WHEN AM I EVER GOING TO USE THIS?

TEACHERS' INSTRUCTIONAL PRACTICES WITH CONTEXTUAL PROBLEMS

by

HOLLY PORTIA LOUISE GARRETT ANTHONY

(Under the direction of Jeremy Kilpatrick)

ABSTRACT

The goal of this study was to investigate high school mathematics teachers' instructional practices with contextual problems and their notions of terminology associated with such problems. The study was motivated by the literature on research and teaching that calls for the incorporation of contextual problems in high school mathematics. Teaching with such problems is not easy. Yet little research has focused on teachers' practices with these problems. In addition, the increased inclusion of contextual problems in the secondary mathematics curriculum has introduced terminology associated with such problems. Three such terms—*mathematics in context*, *applications of mathematics*, and *mathematical modeling*—have been defined and redefined in the literature, and their meaning has been confounded by disputes about what constitutes the *real world* and *reality*. As a result, mathematics educators and teachers alike have somewhat ambiguous notions about these concepts.

Six teachers who taught with contextual problems on a near-daily basis in two schools in the southeastern United States agreed to participate in this study. All teachers were interviewed concerning their notions of terminology associated with contextual problems and their practices with such problems. Three of them were also observed teaching lessons that incorporated contextual problems and interviewed about those lessons. A grounded theory approach and constant comparative analysis were applied to the data.

The results of this study highlighted that these teachers' notions of terminology varied, but that they defined and differentiated the terms along a number of dimensions including the *degree of reality* and the *role and complexity of the mathematics* in the problem. The data also showed that three features of the teachers' instruction with contextual problems were important in shaping the lessons—how they (a) adapted and used problems from other sources, (b) helped the students formulate the problem, and (c) balanced time and attention to the context and the mathematics. Four conditions were identified that enabled the teachers to do this work: technology, commitment, community support, and beliefs. The results of this study have implications for the preparation and support of high school teachers who incorporate contextual problems in their teaching, and for mathematics educators who engage in scholarly writing on this subject.

INDEX WORDS: Contextual problems, Secondary mathematics teachers, Instructional practices, Notions of terminology, Enabling conditions, Beliefs, Mathematics in context, Applications of mathematics, Mathematical modeling, Qualitative research, Graphic organizer

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DEDICATION

I would like to dedicate this dissertation to my husband, Tim, and my parents, Joe and Portia Garrett.

To Tim – You have been supportive and encouraging throughout this process. You continue to inspire me.

To Mama and Daddy – I owe so much of who I am today to the wisdom, strength, and love I always saw in both of you. You believed in me, even when I did not. Your support over the years has been invaluable. You've been present for every accomplishment and milestone along the way—thank you for that. I am proud to be your daughter and I hope I have made you proud of me. This is dedicated to you.

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CHAPTER 1

INTRODUCTION AND OVERVIEW

This report concerns a study of six high school mathematics teachers' notions of terminology associated with contextual problems in mathematics and their instructional practices with such problems. These teachers were well known for their work and had published curriculum materials and articles centered on that work in addition to leading workshops. Interviewing and observing these teachers provided an informative look into how they viewed and taught with contextual problems.

Background

As the changing economic picture in the United States has required people entering the workforce to be educated at higher levels, the public has responded with demands that standards be raised and that schools and educators be more accountable for their performance. In mathematics education, the National Council of Teachers of Mathematics (NCTM) has recommended reforming the curriculum and teaching of school mathematics (NCTM, 1989, 2000). A hallmark of this effort has been to make mathematics more practical, useful, and meaningful to students by using problems embedded in realistic situations or contexts.

The National Science Foundation has supported the development of middle school materials (e.g., Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002) and high school materials (e.g., Coxford, Fey, Hirsch, Schoen, Burrill, & Hart, 1997) that emphasize uses of mathematics and that also ask teachers to teach in unfamiliar ways. For example, when introducing exponential functions, high school teachers have typically defined the relevant terms, examined properties of the functions and their graphs, and supplied students with opportunities to "do the math." With the new materials, teachers can use a realistic context to introduce the topic. Problems based on compound interest, plant growth, or population growth provide students with opportunities to explore exponential functions in practical situations that can make the mathematics more meaningful. Yet many teachers lack the experience and knowledge necessary to comfortably teach with contextual problems or are uncertain how to use such problems in their teaching (Berry & Houston, 2004). Mathematics students often ask, "When am I ever going to use this?" But their teachers are typically not prepared to give them a convincing answer. The use of contextual problems has been proposed as a way to change that.

That proposal and the development of new curriculum materials emphasizing the use of contextual problems has introduced a range of terminology and perhaps instructional practices previously given little attention by teachers. Teachers are forced to sort through terms—*mathematics in context*, *applications of mathematics*, and *mathematical modeling*—and incorporate problems, technologies, and mathematical ideas that may be unfamiliar or challenging. Among the myriad of instructional decisions teachers must make are those about how much time to devote to contextual problems in lieu of the required topics mandated in state curriculum documents and how best to facilitate students in developing competence with such problems. The present study was based on my claim that teachers' notions about the three terms *mathematics in context*, *applications of mathematics*, and *mathematical modeling* and their instructional practices with contextual problems warrant investigation. In this chapter I provide a rationale for that claim.

Using Contextual Problems

Attention to teaching mathematics as a useful tool began in some countries (England, the Netherlands, and the United States) in the 1970s and 1980s (Pollak, 2003). The move towards making mathematics more relevant emphasized the introduction of applications and mathematical modeling into the secondary school curriculum. Proofs of the effort to introduce those topics are numerous national and international conferences; for example, the intercollegiate competition for mathematical modeling sponsored by the Consortium for Mathematics and Its Applications (COMAP) (see http://www.comap.com) beginning in 1985, the International Conferences on the Teaching of Modeling and Applications (ICTMAs), and the 2004 conference of the International Commission on Mathematical Instruction's Study Group 14 on Applications and Modelling in Mathematics Education. These conferences emphasized applications and modeling, as have books and papers written in support of the effort (e.g., Blum, Berry, Biehler, Huntley, Kaiser-Messmer, & Profke, 1989; Breiteig, Huntley, & Kaiser-Messmer, 1993; de Lange, Keitel, Huntley, & Niss, 1993; National Research Council [NRC], 1998).

At these conferences and in these publications, a number of arguments have been made in favor of including contextual problems and modeling activities in mathematics curricula and instruction (Blum & Niss, 1989, 1991; de Lange, 1996). Fifteen years ago, Blum and Niss (1989) outlined five arguments in favor of using applications, modeling, and problem solving. Versions of these arguments are still used today by advocates for teaching with contextual problems.

According to Blum and Niss (1989), the "formative argument"

> emphasizes the application of mathematics and the performing of mathematical modeling and problem solving as suitable for developing *general competences* and *attitudes* with students, in particular orientated towards fostering overall explorative, creative and problem solving capacities, as well as open-mindedness and self reliance. (p. 5)

This argument supports the claim that students need opportunities to apply mathematics and recognize its uses in the world. The intent is that through engagement with contextual problems students will develop enthusiasm and competence with applying, using, and connecting mathematics within and across disciplines. In this way, contextual problems form or shape students' competences and attitudes towards mathematics and, more generally, problem solving.

The second argument is the "critical competence argument" (Blum & Niss, 1989). This argument

> focuses on preparing students to live and act with integrity as private and social citizens, possessing a *critical competence* in a society which is being increasingly influenced by the utilization of mathematics through applications and modeling. The aim of such a competence is to enable students to see and judge independently, to recognize and understand representative examples of actual uses of mathematics. (p. 5)

The critical competence argument for teaching mathematics through contextual problems is in part a response to the needs and demands of the workforce and of society (Howey, 1998). In our ever-changing and technologically progressive world, students need experiences in applying and using the mathematics learned in school to gain an edge in workplace settings. This argument is similar to the formative argument since it argues that students need specific experiences *applying* particular mathematical concepts if they are to be prepared for the workplace—or to be competent citizens. In working with contextual problems, students gain experiences with problems that illustrate the actual uses of mathematics in the world.

A recent branch of the critical competence argument—in response to calls for equity in mathematics teaching—is based on the potential of contextual problems to develop students' sociopolitical awareness and thus their critical competence. Gutstein (2003) and Ladson-Billings (1995) have developed the concepts of "pedagogy for political justice" and "culturally relevant pedagogy" as alternative and more equitable teaching approaches that use contextual problems.

In Gutstein's pedagogy for political justice contextual problems facilitate three main goals: helping students develop sociopolitical consciousness, a sense of agency, and positive social and cultural identities. In Ladson-Billings' approach, the teacher selects problem contexts that are culturally relevant to students, thereby connecting the mathematics being taught with aspects of their daily living. Contextual problems utilized in both approaches not only make mathematics more accessible and exciting for students but also foster their development as critical, competent members of a larger sociopolitical community.

The third argument described by Blum and Niss (1989) is the "utility argument." This argument

> emphasizes that mathematics instruction should prepare students to *utilize mathematics* for solving problems in or describing aspects of specific extra-mathematical areas and situations, whether referring to other subjects or occupational contexts (mathematics as a service subject) or to the actual or future everyday lives of students. In other words, mathematics instruction should enable students to *practice applications*, *modeling and problem solving* in a variety of contexts. (p. 5)

The first part of this argument is practical. Contextual problems provide students with opportunities to use mathematics and to begin to understand its uses in the world and other fields of study (e.g., physics and other sciences). In addition, students can be prepared to recognize and use mathematics in their own lives and future.

> The motivational benefits that can be provided by workplace and everyday problems are worth mentioning, for although some students are aware that certain mathematics courses are necessary in order to gain entry into particular career paths, many students are unaware of how particular topics or problem-solving approaches will have relevance in any workplace. (NRC, 1998, pp. 10–11)

Part of the utility argument is focused on students working mathematics in a variety of contexts and is supported by studies of ethnomathematics (D'Ambrosio, 1985) and studies in developmental psychology (studies of mathematics learning in out-of-school contexts versus in-school contexts; studies of knowledge transfer) that have explored the mathematics of different groups in everyday settings, showing that mathematical knowledge is generated and used in a wide variety of contexts by both adults and children. This research has focused on the connections between cognition, culture, and context. In particular, the everyday practices of different groups have been investigated: for example, dairy workers (Scribner, 1984), construction foremen (Carraher, 1986), farmers (Abreu & Carraher, 1988), child street vendors (Carraher, 1988), carpenters (Millroy, 1992), candy sellers (Saxe, 1990), shoppers (Lave, 1988), and fisherman (Schliemann & Nunes, 1990). All these studies showed that the groups developed efficient strategies for solving mathematical problems in everyday situations—often in ways differing from school-taught strategies. Thus contextual problems can provide students' practice with using mathematics in different contexts, thereby connecting their mathematical practices in and out of school.

Blum and Niss (1989) describe the fourth argument for the inclusion of contextual problems in the curriculum as the "picture of mathematics argument," which

> insists that it is an important task of mathematics education to establish with students a *rich and comprehensive picture of mathematics* in all its facets, as a science, as a field of activity in society and culture. Since applications, modeling and problem solving constitute an essential component in such a picture, this component should be allotted an appropriate position in mathematics curricula. (p. 5)

This argument, in my review of the literature, seems to be the least used. It is probable that advocates deploy this argument the least because it is difficult to support with evidence. Theoretically, students should develop a rich and comprehensive picture of mathematics, but practically, it is difficult to argue why.

The final argument described by Blum and Niss (1989) and supported in the literature is the "prompting mathematics learning argument" that

> emphasizes that the incorporation of problem solving, applications and modeling aspects and activities in mathematics instruction is well suited to assist students in acquiring, learning and keeping mathematical concepts, notions, methods and results, by providing motivation for and relevance of mathematical studies. Such work also contributes to exercise students in thinking mathematically, and in selecting and performing mathematical techniques within and outside mathematics. (p. 5)

The first part of this argument is motivational. It supports the idea that applications and modeling problems have the potential to engage and motivate students since such problems connect mathematics with the world and thus to students' lives. The second part of the argument can be thought of as cognitive in its focus. It is centered on the idea of generalization. To prepare students to deal with novel problems (both realistic and otherwise), and to help them acquire the concepts and skills needed to address many of the dilemmas encountered in life, students must work with concepts and procedures that they can generalize (Masingila, Davidenko, & Prus-Wisniowska, 1996). Thus "knowing and using students' out-of-school mathematics practice is important in school situations because it provides contexts in which students can make connections" (p. 194) and can acquire knowledge they can generalize to other settings.

According to Blum and Niss (1989), the utility argument and the prompting mathematics learning argument were the two most often invoked by advocates for teaching with contextual problems, but the other three arguments were gaining momentum. In my view, each of the five arguments is being used (the fourth to a lesser degree), and research centered on these arguments both supports and refutes the various claims.

Teachers' Notions of Terminology

The increased inclusion of contextual problems in secondary mathematics curricula and teaching in response to the aforementioned arguments has introduced terminology associated with such problems. The terms of interest in the first facet of the present study—*mathematics in context*, *applications of mathematics*, and *mathematical modeling*—have been defined and redefined in the literature and confounded by disputes about what constitutes the *real world* and *reality*. In my review of the literature (see chapter 2), it was difficult to separate the discussion and research on contextual problems from the discussion and research on word problems and other such terms. Overlap between the different concepts and the use of one term in the definition of another made the terminology murky and somewhat ambiguous (e.g., using the term *application* in a definition of *mathematical modeling*, or the term *model* in a definition of *application*, or the terms *contextual* and *real* as modifiers in definitions of many terms). As a result of these practices, mathematics educators and teachers alike may have somewhat ambiguous notions about these concepts and what they encompass.

Many writers recognize the ambiguity associated with these terms and therefore define them in ways that make sense for their work. Even so, one may wish for less ambiguity and argue for consensus on a definition for each term. But before making that argument, one must consider why (or whether) consensus is necessary if useful work can be done without full agreement on precise definitions. One important reason pointed to by Törner (2002) is that clarification of terminology helps determine research focus. Terms play a functional role in educational research: They help us to define areas of needed research and to pose pertinent research questions. Research studies can build on other research and help the field come to cumulative understanding only if there is agreement about what the research objects are. If the terms *mathematics in context, applications of mathematics, mathematical modeling, word problems, contextual problems, context-based problems, real-world problems, experientially real problems, authentic problems*, and *realistic problems* are used almost interchangeably in the literature but with different intended meanings, then research studies using these terms will have little to contribute to one another.

According to McLeod and McLeod (2002), one of the difficulties involved in coming to agreement about a definition may be that the types of definitions as well as the definitions themselves differ. They argue that some authors distinguish among several separate types of definitions, each appropriate for a particular kind of audience. The three types of definitions described by McLeod and McLeod are also appropriate for the present discussion:

> 1. Informal definitions—these are "rule of thumb" definitions (often used parenthetically) for a general audience.
>
> 2. Formal definitions—these consist of three parts: the term to be defined, the class of objects or concept to which it belongs, and the distinguishing characteristics that separate it from all other objects or concepts in the class. (The class should be small, just large enough to include all members of the term, but no larger . . . .[Hence it is not helpful in a formal definition to use the broad term *mathematics in context* as the class to which applications belong.]) The intended audience is more sophisticated in its understanding of technical terms than a general audience, but it is still a broad one.
>
> 3. Extended definitions—these start with a formal definition, but continue in more technical language to include more complete characteristics and instantiations of the term. The intended audience is specialists in a particular field (e.g., mathematicians writing for other mathematicians). (p. 118)

With this classification in mind, one can see that many of the definitions proposed for the terms *mathematics in context*, *applications of mathematics*, and *mathematical modeling* may in fact be useful for different audiences and purposes. What I claim is that little attention has been paid to such considerations when the three terms are used in the literature. Furthermore, little consideration and research has been devoted to how teachers think about and use these terms in their teaching.

<u>Teachers' Instructional Practices</u>

The second facet of the present study focused on teachers' instructional practices with contextual problems. I approached this study with a broad perspective that allowed for contextual problems to be defined by the participants. *Contextual problems* in this study were therefore taken to be problems the teachers identified as such. I sought teachers who used problems on a near-daily basis that in some way incorporated the world outside of mathematics—applications, modeling problems, word problems, and so on. This characterization, however, was never given to the teachers. Once I had invited a teacher to participate in the study, I asked him or her to select two lessons for observation that incorporated contextual problems. The lessons they selected were considered reflective of how they defined *contextual problems*.

The teachers in this study used contextual problems on a regular basis despite formidable obstacles to using modeling and applications in mathematics instruction. Blum and Niss (1989) claim that "many teachers are afraid of not having enough *time* to deal with [such problems] in addition to the wealth of compulsory mathematics included in the curriculum" (p. 11). Similarly, Silver, Kilpatrick, and Schlesinger (1990) assert that teachers often neglect applications of mathematics because they are not familiar with how the mathematics they are teaching might be applied or because they do not want to spend valuable class time on activities seen as "outside mathematics." Blum and Niss further comment that using modeling and applications makes mathematics lessons "unquestionably more *demanding* and less predictable for learners" (p. 11) and that "references to the world outside mathematics make instruction more *open* and more *demanding* for teachers" (p. 12) as well. Even so, Silver et al. argue that students who develop mathematical models of practical situations gain valuable experience in putting mathematics to use. "Students come to see the relevance of the mathematics they are learning, and when they do, it can be a powerful force in motivating further study of mathematics" (p. 9). Thus despite the obstacles, many teachers choose to incorporate contextual problems into their mathematics classes. The inclusion of such problems in the curriculum, however, is often at the expense of other problems and therefore requires a strong commitment by the teacher if contextual problems are to become a regular part of his or her instruction.

One cannot simply insert an application or modeling problem into a mathematics curriculum and expect students to create deep meaning. "How students construct meaning depends much on the teacher's pedagogy and on the classroom environment co-created by the teacher and students" (Gutstein, 2003, p. 48). And teachers "come into teaching with distinct training, motivations, and competence" (da Ponte, 1993, p. 225). Reformers and educators have often failed to consider these points when asking teachers to incorporate contexts in their teaching of mathematics. And incorporation is no easy task. Nunes (1992) reports a study in which teachers created contexts for mathematical topics and integrated them into their instruction. She notes, "After identifying an everyday situation in which the concept is used, the teacher needs to know how to use the situation to promote an awareness and understanding of this concept" (p. 571). Thus much responsibility resides with the teacher.

In support of this claim, several researchers point to the importance of the teacher when using contextual problems (Cooper, 2001; Gutstein, 2003; Ladson-Billings, 1995; Verschaffel & de Corte, 1997). The teacher must help students learn how to work with contextual problems—when to ignore features of the context and when to pay attention to them. Teachers are responsible for selecting problems that will be culturally or socio-politically relevant to their students' lives, or that facilitate other curricular goals. To this end, a mathematics teacher may develop, collect, and cultivate a set of practical situations in which any of several previously learned concepts and techniques might be applied (Silver et al., 1990). Developing such a problem bank, however, is no easy task: "One of the major challenges to more widespread and intensive use of modeling and applications in secondary schools is teachers' lack of experience with and knowledge about it" (Berry & Houston, 2004, p. 35).

In general, the teacher determines how the mathematics curriculum is interpreted and taught, and students' success with contextual problems is heavily dependent on the teacher's instructional practices with such problems. Thus it is important to learn from teachers what they do in the classroom and how they make sense of what they do. With few exceptions (e.g., Chapman, 2004; Kilpatrick, Hancock, Mewborn, & Stallings, 1996), researchers have not studied how mathematics teachers use contextual problems in their instruction.

Situating Teachers' Notions of Terminology and Instructional Practices

In the process of studying teachers' instructional practices with contextual problems, I investigated their beliefs about mathematics, its teaching and learning, and contextual problems. These teachers' beliefs are described in chapter 4 to introduce the teachers under investigation and to situate their instructional practices.

By the term *beliefs*, I refer to an aspect of teachers' knowledge that has been given many labels: perspectives, personal theories, frames of reference, conceptions, and constructs (see, e.g., Calderhead, 1991; Calderhead & Robson, 1991; Carter, 1990; Clark & Peterson, 1986; Peterson & Comeaux, 1987). Beliefs include the frames of reference or the perspectives that teachers use to make sense of their practice and its effects on their students. Beliefs are the incontrovertible personal "truths" held by an individual, deriving from experience or from fantasy and having a strong affective and evaluative component (Pajares, 1992). According to Thompson (1992), belief systems do not require social consensus regarding their validity or appropriateness, and one's personal beliefs do not require internal consistency. Consequently, beliefs are quite disputable, more inflexible, and less dynamic than other aspects of knowledge (Pajares, 1992). Although we cannot live and act without beliefs, one of the most important goals of education is to discuss and promote our awareness of them.

In the present study, I assumed that all our knowledge, including our beliefs and conceptions, has roots in our social activity and is shaped by our experience. I did not view beliefs as determining practice, because I believe the social institutions in which we live and their constraints—including schools—mostly shape practice. Instead, I took beliefs as important in understanding teachers' practices and therefore investigated those beliefs so as to further contextualize and ground the knowledge and theory produced in the study.

Problem Statement and Research Questions

This study was focused on high school mathematics teachers with reputations for using contextual problems. In the first facet of the study, I investigated how these teachers thought about and defined the terms *mathematics in context*, *applications of mathematics*, and *mathematical modeling*. My hope was to begin a conversation in the field (similar to the one about defining *beliefs*; see the articles published in Leder, Pehkonen, & Törner, 2002) about the definitions of these terms and whether the ways in which they have been and continue to be described in the literature are sufficient. I thought that perhaps these teachers could shed light on practical dilemmas resulting from the somewhat imprecise definitions of these terms used in textbooks and in conversation with teachers. In the second facet of the study, I investigated the teachers' instructional practices with contextual problems. My assumption was that teachers are asked to use contextual problems but that many encounter the obstacles described by Blum and Niss (1989) and Silver et al. (1990). To begin to think about how to prepare U.S. teachers more broadly to use contextual problems in their teaching, I studied teachers who were already doing that work on a regular basis.

In this qualitative study, framed by interpretivist and interactionist theories, I interviewed six high school mathematics teachers with reputations for using contextual problems on a near-daily basis, and I observed three of them. My purpose was to build descriptions of: (a) the teachers' notions of and the relations between terms associated with contextual problems, (b) the instructional practices of these teachers, and (c) the conditions that enabled these teachers to teach with such problems. The teachers' beliefs were also investigated as a way to introduce the teachers and situate their instructional practices. Specifically, in this study, I addressed each of the following questions:

1. How do these teachers define and relate the terms *mathematics in context*, *applications of mathematics*, and *mathematical modeling*?
2. What are these teachers' instructional practices with contextual problems?

To address these questions in the chapters that follow, I provide an overview of the relevant literature (chapter 2), detail the theory and methodology employed (chapter 3), and present my findings (chapters 4 to 7). In chapter 8, I discuss the implications of my study and make suggestions for future research.

CHAPTER 2

SURVEYING THE LANDSCAPE: A REVIEW OF RELEVANT LITERATURE

The present study concerned teachers' notions of terminology associated with contextual problems and their instructional practices with such problems. I organize this chapter according to these foci. I first describe how the terms *mathematics in context*, *applications of mathematics*, and *mathematical modeling* have been defined and used by teachers and researchers as reported in the literature. I then discuss research focused on teachers' instructional practices with contextual problems and the challenges they face when incorporating such problems into their teaching.

Muddy Waters—Ambiguity of Terminology

To begin to understand teachers' notions of contextual problems and how these might be "messy," it was important to examine how various terms associated with contextual problems and teaching are defined and used in the literature. It was also important to look at how work centered on such problems is confounded by disputes about what constitutes the *real world* and *reality*. I present relevant literature on these points below.

Teachers' Notions of Terminology

In my review of the research literature, I did not locate any studies specifically investigating mathematics teachers' notions of the terms *mathematics in context*, *applications of mathematics*, or *mathematical modeling*. In the course of studying other aspects of teachers' work, however, some researchers hinted at teachers' knowledge of these terms. For example, in Kilpatrick et al.'s (1996) study of an innovative teacher-developed precalculus curriculum, they reported that the teachers who were developing the curriculum made the decision that "if we can't introduce a concept with an application, then we won't teach it" (p. 154). In the teachers' development of the curriculum—using this criterion for the selection of problems—Kilpatrick et al. reported that the teachers selected "big problems" often based in real data that were "deep, yet manageable questions arising from a realistic situation and offering ample opportunities for exploration" (p. 159). They also reported that "grounding the [precalculus] course in data analysis and modeling, together with [the criterion above], shifted the instructional sequence in precalculus from definition-theorem-proof-example-practice to situation-problem-data-model-solution" (p. 160). From the problems selected by the teachers for inclusion in the curriculum and the shift made in their instructional sequence, one can conclude that these teachers' notions of applications resonated with the definition given by Silver et al. (1990): An application begins in a situation or context, often ill-defined, and solving it requires the creation of a mathematical model. The modeling involved in the application problems selected and designed by the teachers in the study (i.e., Kilpatrick et al.) was unique and relevant to the present study, but how the teachers were thinking of *modeling*, *applications*, or *realistic* was not explored by Kilpatrick et al.

Similarly, in a study by Chapman (2004), explicit attention was not given to the teachers' notions of terminology. Much can be inferred, however, from her research report. Chapman stated that she studied teachers' thinking and practice as it related to their teaching of word problems. She also sought to answer the question: What is the thinking of teachers who include modeling in their teaching, and what teaching strategies do they use that facilitate modeling?

From these statements, one can infer that Chapman viewed modeling as related to word problems, although it is not clear how. She stated that "modeling was considered to include situations that involved general problem solving strategies but not situations that emphasized translating key words to mathematical procedure in an instrumental way" (p. 66). From this statement, one can conclude that she saw modeling as requiring somewhat more sophisticated methods than solving typical word problems.

Chapman's (2004) report also suggests how the teachers in her study were thinking about *modeling* and *word problems*. She reported that the teachers "conceptualized word problems as being most valuable and meaningful when they [were] located in the world in terms of actual or realistic situations" (p. 66) and that the "teachers' practice included specific strategies that facilitated modeling in a variety of situations" (p. 67). Again, one sees that Chapman is connecting *word problems* with *modeling*. The teachers had specific ways of thinking about the connection between reality and word problems, problems that I conjecture could be categorized more specifically as applications or modeling problems.

The question remains: How do teachers think about terms associated with contextual problems? How might different notions of these terms affect practice, if at all? The mixture of informal, formal, and extended definitions (McLeod & McLeod, 2002) of these terms to be found in the literature on teaching and research may have led to ambiguity about their meaning, and teachers' notions of these terms have not been explored.

<u>Researchers' Entanglement of Terms</u>

It is yet to be seen whether teachers have ambiguous notions of the terms *mathematics in context*, *applications of mathematics*, or *mathematical modeling*, but close inspection of the ways in which these terms are defined and used in the research literature supports such a conjecture.

The terms have been defined, redefined, and used to support a number of different arguments and theoretical perspectives. Much of this definition and use has led to fuzzy notions of what the terms mean.

According to Henn and Blum (2004), applications and modeling have been—and continue to be—a central theme in mathematics education because "very many questions and problems concerning human learning and the teaching of mathematics affect, and are affected by, relations between mathematics and the real world" (p. 7), where *real world* denotes the "'rest of the world' outside mathematics, that is, everything that has to do with nature, society or culture, including everyday life, as well as subjects or disciplines different from mathematics" (p. 7). In this description, the real world is broadly defined, but further explication of what is meant by applications and modeling is necessary so that an understanding can be taken as shared by mathematics educators.

> *Mathematical modelling* often denotes the process leading from a real world problem situation to a mathematical model of this situation. In a broader sense, this notion is used here to describe the entire process of linking mathematics and the real world, that is, structuring and mathematising a problem situation, working within mathematics to derive useful results, and validating as well as interpreting these results in the original situation (perhaps involving several iterations round the loop). The term *applications* of mathematics is used to denote segments of the real world that can be, or have been tackled by means of mathematics, and for which there exist corresponding mathematical models. In an application, we know the mathematics and apply it to the real world. In modelling, we begin with the real world situation and look for some mathematics that we might apply to the situation. The term "modelling" emphasizes more the processes involved in the interplay between the real world and mathematics, whereas the term "application" focuses more on the objects involved. During the last decade or so it has become common to use the combined term *applications and modelling* in order to encompass the many kinds of problems and relationships linking the real world and mathematics. (p. 7)

The above quotation implies that to some degree, two of the three focus terms have clearcut formal definitions. Yet these meanings have evolved over time and have been, and continue to be, changed and redefined.

The most familiar, and to some extent, the least fuzzy of the three terms is *applications of mathematics*. According to Silver, et al. (1990),

> applications typically begin with an ill-defined situation outside mathematics—in economics, physics, engineering, biology, or almost any field of human activity. The job is to understand this situation as well as possible. The procedure is to make a mathematical model which we hope will shed some light on the situation we are trying to understand. (p. 9)

In subtle contrast, Pollak (2003) states that an application of mathematics is typically an idealized version of a real-world problem given to a solver "who must then translate it into mathematical terms, carry out appropriate mathematics, and restate the results in the vocabulary of the real-world situation" (p. 650). According to Pollak, every application of mathematics uses mathematics to understand, or evaluate, or predict something in the part of the world outside of mathematics.

Differences like those detected in the definitions of *application* given by Pollak (2003) and Silver et al. (1990) may contribute to the fuzziness of the term. For Silver et al., an application begins with an ill-defined situation, whereas for Pollak, it begins with an idealized, somewhat simplistic, version of a real-world problem. Also note that although Pollak avoided the use of the word *model* in his definition of the term, he described a process that paralleled the modeling process as described by Henn and Blum (2004) and others. The introduction of *model*, explicitly or implicitly, into the definition of *application* complicates a shared understanding of both terms by connecting the concepts—each of which has its own definitions—in implicit ways. Such discrepancies in the definitions may seem insignificant but are important for teachers, curriculum developers, and researchers who are trying to select, design, use, and study applications. A notion shared in both definitions of an application is its connections that lead outside of a mathematical domain. According to these definitions, applications involve situations from the real world or another discipline.

Mathematics educators often describe applications in ways similar to Henn and Blum (2004), Silver et al. (1990), and Pollak (2003). Despite the discrepancies cited above, defining applications in such limited ways is problematic for Robert and Treiner (2004), who remind us of the limitations of language:

> The proposed definition of an "*application of mathematics*" [given by Henn and Blum] does not seem to be satisfactory. . . . Indeed, tackling a world situation by means of mathematics may often be called *creating new mathematics* rather than *applying* them! The word group *applying mathematics* implicitly assumes that mathematics preexist to their application and have the status of a *tool* for other disciplines. This masks the role of other disciplines in the creation of new fields of mathematics. (p. 223)

Thus, in defining applications of mathematics (or any other concept) in particular ways, we limit the possibilities of the concept in some ways and potentially open it other ways. We must always ask: What does this definition allow and what does it prohibit? What might be different if I re-described this concept in a new way? How might I define this term so that others and I can work with taken-as-shared meanings?

The second term, *mathematical modeling*, or simply modeling, while familiar to mathematics educators, is less familiar to many teachers and is easily misinterpreted or misunderstood. The term *mathematical modeling* is used to describe or interpret two related types of activity. The first type is the process of translating the real world into mathematical terms (Gravemeijer, 1997) for the purpose of solving a problem or analyzing a situation (Dossey, 1996). The second type involves the various steps by which one accomplishes the first, loosely organized around a heuristic framework (often called the *modeling process*), and "the process of using these steps to solve a real world problem" (Berry & Houston, 1995, p. 24). Differentiation between these two activities is a matter of attention to language and is seldom done explicitly.

Johnson's (1979) argument for students to model real-world phenomena using mathematics resonates with the former of the two types: translation.

> A mathematical model can be thought of as an equation or set of equations that can be used to explain known phenomena and to make predictions. This, in a sense, differentiates between modeling and the typical translation, of "write an equation and solve for *x*" settings. Translation exercises tend to emphasize a specific case, whereas modeling usually involves some notion of generalization. (p. 138)

Alternately, modeling as a process is often characterized as a cyclic one in which one starts with a "real problem set in words," formulates a mathematical model and mathematical problem by making assumptions and simplifications, solves the mathematical problem and interprets the solution. The cycle is repeated until revisions generate a satisfactory solution to the problem under investigation. This process is complex and often involves creating a miniature problem that is analogous to the larger problem but that enables the modeler to draw more precise conclusions. The miniature problem can be extrapolated to the original real-life problem. Although the model attempts to simulate the original problem, it cannot truly replicate all the constraints that might be imposed by the problem itself.

> What distinguishes modeling from other forms of applications of mathematics are (1) *explicit* attention at the beginning to the *process* of getting from the problem outside of mathematics to its mathematical formulation and (2) an explicit reconciliation between the mathematics and the real-world situation at the end. (Pollak, 2003, p. 649)

Modeling is usually a form of metaphor; a model serves as a simplified representation of a portion of reality (of interest in a certain problem) used for improving the visualization of some aspects, generalizing properties (grammar rules), or allowing comparisons (Dapueto & Parenti, 1999). The process of modeling involves focusing on essential aspects of the problem, identifying relations among the aspects and reality, and associating these aspects to the model. In mathematical modeling this process is called *mathematization*.

