ELEMENTARY PRESERVICE TEACHERS' WORKING MODELS OF CHILDREN'S MATHEMATICS

by

ANGEL ROWE ABNEY

(Under the Direction of Patricia S. Wilson)

ABSTRACT

Four elementary preservice teachers participated in a field-based mathematics methods course about children's mathematics with respect to numbers and operations and discussed an Experiential Model of Children's Counting and Whole Number Development proposed by Steffe, Von Glasersfeld, Richards, and Cobb (1983). Each participant worked one-on-one with a child for eight weeks. The purpose of the study was to investigate the working model that each preservice teacher constructed of her child's mathematics and how her model informed her instructional decisions during the field experience. Data were collected in the form of interviews, videos and observations of the field experience sessions, and course products. The data were analyzed using qualitative case study methods, including micro-analysis (Strauss & Corbin, 1998).

The activities in which the preservice teachers engaged during the course were specifically designed so that the preservice teachers would learn to listen to and learn from children and think about how the children's mathematics informed their instructional decisions. Thus, this course provided the necessary context for my study in that it made it likely that the preservice teachers would attempt to construct and use models of children's mathematics.

I found that the participants engaged in a Mathematics Teaching Cycle similar to that described by Simon (1997). Each participant constructed a model of the child's mathematics based on her interpretation of the experiential model from class. They then used their working models of their children's mathematics to determine learning goals and eventually designed or chose activities to meet those learning goals.

In order to continuously inform the preservice teacher's working model and to extend her child's mathematics she asked many questions, which were classified as probing, prompting, and prodding. Through learning the mathematics of children, the participants also learned mathematics for themselves, including learning to reason strategically rather than rely on traditional algorithms. The participants also saw the need to redefine their conception of what it means to teach mathematics.

INDEX WORDS: Preservice Teachers, Mathematics, Elementary, Children's Mathematics, Constructivism, Model, Field Experience, Hypothetical Learning Trajectory, Mathematics Teaching Cycle

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DEDICATION

This dissertation is dedicated to the memory of my grandmother, Eolia Davidson, “Mama Yoya” and my grandfather, Roscoe Rowe, “Papa Roscoe.”

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CHAPTER 1

BACKGROUND AND STATEMENT OF PROBLEM

When I first began my doctoral studies, my teaching experience was in secondary and college mathematics. I had no specific interest in elementary school mathematics; however, my first assistantship involved assisting the instructor of the introductory mathematics methods course for elementary preservice teachers. This course focused on children's construction of numbers and operations, including whole number concepts and fractions. A core component in the course was a field experience called the School Buddy Experience. Mewborn (2000) explained that "The purpose of the field experience is to provide preservice teachers with opportunities to listen to and make sense of the mathematics that children generate and to consider how children's mathematical thinking impacts their instructional decisions" (p. 54). The field component was eight to ten weeks long, and was set up so that the preservice teachers were working either one-on-one with a child, alongside another preservice teacher with one child, or alongside another preservice teacher with two children. Depending on the instructor of the methods course, the preservice teachers have the opportunity to be engaged in some components of a teaching experiment, namely making hypotheses and testing those hypotheses. The instructor of the course and graduate assistants in the course observe the interactions between the preservice teacher and the child or children during the field experience in order to offer feedback that will assist in their ability to do this work.

While I was assisting in this methods course for elementary preservice teachers I was taking a course called Advanced Studies in Mathematical Learning, in which I was first exposed to constructivist theories in mathematics. We read many articles about the epistemology of constructivism and specifically about children's development in mathematics, which I later integrated with my observations of the children in the field experience for a class project. It was this project that hooked me onto children's mathematics and, for five semesters afterward, I worked with preservice teachers (PST) in this introductory mathematics methods course. Throughout my work with this course, I became particularly interested in the field component. Intuitively, there seemed to be something particularly beneficial about this experience. I recall that my field experiences as a prospective teacher were generally not focused on students' mathematics; rather they were geared toward teacher actions such as writing behavioral objectives, writing lesson plans, enacting those lesson plans, and assessment that generally mandated some worksheet, quiz, or test. The School Buddy Experience had no curricular objectives. The objective was for the preservice teacher to learn about the child's mathematics. As I read more about different ways of getting preservice and inservice teachers to focus on students' mathematics as a basis for instructional decisions, I realized why this experience seemed so beneficial; when PSTs are in field experiences, they typically focus on classroom management concerns to the exclusion of content and making sense of students' mathematics (Mewborn, 1999). Thus, this field experience, where PSTs were working one-on-one with a child, worked to eliminate or at least minimize these management issues so that PSTs were free to concentrate on listening to and learning from children.

Since the purpose of the field experience was for PSTs to learn children's mathematics, I would argue that the preservice teachers were engaged in a partial teaching experiment whose purpose, as suggested by Steffe and Thomson (2000), was to experience children's construction of mathematics and reasoning. This form of teaching differs from a traditional form of mathematical instruction, which has typically been "algorithmic in nature and lecture driven"

(D'Ambrosio & Campos, 1992). In this traditional form of teaching, the goal of the teacher is to bring the child into their understanding of the subject. However, the goal of a teaching experiment, based on a constructivist philosophy, is for the teacher to come into the understanding of the child citation. While it is impossible to know for sure, the teacher's goal is to construct a model of the child's thinking. I wondered if it was realistic to think that preservice teachers could do this work; could they de-center enough to listen to and learn from children? Furthermore, could they use their growing understanding of a child's mathematics to make appropriate instructional decisions? I decided to operate under the assumption that they could build a model of children's thinking, and thus, am interested in how preservice teachers in this field experience make sense of children's mathematics. In particular I'm interested in the *working models* (i.e. useful models) they construct of children's mathematics. I also want to know how they use such working models to form their next tasks or questions. In other words, how do the PSTs' working models inform their instructional decisions?

