diagonal. However, as the next Theorem will illustrate, we do get a total set in $(\mathfrak{T}_T)_\perp$ by also considering F, G of Proposition 5.4 corresponding to $n = 2$.

Since T generates a striped space, the diagonals of rank one operators in the preannihilator of $\mathfrak{T}_T$ will be of particular interest. Recall that the nth diagonal of an operator refers to a superdiagonal when $n > 0$ and a subdiagonal when $n < 0$. The following lemma will be of use:

Lemma 5.5. *Suppose A is a bounded linear operator on $\mathcal{H}^2$ supported on its nth diagonal, and write $k(z) = \sum_{j=0}^{\infty} c_j z^j$, where (c_j) are the entries on that diagonal. Then*

(1) $\langle A, z^n \otimes 1\rangle = k(0)$ when $n \geq 0$, and $\langle A, 1 \otimes z^n\rangle = k(0)$ when $n \leq 0$.

(2) For each $a, b \in \mathbb{D}$, $\big\langle A, 1/(1-az) \otimes 1/(1-\overline{b}z)\big\rangle = a^n k(ab)$ when $n \geq 0$, and $\big\langle A, 1/(1-az) \otimes 1/(1-\overline{b}z)\big\rangle = b^n k(ab)$ when $n \leq 0$.

Proof. (1) Clear.

(2) The ij-entry of the matrix for $1/(1-az) \otimes 1/(1-\overline{b}z)$ is $a^i b^j$. Therefore, when $n \geq 0$, $\big\langle A, 1/(1-az) \otimes 1/(1-\overline{b}z)\big\rangle = \sum_{j=0}^{\infty} c_j a^{n+j} b^j = a^n k(ab)$. The conclusion is similar when $n \leq 0$. $\square$

Theorem 5.6. *Let $T = S + (c-1)z \otimes 1$ be a perturbation of the shift. If $0 < c < 1/\sqrt{2}$ then T is star-reflexive.*

Proof. For convenience, set $\xi = \frac{c^2}{c^2-1}$. Write M for the closed linear span of rank one operators in the preannihilator of the star power span of T. It will be our task to show that $M_\perp \subseteq \mathcal{T}_T$.

Let F and G be functions from Proposition 5.4 with $n = 2$, and set

$$f(z) = 1 - c + F(z)$$
$$g(z) = 1 - c + G(z)$$

so that $f \otimes g \in M$. Applying a partial fraction decomposition to f and g and multiplying each by a nonzero scalar, we may write

$$f(z) = \frac{1}{1+c} + \frac{\alpha}{1-b_1 z} + \frac{\beta}{1-b_2 z}$$
$$g(z) = \frac{1}{1+c} + \frac{\overline{\gamma}}{1-\overline{a}_1 z} + \frac{\overline{\delta}}{1-\overline{a}_2 z}$$

where

$$\alpha = \frac{b_2(1-a_1b_1)(1-a_2b_1)}{b_1-b_2} \qquad \beta = -\frac{b_1(1-a_1b_2)(1-a_2b_2)}{b_1-b_2}$$
$$\gamma = \frac{a_2(1-a_1b_1)(1-a_1b_2)}{a_1-a_2} \qquad \delta = -\frac{a_1(1-a_2b_1)(1-a_2b_2)}{a_1-a_2}$$

Therefore $f \otimes g$ expands to

$$\begin{aligned} f \otimes g = & \left(\frac{1}{1+c} \otimes \frac{1}{1+c}\right) + \left(\frac{\gamma}{1+c} \otimes \frac{1}{1-\overline{a}_1 z}\right) + \left(\frac{\delta}{1+c} \otimes \frac{1}{1-\overline{a}_2 z}\right) \\ & + \left(\frac{\alpha}{1-b_1z} \otimes \frac{1}{1+c}\right) + \left(\frac{\alpha\gamma}{1-b_1z} \otimes \frac{1}{1-\overline{a}_1 z}\right) + \left(\frac{\alpha\delta}{1-b_1z} \otimes \frac{1}{1-\overline{a}_2 z}\right) \\ & + \left(\frac{\beta}{1-b_2z} \otimes \frac{1}{1+c}\right) + \left(\frac{\beta\gamma}{1-b_2z} \otimes \frac{1}{1-\overline{a}_1 z}\right) + \left(\frac{\beta\delta}{1-b_2z} \otimes \frac{1}{1-\overline{a}_2 z}\right) \end{aligned} \tag{5.4}$$

We wish to show that an arbitrary operator A orthogonal to every $f \otimes g \in M$ must live in $\mathcal{T}_T$. In light of Corollary 3.23, we simplify things greatly by assuming that A is supported on only a single diagonal.

Let $A \in M_\perp$, and assume that A is supported on its nth diagonal. For clarity, take $n \geq 0$ (a symmetric argument works when $n < 0$). We wish to show that A is a scalar multiple of $(T^*)^n$. The path to this conclusion is indirect. Since $(T^*)^n$ is supported on the nth diagonal and orthogonol to M, we will establish this fact:

> *The space of operators supported on the nth diagonal, and orthogonol to the space M is of dimension one.*

So assume that A is supported on its nth diagonal and is orthogonal to $z^n \otimes 1$ and all of the rank one operators of Display (5.4). Adopt the notation of Lemma 5.5, where (c_j) are the entries on that diagonal. We will complete the proof by showing that $A = 0$. From Lemma 5.5 we know that A is orthogonal to the first four terms and to the seventh term of Display (5.4). Applying Lemma 5.5 to the remaining terms, we obtain

$$\alpha\gamma b_1^n k(a_1b_1) + \alpha\delta b_1^n k(a_2b_1) + \beta\gamma b_2^n k(a_1b_2) + \beta\delta b_2^n k(a_2b_2) = 0, \tag{5.5}$$

whenever $a_1, a_2, b_1, b_2 \in \mathbb{D}$ have product ξ.

If we expand $\alpha, \beta, \gamma, \delta$ in Equation (5.5), and clear the common denominator $(a_1 - a_2)(b_1 - b_2)$, we obtain

$$\begin{aligned} 0 &= a_2 b_2 (1 - a_1 b_1)^2 (1 - a_2 b_1)(1 - a_1 b_2) b_1^n k(a_1 b_1) \\ &\quad - a_1 b_2 (1 - a_1 b_1)(1 - a_2 b_1)^2 (1 - a_2 b_2) b_1^n k(a_2 b_1) \\ &\quad - a_2 b_1 (1 - a_1 b_1)(1 - a_1 b_2)^2 (1 - a_2 b_2) b_2^n k(a_1 b_2) \\ &\quad + a_1 b_2 (1 - a_2 b_1)(1 - a_1 b_2)(1 - a_2 b_2)^2 b_2^n k(a_2 b_2) \end{aligned} \tag{5.6}$$

Take the limit of Equation (5.6) as $a_1 \to 1$, and make the substitution $b_1 = v, b_2 = w, a_2 = \xi/(vw)$ to obtain

$$\begin{aligned} 0 = &\frac{\xi}{v}(1-w)(1-\frac{\xi}{w})(1-v)^2 v^n k(v) - w(1-v)(1-\frac{\xi}{v})(1-\frac{\xi}{w})^2 v^n k(\frac{\xi}{w}) \\ &- \frac{\xi}{w}(1-v)(1-\frac{\xi}{v})(1-w)^2 w^n k(w) + v(1-w)(1-\frac{\xi}{w})(1-\frac{\xi}{v})^2 w^n k(\frac{\xi}{v}) \end{aligned} \tag{5.7}$$

for $\xi < |v|, |w| < 1$.

Applying Function Theory

Set $p(z) = \frac{\xi}{z}(1-z)^2 k(z)$ and $q(z) = (1-z)(1-\frac{\xi}{z})$. Note that $q(z) = q(\frac{\xi}{z})$. Then Equation (5.7) becomes

$$q(w)p(v)v^n - q(v)p(w)v^n - p(\frac{\xi}{w})q(v)w^n + p(\frac{\xi}{v})q(w)w^n = 0$$

From this point, there are two cases.

Case 1:

> If $n = 0$ then we learn that the function $r(z) = p(z) + p(\frac{\xi}{z})$ is a constant multiple of $q(z)$ on the annulus $\xi < |z| < 1$. But p has at most a pole of order one at the origin, so the Laurent expansion of p takes the form $p(z) = \sum_{i=-1}^{\infty} a_i z^i$. Now $a_i = 0$ for all $i > 1$ since that is the only way r can be a constant multiple of q. But then $k(z) = \frac{a_1 z^2 + a_0 z + a_{-1}}{\xi(1-z)^2}$ in the annulus and hence throughout $\mathbb{D}$. In order for the coefficients of k to be

bounded, it is necessary that its numerator vanish at 1, so k is a scalar multiple of $\dfrac{z}{1-z}$. This means that A is some scalar multiple of the diagonal matrix $(0, 1, 1, 1, \dots)$. Since A must also be orthogonal to the matrix of Display (5.3), we conclude that $A = 0$ as desired.