Enacting this process is not so simple, because the phrase *mathematical modeling* covers a broad range of theoretical and practical orientations; it is not clear what *enactment* of this process entails. In addition, De Villiers (1993, p. 3) describes three different categories of model application: namely, direct application ("immediate recognition of a model to be used"), analogical application ("development of a model that is similar to an existing model"), and creative application ("a completely new model is created using new techniques and concepts"). Not only does De Villiers's characterization make the modeling waters more murky, but in its use of the word *application*, it makes that concept less clear as well. *Mathematical modeling* as a phrase thus encompasses such a range of activity and application that researchers and readers are often forced to guess what speakers might mean when they say their students will be "doing a modeling problem." Will they create a mathematical model that explains a phenomenon in the world? Will they fit a pre-existing model to a set of real data? Or will they apply mathematics previously learned in a new setting?

The last term I explore is the notion of *mathematics in context*. This term gained recognition among mathematics educators in the 1990s with the development of the U.S. middle school mathematics curriculum of the same name, Mathematics in Context (Romberg, 1997–1998). This curriculum was developed at the University of Wisconsin– Madison and the Freudenthal Institute in the Netherlands and is underpinned by the theory of Realistic Mathematics Education (RME). Thus the term *mathematics in context* is a referent that has developed over time from interpretations of RME and from informal usages of the term. For example, a teacher may be said to be teaching mathematics in context if his or her students are solving problems connected in some way to the real world, or if they are doing mathematics in settings outside of school (e.g., using mathematics to design a playground for a city park).

At the heart of *mathematics in context* is the word *context*, and on this word I focused much of my attention and that of the teachers in this study. Treffers (1978/1987) expounds the notion of context in RME, the theory through which the term *mathematics in context* evolved:

> With regard to form and function context problems is a broader conception than traditional word or text problems. A context problem looks like a word problem, or may be embedded in a play, game, described in a story, offered as a press-cutting, represented by models or graphs, or as a combination of such information resources. They can be part of themes or projects. . . . As far as content is concerned, 'context' means: (a) that, in the case of an isolated, self-contained problem, one looks for the 'surroundings' of the text or the presentation of the problem, in other words, for that which is not explicitly formulated or presented, but does belong to the background assumptions; (b) that, in the case of a non-self-contained problem, one looks for the surroundings explicitly evoked by the story, the theme, the location, which, as a matter of fact, also carry their own background assumptions with them. (p. 255)

In this way, *mathematics in context* refers to mathematics taught in real situations (contexts) or within real (context-based) applications. Contexts are elements of each of the constructs: applications and models. The context is the real situation or setting embedded in the application or modeled by the mathematics. Contexts are "domains of reality disclosed to the learner in order to be mathematised" (Freudenthal, 1991, p. 75). These domains vary in form and include: locations (students work mathematics in a real setting outside of school), stories (mathematics presented in the context of a story), projects (mathematics projects in communities and workplaces), themes (mathematics in the context of other mathematics), and clippings (mathematics as in newspapers, television, and commercials). In all of these instances, the goal is to relate mathematics to some aspect of the world or to other aspects of its own domain.

The word *context* has also been associated with notions of "teaching or learning in context" (also known as "situated teaching or learning"). In terms of a given set of disciplinary concepts, the word *context* is used to "indicate a situation or activity in which these concepts are *introduced* and applied in a *meaningful* manner (that is, meaningful as regards to the situation or activity itself)" (Dapueto & Parenti, 1999, p. 2, italics in original).

For Wedege (1999), a differentiation between *task context* and *situation context* is important:

> According to the dictionaries, the word 'context' has two fundamental meanings. The one is linguistic, meaning words that come before and after a word, phrase, statement etc., helping to show what the meaning is. The other meaning has to do with historical, social, psychological etc. circumstances in which, a) something happens, or, b) something is to be considered. Mathematical didacticians use the term 'context' in both fundamental meanings. 'Context' representing reality in tasks, word problems, examples, textbooks, teaching materials, is closest to the linguistic fundamental meaning. . . . I call this type *task-context*. . . . In the other fundamental meaning, . . . researchers in mathematics education speak of a context for learning, using and knowing mathematics (school, everyday life, place of work etc.), or context of mathematics education (educational system, educational policies etc.). . . . I call this type *situation context*. (pp. 206–207)

I applaud Wedege's attempt to clear some of the waters by defining context in these ways; practitioners and researchers can begin to overcome the ambiguity of the terms and work from a common ground in terms of the word *context*. Still, a shared understanding of *mathematics in context* is fuzzy. Is the term similar to Wedege's definitions of the two types of contexts? Neither? Both, in different ways? The answers to these questions are not clear.

In the research literature, scholars have argued that there is a need for common definitions across terms. "To arrive at a definition," however, may mean quite different things depending on the intent of the one who defines it. Without clearly established common referents (Lampert & Ball, 1998), we run the risk of reaching apparent agreement but not really. Thus the use of certain key expressions—in education as in other fields—does not necessarily mean that the speakers or writers are thinking about the same thing. They may be using the same words but defining them in very different ways—in essence, they are talking at each other, not with each other.

Throwing Reality Into the Water

Scholars have struggled to find words to describe the kinds of problems intended to resonate with students—those problems that are meaningful to students because somehow they are more real than typical textbook problems. But what is real varies with the theory and the writer. For some scholars, "real" has been a focal point of their work and has become a well-developed and well-defined concept. For example, the work at the Freudenthal Institute on the RME theory has resulted in a broad interpretation of *real*.

> The word "reality" can be interpreted two ways. First, it can refer to real-life contexts that offer opportunities for concept-building, model-building, application, and exercising (de Lange, 1987). However, reality is not synonymous with real life. It can also refer to mathematical situations that students experience as realistic. The essential point here is the word "experience." Therefore, Gravemeijer (1999) speaks about "experientially real" situations, which can refer both to the real life and to mathematics. For example, if students have developed a mathematical reality in which linear functions are meaningful objects, an assignment that starts with "A linear function *f* has the property…" may be perceived as realistic. Essential in the word "realistic," therefore, is that the activities and concepts involved are meaningful and natural to the students, no matter whether the meaning is derived from a real-life situation, from mathematics, or from another topic. (Drijvers, 2003, p. 53)

Similarly, Brown (2001) proposes that problems need not draw on applications as their sole source of content. "Real is not merely what we can touch. It is important to see it as what touches us" (p. 31). Fictional characters like Barney are products of human imagination and can be very real for children. Thus *realistic* means

> that the context of the problem is imaginable for students. . . . The fantasy world of fairy tales and even the formal world of mathematics can provide suitable contexts for a problem, as long as they are real in the students' minds and they can experience them as real. (van den Heuvel-Panhuizen, 2004, p. 3)

Brown (2001) and van den Heuvel-Panhuizen (2004) claim that problems framed in this way can be engaging and can lead to learning.

Brown (2001) also comments on *real* in terms of mathematical modeling. He notes that the focus in modeling is not on imagination or creativity but rather on seeking a correspondence between the elements of the real world and associated mathematical concepts and operations. In the modeling process, one of the first things students are encouraged to do is to identify and eliminate “irrelevant” information in the name of “making simplifying assumptions.” This step is not always appropriate in real problems. Many so-called real problems involve ethical and moral issues that cannot simply be discarded; they must also be considered. Mathematics cannot solve real problems on its own; rather, people can use mathematics to inform their decisions, such as by learning how confident one can be when making inferences from a small data set.

Although Brown (2001) concedes that the modeling connection is sometimes useful, he proposes an alternative and much broader view of connections between mathematics and the world. He argues that general human devices including humor, metaphor, error making, confusion, morality, and self-understanding connect the mathematics world and the nonmathematics world. Mathematics “is already ‘in’ the real world, and it is only by making believe that it is severed from the world that we arrive at some artificial notion of ‘application.’ That is, the application is there by virtue of the very way we speak about events in the world” (p. 153). Brown’s point is well taken, particularly if one accepts Lakoff and Nunez’s (2000) thesis that mathematics is a mental creation of human minds and that people’s understanding of central ideas in mathematics evolves and continues to evolve through their use of metaphorical thought. There are certainly important connections between mathematics and the world other than those that arise from mathematical modeling or applications.

Mokros, Russell, and Economopoulos (1995) identified the focus on making problems "real" for students in elementary grades as one of the red herrings of the reform movement in mathematics education. *Real* in this sense was taken to mean relevant, familiar, about the world that students know, and built on and applicable to their everyday experience. Russell (1996) cautioned that teachers and mathematics educators should not put too much stock in such characterizations of real contexts. "It is not the context of the problem itself, but the context of mathematical inquiry that determines whether or not students are engaged with mathematics" (p. 159). Students do not perceive school mathematics tasks as real merely because they have been given a real world "veneer" (Maier, 1991), yet their mathematical procedures and performance are largely determined by the context. Hence such problems may not simply be unrealistic; they may affect students' performance.

The following excerpt from Robert and Treiner (2004) points to another danger in defining *real*:

> One can always redefine the meaning of a word, and work with a new convention, but it is dangerous to do this with the word "real," when the subject matter is physical and/or mathematical modeling. "Real world," in the context of sciences, is usually opposed to "abstract world." This opposition is fruitful, since one is interested in various *phenomena* which we would like to *understand*, and understanding always goes with an "abstract" (mental) reconstruction of the "real world," the world out there. (p. 223)

Summary

I have pointed to the multiple ways in which the terms *mathematics in context*, *applications of mathematics*, and *mathematical modeling* have been used in the research literature. By paying little attention to the type of definition being used—informal, formal, or extended (McLeod & McLeod, 2002)—and by using these terms to define one another, teachers and researchers may exhibit fuzzy notions about what the terms reference. Furthermore, the multiplicity of meanings and uses associated with these terms have led some researchers to make explicit distinctions between kinds of modeling and types of contexts. Yet these efforts have had little effect because others have not taken up these ideas and employed them in their work. In addition, disputes about *what is real* and *how real a problem's context is* have further muddied notions about these terms.

It is important to note also that teachers, researchers, and mathematics educators have not explicitly defined the term *mathematics in context*. It is employed in the literature and used informally in conversation as if teachers and researchers have a shared understanding of its meaning. In my view it is the least articulated of the three terms in the present study.

Finally, I think it is important to recognize that the three terms of interest in this study are only a few of many that could have been studied. Sorting through the research literature on teachers' and students' work with word problems, contextual problems, applications, work-based problems, and modeling problems proved difficult. Many of the ideas in that research seem to overlap. Yet when one considers how these terms are differentiated by scholars who specialize in studies of, say, word problems—if one can detect how they are differentiated—one is not sure how or to what extent the research on one problem type informs research on another type. A more explicit distinction between definition types and problem types is needed. This study sought to address that need by investigating teachers' notions of the three terms and their relations.

Teachers' Instructional Practices With Contextual Problems

Little attention has been given to teachers' notions of terminology associated with contextual problems. Likewise, little research has focused on teachers' instructional practices with such problems. In this study, teachers' instructional practices were taken to include both teachers' talk and action, what they say as well as what they do. These forms of practice shape mathematics lessons and students' opportunities to learn.

Chapman (2004) studied teachers' thinking and practice as it related to their teaching of word problems. Part of her study focused on the teaching strategies the teachers used to facilitate modeling. Chapman and two research assistants interviewed and observed 22 mathematics teachers from junior and senior high schools. Using inductive analyses, they described teachers' conceptions that influenced modeling pedagogy and offered examples of teaching strategies. Chapman reported excerpts from the teacher interviews that "are representative of the teachers' conceptions of mathematics" (p. 66) and gave examples of modeling lessons taught by the "exemplary" teachers. Her findings focused on the teachers' conceptions (e.g., these teachers had very "logical" conceptions of mathematics, thought word problems were most valuable and meaningful when located in actual or realistic situations, and viewed problem solving as a thinking process and life skill) and teaching strategies.

Chapman (2004) concluded that the teachers' conceptions were important factors in creating a classroom culture that supported modeling and application. "With the appropriate conceptions, teachers are able to implement teaching strategies that allow students to experience modeling in a realistic way" (p. 70). But what were those teaching strategies? In four examples, Chapman described teachers whose instructional practices included focusing on problem solving and mathematical assumptions; questioning students' work, ideas, and assumptions; facilitating students as they worked collaboratively on tasks; leading whole-class discussions of the mathematics and contexts in the given problems; and incorporating real-world data from newspapers and elsewhere. In the report, Chapman did not elaborate on these practices.

Chapman's (2004) study focused on the modeling pedagogy of teachers selected because of evidence that they used "real or realistic situations to develop and apply mathematics concepts" (p. 66). Her findings concerning teachers' practices were informative but lacked a degree of detail that could differentiate the teachers' practices from those of almost any reform-oriented mathematics teacher. For example, many teachers question students and facilitate classroom discussions. But how were these practices specific to teaching with "word problems" or using "a modeling pedagogy?"

Unlike Chapman, I did not restrict the present study to teachers who were using a modeling pedagogy. My goal was to draw attention to the negotiations made by the teachers (and their students) in their work with contextual problems—how the teachers organized the lessons to facilitate students' learning, and how they adjusted their teaching (and the mathematics) to account for both contextual and mathematical considerations. In my reading of Chapman's report, I realized that I was interested in the finer details of the teachers' practices—details missing from the report. I therefore zoomed in to investigate the teachers' practices—*how* they were questioning students, *what kinds* of questions they were asking and for *what purposes*, and *how* they were facilitating discussions. For me, it was insufficient to say that teachers were facilitating discussion without describing *how* and *toward what goal* they were working.

The present study was also informed by a study conducted by Kilpatrick et al. (1996), which focused on teachers' development of a precalculus curriculum based on the idea that all mathematical concepts would be introduced to students through an application. In this way, the students would select the mathematics to use to solve a given problem, or they would develop new mathematics and models as needed. Kilpatrick and his colleagues reported that the teachers who attempted to implement this curriculum in their courses faced challenges but were able to gradually change their teaching practices.

Initially, some teachers in the study (Kilpatrick et al., 1996) feared that the students would not be able to solve the problems and often gave too much information and assistance. Over time, and with support and collaboration, the teachers became more comfortable with allowing the students to struggle and work together on "big" ill-defined problems that yielded multiple correct answers. Their instruction incorporated the use of technology in the form of graphing calculators, and much emphasis was placed on data analysis and modeling. The teachers in all five schools in the study were involved in the mathematics education community at various levels: national, local, and at the school where the curriculum had been developed. Through the collaborations within these communities, the teachers' instruction evolved as the curriculum did. Central themes in the precalculus text—data analysis and modeling—began to spill over into other subject areas. Not only had precalculus changed but so had the teachers and their approaches to teaching. For example, the teachers no longer followed a detailed syllabus for their courses. They became much more interested in connecting themes of ideas across mathematics. "They felt comfortable modifying, reordering, or dropping units that did not seem to work as they should and adding others" (Kilpatrick et al., 1996, pp. 220–221). In addition, many of the teachers studied by Kilpatrick and his colleagues were willing to take risks in their teaching and move away from traditional modes of teaching. This move required knowledge, courage, and support. As a result, the teachers in the study were able to make change not only in their school but across schools and states.

Because the teachers in Kilpatrick et al.'s (1996) study evidenced a commitment to contextual problems and a willingness to use such problems regularly in their teaching—so much so that they were willing to discard mathematical topics they could not introduce through an application—I wondered what the practices of such teachers might be. I was interested in studying teachers' practices as a way to inform mathematics educators seeking to prepare teachers to teach with contextual problems. The most fruitful site for informing this interest was with teachers already doing that work. Similarly, Kilpatrick et al. talked with teachers, reviewed historical documents, observed some teaching, and took into account factors at local, state, and national levels that affected the teachers' development of the precalculus curriculum. I, too, felt it was important to interview and observe teachers, to look both closely and broadly at their instructional practices, and to account for the conditions enabling them to do their work.

Challenges of Using Contextual Problems

In spite of the arguments presented in chapter 1 for the inclusion of contextual problems in the teaching of mathematics, counterarguments and challenges to using such problems are numerous. For example, one debate alluded to in chapter 1 asks whether the context of a problem hinders or helps learning. One side argues that students should work predominantly with abstraction and that competence with abstract mathematical thinking is necessary if they are to generalize and apply their knowledge in any situation or setting. Advocates of this argument are concerned that students who are taught mathematics predominantly in context will not be able to think abstractly—using contexts hinders learning. Advocates on the other side of the debate claim that students need some practice applying mathematics and recognizing its uses in the world. If students work solely with abstract mathematics, they may become disengaged from the content and disinterested in mathematics because the subject will seem disconnected from their lives—using contexts helps learning. The debate may lead teachers of mathematics to question whether they should use contextual problems in their teaching. Perhaps a balance of the two perspectives would be best. Students should engage in both mathematical abstraction and application. Thus, alongside the learning of pure mathematics, contextual problems are worth consideration.

Another reason for teachers to question the place of contextual problems in the curriculum has to do with research that shows many students ignore the context of a problem focusing only on the numbers. One reason students ignore the context may be its artificiality. The context may be so artificial that, for the student, it is really not a context at all. Or the context may not be experientially meaningful or real to the student and is therefore seen as unimportant or confounding. Blum and Niss (1991) note that

> there are abbreviated and restricted links between mathematics and reality which are much more frequently found: On the one hand a direct application of already developed "standard" mathematical models to real situations with a mathematical content, on the other hand a "dressing up" of purely mathematical problems in the words of another discipline or of everyday life. Such problems often give a distorted picture of reality. (p. 40)

Students may also ignore the context because of established practices of school mathematics. Many students have learned that mathematics as presented in school is about numbers and getting correct answers. The problem's context may appear to be of little concern to the teacher compared with the problem's solution. Thus many students have learned that there is no need to pay attention to context when solving a problem. In fact, attention to the context may lead to an answer deemed unacceptable by the teacher. The student may account for too much "reality" when solving the problem. According to Schwarzkopf (2004), solving word problems demands an interplay between at least two very different framings: namely, a framing based on "everyday understanding" of the problem's real-world context and a "mathematical" framing:

> The interplay is complex in nature, because within different framings the participants are acting in different ways concerning the relevance of facts, the meaning of assertions, the acceptance of statements, the rules for correct reasoning and many other aspects. Hence, solving [contextual] word problems demands a switch between framings that are not only different, but sometimes contradictory concerning their rule for rational acting. (p. 242)

This quotation points to the complexities encountered by students when working with contextual problems. On the one hand, students must understand the framing within which they are working and make decisions based on that framing. On the other hand, contextual problems may give students a distorted picture of reality because particular features of the problem are often irrelevant in a mathematical sense. Students may view all contextual problems within a mathematical frame because the problems are encountered in a school mathematics context.

Cooper and Dunne (2000) provide further evidence that students may have learned to ignore a problem's context in a school setting. They report that in the context of realistic assessments, to be successful, students are often required to consider some, but not too many, features of the real situation described in a task. Such findings have led Cooper and Dunne and other researchers (Boaler, 2000a; Schoenfeld, 1991; Verschaffel, Greer, & De Corte, 2000) to express concern about students' suspension of sense-making when working with contextual problems, particularly when connecting school mathematics and reality. Students may use solution strategies with contextual problems that do not include any consideration of the realities of the situations described in the problems. Boaler (1993) suggested that

> contexts *may* be useful in relation to learning transfer even though contexts as they are generally used are *not* useful, and that the factors which determine whether a context is useful or not are numerous and complex and have little to do with a description or depiction of real world events which students will eventually encounter. (p. 13, emphasis in original)

Thus, the selection of contextual problems for use in instruction and the classroom practices established when working with such problems require deliberation by the teacher.

To counteract the tendency of some students to strip away and ignore real-world constraints, Verschaffel and de Corte (1997) designed instruction that would, they hoped, encourage fifth-grade students to be more sensitive to real-world constraints when formulating solutions. In a comparative study, they found a strong tendency for students to exclude realistic considerations on the pretest, but instruction in the experimental class was effective in significantly improving realistic responses on the posttest. In the control class, students performed similarly on both the pretest and posttest. Thus simply including contextual problems in the curriculum was not sufficient; students had to be taught how to work with such problems. As a result, Verschaffel and de Corte suggested that interventions aimed at establishing new social and socio-mathematical norms in line with the mathematical modeling perspective need to occur often and be explicit.

On the one hand, students who have not been taught how to work with contextual problems may ignore a problem's context. On the other hand, they may place too much importance on the context. Research has shown that in settings where ignoring the context is desirable (e.g., standardized assessments), students who pay attention to the problem's context will be less successful. And not all students appear to be equally adept at ignoring the context. For example, Gipps and Murphy (1994) found that the students who tend to be successful with contextual problems are those who can take a contextualized problem and strip away (or ignore) the context to focus on abstracted techniques—in their study, boys tended to be more successful than girls at this process. Cooper (2001), using Bernstein's theory of pedagogical practices (e.g., Bernstein, 1996), argued that those students who are more likely to be comfortable ignoring the everyday reality, focusing instead on the esoteric decontextualized nature of mathematical problems, tend to be from the higher social classes. Together these studies highlight that mathematics—and mathematics situated in contexts—can exclude subgroups of students from "success" in school mathematics.

Contextual problems themselves also pose challenges. No one task context can offer an application that is familiar and, more importantly, meaningful to all students. It is equally difficult to create contextual tasks that are faithful to both the context and the way mathematics is used in the world. Contextual tasks are often simplified so that they are manageable for students or so that they emphasize a particular mathematical point. In this simplification, the context may be distorted so that it is incorrect, or so that it introduces contextual issues that will frustrate and possibly confuse students (e.g., pizza cut into 9 slices). Such problems are challenging for teachers and students who are trying to use contextual problems.

Despite the refutations and documented challenges of using contextual problems, teaching with such problems continues to be strongly valued (Pollak, 2003; van den Heuvel-Panhuizen, 2004) and promoted (NCTM, 2000). It can be observed in middle school and secondary mathematics curricula such as Mathematics in Context (Romberg, 1997–1998), the Connected Mathematics Program (Lappan, et al., 2002), and the Applications/Reform in Secondary Education (ARISE) project of the Consortium for Mathematics and Its Applications (COMAP, 1998–2000; see http://www.comap.com/highschool/projects/). The reason is that using such problems can potentially confer on students a variety of benefits that outweigh the challenges (e.g., to engage, motivate, prepare for workplace, enhance mathematical reasoning, increase ability to generalize mathematics to new or different situations, and increase knowledge transfer; see chapter 1). Contextual problems were therefore the focus of the present inquiry.

CHAPTER 3

METHODOLOGY AND RESEARCH DESIGN

In light of the research questions, I used a qualitative research design and interpretive approaches to data collection and analysis. Interviews with and observations of teachers were the main sources of data. Artifacts (mathematical task handouts, teachers' curriculum vitaes, and teachers' graphic organizers) were also important data sources. Interpretive analyses and a grounded theory approach were used to generate descriptions of teachers' beliefs about, notions of, and instructional practices with contextual problems.

My Position

In this study I entered the classrooms of the teacher participants and positioned myself as "a fly on the wall." Though such a position is never neutral—the researcher's presence inevitably changes the context and brings with it a set of beliefs about the world and how it should be understood and studied (Crotty, 1998; Denzin & Lincoln, 2000; Lerman, 2000)—my presence in the classroom consisted only of operating the video camera and sometimes walking about to zoom in on teachers' and students' conversations and work. I did not directly interact with the students or teachers in the classroom. Outside the classroom, I spent considerable time with the teachers: attending math team meetings, attending sporting events, eating meals, walking for exercise, and conversing. Such interactions strengthened the trust and increased the comfort levels between me and the participants. In all interactions, the participants were positioned as the “experts”—those from whom I hoped to learn something. I maintained the position of “nonexpert,” deliberately refusing to define terms such as *mathematics in context* and consciously seeking their meanings for such terms. I tried to work within their frame in an attempt to get at the meanings they attached to their work as teachers. I did not directly interact with their students on any level.

In a qualitative research study, it is important to account for the “subjectivities” of the researcher. I am a middle-class White woman[1] working from a number of theoretical positions with participants who were similar both economically and racially. Philosophically, I do not believe there is an objective truth waiting to be discovered or that truth can be found in the meanings I (or anyone else) attribute to the classrooms and teachers in this study. Truth and meaning are not discovered, but constructed and produced. In all aspects of this study, sense-making and meaning-production were my constructions or productions. To describe the difficulty in accounting for the theories impacting those constructions, I offer the following rambling notes from my researcher log (October 8, 2004):

> In the analyses conducted in this study, it is important to note that the researcher approached the data with a number of theories embedded in her own thinking in ways that cannot be articulated. Informed by socio-cultural theories of mind and interpretive methodologies—all laced with post-structural theory—to name a few, I have approached my data with a “theoretical gaze” that is overlapping, “folded” within itself, and inexplicable. Yet, somehow, the multiplicity of my mind and self opened paths into the data that “made sense.” Perhaps it is our nature (though I don’t believe in such a thing) to want to “make sense” of things, to rationalize them in some way and “explain” away all incoherence. In fact, research methodologies have taught us that analysis of data must be articulated, accounted for, and theorized. We must account for data that are unusual, that do not “fit”—we must adjust and adapt our theories (particularly when developing

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[1] I have categorized myself in ways recognizable in our current socio-historical setting (though I may not think of myself in these ways). In response to the question, “is there a realm of personal identity possible apart from social constraint?” Scott (1991) reminds us that “the social and personal are imbricated in one another and that both are historically variable. The meanings of the categories of identity change and with them the possibilities for thinking self” (pp. 794–795).

> grounded theory) to encompass and subsume “negative cases.” It is in this way that the research is “reliable” and “trustworthy” —and rigorous. And yet, in doing this work, in explaining (or explaining away) the interesting nuances, in trying to verbalize and analyze your “headwork” as a researcher, one thing becomes perfectly clear: This work is an illusion, a powerful myth, something to strive for, but impossible to obtain. I cannot account for all of the theory that was put to work in my analyses; there are too many influences and no sense or logic to their employment. What I can offer the reader is the assurance that interpretation in this study was grounded (in the sense of, close to the data) and it was theoretical. It is impossible to think outside of theory. And it is impossible to account for it in any coherent and complete way. What I offer is the best description of *what I think I did*. Should this be considered a lack of attention to rigor? I think not. I think such acknowledgement is the most sincere and “honest” depiction of the analyses conducted here. I leave it to the reader to judge the product.

Issues of Validity

A number of procedures can be applied in qualitative research to enhance the trustworthiness of a study’s findings: prolonged engagement and persistent observation, triangulation, peer review and debriefing, negative case analysis, clarification of researcher bias, member checking, thick description, and external audit, just to name a few (Glesne, 1999; Lincoln & Guba, 1985; Merriam, 1995). In this study, prolonged engagement (over the course of 3 months) and multiple observations (16 in all) enhanced the trustworthiness of the findings. With extended engagement, the tendency for participants to exhibit contrived behaviors was reduced. Triangulation of data sources—observations, interviews, and artifacts—increased the probability that emergent conceptions were consistent with a variety of data. Triangulation was done by looking across the data and highlighting the apparent convergences, inconsistencies, and contradictions in what the teachers said and did (Mathison, 1988). This process allowed for a more holistic view of the data and a more “reliable” interpretation of the data. Thick description was important in terms of describing the teachers, their teaching contexts, and the findings, as well as the methodological decisions, concerns, and analyses. Thick description helps increase the trustworthiness of the study because any researcher can trace my steps in all aspects of the research process. In addition, interpretations were scrutinized by me and members of my doctoral committee. I also maintained a researcher's log that documented my shifting interpretations, theoretical considerations, and research activity.

Crotty (1998) noted,

> In the end, we want outcomes that merit respect…. Our outcomes will be suggestive rather than conclusive. They will be plausible, perhaps even convincing, ways of seeing things – and, to be sure, helpful ways of seeing things – but certainly not any 'one true way' of seeing things. (p. 13)

In the sections that follow, I lay out the methods used in this study to make explicit how the outcomes were produced.

Theoretical Perspectives

Because situations of teaching practice have unique characteristics of complexity, specificity, instability, disorder, and undetermination (da Ponte, 1994), Boaler (2000b) advocates situated theoretical perspectives that focus on the culture, practices, and expectations within the classroom, and that emphasize the communities within which the students and teachers work, the practices that are central to their classroom communities, and the relationship of these practices to students' cognitive development. She argues that such an interpretation is important because it accounts for the learning of practices in the school community—the "patterned participations, systematic dances" (Birdwhistell, quoted in McDermott, 1993, p. 276) of practice. Borrowing from Greeno and the Middle School Mathematics Through Applications Project Group (MMAP) (1998), Boaler (2000b) points out that a situated perspective

> does not negate the importance of depth of understanding or of practicing methods; it includes both features within a focus on the broader patterns of

> participation in communities and the various constraints and affordances to which students become attuned in the mathematics classroom. (p. 116)

In the present study, I took Boaler's recommendation and adopted complementary perspectives that centralize the communities within which teachers work and the practices central to these communities: interpretivist and interactionist perspectives. The following descriptions of these perspectives should be taken as reflective of my theoretical gaze regarding the research design and data analyses.

Interpretivist Perspective

Mathematics is a part of students' social and cultural lives, and the mathematics classroom has its own social and cultural life. To portray teachers' notions of terminology associated with contextual problems and their instructional practices with such problems, this study was framed from the philosophical position of interpretivism. This position allowed me to focus on the social and cultural aspects of these teachers' classrooms.

> Central to interpretivism is the idea that all human activity is fundamentally a social and meaning-making experience, that significant research about human life is an attempt to reconstruct that experience, and that methods to investigate that experience must be modeled after or approximate it. (Eisenhart, 1988, p. 102)

From this perspective, meanings and actions, context and situation are inextricably linked and make no sense in isolation from one another. The "facts" of human activity are social constructions; they exist only by social agreement or consensus among participants in a context and situation. Thus what counts as teaching, reasoning, or whatever depends on the ways (and whether) these things are defined and used in human groups (Bredo & Feinberg, 1982, p. 116). In other words, it makes little sense for an interpretivist to, for example, catalog beliefs about mathematics without also considering the contexts in which these ideas are important.

The primary purpose of this study was to *describe* and *understand* various aspects of teaching with contextual problems. Interpretivist research was therefore appropriate since it is a "research that seeks merely to understand, . . . that reads the situation in terms of interaction and community" (Crotty, 1998, p. 113) and that is "oriented towards an uncritical exploration of cultural meaning" (p. 60). Furthermore, it is a perspective that "does not concern itself with the search for broadly applicable laws and rules, but rather seeks to produce descriptive analyses that emphasize deep, interpretive understandings of social phenomena" (Scott, 2001, ¶2). This perspective involves the denial of an objective reality independent of the frame of reference of the observer; reality is mind dependent and influenced by the process of observation. Interpretive research thus generally enables the researcher to gain a descriptive understanding of the values, actions, and concerns of the subjects under study. It also provides an avenue through which the researcher can offer a "thick description" (Geertz, 1973) of the phenomena at hand.

The interpretivist perspective in the present study looked "for culturally derived and historically situated interpretations of the social life-world" (Crotty, 1998, p. 67)—in this study, the teachers' mathematics classrooms. A primary goal of interpretivist research is to understand meaning in the context in which it is "produced and received" (Moss, 1996; Thompson, 1990). I therefore found this broader theoretical perspective appealing as I tried to understand the meanings created by the teachers when using contextual problems in the culture or community of their classroom.

Interactionist Perspective

In this study, knowledge was taken as both situated and socially mediated. Knowledge was also viewed as at once tacit and conscious, embodied and discursive (Adler, 1996). In other words, what one knows often lies in what one says and what one does in a specific situation.

With this understanding of knowledge, I adopted an *interactionist* view wherein the individual and the society are seen as inseparable units, having a mutually interdependent relationship, as "teacher and students interactively constitute the culture of the classroom" (Bauersfeld, 1994, p. 139). The interactionist perspective, requiring the consideration of both psychological and sociological theories, takes into account factors such as the social context of learning (Balacheff, 1990) and the nature of the knowledge being learned—also a social construct. The teacher is certainly one of the most important elements of the learning context and a key person in the definition of knowledge. In the interactionist perspective, mathematical activity is inherently social and cultural (Brown, Collins, & Duguid, 1989; Greeno, 1991; Lerman, 1996; Schoenfeld, 1987; Sfard, 1994). From this perspective, "what we are able self-consciously to articulate, as well as how we act, form and are formed by our developing identity and our activities and practices in a social, cultural, and historically contingent world" (Adler, 1996).

I approached the present study informed by this perspective. Knowledge was taken as situated and as culturally and socially mediated; and the individual and society were taken as inseparable units. Knowledge (produced by the teachers through description and practice) was considered within the social and cultural activity of their classroom and school. And the knowledge produced in the present study should likewise be taken as mediated by the cultural, social, and historical world at present.

Participants and Participant Selection

To study teachers' notions of and instructional practices with contextual problems, I sought secondary mathematics teachers who were using contextual problems on a near-daily basis. In a pilot study in 2003, I had focused on public school teachers who were using curricular materials that claimed an emphasis or theme on applications and contexts (Lappan et al., 2002; Coxford et al., 1997). But because of pressures imposed by administrators and state examinations, these teachers were paying little attention to the contextual aspects of the materials. They were extracting the mathematics to be tested on exams and orchestrating practice opportunities for their students. Although this approach should not be condemned—and is understandable—these public schools proved unfruitful sites for my study.