I chose to use the language *working model* to imply that the purpose of a teacher building such a model is to work with it to inform instruction. Steffe and D'Ambrosio (1995) contend that "working models are not scientific models" (p. 147); rather working models are considered an observer's useful interpretations and descriptions of another's general ways of operating. The measure of a working model is its viability; meaning it is valued depending on the extent to which a teacher is able to use it in order to make appropriate instructional decisions regarding a child's mathematics (von Glasersfeld & Steffe, 1991). Another aspect of my concept of working model is taken from Steffe and D'Ambrosio's (1995) notion of living models, in the sense that such a model is ongoing, ever changing, and never quite complete or accurate.

Background and Assumptions

This research study is based on my view that learning is a constructive activity in which individuals build meaning through making sense of their world. My underlying assumptions include my belief that children can construct powerful mathematical structures through problem solving and that perturbation is a necessary component of learning. I also believe that preservice teachers need to be engaged in partial teaching experiments in order to become aware of using children's mathematics to inform their instructional decisions. With guided performance, preservice teachers are capable of formulating models of children's mathematics in that they do not have access to a child's reality. Whether their models are viable and whether they can use their models to inform their instructional decisions is an entirely separate matter. My questions are:

1. What models do the preservice elementary teachers construct of children's mathematics? Do they construct working models?
2. How do preservice teachers' models inform their instructional decisions?

Description of Field Experience

Every semester, the department of elementary education accepts two cohorts of about 30 preservice teachers into the early childhood teacher education program. These PSTs progress together in their cohort during their junior and senior years. Before being admitted to the teacher education program, PSTs take a mathematics content course designed for prospective teachers, which includes arithmetic and problem solving. During their first semester in the program, they take their introductory mathematics methods course, which focuses on children's construction of mathematics with respect to numbers and operations and their second content course, which focuses on teaching and curriculum with an emphasis on geometry, algebra, probability and data analysis, and problem solving. Then they take their third content course focusing on algebra and problem solving.

Their introductory mathematics methods course is when their field experience, namely the School Buddy Experience. The PSTs have other field experiences during their programs but this is the only mathematics-specific experience focused on learning children's mathematics. In the course under study, the instructor's goals were to connect this experience with the rest of the course by providing language and theory to discuss what PSTs had seen children do with the problems that they posed. The PSTs worked one-on-one with their buddies once a week for eight consecutive weeks during the course, starting the fifth week of class. During the eight weeks of the field experience, PSTs met twice a week: one day at a local elementary school and one day at the university. The PSTs pulled their school buddies out of the classroom and spread around the hallways to work with the children on mathematics for 50 to 60 minutes. The school in which the field experiences were held had a diverse population with 17% black, 68% white, 7% Asian/Pacific Islander, 4% Hispanic, and 4% multi-racial.

The instructor and graduate assistant of the course usually observed the teaching episodes , participating occasionally. The teacher educators were involved in the experience by offering guided performance to the PSTs, often making notes to provide feedback. These notes included ideas such as tasks to try, questioning techniques, or hypotheses about the child's mathematics. At times the teacher educators intervened in order to model questioning techniques to make the child's thinking public (Mewborn, 2000).   However, the teacher educators involved in the course during the time of my study did not choose to intervene during the interactions of my four participants.

Rationale and Statement of the Problem

In the teaching principle, the National Council of Teachers of Mathematics (2000) called for teachers to have not just a deep understanding of the content that they teach, but also knowledge of their students. In order to make this type of reform in mathematics education possible, Thompson and Thompson (1994) contended that classroom discourse and communication are essential elements. They suggested that teachers “must be sensitive to children’s thinking during instruction and shape their instructional actions accordingly—to ensure that children hear what they intend them to hear” (p. 279). They studied one inservice teacher’s struggle to help a student understand rates conceptually. They found that, while this teacher had an extensive knowledge of the relationship between rates, distance, and time, he did not know it in a way that helped him to make it accessible to his student. They concluded that his lack of mathematical knowledge for teaching (MKT) caused his teaching experiment to fail (A. G. Thompson & Thompson, 1996).

Through Thompson and Thompson’s (1996; 1994) case study, it seems clear that this type of reform is not easy for teachers. It was difficult to construct viable models of a child’s mathematics and use these models to inform instruction with one child, much less with an entire class. Despite extensive education, most teachers have never experienced the type of mathematical instruction that reform movements such as NCTM have asked of them. Furthermore, many teacher educator programs assume that the ideas of reform, such as beginning mathematics instruction with students’ current ideas, are being assimilated by preservice teachers. However, research suggests that this does not naturally occur (Cobb & Macclain, 2001; Franke, Carpenter, Levi, & Fennema, 2001). Franke, Carpenter, Levi, and Fennema (2001) concluded that being sensitized to students’ mathematical thinking invokes generative change in teachers' ways of operating while teaching experience alone was shown not to be sufficient. This raises the question of how teachers learn to be sensitized to students' mathematical thinking, which is essential for the type of teaching advocated by NCTM (2000). Teachers need to have opportunities in their preservice training to engage with a child in one-on-one in order to learn children's mathematics. Therefore, I chose to study one such opportunity.