Case 2:

If $n \neq 0$, it follows that

$$\begin{aligned} q(w)p(v) - q(v)p(w) &= \left(\frac{w}{v}\right)^n \left[p(\frac{\xi}{w})q(v) - p(\frac{\xi}{v})q(w)\right] \\ &= \left(\frac{w}{v}\right)^n \left[p(\frac{\xi}{w})q(\frac{\xi}{v}) - p(\frac{\xi}{v})q(\frac{\xi}{w})\right] \\ &= \left(\frac{w}{v}\right)^n \left[\left(\frac{w}{v}\right)^n [p(v)q(w) - p(w)q(v)]\right] \end{aligned}$$

In particular this is true for $w/v \neq 1$ so we learn that $q(w)p(v) - q(v)p(w) = 0$, and it follows that $p(z)$ is a scalar multiple of $q(z)$. Thus $k(z)$ is a scalar multiple of $(z - \xi)\dfrac{1}{1-z}$. Recalling that $k(0) = 0$, we again conclude that $A = 0$ as desired.

$\square$

Chapter 6

Star-stability and Normality

In this section we investigate the relationship between star-stability and normality. All normal operators are star-stable; however there are many operators that are star-stable, but not normal. For example, all of the non-normal Jordan matrices are star-stable.

We conjecture that all star-stable operators on $\mathbb{C}^n$ with n distinct eigenvalues are normal. In a sense, the conjecture is an assertion about problems that occur with star-reflexivity with respect to similarity. Every operator A with distinct eigenvalues is similar to a diagonal matrix D, and since basic reflexivity is invariant under the diagonalization process, A is reflexive, just as D is. The normal matrix D is also star-stable, so to claim that A is not star-stable unless it is normal is to assert that star-stability behaves badly with respect to the similarity between D and A.

6.1 Introduction and notation

Throughout this chapter, let A denote a finite dimensional operator with distinct eigenvalues. If A is star-stable, then we have the containment:

$$\mathrm{ref}(\mathcal{A}_A + \mathcal{A}_{A^*}) \subseteq \mathrm{ref}(\mathcal{A}_A) + \mathrm{ref}(\mathcal{A}_{A^*})$$

In this case, since A is similar to a diagonal operator, it is reflexive and the containment simplifies to the criterion:

$$\mathrm{ref}(\mathcal{A}_A + \mathcal{A}_{A^*}) \subseteq \mathcal{A}_A + \mathcal{A}_{A^*}$$

That is, the linear space $\mathcal{T}_A = \mathcal{A}_A + \mathcal{A}_{A^*}$ is reflexive. So a restatement of the conjecture is that if A is a star-reflexive operator with distinct eigenvalues, then A is normal. This appears to be true, and an analysis has been carried out for operators on $\mathbb{C}^2, \mathbb{C}^3,$ and $\mathbb{C}^4$.

Since we are dealing extensively with $\mathcal{A}_A$, the following characterization is quite useful.

Lemma 6.1. *Suppose $A : \mathbb{C}^n \to \mathbb{C}^n$ has n distinct eigenvalues. Let $\{f_i\}$ be a basis of eigenvectors for A, and write g_i for the dual basis. Then $\mathcal{A}_A$ is spanned by the mutually orthogonal idempotents $\{f_i \otimes g_i\}$.*

Proof. Let $\lambda_1, \dots, \lambda_n$ denote the distinct eigenvalues of A. Let P_i be a polynomial such that $P_i(\lambda_j) = \delta_{ij}$. Then $f_i \otimes g_i = P_i(A) \in \mathcal{A}_A$. Since $A = \sum_{i=1}^n \lambda_i f_i \otimes g_i$, the set $\{f_i \otimes g_i\}$ spans. $\square$

From now on, $\{f_i\}$ denotes a basis of unit eigenvectors for A with dual basis $\{g_i\}$. Define the matrices

$$B = \sum_{i=1}^n f_i \otimes e_i \text{ and } C = \sum_{i=1}^n e_i \otimes g_i\,. \tag{6.1}$$

Then $CB = I$ since the rows of C are vectors g_i and the columns of B are vectors f_i. Also, note that $B^{-1}AB$ is a diagonal matrix with the same eigenvalues as A; that is, B implements a similarity between A and a normal operator.

Proposition 6.2. *Let I be a subset of $\{1, \dots, n\}$. Set $J = \{1, \dots, n\} \setminus I$, and write P for the orthogonal projection whose range is spanned by $\{f_i \mid i \in I\}$. Then the following are equivalent:*

(1) P is a polynomial in A.

(2) $f_i \perp f_j$ for each $i \in I, j \in J$.

(3) $g_i \perp g_j$ *for each* $i \in I, j \in J$.

Proof. (1) $\Rightarrow$ (2). Since P is a polynomial in A, it must be in the span of $\{f_i \otimes g_i\}$, by Lemma 6.1. It is clear that $P = \sum_{i \in I} f_i \otimes g_i$. Let $i \in I, j \in J$. Then $\langle f_i, f_j \rangle = \langle P f_i, f_j \rangle = \langle f_i, P f_j \rangle$. However, $P f_j$ is zero, so the inner product is zero.

(2) $\Rightarrow$ (1). Let $\tilde{P}$ be a polynomial such that $\tilde{P}(\lambda_i) = 1$ for $i \in I$ and $\tilde{P}(\lambda_j) = 0$ for $j \in J$. Since $\operatorname{span}\{f_i\}_{i \in I}$ and $\operatorname{span}\{f_j\}_{j \in J}$ are orthogonal, $\tilde{P}(A)$ is the projection P.

(2) $\Leftrightarrow$ (3). Sufficiency: Let $i \in I, j \in J$. Then $g_i \perp \{f_k \mid k \neq i\}$, and in particular $g_i \perp \{f_k \mid k \in J\}$. This means that g_i is in the span of $\{f_k \mid k \in I\}$. Similarly g_j is in the span of $\{f_k \mid k \in J\}$. It follows that $g_i \perp g_j$. For necessity, swap the roles of f and g. $\square$

The significance of this proposition is that if there exists a non-trivial partition $\{1, \ldots, n\} = I \dot{\cup} J$, then A can be divided into a direct sum $A = A_1 \oplus A_2$ where $A_1 = \sum_{i \in I} \lambda_i f_i \otimes g_i$ and $A_2 = \sum_{j \in J} \lambda_j f_j \otimes g_j$. In addition, the algebra $\mathcal{A}_A$ is a direct sum of algebras $\mathcal{A}_{A_1} \oplus \mathcal{A}_{A_2}$, where $\mathcal{A}_{A_1}$ and $\mathcal{A}_{A_2}$ are spanned by the idempotents $\{f_i \otimes g_i\}$ and $\{f_j \otimes g_j\}$, respectively. The star power spans also split in the same way. Reflexive closures respect direct sums of spaces (including algebras), so any time such a partition is found during the analysis of $\mathcal{A}_A$ or $\mathcal{T}_A$, we may pass to the similar analysis for the (simpler) operators A_1 and A_2.