Using network sampling (Patton, 1990), I located potential participants through recommendations from university professors and by attending a national conference for high school teachers in February 2004. The conference brought together both public and private school teachers who were immersed in work with contextual problems and mathematical modeling and thus offered a rich site for recruiting participants. An important criterion for participation in this study was geographic location: Participants needed to be within driving distance of northeast Georgia or middle Tennessee. At the conference, I attended every presentation based on contextual problems given by teachers who taught in this geographic region. Based on their presentations and my conversations with them about their teaching and contextual problems, I identified 8 to 10 teachers that I thought might be using contextual problems on a near-daily basis. The teachers were from three schools: a public school, a private school, and a public school of mathematics and science. The potential sites were narrowed to two schools when the teachers at the public school did not respond to my e-mail requests to visit and observe their classes.

Before formally inviting the remaining teachers to participate in my study, I visited their schools (in two different states) and observed their teaching over 2 days. I also observed other mathematics teachers in each school based on teacher recommendations and because the teachers using contextual problems regularly (in the geographic region of interest) seemed to be clustered in these schools. I selected six teachers for participation based on their interest in participating and their self-identification as using contexts in their mathematics instruction consistently and regularly—that is, contexts played a role on a near-daily basis. High school teachers were appropriate for study since (a) demands for the relevance of school mathematics content are more often aimed at the high school curriculum, and (b) high school mathematics topics lend themselves to richer and more diverse contexts than do topics in the lower grades (J. Kilpatrick, personal communication, November 15, 2003). These six teachers were appropriate for study because they were (a) interested in the research topic, (b) immersed in curriculum development work and workshops with contextual mathematics, (c) seasoned in teaching with such problems (14 or more years), (d) experienced in sharing their work with colleagues via publications and in professional development settings (they could be articulate about their work), (e) situated in schools that allowed for academic freedom with regards to curriculum decisions, and (f) located geographically within driving distance. Furthermore, the teachers were highly qualified academically (2 with doctorates, 4 with master's degrees or beyond), and thus there could be little question as to the soundness of their mathematical knowledge.

In these ways, the sample in this study was small and purposive (Denzin & Lincoln, 2000), which may raise questions about its representativeness. As regards representativeness, the issue concerns the generalizability of cases to theoretical propositions and not a population. That is, the issue is tied to building a theory, which can then be applied to other cases and contexts (Silverman, 1993, p. 160). The teachers in this study were not meant to represent all teachers; they served as a sample from which I could build a theory whose generalizability the reader can judge.

Schools

Watercliff Academy

Watercliff was a private pre-first-grade to 12th-grade school located in a large city in the southeastern United States. (All school and teacher names are pseudonyms.) The school dated from the 1950s and sat on large grounds with surrounding sports fields, trees, and streams. It was a beautiful and secluded campus. Watercliff enrolled approximately 1700 students (Grades 9 to 12: 750 students) who came from communities around the city. The school was highly selective; according to its descriptive literature, it sought "students who demonstrate excellent academic promise and personal integrity and who represent various racial, ethnic, religious, and economic backgrounds." The gender distribution was about equal, with a minority enrollment of about one-sixth. The school employed around 250 teachers and administrators (72% with graduate degrees).

High school mathematics courses at Watercliff were taught in two buildings. Classrooms varied in size but all had whiteboards, chalkboards, overhead projectors, and computer projector equipment, and most were decorated with students' work and posters, often even on the ceilings. Computers were accessible in labs that were always open during school hours, and students could use them between classes and during breaks to type papers, conduct research, or check e-mail. All high school mathematics students were expected to purchase graphing calculators, which were used regularly in all courses. Classes were from 15 to 20 students each, and met four out of five days a week for 55 minutes each on a rotating schedule. This schedule allowed for the inclusion of assemblies and other programs during the school week. Technology such as graphing calculators, computers, and Calculator Based Laboratories (CBLs) were regularly integrated into lessons, and students utilized these and other tools in experimentation and simulation activities.

The mathematics faculty in the high school (Grades 9–12) consisted of 14 teachers, one of whom served as department chair. Teachers met two to five times a year as a department to discuss matters of significance. Smaller course groups met intermittently. Teachers of the same course strove to teach their lessons at the same pace but were free to adjust the syllabi as they saw fit. For example, one teacher reported hurrying through particular topics to allow time for the inclusion of week-long modeling projects throughout the year, projects his fellow teachers did not conduct. Extensive collaboration among teachers was rare; most collaborated only when newer teachers wanted to discuss a particular topic or lesson they were preparing to teach.

Constantia Ridge

Constantia Ridge was a public residential Grades 11–12 high school that emphasized mathematics and science. It was located in a city in a southeastern state. It enrolled approximately 600 students (almost equal gender distribution; about one-sixth minority, including African American, Hispanic, and Native American), who came from all over the state. The school was selective and actively recruited students who showed interest in mathematics and science. The students varied in economic and racial backgrounds. According to a school pamphlet, "On average there are over 80 of the 100 counties in the state represented in each graduating class." Students were selected based on their SAT scores, their scores on an entrance exam, and an interview with teachers and students from the school. Attendance was free, and students were provided with accommodation, food, books, and opportunities to participate in typical high school courses and extra-curricular activities. The school opened its doors in 1980 in a renovated hospital; thus, the school grounds were larger than most urban public high schools, with beautiful old trees lining the entrance.

The mathematics courses at Constantia Ridge were taught on two floors of the main building. The classrooms were small yet well equipped, with three to five chalkboards, overhead projectors, and computer projector equipment in each room. Decoration in classrooms was sparse, with blank walls and ceilings. Computers were accessible in labs that were always open, and students could stop by between classes, during breaks, and in the evenings to type papers and conduct research. As a residential school, the science and computer labs were often open late into the evenings, with students and teachers working on projects and experiments. Though it was not a requirement, most students had purchased graphing calculators, as these were used regularly in all mathematics courses.

The mathematics classes included from 10 to 18 students each and met for 40 minutes one day and 50 minutes on two days each week on a rotating schedule with an additional 90-minute lab period one day each week. This schedule allowed for the inclusion of modeling problems and other in-depth investigations that could not be sufficiently tackled in 40 or 50 minutes. Technology such as graphing calculators and Calculator Based Laboratories (CBLs) were regularly integrated into lessons, and students often utilized these and other tools in experimentation activities and modeling projects.

The school employed approximately 70 teachers and administrators who taught full and part-time (all with master's degrees and over a third with a doctorate in their field), 13 of whom were mathematics teachers. All of the mathematics teachers either had or were working on getting National Board for Professional Teaching Standards (NBPTS) certification, and all had participated in outreach programs to service teachers across the state. The precalculus teachers (a team of 9) met every week to discuss the topics they were teaching, to maintain parallel pacing, and to develop and work mathematics problems. These collaborations were central to their work as a department and as individual teachers. As a department, they had written precalculus and calculus textbooks that developed mathematics through applications and mathematical modeling. Thus the teachers were always looking for problems from the newspaper, television, and other media that could be adapted and developed into rich mathematical investigations. The hallways that surrounded their offices and classrooms were continually adorned with new clippings from the newspaper and with challenging "Problems of the Week." Teachers were not required to participate in after-school activities, but many worked until six or seven o'clock every day tutoring students, helping with labs, or attending meetings. The four participants in this study were dedicated in that way and were all part of the precalculus curriculum development initiatives that had taken place at Constantia Ridge from the late 1980s through the late 1990s.

Participants

Six teachers participated in this study: two from Watercliff Academy and four from Constantia Ridge. I first visited Watercliff Academy in January 2004 and spent 2 days observing the mathematics teachers and discussing their instructional practices. After observing 11 classes and 7 teachers, I invited 2 teachers to participate in my study. I selected Gary for three reasons: He had been recommended by university professors and was known for using modeling and contextual problems regularly in his teaching; he was planning two "modeling lessons" for later that spring; and he was interested in my research and willing to participate. Hank was invited to participate in an interview only. He was not planning any lessons for the spring that he thought would use contextual problems but was interested in discussing how he selected and designed contextual problems for classes.

I first visited Constantia Ridge in February 2004 when it hosted a conference for teachers of high school mathematics. In March 2004 I returned to the school and spent 2 days observing nine classes and seven teachers across subjects. I invited four to participate in my study. I invited Tom for an interview only, because he did not anticipate teaching any lessons during April and May that would use contextual problems. He did, however, want to share how he selected, adapted, and used contextual problems in courses. I invited Rhonda and Diane to participate fully in the study because each had been recommended as a teacher who used modeling approaches and contextual problems regularly, each was excited and willing to participate in the study, and each anticipated teaching upcoming lessons that would use contextual problems. I also invited Cathy to participate fully in the study, but later in the spring she said she would not be teaching any lessons that would use contextual problems, so I conducted an interview only. Table 1 provides an overview of the teachers' professional backgrounds and community involvement.

Hank

Hank was in his 14th year at Watercliff Academy and chaired the mathematics department. He had received a number of teaching awards and was proactive in his ongoing professional development as a teacher. He was also active in publishing and presenting ideas and problems for fellow mathematics teachers. In addition to teaching at Watercliff, Hank served as a mathematics professor at a local junior college and as a Scholastic Assessment Test (SAT)/College Prep instructor for a national organization. At Watercliff, he taught three classes each day and spent two class periods tending to departmental responsibilities and administrative

Table 1: Participants' Professional Backgrounds and Activities

Teacher	No. Years Teaching	Highest Degree Attained	Community Involvement
Watercliff Academy			
Hank	14	Ed.D. NBPTS (2003)	Pub/Present Professional org.
Gary	30	Ed.S.	Author/Pub/Present Professional org.
Constantia Ridge			
Tom	29	Ph.D.	Author/Pub/Present Professional org.
Diane	28	M.A. NBPTS (1997)	Author /Pub/Present Professional org.
Rhonda	16	M.A. NBPTS (1997)	Author /Pub/Present Professional org.
Cathy	14	M.S. NBPTS (current)	Author /Pub/Present Professional org.

Note: NBPTS (year) = National Board for Professional Teaching Standards certification and year obtained; Author = involvement in writing textbooks focused on contextual or modeling problems; Pub/Present = publications and presentations at a national level; Professional org. = involvement in professional mathematics teaching organizations.

duties. Once a week, he went to Watercliff Elementary School to conduct "Math Recess" with Grade 3–5 students. During those times, he provided those students with opportunities to play pattern or number games, or to solve mathematical puzzles—activities that were active, hands-on, and mathematically rich. He often played the games himself and used his enthusiasm and passion for mathematics to fuel students' interests. This enthusiasm was evident in all of his teaching.

Gary

Gary was in his 30th year of teaching. He served as school sponsor of Watercliff's math team, which had won a number of awards throughout the years. The school had four "National Outstanding" teams in an annual secondary level national modeling contest, one honorable mention in an annual international undergraduate modeling contest, and several regional and state American Mathematics Contests (AMC) recognitions. Gary had an M.S. in applied mathematics and an Ed.S. in secondary mathematics curriculum and instruction. He had received a number of teaching awards and some recognition for his work with students in Advanced Placement Statistics. He participated in professional mathematics education organizations and regularly attended national conferences and meetings. By invitation, he had attended the first Woodrow Wilson Summer Institute (on statistics) in 1984 and had served in a leadership role for the institutes from 1988 to 1993, an experience he cited as integral in shaping the teacher he had become. In addition, he was active in curriculum development projects, including a secondary mathematics curriculum project focused on applications.

Tom

Tom was in his 29th year of teaching and was responsible for "Problems of the Week" at Constantia Ridge. He had his M.Ed. with a specialization in mathematics and his Ph.D. in mathematics education. He had received National Board certification and a number of teaching awards. Tom participated in national professional mathematics education organizations, often in leadership capacities, and was proactive in his professional development as a teacher and leader in the mathematics education community. In addition, he was active in publishing and presenting ideas to fellow mathematics teachers, though most of his energies were spent on and with his students. In the 1980s and 1990s, Tom served in a leadership role for the Woodrow Wilson Summer Mathematics Institutes—an opportunity that led to ongoing collaborations and friendships. He felt that it was "essential [for teachers] to have a community" and spoke of the Woodrow Wilson experience as integral in fostering his growth as a teacher and his continued involvement in the national mathematics teaching community.

Diane

Diane was in her 28th year of teaching and for the past 10 years had served as sponsor for Constantia Ridge's participants in the yearly Mathematical Contest in Modeling sponsored by COMAP. She had both her B.S. and M.A. in mathematics and had completed additional graduate coursework to obtain a teaching certificate in the late 1970s. Diane had received National Board certification and a number of teaching awards. She participated in professional mathematics education organizations and was active in organizing and leading professional development activities for colleagues and teachers. In addition, she had worked on a number of textbook development projects and had served as the principal investigator on four other projects funded by the Eisenhower Grants Program and the National Science Foundation from 1987 to 1998.

Rhonda

Rhonda was in her 16th year of teaching at Constantia Ridge and had devoted part of the previous 12 years to teaching mathematics courses to students and teachers in remote sites throughout the state via cable television as part of a distance learning and outreach project. She had both her B.S. and M.A. in mathematics as well as National Board certification. She participated in professional mathematics education organizations and actively participated in professional development activities. During the summer and other school breaks, and over some weekends, Rhonda and her colleagues organized and led a variety of professional development opportunities: conferences, workshops, institutes, and so on. Rhonda found the outreach opportunities through distance learning and professional development activities to be among the most rewarding aspects of her job.

Cathy

Cathy was in her 14th year of teaching, though she had been in and out of teaching and computer programming as careers over the preceding 27 years. She had both her B.S. and M.S. in mathematics and was currently working to obtain National Board certification. Cathy had been involved in some curriculum development projects at Constantia Ridge but had devoted much of her efforts to technology and some distance-learning projects. She participated in professional development opportunities and regularly co-organized and co-led sessions at conferences (and workshops) for teachers. Because of her background in computer programming, Cathy was passionate about incorporating technology (graphing calculators and MathCad—a commercial software package) into her mathematics teaching at every opportunity.

Instruments and Methods

Background and Beliefs Interview

All six teachers in this study participated in a semi-structured (Bernard, 1994) Background and Beliefs (B/B) Interview. I asked the teachers for a copy of their curriculum vita, and using questions piloted in Spring 2003 and revised for the present study (see Appendix A), I met with each one individually for 60 to 90 minutes either during his or her planning period or after school. In the interview, I asked general questions about their teaching background, their school and curriculum, the purpose they saw for teaching mathematics, and how they prepared and delivered mathematics lessons. I also asked them to react to statements about teaching with contextual problems and asked specific questions about terminology—*mathematics in context*, *applications of mathematics*, and *mathematical modeling*—to get their interpretations of these terms and what they meant in the context of teaching and learning. Each interview was audiotaped and transcribed by me as soon as possible so that I could ask missed questions and seek clarifications of the teacher's comments in a subsequent interview. In addition, three teachers were e-mailed two clarification questions in fall 2004.

Graphic Organizer

The concluding task in the B/B Interview involved the teacher in drawing a graphic organizer relating the three terms: *mathematics in context*, *applications of mathematics*, and *mathematical modeling*. This task followed teachers' descriptions of the terms and how they were related to the real world, if at all. Specifically, I gave each teacher a blank sheet of paper and asked, "Can you draw a graphic organizer that shows how you think about the three terms and how they are related or not?" If the teacher asked for clarification of what I meant by "graphic organizer," I responded with the question, "Is there a way you can organize the three terms that shows how you are thinking about them?"

Hank, Gary, and Tom drew their organizers immediately. Cathy and Rhonda asked to work on their organizers and give them to me later. I agreed on the condition that they would not discuss the question, or their organizer, with anyone until after I had completed my data collection. (Cathy gave me her organizer in the afternoon following her B/B Interview. Rhonda gave me her organizer the day after her B/B Interview.) Diane was also asked to draw an organizer, but time constraints prevented her from doing so. Each teacher's description of his or her graphic organizer was audiotaped, and I asked questions to clarify aspects of the organizer and its description. The organizers served to clarify for the participants—and for me—the ways in which they had previously described the three terms in the interview.

Pre-lesson and Post-lesson Interview

Three of the six teachers were observed and videotaped teaching lessons that they thought incorporated contextual problems. Before each of the observed lessons, I conducted an interview with the teacher concerning the nature of the upcoming lesson: its organization, its goals, the mathematical foci, the teacher's and students' anticipated activity during the lesson, and its location in their curriculum sequence (what students already knew and what was yet to come). Upon completion of the lesson—sometimes spanning 2 days—the teacher was interviewed again. The post-lesson interviews provided opportunities for the teachers to comment on the lesson: whether they met their goals and their impressions of how it went, among other things. During these interviews to gain access to the teacher's perspective, I often cited particular events, or segments, in the lesson and asked both broad and targeted questions about the mathematics, the context, and the pedagogical decisions in that lesson segment. I also used the teacher's oral lesson plan from the pre-lesson interview to focus the post-lesson interview questions (see Appendix B for a sample of pre- and post-lesson interview questions).

Classroom Observation

Informal observations from January to March 2004 preceded the onset of data collection to help identify participants and to portray the larger school context within which these teachers worked. The informal observations were also part a concerted effort to acquaint myself with the school, the teachers, and the classes that would later be videotaped so that all would be accustomed to and comfortable with my presence—an attempt at minimizing observer effect. The informal observations were not videotaped, but I did take fieldnotes.

The videotaped classroom observations served three purposes. First, they provided windows into teachers' "enacted" practices and served as a measure against which to relate their “espoused” practices during the interviews. The enacted-espoused dichotomy has been proposed by Ernest (1991) to explain the discrepancies found in what teachers say and believe and what is observed in their practice. Second, the observations allowed access to teachers’ instructional practices with contextual problems, including their organization of a lesson, their questioning practices, and their interactions with students. Finally, the use of video captured the complexities of teaching with contextual problems and allowed for in-depth analyses of the complex interactions that occurred in their classrooms over time.

In this study, the interactions of concern were those between teachers and students engaged in mathematical lessons centered on contextual problems. Teachers were asked to choose two lessons for observation that they thought incorporated *contextual* problems—a word I refused to define a priori (so that the teachers’ lesson choices reflected how they interpreted the meaning of the term). Three of the six teachers who participated in the study were not formally observed, because they determined they would not be teaching any lessons that fit that description. The other three teachers (across two schools) all chose precalculus lessons for observation even though I had not specified a particular mathematical domain. For each of the one or two lessons chosen, the teacher was observed for the duration of the lesson, 1 or 2 consecutive days. Each lesson was observed being taught to more than one precalculus section, except for two lessons that were observed only once. As a result, 16 videotaped class period observations in April and May 2004, distributed across three teachers (see Table 2), documented the nature of the teachers’ instructional practices with contextual problems and provided the stimulus for post-lesson interviews. In addition to the video records, I collected handouts, assessments, and assignments during observation days.

Table 2: Levels of Participation

Watercliff Academy		
Teacher	Interviews	Observations
Hank	B/B Int. & graphic organizer	2 (Informal)
Gary	B/B Int. & graphic organizer Pre & Post Interviews of L1 Pre & Post Interviews of L2	3 (Informal) Bird Problem (L1): 3 precalculus classes □ 2 days Wind Chime Problem (L2): 3 precalculus classes □ 2 days
Constantia Ridge		
Teacher	Interviews	Observations
Tom	B/B Int. & graphic organizer	2 (Informal)
Cathy	B/B Int. & graphic organizer	2 (Informal)
Rhonda	B/B Int. & graphic organizer Pre & Post Interviews of L3 Pre & Post Interviews of L4	3 (Informal) Swing Problem (L3): 1 precalculus class □ 1 day CO_2 Problem (L4): 1 precalculus class □ 1 day
Diane*	B/B Interview Pre & Post Interviews of L5	2 (Informal) Foul Shot Problem (L5): 2 precalculus classes □ 1 day

* Diane was asked to complete a graphic organizer but did not return it to me.
Note: B/B Int. = Background and Beliefs Interview; L# = Videotaped lesson number.

To summarize my data across both schools (see Table 2):

- Background and Beliefs Interviews with 6 teachers, 5 of whom produced graphic organizers.
- Pre- and post-lesson interviews with 3 teachers for the 5 observed problems.
- Videotaped observations of 6 different precalculus sections (2 once, 1 twice, and 3 four times) for a total of 16 class periods.
- Videotaped observations of 10 lessons focused on different problems (the Bird Problem, Wind Chime Problem, Swing Problem, CO_2 Problem, and Foul Shot Problem; see chapter 6 for a description of the 5 problems).

A Grounded Theory Approach

A grounded theory approach (Glaser & Strauss, 1967) guided the analysis of the interviews and videotapes. This approach is a systematic set of procedures to develop an inductively derived grounded theory about a phenomenon (Strauss & Corbin, 1990). It assumes that the processes of data collection, coding, analysis, and theorizing are simultaneous, iterative, and progressive. The researcher is responsible for developing theories based on observations of and interactions with the individual or group. The theories are "grounded" in the data, but the researcher adds her or his own insight into why those experiences exist and thus creates the theory.

Often grounded theory attempts to reach a theory or conceptual understanding through inductive processes (Strauss & Corbin, 1994). Inductive analyses, consisting of a series of coding activities, are used to develop theories by closely inspecting data to reveal commonalities and themes. From the coding activities, concepts (names given social patterns in the data) are generated, allowing the researcher to build descriptions and interpretations. Subsequent coding serves either to confirm the created categories or to refine, extend, or modify them to fit the new data. As the research study progresses, coding subsides and analysis and theory building become more dominant. In this study, I devoted many hours to coding the data, categorizing those codes, and reflecting on their implications. This process was important for identifying concepts and locating supportive excerpts from the data. My goals in using inductive methods were to identify concepts, describe them, and compare them across participants (Charmaz, 2000).

Glaser (2002) cautions that many who claim to use grounded theory stop once they have rich descriptions of the concepts identified in their data; they fail to conceptualize their data. "Conceptualization in grounded theory must be done as a careful part of theory generating and emergence, with each concept earning its way with relevance into the theory" (p. 24). In this study, I sought to move beyond mere description and to conceptualize the concepts in terms of the literature and theories available, thus producing new theory.

Data Analysis

Qualitative data analysis is often explained as a process of organizing data, breaking data into manageable units through codes, synthesizing the data through categories, and finding patterns and irregularities among all data collected. This reading and rereading process occurs both during and after data collection (Bogdan & Biklen, 1998; Glesne, 1999; Wolcott, 1994). In this study, analysis was not one step in a linear process but was instead embedded in all aspects of the study, from data collection to writing. Writing is itself a "method of inquiry" (Richardson, 2000; E. A. St. Pierre, personal communication, September 9, 2002), and I considered it an important part of the analysis process. Writing as a method of inquiry "provides a research practice through which we can investigate how we construct the world, ourselves, and others, and how standard objectifying practices of social science unnecessarily limit us and social science" (Richardson, 2000, p. 924). Through synthesizing, organizing, and representing my data in writing, new interpretations and realizations surfaced and further informed the study. Thus writing practices throughout the research process not only allowed me to document and report events and observations but also fueled my "headwork" (Van Maanen, 1995) with the data and theories.

In this study, the analysis included: (a) transcribing and reading transcripts during the data collection to shape subsequent data gathering and the direction of the study, (b) coding and categorizing data from transcripts using a constant comparative method of inductive analyses, (c) analyzing teachers' graphic organizers for structure and content, (d) watching and rewatching videotapes of teachers' classrooms and identifying their practices, and (e) writing. Each of these analyses is detailed below.

Background and Beliefs Interview

Analysis of the Background and Beliefs Interviews began with the reading and rereading of transcripts highlighting those phrases that seemed to be important. I used a constant comparative method of inductive analysis that required "constantly comparing" pieces of data to generate "explicit categories which help provide an understanding of the data" (Groves, 1988, p. 277). The first phase of the process involved coding the data set and comparing those codes. Initial codes included belief, technology, mathematics, pedagogical practice, learning, and so on, as well as a number of "in vivo" codes (Glaser, 1978). In addition to coding the transcripts, I highlighted all instances in which a teacher used the following terms: *context*, *real world*, *applications*, and *modeling*. As I coded and highlighted the interviews in this way, I listed the codes that were recurring across the participants. In other words, as I noticed "similar talk" among the teachers, I made a note of it. With this list, similarities and differences between what the teachers emphasized became apparent. It became possible to attend to presences and silences within and across the six participants. Subsequent coding served either to confirm the created themes or to refine, extend, and modify them according to the new data. After completing this initial process, I returned to the transcripts and highlighted the words that the teachers had used when talking about mathematics in context, applications, and modeling: *use*, *practical*, *useful*, *motivate*, *interesting*, and so on. Because a significant part of this study involved the teachers' interpretations and understanding of the three terms above, paying attention to their descriptions was important. After completing this second coding process, I grouped the codes into categories that aided in the description of the participants, their beliefs, and their instructional practices with contexts.

Graphic Organizer

I analyzed each organizer in terms of its structure and content. In analyzing the structure, I paid attention to links that were drawn or not drawn between terms; the use of linear, cyclic, or overlapping structures; and the placement of words in the structure. The content of each graphic organizer was analyzed in terms of the teacher's verbalizations concerning the content (at a local level) and in terms of domains—mathematical, pedagogical, and so on (at a global level). Of each organizer I asked, "What, in general, is this representation about?" Asking this question led to a "free-association" exercise wherein I jotted down ideas that seemed associated with the representation. When I did this exercise for each of the five organizers, similarities and differences became apparent. In making comparisons of the organizers across participants, I did not make judgments of right or wrong, or good or bad.

Pre-lesson and Post-lesson Interviews

I specifically analyzed the pre- and post-lesson interviews for teachers' goals and reflections on lessons. Identifying teachers' intended goals for the lessons taught, their lesson plan, and their overall reactions to the lesson in post-lesson reflection helped shape the analysis of the videotaped lesson observations and pointed to segments identified by the teachers in lessons in which either positively or negatively "things did not go according to plan." Other details in these interviews—the location of the lesson in the curriculum sequence and the anticipated activities of students and teachers—aided in the description of the lessons but were not otherwise analyzed.

Videotaped Observations

Five problems were observed (and videotaped) being taught, some across multiple class periods. I produced a detailed lesson outline (and selectively transcribed segments) noting the work and talk of both the teacher and students for each lesson to lay out the overall organization and implementation of the problem. Problems whose teaching was observed multiple times were reviewed for significant shifts in lesson delivery, and identified differences were noted on the lesson outlines. To focus my analysis of teachers' classroom practices in the videotaped lessons, I began by segmenting each lesson into manageable chunks. This segmenting was done according to the sequencing of the lesson: introduction of task, or the *launch*; teacher-guided instruction, or *direction*; student activity, or *exploration*; culminating activity, or *regrouping*; and *summary* or conclusion of the investigation. These segments were used in organizing the lesson outlines and for framing analyses; some lessons did not include all of these segments (e.g., not all lessons were summarized and concluded in the classes I observed).

Lerman (1998) proposed the metaphor of a zoom lens whereby what the researcher chooses as the object of study becomes

> a moment in socio-cultural studies, as a particular focusing of a lens, as a gaze which is as much aware of what is not being looked at, as of what is. . . . Draw back in the zoom, and the researcher looks at education in a particular society, at whole schools, or whole classrooms; zoom back in and one focuses on some children, or some interactions. The point is that research must find a way to take account of the other elements which come into focus throughout the zoom, wherever one chooses to stop. (p. 67)

I adopted Lerman's approach, zooming back to look at the classroom as a whole, paying attention to the organization and practices in use. I also zoomed in to analyze in each lesson segment the teacher's pedagogical actions and verbal instruction. This close analysis was based on two assumptions: (a) teachers' practices are taken to include both teachers' talk and action, what they say as well as what they do; and (b) teachers' talk is observable and manifests itself in action; what teachers say changes the context in which they are working and ultimately their actions in that context. In this way, teachers' utterances are actions—part of what teachers do. The utterances shape mathematics lessons and, in part, constitute teachers' instructional practices. Thus, teachers' verbal utterances and shifts or stases in the videotaped lessons as a result of the utterances were central. The verbal utterances were not interrogated on a linguistic level; they were viewed only in terms of their impact on the mathematical investigation. For example, I analyzed whether the utterance *opened* or *closed* the problem or the student's question; whether the focus of the utterance was on the *mathematics* or the *contextual* features of the problem; and whether the utterance was intended to assist students in formulating the problem. Thus, in analyzing teachers' instructional practices, I specifically looked for these aspects, noting which dimensions a specific lesson segment embodied. For example, in an *exploration* segment a teacher's practice might be described as: using technology, engaged in talk focused on the mathematical components of the investigation (ignoring contextual features), and questioning in ways that allow the students to formulate the problem.

CHAPTER 4

THE TEACHERS AND THEIR BELIEFS

> I have made a conscious effort to be "the researcher I would want researching me." As such, I have stayed close to the data, using teachers' quotes in contexts that complement their original use. I have wrestled with description, grappled with theory, and considered its implications. I have thought about the knowledge I am producing: "Where it will go, and what it will do." Where might it lead me? Where might it lead others? (Researcher's Log, September 6, 2004)

In this chapter, I introduce each participant, describing his or her goals for teaching and highlighting some of his or her beliefs. Ernest (1989) and Raymond (1997) have proposed that to understand teachers' knowledge and practices, one must consider their beliefs about mathematics and its teaching and learning. I also considered the teachers' beliefs about contextual problems since such problems were the focus of the present study. These teachers' beliefs are reported so as to orient the reader to the chapters that follow and to provide a foundation for thinking about the teachers' notions of terminology and their instructional practices. I introduce each teacher and then present the teacher's beliefs according to the following order: beliefs about mathematics and its teaching and learning, and beliefs about the nature and role of applications, modeling, and contextual problems.

The Teachers

Hank

Hank was energetic and enthusiastic when doing and teaching mathematics. In his class, every activity was exciting and rewarding; wrong paths were opportunities to learn something unexpected. For Hank, a student's interest and perseverance in mathematics were of greatest importance.

> The biggest thing I want is for kids to have experienced some really tough problems, really tough situations . . . and look back and say, "that was a hard class. I learned a lot. I'm not going to be able to remember it all. But what I do remember most of all is that I was able to get through some tough situations." (B/B Interview; May 19, 2004)

Hank believed the teacher was responsible for presenting complicated problems so that students could tackle them: "What I can teach you is how to think, how to persevere." The students were responsible for fueling the class with their curiosity and interest in mathematics and the world around them. Hank's desire to encourage curiosity in his students was not surprising given his own curiosity and love for mathematics. He loved the subject and loved to talk about it. He repeatedly stated that he put stock in Galileo's claim: "Mathematics is the language with which God has written the world." For Hank, teaching mathematics was about helping students learn to "translate into and out of the language of mathematics—between the language of the world and the language of mathematics"—and thus to make connections between the two. "Truly understanding" a concept was a result of "making connections between bizarre things," such as the connection between Pascal's triangle and the number of handshakes in a crowd. Hank believed that through careful "organization," the teacher could "bring some order into [students'] understanding [of] what's going on in the world." After all, mathematics could be viewed as the "science of patterns" and thus spawn curiosity if students began to notice changes in the patterns around them in the world. In this way, mathematics was inextricably connected to the world and the world to mathematics: Mathematics is everywhere if we are sensitive enough to recognize it.

Hank's views about the use of contextual problems, in terms of applications and modeling, were positive. He had co-taught a mathematical modeling course at Watercliff Academy in the mid 1990s and had found the experience "fun" and exciting: "I just thought it was very, very cool that you could tackle something like [complex modeling problems] on the high school level." Though the course was no longer offered at his school, Hank was committed to contextual projects and assigned six to eight such projects in each of his mathematics courses each semester. He described his courses as "scattered, spastic, and high energy" and strove to create energy for his students through their engagement in contextual projects. Hank spoke of "contextual stuff" as "anything that's not textbook, and you just have to sink your teeth into it and play." He actively sought ideas for projects and kept stacks of resources and problems in his office. He commented, "If it's been a month since I did something, I know it's time for me to start flipping through my stacks and try to find something." His ideas for projects were not systematic. He did not have a set of problems he used repeatedly in his classes; he simply waited for inspiration from something going on in the world or from something he had read somewhere. For Hank, the world was filled with mathematics:

> When you watch a car commercial or an old western, . . . you see that wagon or car going down the road, and the wheels are spinning the wrong way. And why? Why? Why is that? And it has everything to do with how quickly the wheels are spinning and the fact that you're filming at one rate and the wheels are spinning at another rate. And it's actually the interference of two trig functions. [You are] sitting at a concert, and you hear the musicians warming up. And you know they're really close to being in tune with each other when you start to hear the beats. And it's the interference—destructive and constructive interference—of sound waves, which is nothing more than trigonometry. . . . So it's just knowing math and then having an eye for what I call projects. When you hear something or when you see something different, your mathematical curiosity kicks in, and that helps you understand why it is. (B/B Interview; May 19, 2004)

Hank commented that it would be difficult for him to teach a course without helping his students make connections between mathematics and the world.