Mewborn (2000) explained that the School Buddy Experience had not been formally researched. Although my purpose was not to study the effectiveness of the field component, I learned how the experience affected four PSTs in this program and how their experiences could inform teacher preparation programs' field experience models. My study investigated ways in which PSTs interpreted the realities of children with respect to their mathematics. Through this interaction, PSTs determined the "mathematics of children," which refers to the models that they construct of the mathematical realities of children (Steffe & Thompson, 2000). My goal was that the PSTs would use the mathematics of children to determine the mathematics for children as they were teaching in the field experience. I wanted to know how they would use mathematics of children in their planning during the field experience. Steffe states that "we regard the mathematics of students as a legitimate mathematics to the extent we can find rational grounds for what students say and do" (Steffe & Thompson, 2000, p. 269). The PSTs should be engaged in making sense of what they see students do including changes in their current ways of operating.

Pelligrino (2002) suggests that designing assessment based on a model of students' thinking should be an essential element of school education and that the purpose of assessment should be to make inferences and provide clues about the type of tasks in which students should be engaged. He argues that assessment is a cyclical process which

> must meld three key components: cognition, which is a model of how students represent knowledge and develop competence in the domain; observations, which are tasks or situations that allow one to observe students' performance; and interpretation, which is a method for making sense of the data relative to our cognitive model (Pellegrino, 2002).

He further suggests that current practices of assessment are highly limited and typically based solely on narrow interpretations of what students know.

If this practice of assessment and instruction is to change, how might this change occur? Where does such reform begin? I believe it should begin with preservice teacher education. In this study I investigated one such experience where preservice teachers were engaged in the practices described by Pelligrino. They were provided with a cognitive model, which I have called an experiential model, of how children develop competence in the domain of whole number. The experiential model is a scientific model and should be distinguished from the PSTs' working models. The PSTs provided children with tasks in order to observe the student's mathematical actions, then described and interpreted their children's mathematical actions within the framework of the experiential model of whole number development. Each preservice teacher used her interpretation in order to plan activities for subsequent sessions with her school buddy. This is precisely the type of practice Pelligrino described as essential for reform.

CHAPTER 2

REVIEW OF LITERATURE

> In my haste to supply the children with my own bits and pieces of neatly labeled reality, the appearance of a correct answer gave me the surest feeling that I was teaching. Curriculum guides replaced the lists of questions, but I still wanted most of all to keep things moving with a minimum of distraction. It did not occur to me that the distractions might be the sounds of children thinking (Paley, 1986, p. 78).

Paley's quote above highlights the kind of teaching that tends to be void of children's thinking. As a kindergarten teacher, she began to realize the need to change through "the painful recognition of [her] own vulnerability" (1986, p. 78). Through observing an outsider coming in to spend time with her class, she noticed that "…he listened to their responses with the anticipation one brings to the theater when a mystery is being revealed" (p. 78). She began to mimic the outsider's open-ended questions in order to bring out children's intuitive ways of approaching problems and used their responses to "build a chain of ideas without the need for closure" (p. 78). In doing so, Paley was learning to listen to children and act upon her listening to inform her instructional decisions. In my literature review, I discuss research that has investigated preservice and inservice teachers learning to listen to children and act upon their listening to make instructional decisions. I also discuss literature on constructivism, children's mathematics, and building models of students' mathematics.

Constructivism and Model Building

Ball (1997) suggests that figuring out what students know is central to the task of teaching. When teachers simply ask students to fill in the blanks on a given worksheet they are blocking their access to interpreting the ways in which students think. However, when teachers open up assessment to include children's explanations and solution methods to open-ended tasks, they are left to admit that learning is a complex endeavor including many subtleties that cannot be measured. "At the same time, the open-endedness increases the uncertainty of interpreting and appraising student progress and makes it more difficult to share common standards across students, groups, and settings" (Ball, 1997, p. 771). Ball asserts that while knowing what students know is not possible, it is still a worthwhile pursuit claiming that , "issues like these challenge core epistemological and psychological assumptions about what it means to 'know' something" (Ball, 1997, p. 771). Since knowing what students know in mathematics is not possible, teachers are left to construct a model of a student's mathematics in order to determine what action to take next (Steffe & Thompson, 2000).

Steffe and Thompson (2000) state that "at a very minimum researchers in a teaching experiment who make a claim about what students know are obliged to make records of the living models of students' mathematics that illustrate aspects of the claim available to an interested public" (p. 303). While I do not claim that the preservice teachers involved in my study were engaged in a teaching experiment, they were recording their interpretations of the mathematical actions of the child with whom they were working. This work of describing and interpreting the mathematical actions of their students in order to work more productively is what I have called constructing working models.

Constructivism is an epistemology which differs from other theories of learning in that it rejects the notion of being able to determine the validity of knowledge by matching it to an absolute reality. Rather, knowledge is seen as functional as long as it is useful or viable within the framework of a subject's experience. However, when new experiences arise for which knowledge is no longer viable, accommodations must be made in order to maintain viability in order to internalize the new experiences (Steffe & Thompson, 2000). Perhaps the most relevant conviction of constructivists is that “conceptual knowledge cannot be transferred ready-made from one person to another but must be built up by every knower on the basis of his or her own experience” (Steffe, von Glasersfeld, Richards, & Cobb, 1983). If we apply these ideas to mathematical knowledge, then “mathematics is viewed as an ongoing creation of human minds, not an aspect of the external world waiting to be discovered” (Simon, 1997, p. 58).