A small example of this principle is:

Proposition 6.3. *If* $A : \mathbb{C}^2 \to \mathbb{C}^2$ *is a star-stable operator with distinct eigenvalues, then* A *is normal.*

Proof. Let $\mathcal{A}_A = \operatorname{span}\{f_1 \otimes g_1,\ f_2 \otimes g_2\}$, and write $\mathcal{S} = \mathcal{T}_A$. Our method is to show that A decomposes by establishing that $f_1 \perp f_2$. By assumption, A is star-stable so $\operatorname{ref} \mathcal{S} \subseteq \mathcal{S}$. Now the dimension of $\mathcal{S}$ is no larger than 3 since $\mathcal{A}_A$ and $\mathcal{A}_{A^*}$ both contain the identity. By definition of reflexive there must exist some nontrivial rank one operator in the orthogonal complement of $\mathcal{S}$. A rank one operator $a \otimes b$ is orthogonal to $\mathcal{S}$ if and only if it is orthogonal to each of $\{f_1 \otimes g_2,\ f_2 \otimes g_2,\ g_1 \otimes f_1,\ g_2 \otimes f_2\}$, i.e. if the four equalities hold:

$$\begin{aligned} \langle a, g_1\rangle\langle f_1, b\rangle &= 0 \qquad & \langle a, f_1\rangle\langle g_1, b\rangle &= 0 \\ \langle a, g_2\rangle\langle f_2, b\rangle &= 0 \qquad & \langle a, f_2\rangle\langle g_2, b\rangle &= 0 \end{aligned}$$

The first row of equalities is sufficient for our purpose in this case. Without loss of generality, we may assume that $\langle a, g_1\rangle = 0$. This means that $a = \lambda f_2$ for some nonzero scalar λ, but by scaling we may assume $\lambda = 1$. In the second equation, if $\langle a, f_1\rangle = \langle f_2, f_1\rangle \neq 0$ then $\langle g_1, b\rangle = 0$, yielding (again scaling if necessary) $b = f_2$. But then $\langle a, b\rangle = \langle f_1, f_2\rangle \neq 0$, contradicting the fact that $\mathcal{S}$ contains the identity, so we are forced to concede $f_1 \perp f_2$ as desired. $\square$

6.2 Matricial Translation

In each case, we wish to proceed along the same outline: Starting from an operator A having distinct eigenvalues, we use star-stability to assert that $\mathrm{ref}(\mathcal{T}_A)$ has dimension no more than $2n - 1$ (spanned by the rank one operators $f_i \otimes g_i$ and their adjoints, and containing the identity). It follows that $\dim((\mathcal{T}_A)_\perp \cap \mathfrak{F}_1)$ has dimension at least $n^2 - 2n + 1$. If we assume that A cannot be written as a nontrivial direct sum, and then determine that a spanning set for $(\mathcal{T}_A)_\perp \cap \mathfrak{F}_1$ has less than $n^2 - 2n + 1$ members, it is forced that A must split and we continue inductively with its direct summands to conclude that A is normal.

Finding a minimal spanning set for $(\mathcal{T}_A)_\perp \cap \mathfrak{F}_1$ is a difficult task. For this reason, the problem has been translated into a matricial counting task.

Write

$$X = B^*B \text{ and } Y = CC^* \tag{6.2}$$

where B, C are from Equation (6.1). Note that X is the matrix whose entries are the inner products $\langle f_i, f_j\rangle$. The matrices X and Y are positive definite, X has ones on the diagonal, and $Y = X^{-1}$. Every positive definite matrix with ones on its diagonal arises as the X corresponding to some operator A.

Recall that since $\{g_i\}$ is a dual basis to $\{f_i\}$, we always have $f = \sum_{i=1}^{n} \langle f, g_i\rangle f_i$. Define the *support* of a vector $f \in \mathbb{C}^n$ to be the set $\operatorname{supp} f = \{i \mid \langle f, g_i\rangle \neq 0\}$. Also, given subsets I, J of $\{1, \ldots, n\}$, write $I \times J$ for the matrix $(X_{ij})_{i\in I, j\in J}$.

We will call a (possibly non-square) matrix M *bisingular* if M and M^* have non-trivial kernels.

The following lemma gives a crude first attempt at finding a spanning set for $(\mathfrak{T}_A)_\perp \cap \mathfrak{F}_1$. It says that bisingular submatrices of X correspond directly to rank one operators $a \otimes b \perp \mathfrak{T}_A$.

Lemma 6.4. *(1) If $a \otimes b$ belongs to $(\mathfrak{T}_A)_\perp$, then the matrix $\operatorname{supp} a \times \operatorname{supp} b$ is bisingular. (2) Conversely, if the matrix $I \times J$ is bisingular, then there are vectors $a, b \in \mathbb{C}^n$ with supports contained in I, J, respectively, such that $a \otimes b$ belongs to $(\mathfrak{T}_A)_\perp$.*

Proof. Let $a \otimes b \in (\mathfrak{T}_A)_\perp$. Let $I = \operatorname{supp} a$ and $J = \operatorname{supp} b$, and write $a = \sum_{i\in I} \overline{\alpha_i} f_i$ and $b = \sum_{j\in J} \overline{\beta_j} f_j$. For convenience, take $l = |I|$ and $m = |J|$.

Since $a \otimes b \perp \mathfrak{T}_A$, we have $a \otimes b \perp f_i \otimes g_i$ for each i, and in particular, $a \otimes b \perp f_i \otimes g_i$ for $i \in I$. However, for $i \in I$, $\langle a, g_i\rangle \neq 0$, by assumption. Therefore, since $0 = \langle a \otimes b, f_i \otimes g_i\rangle = \langle a, g_i\rangle\langle f_i, b\rangle$, we must have that $\langle f_i, b\rangle = 0$ for $i \in I$. This means that $\sum_{j\in J} \beta_j \langle f_i, f_j\rangle = 0$ for each $i \in I$, and this may be written as a matrix product:

$$M\begin{bmatrix}\beta_{j_1}\\ \vdots\\ \beta_{j_m}\end{bmatrix}=\begin{bmatrix}0\end{bmatrix}\text{ where } M \text{ is the } l\times m \text{ matrix } \Big(\langle f_i,f_j\rangle\Big)_{\substack{i\in I\\ j\in J}}$$

Notice that M is the matrix $\operatorname{supp} a\times\operatorname{supp} b$.

Since $a\otimes b\perp g_j\otimes f_j$ as well, we have $\langle a,f_j\rangle=0$ for $j\in J$. Expanding and taking conjugates, this is $\sum_{i\in I}\alpha_i\langle f_j,f_i\rangle=0$ for each $j\in J$, which corresponds to the matricial product:

$$N\begin{bmatrix}\alpha_{i_1}\\ \vdots\\ \alpha_{i_l}\end{bmatrix}=\begin{bmatrix}0\end{bmatrix}\text{ where } N=\Big(\langle f_j,f_i\rangle\Big)_{\substack{j\in J\\ i\in I}}=M^*$$

Therefore $\operatorname{supp} a\times\operatorname{supp} b$ is bisingular.

For the converse, begin with the singular matrices M,M^* and work backwards from vectors in each kernel. □

Knowing that rank one operators correspond to bisingular submatrices of X is a place to begin, and Lemma 6.4 could be used to find a spanning set for $(\mathcal{T}_A)\perp\cap\mathfrak{F}_1$. However, our desire is actually to find a <u>minimal</u> spanning set. There are too many bisingular submatrices in X for this purpose, however the next three Lemmas show that not every bisingular submatrix needs to be considered.

Lemma 6.5. *A spanning set for $(\mathcal{T}_A)_\perp\cap\mathfrak{F}_1$ may be found from operators $a\otimes b$ where $M=\operatorname{supp} a\times\operatorname{supp} b$ is a bisingular submatrix of X with* $\operatorname{nullity} M=1$.

Proof. Let $a\otimes b\perp\mathcal{T}_A$. We know from before that $M=\operatorname{supp} a\times\operatorname{supp} b$ is bisingular so assume that $\operatorname{nullity} M=t>1$. Then the system $M\begin{bmatrix}\beta_{j_1}\\ \vdots\\ \beta_{j_m}\end{bmatrix}=\begin{bmatrix}0\end{bmatrix}$ has t free variables and a basis for the kernel can be constructed of the vectors determined by setting $(t-1)$ of the free variables to zero and the remaining free variable to 1 in turn. Denote such a basis by $\left\{ \begin{bmatrix} \beta^{(1)}_{j_1} \\ \vdots \\ \beta^{(1)}_{j_m} \end{bmatrix}, \dots, \begin{bmatrix} \beta^{(t)}_{j_1} \\ \vdots \\ \beta^{(t)}_{j_m} \end{bmatrix} \right\}$ and set $a_j = \sum_{i=1}^{m} \beta^{(j)}_{j_i} f_{k_i}$. Note that a is contained in the linear span of $\{a_j\}$ and more importantly that $a \otimes b$ is contained in the span of $\{a_j \otimes b\}$. Now since the sum for each a_j has $t-1$ zero terms, $\operatorname{supp} a_j \times \operatorname{supp} b$ is a new matrix M_j formed from M by removing those respective columns. (Notice that b is still in the kernel of each M_j^*.) Since $\operatorname{nullity} M_j = 1$ we are finished. □

An important consequence of this lemma is that only square submatrices of X must be considered. An operator $a \otimes b$ is orthogonal to $\mathfrak{T}_A$ if and only if $b \otimes a$ is, and the matrix $\operatorname{supp} b \times \operatorname{supp} a$ is the adjoint of the matrix $\operatorname{supp} a \times \operatorname{supp} b$. For any non-square, bisingular matrix M, either M or M^* would have nullity greater than one and could, by the lemma, be replaced by smaller submatrices.