> I'm not a pure theorist. It just doesn't make sense to me unless I'm using situations to help drive my students' learning. . . . One of the great disservices that we do to our students is ask them to learn this and this, and learn this and this, and they all come as these separate towers of knowledge. We've got to find a way to help kids see the connections between things. . . . I see the connections, and then my job becomes to help my students see those connections. (B/B Interview; May 19, 2004)

Hank believed that teaching mathematics using many different contexts and showing students many different solution strategies helped them "hold on to [the concept] even though they don't remember the details—[they] remember the pattern." To this end, he believed that the best classes he had taught were those in which he had taken one problem and solved it a number of different ways instead of teaching five different problems during the hour. He believed the context in a problem pointed to the "actual utility of the mathematics." Therefore, exposing students to a variety of contexts and showing them multiple perspectives was central to the work of teaching. He commented, "It's a crime not to go there."

<u>Gary</u>

At Watercliff, Gary was highly regarded by his fellow teachers and had earned the right to make decisions about curriculum and instruction. Though Gary seldom used the textbook to support his teaching—in an interview he was unable to name the precalculus text he was using—his students excelled. His favorite course to teach was AP Statistics "because it's always applicable. There's never the question about why in your life you'll probably do this. It's all around you all the time." He also commented that in teaching statistics "you always have the ability to use simulation or hands-on kind of stuff to get that understanding more concrete; so you're still closer to reality in some sense." Thus the relevance of mathematics to students' lives was at the heart of his teaching philosophy. For Gary, learning how to ask good questions was "the heart of learning," and he worried that he did not provide enough opportunities for students to cultivate that skill, which was part of becoming a good problem solver. Gary insisted that students write and discuss mathematics in precise, “mathematically correct” ways. Gary commented, “I am very, very, very picky about the language. I will not let them use pronouns, and they get really mad at me for doing that.” His purpose was to help the students gain a better command of the language of mathematics, “to be able to make quantitative statements precisely,” because he believed much of mathematics depended on understanding and using its language. “The inability to use the technical language correctly, and to speak carefully, hurts the idea development.”

In Gary’s larger goals for students, he wanted them to be able to learn on their own, “to look at the world with their eyes open and ask questions, and have reasonable skill at facing an unknown situation without panic.” To this end, he sought problems from the world and orchestrated lessons in ways that facilitated “Socratic discussion” and questioning. He believed that mathematics was “inherently about structure—noticing structure and then utilizing structure to take advantage.” Still, Gary wanted students to be able to recognize and account for “wiggle” in the world, the idea that there is some uncertainty associated with everything, and that “your job is to minimize it, to measure it” and make sense of it.

Gary noted that most of the facts taught in school were useless to students. He believed, however, that facts were extremely useful when taken as a body of knowledge that illustrated how to approach thinking. “That’s why we spend our time doing this.” He believed teachers improved their instruction by teaching courses multiple times and by trying new ideas. He commented that few problems were “directly importable” into classes, but that in adaptation and revision, teachers could bring rich problems to their students. Teaching, like modeling, was a “give and take,” and therefore the teacher must be flexible and adaptable as well.

When talking about and discussing ideas concerning contextual problems—especially, mathematical modeling—Gary spoke knowledgeably and confidently. He had worked with Henry Pollak (a well-known figure in the mathematical modeling world and in mathematics education in general) through the years and often quoted Pollak's work. Other colleagues had also influenced the "modeling" direction Gary had taken as a teacher, and he was surprised at the emotion he expressed when referencing them—my perception was that he had not realized their importance in his life until the interview. Always open to new ideas and problems, Gary had gathered, adapted, and refined a number of modeling problems that he used every year in particular courses. He believed such problems were motivating for some students and that for some they provided a concrete link to another idea and thus helped them remember things better.

<u>Cathy</u>

Cathy was an enthusiastic teacher who truly loved the beauty of mathematics. Her eyes sparkled as she spoke of "cool" mathematics and those teaching moments in which she could see students "get it." She loved mathematics and loved teaching it, particularly with the more advanced and enthusiastic students.

> What I love about teaching is—I love the material. I like the self-containedness of starting at the beginning, knowing it's going to end, [and] having a finite amount of time to make things happen. So if it goes really bad, it ends! If it goes really well, it's kind of sad that it ends, but [next year] it starts over again. (B/B Interview; May 12, 2004)

For Cathy, mathematics learning was about developing new "tools" and mastering different "techniques" —technological, verbal (words), analytical, numerical, and graphical (TWANG, her students' acronym) —and she worked diligently to provide the students with opportunities to practice and use those tools. In this way, mathematics teaching was a means for providing students with "toolkits" —a term used by many teachers in her department—that they could use to solve problems both in and out of school. Cathy wanted students to motivate their own study by recognizing the inadequacies of their current toolkit. She organized lessons so that students would realize that the new task at hand required something they did not know, something more powerful than the tools they already had, and thus they would deduce what the new, needed "technique" might be. Central to this organization was a balance between providing enough information so that students could tackle a given problem and yet withholding enough information that the problem was challenging and interesting; Cathy wanted students to "figure it out rather than [her] telling them."

Most important, and evident in Cathy's teaching, was her belief that students needed to have adequate time to learn: time to make mistakes and try again, time to think and reason, time to try out different tools and select appropriate ones, and time to make connections and applications. "The only way you get good at [contextual problems] is doing them. And the only way to get good at them, by doing them, is to get enough time to really just do them wrong, get some help, [and then] do it right." Cathy believed that many applications were artificial but still useful in that they could help students understand mathematics as a body of knowledge related to the real world. And in this way, applications served to attract students to the theoretical side of mathematics.

In discussing contextual problems and their role in teaching, Cathy believed such problems were critical. "I don't think there is very much need at the high school level to study math for math's sake. I think if we can't tell you why anyone would ever want to do this, then we shouldn't be doing it." She observed that there were exceptions to her comment; for example, infinite series and limits of functions in calculus. She noted, "I include functions as part of the real world even if they don't describe anything real," explaining that such topics in mathematics were objects worth thinking about for their own sake because "eventually functions are used to describe real things." In this way, she believed that mathematics could be real and powerful in the absence of contexts. Finally, Cathy noted that using contextual problems in her teaching was important. "I think [students] need to keep being tied back to the real world or they lose their motivation."

Diane

Diane was a confident teacher who was very dedicated to teaching her students. Her dedication was fueled by her love of mathematics. She enthusiastically taught a variety of courses (preferably three different ones each year), including a mathematical modeling course, and found teaching to be both fun and rewarding. She believed mathematics was a useful subject. Even if her students did not see its direct relevance to their lives, she sought to instill an appreciation that would continue to engage and motivate them in further study. Using what she called interesting and challenging problems, Diane worked toward her goals as a mathematics teacher:

> I wish they would persevere so that when they look at a problem, they won't just say, "I can't do this" and move on to the next problem. That they understand the value of writing things down and mucking around and making progress. And that sometimes when you write things down, something good turns out, and sometimes it doesn't. But that they can reason and that they don't want to just be able to do something, but that they want to understand why. (B/B Interview; May 11, 2004)

Diane spoke of collaborations with colleagues as integral to her work as a teacher and relied on those collaborations (and interesting applications) for inspiration on ways to

> bridge what [students] learn in mathematics and what they learn in their science classes so that their overall learning experiences are more rich, and they can more easily take what they learn in one place and apply it someplace else. (B/B Interview; May 11, 2004)

Diane very rarely taught "math for the sake of math" and believed that the students needed to understand how mathematics was useful, even if it was not useful for them. The students also needed to be able to manipulate symbols, and Diane believed it was a disservice not to teach those skills regardless of the students' skills with applications and modeling. She remarked that students needed to use mathematical language properly and share their ideas in class discussion. She believed these practices kept the students accountable for their own learning.

Diane spoke enthusiastically about the use of contextual problems in her teaching. She believed contextual problems could motivate students and help them see its usefulness.

> I think it captures the imagination of the students and keeps them interested, and they can see some use to it. . . . I think what we try to do is make connections. And I think applications help us make those connections better, which helps math hang together better. And I think it also helps the kids make math relate to what they're doing in their other subjects . . . to have those "aha" experiences. (B/B Interview; May 11, 2004)

She elaborated that it was the teacher's primary job to provide contexts for mathematical explorations. Students were expected to understand and make interpretations from those contexts; the teacher could aid students, but not make the interpretations for them. Diane believed that some mathematics could not be taught in contexts, but that doing so was desirable.

Tom

Tom was a dedicated and hard-working teacher. He spent hours developing problems and projects for students and sharing ideas with his colleagues. He was always available to help students and tried to challenge them at all times. For Tom, central to teaching was "posing problems [or] fairly controlled questions that try to elicit some sort of response." The teacher should always have specific goals in mind, and Tom believed that the teacher's job was to help the students figure out the mathematics they needed to accomplish those goals or to develop new mathematics when necessary. Tom commented,

> Oftentimes [students] see mathematics presented as a completed idea already. So all the decisions have been made, and they don't recognize that there are all these points at which you have to make a decision. . . . So, in my teaching, I try to point those out. That here we have to make a decision, but unfortunately we don't know enough right now to make a good decision. But we need to recognize that we're making a decision so that if we don't get things we like, here is a point at which we can come back and think. (B/B Interview; May 11, 2004)

Tom believed that thinking about, reasoning with, and discussing mathematics were important for fostering students' learning, and he sought opportunities to "set them up" and "push the edge" towards those ends. He held high expectations for students and expected them to successfully tackle problems they had not been taught to do. He wanted the students to share their ideas but also to be able to write about the mathematics in correct mathematical forms including equations, and in terms of what mathematics meant within particular problems or contexts. Tom's larger goal for students was "for them to be able to take what they've learned and move forward with it; to be able to use it and to be able to recognize it in different contexts; to make that transfer." He acknowledged that such transfer did not happen automatically, and he worked hard to provide opportunities to help students make such leaps.

Tom held very specific views of applications and modeling and spoke extensively about their incorporation in his teaching. He had developed a modeling course nearly 15 years earlier and had found much satisfaction in teaching it. "It's the first time, and maybe the only time, the students get a chance to show what they know and what they can do. . . . They really get an opportunity to create their own mathematical ideas." Tom also believed that the course prepared students for formal mathematics and proof since it required them to start with conjectures, something he felt was missing in most traditional high school mathematics courses.

> When you create a mathematical model, you are making some conjecture. "I think this is a good representation and I need to convince you that it is. The results I am drawing from this are valid." And I think the logic of that is the same kind of logic in the more formal proof, theorem proof, [except] the conjecture is the missing piece [in non-modeling courses]. (B/B Interview; May 11, 2004)

In further discussion of contextual problems, Tom noted his belief that such problems were beneficial because the students could get excited about mathematics, and in open-ended problems, find solutions that were their own: They could take ownership of the mathematics. He commented that his goal was for mathematics to make sense to students, "that they can see where things come from and why they work. . . . I want them to be able to see the mathematics in a situation and to use it properly." His job was to put the students in the right settings so that they could make connections to other mathematics, science, and the world. One way to accomplish that goal was through contextual problems. Such problems could be challenging for students: "They don't do it well initially—you're always disappointed. But it's a hard task [for students], and you don't get good at it unless you do it." Tom believed that teaching with such problems was "very worthwhile."

<u>Rhonda</u>

For Rhonda, teaching was a privilege and a passion. She had stumbled into the profession and loved her work. "I enjoy the courses that I've been able to teach and look forward to seeing the kids everyday. You always wish there was more time you could spend [with the kids] rather than the political things." Rhonda came to Constantia Ridge during the time in which the department was moving towards a modeling emphasis in their courses, and she had enthusiastically joined the effort. She spoke of the freedom within her department to try new things, and she actively sought rich problems to use in her courses. Modeling and applications were an ever-present thread in her teaching, and her students always worked in groups except during major assessments. Rhonda described her classes as noisy and active with "a questioning sort of atmosphere rather than a lecturing one." She believed that students should be actively engaged in mathematics "so that they feel ownership of what they've learned, so it's not just something 'she told me to remember.'"

Rhonda acted as a facilitator or moderator in her courses and worked to ask the "right questions" so the students could get started and work on their own. She often answered students' questions with a question and regularly deviated from her plans to pursue questions or suggestions from students. She believed such deviations were worth the time if they helped the students learn. Rhonda was interested in students discussing and writing mathematics. She did not want them to "rehash the math" in their writing. Instead, they were to talk about what they understood and how they came to that understanding. Rhonda believed strongly that students should receive credit for doing mathematics regardless of their outcomes; thus she was very focused on students' thinking processes as well as their mathematical products. She gave grades to students "if they explained something to the class, if they asked a good question, [or] if they were productive when they had time to work on their own." When asked about her goals in teaching mathematics, Rhonda commented,

> There's certainly a basic amount of content. But I guess what I really want them to leave with is the confidence that they can use the math that they have learned, that they can use math outside the math classroom. [That] they're confident in their ability to do that. That they have the technology tools that they have built up. That they have learned something about modeling skills. That they know how to ask questions. [And] that they know how to talk about math and write about math. (B/B Interview; May 4, 2004)

Rhonda held high expectations for her students and believed that if they had really learned a concept they would be able to use it in other places. In this way, mathematics would be useful to them, and perhaps they would pursue a technical career after high school. To that end, Rhonda used “rich” problems and engaged students in doing experiments, analyzing data, using calculators, writing solutions, and questioning their assumptions. She believed that such problems made mathematics seem more useful to students and helped them remember what they were learning. Besides, they made mathematics more enjoyable. But students needed time to work with such problems.

> If we do all these different things [mathematically], and do it in a context, and do it a lot—that seems more natural. And it’s just something that they get good at, which, I guess, is a focus of mine. (B/B Interview; May 4, 2004)

Rhonda also encouraged the use of correct vocabulary in her classes. Allowing students to use “silly terms” was not useful in her opinion. “We use the words that everybody is going to understand when [they] leave and are talking about math in other places.”

Teachers’ Beliefs About Mathematics

The teachers who commented on mathematics held different beliefs about it. Hank believed mathematics was a language that described the world and also the science of patterns. Gary believed mathematics was inherently about structure and using that structure to understand things. Cathy believed that mathematics was a source of tools that could aid one in explaining the world. All of the teachers believed that mathematics was a body of knowledge that included content, processes and procedures, and skills requiring practice. Mathematics could be useful in explaining the world and other mathematics. It was complicated and complex, and students needed time and practice in developing both confidence and proficiency with it.

The teachers also shared the belief that mathematics was powerful. What varied were their interpretations of how it was powerful. Tom noted,

> The power of mathematics is the context-free nature of it. That the quadratic equation has the same solution regardless of what it’s describing [in the world]…

> that once I get [the problem] into a mathematical form, the mathematics operates in a context-free environment. That's its power. (B/B Interview; May 11, 2004)

Likewise, Gary commented,

> The difference that math brings is that it focuses on structure much more than some of the other disciplines are forced to . . . the fact that algebra is simultaneously a language, and a set of computational rules, means that I can convert sentences that do not appear to be equivalent. I can convert one into another and know the equivalence by virtue of the structure of the language I am dealing with. I couldn't do that in English. I can't transpose a sentence, and restate that sentence, and guarantee that it means the same thing. In math, I can. That's a luxury and a burden, but it's a fact. (B/B Interview; April, 16, 2004)

Still, Gary noted that mathematics allowed for "wiggle room" where interesting things could happen. Cathy and Hank, unlike Tom and Gary, believed that the power of mathematics lay in its ability to describe the world—in its potential for explaining real things. Mathematics allowed one to understand their surroundings, to recognize patterns, and discern changes in those patterns.

Teachers' Beliefs About Mathematics Teaching and Learning

The teachers also expressed beliefs about teaching mathematics and how students learned. The most prevalent belief they held was evident in their goals for mathematics teaching. All of them spoke of their work as aimed at instilling confidence and perseverance in their students. Every teacher wanted his or her students to tackle complicated problems and make mistakes, but also to realize that they could do mathematics. The teachers worried little about their students remembering every mathematical formula or rule; they were much more concerned with the students gaining confidence. Thus teaching was about putting the students in settings where they could develop confidence, struggle and persevere, and be successful.

These teachers did not believe that teaching could be accomplished in a recipe-like fashion. One must have goals—for example, teach precise language and specific content—but one must also be open to following students' paths and trying new things. The teacher and the instruction needed to be flexible and adaptable. The students needed to feel ownership of the mathematics.

These teachers believed that connecting mathematical ideas—across other disciplines (physics, other sciences, etc.), across topics within the precalculus course, and across topics in other mathematics courses—was central to teaching. The teachers discussed the potential that mathematics offered in connecting ideas and believed students needed opportunities to make those connections. Choosing problems that accomplished this goal and that engaged the students was important. Students learned mathematics by engaging in problems and struggling. Important for learning was making mistakes and having time to rethink one's work and try again. The teachers believed that students needed multiple opportunities to work with rich problems and that "real learning" was evident when students could take mathematics and use it in different settings—in science class, in another mathematics class, or in the world.

Teachers' Beliefs About Contextual Problems

Selecting Contextual Problems

Central in planning a lesson is the selection of a good problem. A teacher's instructional practices are largely dependent on the nature of the problem or topic being explored and the mathematics to be worked with. All the teachers in this study believed that good contextual problems were rich mathematically. Such problems allowed for multiple solutions and multiple solution paths. Good problems were those that students could get started with. Diane commented that she selected problems “related to mathematics they can do pretty well and at least make a start, . . . but I’d like there to be a challenge without it being so frustrating that you lose the value.”

Cathy believed that it was important for a given contextual problem to be “understandable.” She commented, “Some problems are too esoteric, [and] the kids can’t wrap their mind around [it].” As an example, she described a favorite problem about an undershot waterwheel that she did not use in her teaching, because her students had no concept of how a waterwheel worked. Cathy thus tried to find contextual problems that the students could understand and that they were interested in— “it helps if they care.” She also stressed that the problem needed to be challenging or involve a “kicker in the end.” Otherwise, the problem would be a waste of the students’ time. She suggested that “the stronger the kid, the more ill-defined [the problem] should be.” “So, you’ve got to have something in there that’s worth their time, that they care about the answer one way or the other, or they can just appreciate the beauty of it.”

Gary believed that good contextual problems needed to first of all be fun, but also rich: “something that offers the opportunity for more than one path, or if not more than one path, varying results through that path.” Likewise, Tom stressed that good problems “should be at least somewhat interesting to the students” and should “lend themselves to a lot of different approaches. That students don’t have to know a certain piece of mathematics to be able to do it.” He elaborated,

> The main requirement is that it allows for a number of different approaches from students with different levels of mathematics so that you’re not getting everyone doing the same thing and getting the same results. What makes [a contextual problem] interesting for the students is that their solutions are quite different [from each other] because their models are different. (B/B Interview; May 11, 2004)

Tom believed that contextual problems yielded different results depending on how they were posed by the teacher. When the teacher is interested in students learning a particular piece of mathematics, the problem and its context might be posed so they are "very controlled and very specific." At other times, as in the modeling course, the problem is posed as "very open-ended," and students are challenged to see what they can do. Thus, the potential of a problem and how "good" it was depended on how well it was posed and whether it accomplished the goals of the teacher. Rhonda added that good problems encouraged classroom discussion. Students thought about contexts and problems differently, and good problems encouraged the students to share their approaches and ideas with one another.

Finally, good contextual problems were those that allowed for teaching more than one concept—problems that accomplished many goals. For example, the CO_2 Problem (described in chapter 6) was considered a good problem because it (a) was mathematically rich (it involved aspects of trigonometry, exponentials, sums, linear regression, residuals, graphical representations, logarithmic re-expression, and so on); (b) was based on a large amount of real data, yet students could work with it; (c) required students to connect mathematics in ways new to them; (d) highlighted features of the graphing calculators and gave students practice using that technology; and (e) provided a context for reviewing other minor mathematical points, like the importance of data lists being in parallel forms to facilitate comparison.

In sum, the teachers unanimously agreed that it was the teacher's job to select good problems for use in instruction. They believed that the teacher should actively seek contextual problems that could be of use. If they tried a new problem and it proved beneficial to students' learning, it was shared with colleagues. If it needed adaptation, it was modified and tried again. If it "flopped," it was thrown out. All of the teachers (Hank, to a lesser degree) had contextual problems they had developed over time and used each year in their classes, but new problems were continually being developed. Thus their problem sets were always evolving and expanding. The teachers reported that this innovation—particularly the teachers at Constantia Ridge—kept them enthusiastic about the mathematics content they taught.

Using Contextual Problems Often

The teachers in this study believed that many teachers hesitated to use contextual problems regularly because they had "tried it once and it didn't work." The teachers spoke of the danger of trying something only once. If one's goal was for the students to be skilled at tackling large modeling problems, then they had to be given numerous opportunities to tackle such problems. Tom and Cathy both stressed that the incorporation of contextual problems had to happen regularly and often if students were to cultivate an ability to solve them. They cautioned that inserting such problems "once in a while" would only leave teachers disappointed and students frustrated. Teachers needed to be patient and persistent in their use of such problems to provide students ample opportunity to engage with the problems, to make mistakes, and to try again. Hank and Rhonda also commented that when pursuing one's goals for students' perseverance or further development of mathematical language use, one must consciously work toward that goal every day. They believed it could not be the focus of one's attention part of the time or one day a week. If a goal was important, it needed to be addressed daily.

The belief held by these teachers that contextual problems should be used often may surprise readers since three of the six—Hank, Tom, and Cathy—were not observed teaching lessons that involved contextual problems, because they did not plan to teach any such lessons during April and May 2004. The teachers, however, were, in fact, teaching such lessons during that time but later explained how they reasoned that the lessons would not be appropriate for my study. Hank explained that he was only teaching the Advanced Placement (AP) Calculus course that semester, and since the AP exam was in the first week of May, he planned to alter his instruction for that month to prepare students for the exam. He planned to return to regular instruction following the test but was concerned that the students would not be performing at their potential since they viewed the exam as the end of the year. Tom reported that he only taught statistics in the spring, a course he defined as “always contextual,” but he commented that the course was “different enough from ‘traditional mathematics’ that [he thought] it would not be appropriate [for my study].” And, since there was ample opportunity for me to visit more traditional courses (precalculus and calculus) taught by the other teachers in his department, he reported that he would not be teaching any lessons relevant to my study. Cathy was teaching a precalculus course that was rich with contextual problems, but she noted that in late April and early May the students were completing a unit on trigonometry and were not doing much “in context,” by which she meant “modeling some real world phenomena.” She elaborated on the unit and its lack of contextual problems:

> Trig identities are just cool and useful, but they don’t help you model a Ferris wheel! When we do the unit circle [in the same period of time], again, there isn’t much reference to physical reality. So during those lean weeks, I keep class interesting in other ways: by using technology a lot (both calculator and other software), by sending the whole class to the board to work problems, by making up games (Jeopardy-like, or the “unit circle race”) that give them some reason to care. . . . We do have interesting modeling problems in trig (predator/prey, the swing lab, the Ferris wheels (single and double)), but there are weeks at a time when we aren’t doing that. (e-mail communication; December 8, 2004)

Cathy was also aware that Rhonda was planning to teach “the swing lab” while I was there to observe, and she did not think I would be interested in seeing it twice even though she, too, would be teaching that lesson. Obviously, Tom and Cathy had particular ideas about what I meant by “contextual problems” and about what would be appropriate for my study. As a result, even though they taught with contextual problems during that time, each determined that he or she would not be teaching a lesson appropriate for my study, as did Hank. So although I did not formally observe these three teachers teaching with contextual problems, their commitment to and regular use of such problems supports their belief that contextual problems needed to be taught often if students (and teachers) were to become successful at working with them.

Closing Comments

Even though teachers' beliefs were not the focus of this study, research indicates that to some degree teachers' practices are influenced by their beliefs and vice versa (Carpenter & Fennema, 1991; Cooney, 1994; Ernest, 1989; Gamoran, 1994; Leder, Pehkonen, & Törner, 2002; Raymond, 1997; Thompson, 1992). Therefore, I took teachers' beliefs to be important in framing and understanding their practices. This chapter highlighted these teachers' beliefs about mathematics, its teaching and learning, and contextual problems. The teachers were excited about their work as mathematics teachers, and their conversation highlighted their beliefs about teaching with contextual problems and their thinking about such problems because such problems were central in their work and because they knew the focus of this study. These teachers' beliefs orient the reader to the chapters that follow and provide a foundation for thinking about the teachers' notions of terminology (chapter 5) and their instructional practices (chapter 6).

CHAPTER 5

TEACHERS' NOTIONS OF TERMINOLOGY

In this chapter, I describe the teachers' notions of the terms related to contextual problems. I also present and discuss the graphic organizer drawn by each teacher. (One teacher, Diane, did not produce a graphic organizer.) Finally, I synthesize the teachers' notions of the three terms and identify the dimensions along which they described the terms. This chapter addresses the first research question in this study: How do teachers with reputations for using contextual problems define and relate the terms *mathematics in context*, *applications of mathematics*, and *mathematical modeling*?

Muddy Waters: Teachers' Notions of Terminology

Hank's Notions

Hank began by discussing applications:

> A flat-out application—that's in action, it's math touching the world. Where mathematical modeling can touch the world, it doesn't need to . . . that's the backbone, that's where the most pure logic comes from—in the modeling. I can see building mathematical models as intellectual exercises, understanding what's going on. Yeah. I get a real world, but am I *using* it? Mathematics in context is what a lot of textbook authors use as a cheap way out. Let's learn a lot of math theory. And on the side, okay, we've learned it; let's actually put it some place and say, "Okay. Here it is in context. We've learned a theory and now here we are in context." Whereas *application* to me says, "I've got something that I need to figure out, and I'm learning the math to try to go along with it." . . . Mathematical modeling could be either end of this. . . . It is the creation of the mathematical framework. It is the mathematical structure that allows you to understand the application or allows you to go into context. (B/B Interview; May 19, 2004)

To bring more clarity to Hank's notions of the three concepts, I asked him to draw a graphic organizer relating them. He immediately produced a Venn diagram. A reconstruction of Hank's graphic organizer is given in Figure 1. Verbally relating the three phrases proved challenging for Hank because he felt "that they were all shades of exactly the same thing." He was not able to account for all of the overlaps in the diagram (e.g., the small spaces created outside the intersection of applications, APPS, and mathematics in context, CONTEXT, in the MODELING oval).

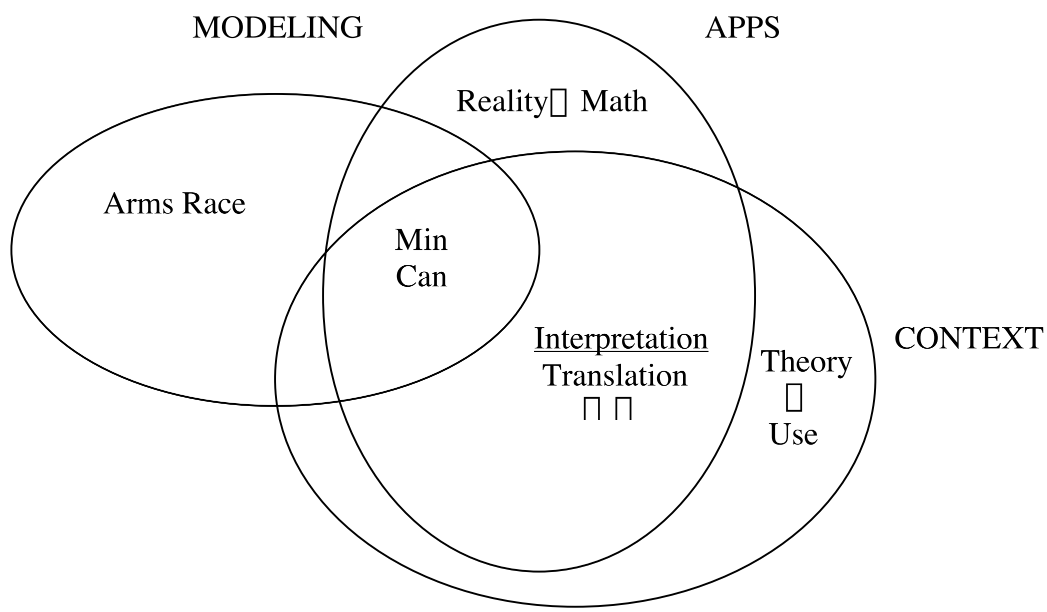

Figure 1. Hank's graphic organizer for mathematics in context, applications of mathematics, and mathematical modeling.
Note: Arms Race and Min Can are abbreviations for mathematical problems (see p. 88).

Hank pointed to the "massive overlap" in the figure between applications and mathematics in context as important in his understanding of the two: "One drives the other," and they differ only in which one is driving at a particular time. He further clarified that the non-overlapped part of applications "is [where] reality drives math" but noted that both applications and mathematics in context "end up using a lot of the same ideas" since the student is translating back and forth between the language of the world and the language of mathematics.

> You're still doing nuts and bolts here [applications outside the overlap], but what starts it is the reality. Whereas math in context, to me . . . is "I've learned the theory, now I'm finding a way to use it." (B/B Interview; May 19, 2004)

In other words, for Hank, math in context meant starting from the theory and "driving into the common" where the interpretation and translation happens both ways: into mathematics and out of mathematics.

On the other hand, mathematical modeling intersected applications and mathematics in context because translation and interpretation with respect to the context or application in the problem were important in creating a mathematical model. In many respects, however, modeling stood alone. Hank explained:

> You've got a situation that needs to build in, and it could have started from reality and gotten into here [applications outside the overlap]. And then I had to build a math model, and I'm doing all my translations back and forth [into the interpretation and translation area of the diagram and out of the modeling area outside the overlap]. All of this can overlap, but it's also possible that you could be doing—there is some translation outside of math, but it doesn't necessarily have to tie into reality. And there is some modeling that doesn't have to deal with application. (B/B Interview; May 19, 2004)

Clearly for Hank the three terms were in fact "shades of the same thing."

A striking feature of Hank's graphic organizer is his self-imposed restriction of these terms to a mathematical domain. For him, the three terms referred to mathematics; they were about problems that might or might not incorporate the real world. They were interconnected and difficult to delineate. Unlike Rhonda's model (presented below), the three terms did not seem to have pedagogical implications. Instead, they were about structuring and deriving mathematics: developing applications from theory, using theory in contexts, or building mathematical models or frameworks. He offered examples of problems that fit within these various categories and chose to include some in his graphic organizer (Arms Race, a modeling problem about the accumulation of nuclear weapons, and Min Can, an application problem concerning optimization of volume in canning drinks). His organizer showed movement—translating and interpreting, moving in and out of the categories seamlessly and constantly. *Theory* and *use* were important, though he was not clear how. *Reality* could be everywhere but did not have to be; modeling could just be about pure theory and logic.

In general, Hank's model exemplified the instability and flexibility of the three terms—that they overlapped and were "folded" within themselves. Yet Hank thought that they must somehow be distinct at times; surely a problem could be classified as one or the other. How could one problem be both? A problem could cross boundaries—be multiple and open—only in the sense that some of the processes used in solving the problem, interpretation and translation, were shared. Some shared features were unexplainable; some overlaps held no meaning in the model. And yet, an overlap model offered Hank flexibility in relating the three terms.

Gary's Notions

Though Gary talked of mathematics investigations in terms of modeling, he clarified that "really most of what you see in my classes are applications. . . . We try to do some modeling as part of that." He viewed modeling in terms of a process that could be applied to either application or modeling problems:

> When I refer to *application,* generally speaking, what I am assuming is that the mathematics is already known, or a body of mathematics is already known, and the identification of what mathematics is appropriate to bring to bear on a particular problem is also already known. Minimal understanding is necessary to start the problem. *Modeling,* to me, is when you're faced with a situation, and maybe there's not even a question yet or a problem, and you have a situation you're trying to understand. And you fiddle with it. And you look at it, and you fiddle with it in the real world for awhile. . . . I've got to figure out what matters and what doesn't matter. And none of the decisions I make necessarily are correct. They're all subject to "review and come back later and modify." In

> modeling there's a give and take, a back and forth. (B/B Interview; April 16, 2004)

Thus, Gary viewed applications and modeling as very different constructs. In the former, you know the structure, and you apply it to a situation that you know it will work for:

> It is going to go one way. I'm going to look at the context and say, "Oh, I know the math that goes with that." I'm going to patch it over the top and make it fit, circle the answer and go on to the next problem. (B/B Interview; April 16, 2004)

In the latter, you have a situation and you are trying to create its structure recognizing that you may need to make modifications and revisions a number of times before finding one that best describes the given situation:

> [Modeling] is really looking at and letting the context inform the mathematics and letting the math inform the context. And sometimes one will drive, and sometimes the other will drive. And I think that to me is the distinguishing characteristic of a model. (B/B Interview; April 16, 2004)

With reference to *mathematics in context,* Gary commented, "I don't have a notion of what I mean by that. . . . I guess I would probably consider it as a super-heading for both of the other two." *Context,* however, is "all the stuff that is not math—all the things that someone who is not a mathematician would care about." And for Gary, there was only one context: the world.