When teachers or researchers seek to understand the mathematics of a student in order to inform their instructional decisions, constructivists maintain that such constructs are necessarily hypothetical since another person’s conceptions are not observable. Teachers would be interpreting their students’ mathematical actions in terms of their own experience. Thus, Steffe and colleagues suggest that the work of making sense of another’s conceptions involves making hypothetical models, which are constructed to order, comprehend, control, and explain their experience (Steffe et al., 1983). These models may also serve to help teachers predict what students might be able to do in order to choose or create appropriate tasks for a child. Since the purpose of the models is to work with students in a more productive manner, I chose to use the term working model.

Mathematics Teaching Cycle

Simon’s notion of Mathematics Teaching Cycles serves as a conceptual framework for teachers acting upon their listening to children. As noted by Simon (1997), “many teachers have developed their models of teaching in the context of thousands of hours as students in traditional classrooms” rather than being based on explicit models of learning (p. 57). As a result, teachers’ expectations, values, suppositions, and assumptions about learning and teaching of mathematics tend to be based solely on a lecture-demonstration model in which teaching is predominantly telling and showing. Simon proposes a model of teaching that is based on models of student learning, a cyclic process that he calls Mathematics Teaching Cycle (MTC). MTC "refers to a conceptual framework that describes the relationships among teacher's knowledge, goals for students, anticipation of student learning, planning, and interaction with students. In the MTC Simon (1997) points out two goals for the teacher: to be responsive to students' mathematics while at the same time she "poses tasks and manages discourse to focus on particular mathematical issues" (p. 76). Below is Simon's diagram of the MTC.

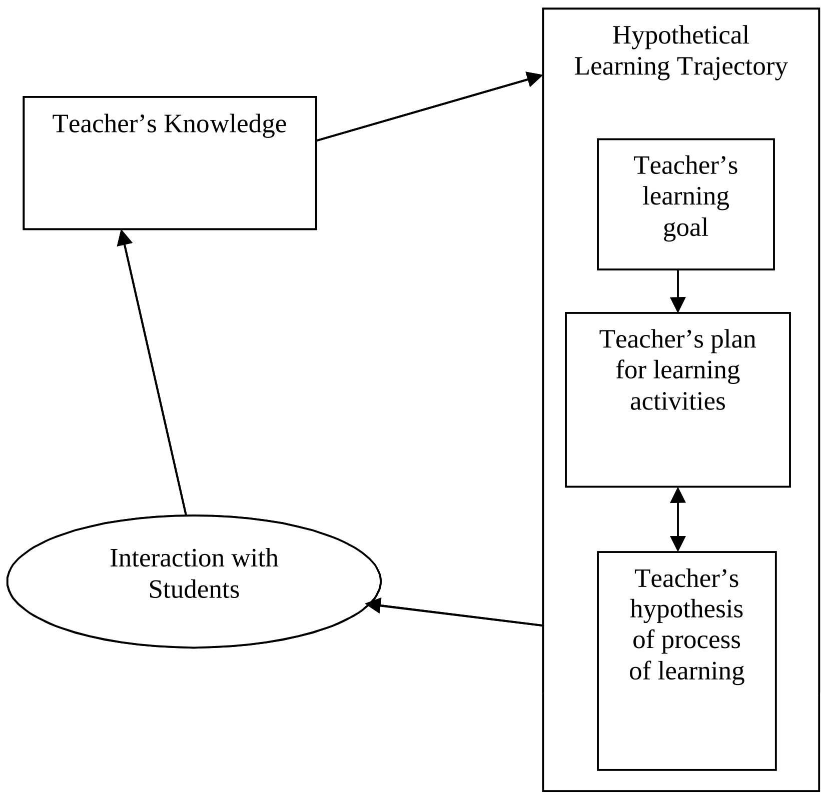

Figure 1. Mathematical Teaching Cycle (Simon, 1997, Figure 3.1, p. 77)

Simon suggests that the interaction with students contributes to the knowledge of teachers, however, since the students' mathematics can never be known precisely, teachers must construct models of the students' mathematics. This model is subject to constant change upon a teacher's interaction with students and other tasks, activities, reading and other teachers. This process of teaching is an inquiry process of listening to and observing "the spontaneous mathematical activity of students, eliciting mathematical activity through selected tasks and questions, developing hypothetical models of the students' mathematics, and further inquiring to refine these developing models" (p. 77). In the MTC, the teacher's models of students' mathematics serves as a basis for making instructional decisions, including choosing appropriate goals for learning and planning for learning activities. Simon states, "According to the MTC, the mathematics teacher's actions are at all times guided by his or her current goals for student learning, which are continually being modified based on interactions with students" (p. 77). It is important to note that these goals may be modified spontaneously during the interaction or during planning.

The Hypothetical Learning Trajectory has three parts: the teacher's learning goal, her plan for learning activities, and her hypothetical learning process. This hypothetical learning process refers to "a prediction of how the students' thinking and understanding will evolve in the context of the learning activities" (p. 78). Simon notes that the teacher's development of a hypothetical learning process and the development of the learning activities happen simultaneously and necessarily depend on each other. That is, the sequence of learning activities provided by the teacher influences the generation of students' thinking and learning which would then influence a teacher's hypothetical learning process. However, her hypothetical learning process influences the initial sequence of learning activities provided. This symbiotic relationship is indicated by a bidirectional arrow in the diagram above.