As the next lemma indicates, it is possible to be even more selective:

Lemma 6.6. *A spanning set for* $(\mathfrak{T}_A)_\perp \cap \mathfrak{F}_1$ *may be found from operators* $a \otimes b$ *where* $M = \operatorname{supp} a \times \operatorname{supp} b$ *is a singular square submatrix of* X *such that no proper subset of rows or columns are dependent.*

Proof. Let $a \otimes b \in (\mathfrak{T}_A)_\perp$, and let $M = \operatorname{supp} a \times \operatorname{supp} b$ be an n-by-n submatrix of X with $\operatorname{nullity} M = 1$. Write $a = \sum_{i=1}^{n} \overline{\alpha_i} f_i$ and $b = \sum_{j=1}^{n} \overline{\beta_j} f_j$, with $\vec{\alpha}, \vec{\beta}$ in the kernel of M, M^* respectively. Assume M has a proper subset of rows or columns that are dependent. By symmetry we consider columns, and for simplicity, assume that the first $n-1$ columns of M are dependent. If we were to row-reduce M we would find that the lth column would not contain a pivot element for some $l < m$ and since $\operatorname{nullity} M = 1$, the remaining $n-l$ columns would all contain pivots. We conclude that the last $n-l$ variables in the system $M\vec{\alpha}=\vec{0}$ must all be 0. This would contradict the assumption that the columns of M are $\operatorname{supp} a$, so M must have no proper set of dependent columns. □

Proposition 6.7. *Let $M = I \times J = (X_{ij})_{i\in I, j\in J}$ and $\tilde{M} = (Y_{ji})_{j\in\tilde{J}, i\in\tilde{I}}$, where $|I| = |J|$, $\tilde{I} = \{1,\dots,n\} \setminus I$, and $\tilde{J} = \{1,\dots,n\} \setminus J$. Then M is singular if and only if $\tilde{M}$ is singular.*

Proof. This is a generalization of the (classical) adjoint formula for the inverse of a nonsingular matrix. Up to sign, the determinant of a submatrix of X is the determinant of X times the determinant of the complementary submatrix of $Y = X^{-1}$. That is, $|M| = \pm|X||\tilde{M}|$, so M and $\tilde{M}$ are simultaneously singular or nonsingular. □

This result is very helpful. In general it is much easier to recognize when small matrices are singular, than large ones. The extreme of this is that singular one-by-one matrices are trivial to identify. Proposition 6.7 shows that A may replaced by A^*, effectively switching the roles of X and Y, and a spanning set of rank one operators can be found corresponding to singular submatrices of the new X.

The following result makes use of this technique.

Proposition 6.8. *If $A : \mathbb{C}^3 \to \mathbb{C}^3$ is a star-stable operator with distinct eigenvalues, then A is normal.*

Proof. The dimension of $\mathcal{T}_T$ is at most 5, and if A is star-stable then $\dim \operatorname{ref}(\mathcal{T}_T)$ is also no more than 5. So we should be able to find 4 independent rank one operators in $(\mathcal{T}_T)_\perp$.

Write:

$$X = \begin{bmatrix} 1 & a & b \\ \overline{a} & 1 & c \\ \overline{b} & \overline{c} & 1 \end{bmatrix}$$

Since a spanning set for $(\mathfrak{T}_A)_\perp \cap \mathfrak{F}_1$ corresponds to singular submatrices of X, we proceed by cases, starting with one-by-one submatrices (zeros).

(4 zeros) If X has four (or more) zeros in it, then it is a quick check that one row has a single one in it. Since X is the matrix of inner products of the members of $\{f_i\}$, some $f_i \perp \{f_j \mid j \neq i\}$, so A splits as a direct sum, and we're finished inductively.

(2 zeros) If X has (only) two zeros, we may as well assume $f_1 \perp f_3$, so

$$X = \begin{bmatrix} 1 & a & 0 \\ \overline{a} & 1 & c \\ 0 & \overline{c} & 1 \end{bmatrix}$$

But X clearly has no singular two-by-two submatrices, so the dimension of $(\mathfrak{T}_A)_\perp \cap \mathfrak{F}_1$ is no more than 2, contradicting the star-stability assumption.

(0 zeros) If X has no zeros in it, then it has four singular two-by-two submatrices. However, Proposition 6.7 allows us to consider A^* and Y instead, which has four zeros in it, and we appeal to that case. $\square$

Note: none of the results of this section depend on the normalization of the vectors f_j (i.e. the assumption that the diagonal of X contains all ones). Indeed, multiplying X on the left or right by an invertible diagonal matrix does not affect which submatrices are singular.

6.3 Reflexive, Star-Transitive Operators

The matricial results from the previous section may be used to demonstrate a relative abundance of operators that are simultaneously reflexive and star-transitive. The method involves perturbing operators known to be reflexive in order to guarantee their star-transitivity. This exposes a subtleness of star-reflexivity, since for star-reflexive A it is always possible to find star-transitive operators close to A without breaking the regular reflexivity of A.

Lemma 6.9. *Fix a positive integer n. Let $\mathcal{D} \subset M_n$ be the set of diagonal matrices with distinct diagonal entries $\{c_1, \dots, c_n\}$, $\mathcal{G} \subset M_n$ the set of invertible $n \times n$ matrices, and $\mathcal{V} \subset M_n$ the $n \times n$ matrices with n distinct eigenvalues. Define $\phi : \mathcal{D} \times \mathcal{G} \to \mathcal{V}$ by $\phi(D, G) = GDG^{-1}$. Then ϕ is continuous, surjective, and open.*

Proof. A variant of this is Theorem 16.1 of [1]. For a self-contained proof, note that the map $G \mapsto G^{-1}$ is continuous, as is matrix multiplication, so ϕ is continuous. If V has distinct eigenvalues, then $V = GDG^{-1}$ for some diagonal matrix (of eigenvalues) D, so ϕ is surjective.

It remains to show that ϕ is open. This is done by finding local right inverses for ϕ. We use the Dunford-Riesz functional calculus: if f is a complex-valued function which is analytic in a neighborhood of the spectrum of an operator A, then $f(A)$ may be defined in terms of a contour integral. Let Γ be a chain in the domain of analyticity of f which has a winding number of one around every point of the spectrum of A; then

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(zI - A)^{-1}\, dz$$

This definition is independent of the choice of Γ, and is multiplicative in the sense that $(fg)(A) = f(A)g(A)$ when both f and g are analytic in a neighborhood of the spectrum of A. (A development of this functional calculus can be found in Chapter VII(4) of [7].)

In particular, when $f = \chi_\Delta$ is a characteristic function, then $f(A)$ is an idempotent commuting with A. Note also that if $||B - A||$ is sufficiently small, then $f(B)$ is well-defined and close to $f(A)$. That is, at least locally, $f(B)$ depends continuously on B.

Now fix operators $D_0 \in \mathcal{D}$, and $G_0 \in \mathcal{G}$. Write $A_0 = \phi(D_0, G_0)$. Denote the diagonal entries of D_0 by $c_1, \dots, c_n$, and the columns of G_0 by $x_1, \dots, x_n$. Choose disjoint small open balls B_j about the c_j and let $f_j = \chi_{B_j}$. Then for A sufficiently close to A_0, each $f_j(A)$ is an idempotent close to $f_j(A_0)$. This implies, in particular, that none of the $f_j(A)$ vanish. Since the product of any pair of the $f_j(A)$ is zero, their ranges must be independent, and since there are n of them, each must have rank precisely equal to one. Since $f_j(A)$ commutes with A, we see that in fact $\big(A\big)\big(f_j(A)\big)$ is a scalar multiple of $f_j(A)$, which means that the vectors in the range of $f_j(A)$ are eigenvectors of A. In particular, for A sufficiently close to A_0, $f_j(A)x_j$ is a nonzero eigenvector of A.

Let U be a sufficiently small neighborhood of A such that all of the preceding paragraph is valid. For $A \in U$, define $G(A) = \sum_{j=1}^{n} f_j(A)x_j \otimes e_j$, that is $G(A)$ is the matrix whose jth column is $f_j(A)x_j$. This makes $G(A)$ invertible, and $D(A) = \big(G(A)\big)^{-1}A\big(G(A)\big)$ diagonal. In particular $G(A_0) = G_0$ and $D(A_0) = D_0$. Since $D(A)$ depends continuously on A, by shrinking U if necessary, we may assume the eigenvalues of $D(A)$ are distinct for each A in U. Then we have $\phi\big(D(A), G(A)\big) = A$ for each A in U, and we have constructed the desired continuous local right inverse for ϕ. $\square$

Theorem 6.10. *There is a dense open subset of M_n consisting of transformations which are reflexive, but star-transitive.*

Proof. Write $\mathcal{V} = \{A \in M_n \mid A \text{ has } n \text{ distinct eigenvalues}\}$ and $\mathcal{W} = \{A \in \mathcal{V} \mid A \text{ is star-transitive}\}$.