Gary also had strong beliefs about the role of the "real world" in these kinds of problems. He began by explaining his notion of "whimsical" problems that he had learned from Henry Pollak. Whimsical problems were those in which

> the words around [a problem] are realistic. They allow you to envision something taking place, but it isn't going to happen. It isn't reality. It would be something I would qualify as an application, [though] I would like an application to be more realistic as well. (B/B Interview; April 16, 2004)

As an example, Gary described a problem in which two people leave two different towns at the same time driving towards each other, and the task is to find where they meet. It is whimsical in the sense that it assumes constant speed with no interruptions, no traffic, no traffic lights or stops, and so on. It involves real things, but it would never really happen. The closest problems to reality were modeling problems. Such problems were "big" problems, requiring multiple days to solve, wherein the situation is "really, real." It could happen. The task then is to understand it and explain it—to try to predict it and evaluate it. Gary explained,

> The modeling process can produce an explanatory theory if you don't have one or it doesn't hold, or it can look for ways to extend an already existing explanatory theory and verify it. . . . So you can start with a theory that comes from the real world side and have it tell you what math ought to happen. Or you can start with observed mathematical realities and ask, "What do they tell you about the world?" (B/B Interview; April 16, 2004)

The model you make will not "capture reality," but it "ought to capture as much as you want to capture, and you have to decide what your tolerance is. A mathematical model is an idealization; the world isn't ideal. Sometimes [your model] just don't work. . . . All models are wrong; some are useful."

A reconstruction of Gary's graphic organizer relating the three terms—mathematics in context (Context), applications of mathematics (Appl), and mathematical modeling (Model)—is given in Figure 2. Gary used two illustrations to make his point. He drew the concentric circle configuration first in order to explain why such an organization would not work. He explained that applications are part of the modeling process, but they are not a "kind of modeling" as this picture implies. It is not that "some models are applications." Instead, the modeling process is embodied in the organizer Gary drew on the right in Figure 2 (partly derived from his work with the ARISE materials (see http://www.comap.com/highschoolprojects/mmow/introduction.html)). This configuration organizes the processes involved in modeling wherein the problem starts in the real world (RW), and you decide to explain it in terms of mathematics (Math). So you move to the right in his illustration and do some calculations (Calc) that then allow you to make some predictions (Predict) about the situation in the real world. Gary commented that the process did

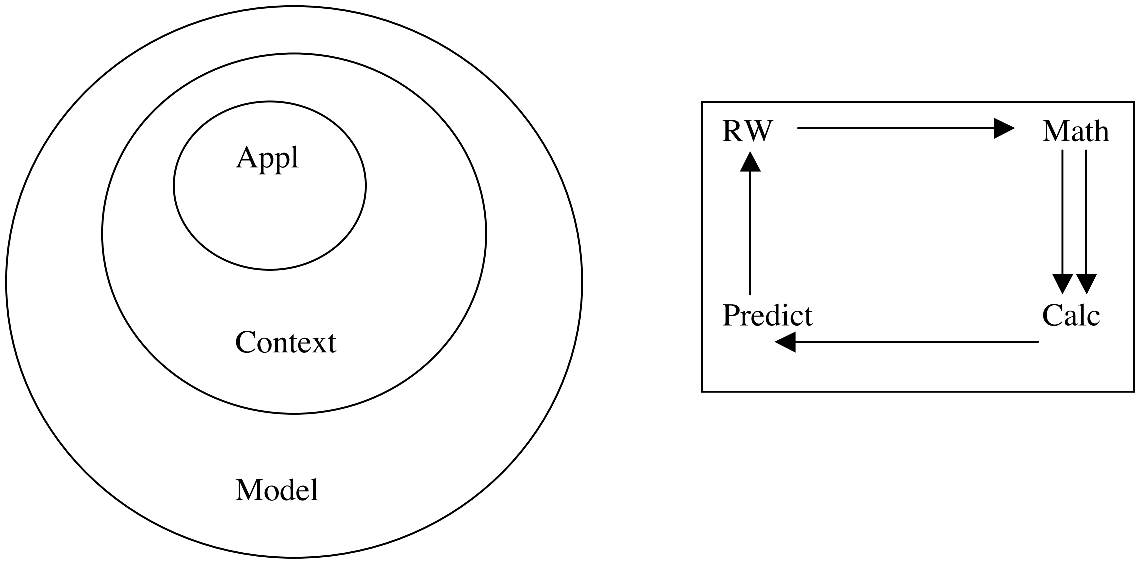

Figure 2. Gary's graphic organizer for mathematics in context, applications of mathematics, and mathematical modeling.

not stop there; next, you must check your predictions against the real world (RW) to see if your model is reasonable. If it is not, then you may have to repeat various parts of this process again until you arrive at a model that works within your tolerance.

Gary also explained his notions about applications in terms of this cyclical process. He thought that applications were represented by the connection between Math and Calc in the illustration though he acknowledged that they probably had elements of the real world. "It's a narrowly formulated problem, very specific, no excess information. You've got exactly what you need . . . typical textbook word problems," and you do some mathematics to arrive at an answer or calculation, and you stop. You do not continue in the cycle to predict, nor do you look back at the real world. Taking the notion of applications one step further, Gary considered mathematics in context to involve the same work as applications plus the move "a little bit along this path, probably not much" towards making predictions. In other words, you might connect your calculations to the real-world situation, but you would not actually test them to see if they were reasonable descriptions of the world.

Gary's relation of the three terms was couched in terms of modeling as a *process*. He rejected a configuration that implied they were nested. Yet in his alternate configuration, a cyclical process, both *applications* and *mathematics in context* were "stops" within the larger process of *modeling*. The real world served as a beginning or an end, and often both. Actions were required; they allowed for movement to the next stop. The terms were not hierarchical, yet some degree of linearity was implied. One moved deliberately and in an orderly way through the process; there was no path from *calculations* to the *real world* or from *prediction* to *mathematics*, and one did not move backwards in the progression. Additionally, the terms were not about mathematics per se, and they were not pedagogical. Particular problems or examples were not accounted for in the organizer. The terms were about processes and the ways in which one solved and resolved problems. They were fixed in a loop with sketchy beginnings but definite endings.

<u>Cathy's Notions</u>

When first asked to talk about the three terms, Cathy commented that "for some people they are just three ways of saying the same thing," but that was not the case for her. Mathematics in context was finding mathematics "inside of other things." Examples she offered included walking in the woods and finding an excuse to talk about mathematics—often in the context of parents and children, not in the context of schooling. Applications moved in the opposite direction; you had some "cool mathematics, and you're trying to find something to apply it to." And mathematical modeling was taking applications "to an extreme." For her, modeling was a combination of the other two terms in which you have a "very large complex problem that can be visualized mathematically. And it isn't always obvious." Yet, modeling differed from the other two:

> Modeling is more driven by the problem. The problem comes to you, and then it's your job to decide if you know any math that would model it. And if not, you go find some new math or you develop new math—that's where math really gets happening in the real world. . . . "Math in context" is just making sure kids see math every time you can find an excuse to talk about it. "Applications of math" is finding interesting enough problems for the particular topic you're doing in class. The "math modeling" is really the real world—it's real big actual problems that might or might not succumb to a mathematical model. And the power is, if you can model it with math well enough, then you have a chance of reaching a solution. (B/B Interview; May 12, 2004)

In the way Cathy was thinking about the three terms in the above description, she noted an increase in difficulty and complexity of the problem types as she moved from math in context to mathematical modeling. She also sensed minor differences in terms of the role of the real world in each—modeling "really is the world," applications were the real world being "brought in," and math in context was seizing opportunities to "see math out there." Cathy was adamant that using these kinds of problems with students was necessary: If the teacher does not take time to do

> some real problems, [students] don't understand that [mathematics] has a use. It's got to happen, it's got to happen regularly, but it can't happen with every topic. [Students] need to keep being tied back to the real world, or they lose their motivation. And I lose my honesty as a teacher. (B/B Interview; May 12, 2004)

Cathy emphasized: "If we can't tell [students] why anyone would ever want to do this [in school or out of school], then we just shouldn't be [teaching] it."

Figure 3 is a reconstruction of Cathy's graphic organizer. Two features of Cathy's relation of the three terms are evident. First, the terms were ways of classifying mathematics problems. Problems could be "typed" and fit into separate categories using these labels. The terms were distinct and could be defined; they were not overlapping or interwoven in the ways

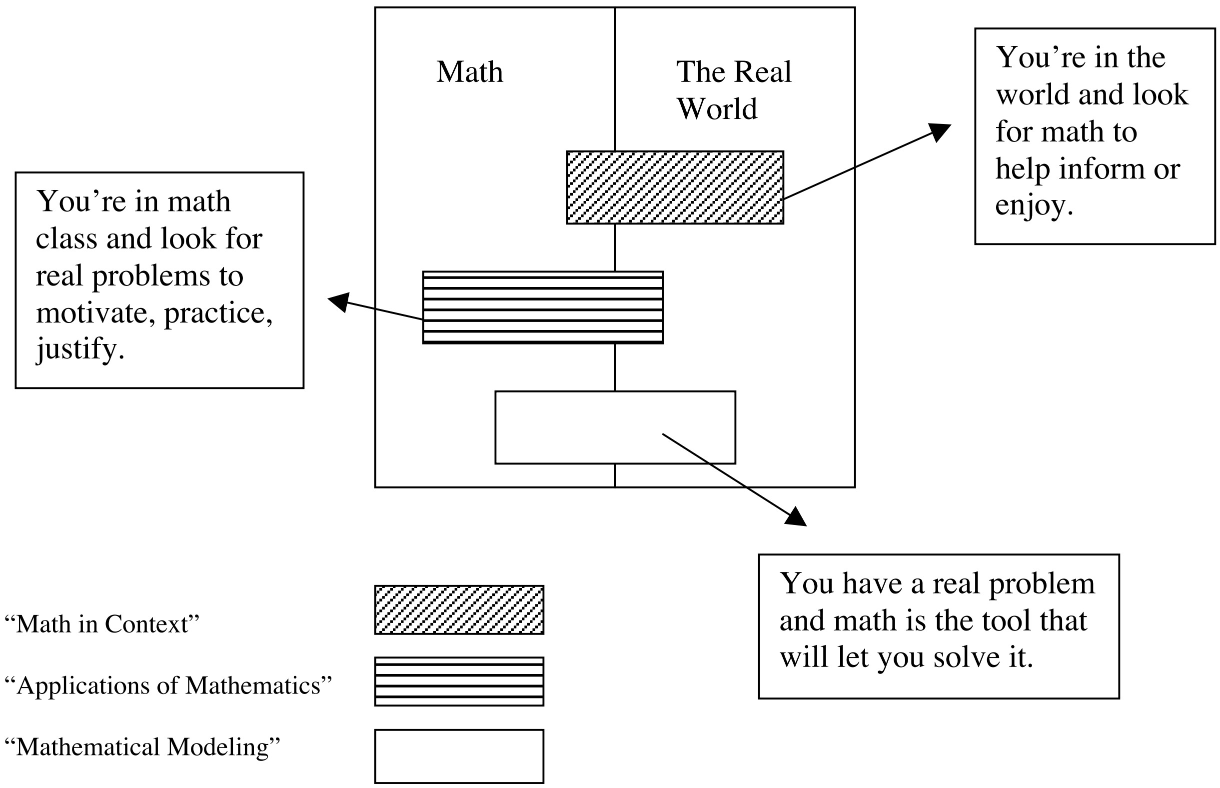

Figure 3. Cathy's graphic organizer for mathematics in context, applications of mathematics, and mathematical modeling.

that Hank organized them. Second, the relation of these terms to the real world was central in Cathy's organizer. Each construct interacted differently with the real world. This interaction was depicted in both the descriptions Cathy wrote (in the arrowed boxes of her organizer) and in the placement of the rectangular representations of the terms in the realms of mathematics and the real world (Figure 3). Mathematics in context was predominately about the world, applications were predominantly about mathematics, and modeling was equally important in both realms. Cathy's written comments also denoted location; the different kinds of problems were worked in different contexts. Mathematics in context problems took place "in the world," application problems occured in "the math class," and modeling problems could occur in either context. The terms were hierarchical in the sense of complexity: mathematics in context was at the bottom of the organizer and the least complex, whereas modeling was at the top of the hierarchy and most complex. One might even argue that the shading of each rectangular representation of the term supported this increase in complexity.

Diane's Notions

Diane did not produce a graphic organizer, but she did discuss the ways in which she was thinking about the three terms. Her initial reaction, like Cathy's interpretation, was that in moving across the terms from *mathematics in context* to *mathematical modeling*, "we're going from something that is littler to [something that is] bigger." Diane spoke of the three terms in following way:

> Modeling is when you have a big, open-ended problem, and you are trying to grab some mathematics to help explain what's going on or describe what's going on. Applications are, here's a body of knowledge that directly connects with, [that] applies to this particular scenario. And then, math in context is closer to applications. . . . It is at least loosely tied to the real world. (B/B Interview; May 11, 2004)

Later in clarifying her definitions, Diane offered that modeling was a "bigger" concept in the sense of being more nebulous than the others both in yielding many different answers and allowing for many different approaches to arrive at those answers. Modeling was more than teaching students what mathematics to use in various situations. For Diane, those practices were "applications [of mathematics] to particular scenarios" because the students already knew the mathematics involved. Modeling required students to develop the mathematics and models; it involved "students making stuff up themselves" in order to model a phenomenon. The real world was "the whole point" of modeling. For Diane, applications involved the real world to some extent, but often they were "more hokey" than modeling problems because the students were forced to suspend reality to work the problem. Finally, without elaborating on how she was thinking about mathematics in context, Diane suggested that the role of the real world in the three terms increased as one moved from contexts to modeling.

From these descriptions, one might guess that if Diane had produced a graphic organizer, it would have been similar to Cathy's. Both saw an increase in complexity and in the role of the real world as one moved from mathematics in context to mathematical modeling, and both recognized applications as sometimes being artificial or hokey. Both viewed modeling in terms of large, nebulous problems that involved students in developing mathematics. And both defined applications as problems wherein students already knew the mathematics and were learning where it was applied or used in the real world. A difference in the descriptions offered by Diane and Cathy lay in what they said about *mathematics in context*. Cathy articulated what that phrase might mean and situated it in settings outside the school context. Diane connected it to applications and the real world and offered examples of contexts: mathematics, situations, future coursework, and so on.

<u>Tom's Notions</u>

During Tom's participation in curriculum development initiatives focused on applications and modeling and during his development of a modeling course, he had devoted time to reading and studying articles about problems based in applications and modeling. He described how he defined these terms:

> Application problems are problems in which the mathematical model has been defined for the student, and it illustrates how the mathematics can be used. It is mathematics in context, but there is no creative modeling on the part of the student. [Students] may have to creatively use the model that they're given, but they don't have to create anything. The modeling context is where the model is not given; the purpose of the activity is to create the model. . . . [It could be] a controlled context, or it could be very open ended. But the creation of the mathematical representation is the purpose of the activity. . . . An application activity is after the model has been created: What can we do with it?. . . You can't have the application without the model. (B/B Interview; May 11, 2004)

Not only did Tom describe modeling in terms of a mathematical model or structure, he also spoke of the “iterative” modeling process wherein “the operating mantra is to create the simplest version of the situation that still contains the essence of the problem” and to find a solution that may have to be improved upon by changing the assumptions of the problem and doing it again. The goal of this process is to “make it better, make it better, make it better—make it more realistic, make it more realistic” until you are satisfied that it models the “portion of reality” you are seeking to emphasize.

He described the modeling process as more advanced and more difficult for both the teacher and the students than working with applications. Because modeling problems are more open and require creativity and decision-making by the students in pulling together ideas to find a mathematical representation, the teacher must spend time thinking about the mathematics, where students may go with it, and what questions will control the problem so that students are directed toward fruitful outcomes. With applications, the work by the students and the teacher is more straightforward and thus less demanding on both. When asked about mathematics in context, Tom commented that he had not thought about the phrase as a descriptor but noted that he saw applications and modeling “as two prongs coming out of mathematics in context.”

The role of the real world in these kinds of problems was important. “If you’re trying to describe something in the real world, then you are limited by reality.” Tom explained that students must always measure their model against reality (i.e., you cannot have infinite populations, frictionless masses, no air resistance, etc.) and recognize that a model is not reality. Instead, he described a model as a “caricature of reality”:

> You emphasize the nose and the ears and hide the chin. There are things that you bring out for focus, and you ignore other pieces. So that’s what you’re trying to do. You’re trying to create a mathematical representation that emphasizes some

piece that you're interested in while ignoring some others. . . . Mathematics is not the real world. It's a description of some portion of the real world but is always less than the real world. That always puts some constraints on the kinds of things that you can do. (B/B Interview; May 11, 2004)

In terms of applications and mathematics in context, Tom did not elaborate on the role of the real world. But in describing applications he used the example of coffee cooling to room temperature, obviously a portion of the real world in which we live. To get further insight into Tom's ways of defining and relating mathematics in context, applications, and modeling, he was asked to produce a graphic organizer (see Figure 4).

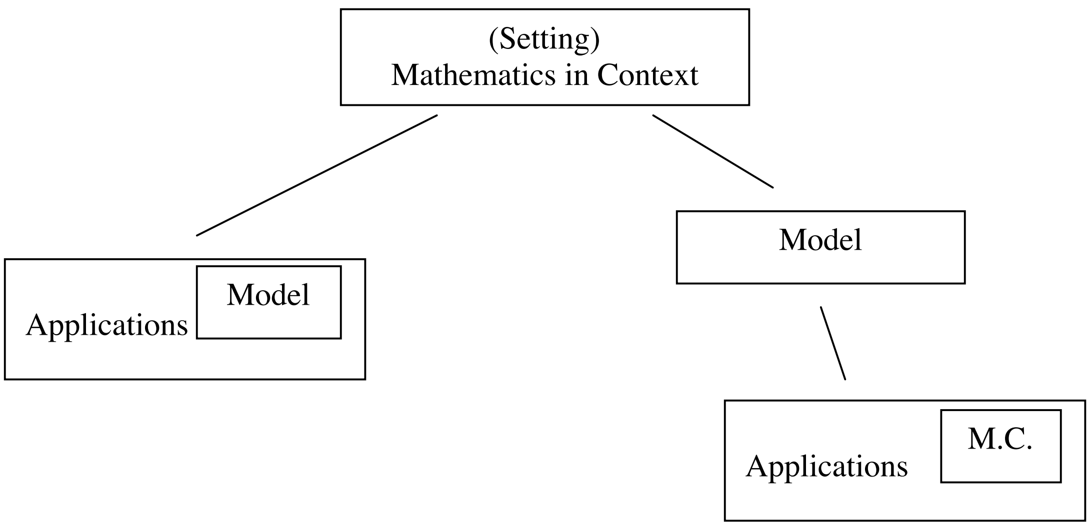

Figure 4. Tom's graphic organizer for mathematics in context, applications of mathematics, and mathematical modeling.

In accordance with the structure Tom had described earlier, he drew applications and modeling as two prongs coming out of mathematics in context—the box that represented the problem's setting or situation. Given the setting or situation, Tom noted two possibilities. On the left, you can have applications wherein the model is already given, and "you can ask questions about the setting." Thus he placed *model* inside the applications box on the left side of his organizer. On the right, "there's a situation [and] I develop a model. And the applications are really me finding out about the situation." So, Tom placed *mathematics in context* (M.C.) inside his applications box on the right side of his organizer. In essence, he saw applications and models as embedded within each other but in different ways. Sometimes the model is given, and you are concerned about its application. And sometimes you have to create the model and then ask questions of applicability in a setting. "That's the real difference: whether you're given the model."

In Tom's organizer, mathematics in context served only as a descriptor for the real world setting or situation embedded in a problem. Mathematics in context was not part of an iterative process as in Gary's model, nor was it a way to type problems as in Cathy's model. Application and modeling, however, were ways to classify problems. In the former, the model is given, and in the latter, the model is to be created. Applications do not stand alone; they are possible only when you have the model. And applications are the endpoint for all problems derived from a real setting. The purpose is to learn something about the setting—how the mathematics applies to the setting. The difference is in whether you are required to create the mathematical model or whether it is already given. Once you have a model, you can make a statement about context. The organizer does not highlight anything pedagogical, nor does it illustrate modeling as a process. It differentiates two problem types and the processes involved in each (whether or not you have to create the model). It connects all of the terms in some way: Mathematics in context is connected both to applications and mathematical modeling; applications are connected both to mathematical modeling (in the left prong) and mathematics in context (in the right prong); and mathematical modeling is in the midst of movement from mathematics in context to applications (in the right prong).

Rhonda's Notions

When Rhonda was asked about the three terms, she commented that she had never considered how she might describe them. Yet, she had specific notions about what the terms *modeling* and *applications* might mean, and about teaching content "in a context."

> "In context" makes me think of, "I am learning something new but I want to know why I am learning it." So I want to put it into some sort of context . . . because I don't want [the math concept] to just be a skill. . . . An application makes me think of a bigger picture and something where I might use more than one skill. Well, you often use more than one skill. But it's something where I use lots of things that I know, and it's not necessarily connected to something new. . . . I don't think I ever realized this, but [in context] seems more constrained to me. (B/B Interview; May 4, 2004)

Rhonda explained what she meant by *constrained* as she drew a graphic organizer of the three terms later in the interview. But first she defined *modeling* as a "very big, very open-ended, almost vague kind of question that you want to answer" that often requires simplifying assumptions in order to get to mathematics you can work with. In comparison with the other two terms, Rhonda clarified her view of modeling:

> "In context" seems like a single thing where you want to learn this skill, but not for the sole reason of learning a skill, but because you need it for something. Then you get to put several of [these] things together in an application, which is still pretty specific. But then modeling is sort of all over the place. You could solve it several different ways. It's not at all clear what you need to do to begin with, probably. You've really got to muck around with it for awhile before you can figure out what's going on. (B/B Interview; May 4, 2004)

In further discussion, Rhonda noted that modeling problems were rich and required a minimum of 90 minutes in class, whereas it would be "boring to do a bunch of applications and 'in context' problems in 90 minutes." She also commented that students needed less guidance when working with application problems as compared with modeling problems, wherein deciding which mathematics to use was often the most challenging step. Finally, she discussed real problems versus realistic problems, pointing out that real problems were those that used real data. Real data were either collected by students or located in resources in which others had published data they had collected. Realistic problems were based on realistic data—data that were "made up" to reflect a certain mathematical phenomenon like exponential growth. The solution would be realistic but not real. Real solutions were those that "our science colleagues would agree with . . . [solutions] that could be replicated."

Figure 5 is a reconstruction of Rhonda's graphic organizer relating the three terms. Rhonda explained its organization:

> If we're talking about "in context," I generally think about that as being a particular skill or concept. So this (points to lowest funnel) sort of funnels, narrowly funnels, into [mathematics in context]. And if we put several of these skills and things [together], they funnel in a broader way (pointing to the middle-sized funnel neck) into the applications and the real world problems. Now these things (points to clouds), the writing, the talking, the technology, the group work, and the assessment are sort of always throughout. They're into all of it. And then all of these things (points to everything in organizer except the phrase "mathematical modeling"), in a very wide way (points to wide funnel at top of organizer), funnel up into the model. (CO_2 Problem Pre-lesson Interview; May 12, 2004)

In accordance with Rhonda's verbal descriptions, the level of constraint from mathematics in context to mathematical modeling in her organizer is illustrated in the broadening of the funnel neck as one progresses upward—the funnels go up in her organizer, even as they seem to go down. Similarly, the number of skills and concepts involved increases as one funnels into mathematical modeling. The organizer is predominantly about teaching mathematics. It depicts a number of pedagogical aspects shared by mathematical modeling, applications of mathematics, and mathematics in context, as the "clouds" illustrate. The terms are hierarchical in complexity and breadth of focus. As you funnel up from mathematics in context, the problems become increasingly complex and broader in scope. Modeling encompasses everything in the organizer. The terms are not separate or distinct but embedded within bigger, more complex

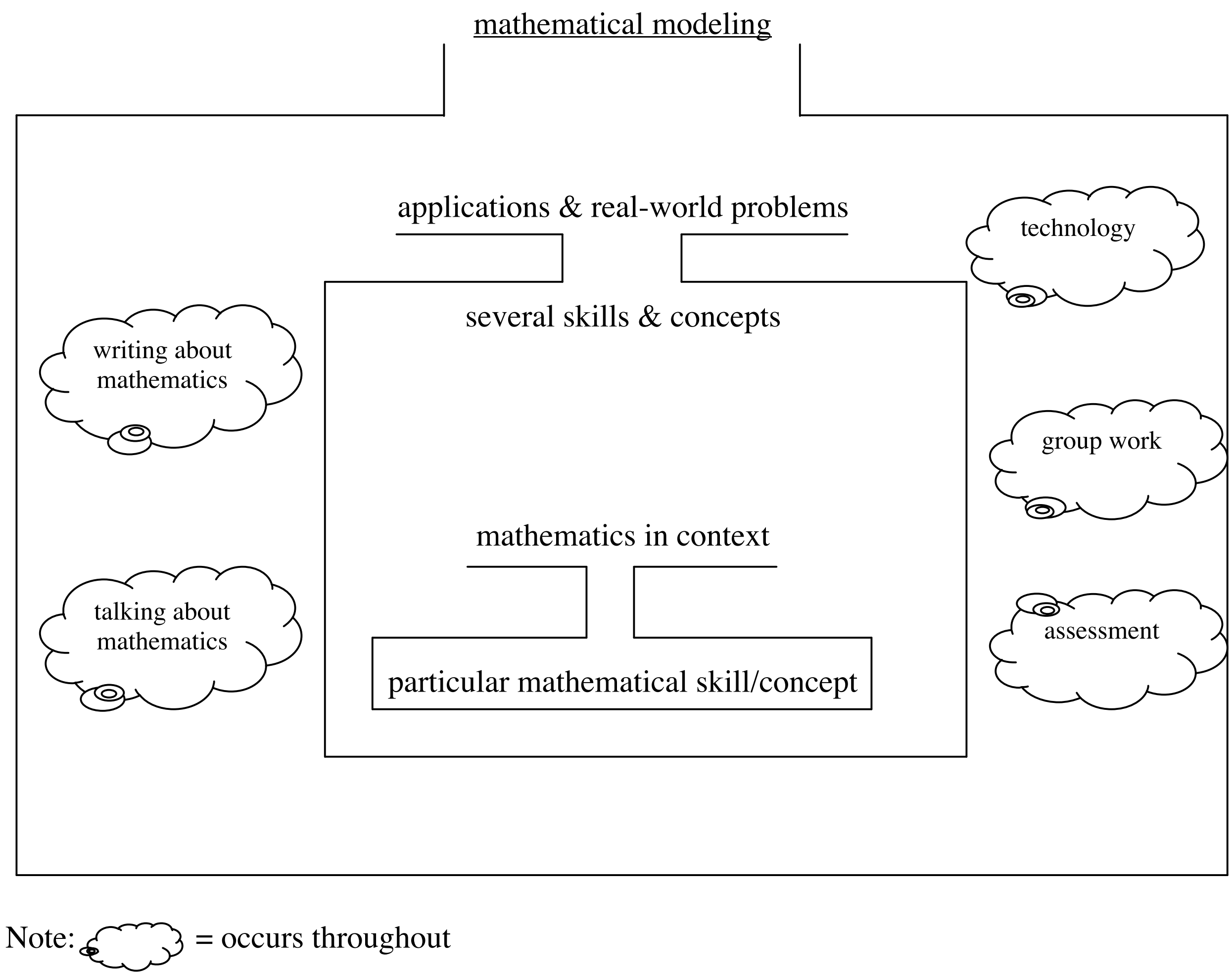

Figure 5. Rhonda's graphic organizer for mathematics in context, applications of mathematics, and mathematical modeling.

structures. Nothing falls outside of the organizer, and the roles of mathematics and the real world are not emphasized or evident. The three constructs are about teaching practices and problems. They are also about a classification of problems based on the number of skills and concepts involved in solving them. This view is very different from the distinctions made by the other teachers in their classifications. Tom classified problems according to whether or not the model was given in advance. Gary classified the problems in terms of the processes involved in solving them. And Cathy classified problems by the extent to which they incorporated the real world and by the context in which they were encountered (real world or mathematics class).

Synthesis of Teachers' Notions

In this study, all six teachers were asked the interview question (Appendix A) in the same way with regard to the sequencing of the three terms, *mathematics in context*, *applications of mathematics*, and *mathematical modeling*. This ordering was crucial for some teachers, as they felt the sequence held meaning. For example, Rhonda and Cathy noted that the three were ordered so that they increased in breadth of focus and complexity. For Gary, the three were ordered from general to specific (i.e., mathematics in context was a general heading that encompassed the other two). And similarly Tom structured applications and models as "two prongs" coming out of mathematics in context. Also important to note is that the teachers did not necessarily address the terms in the order in which they were given. Four of the six teachers defined first the term *applications of mathematics*. Most of the teachers were more confident with their definitions and explanations of mathematical modeling but chose to address applications first because of the brevity with which they could describe it. The teachers unanimously reported that mathematical modeling was the most complex of the three terms and therefore that explaining and defining it required more time and words. Three teachers (Gary, Tom, and Diane) sidestepped discussion of the phrase *mathematics in context*, but Gary and Tom later incorporated it into their descriptions and accounted for it in their graphic organizers. The other three teachers tackled the phrase head on, offering their understanding of what it might mean. In the following paragraphs, teachers' notions about the three terms are synthesized according to the order used in the interview question.

Mathematics in Context

The teachers' ways of describing *mathematics in context* were more dissimilar than their descriptions and definitions of the other two terms. This variation was not surprising since all of the teachers reported unfamiliarity with the descriptor or phrase. Still, "in context" was meaningful for the teachers, and even those who avoided lengthy elaboration on what the phrase might mean could not avoid the use of *context* in describing the other terms. Tom, Hank, and Gary associated "in context" with the real-world setting or situation reflected in a particular problem. But Hank and Rhonda described mathematics in context in ways similar to descriptions of *applications* and *word problems* in the mathematics education literature. For example, Hank noted that mathematics in context problems were "what a lot of textbook authors use as a cheap way out" (similar to common descriptions of word problems); and Rhonda noted that the term was basically a one-step "application" narrowly focused in mathematics. Like Rhonda, Diane described mathematics in context as closer to applications, related loosely to the real world, but she did not explain how that was so. Cathy took the phrase "in context" literally and described it as denoting mathematics "in the world," where students observe mathematics around them and take notice of it. Thus, the teachers' descriptions were varied and often vague perhaps because of their unfamiliarity with the phrase coupled with the limited ways in which the phrase has been employed in the literature (most often in reference to a particular curriculum). Even so, the teachers unanimously discussed *context* (omitting the word *in*) in terms of the "setting" or "situation" in which the problem was embedded. They often discussed *context* as a way to make the mathematics seem useful for students. That description meant that sometimes more advanced mathematics or upcoming assessments (Advanced Placement tests) were themselves contexts.

Applications of Mathematics

Applications held somewhat similar yet subtly different meanings for the teachers. Gary, Cathy, Diane, and Tom thought of applications as problems in which the mathematics or a body of mathematics was already known, or in which the mathematical model was already given, and the setting or situation in the problem served to illustrate where and how the particular mathematics could be applied or made to seem useful. Cathy situated such problems in the mathematics classroom, and like Diane and Rhonda, defined them as "real problems" that held potential in motivating and captivating students. Rhonda viewed applications as multi-step problems that required a number of skills and understanding of related concepts to solve. And Hank viewed applications as starting in reality; one's charge was to learn the mathematics that went along with it. He saw "reality [as] driving the mathematics." In this way, his description varied from those of the other teachers, who thought the mathematical structure pre-existed its application. Hank's description allowed for the creation of new mathematics in light of a need for something in the real world; the application could exist before the mathematics.

When speaking about applications in terms of reality, the teachers were skeptical about "how real" such problems were. Gary and Rhonda in particular, commented that many applications were whimsical or realistic but not real. For Gary the distinction was in whether the setting or situation embedded in the application could or would really happen; and for Rhonda, the distinction was in whether the data given in the problem were real data that students (or someone else) had actually collected, or whether the given data were contrived, made up to illustrate a particular mathematical concept. Both saw value in such problems but thought the connections to reality were looser than those made in modeling problems.

Mathematical Modeling

The teachers used a number of adjectives to describe mathematical modeling (*big*, *nebulous*, *complex*, *vague*, *open-ended*, *difficult*), and for some teachers *modeling* was used both as a referent for a mathematical structure and for an iterative process. In terms of a mathematical structure, Hank characterized a model as the backbone—the place where pure logic resided. Tom and Gary clarified that a model was not reality but a representation or caricature of reality. Yet, for Cathy modeling was really the real world—modeling problems were as close to reality as one could get. In terms of modeling as a process, the teachers described it as *cyclical*, *iterative*, *involved*, *complex*, and *time consuming*. The process began in reality, from some context originating in the real world, then moved into a mathematical realm to create a model, and returned to the context or reality from which one started. In this process it was necessary to ask questions of the solution: Is our answer intuitive? Does it make sense in terms of the context? Did our assumptions yield expected results? Not only did this process apply to the solving of particular modeling problems, but for all of the teachers except Hank, the process also indicated a way to organize classroom activity. In mathematics class, decisions were made about assumptions about the real world, a model was created and discussed, questions of the model in terms of the context were asked, new or different assumptions were made, and the process was repeated. Thus the modeling process served as both a recipe for solving complex mathematical problems and as a way to organize mathematics learning. The latter of these two meanings is not explicated in my reading of the mathematics education literature.