The Hypothetical Learning Trajectory includes a teacher's plan for instruction before interacting with students. However, the plan alone is not what constitutes the experience during the classroom. Simon (1995) states, "as the teacher interacts with and observes the students, the teacher and students collectively constitute an experience" (p. 137). As the teacher seeks to understand what is happening and what has happened during the interaction, her knowledge is changed which then constitutes a modification in the Hypothetical Learning Trajectory. Simon suggests that a teacher's interaction with students involves inquiry into students' mathematics, facilitation of discourse, problem posing, and interactive constitution of classroom practices. Teacher's Knowledge consists of, but is not limited to, her knowledge of mathematics, knowledge of mathematical activities and representations, conceptions of mathematics learning and teaching, knowledge of student learning of particular content, and models of students' knowledge.

In my study, I am investigating the working models preservice teachers construct of children's mathematics and how these models informed the teachers' instructional decisions. Simon's MTC provides a way for me to frame my study. The PSTs planned an interaction with the child. After each participant's initial interview with her child, she developed goals for her child's learning, planned appropriate activities that were geared toward her learning goals, and hypothesized about her child's process of learning. During the interactions, the preservice teachers were striving to be sensitive to their buddies' mathematical thinking.

The following diagram is an elaborated version of the one of the Mathematics Teaching Cycle above. Each box outside of the HLT is a part of the Teacher's Knowledge, located in one bubble in the previous diagram. These aspects of the participants' knowledge, as well as the skills suggested in the interaction with students bubble, are my focus in this review of literature because I believe that these are essential knowledge and skills for constructing a working model of a child's mathematics.

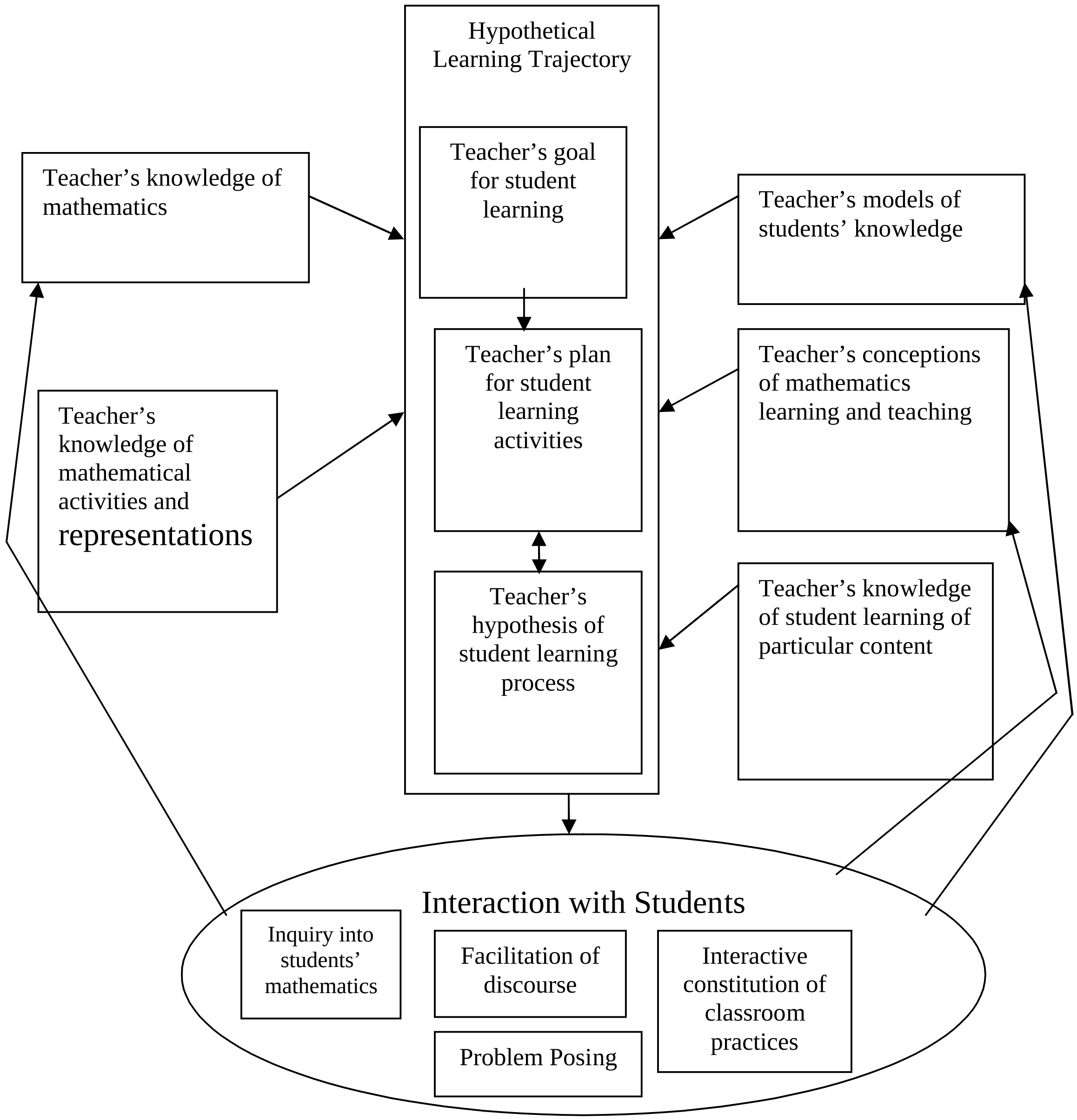

Figure 2. Extended Mathematical Teaching Cycle _____(Simon, 1997, figure 3.2, p. 79)

My research questions are:

1. What working models do preservice teachers construct of children's mathematics?
2. How do those models inform their instructional decisions?