The openness of the map ϕ of Lemma 6.9 shows that $\mathcal{V}$ is open. Moreover, given $A \in M_n$, we can find an invertible matrix G so that $J = G^{-1}AG$ is lower triangular.

Perturbing J slightly, we obtain a lower triangular matrix K whose diagonal entries are distinct. Then GKG^{-1} is a member of $\mathcal{V}$ which is close to A. Thus $\mathcal{V}$ is dense and open in M_n and we know all of its members are reflexive.

Now suppose $D \in \mathcal{D}$ and B is an invertible matrix in M_n. It is the content of Section 6.2, that $\phi(D, B)$ is star-transitive if and only if none of the square submatrices of $X = B^*B$ is singular. Continuity of determinants, matrix multiplication, and adjunction shows that $\mathcal{Z} = \{B \in \mathcal{G} \mid \text{no square submatrix of } X = B^*B \text{ is singular}\}$ is open in M_n. Since the finite intersection of dense open sets remains dense, we can establish density of $\mathcal{Z} \subset M_n$ by considering individual submatrices of X. Suppose that some square submatrix $(X)_{I,J}$ has a one-dimensional kernel. Since we know that this cannot happen for a principal submatrix, we can chose some $i \in I$ which does not belong to J. By slightly perturbing the ith column of B, we may adjust the ith row of $X = B^*B$ and spoil the linear dependence of the rows of $(X)_{I,J}$.

We complete the proof by appealing to the openness of ϕ to conclude that $\mathcal{W} = \phi\{\mathcal{D} \times \mathcal{Z}\}$ is open, and the continuity of ϕ to conclude that $\mathcal{W}$ is dense in M_n. $\square$

6.4 Zeros in X

Without question, the singular submatrices of X that are the easiest to work with are the one-by-one submatrices; that is, the zeros. Because of this, it is helpful to know just how many zeros can be in X.

Proposition 6.11. *Assume for any nontrivial partition $I \dot\cup J = \{1, \dots, n\}$ there exists $i \in I$ and $j \in J$ with $\langle f_i, f_j \rangle \neq 0$.*

(1) There are no more than $n^2 - n - 2(n-1)$ zero entries in X.

(2) For some permutation of the $\{f_i\}$ the matrix X has a nonzero entry above the diagonal in each column. If the first nonzero entry above the diagonal in the

kth column is in the lth row, then the first nonzero entry above the diagonal in the $k+1$ column will be in row l' for some $l' \geq l$.

Proof. (1) Let G be a graph and label its vertices $f_1, \ldots, f_n$. For each pair of vertices, place an (undirected) edge if $\langle f_i, f_j \rangle \neq 0$. Then having G be disconnected is equivalent to finding index sets I, J with $\{f_i \mid i \in I\} \perp \{f_j \mid j \in J\}$. Since by assumption this cannot be, G must be connected, and hence must have at least $n-1$ edges. Each edge in G corresponds to a pair $\langle f_i, f_j \rangle = \overline{\langle f_j, f_i \rangle} \neq 0$, that is to two nonzero entries of X. Include the diagonal $\langle f_i, f_i \rangle \neq 0$ and we have accounted for $2(n-1)+n$ nonzero entries.

(2) Let the initial ordering be denoted $f_1^{(0)}, \ldots, f_n^{(0)}$. Fix $f_1 = f_1^{(0)}$ and reorder the remaining vectors to form a new ordering $f_1, f_2^{(1)}, \ldots, f_n^{(1)}$, placing first the vectors whose inner product with f_1 is nonzero. Now fix $f_2 = f_2^{(1)}$. Continue inductively. When $f_1, \ldots, f_k$ are fixed, reorder the remaining vectors $f_1, \ldots, f_k, f_{k+1}^{(k)}, \ldots, f_n^{(k)}$, choosing first any vectors whose inner product with f_1 is nonzero, then vectors whose inner product with f_2 is nonzero (a kind of dictionary order), and so on. Choose $f_{k+1} = f_{k+1}^{(k)}$. Since we have assumed there is no partition of the $\{f_i\}$ into mutually orthogonal sets, it is assured that f_{k+1} has a nonzero inner product with at least one of the previously chosen vectors (otherwise the set of remaining vectors are orthogonal to the set chosen so far), guaranteeing a nonzero entry in the $k+1$ column. Under the final nth ordering, the matrix has the form indicated. ☐

This theorem gives a constraint on the dimension of $(\mathcal{T}_A)_\perp \cap \mathfrak{F}_1$, at least limiting how much can be contributed by the operators associated to singular one-by-one submatrices of X. For some operators, this will allow us to immediately assert normality from star-stability. For example if $(\mathcal{T}_A)_\perp \cap \mathfrak{F}_1$ is completely spanned by operators associated to singular one-by-one submatrices then A is normal. This is because by Lemma 6.1, $\dim \mathcal{S} \leq 2n-1$, so if A is star-stable then $\dim \operatorname{span}((\mathcal{T}_A)_\perp \cap \mathfrak{F}_1) \geq$

$n^2 - 2n + 1$ but only $n^2 - n - 2(n-1)$ can be accounted for by zeros unless $A = A_1 \oplus A_2$. Since the spaces associated to A_i are also spanned by operators associated with singular one-by-one submatrices of the corresponding X_i, we conclude inductively that A is normal.

Given this line of reasoning, it is helpful to know situations when $(\mathfrak{T}_A)_\perp \cap \mathfrak{F}_1$ is spanned by operators associated to singular one-by-one matrices. One situation is when X attains the maximal number of zeros from Proposition 6.11.

Theorem 6.12. *If X contains $n^2 - n - 2(n-1)$ zeros, then either there is a nontrivial partition of $\{f_i\}$ into mutually orthogonal sets, or every singular m-by-m submatrix of X with $m > 1$ has a proper subset of dependent rows or columns.*

Proof. Assume there is no nontrivial partition of $\{f_i\}$ into mutually orthogonal sets. We may assume that the $f_1, \ldots, f_n$ are in the order from the proof of Proposition 6.11. If not, the permutation required to attain this ordering has the effect of permuting the rows and columns of X, but every submatrix of the new matrix corresponds exactly to a submatrix in the original problem so there is no loss of generality. Since we have assumed that X has the maximum possible number of zeros, we know that above the diagonal each column contains precisely one nonzero entry (and consequently each row below the diagonal contains precisely one nonzero entry). Part (2) of the proposition tells us that if the nonzero entry above the diagonal in column k is in row l, and if the nonzero entry in column $k+1$ is in row l' then $l' \geq l$.

If $M = (m_{ij})$ was formed by selecting the same numbered rows and columns from X, then M is principal hence invertible so there is nothing to show. If M is not principal, then there is a highest index k so that M includes the kth row from X but not the kth column. There is also a last column l included in M such that the lth row was not part of M. Since M^* is also a submatrix of X, we may assume that $k > l$.

So M has the form: $M = \left[\begin{array}{c|c} W & Y \\ \hline Z & T \end{array}\right]$ where W is $s \times s$, and $T^* = T$ is a (potentially 0×0) principal submatrix of X. If it were possible to show that $Z = 0$ we would be done, since we would obtain $\det M = \det W \det T$. This would force $\det W = 0$, and imply that the columns of W are dependent. Since $Z = 0$ this means the first s columns of M are dependent. More generally, if we can show that there exists a nonempty set J of rows such that for $i \in J$, $m_{ij} \neq 0 \implies j \in J$ then either the columns $\{j \mid j \notin J\}$ or the rows $\{i \mid i \in J\}$ are linearly dependent.

We are also finished if M contains a row of zeros. And if M contains a row with only a single nonzero entry, the columns not containing the nonzero entry are dependent, as can be seen by expanding the determinant along that row. Of course similar statements are true of the columns as well.

So to achieve the result, we will show that M contains one of the following:

1. A column having at most one nonzero entry.

2. A row having at most one nonzero entry.

3. A proper subset J of rows such that for $i \in J$, $m_{ij} \neq 0 \implies j \in J$

If T is 0×0 then we are done immediately since the last row of M contains elements that live strictly below the diagonal of X (since we assumed above that $k > l$). Below the diagonal a row has precisely one nonzero entry, so the last row of M has at most one nonzero entry.