Dimensions of Descriptions

In this study I found that the teachers' ways of thinking about and describing the terms *mathematics in context*, *applications of mathematics*, and *mathematical modeling* were based on a number of dimensions. The teachers described mathematics in context two ways: as the setting or situation within a given problem, or as a referent for the place in which one was engaged in mathematics ("in the world" or "in math class"). Two dimensions were noted: the degree of reality and the nature and complexity of the mathematics. An application of mathematics was described along four dimensions: the role and complexity of the mathematics (and whether it was already known or had to be created), the degree of reality (whimsical or real), its status as a category for classification of problems, and its purpose in terms of course goals. Mathematical modeling was classified in two ways: as a mathematical structure representative of "a portion of reality" and as an iterative, cyclical process. The former was described along three dimensions: the role of the real world, the nature of the mathematics ("context-free" or "pure logic"), and the role of the model in the modeling process. The latter was described in two ways (as a recipe for solving complex problems and as a way to organize classroom activity and learning) along four dimensions: the role and complexity of the mathematics, the role of the context or setting, the pedagogical moves involved (questioning assumptions, using technology, etc.), and the time and activity involved. Some of the dimensions were shared across the terms and used by the teachers to differentiate between them. Table 3 summarizes the shared dimensions and highlights how each dimension allows one to make explicit distinctions between the terms.

The dimensions of the descriptions given by the teachers denote attention to different aspects of each term. The teachers did not simply recite textbook definitions, but discussed the terms in relation to each other, highlighting dimensions of the terms that differentiated them. For

Table 3: Shared Dimensions of Teachers' Descriptions

	Mathematics in Context	**Applications of Mathematics**	**Mathematical Modeling**
Dimensions:			
Degree of Reality	real: math in the world	whimsical or realistic, but not real	"really real"
Complexity of Mathematics	straightforward, least complex	some complexity but less than modeling	nebulous and highly complex
Role of Mathematics or Model	mathematics seen in the world	mathematics and model known	mathematics and model unknown
Purpose in terms of Goals	to help students see mathematics in the world	to help students see how known mathematics is used (apply it)	to create mathematics or a model to explain a phenomena
Time and Activity Required	Minimal time and activity	some time and activity	lots of time and activity

example, both mathematics in context and mathematical modeling were considered by the teachers to be directly linked to the real world (see Table 3), but by using another dimension they were able to differentiate between the terms. Another example comes from the teaching and research literature. Applications tend to be defined as problems in which previously known mathematics is applied in a setting or context. These teachers might think of mathematics in context the same way, but add that what sets the two terms apart is the degree of reality in the problem and where the problem is being solved (e.g., if the problem is being addressed in math class, it is most probably an application; if the student encounters the problem while walking in the woods, he or she is most probably doing mathematics in context, even if the mathematics is previously known and is being "applied"). In other words, these teachers did not define a term by paying attention to only one dimension. It was in looking across dimensions that they began to differentiate between the terms.

The dimensions I identified in the teachers' descriptions are reflected (to some degree) in the definitions and descriptions given by mathematics educators (see chapter 2). Recall, however, that in my review of the literature I highlighted discrepancies in some of the definitions offered by mathematics educators. These discrepancies led me to question whether teachers' (and mathematics educators') notions of the terms might be blurred. In some ways, the teachers struggled to define and differentiate between the terms—they were often seen as shades of the same thing. Yet, there were some dimensions along which they began to detect differences. By focusing on those aspects they were able to bring some clarity to their ways of thinking about the terms. This was evident in their graphic organizers and in their conversational resolutions. That is not to say that these teachers' notions were clear or that they were more articulate than those offered by mathematics educators. In fact, the teachers' ways of relating the terms and describing them were quite informal, sometimes messy, and closely tied to classroom practice. Even so, I did not perceive that these teachers' knowledge and practices were affected by the ambiguity of the terminology. For me, new questions arise: Does the ambiguity of terminology affect the research or teaching practices of mathematics educators? Could mathematics educators benefit from thinking along the dimensions noted in the teachers' descriptions and detectable in some definitional work of mathematics educators? In chapter 8, I discuss the implications of mathematics educators beginning to think more explicitly about the three terms as these teachers did—along multiple dimensions.

CHAPTER 6

TEACHERS' INSTRUCTIONAL PRACTICES

> This study was about the work that teachers do: teaching, molding, shaping, educating, liberating, creating, disciplining, organizing, orchestrating, designing, developing, thinking, speaking, acting, inspiring, motivating, celebrating. How can such work be theorized? How can it be captured and described and therefore "inscribed?" All metaphors fail. Words leave me empty. Is it a practice? Is it a profession? Both? Neither? What are "pedagogical moves" and "instructional practices"? What is "the art of teaching," "the science of teaching," "pedagogical content knowledge"? What must be focused on? What must be overlooked? How can you look at everything and see nothing? How can you "see" nothing and find everything? What gets privileged? Why? (Entry from Researcher's Log, September 1, 2004)

This chapter addresses the second research question in this study of teachers with reputations for using contextual problems: What were these teachers' instructional practices with such problems? To inform this question, I observed Gary, Diane, and Rhonda as they taught lessons they identified as focused on contextual problems. Five contextual problems were observed across multiple precalculus sections for a total of 16 videotaped class period observations. To organize this chapter, I describe each of the five problems in the lessons taught by the teachers. I then zoom back to identify and describe general practices in these teachers' classrooms, and zoom in to describe their instructional practices with contextual problems.

Problem and Lesson Descriptions

Bird Problem

Presentation. The first lessons I observed dealt with the Bird Problem. The lessons took place in Watercliff Academy in April 2004, with Gary teaching. I observed the problem being taught in lessons in three different precalculus sections for the duration of each lesson, in this case, two 50-minute periods. Gary presented an adapted version of a problem that had appeared in the *Mathematics Teacher* (Keller & Thompson, 1999). The lesson followed the study of rational functions and was used as a culmination for that topic even though the students had been working with trigonometry during the previous week. The goal for the lesson was for students to see an application of rational functions. The problem was concerned with the minimum energy expenditure of a bird trying to get food out of a shellfish by dropping it onto rocks to break it open.

To introduce the problem, Gary announced to the class that they would be doing an "in-class project" and said

> Here's the context we're thinking about. We have a bird that likes to eat shellfish and has a problem of opening the shellfish. What we want to do is find out if there's a good strategy the bird could use to open the shellfish. So, let me put you in his place first. If you were a bird, how would you go about opening the shellfish? (Bird Problem; April 26, 2004)

Students offered a number of possibilities: "let someone else do it," "use my beak," "use my teeth," "put a rock in my beak and pry [the shellfish] open," and "drop it on a rock, but that would be messy." Gary pursued each of these possibilities in discussion with the students. The first three suggestions were dismissed because they introduced a competition between the hardness of the shellfish and the bird's beak. Two other students then suggested putting the shellfish on a railroad track or letting a shark bite it. Gary closed this part of the discussion by suggesting they pursue the only 'plausible' suggestion that had been made: dropping the shell onto rocks. He then asked the student who introduced the idea to "describe how dropping [the shellfish] onto rocks might proceed." She described the bird as diving toward the rocks and dropping the shell. Several other students added what Gary later called "interesting wrinkles" to the conversation: wrapping the shellfish in leaves to prevent the fish from "splattering" and being lost in the rocks, and flying toward a rock wall. Gary pursued these, asking what the disadvantages of each might be. The students noted that the leaves would "pad" the shellfish and might therefore make it more difficult to open, and that flying toward a rock wall would be less efficient than flying in general—besides, if the shellfish "stuck" to the wall, it would be difficult for the bird to eat.

Gary latched onto the students' mention of efficiency and asked, "What is the purpose of dropping the shellfish?" Several students answered that the purpose was for the bird to eat the fish inside. Gary asked, "And what is the purpose of eating?" to which students replied, "Energy." Gary commented, "Exactly. It shouldn't cost you more energy to open [the shellfish] than you get from it. You need to get more out of it than you put into [opening] it." He commented that energy expenditure would certainly be important to the bird, then directed the students back to the idea of dropping the shellfish, and asked them to think about the trajectory of the bird's flight, or how it might fly to drop the shellfish. The students discussed "hovering" and "diving." Hovering was quickly dismissed, but the students thought that diving would be most effective for the bird. Gary agreed but said that to "standardize" the experiment the students would be doing later, they would assume that the bird simply flew horizontally and dropped the shell. (In other classes, Gary also provided reasons for assuming there was no wind and that the terminal velocity of the dropped shellfish was unimportant.) To aid students in further formulating the problem, Gary asked, "What matters to the bird in terms of efficiency? What variables?" Two students formulated a question that concerned height (one variable) but not the number of drops, "What is the lowest height from which the bird can release the shell using the least energy and crack it open?" In one class, there was a discussion of whether one drop from a higher height would use less energy than two drops from a lower height, since the bird would have to "go up and back down" more times in the latter scenario. Gary used this discussion to bring students to the idea that a rough measure of energy would take into account those variables: height and number of trips (or drops). After some discussion and reminders by Gary about what the students had already said, the students identified the variables of importance: the height from which the shell was dropped (independent variable) and the number of drops (dependent variable). Reaching this point in the lesson required 20 minutes, 25 minutes, and 15 minutes, respectively, in the three classes observed.

Gary asked the students to visualize individually what the relationship between the two variables might look like and to sketch a graph. The students' graphical depictions varied. Some drew lines with negative slopes, some drew rational function curves, and others drew step-wise functions. But all agreed that the graph would have both a vertical and a horizontal asymptote. Gary asked the students to explain the reasoning supporting their graphs, and after much discussion they decided that there were indeed asymptotes, but that they did not yet know exactly where. (Students offered persuasive arguments for a vertical asymptote at zero or one.) They also agreed that the graph should be a rational function. After Gary and the students had created the skeletal form of a rational function (number of drops = $1 + (A/(x - h_0))$), Gary paired students and assigned each pair two heights in centimeters from which to drop peanuts. This experiment was meant to simulate how a bird might drop shellfish. The students recorded the height and the number of drops required to break open the shell (split the peanut into halves) for eight trials at each height. Gary put a table on the board listing all the heights assigned and asked the students to record the *average number of drops* for each height on the board when they were finished with the experiment. As the class period ended, Gary directed the students to "transcribe all the data off the board" and said that their homework assignment was to come up with the two numbers, A and h_0, based on data from the class. He hinted that part of finding those numbers lay in figuring out how to get them. He suggested they think of a transformation that would turn the rational equation into a linear one. Doing so would allow them to do a linear regression (using their graphing calculators) to find the values of the variables. He referred the students to the problem handouts he had given them from the *Mathematics Teacher* (Keller & Thompson, 1999) for more help.

The next day, Gary and the students worked through the homework assignment, applying the linear transformation and finding values for A and h_0. In this process, the students noticed that for the height of 90 cm, the number of drops recorded was 1. Using this value would yield zero in the denominator, so as a class the decision was first made to "fudge" the number to 1.01. Later they omitted the data points at 90 cm and 70 cm to make their transformation "look better."

After the students, with some assistance from Gary, linearized the data and found the equation for the rational function, Gary directed them back to the context of the problem and reminded them that, in reality, the bird would be "most interested" in minimizing energy expenditure. He challenged the students, "How do you measure energy if you've got these multiple flights involved?" Gary reminded the students that "originally they said the number of flights, but they weren't counting on dropping [the shellfish] 15 times…. What should [we] use as a measure of energy now?" One student suggested that the energy expenditure could be thought of as the product of the height and the number of drops (this idea may have come from the *Mathematics Teacher* handouts). Gary asked the student to explain his reasoning (in case he read it in the handouts) and polled the class to see if they bought the student's argument. The class agreed that the argument was reasonable.

Using parts of the equations they had just found, Gary and the students wrote the energy function and graphed it to find that it had a minimum. They discussed the implications of the minimum in terms of the context: There was an optimum height from which to drop the shellfish that minimized the number of drops required and thus the energy expended by the bird. Gary emphasized that the minimum was "flat," and therefore that the bird had some margin for error in the drop height: A drop within a few feet of the optimum drop height would be almost as good as the optimum. To conclude the lesson, Gary asked the students to complete for homework a one-page summary of the lesson so that someone who was not there would be able to understand how to solve the problem, why certain procedures were done (where equations came from and what they meant), and what the solution meant in the context of the problem: the "punch line."

As the Bird Problem unfolded in class, one difficulty encountered by the students and managed by Gary is worth mentioning. At times it was difficult to keep track of the data in the calculator lists and graphs. Gary had put a table on the chalkboard with the headings x, y, and $1/(y - 1)$ but did not list values underneath. In addition, Gary stored his data in different lists from those used by the students, so the students were constantly translating Gary's instructions into their own organization. In one class, Gary stored his re-expressed data on top of the raw data in his list because of insufficient space. Later when he translated the re-expressed data back into the raw data for his own purposes, some students questioned whether they should be doing the same. They quickly realized, however, that it was not necessary for them to do the translation since they already had the raw data stored in another list.

Reflection. Gary was pleased with how the problem had gone in two of the three classes. In the third class the problem had been approached well, but the students' data had been more "fuzzy" than those of the other two classes. Gary attributed the problem to sloppy data collection. He was disappointed that students across all the classes had not been able to "come up with the transformations they needed to get linearization" without heavy reliance on the handouts and instruction, but he thought that everyone had been engaged and had done "the right things" once they had direction. In all three classes, much of the discussion had hinged on one or two students recognizing the processes involved in the problem and explaining it to their classmates.

Gary commented that he felt the time constraint of 2 days—all the time he was willing to take for this problem since it was simply a wrap-up application lesson—and worried that he had given too many hints, had been unwilling to pursue students' side trips outside of the main discussion, and had not allowed the students to work as independently as he would have liked. Even so, he valued the time he had taken at the beginning of the lessons to talk with students about the context, the variables, and the graph. Without that time, Gary believed the students would not have been as successful as they were. Gary's biggest regret after having taught the lesson was in what he believed to be a missed opportunity. He explained,

> The idea is that if I look at one measure of energy and you look at another measure of energy that is double of mine, then the minimum for mine and the minimum for yours occurs at the same time. They are not the same value, but they occur with the same parameter choices. And that's a subtle idea. And I don't know that it came up [in all three classes]. (Bird Problem Post-lesson Interview; April 29, 2004)

Gary did not mention this idea in his goals for the lesson, but he had made a point about it in one class. From my perspective, this was indeed a missed opportunity for the other classes.

The idea was subtle, and I did not perceive that the students had grasped it. However, the idea was a “bonus” in the Bird Problem. It was a reasonable extension for Gary to want to make, and it was unfortunate that he was not able to do so, but its omission did not affect the success of the problem in reaching Gary’s goals.

Wind Chime Problem

Presentation. The Wind Chime Problem was the second problem I observed being taught by Gary (to the same precalculus sections as the Bird Problem). The lessons took place in May 2004 in 50-minute periods over the course of 2 1/2 days. Gary had adapted and developed the Wind Chime problem over the past 3 years after seeing a similar version presented at the T^3 (Teachers Teaching with Technology) International Conference in 2001. The goals for the lesson were for students to measure the frequencies of sound waves by “banging” on copper pipes, to explore the relationship between pipe length and pitch, and to use what they learned to build a wind chime whose notes formed a chord of their choosing. Gary viewed this lesson as an assessment, a project that pulled together ideas from across the semester—geometric sequences, power relations, logarithmic re-expression, exponential functions, trigonometric functions, periodicity, and frequencies—and hoped it would serve as a “wakeup call” before the final examination.

The first part of this lesson introduced the students to musical chords and chord frequencies and asked them to find the common ratio for any pair of successive notes given that the frequencies of the notes of the Western (equal temperament) musical scale form a geometric sequence. The students were also expected to complete a table of note numbers, note names, and frequencies for three full octaves and to investigate a chord with at least four notes in it whose sound they liked. They were asked to list the note names and frequencies in the chord they chose, giving the name of the chord if it had one. Over the course of the next two days, the students collected data using CBL (Calculator Based Laboratory) microphones connected to computers loaded with the CBL2/TI-InterActive! software (Texas Instruments, 2000–2004; see http://education.ti.com/educationportal/). Student pairs selected half-inch copper pipes of various lengths, used different techniques to "bang" them (they often tapped the side of the desk or used a pencil to tap the pipe), and used microphones to capture the sound waves produced. The students were expected to produce and save three "good" connected scatterplots of data in TI-InterActive! for each of three pipe lengths. Next, the students were to use the sound wave graphs produced by TI-InterActive! to determine the period of the wave and subsequently the frequency of each note—defined as the reciprocal of the period. The students were to answer questions that led them to use logarithmic re-expression to write an equation (using the regression package on their calculator) for their data, to graph the re-expressed data, and to discuss how well their equation fit their data. Finally, the students were to use their equation to compute the lengths of pipes they needed for their chosen chord and with Gary's help build a set of wind chimes the following week.

Gary first introduced the Wind Chime Problem to students in the last 10 minutes of the class period the day before I began observation. Gary reported that he had given the students a handout, briefly described the problem and the tasks it contained, and asked them to complete three tasks (see the first three sentences of the previous paragraph) that night.

> I really didn't do much by way of introduction. I told them kind of what the scenario was, kind of an overview of what we were going to be doing. They had seen the wind chimes [from previous classes] hanging in here all year long. So they knew that they were there, and that was some sort of project. (Wind Chime Problem Post-lesson Interview; May 19, 2004)

In presenting the Wind Chime problem to students in the computer lab the next day, Gary began with a demonstration of how the CBL equipment was to be connected to the computer and how the data collection was to be conducted. A student volunteer demonstrated how to hold the microphone and follow the pipe as Gary held a pipe by a paperclip through a hole near its top and "banged" it along the desk edge. As TI-InterActive! captured the data, Gary verbally evaluated the graphs of sound waves, describing why particular graphs were not "good." Gary made it clear that the students were trying to capture the hum of the pipe, not the noise made when hitting the pipe, and thus their graphs should be somewhat smooth wave graphs. He pointed out all the menu and window options in TI-InterActive! that the students needed to work with, and explained their goal of making and saving three good graphs for each of three different pipe lengths. After six iterations of banging the pipe and checking the graph, Gary obtained a graph with which he was satisfied and demonstrated how the students could capture, label, and save the graph in a TI-InterActive! worksheet. Gary noted that it might take several attempts to capture three good graphs for each pipe, so the students should use their time wisely. He suggested that they start with longer pipes as the graphs would tend to be smoother than for shorter pipes. With 25 minutes left in the period, the student pairs began their data collection. As the students worked, Gary circulated around the room, checking their work from the previous night (he reported later that less than half of the students across the three classes had used a geometric sequence in their frequency table; most had used an arithmetic sequence) and offering evaluative comments about the wave graphs they were capturing. Occasionally Gary commented on how students were banging their pipes or on how they were using their microphones. Some student pairs asked questions about their graphs, but most worked independently, deciding for themselves which were good wave graphs and which were not. At the end of the period, none of the students was finished with data collection, so work continued the following day in the computer lab. Gary directed the students to “pick up where they had left off” and said they should be able to finish data collection and at least find the period and frequency for each graph by the end of the period. As pairs finished their data collection (25 to 30 minutes into the period) and began work with the data, difficulties began to surface.

One pair had successfully calculated the period and frequency of one pipe, found the power relation, and re-expressed it using logarithms. But when they plotted their three data sets, they reported that the “curve was bending the wrong way.” Gary questioned the pair first on their calculations and data collection and found no problems with the “mechanical” aspects of their problem. Next he questioned them to consider other sources of error. With Gary’s help, the pair hypothesized that frequencies change because pipe length differs—the point of the project—and perhaps also because the different pipes were not cut from the same “stock.” So with different pipes of similar lengths, Gary pointed out, variation in material due to “wall thickness, or that kind of stuff, [would] seem bigger, because the differences you’re trying to measure are smaller.” Gary indicated that such variation would be negligible given lots of data points, “but when you have only three points, that’s the pattern. It’s not noise.” The students selected a fourth pipe whose length differed greatly from the other two they had used and collected more data. In this iteration, they were able to see the desired pattern in their data.

Another student pair struggled to calculate the period of their wave graphs because they could not easily identify peaks and valleys. Gary questioned their criteria for selecting the graphs during data collection and discovered that their criteria had differed from his. They were looking for smooth connections between points, whereas Gary had been concerned with “clean periodicity.” Gary challenged them to think about what they were trying to measure and to "make sure it's there before you try to measure it." This student pair collected further data and selected their graphs according to Gary's advice.

Finally, two student pairs working together questioned their equations when they overheard other groups talking about subtraction and realized they had not been subtracting anything. It turned out that in "counting" to find the period, they had used the last peak of their wave as an endpoint but had used zero instead of a measured value on their graph as the starting point. Without much delay, they were able to rework their equations using correct measures for the period.

At the end of the second day, Gary asked the students to finish their calculations and the remaining tasks on the handout for homework (i.e., use logarithmic re-expression to write an equation for the data, graph the re-expressed data, discuss the fit of the equation to the data, and compute the lengths of the pipes needed for their wind chimes to produce their chosen chord). He said that they would have 2 days to do so and that their work was to be accompanied by typewritten descriptions and explanations of the data collection, their solution process, and their solutions.

Reflection. Gary had not anticipated the levels of background and group noise in the computer lab during data collection and was disappointed that most students did not finish their collection more quickly. (Noise interference had not been so great in the past.) He also reported that during data collection, he realized there were "good ways and bad ways to hit the pipe. And it actually takes some practice with it . . . and also a little bit of instruction" for students to get "a feel for it." When the students were hitting their pipes too hard, they were getting overtones in their graphs, which was evident as "a sort of thick looking graph [where one could] see the main wave, [but] it has these sort of bristles on it as you go along." In those cases the data were not as smooth, and the students had more difficulty getting good graphs. Still, Gary thought the lesson had been successful in forcing the students to review content from earlier in the year, and he thought the students had enjoyed the project.

Gary was planning to have the students build their wind chimes during the next week. They would give him their cut pipes with markings, and he would drill holes and return the pipes to the students, who would then add the clapper. This process would not be done in class, and therefore I did not see the completed chimes. Gary reported that he had "one bit of reality [that he was going to] spring on" the students at the completion of the project. If the students completed their calculations correctly, the chimes they built would produce the chord they chose early in the project and worked with in their calculations. Gary noted, however, that "the truth is, when you're listening to a wind chime, the tone we're recording [the hum] isn't the one that you really hear the most. You can hear this tone, but what you really hear is the clank." Thus the students would have to listen after the clank to hear their chord, provided they chose a wooden clapper or something similar; metal clappers confounded the problem further.

In considering the problem for use next year, Gary reported that he wanted to rewrite it so that the nature of the problem changed. He thought the students were spending too much time collecting data for a wave, "and they don't really study the wave they get." He planned that student pairs the following year would be asked to collect three good graphs for the frequency of just one pipe. And instead of stopping there and using the regression package on the calculator or computer, he would ask them to construct a sinusoidal curve and write an equation to fit the data. In this way, the students could

> superimpose their own wave and let it fit, or not fit, and adjust it so they [could] get some practice with sinusoidal stuff too. . . . And then we pool all the data from everybody's. So everybody would have the same data set to work with when they

did the re-expression and power function. (Wind Chime Problem Post-lesson Interview; May 19, 2004)

Foul Shot Problem

Presentation. Diane taught the Foul Shot Problem in May 2004 in one 90-minute precalculus class period. This was the only problem I observed Diane teach, but fortunately I observed it being taught in lessons with two different classes. Diane described four goals for this lesson: (a) to make a point about mathematical modeling, that the mathematics one does is influenced by the assumptions one makes; (b) to practice making "qualitative graphs," sketches of graphs that illustrate proper orientation and "slant" without using actual tick marks or points; (c) to understand that a problem may have multiple answers that are correct, and (d) to see that when a solution method does not "work," one must try something else, the notion of perseverance.

In using this problem in class, Diane presented the students with a half sheet of paper with the following problem typed on it:

> Imagine that you are sitting in front of the television one Saturday afternoon watching the Chicago Bulls play the Boston Celtics. The Bulls' star player drives to the basket and is fouled. As he stands at the free throw line, the announcer states that he is hitting 78 percent of his free throws this year. He misses the first shot but makes the second. Later in the game, the player is fouled for the second time. As he moves to the free throw line, the announcer states that he has made 76 percent of his free throws so far this year. Can you determine how many free throws this player has attempted and how many he has made this year?

After students read the problem to themselves, Diane read it aloud and asked students to work in their groups of four to think about how they were going to solve this problem. As they worked, she moved from group to group asking questions about their solution ideas. After 15 minutes, she asked a student for his group's solution process. The group had used ratios and simple substitution to solve the problem. (Let m = no. of shots made, let a = no. of shots attempted. The two equations were $m/a = 0.78$ and $(m + 1)/(a + 2) = 0.76$.) The group had found that $a = 26$ and $m = 20.28$. Diane solved their system on the board and arrived at the same answers. Several students, however, identified a problem. One cannot make 28 hundredths of a foul shot, so they questioned where that number came from. The class decided that the announcer had probably rounded his announced percentages. Diane identified that as an assumption the students had started with and suggested that they needed new assumptions. She asked the students to identify the range within which the announcer would have rounded the percentage to 76 or 78 and wrote those inequalities on the board. She pointed out that *m* and *a* were positive integers, so the solutions to the inequalities should be positive integers. After some discussion, Diane separated the two compound inequalities into four separate inequalities, simplified them algebraically, and asked the students to graph them "qualitatively" by hand, without worrying over exact coordinates. Diane then graphed the four lines in Quadrant 1 (since they were limited to positive integer solutions) and shaded regions according to the inequalities. She pointed out that the resulting shaded region contained an infinite number of points but that the only points of interest in this problem were the integers in that region, or the lattice points (if one visualized graph paper underneath the shaded region). Diane pointed out that visually the students could not tell where the lattice points were on this graph and asked them to use their graphing calculators. When Diane asked how the calculator might be helpful, two or three student groups suggested graphing the lines and looking at the region of interest. Diane pursued this suggestion and showed that one cannot choose a calculator window that makes that region visible. With some guidance from Diane, the students realized that they should look in the table on the graphing calculator for an integer between the *y* values of the first two lines and the same integer value between the *y* values of the second two lines. Diane and the students looked for such integers together and identified $m = 18$ and $a = 23$ as one possible answer. They verified this answer by checking the ratios 18/23 and 19/25. Diane challenged the students, in their groups, to see if there were other answers. The students identified $m = 21$ and $a = 27$ as a second answer. As the period came to a close, Diane pointed out that the key idea to recognize in working this problem was that "different assumptions yielded different mathematics" and asked the students to find six more lattice points for homework.

Reflection. Diane remarked that the lesson had been rushed and that she thought the second class had done much better than the first. The students in the first class spent more time guessing ordered pairs than finding ratios, and that had slowed their progress. Understanding the need for an integer value between the *y* values of the first two lines and the same integer value between the *y* values of the second two lines had proved very problematic for both the students and Diane. The students struggled to see the lattice points on the table, and Diane struggled to draw the lines so that the students could follow the argument about integers and their locations in terms of the lines. She talked through this part of the problem more extensively in the second class, which she thought had facilitated the students' understanding. Overall, Diane was pleased with the lesson and pointed to the way in which the context of the problem had forced the students to change their original assumptions (one cannot make a fraction of a foul shot). She thought that making this transition and seeing the lattice points were the two turning points in the problem and essentially that both had gone smoothly.

Swing Problem

Presentation. The Swing Problem was the subject of the first lesson I observed being taught by Rhonda. This precalculus lesson was taught in May 2004 in one 90-minute class period. In working the Swing Problem, the students collected data of a person swinging on a playground swing and wrote parametric equations that described the horizontal and vertical positions of the swinger as a function of time. The students then graphed their equations on the graphing calculator to verify that their equations were correct.

This problem began with Rhonda explaining the task to the students, “You need your calculator and some paper and a pencil. Our ultimate goal is going to be to try and model what it looks like for a person to swing on a swing.” She then asked them to think individually about what an equation and graph of a person swinging might look like. She asked them to sketch their graph. After 5 minutes, Rhonda took suggestions from the students and sketched a sine wave on the board, asking students to identify the labels for the axes. Two or three students suggested “time” for the *x*-axis and “height from the ground” for the *y*-axis. One student was thinking about “position” instead of “time.” Based on that suggestion, another student introduced the word “parametric” into the conversation. Rhonda pursued this, asking for clarification of how parametrics might help. With input from four students, Rhonda noted on the board that one could think about time and vertical position, as well as time and horizontal position. She commented that students could then graph these as a parametric pair and “see the swinger swinging.” But first,

> You need to think about both things separately. You need to think about what this vertical height is going to look like over time. You want to think about what this horizontal displacement and time look like. So you can write those two independent equations before you write them as a parametric pair. (Swing Problem; May 5, 2004)

Rhonda suggested that the students try to sketch separate graphs for the two positions to help them think about the kinds of measurements they would need to collect later. She asked the students to discuss, in groups of six, their sketches and the measurements they needed to gather at the swings. She announced that they should assume the swinger goes from rest to steady motion (instead of building up to a steady motion), and suggested it would be good to collect more measurements than needed. Each group nominated a swinger, gathered meter sticks and stopwatches, and walked outside to the swings. As the groups developed ways to measure the motion of the swinger, Rhonda moved from group to group checking that all members of the group agreed on the needed measurements and on how they would be collected. She moderated disagreements between group members whose opinions differed. One group measured the maximum height of the swinger by using an indicator (a students' arm) to mark the height, removing the swinger from the swing, and extending the swing to the indicator for more accurate measurement. Other groups collected more approximate measurements. After 15 minutes, the students returned to the classroom and split into groups of three to write equations that would model the motion of their swinger.

In the classroom, Rhonda reminded the students that they were expected to turn in their measurements, their equations, and an explanation of where their numbers came from. They were also expected to graph their equations in parametric mode on their calculators to see if their equations indeed modeled a swinger. Rhonda walked around observing the student work and responding to questions the students had. She never answered a question directly but instead asked questions of students to help direct or clarify their thinking. The students encountered a number of difficulties. For example, two groups grappled with whether a sine or cosine function was more appropriate. And all the groups struggled to find the periods of their functions.

After 30 minutes of group work, Rhonda addressed the class: "Do you think the periods are the same for [the] horizontal and vertical [motion]?" Some students answered yes. Others said no. Rhonda selected two students to participate in an illustration. She asked one student to call out either "up" or "down," and the other student to call out either "front" or "back," as

Rhonda moved her hand in a swinging motion (tracing a concave up arc in the air). As a result of this illustration, the students were able to hear the difference in the periods for each motion—the student chanting “front” and “back” (horizontal motion) did not switch words as often as the other student (vertical motion). Rhonda allowed the students to continue to work in groups and encouraged them to consider the illustration in their work. The class ended with one student from each group of three turning in the group’s paper. None of the groups was able to complete the assignment, though some had written equations they had not yet tested on the calculator.

Reflection. Rhonda was disappointed that the students did not complete the problem. She thought they had met some intermediate goals—selected reasonable phenomena to measure and remembered the mechanical skills, such as incorporating amplitude, shifting graphs, and finding the period—but had not reached the ultimate goal: modeling their swinger. Rhonda reported that the data collection had required more time than usual for this problem, which had contributed to the lack of time for a solution. She also identified two areas of difficulty that had slowed the students’ progress. First, the students struggled to “interpret from the physical into the mathematical,” or to “connect the real world to the numerical stuff on paper to get the numbers to come out right.” Second, all of the students struggled to find the periods for the horizontal and vertical motion given their data. Part of the difficulty lay in the language and keeping the motions separated in their thinking. Rhonda noted,

> It’s hard when you’re talking about one not to introduce the other one. And it’s hard to only think “up/down” and then “back/forward” and then, depending on the words you use, you can make one of the words sort of drift into either category. So it’s hard to get things right [when talking to each other]. (Swing Problem Post-lesson Interview; May 5, 2004)

Neither of these struggles surprised Rhonda. She reported that finding the periods was always what students struggled with the most. And unless students have had multiple opportunities to make interpretations from the physical world into the mathematical world, and to make mistakes, such interpretations are difficult.

Because the students had worked hard during the entire 90-minute period, Rhonda decided she would allow time during another class period to complete the problem. I did not observe that period (because it fell on a day when I was not able to observe the class), but I asked Rhonda to report on it in our next interview. She had given the students 20 minutes to finish the problem in class and reported that by the end of that time, most of them were able to show their swinger swinging on the calculator, and many had written explanations of their equations.

CO_2 Problem

Presentation. The CO_2 Problem was the second problem I observed being taught by Rhonda. She taught the lesson to a precalculus class in May 2004 in one 90-minute period. It was the same precalculus class in which I had previously observed the Swing Problem. The problem was used as a culminating activity to pull together exponentials, sums, and trigonometry before students took their final examination. The problem was centered on finding a mathematical model (or equation) that fit data of the CO_2 concentration in the atmosphere collected on a monthly basis above the island of Mauna Loa in Hawaii. Data were given for each month from 1958 to 1989 and from 1976 to 1995, the yearly averages for the same years, and a dot graph of the data. The graph showed that the concentration of CO_2 was generally increasing with time but that it oscillated between years.