Simon's framework of Mathematics Teaching Cycles (MTC) offers a way for me to situate my findings in my study with respect to these guiding questions. Certainly the first question is related to the first box on the right side of the diagram, labeled Teacher's Models of Student Knowledge. An arrow in the diagram indicates that the teacher's models directly affect her Hypothetical Learning Trajectory which, in turn, affects her interaction with the child. Since the Hypothetical Learning Trajectory is co-constructed by the student and the teacher, the HLT necessarily involves a teacher's planned instructional decisions as well as those modified in the act of teaching or her spontaneous instructional decisions (Simon & Tzur, 2004). Thus, the second question can be situated in the box containing the HLT as well as in each of the components of the bubble called Interaction With Students. The teacher's interaction with the student also directly informs the teacher's model of a student's mathematics.

The preservice teachers involved in the study were in a course that introduced them to Steffe and colleagues' descriptive experiential model of children's mathematics with respect to whole number development involving counting and operations. I have situated the preservice teachers' knowledge of the experiential model in the last box on the right, labeled Teacher's Knowledge of Student Learning of Particular Content. According to the diagram, a teacher's knowledge of how children learn particular content also directly affects the HLT, which is a major part of what I have called the participants' instructional decisions. However, each preservice teacher's interaction with the child necessarily informs her construction of the experiential model from class.

In light of Simon's framework for teaching, MTC, I questioned whether the preservice teachers adapted their working models of the children's mathematics as they interacted with the children. This suggests that I am studying the arrow from the interaction with the child to the knowledge of the teacher. Since another question that I have concerns how the participants' working models inform their instructional decisions, it would seem that I am also investigating the arrow from the teacher's knowledge to the HLT, which I have called instructional decisions, and from the HLT to the interaction with the child. This teaching cycle will be further revealed and elaborated in chapter five, which describes the findings in detail.

Knowledge and Skills that Enhance Building Working Models

There are many skills and domains of knowledge involved in constructing a working model of a child's mathematics and each model would be valued to the extent with which the teacher could work with it, requiring the ability to design and choose appropriate tasks and goals with respect to her model of the child's mathematics. I have chosen to discuss only a part of such knowledge and skills. The skills and knowledge I have chosen to focus on with respect to creating such a viable working model are listening, questioning, choosing appropriate learning goals and tasks, and a teacher's knowledge of student learning of particular content.

Some of these skills and knowledge domains are explicit in Simon's diagram of the MTC. Others are more implicit in this diagram. The knowledge and skills that I have chosen to bring to the surface are included in Simon's diagram of the MTC in both the boxes that indicate a Teacher's Knowledge and in the bubble labeled Interaction With Students. I believe that these knowledge and skills are both applied and developed during the interaction with students and in planning for the interaction with students. I have chosen to focus on teachers' listening, questioning, and choosing appropriate tasks. I believe that while listening and questioning are not explicit in Simon's MTC, they are an implicit part of the Interaction with Students, particularly in the inquiry into students' mathematics and the facilitation of discourse. The knowledge on which I chose to focus includes the knowledge of student learning of particular content and mathematical knowledge, both of which are explicit components of the Teacher's Knowledge in Simon's MTC.

Interaction with Students

If any teaching is to be considered legitimately constructivist, the constructivist teacher should be regarded as a learner during the interaction with students (Steffe & D'Ambrosio, 1995). This learning requires a spirit of inquiry, a desire to facilitate discourse among the students, and the ability to choose appropriate tasks for particular students. Since the context of my study includes preservice teachers working with only one student each, one might be convinced that some of these skills may not be necessary, but I would argue that each one is essential. For instance, while the preservice teacher was not be facilitating discourse among several students, she was trying to facilitate discourse between herself and her student. In order to inquire about a child’s mathematics, keep a child acting mathematically, and stretch the child’s current schema, the preservice teacher had to ask questions and choose appropriate activities. These are the skills essential to constructing a viable working model on which I focus the discussion of the interaction with students.

Listening

A necessary part of constructing working models is listening to students. Some researchers have chosen to focus exclusively on the act of listening in order to investigate teachers’ constructions of models of students’ mathematics (D'Ambrosio, 2004; Davis, 1996, 1997). Davis argues that “an attentiveness to how mathematics teachers listen may be a worthwhile route to pursue as we seek to understand and, consequently, to help teachers better understand their practice” (Davis, 1997, p. 356). He suggests that listening is a fully human endeavor in which all senses should be engaged. It is a process that involves both a physical and emotional response.

However, it is important to note that we are always listening through the constraints of our own perception. Davis (1996) believes that we need "to acknowledge that our listening always and inevitably occurs against the backdrop of personal histories that are set in and shaped by cultural, historical, social, and environmental factors" (p. 47).

While Davis (1997) identifies three types of listening in which teachers are engaged as a framework, he also acknowledges that listening cannot be reduced simply to a set of skills and guidelines. The manners of listening that Davis identifies are evaluative listening, interpretive listening, and hermeneutic listening. In the next three paragraphs, I discuss how Davis defines these three modes of listening.

A person engaged in evaluative listening most likely sees hearing and listening as synonymous. He suggests that evaluative listening is listening without acknowledging the constraints of our own perceptions. In the mathematics classroom,

> this manner of listening is manifested in the detached, evaluative stance of the teacher who deviates little from intended plans, in whose classroom student contributions are judged as either right or wrong (and thus have little impact on lesson trajectories) (Davis, 1996, p. 52).