Now, consider the row of M lying just above T. By choice, we have guaranteed that the number (in X) of this row is greater than the column number of each of the columns lying to the left of T.

$$M = \left[\begin{array}{c|c} W' & Y' \\ \hline b_1 \dots b_s & a_{s+1} \dots a_m \\ \hline Z & T \end{array}\right]$$

Here we have indicated elements that originally came from below the diagonal of X by bs and elements that came from above the diagonal with as. The elements of Z are all from below the diagonal, and the elements of Y' are from above. No such characterization can be made of the elements of W'. Our first note is that if $a_{s+1}, \dots, a_m$ are all 0, then we are finished, since at most one of the $b_1, \dots, b_s$ can be nonzero.

So we may assume that that there is some largest index t such that $a_t \neq 0$. Let $J_1 = \{i > t \mid m_{it} \neq 0\}$. If J_1 is empty, then a_t is the only nonzero element on the tth column of M. Otherwise, define inductively $J_n = \{j \mid m_{ij} \neq 0 \text{ for some } i \in J_{n-1}\} \supseteq J_{n-1}$ and set $J = \cup J_n$. Clearly J is a nonempty set of rows such that for $i \in J$, $m_{ij} \neq 0 \implies j \in J$. Because of the symmetric nature of T and the fact that below the diagonal there is at most one nonzero entry in each row, J will contain only rows greater than t so it is a proper subset of rows as well. ☐

Remark: This result actually shows that if $\dim \mathcal{T}_A = 2n-1$ and X has $(n-1)(n-2)$ zeros, then $\mathrm{ref}(\mathcal{T}_A)$ has dimension $3n-2$.

Corollary 6.13. *"Staircase" operators with distinct diagonal entries, i.e. operators of the form* $A = \begin{bmatrix} \lambda_1 & * & & \\ & \lambda_2 & & \\ & * & \lambda_3 & * \\ & & & \ddots \end{bmatrix}_{n\times n}$ *are star-stable if and only if they are normal.*

Proof. For this choice of operator, X is a tridiagonal matrix, and has $n^2-n-2(n-1)$ zeros. By Theorem 6.12, we know that $(\mathcal{T}_A)_\perp \cap \mathfrak{F}_1$ is spanned by operators that correspond to the zeros in this matrix. If A is star-stable we are forced to conclude that it breaks into an orthogonal direct sum $A = A_1 \oplus A_2$. We then finish inductively, considering A_1 and A_2. $\square$

6.5 The 4-by-4 case

Although the two-dimensional and three-dimensional cases were relatively, straightforward, proving that star-stable operators on $\mathbb{C}^4$ with distinct eigenvalues are normal is more difficult. The following lemma will be of some use.

Lemma 6.14. *If* $M = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,k} \end{bmatrix}$ *and* $N = \begin{bmatrix} a_{1,2} & \cdots & a_{1,k} & a_{1,k+1} \\ \vdots & \ddots & \vdots & \vdots \\ a_{k,2} & \cdots & a_{k,k} & a_{k,k+1} \end{bmatrix}$ *are singular k-by-k matrices (agreeing in $k-1$ columns), then one of the following must happen:*

1. *The common $k-1$ columns of M and N are dependent.*
2. *Each matrix* $M_j = \begin{bmatrix} a_{1,1} & \cdots & \widehat{a_{1,j}} & \cdots & a_{1,k+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k,1} & \cdots & \widehat{a_{k,j}} & \cdots & a_{k,k+1} \end{bmatrix}$ *is singular for* $1 \leq j \leq k+1$.

Proof. Note that the order of the columns is unimportant, and that the result rests on the condition that two singular matrices agree on $k-1$ columns. The stated ordering will be used for convenience.

Assume that the common $k-1$ columns of M and N are linearly independent. Then the first column of M is in their linear span, as is the last column of N. This means the span of all $k+1$ columns has dimension only $k-1$, and any choice of k columns is linearly dependent, yielding a singular matrix. $\square$

Proposition 6.15. *If $A : \mathbb{C}^4 \to \mathbb{C}^4$ is a star-stable operator with distinct eigenvalues, then A is normal.*

Proof. The dimension of $\mathcal{T}_A$ is no more than 7, so if A is star-stable, then $\operatorname{ref}(\mathcal{T}_T)$ is also at most 7, and we should be able to find 9 independent rank one operators in $(\mathcal{T}_T)_\perp$. As with the three-by-three result, we proceed by cases, based on the number of zeros in X. By Proposition 6.11, if X has more than 6 zeros then A splits into a non-trivial direct sum, and consequently is normal. In every other case, we will show that the star-stability assumption is contradicted by showing that $\dim\big((\mathcal{T}_A)_\perp \cap \mathfrak{F}_1\big) \leq 8$.

(6 zeros) If X has 6 zeros, then Theorem 6.12 implies that the dimension of $(\mathcal{T}_A)_\perp \cap \mathfrak{F}_1$ is 6, thus contradicting the star-stability assumption.

(4 zeros) If X has 4 zeros, there are two subcases to consider. The first is when some f_i is orthogonal to two other f_j; by symmetry we may assume $f_4 \perp \{f_1, f_2\}$. The second subcase, up to symmetry, is when $f_1 \perp f_2$ and $f_3 \perp f_4$.

Assume $f_4 \perp \{f_1, f_2\}$. Then we may write

$$X = \begin{bmatrix} 1 & a & b & 0 \\ \overline{a} & 1 & c & 0 \\ \overline{b} & \overline{c} & 1 & f \\ 0 & 0 & \overline{f} & 1 \end{bmatrix}$$

In this matrix, the only possible singular two-by-two submatrices without an obvious proper subset of dependent rows or columns are $\begin{bmatrix} 1 & b \\ \overline{a} & c \end{bmatrix}$, $\begin{bmatrix} a & b \\ \overline{c} & 1 \end{bmatrix}$, $\begin{bmatrix} a & b \\ 1 & c \end{bmatrix}$ and their adjoints. However whenever more than one of the submatrices (and their adjoints) are singular simultaneously we are forced to admit a singular principal submatrix by Lemma 6.14, in contradiction to the positive definiteness of X.

To obtain three-by-three submatrices of X we must eliminate one row and one column. If we eliminate the 4th row and 4th column, the submatrix is principal, so we may assume that we have not removed the 4th column. This column has two zeros in it, and if we do not remove one of them a column with a single nonzero will result. If there is a column with a single nonzero, then the rows containing the zeros will be dependent already and we may forego counting this matrix in lieu of the smaller submatrices, by Lemma 6.6. So we must remove row 1 or 2. This leaves row 4 in the submatrix and analogously we must remove column 1 or 2. Since we cannot remove the same row and column, our submatrix can only be $\begin{bmatrix} a & b & 0 \\ \overline{c} & 1 & f \\ \overline{0} & \overline{f} & 1 \end{bmatrix}$ or its adjoint.

At most our tally is now 4 one-by-one submatrices, 2 two-by-two submatrices, and 2 three-by-three submatrices. This yields a spanning set of only 8 vectors, so we see that A cannot be star-stable in this subcase. We have also established these facts:

Corollary 6.16. *If $X \in M_4$ contains a two-by-three submatrix M of rank one, then A cannot be star-stable.*

Proof. The two three-by-three submatrices that contain M are both singular. By Proposition 6.7, Y has two zeros in the same column. Therefore A^* (hence A) cannot be star-stable, by the subcase just finished. □

.

Corollary 6.17. *If $X \in M_4$ contains singular submatrices $\{a, b\} \times \{c, d\}$ and $\{a, b\} \times \{c, e\}$ (a, b distinct and c, d, e distinct) then A cannot be star-stable.*

Proof. The submatrix $\{a, b\} \times \{c, d, e\}$ either contains a column of zeros, or has rank one by the argument in Lemma 6.14. □

We continue with our analysis of the 4 zeros case. Assume $f_1 \perp f_2$ and $f_3 \perp f_4$. Then we may write

$$X = \begin{bmatrix} 1 & 0 & b & d \\ 0 & 1 & c & e \\ \bar{b} & \bar{c} & 1 & 0 \\ \bar{d} & \bar{e} & 0 & 1 \end{bmatrix}$$

In this matrix, the only possible singular two-by-two submatrices are $\{1,2\} \times \{3,4\} = \begin{bmatrix} b & d \\ c & e \end{bmatrix}$ and its adjoint. By Proposition 6.7, the submatrix in the same position in Y is also singular; call this matrix M.