In introducing the problem to the class, Rhonda reminded them that she had linked the data into their calculators at the end of the previous class and thus they did not need to type them in manually. She cautioned that the students needed to take detailed notes to keep track of the data and where things were stored in the calculator, as well as to "think." She also gave the students five handouts with information related to the problem (articles from the Internet about Mauna Loa and the CO_2 concentration, the data lists, and the graph) and described the task of finding a mathematical model.

> The programs, that you have, have a lot of data about CO_2 concentration over time. So what we are ultimately going to end up doing is try to see if we can fit a model to this CO_2 concentration data over a pretty long span of time. (CO_2 Problem; May 12, 2004)

Rhonda directed the students to the handouts she had given them, and asked them to consider what was happening in the given graph of CO_2 data. After noting that the graph was both increasing and oscillating, she gave an overview of the information in the handouts and pointed out that the data in the students' calculators included measurements for each month, and averages for each year, for two different time spans. Rhonda then instructed the students to graph the data from 1956 to 1989 together with the averages for those years. She demonstrated the steps on a calculator connected to a projector and reminded the students how to call up stored data and graph them. In addition, she wrote notes on the chalkboard throughout the lesson to keep track of where data were located and what equations the students were working with. To begin, the students decided that most likely a sum would model the data. Rhonda asked them to think about what they might be summing, and together they wrote the general form "$y = \sin(x)$ or $\cos(x)$ + toolkit." After looking at the data, the class—with input from Rhonda—decided to focus on fitting a model to the averages first, or finding the "toolkit function." Several students hypothesized that an exponential might work. Rhonda agreed to try that first and said they could adjust it later if they needed to. She asked, "If we believe it is exponential, what must we do to find it?" After hints from Rhonda, the students decided to "log the *y*s." Rhonda pursued this idea with the class and graphed the resulting data. They were satisfied that their graph was linear. Rhonda suggested that they try a linear regression to make sure. This time the class was not convinced that the line fitted the data and suggested that a shift might "fix it" since the calculator automatically assumed the asymptote was at zero. To decide how much to shift the data, Rhonda referred the students to the table of data and asked them to make a guess about where the data were leveling off. Three or four students suggested a shift of 310, and accordingly, Rhonda subtracted 310 from the *y* values in the data list, took their natural log, and graphed the new data. The result was not linear, so the class decided to adjust their shift. First, they tried 314 but realized they had gone in the wrong direction. Next, they tried 300 and were satisfied. To verify their choice, Rhonda, together with the students, did a linear regression and looked at the residuals. Using the line obtained in the linear regression and making the appropriate substitution for the *y* given by the calculator (recall $y = \ln(y - 300)$ with the shift), they found the equation for the exponential part of the sum. To verify their equation, they graphed it along with the data and checked the residuals. The students and Rhonda were satisfied with the fit.

After a 10-minute break, the class continued by trying to find the model for the oscillating part of the graph, that is, the trigonometric part of the sum. After turning off the equations and lists they had been using on the calculator, they looked at the graph of the oscillating data along with the equation they found for the exponential part of the sum. Rhonda pointed out that they did not know the period or amplitude of the oscillating data and that the entirety of the graph was "data overload." After discussing the advantages of zooming in on the function or viewing it in a smaller window, the class chose the latter option. As the students looked at the data from 1968 to 1972, Rhonda reminded them that to apply a cosine or sine function they needed the data to be along a horizontal line since, at this point, that was the only way they knew to work with trigonometric functions. She asked, " How can we make [the data points] relative to something horizontal?" A student made a guess and suggested they try inverse sine or cosine. Rhonda explained that trigonometric functions did not work in the same way as exponentials and roots and therefore that taking the inverse would not help. With no further suggestions from the students, Rhonda recommended they make the data into residuals, and they did so. From there, the students wrote a general form for the function of the oscillating curve—with a period of one year—and left space to insert a coefficient and shift later: __$\sin(2\pi x)$. Using the Trace function on the calculator, Rhonda and the students decided the coefficient should be 3. And Rhonda announced that the shift should be 1/11. When students commented that this shift "came out of the blue," Rhonda explained that a shift of 1/12 made sense to account for a shift of one month; in putting the data into the calculator lists, however, the data had been distributed so that 1/11 worked better. Students accepted that explanation and inserted the coefficient and shift into their equation. Satisfied with their trigonometric function and their exponential function, the class wrote the equation of the sum, graphed it with the data, and decided it was a good fit.

Rhonda ended the lesson by asking the students to hypothesize whether the sum equation they found would fit the other data set (1976 to 1995). Most students thought that it should, so Rhonda graphed the equation with the data set. For the late 1980s and 1990s, the equation no longer fit the data—the data were increasing at a slower rate. Rhonda pointed out that for the CO_2 concentration in the atmosphere, these results were positive (pollution of the atmosphere was increasing more slowly than the model predicted), and she suggested that perhaps the part of the sum they modeled with an exponential function was instead linear.

Reflection. Rhonda and the students worked together throughout the solution of this problem. Rhonda demonstrated and directed work on the TI-89 graphing calculators and kept track of data and equations on the board, thus alleviating some of the students' bookkeeping responsibilities. Rhonda commented that this was not a problem she would expect students to solve on their own but believed that it "was good for them to see an application of that kind of sum. It is hard to imagine what it would really be like. But I think this [problem] is one that they can all understand and believe is real."

Rhonda remarked that she felt a bit rushed and would have liked more time, but she was not sure what would be gained by distributing the problem across 2 days. She acknowledged that the nature of the lesson limited the students' activities and independence, but she thought the students had been engaged in, and benefited from, the problem. The problem required that the students recall particular calculator skills and mathematical content—in particular, sums, residuals, exponentials, trigonometric functions, linear regression, algebra, and the shifting of functions—that they had not used for awhile. Rhonda was favorably impressed by how well the students had responded to the problem and had worked with the mathematics, with the exception of failing to realize that they needed to make the oscillating data into residuals. She was disappointed that she had given so many hints about the oscillation, but she commented that this realization was the most difficult aspect of the problem.

The Problems Classified According to Dimensions

There exists a wealth of applications and modeling problems and materials for use in mathematics classrooms at various grade levels. These materials range from mere "dressed up" mathematical problems to "really real" authentic problem situations. The teachers in this study reported using a variety of problems in their classes—from straightforward applications to complex modeling problems. The five problems described in this chapter were representative of these different problem types. I did not ask the teachers to classify the problems, but they offered their interpretations nonetheless. The ways in which they described the problems further support the dimensions I described in chapter 5.

Gary classified the Bird Problem as an application because the students were applying mathematics that they already knew. He noted, however, that the problem required the students to recall known mathematics and incorporate it in ways they had not previously done. He also commented that even though the problem was not a modeling problem—the students were not creating new mathematics—the problem did incorporate aspects of the modeling process. In the Bird Problem, the students were required to make some assumptions: the bird would not dive to increase the speed with which the shellfish hit the rocks, wind would not be a factor, all shellfish were of the same weight, and so on. The students used the context to think about the mathematics and the variables that were important for their consideration. They then worked independently of the context in the mathematical domain to find that an optimal height existed. Finally, the students interpreted the mathematics in terms of the context and wrote explanations for how the bird could minimize its energy expenditure. The students did not complete the modeling process since they did not rethink their assumptions and repeat the process. Still, Gary felt that their experience with the problem was valuable and that they were learning about modeling, whether they knew that explicitly or not.

The Wind Chime Problem was classified by Gary as "math in context." He did not consider it a modeling problem because the students were given a specific problem to work on. He did not give them the freedom of, say, "Here's a bunch of pipes, go do something." He specified that they were to capture sound waves, write equations to model them, decide how frequencies were related to pipe length, and build a wind chime that sounded a particular chord. As with the Bird Problem, the students were familiar with the mathematics involved, and they were learning how it could be used. In my view, this description of the Wind Chime Problem parallels Gary's description of the Bird Problem, and he could have easily classified the former as an application. Recall, however, that Gary viewed applications as somewhat more advanced than mathematics-in-context problems because in solving an application, one moved closer to completing one cycle of the modeling process. Thus, I suspect the aspects of the modeling process embedded in the Bird Problem led him to classify it as an application, whereas the Wind Chime Problem hardly incorporated the modeling process at all. The students found the equation that modeled the wave graphs, but the problem did not require them to consider the context, make simplifying assumptions, or test their equations in terms of the context. Therefore Gary had classified the Wind Chime Problem as mathematics in context.

Like Gary's description of the Bird Problem, Diane's classification of the Foul Shot Problem was as an application that introduced aspects of modeling. The students were already familiar with the mathematics involved. Diane had commented to her students that they could have tackled the problem before they took precalculus and been successful with it. Still, she preferred to introduce the problem in precalculus because it gave the students the opportunity to learn a valuable lesson about mathematics and modeling—that is, the assumptions one makes determines the mathematics one does and the solution one gets. Changing those assumptions changes the mathematics. The Foul Shot Problem forced the students to think about the context, recognize that their answers were inadequate, and therefore realize the assumptions they had started with. Many problems do not force that issue—the students make assumptions but are often unaware that they are doing so. The Foul Shot Problem brought attention to those assumptions and required that new ones be made. The mathematics in the problem (already known and relatively noncomplex) led to Diane's classification of it as an application though it taught a lesson about modeling.

Rhonda classified both of the problems she taught—the CO_2 Problem and the Swing Problem—as modeling problems. These classifications were based on the nature of the problems (to create a model for a situation), the role of the real world (both used real data), the mathematics involved (complex and required integration in new ways), and the time required in the solution process. In the CO_2 Problem, the students were given real data and a graphical representation, and asked to find an equation that modeled the data. In the Swing Problem, the students collected their data, worked to write the parametric equations that modeled the swinger, and tested their model by graphing their equations on the calculator. It should be noted that the problems involved the production of mathematical models but that the students did not necessarily use the modeling process. Rhonda, however, organized the lesson on the CO_2 Problem so that it mimicked the modeling process.

In summary, the teachers classified the problems in this study differently. Their classifications were based on a number of dimensions—some pedagogical, some mathematical, and some contextual. Some dimensions seemed more important than others. For example, the Bird Problem could easily be characterized as a modeling problem according to Gary's descriptions of the problem types, yet the fact that the students already knew the mathematics relegated it to classification as an application.

The Lessons

Some teachers doubt that a single problem has the potential to engage students for extended periods of time and to fill 50-minute class periods. Every lesson observed in this study was based on a single problem that required at least 90 minutes—some required two 50-minute periods—and the teachers reported they were rushed to finish three of the five problems. The lessons were rich in mathematics and rich in technology. Some problems (the Bird Problem, Wind Chime Problem, and Swing Problem) incorporated simulation, experimentation, and data collection. The others did not. In four of the five lessons, students worked in either pairs or "pods" (groups of three or four); in the fifth (the CO_2 Problem), students were seated in pods but worked mostly as individuals interacting as a class with the teacher. Questioning, argumentation, discussion, visualization, reasoning, and verification were central in all of the lessons. And three of the five problems required that the students write about mathematics and their reasoning.

Each of the problems covered a range of mathematical ideas and content, not just a single topic. Some problems connected different aspects of mathematical content, some were used as assessments, and some developed problem-solving processes. Because several problem solutions relied on the data that students collected, there were often multiple correct answers and different approaches to a solution. Aspects of modeling could be identified in all of the problems; some were more focused on model production, and others on the modeling process.

The problems, with the exception of the Foul Shot Problem, required students to incorporate mathematics content in ways that were unfamiliar. Attention to the context in the problems varied and seemed most important in the Bird Problem and the Foul Shot Problem. Finally, the degree to which students worked independently also varied with the problem. Some lessons (Wind Chime Problem and Swing Problem) were more open, and the students worked independently for the bulk of the time. Other lessons were very directed (the Bird Problem, Foul Shot Problem, and CO_2 Problem), and the teacher's role was central in the unfolding of the lesson.

Classroom Practices

In this section, I zoom back and take a wide-angled look at these teachers' classrooms to describe features of their instruction in general. I focus on the practices of both the students and the teachers, namely their roles in classroom discussion and problem solving.

The teachers in this study took up roles as guides or facilitators. The teachers designed, adapted, or selected instructional activities; prompted initiating and guiding discussions; and reformulated or clarified aspects of students' mathematical contributions. The teachers maintained an expert role in the classroom but shared that role with students. For example, all of the teachers were open to students' suggestions and often pursued tangents and made side trips while solving a problem if they thought it would facilitate student learning and if they had time. The knowledge contributed by students was respected and valued. In addition, the teachers expressed enthusiasm and a passion for mathematics and worked to stimulate and incite passion in their students.

The teachers in this study had established practices that forced the students to be more self-reliant. The students could not turn to the teacher for validation of their answers or for direct answers to their questions. The teachers often answered questions with questions—though, sometimes leading questions—and forced the students to "figure it out" for themselves. They were not expected simply to produce correct answers quickly by following prescribed procedures. In these teachers' classrooms, the students had other obligations such as explaining and justifying their solutions, trying to understand the solutions of others, and asking for explanations or justifications if necessary. The students worked cooperatively in pairs or "pods" and shared their ideas and misunderstandings openly.

The teachers and the students alike practiced correcting each other and providing argumentation for claims. These practices were so natural that, as an outsider, I wondered whether the students might be offending one another. The students debated each other's suggestions, questioned each other's assumptions, and critiqued each other's ideas and answers. The level of maturity with which these negotiations were handled indicated that these mathematical practices had been well established and that the students were accustomed to such activity.

In addition, whole-class discussion of solution procedures and problem situations was a core practice in these teachers' classrooms. Many class discussions involved mathematical discourse—conjecturing, justifying, and challenging ideas and solutions. In addition, the whole-class discussion often involved the interpretation of the situation expressed in the contextual problem, and explored the reasonableness of the solution processes in light of the given context. For example, in the Foul Shot Problem, the first transition point in the lesson depended on an interpretation of the solution as unreasonable in the problem context. In recognizing that one cannot have a fractional part of a foul shot, one is forced to make new assumptions or conjectures and to approach the problem from a different perspective. This shift demands a reinterpretation of the situation expressed in the problem—instead of assuming that the announcer gave exact percentages, one assumes the percentages were rounded according to conventional criteria. Such discussions shift attention towards reflection on the solution processes as well as the solution itself.

Across all of the classes I observed, the teachers required the students to give explanations with their answers. For example, when his students were working on the Wind Chime Problem, Gary insisted that they tie their descriptions to the particular characteristics of the graph from which they drew their conclusions. Related was an emphasis on using and understanding the language associated with mathematics. Gary did not permit his students to speak with pronouns and insisted that their language be precise and accurate. Similarly, Rhonda and Diane paid attention to their students' language, making corrections and asking for clarifications and elaborations as necessary.

Finally, visualization of problem situations and graphs was a common practice in these classrooms. The teachers asked the students to think by themselves about various aspects of a problem and how they might solve it. To help the students focus their thoughts, all three teachers asked for qualitative graphs—graphs that represented the features of an equation (slant, intercepts, and so on) without being exact. The teachers cited potential benefits of this activity—improved understanding of graphs and how they relate to features of equations, better articulation of the dynamics at play in a given problem, and increased "thought time" on task.

Teachers' Instructional Practices With Contextual Problems

In this section, I zoom in to describe the features of the teachers' instruction that I identified as important in shaping the unfolding of contextual lessons—how they (a) adapted and used problems from other sources, (b) helped the students formulate the problem (by their talk, and by opening or closing students' ideas), and (c) balanced time and attention to the context and the mathematics.

Adaptation of Problems and Problem Formulation

The teachers in this study reported that media sources—newspapers, books, television, and the Internet—were abundant with data and descriptions that were adaptable as problems and projects. Practitioner journals were seldom read by these teachers and rarely a source of contextual problems. More often, a news report or data set sparked an idea that was developed and implemented as a problem in the classroom. The problems I observed were derived from a number of sources. The Bird Problem and the Wind Chime Problem were adaptations of problems Gary had seen elsewhere. The Bird Problem had appeared in the *Mathematics Teacher* (Keller & Thompson, 1999), and the Wind Chime Problem had been motivated by a presentation he had attended at the T^3 International Conference in 2001. The Foul Shot Problem resulted from an announcement heard by a teacher at Constantia Ridge when watching a Chicago Bulls basketball game in the mid 1990s, and the other two problems had been developed by the teachers at Constantia Ridge as part of a curriculum development project.

In the discussion that follows, I make two points about how these teachers used contextual problems. First, the lesson they taught rarely reflected the problem as it was given in their textbook or the handouts they used. The teachers had a "big picture" sense of what the problem was about and the goals they wanted to accomplish. They did not adhere to the problem as it was presented in the materials; they used the problem as a starting point for their work with their students and allowed the problem to unfold in ways that were their own. Second, the opening question in the problem presented to the students was seldom the question they would be answering. The teachers often orchestrated the lesson so that the students were forced to formulate the mathematical question and decide what the problem to be solved was. Even when it seemed that the students had been given the problem from the beginning (the Foul Shot Problem), that was not the case.

For example, the Bird Problem as it appears in the *Mathematics Teacher* (in five handouts that Gary "used") presents a paragraph about biologists observing the behavior of crows when dropping mollusks (or whelks), and asks, "Why do crows consistently fly to a height of about 5 meters before dropping a whelk onto the rocks below?" The question is followed by a series of five other questions about the possible flight patterns of crows when dropping mollusks and the relationship between the number of drops and the height of the drops. As the problem unfolds in the handouts, the students are directed through the peanut experiment and given the form for the equation that models the peanut data. They are also given the equation for the linear re-expression. Without any commentary on why one might be interested in a formula for work or energy, the formula is given with the appropriate substitutions already made from the earlier equations, and the students are asked a series of questions related to the formula. The problem concludes with the students writing a reflection and exploring the conjecture that crows select a height that minimizes the work needed to break open the whelks. Gary's implementation reflected aspects of the published problem but was distinctly different.

Gary introduced the Bird Problem by asking the students to think about how a bird might open a shellfish. He pursued their ideas and in questioning led them to decide that dropping it onto rocks would be the best method. Gary then asked the students to think about how a bird might drop the shellfish. Latching on to a student's mention of efficiency, Gary led the students to formulate the mathematical question (though they did not yet know it): Find the optimum height from which the bird should drop the shellfish that minimizes the energy expenditure of the bird.

Gary never mentioned the paragraph given on the handouts he used from the *Mathematics Teacher* (Keller & Thompson, 1999), and he did not ask the questions presented there. In his implementation, the students realized that the relationship of importance was that between the number of drops and the height from which the shellfish was dropped. Gary did not introduce these variables—the students identified them on their own. Similarly, the students relied on their knowledge of mathematics and developed all of the formulas and equations used in the problem. Through discussion, they decided that a bird would be most interested in energy expenditure and discussed why. They conjectured how they might quantify energy and looked for a minimum. They then interpreted the minimum in terms of the context—there was an optimal height from which the bird could drop the shellfish so that energy expenditure was minimal. The only aspect of the Bird Problem Gary introduced as presented in the *Mathematics Teacher* was the peanut simulation. He led the students to formulate the mathematical problem for themselves by requiring them to think and conjecture, by questioning their ideas and asking for clarification, and by asking questions that forced them to pursue an idea or abandon it for a different one. Gary was adept at opening the problem up to possibility and closing down what he determined to be nonproductive routes without obviously doing so. The students were in control and were formulating the problem in their discussion; Gary was also in control but in subtle ways. As is evidenced, Gary's implementation of the Bird Problem was very different from its presentation in the *Mathematics Teacher*.

With the Foul Shot Problem, the Swing Problem, and the CO_2 Problem, the teachers stated the problem from the beginning in terms of the context. Unlike the Bird Problem, the students spent little time deciding what question to ask of the context. However they were responsible for deciding how to accomplish the given task. They were required to develop the mathematics they needed. The teachers assisted by questioning the students and directing their efforts towards graphical sketches and needed variables but the students formulated the problem.

In the Foul Shot Problem, the student groups went to work immediately to find the number of shots taken and made by the basketball player. When they thought the problem was finished (and after discussion guided by Diane's questions), they realized what the actual problem was—finding a range of solutions for the number of shots taken and made. This realization required a reformulation of the problem by the students and a different solution strategy. This reformulation was no accident; Diane had planned it. In Diane's implementation of this problem, she did not work from a written lesson plan, and to my knowledge the problem is not published in any text, so I cannot comment specifically on whether or how her implementation differed from a suggested plan.

In the Swing Problem, Rhonda stated, "Our ultimate goal is going to be to try and model what it looks like for a person to swing on a swing." To begin, she simply asked the students to sketch a graph of a person swinging. A discussion of a proposed sketch in turn prompted a student to introduce the idea of parametric equations. Rhonda asked for further clarification, and the mathematical problem was formulated: Write the parametric equations that together will model a person on a swing. Again, we see that Rhonda allowed the students to develop the problem by asking them questions and to offer their ideas. Together the students decided what the problem involved; Rhonda did not have to tell them. Also, in Rhonda's implementation of the Swing problem, she did not work from a written lesson plan. The problem appears in a precalculus textbook used at the school, but she did not implement it in the way described. In the textbook, the problem provides measurements taken by students on a fieldtrip and states for students to (presumably as individuals), "Write parametric equations that describe the horizontal and vertical positions of the swinger as a function of time. Graph the equations." Rhonda, however, required that her students work together to formulate that problem by reasoning about the task. She also required them to decide what measurements they needed and to gather those measurements. The students were never asked to read the problem in their textbook.

A similar analysis can be offered for the CO_2 Problem. The students were given data lists (on paper and in their calculators) and other background information from the Internet. The problem appears in their precalculus textbook but was not referred to. The problem in the textbook presents the scatterplot the students fitted a model to, but also describes that a sum equation is needed and states that a sinusoidal function will model the oscillating data. The textbook problem then asks the students to create the function that models the data and answer questions about it. Rhonda's implementation differed from the textbook presentation. She gave the students a graph of the data and stated their task: "To see if we can fit a model to this CO_2 concentration data over a pretty long span of time." She did not tell them the mathematics needed in their model. In a class discussion Rhonda facilitated, the students put their ideas together and formulated the problem for themselves.

The Wind Chime Problem was implemented in the most step-wise manner I observed. The students were given handouts outlining each step in the problem, and they spent their time putting those steps into action. The students worked independently for much of this problem. Gary spent little time helping them develop the problem. He had already done so in the handouts that provided the step-by-step instructions for carrying out the problem.

Balancing the Mathematics and the Context

In using contextual problems in instruction, the teachers attended to the development of both purely mathematical skills (symbol manipulation, algebraic formulas and computation) and process skills (modeling, writing, explaining, and justifying). To develop these skills through contextual problems, the teachers (and the students) had to balance the attention they gave to the context and to the mathematics. It was the teacher's task—if this was to be a mathematics lesson—to see to it that the problem became (or remained) mathematical. Each teacher organized the lesson so that it contained all the aspects of the activity considered necessary to develop the mathematics of interest. If the students were to learn a specific skill or develop a particular process skill, then the teacher had to make sure that these skills were facilitated by the problem or taught separately. The teachers had learned to take an application or a phenomenon in the world and develop rich mathematics out of it. Thus attention to pure mathematics was couched within contextual problems. Even problems that required heavy consideration of the context remained mathematical. The lessons were not “fuzzy and warm” opportunities to talk about the world. They were about mathematics. This focus sometimes resulted in near disregard for the context altogether.

For example, in the CO_2 Problem, the students were asked to find an equation that modeled a given data set. The students were not required to make simplifying assumptions or consider the context of the problem in order to work with it. Interpretation of the context happened only at the end of the class period, when the students thought about what a general increase in CO_2 concentration across time meant for the environment. Why the CO_2 data were oscillating between years was never discussed and was irrelevant in terms of the goals. The students were able to find a sum equation that modeled the data without thinking about the data in the real world. Mathematical considerations sufficed and were most important. The teacher did not spend any class time talking about CO_2 concentration in the atmosphere or any other aspect of the world. It was either assumed that students understood the contextual aspects of the problem, or that these aspects were irrelevant.

Likewise, the Swing Problem required little, if any, consideration of the real world. The introduction of the problem by the teacher required that the students think about the context but within a mathematical frame (what the graph of a swinger might look like). There was no conversation about swinging in real life. To solve the problem, the students collected their data, wrote the parametric equations that modeled the swinger, and tested their model by graphing their equations on the calculator. Certainly their data were real—they collected them themselves—but in their subsequent work, the students worked exclusively within the realm of mathematics. They had a sense of how the graph should look, and they knew that swinging involved both vertical and horizontal motion. Their goal then was to incorporate real numbers and write the equations to model the combination of those motions. The students paid attention to the motions of a swinger in their mathematical work, but they did not rely on the context for interpretations. The swinger was the source of the numbers to be used and provided a visual image of the dynamics involved in the problem—nothing more.

Teaching the Wind Chime Problem also required little balance between mathematics and the real world. The problem was about using mathematics. The students were told that the frequencies of sound waves corresponded to notes in the Western scale and were instructed to investigate the relationship between pipe length and frequency so they could build a wind chime that sounded a particular chord. This problem was certainly connected to the world and contextual in nature; but in working the problem, little attention was given to the context. The students collected data, focused on the mathematics involved, and built their wind chimes with no consideration of the context—beyond the connection of frequencies to musical notes. The teacher did not spend time discussing wind chimes or how they were built.

Balancing attention to context and mathematics was more central in the Bird Problem and the Foul Shot Problem. The teachers worked hard to balance the amount of time and attention the students spent in consideration of the different aspects of these problems. They also worked to keep the problems mathematical. Gary spent half of the class period orchestrating a discussion with the students that centered on the context of the Bird Problem. The discussion covered a range of contextual issues (how to open the shell, how to fly to drop it, the weight of the shell, terminal velocity, and so on), some that merited pursuit, and others that were dismissed in conversation. After experimentation and work in the mathematical domain—half of another class period—more attention was given to the context in consideration of the energy expenditure of the bird. Yet Gary steered the conversation so that it was mathematical in nature and kept the problem focused on the mathematics even though it relied heavily on contextual considerations. Likewise, Diane's work with the Foul Shot Problem required a balance between the context and the mathematics. It was in exploring the context that the students were able to develop different mathematics than they had started with. Thus attention to the context was not a discussion of basketball and favorite players; it was about mathematics and the assumptions that had to be reconsidered.

Closing Comments

The teachers in this study were masterful coordinators of lessons based on contextual problems. They had developed skill at questioning students and facilitating classroom discussion so that the students were led to formulate the mathematical problem and articulate solution strategies—an uncommon practice for many high school teachers. Contextual problems were used regularly and the students had become accustomed to providing explanations, justifying solution strategies, and using mathematical language. The teachers chose rich problems that incorporated mathematics in new ways, and allowed the students to pursue multiple paths (sometimes unhelpful) in the solution process. Careful thought was given to balancing time and attention to the context and the mathematics of the problem (and the use of technology). In chapter 8, I connect these teachers' practices to the literature in chapter 2 and draw implications for researchers and professional developers interested in teachers' practices with contextual problems.

CHAPTER 7

ENABLING CONDITIONS

One focus of inquiry in this study was teachers' instructional practices with contextual problems. In analyzing the data and writing about the teachers' instructional practices, I decided it was insufficient to only describe these teachers' practices—their work was not as simple as using contextual problems regularly and balancing time and attention to the context and the mathematics. Their work was complex. It was no easy task to lead students to formulate the problem for themselves. It was no easy decision to choose to use multiple class periods to solve one problem. The contextual problems they selected were mathematically and technologically complex—more complex than typically used in a precalculus course. I realized that these teachers' work with contextual problems was not unique only at a classroom level, but also at a structural level. I became aware that particular conditions were enabling these teachers to make decisions to use complex contextual problems regularly. I had approached this study with an underlying question about how U.S. teachers could more widely incorporate contextual problems into their mathematics teaching. What would it take for that to happen? I did not explicitly seek to answer that question in this study but had hoped that an inquiry into teachers' practices would begin to inform that question. But as the teachers talked in their interviews, they attributed their work with contextual problems to a number of factors they recognized as important in shaping their work. I have labeled these factors as *enabling conditions* and describe them in this chapter: technology, commitment, community support, and beliefs.

The Role of Technology

Many technological devices are available today, and many of them are highly relevant for applications and modeling. In a broad sense these technologies include calculators, computers, the Internet, and all computational or graphical software as well as all kinds of instruments for measuring, for performing experiments, and so on. These devices not only provide increased computational power but also broaden the range of possibilities for approaches to teaching, learning, and assessment. On the other hand, the use of calculators and computers may also bring inherent problems and risks.

Tom, Cathy, and Rhonda explicitly mentioned that technology had allowed for the use of more complex problems in their teaching. Using the regression packages on the calculators and the graphing features, students could work with large data sets to understand phenomena in the world. Graphing calculators minimized the amount of time taken to input data (via linking capabilities) and to draw graphs. These savings allowed more time for students to think about the mathematics and to make sense of what the mathematics was doing. Technology played a central role in all five of the observed lessons.

In the Bird Problem, the students used the graphing calculators to (a) input the data gathered in the peanut simulation, (b) study the graph of the data, (c) linearize the data and re-express them, and (d) study the graph of the re-expressed data to locate the minimum. Using technology allowed the students to complete the problem in 2 days, and instead of focusing on graphing the data and trying to re-express them by hand—a complicated task—the students were able to focus on the linearization and what the re-expressed data meant in the problem. Using the calculators was not, however, unproblematic. Recall that Gary taught this lesson and did not have list space available in his calculator for all of the data. There was some confusion for students about where different data were stored and why Gary was translating them. Additionally, the nature of the data—a 1 in the data set gave a 0 in the denominator—gave the students calculator errors whose cause they did not initially recognize.

In the Wind Chime Problem, the students used their graphing calculators, the CBL unit and sound probe (microphone), and computers with TI-InterActive! software. The CBL unit allowed the students to capture the sound waves produced when banging their pipes. The TI-InterActive! software interpreted the wave data inputted into the computer through the CBL unit and stored them in a tabular and graphical form. The software also allowed the students to select and save data in a word-processing program. Thus students could, in one place, display their tables and graphs, label their data, and write sentences explaining the data and their reasoning. Finally, the students used their calculators to find the period and amplitude of their wave graphs. The students did not encounter very many technology obstacles. One group accidentally forgot to change a setting in TI-InterActive! and on one occasion collected sound data that the software interpreted as temperature. Gary informed the students that they could still use the data but that they should write notes explaining the units assigned by the software.

The Foul Shot Problem and the Swing Problem used technology minimally in comparison with the other three problems. In the Foul Shot Problem, the students wrote equations and solved them, made new assumptions, wrote new equations in terms of inequalities, simplified those equations, and graphed them by hand. When the students suggested graphing the lines on the calculator (TI-89) to locate the lattice points, they learned that the technology was limited—it was impossible to see the region between the graphed lines, much less the lattice points in the calculator window. Still they used the table of values in the calculator to locate the solutions to the problem. In the Swing Problem, the students used their graphing calculators to calculate the periods and amplitudes in the model they were developing, and used the graphing features to "check" their equations. As with the Foul Shot Problem, the students mostly worked the problem on paper.

Finally, technology was central in the CO_2 Problem. The graphing calculator was in hand for the duration of the 90-minute problem. Students first linked their calculators and shared the data, thus saving time inputting them. Once students had the data, they were able to graph the different data sets, single out portions of the data to look at, apply techniques of linear regression and look at residuals, trace their graphs to locate important values, and so on. The success of this problem hinged on students' adeptness with the calculators and their understanding of calculator functions. Rhonda paid careful attention to the location and organization of data and helped students keep track of their work. She worked through the whole problem with the students, reminding them of calculator functions and demonstrating the calculator work. Her guidance eliminated a number of technological problems that could have developed if students had been working on their own. Still, some students encountered difficulties (oddities in their graphs because of calculator settings), and Rhonda offered immediate assistance to get them back on track.

The Wind Chime problem incorporated a number of technologies, whereas the other problems mostly relied on the students' work with graphing calculators. Using technology allowed the students to tackle complex problems in a reasonable amount of time (one or two class periods) and work with mathematics they would not have been able to do otherwise. They were able to focus on *what the mathematics was doing* and pay less attention to *how to do the mathematics*. In particular, with the CO_2 Problem, technology enabled the students to find a mathematical model for a large and complicated data set. There were data for every month from

1958 to 1995 (with an overlap of data from 1976 to 1989) and averages for every year. The data trend was increasing over time while oscillating between years. Without the use of technology, the problem would have been too involved and too complicated for use in a precalculus course. Thus, technology enabled the teachers to introduce more complex mathematics than such courses usually contain.

Commitment to Contextual Problems

Teaching applications and modeling activities is always time consuming because attention has to be paid to several crucial phases of the modeling process, including work on extra-mathematical matters. Applications and modeling components of general mathematical curricula have to compete with other components of the curriculum; in particular, work on pure mathematics. Thus I briefly describe the commitment of these teachers to use contextual problems and how they balanced attention, time, and effort between applications and modeling activities and other mathematical activities in their classroom.