When a teacher is listening evaluatively, listening tends to be primarily the responsibility of the learner rather than the teacher.

He describes interpretive listening as the listening in which radical constructivist teachers would engage (Davis, 1996, p. 52). In this type of listening, teachers attempt to understand what students are thinking. Thus, in a mathematics classroom where a teacher is listening interpretively, there tends to be more opportunities for students to demonstrate and articulate their mathematical thinking. Rather than listening to merely assess what students are learning, teachers listening interpretively are deliberately listening and interpreting what students are doing. "All of this is based on the realization that the teacher and learner alike are actively constructing conceptualizations: the learner constructs the mathematics; the teacher constructs the learner" (Davis, 1996, p. 52). Teachers engaged in interpretive listening are aware of the fallibility of the sense that is being made. In other words they are aware that their listening is through the constraints of their own perceptions.

Davis (1996) contends that "both 'evaluative' and 'interpretive' listening are premised on conceptions of human identity and agency as essentially subjective, autonomous, isolated, and insulated" (p. 53). He further suggests that, when it comes to mathematics teaching, both of these types of listening demand a distinct role of the teacher and for the student. In hermeneutic listening, the boundaries of teacher and student are blurred, suggesting that "this manner of listening is more negotiatory, engaging, and messy, involving the hearer and the heard in a shared project" (p. 53). In this type of listening, not only does the hearer recognize the constraints of her own perceptions, but interrogates "the taken-for-granted and the prejudices that frame perceptions and actions" (p. 53). There is no autonomous or radical knower. In summary, he states, "hermeneutic listening is a participation in the unfolding of possibilities through collective action" (p. 53).

Since listening is such an important part of constructing working models of students' mathematics, these concepts of listening provide a way of framing the invisible, inaudible ways in which PSTs engage in the field experience in order to make sense of children's mathematical ways of operating (Davis, 1997). The sense of listening to students hermeneutically changes a teacher's ontological way of reasoning from seeing mathematics as "objective truth" to seeing mathematics as a structure constructed by humans (Davis, 1997). Assessment also changes from seeing if the child's reasoning "accurately matches the world as it might be 'in itself'—a match that, as the skeptics have reiterated, we could never check out—but whether or not it fits the pursuit of our goals, which are always goals within the confines of our own experiential world" (von Glasersfeld, 1990).

D'Ambrosio (2004) presents a theoretical paper to discuss the three types of listening suggested by Davis (1996) in order to consider the challenges of trying to get preservice elementary teachers to embrace a constructivist view on mathematics teaching. She describes listening to include "attending to all types of 'utterances' or products produced by students, such as student talk, student work, and student actions" (D'Ambrosio, 2004, p. 139). Rather than focusing on the constraints of one's own perception to describe these conceptions of listening, she focuses instead on the notions of the "voice of the discipline," the "voice of the child," and one's "inner voice" to distinguish these modes of listening (p. 140).

She notes that, traditionally, teachers have been engaged in evaluative listening–listening for students to give particular responses guided by their own understanding of mathematics, the voice of the discipline. She identifies interpretive listening as a type of listening where teachers strive to hear the voice of the learner, but still have a goal of pushing their students towards a more appropriate mathematical response. In other words, the teacher is still pushing "the voice of the child" to be the "voice of the discipline."

Hermeneutic listening is the third level of listening that Davis (1996) identifies. D'ambrosio suggests that constructivist teaching necessarily involves the use of hermeneutic listening (D'Ambrosio, 2004), where as Davis (1997) states, "hermeneutic listening rejects the assumption of the radical subject that underlies both the constructivist's concern for individual sense-making and the critical theorist's concern for 'voice' or personal empowerment" (p. 370).

D'Ambrosio suggests that, in the case of hermeneutic listening, teachers are constructing the learners as they are constructing the mathematics. I believe that this is the type of listening necessary for productive interaction in the model building described by Steffe and Thompson (2000). Teachers are engaged in constructing the mathematics of children while at the same time renegotiating their own mathematics in order to come up with mathematics for children (D'Ambrosio, 2004). In my mind, mathematics for children refers to the mathematical tasks that teachers find or design for students in order to create perturbations so that students will modify their current schema. However, these learning activities should be based on the mathematics of children (Steffe, 1988).

"The hermeneutic listener views her own learning as a process of inquiry" (D'Ambrosio, 2004, p. 147). D'Ambrosio suggests that it has never been documented that a PST listened hermeneutically in order to construct a viable model of a student's mathematics. She further notes that there are relatively few examples of PSTs who successfully used interpretive listening. I believe that my study may reveal such listening by preservice teachers.

But how do we see the invisible? How do we hear the inaudible? Davis (1997) suggests that we can infer the type of listening in which a teacher is engaged with regard to the types of questions she asks. However, D'Ambrosio (2004) states,

> the different forms of listening are not a result of the questions asked by the teacher, but rather a result of how the teacher attends to the children's responses, regardless of the types of tasks or questions posed or the dynamics of the class discussion (p. 148).

How will I see listening? In my study I found three types of questions which preservice teachers were asking their buddies: prompting, prodding, and probing. I distinguish these questions as categories in my findings in chapter five. Like Davis, I believe that it is in these questions that the preservice teachers' listening can be inferred.