Of the sixteen three-by-three submatrices of X, only the four principal submatrices are obviously nonsingular. The remaining submatrices are

$$\begin{array}{ccccc}
 & \begin{bmatrix} 1 & 0 & d \\ 0 & 1 & e \\ \bar{b} & \bar{c} & 0 \end{bmatrix} & \begin{bmatrix} 1 & b & d \\ 0 & c & e \\ \bar{b} & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & b & d \\ 1 & c & e \\ \bar{c} & 1 & 0 \end{bmatrix} & \text{(1,2,3)} \\
\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & c \\ \bar{d} & \bar{e} & 0 \end{bmatrix} & & \begin{bmatrix} 1 & b & d \\ 0 & c & e \\ \bar{d} & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & b & d \\ 1 & c & e \\ \bar{e} & 0 & 1 \end{bmatrix} & \text{(1,2,4)} \\
\begin{bmatrix} 1 & 0 & b \\ \bar{b} & \bar{c} & 1 \\ \bar{d} & \bar{e} & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 & d \\ \bar{b} & \bar{c} & 0 \\ \bar{d} & \bar{e} & 1 \end{bmatrix} & & \begin{bmatrix} 0 & b & d \\ \bar{c} & 1 & 0 \\ \bar{e} & 0 & 1 \end{bmatrix} & \text{(1,3,4)} \\
\begin{bmatrix} 0 & 1 & c \\ \bar{b} & \bar{c} & 1 \\ \bar{d} & \bar{e} & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & e \\ \bar{b} & \bar{c} & 0 \\ \bar{d} & \bar{e} & 1 \end{bmatrix} & \begin{bmatrix} 0 & c & e \\ \bar{b} & 1 & 0 \\ \bar{d} & 0 & 1 \end{bmatrix} & & \text{(2,3,4)} \\
(1,2,3) & (1,2,4) & (1,3,4) & (2,3,4) &
\end{array}$$

However, if we apply the duality of Proposition 6.7 again, no submatrix listed here that is dual to to a position in M can be singular. If it were, then M would have a zero, and being singular itself, this would force two zeros in the same row or column of Y, leaving us in the previous subcase.

This reduces the potentially singular 3-by-3 submatrices of X to $M_1 = \{1,2,4\} \times \{1,2,3\} = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & c \\ \overline{d} & \overline{e} & 0 \end{bmatrix}$ and $M_2 = \{2,3,4\} \times \{1,3,4\} = \begin{bmatrix} 0 & c & e \\ \overline{b} & 1 & 0 \\ \overline{d} & 0 & 1 \end{bmatrix}$ and their adjoints. However, neither may be simultaneously with the 2-by-2 matrix. For M_1 this leads to the contradiction: $|d|^2 + |e|^2 = 0$ (simply calculate each determinant, which must be zero). For M_2 the contradiction is: $|b|^2 + |d|^2 = 0$.

Therefore, the count is: either 4 zeros and 2 two-by-two matrices, or 4 zeros and 4 three-by-three matrices. Therefore, $\dim\bigl((\mathfrak{T}_A)_\perp \cap \mathfrak{F}_1\bigr) \leq 8$, contradicting the star-stability of A.

(2 zeros) Assume, by symmetry, that $f_1 \perp f_2$, and write:

$$X = \begin{bmatrix} 1 & 0 & b & d \\ 0 & 1 & c & e \\ \overline{b} & \overline{c} & 1 & f \\ \overline{d} & \overline{e} & \overline{f} & 1 \end{bmatrix}$$

By Proposition 6.7, we may assume that A has no more than 2 singular three-by-three submatrices. Among the two-by-two submatrices of X, 22 are obviously nonsingular, leaving the remaining 14 to check (they are arranged spatially to make applications of Corollary 6.17 more apparent):

$$
\begin{array}{ccccccc}
 & & & & & \begin{bmatrix} b & d \\ c & e \end{bmatrix} & (1,2) \\
 & & \begin{bmatrix} 1 & d \\ \overline{b} & f \end{bmatrix} & & & \begin{bmatrix} b & d \\ 1 & f \end{bmatrix} & (1,3) \\
 & \begin{bmatrix} 1 & b \\ \overline{d} & \overline{f} \end{bmatrix} & & & & \begin{bmatrix} b & d \\ \overline{f} & 1 \end{bmatrix} & (1,4) \\
 & & & & \begin{bmatrix} 1 & e \\ \overline{c} & f \end{bmatrix} & \begin{bmatrix} c & e \\ 1 & f \end{bmatrix} & (2,3) \\
 & & & \begin{bmatrix} 1 & c \\ \overline{e} & \overline{f} \end{bmatrix} & & \begin{bmatrix} c & e \\ \overline{f} & 1 \end{bmatrix} & (2,4) \\
\begin{bmatrix} \overline{b} & \overline{c} \\ \overline{d} & \overline{e} \end{bmatrix} & \begin{bmatrix} \overline{b} & 1 \\ \overline{d} & \overline{f} \end{bmatrix} & \begin{bmatrix} \overline{b} & f \\ \overline{d} & 1 \end{bmatrix} & \begin{bmatrix} \overline{c} & 1 \\ \overline{e} & \overline{f} \end{bmatrix} & \begin{bmatrix} \overline{c} & f \\ \overline{e} & 1 \end{bmatrix} & & (3,4) \\
(1,2) & (1,3) & (1,4) & (2,3) & (2,4) & (3,4) &
\end{array}
$$

Notice that, except in the last column, any two matrices that are displayed in relative vertical position share a common row; similarly, in all but the last row, any two in relative horizontal position share a common column. If two such matrices are both simultaneously singular, then Corollary 6.17 applies and A cannot be star-stable.

This means that no two of the submatrices $\{1,3\}\times\{3,4\}$, $\{1,4\}\times\{3,4\}$, $\{1,3\}\times\{1,4\}$ may be simultaneously singular. Similarly, no two of the submatrices $\{2,3\}\times\{3,4\}$, $\{2,4\}\times\{3,4\}$, $\{2,3\}\times\{2,4\}$ may be simultaneously singular. These two groups can contribute only 4 two-by-two matrices (one from each, and the adjoints). Together with at most one three-by-three adjoint pair, this only accounts for 8 spanning members of $(\mathcal{T}_A)_\perp \cap \mathfrak{F}_1$ unless also $\{1,2\}\times\{3,4\}$ is also singular.

If $\{1,2\} \times \{3,4\}$ is singular, the only compatible matrices from the two groups are $\{1,3\} \times \{1,4\}$ and $\{2,3\} \times \{2,4\}$. With their adjoints, this set of matrices is self-complementing, so the same submatrices of Y are also singular. Since Y has at most 2 zeros, no zero of Y can occur in any of the three two-by-two submatrices listed (or Corollary 6.17 applies). In other words, the only three-by-three singular submatrix of X possible are $\{1,3,4\} \times \{2,3,4\}$ and its adjoint. The singularity of the two-by-two matrices implies that $f = e\overline{c}$ and $b/d = c/e$. Together with the three-by-three, this forces

$$\begin{aligned} b(\overline{c} - \overline{e}f) &= d(\overline{cf} - \overline{e}) \\ (b/d)(\overline{c} - \overline{e}e\overline{c}) &= (\overline{c}c\overline{e} - \overline{e}) \\ (c/e)(\overline{c} - \overline{e}e\overline{c}) &= (\overline{c}c\overline{e} - \overline{e}) \\ |c|^2 - |ce|^2 &= |ce|^2 - |e|^2 \\ |c|^2 + |e|^2 - 2|ce|^2 &= 0 \end{aligned}$$

This may happen only if $|c| \geq 1$ or $|e| \geq 1$, contradicting the positive definiteness of X. Since the matrices may not all be simultaneously singular, $\dim\big((\mathfrak{T}_A)_\perp \cap \mathfrak{F}_1\big)$ is no greater than 8, and the star-stability assumption is contradicted.

(0 zeros) Since X has no zeros, we may assume as well that X contains no singular three-by-three submatrices, since this would imply that Y contains zeros, by Proposition 6.7. Write

$$X = \begin{bmatrix} 1 & a & b & d \\ \overline{a} & 1 & c & e \\ \overline{b} & \overline{c} & 1 & f \\ \overline{d} & \overline{e} & \overline{f} & 1 \end{bmatrix}$$

The only clearly nonsingular two-by-two submatrices of X are the principal ones. The remaining submatrices are grouped into five classes:

Group **abc** contains the submatrices $\{1,2\}\times\{1,3\}$, $\{1,2\}\times\{2,3\}$, $\{1,3\}\times\{2,3\}$, and their adjoints (this group gets its name from the fact that its members are the submatrices of X that contain only letters $a, b, c, 1$ or their complex conjugates).

Group **ade** contains the submatrices $\{1,2\}\times\{1,4\}$, $\{1,2\}\times\{2,4\}$, $\{1,4\}\times\{2,4\}$, and their adjoints.

Group **bdf** contains the submatrices $\{1,3\}\times\{1,4\}$, $\{1,3\}\times\{3,4\}$, $\{1,4\}\times\{3,4\}$, and their adjoints.