Though I observed lessons taught by only three of the six teachers in this study, all discussed how they incorporated contextual problems into their classes and balanced attention between those problems and the mathematics content and skills they were required to teach. Gary and Hank taught at Watercliff Academy and worked under the expectations that all students would perform well on standardized tests (SAT and Advanced Placement exams) and would be accepted into universities upon completion of high school. Working within those expectations, Gary and Hank had stopped offering a modeling course so that students could take calculus-based courses. The perception of the administration at Watercliff was that universities valued calculus more than modeling. Still, Gary and Hank were committed to teaching with contextual problems.

Gary reported teaching three large modeling problems each year. These problems required 2 to 3 days of class time—the Suez Canal problem (taught in the fall and not observed in this study), the Bird Problem, and the Wind Chime Problem. He also reported assigning two or three other large contextual problems each semester and discussed some examples he had used in the past. The students were expected to complete these problems on their own over the course of 1 or 2 weeks. Hank did not report using large amounts of class time on contextual projects, but reported assigning six to eight such projects to students each semester and, in the interview, challenged me to think about some of them. He reported that his projects were somewhat less involved than those given by Gary, but they connected the mathematics being learned to the world.

Tom, Diane, Cathy, and Rhonda taught at Constantia Ridge in a department with the philosophy that precalculus (and other mathematics, as much as possible) should be taught by beginning with a situation and developing the mathematics from it. Thus attention, time, and effort toward applications and modeling activities were highly valued and expected. The school expected the students to perform well on standardized tests and hoped they would pursue degrees in technical fields like engineering, mathematics, and the sciences after high school. The mathematics teachers viewed the use of contextual problems as facilitating those expectations and hopes. By being exposed to applications of mathematics, students would see mathematics as useful and meaningful and would perhaps pursue its further study. The teachers were also confident that through engagement in rich problems, the students would develop the mathematics necessary to be successful on standardized tests and in college. Finally, Tom and Diane taught a modeling course in which students were given a setting and the freedom to choose a question to ask of the setting and pursue it. The focus of this course was for students to gain experience with using mathematics to model aspects of the world—modeling as a process.

All of the teachers in this study held a "big picture" philosophy of teaching mathematics. Their focus was broad and centered on where students would be at the end of the school year or the end of high school. They worried less about the day-to-day pressures of standardized testing and curriculum guides. Instead, while building a classroom culture that fostered students' skills with reasoning, problem solving, questioning, explaining, and writing mathematics through engagement in rich contextual (or modeling) problems, they trusted that their students would succeed.

Applications and modeling activities inevitably compete with pure mathematical activities for time and attention in courses. The teachers in this study, however, valued such problems and were therefore willing to organize their classes so that applications and modeling were central in their work. They made these arrangements, with some exception, in spite of curriculum guidelines and content requirements within the school. They were confident that applications and modeling problems enhanced students' understanding of pure mathematics content and were therefore necessary and worthwhile components of their teaching. Using 2 to 3 days on one problem was acceptable if the problem was rich and provided an opportunity for the students to connect mathematics concepts to each other and to the real world. In addition, a commitment to such problems over time enabled the teachers to incorporate contextual problems more regularly because with repeated exposure, the students gained skill in working with such problems and became more willing to engage with and solve them. Teachers' instructional practices with contextual problems were also refined and improved over time as they interacted with different groups of students (and teachers in workshops and at conferences) to work through the problems.

Community Support

The teachers in this study (with the exception of Hank and Cathy, who began in the 1990s) began to implement ideas of applications and modeling in their teaching in the mid to late 1980s and have sustained that implementation over the years. How was this possible? As I talked with the teachers, I noted one thing: Their work with applications and modeling was sustained by their participation in communities that valued such work. The teachers were constantly referencing their colleagues as sources of motivation and as integral in shaping the teachers they had become. Gary and Tom referred to ongoing friendships they had formed in the mathematics education community at large through participation in the Woodrow Wilson Institutes in the 1980s to 1990s and other such events. They reported that they continued to collaborate with many of those same people on various projects and that those collaborations were central to their daily work as teachers. Gary expressed great emotion when describing the friendships formed and expressed gratitude for the opportunities he had been given. He felt honored to have been able to work with Henry Pollak and other talented mathematicians, mathematics educators, and teachers. Gary's participation in the mathematics education community outside his school had been very influential in his teaching. The community at Watercliff was also important, but it had not shaped his career in the same ways as the external community.

Hank had begun teaching in the 1990s at Watercliff Academy with Gary and reported that the shift in his teaching towards the inclusion of contextual problems had happened during his time at that school. He could not pinpoint whether the influence had come from his previous department head, his co-teaching of the modeling course with Gary, or his own reading and development as a teacher, but he viewed his colleagues and community as sources of encouragement and growth.

Tom spoke about the communities in which he was involved outside of school and cited that work as very influential in his daily practice. He, however, felt very fortunate to also have the department support at Constantia Ridge. He commented, "It's essential to have *a* community" in which everyone works as "coequals." Tom added that being part of two communities—one external to his school and one internal—allowed for two different kinds of collaborations. The internal community's work focused on what was happening in their school—on the philosophy of the department, their commitment to particular kinds of teaching, and their day-to-day activity. The external community served a different purpose. Most of Tom's work outside of Constantia Ridge involved work on curriculum projects or serving on committees. He recognized that many teachers, particularly those "who have come to the Woodrow Wilson things or gone to Exeter [Phillips Exeter Academy's summer workshop], have developed a stronger affiliation with that external community than [with] their internal [community]—I have the luxury of having both."

Diane, Cathy, and Rhonda also spoke about their participation in both an internal and external mathematics education community. Often they led workshops, gave presentations, and worked with teachers. They found this work exciting and rewarding. They reported learning from each other and from the teachers with whom they worked. In addition, they (along with Tom) spoke of the community within their department at Constantia Ridge as one of collegiality and innovation. The teachers at Constantia Ridge had been fortunate enough to develop a precalculus and calculus curriculum together. That experience had served two purposes: to bring them together as a community and to develop a shared philosophy in the department that drove all of their work. The teachers were continually developing new problems, observing each other's teaching and offering critiques, and exploring new mathematics. Working within a structure that valued the use of applications and modeling problems and that provided opportunities for collaboration toward such use, these teachers had been able to sustain their commitment to this collaboration for almost twenty years.

From the conversations with the teachers in this study, I concluded that participation in communities played a role in helping teachers sustain curriculum initiatives. And community involvement must extend beyond the gates of one's school. The teachers at Constantia Ridge had developed a strong internal community with shared philosophies and ongoing innovation in terms of applications and modeling activities. Yet they reported that their work outside of the school also played a central role in their development as teachers. Sharing their work with others led to self-renewal and encouragement—others were interested in what they were doing and that was exciting. At Watercliff Academy, the internal community was not as strong. Teachers met two to three times per year to discuss testing and curriculum, but a collaborative atmosphere had not been established. Thus, Gary relied on collaborations outside his school for support. Hank hardly spoke of collaborations internal or external to his school, yet commented that in his attendance of state and national meetings, he always latched on to some idea he could use in his classroom.

Teachers' Beliefs

In chapter 4, I described the beliefs held by all six teachers' concerning mathematics, its teaching and learning, and contextual problems. These teachers' beliefs are important for consideration in conceptualizing the enabling conditions of these teachers' practices with contextual problems. For example, because these teachers believed that students should struggle with mathematics, make mistakes, and try a different approach, the teachers were willing to use multiple days for the solving of one problem. These teachers also believed that students were capable of working with difficult contextual problems and therefore waited for them to formulate the problem and figure out the mathematics needed to solve it. The teachers did not give this information freely. In addition, these teachers were comfortable using class time to let students think and struggle, so much so that students' questions were often answered with questions. And that these teachers were continually seeking new problems and contexts with which the students could engage in mathematics supported their belief that it was the teacher's responsibility to connect the mathematics being taught to the world.

The teachers also held strong beliefs about the power of contextual problems for reaching more general curriculum goals and for motivating and engaging students. In part, these beliefs enabled the teachers to make decisions about using contextual problems, using them regularly, and remaining committed to them even when it was difficult for students, or when the administration was not completely supportive. An example of the latter was illustrated by Gary and Hank's commitment to teaching with contextual problems after the modeling course was replaced by a calculus-based course. In addition, their beliefs that such problems were important helped sustain their commitment to such problems despite the low level of community support internally at their school. At Constantia Ridge, the teachers' beliefs about the importance of teaching with contextual problems had led to a departmental philosophy that supported those beliefs and encouraged their work with such problems. Their beliefs and shared philosophy encouraged and strengthened their collaborative internal community and inspired their involvement in an external community.

Closing Comments

Many of the instructional practices with contextual problems observed in the teachers in this study were atypical, as were some of their beliefs. Part of what allowed for such practices and beliefs were what I have labeled *enabling conditions*. The facilities of technology had allowed these teachers to introduce more complex mathematics in earlier courses, and also allowed students to engage in interpreting and understanding mathematical ideas before understanding the mechanical aspects. The students were able to use mathematics to model real-world phenomena, and in the process, motivation for the study of particular mathematics was induced. The process was further facilitated by the teachers' commitment to teaching with such problems in spite of curricular guidelines, student difficulties, and time constraints. In addition, teachers' involvement in communities, both internal and external to their school, enabled them to find support for their efforts. The community also served as a source of professional development and self-renewal for these teachers by introducing them to new ideas, problems, contexts, technological improvements, and so on. In this way, community involvement was not only inspirational and motivational but also educational. Finally, these teachers' beliefs were part of what enabled them to teach with contextual problems on a regular basis. Their beliefs also supported other enabling conditions—like their commitment to using such problems and their willingness to be involved in the mathematics education community at large. In chapter 8, I use the enabling conditions identified in this chapter to draw implications for mathematics educators seeking to prepare teachers to work with contextual problems. I argue that these conditions are important for consideration in that preparatory work.

CHAPTER 8

CONCLUSION

> It's not what we read, but what we remember, that makes us learned. It's not what we intend, but what we do, that makes us useful. It's not a few faint wishes, but a lifelong struggle, that makes us valiant. Henry Ward Beecher

Summary and Reflection

The incorporation of contextual problems in mathematics teaching is recommended and encouraged (NCTM, 1989, 2000), but few researchers have studied teachers' practices with such problems. The purpose of this study, therefore, was to investigate high school teachers' notions of terminology associated with contextual problems—*mathematics in context*, *applications of mathematics*, and *mathematical modeling*— and their instructional practices with such problems. In this study, I focused on the practices of six teachers with reputations for using contextual problems on a near-daily basis. As part of that investigation, teachers' beliefs about mathematics, its teaching and learning, and contextual problems were described and used to situate the teachers' practices. The significance of the study lies in its contribution towards a better understanding of how teachers teach with contextual problems—in particular, how they adapted the problems, helped students formulate the mathematical question, and balanced time and attention to the context and the mathematics.

The participants in this study were six secondary teachers—two in a private school and four at a public school of mathematics and science. The teachers were selected based on recommendations from university professors and on their presentations at a conference focused on secondary mathematics teaching. All participants met a range of criteria including more than 10 years of teaching experience, a reputation for teaching with contextual problems, and advanced degrees in mathematics. All six participants were interviewed, and three were observed teaching lessons they selected and identified as incorporating contextual problems. The data from the interviews and observations were analyzed using constant comparative techniques associated with grounded theory methodology (Glaser, 2002; Glaser & Strauss, 1967).

The findings in this study addressed two independent research questions:

1. How do teachers with reputations for using contextual problems on a near daily basis define and relate the terms *mathematics in context*, *applications of mathematics*, and *mathematical modeling*?

2. What are these teachers' instructional practices with contextual problems?

<u>Relating the Findings to the Research Questions</u>

Ernest (1989) and Raymond (1997) have proposed that to understand teachers' knowledge and practices, one must consider their beliefs about mathematics, and its teaching and learning. Thus, in answering the research questions, I first investigated these teachers' beliefs and presented them in chapter 4. Specifically, I asked: What beliefs about mathematics and its teaching and learning do these teachers hold? What beliefs about the nature and role of applications, modeling, and contextual problems do these teachers hold? I noted that these teachers held various beliefs about mathematics and its teaching and learning. They shared beliefs about selecting contextual problems, however, using them often and allowing students to struggle when solving such problems. This finding has implications for the further study of teachers' beliefs as it relates to their teaching of contextual problems. The sample of teachers in this study held differing beliefs about mathematics and its role in the curriculum, yet their classroom practices and beliefs about students' engagement with contextual problems were strikingly similar. Perhaps consideration of how this configuration of beliefs (and others) affects practice could provide further insight into teachers' work with contextual problems.

The first research question was then addressed in chapter 5, where I described the ways in which these teachers thought about and defined the terms *mathematics in context*, *applications of mathematics*, and *mathematical modeling*. I offered an analysis of the graphic organizers drawn by the teachers relating the terms and identified the shared dimensions along which the teachers described them. I noted that the teachers' notions of the terms were at times fuzzy, but when taking into account multiple dimensions of each term they were able to make distinctions between them. I also offered that the teachers' practices seemed unaffected by the ambiguity associated with the three terms. In fact, even when faced with unfamiliar terms (e.g., *mathematics in context*), the teachers did not express concern about whether the descriptions they offered were "right" or "wrong." They were more interested in knowing how their colleagues had answered the question; they were concerned about having and maintaining a shared philosophy and understanding within their department. In fact, the teachers at one school asked for a copy of my interview guide so the questions could be the central topics of their mathematics team retreat in the upcoming summer.

These findings lead me to question whether there is a need for consensus on the definitions of the three terms and whether the dimensions I identified in the teachers' descriptions might benefit mathematics educators conceptions of the terms. Recall that I identified discrepancies in some of the definitions being put forth by mathematics educators (chapter 2) and argued that perhaps the language and associations surrounding the terms had muddied conversation and understanding among both mathematics teachers and researchers in mathematics education. The six teachers in this study, however, were able to make distinctions between the terms and reconcile for themselves how the terms might or might not be related despite their ambiguity. Thinking about the terms along multiple dimensions proved useful in this reconciliation. Perhaps these same dimensions, which are detectable in some of the descriptions given for the three terms in the research literature, warrant more explicit attention from mathematics educators. The literature cited in chapters 1 and 2 suggests that authors often select definitions that work best for their purposes but may fail to consider the different roles that definitions play. They may also fail to realize how the definition they put forth—whether informal, formal, or extended (McLeod & McLeod, 2002)—overlaps with or confuses other definitions of the same or a similar term. As a result, researchers may be unable to determine the degree to which another's research informs their own. Perhaps consensus on definitions is not necessary—different definitions serve different purposes—but maybe explicit attention to multi-dimensional definitions, rather than one-dimensional definitions, could resolve some of the ambiguity currently associated with the three terms of focus in this study.

Besides the dimensions, I identified a distinction in the teachers' notions that seems unarticulated in the literature—the idea of using the *modeling process* itself as a way of structuring and organizing classroom activity and learning. Typically, modeling, when thought of as a pedagogical move, involves modeling an activity, repeating the activity with students, and then asking students to do the activity on their own. The teachers in this study however spoke of the (mathematical) modeling process as a form of classroom organization. Students were given a problem; as a class, particular assumptions were made and discussed; mathematical activity was engaged in without consideration of the context (technology was often used); a model was created; questions of the model and assumptions were made in terms of the context; new or different assumptions were made; and the lesson and process continued. In essence, the teachers used the modeling process to shape the organization of their lessons, and not just in solving problems. This practice might provide a new way for thinking about classroom activity and organization as mathematics educators work with teachers seeking to teach with contextual problems.

The central focus of the second question—teachers' instructional practices—was addressed in chapter 6. In investigating these teachers' practices with contextual problems, I found that on the surface their practices were not very different from those one might expect of many teachers. Many reform-oriented teachers use technology, act as facilitators, and negotiate a classroom culture that involves students in questioning, reasoning, and argumentation. What differed was these teachers' commitment to contextual problems and modeling, and their practices specific to such problems. They valued students' engagement with such problems and therefore adjusted their curriculum to allow for their inclusion. For these teachers, knowledge was continuously recreated, recycled, and shared by teachers and students. Thus, they were not dependent on state curriculum frameworks or textbooks to decide what and how to teach. I identified three practices of these teachers' instruction as important in shaping the unfolding of contextual lessons: how they (a) adapted and used problems from other sources, (b) helped the students formulate the mathematical problem (by their talk, and by opening or closing students' ideas), and (c) balanced time and attention to the context and the mathematics.

First, these teachers were adept at adapting problems and modifying them in action. As students made suggestions and shaped the direction of the lesson, the teachers were able to negotiate their goals in view of the new direction while maintaining a vision of how the problem might unfold and where it would lead students. The teachers were proficient at giving the students ownership of the problem and their work with it. Yet the teachers accomplished the goals they had set for the problem. The teachers were not rigidly tied to the organization and presentation of the problems as they were laid out in textbooks or articles. Even problems the teachers had developed themselves were presented in ways that resembled their original forms, but were significantly different. These adaptations, in part, facilitated the second practice I identified—their ability with leading students to formulate the mathematical problem.

When introducing a lesson, it is typical for the teacher to tell the students the mathematical goal of the lesson detailing the mathematics needed to accomplish that goal. Yet these teachers refrained from that practice. They gave the students a task (e.g., model a swinger in a swing) but it was the responsibility of the students to realize the mathematics involved in completing that task. Finally, I noted that these teachers masterfully balanced time and attention to the context and the mathematics. They devoted class time to the discussion of the contextual features as well as the mathematical features of the problems. They also balanced the teaching of contextual problems with the teaching of more straightforward computational problems. These findings invite researchers to consider a number of important questions: How might we instill these practices in teachers? What levels of comfort and what skills are necessary for a teacher to do this work? How can teachers foster students in taking the responsibility of formulating the problem? What are the advantages for students who formulate the problems for themselves? Are there disadvantages? By focusing on these three practices specific to teaching with contextual problems, in addition to the general teaching practices of these teachers, we can begin to parse out skills and knowledge important for successful teaching with contextual problems.

Finally, in chapter 7, I described four enabling conditions that allowed these teachers to do the work described in chapter 6. These teachers' practices with contextual problems were unique and complex. And their work was unique not only at a classroom level (i.e., their practices with contextual problems) but also at a structural level. Particular conditions were important for enabling these teachers to make decisions to use complex contextual problems regularly. The first enabling condition was the introduction of technology—namely, graphing calculators and computer software—into the mathematics curriculum. Technology allowed the teachers to introduce complex mathematics in earlier courses. The second and third conditions were their commitment to such problems and their involvement in communities both internal and external to their school. Finally, I made the claim that these teachers' beliefs as described in chapter 4 could be considered as enabling conditions, as well as supports for the other enabling conditions. Furthermore, the enabling conditions could be classified into two groups. The teachers' commitment and beliefs could be thought of as *internal* conditions—perhaps not easily changed, motivated, or inspired. The introduction of technology and the teachers' involvement in communities could be considered as *external* conditions—those that could be affected by intervention from mathematics educators. Classifying the enabling conditions in this way provides a way for mathematics educators to begin to think about how they can best serve teachers in their work with contextual problems.

Significance of the Study

Teachers' Notions of Terminology

Describing these teachers' notions of the three terms and how they were related was important for two reasons. First, without an understanding of how teachers and mathematics educators are thinking about and using the terms, we cannot begin to clarify the meanings and understand how different descriptions and definitions affect research and practice. If mathematics educators are not using definitions of terms in a consistent manner so that research can accumulate and inform further study, then the result is pockets of knowledge that may or may not be contributing in significant ways. The teachers in this study did not offer textbook definitions of the terms. They articulated their versions of how the terms were related or not, and I was able to identify a number of shared dimensions along which they considered and differentiated the terms. In the literature reviewed in chapters 1 and 2, explicit attention was not given to these dimensions, although they can be identified in some of the definitions found in the literature. I propose that mathematics educators can gain from giving attention to how they are defining and using the terms *mathematics in context*, *applications of mathematics*, and *mathematical modeling*. One gain would be more explicit and articulated notions of how these terms are related, and how research on one of these concepts informs research on other related concepts. By considering each term along each dimension identified in these teachers' conversations, and perhaps other dimensions not articulated in this study, mathematics educators can begin to more easily differentiate between the terms and more readily recognize their blurred boundaries. As a result, mathematics educators could more readily see how research on contextual problems is related, and our knowledge could accumulate.

Second, discussing how teachers (and researchers in mathematics education) think about and use these terms provides insights into the philosophies with which they approach their practices, whether teaching or research. The teachers in this study were not interested in "correct" definitions and descriptions of the terms. They were interested in how different descriptions might affect their philosophies of teaching, mathematics, and learning. Perhaps as researchers and teachers in mathematics education, we too should be concerned with shared philosophies as well as precision in our employment of terminology. Muddy notions about terminology may result in muddied philosophies and incoherent goals for teaching with contextual problems.

Given the array of literature describing terms such as *mathematics in context*, *applications of mathematics*, *mathematical modeling*, the *real world*, and *reality*, it is not surprising that teachers and mathematics educators have various notions of what these terms might mean. The questions we must begin to ask should take into consideration whether and how different ways of organizing and defining the terms matter in practice. If the terms are defined in a particular way, what might that mean for teaching practices and curriculum design? What might change if they were defined differently? How is *real* described in our schools and in our mathematics classrooms? Who decides what is real? And in making that decision, who is privileged and who is marginalized? Some researchers are beginning to make strides in these directions, but much work is yet to be done.

Other questions for consideration ask whether teachers should be expected to have clear notions of these ideas, or if that is even possible. Should the theories and concepts developed by researchers and other mathematics educators be clear to practicing teachers? Can they ever be clear to anyone? Exactly how much should be clear, and how much is irrelevant for practice? I believe that teachers are more proficient in the classroom and in implementing curricular ideas if they have a shared understanding of what the curriculum is about and how it was designed. The teachers in this study had a clear understanding of what they meant by *modeling* and *applications,* and that understanding was part of a shared departmental philosophy—at least at Constantia Ridge. Certainly differences were evident in their descriptions, but each held a "big picture" philosophy concerning the teaching of contextual problems and what that looked like. This philosophy motivated and sustained their teaching with such problems.

Teachers' Instructional Practices and Enabling Conditions

Teachers' instructional practices with contextual problems warrant continued investigation, and this study has contributed to that process. First, my research informs our general understanding of the relations between the structures and conditions of schooling and teachers' instructional practices. Specifically, in this study I identified enabling conditions that allowed these teachers to engage in their work with contextual problems. I also identified three instructional practices specific to that work. For mathematics educators interested in preparing teachers in the United States (and around the world) to more broadly and commonly use contextual problems in their teaching, much can be learned from my study. For example, part of what allowed the teachers in this study to do their work appeared to depend on their access to and ability with technology, their commitment to using contextual problems even when doing so was not supported by their administration or was absent from their textbooks, their involvement in a community that supported that work even if it was outside their school, and their beliefs about teaching with contextual problems and how students learn mathematics through such problems. This finding points to key (external and internal) conditions that facilitate teachers' work with contextual problems. To prepare teachers for this work, mathematics educators must attend to these—and perhaps other—conditions. The finding also raises questions about teacher induction and support. What happens when teachers enter a closed community? How can communities be built so that teachers are doing authentic work together and developing a shared philosophy? What kinds of materials and professional opportunities support teachers who seek to use contextual problems in their instruction?

Another question raised by this part of the study is related to the fact that the teachers all chose precalculus lessons for observation. One must ask whether the nature of precalculus courses is itself an enabling condition. Precalculus is one of the few high school mathematics courses that does not require a high stakes test be passed upon completion of the course. In addition, it is a course typically taught to sophomores or juniors and is therefore early in the sequence of courses a student will take. As such, the course is low-pressure for both the teacher and the students—it is not considered central in preparing students for standardized exams important for college admission. Perhaps this condition enables teachers to devote more time to teaching with contextual problems and fostering students' work with them. Teaching a course that necessitates more accountability might diminish teachers' efforts with contextual problems. Investigation into teachers' instructional practices with contextual problems in other mathematics courses warrants further investigation.

Second, the findings of this study contribute to the field by beginning a conversation about teacher preparation and students' role in learning. How can we prepare teachers to use contextual problems in their teaching? This study suggests that in addition to the enabling conditions, teachers need to engage in particular practices to facilitate students' learning. They must be adept at questioning students, must allow students to struggle and think (exhibiting patience), and must be knowledgeable in both mathematics and pedagogy. These skills are not easily developed. Many teachers think it is their job to remove struggle and challenge from students' studies of mathematics. The teacher must "help" the student by giving hints, or the teacher must organize his or her lessons and select problems so that students struggle little—after all, students may become frustrated and disengage. The practices of the teachers in this study challenged these ideas about the teacher's role in the classroom. The teachers provided a site of struggle for students and set up a classroom culture that involved elements of risk and discomfort for both themselves and their students. Yet they were successful—the students were learning complex mathematics and its uses in the world, and the teachers were gaining proficiency and recognition as teachers. Perhaps these practices could be reconceptualized as aspects of pedagogical content knowledge that can be developed and taught to teachers. How might that knowledge be obtained? What beliefs must teachers hold before they can allow students to struggle, perhaps for days, with one problem? How can we as mathematics educators facilitate teachers in developing this skill?

Third, the findings have important policy implications for those working in education, especially for those concerned with curriculum development and teacher education. What should be included in the curriculum and how can a spirit of innovation be developed in teachers? What is the role of curriculum guidelines, and how might they address the use of contextual problems? Should use of contextual problems be mandated or recommended at all? When is too much? What is not enough? These debates have already begun among mathematicians and mathematics educators. Perhaps teachers and policy makers should be more involved in this conversation.

Fourth, research such as this should continue to be undertaken because mathematics remains one of the key areas of study within our formal educational institutions. It also continues to be a domain within which students' abilities are measured and judged and used as part of both selection and allocation processes within and between schools. In other words, mathematics remains a gatekeeper; and in further study of teachers' practices, we can begin to recognize ways in which mathematics can be made more meaningful for all.

Finally, more and more national and international mathematics assessments (e.g., Program for International Student Assessment, Trends in International Mathematics and Science Study) incorporate realistic tasks that may or may not be equally accessible to students of different cultural, social-class, and economic backgrounds (Cooper & Dunne, 2000). Research into the pedagogy and the cultural practices of the classrooms in which students are studying mathematics can contribute to a better understanding of students' performance on such assessments.

Limitations and Future Research

One of the major limitations of these findings comes from the lack in diversity in and the representativeness of the participant group. Although this group consisted of secondary mathematics teachers from two schools, neither the teachers, the mathematics departments, nor the schools were at all typical of the United States. Even the department and teachers at the public school were not representative of those at other public schools. The teachers were highly qualified, all with a master's degree or higher, and the department had been involved in a curriculum development project in the late 1980s that had resulted in a strong departmental community with a shared teaching philosophy. In addition, the way in which the teachers were selected for the study resulted in a set of participants who had been involved in many similar workshop and curriculum experiences. They shared common ideas about teaching with contextual problems and mathematical modeling. Their common history limits any generalization of the finding that involvement in a community is a condition that enables teachers to use contextual problems regularly and well. Future research should explore the instructional practices of public school teachers and take into consideration the conditions that enable or disable that work.

Another line of research should investigate the potential for creating for other schools the enabling conditions identified in these schools and departments. What additional conditions might need to be satisfied for public school teachers to use contextual problems regularly and powerfully? How can involvement in communities be promoted so that teachers' participation yields positive outcomes?

A second limitation of the study was that I was unable to gather observational data on the instructional practices of all six participants. Though the three unobserved teachers were teaching with such problems during the time of data collection, they had their own reasons for declining to participate in the observational phase. Further, the data on the three teachers who were observed are limited to some degree because the data were gathered near the end of the school year. The students in the classes observed were already accustomed to working with contextual problems, and the teachers had learned how to facilitate the activities so that they were productive. Future research might explore teachers' instructional practices longitudinally to see how they develop a classroom culture in which students are comfortable tackling nebulous open-ended, contextual problems with little guidance from the teacher. How might that culture be developed in other classrooms? Do teachers' practices differ according to the time of year; for example, is there much more guidance early in the year?

Conclusion

In teaching with contextual problems, the teacher may question students' suggestions for solutions, scrutinize answers as openly as possible, utter objections, and propose ways of symbolizing the different solutions and contexts, as well as encourage symbolic representation of the reasoning process and so on.

> Mathematics is not only a matter of right or wrong, it can also be correct but inappropriate or even abused. It is the teacher's responsibility to establish this insight in pupils by systematically providing the relevant questions in the common mathematical activity. (van Oers, 1996, p. 105)

Through communication between students and teachers, a pool of shared knowledge, strategies, and experiences can be built up. That is precisely what the teachers in this study had done. As I have suggested, much can be learned from these teachers and their practices with contextual problems.

In addition, students need opportunities to use mathematics, to choose methods, to change and adapt methods, to discuss and negotiate directions with other students, to interact with systems in the environment (Greeno, 1991), and generally they need to become attuned to constraints and affordances that are represented in other situations. These teachers provided their students such opportunities and used contextual problems as the means for facilitating students' learning. Many teachers might hesitate to use such complex and nebulous problems with their students, but the work of these teachers demonstrates that students can be successful with such problems when given a chance, and when sufficiently supported. Moreover, in a classroom where contextual problems are a regular part of activity, students need not ask, "When am I ever going to use this?" They already know the answer.

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APPENDIX A

BACKGROUND AND BELIEFS INTERVIEW PROTOCOL

[Ask for resume or curriculum vita]

Ethnographic Information

- Tell me about this school. How would you describe it to someone who has never been here?
- Who makes curriculum decisions in your school or district? Have you ever had any input?

Teaching Information

- Have you always been a teacher?
- Why did you become a teacher? Why a mathematics teacher?
- In what schools have you taught in your career? How long were you there?
- How many years have you taught high school?
- How many years have you taught mathematics?
- What mathematics courses have you taught and for how many years?
- How would you describe yourself as a teacher?
- How is a typical day in your classroom organized? What kinds of things did you do as a teacher?
- What are the school's expectations for students in mathematics?
- Can you describe your goals for mathematics and mathematics teaching?
- Have you attended in-services or workshops about teaching with contexts or mathematical modeling? What kinds of activities did you do at these workshops?
- Have you read any research or articles about teaching with contexts or mathematical modeling? Can you talk about what you read? Does anything particular stand out in your memory?
- Are there any other specific things about your teaching career or mathematics you would like to share with me? (This can be anything you feel it is important for me to know.)

Ideas of mathematics in context

- What does the phrase *mathematics in context* mean for you? Are there many contexts? Examples of contexts?
- What does the phrase *applications of mathematics* mean to you? How is this related to your definition of *mathematics in context*?
- Talk about the idea of *mathematical modeling*. What does this mean to you?

- How are the phrases *mathematics in context*, *applications of mathematics*, and *mathematical modeling* related?
- How does the "real world" play into those three (if at all)?
- Discuss your thoughts on teaching mathematics in context. Worthwhile? Not? Important? Not? Should we do it?
- Tell me about how this curriculum addresses the idea of mathematics in context. What about applications? Models?
- Has a student ever said "Mr./Mrs. X, when will I ever use this?" Can you tell me about a time like that? How did you respond?
- React to the following statements.

  It is the teacher's job to understand, interpret, and provide contexts for the mathematical topics students explore.

  All mathematics should be taught in contexts.

  Mathematics is relevant to students only when it is presented contextually.

  Mathematics is only motivating for students when it is presented contextually.

  Mathematics should never be taught in context. Students should learn the rules and procedures in school and discover its applications through their lived experiences.

Extra questions added during course of study:

Can you talk about the importance of community?

Can you draw a graphic organizer to show how you think about mathematics in context, applications of mathematics, and mathematical modeling and how they are related or not related?

In the phrase, *mathematics in context*, what does the word *context* mean to you?

How do you design problems? Where do you find problems? How are they developed? Can you talk through this process?

What do you think makes a problem "good?"

APPENDIX B

PRE-LESSON AND POST-LESSON INTERVIEW PROTOCOL

Sample of Questions Usually Asked

<u>Pre-lesson Interview</u>

What are your goals for the lesson?

What came just before this lesson in your curriculum? What comes next?

What will you be doing during the lesson? What will the students be doing?

Are any aspects of this lesson particularly difficult for students?

Will any aspect of this lesson be new for students (experimentation, technology, mathematics)?

Is there anything else I need to know about this lesson?

<u>Post-lesson Interview</u>

Talk about how you felt the lesson went? Confusions? Contexts? Math?

Restate the goals of the lesson and talk about if those goals were met.

Given what you know about students and how they do on this problem, when students became confused about/by the problem, do you think it had anything to do with the context or the mathematics?

Across the classes, a lot of students' experiential knowledge came into play in discussing the problem. You always addressed their comments and discussed them but how do you decide what to pursue further in the model? Is it predetermined by the project itself? Or…?

What was your goal in having students draw a graph/think about plan before they did the experiment/explored?

As the problem progressed, the context was stripped away—pure mathematics was manipulated and worked on—and then later they returned to the context. As you think about this and other lessons you've used, can you compare your progression and talk about the unfolding of contextual or modeling lessons?