Group **cef** contains the submatrices $\{2,3\}\times\{2,4\}$, $\{2,3\}\times\{3,4\}$, $\{2,4\}\times\{3,4\}$, and their adjoints.

In each of the preceding groups, only one matrix/adjoint pair can be simultaneously singular without triggering Corollary 6.17. Therefore, if there are no other singular submatrices of X, then $\dim\big((\mathfrak{T}_A)_\perp \cap \mathfrak{F}_1\big) \leq 8$, contradicting star-stability.

So we may assume some submatrix is singular from the remaining group, **other**, which contains $\{1,2\}\times\{3,4\}$, $\{1,3\}\times\{2,4\}$, $\{2,3\}\times\{1,4\}$, and their adjoints. We may reorder $\{f_i\}$ if necessary, and assume that $\{1,2\}\times\{3,4\} = \begin{bmatrix} b & d \\ c & e \end{bmatrix}$ and its adjoint are singular. Having assumed this, if any other matrix/adjoint pair in **other** is also singular, then the groups **abc**, **ade**, **bdf**, and **cef** must contain non-singular matrices or Corollary 6.17 applies. If this happens, $\dim\big((\mathfrak{T}_A)_\perp \cap \mathfrak{F}_1\big) \leq 6$.

So the only possible way for A to be star-stable is if $\{1,2\}\times\{3,4\}$, $\{1,3\}\times\{1,4\}$, $\{1,3\}\times\{2,3\}$, $\{1,4\}\times\{2,4\}$, and $\{2,3\}\times\{2,4\}$ and their adjoints are singular. If this happens then $(\mathfrak{T}_A)_\perp \cap \mathfrak{F}_1$ is spanned by the 10 rank one operators.

$$
\begin{aligned}
r_1 &= (df_3 - bf_4) \otimes (\bar{c}f_1 - \bar{b}f_2) & r_6 &= (\bar{c}f_1 - \bar{b}f_2) \otimes (df_3 - bf_4) \\
r_2 &= (f_2 - \bar{c}f_3) \otimes (f_1 - \bar{b}f_3) & r_7 &= (f_1 - \bar{b}f_3) \otimes (f_2 - \bar{c}f_3) \\
r_3 &= (f_2 - \bar{e}f_4) \otimes (f_1 - \bar{d}f_4) & r_8 &= (f_1 - \bar{d}f_4) \otimes (f_2 - \bar{e}f_4) \\
r_4 &= (ef_2 - f_4) \otimes (cf_2 - f_3) & r_9 &= (cf_2 - f_3) \otimes (ef_2 - f_4) \\
r_5 &= (df_1 - f_4) \otimes (bf_1 - f_3) & r_{10} &= (bf_1 - f_3) \otimes (df_1 - f_4)
\end{aligned}
$$

The singularity of these matrices implies that $|b| = |d| = |e|$, $c = \frac{be}{d}$, $a = d\bar{e}$, and $f = \bar{b}d$. It is readily checked that

$$
\begin{aligned}
r_8 &= -\frac{\bar{b}}{bd}r_1 - \frac{\bar{b}c}{b\bar{c}}r_2 + \frac{\bar{b}c}{b\bar{c}}r_3 + \frac{c}{\bar{c}\bar{d}}r_6 + r_7 \\
r_{10} &= -\frac{\bar{d}}{cd}r_1 - \frac{b\bar{d}}{\bar{b}d}r_4 + \frac{b\bar{d}}{\bar{b}d}r_5 + \frac{b}{\bar{b}\bar{c}}r_6 + r_9
\end{aligned}
$$

so $\dim\big((\mathfrak{T}_A)_\perp \cap \mathfrak{F}_1\big) \leq 8$ and A cannot be star-stable. This finishes the final case of the proof. $\square$

This last case highlights the difficulty that will be encountered if these matricial arguments are applied to the question of star-stable operators on $\mathbb{C}^5$ or larger. Despite the attempt to make the matricial tools as sharp as possible, it is certain that they will be inadequate in $\mathfrak{B}(\mathbb{C}^5)$ and larger; it will be necessary to hand select linear relations to transform a spanning set in $(\mathfrak{T}_T)_\perp \cap \mathfrak{F}_1$ into a minimal spanning set, as it was here.

Chapter 7

Open Questions

Question 7.1. *Is a star-stable operator A with distinct eigenvalues always normal?*

All normal operators are star-stable, and the analysis of Chapter 6 shows that a star-stable A with distinct eigenvalues is normal when $A \in M_n$ for $n = 2, 3, 4$. This would seem to be compelling evidence, at least for operators acting on finite dimensional spaces, for a positive answer.

Although some progress was made, characterizing star-stable operators having the maximum number of eigenvalues appears to be difficult. It might be natural to look at the other extreme and consider members of M_n having only a single eigenvalue. Addition of a scalar multiple of I does not affect star-stability properties so this is equivalent to asking which nilpotent operators are star-stable.

Question 7.2. *Is $A \oplus A$ star-stable for an arbitrary operator A? Is a direct sum of star-stable operators also star-stable?*

These questions are in the same spirit as questions that were raised (and answered negatively in [33],[21]) about the reflexivity of operators. As with the earlier conjectures, there is some positive evidence given by weighted shifts and operators unitarily equivalent to Jordan matrices. Despite the evidence, it is likely that these questions (like their predecessors) will have negative answers.

Question 7.3. *If T is a Toeplitz operator, is T always star-transitive or star-reflexive? Does there exist a star-stable Toeplitz operator that is not star-reflexive?*

The main result of [3] exposes a distinct dichotomy: if T is an analytic Toeplitz operator, then T is either star-reflexive (hence star-stable) or star-transitive. A related question is to determine which functions $f \in \mathcal{H}^\infty$ have the property that $f(S)$ is star-reflexive (or star-stable).

Question 7.4. *Let T be an injective weighted shift. If T^* has a nonzero eigenvalue, is every (weak-star closed) intransitive subspace of $\mathcal{T}_T$ reflexive? Is this true for every reflexive shift T?*

When T^* has an eigenvalue, T is reflexive, so the latter question subsumes the first. A positive answer might be suggested by [3] which answers this question in the specific case of $T = S$, the unilateral shift. For any shift T, the space $\mathcal{T}_T$ is unistriped, and the reflexive closure is striped. When T^* has an eigenvalue, it is these properties that are used in Corollary 4.5 to show that many of the intransitive subspaces of $\mathcal{T}_T$ are reflexive, another fact that might also indicate a positive answer.

Question 7.5. *Do injective shifts exist that are neither star-reflexive nor star-transitive?*

Many of the well-studied injective shifts to date are star-transitive, including the regular unilateral shift and the unicellular shifts. The perturbed shift $T = S + (c - 1)e_1 \otimes e_0$ of Chapter 5 is star-transitive for large values of c but abruptly changes and becomes star-reflexive when $|c| < \sqrt{1/2}$. What happens when $c = \sqrt{1/2}$?

Question 7.6. *Is $\mathcal{T}_T$ hyperreflexive when $T = S + (c - 1)e_1 \otimes e_0$ for $c < 1/2$.*

A space $\mathcal{S}$ is hyperreflexive if for some $\alpha > 0$, the set $\alpha\,\mathrm{ball}(\mathcal{S}_\perp)$ is contained in the trace norm closed convex hull of $\mathrm{ball}(\mathcal{S}_\perp \cap \mathfrak{F}_1)$. In [9] it is shown that $\mathcal{T}_T$ is hyperreflexive (hence reflexive) for shifts with

1. $||T|| = 1$

2. $\lim_j ||T^j||^{1/j} = 1$

3. $\lim_j T^j x = 0 = \lim_j (T^*)^j x$ for all $x \in \mathcal{H}^2$.

Although the shift T does not satisfy property 3, the reflexivity of $\mathcal{T}_T$ may make it a good candididate for hyperreflexivity.

Question 7.7. *What are the reflexivity properties of a shift T that is formed by perturbing only a finite number of weights of S?*

The unilateral shift S shows a surprising change of nature when only its 0th weight is modified. Neither an algebraic nor a matricial view of the spaces $\mathcal{A}_T$ and $\mathcal{T}_T$ would indicate a significant difference between the operator $T_{1/4} = S + (\frac{1}{4} - 1)e_1 \otimes e_0$ and $T_{3/4} = S + (\frac{3}{4} - 1)e_1 \otimes e_0$, yet they have completely different star-reflexivity properties. Only Jensen's inequality (a distinctly analytic result) hints that such a difference should exist. Because the 0th weight of S appears naturally when Jensen's inequality is applied, there is essentially a blank slate with reguard to perturbing any of the other weights of S.

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