While listening does provide a "vital interpretive lens for examining and understanding the teacher-learner dynamic in mathematics classrooms" (Davis, 1997, p. 356), Steffe suggests that listening is not sufficient to a framework on teacher learning; Teachers must also learn to act upon their listening (Steffe, Personal Communication, 2007).

Questioning

Ball (1997) suggests that while figuring out what students know is critical to the task of teaching, no undertaking is more difficult. In their attempt to figure out what a child is thinking, teachers observe, listen, and ask questions. D'Ambrosio (1995) suggests that the role of a teacher should be one of inquiry, implying that the role of questioning is a significant factor in whether a teacher can successfully create an environment where classroom discourse is productive and valued. However, this raises the question of whether or not there are both good and bad questions for teacher to ask and, if so, which types of questions close thinking and discourse and which open up the opportunity for inquiry and discourse?

Pimm discusses a teacher questioning type that is typical of many mathematics classrooms- questions frequently posed in the form of statements with small openings in which students are expected to fill in key blanks (Ball, 1997). The student does little more than play a small role in the teacher's script in this line of questioning (Ball, 1997). The teacher is often simply checking to be sure the students understand in the way that she understands. A teacher engaged in this dialogue is most likely listening evaluatively, leaving students' ways of knowing unprobed. Her understanding of mathematics is her only guide to help her know how to act,, making it difficult to make instructional decisions that are responsive to students' ways of knowing.

According to Davis (1996) Gadamer classified a similar mode of questioning that he calls pedagogical questioning. Davis (1996) criticizes pedagogical questioning, suggesting that it lacks the type of openness "required of a true question" (p. 250). The purpose of such questions is not to probe a student's mathematics but to play a game of "guess-what-I'm-thinking" (p. 251). Davis suggests that the pedagogical question lacks a questioner in that the teacher already knows the answer for which she is looking.

A second type of question identified by Gadamer is the rhetorical question. Davis (1996) criticizes this type of question suggesting that it lacks both a questioner and an answerer. He suggests that the two types of questions described "effectively close down the very possibility of mathematical thought" (p. 252) and further notes that these questions serve to disrupt the power dynamics needed for a classroom discourse. He recommends that if a question is intended as an end to learning, or if the teacher already has the answer in mind, then the teacher is better off not asking the question at all.

The third and only line of questioning regarded by Gadamer and Davis as one of value is hermeneutic questioning. In this line of questioning the teacher is sincere in her desire to learn. A teacher engaged in hermeneutic questioning is trying to gain an understanding of the student and is thus most likely listening hermeneutically. In this line of questioning the teacher and student are involved in reciprocal engagement. For Gadamer, hermeneutic questions are the only "true questions" (p. 250).

Unlike Davis and Gadamer, the work of Driscoll implies that there are a variety of question types with value. He suggests that asking a variety of questions aimed at helping students organize their thinking by responding to prompts is a worthy task for teachers. He gives examples from algebra teachers including “Once you have made a chart, look for an easier way. Pay attention to how the numbers group and how the groupings might suggest an easier way” (Driscoll, 1999, p. 3). He also gives the example “Can you explain what the 3 and the 5 represent in that equation” (Driscoll, 1999, p. 3)? As will be indicated in chapter 5, my participants asked similar questions to elicit particular strategies from their buddies that they felt were in their zone of potential construction. Questions that guide students’ ways of operating are not what Davis or Gadamer would call hermeneutic, therefore, I doubt that they would see them as valuable tools for teachers.

Driscoll highlights five question types that are useful for teachers to be aware of so that they can seek to balance the range of intention and context with respect to their questions. The questions that he identifies are for the purposes of managing, clarifying, orienting, prompting mathematical reflection, and eliciting algebraic thinking. For Driscoll, each of these types of questions serves a valuable purpose in teaching students algebra. In the paragraph that follows I describe each of the named categories of questions.

Managing questions are intended to keep students engaged in a particular task and to help them organize their work. A teacher uses clarifying questions when she is probing the student for information or when she is not clear about what the student means. She may also use these questions when she is trying to clarify a question or problem for the student. Orienting questions are intended to prod students to get started on a mathematical task, or to motivate them to continue to engage in or focus on a mathematical task. Driscoll (1999) states that these questions can be used to “motivate the student toward the correct answer or away from the incorrect answer” (p. 6). Questions intended to prompt mathematical reflection are typically used to get students and to reflect on their own thinking and to explain others' and their mathematical ways of operating. These questions can be used to extend their thinking about the mathematics in a given task. Questions designed to elicit algebraic thinking are questions that are meant to guide students to use particular strategies. These questions may also ask students to justify their generalizations. A question of this type might be "can you decompose this number in a way that will be helpful for you"?

In my study, I recognized a variety of question types that the participants used as valuable to the discourse in the interaction during the field experience. Thus, while I recognize the value in hermeneutic questioning and see the ineffectiveness in playing the "guess-what-I'm-thinking" game, I find my thinking to be more aligned with the way that Driscoll discusses teacher questioning. I can see the value in trying to elicit more sophisticated strategies from students than the ones a teacher has observed him use. I believe that some questions that are intended as an end to learning can be valuable. Even though a teacher may have an answer in mind when asking prompting questions, they can be used as instructional tools to stretch a student's thinking. However, when a teacher decides to use such prompting questions, it is imperative that she be sensitive to the child's mathematical ways of operating.

Choosing Appropriate Tasks

A teacher not only needs to learn to listen to children, but to act on her listening in order to choose appropriate tasks (Steffe, personal communication). In choosing tasks, a teacher has to decide whether her goal is for a child to construct mathematical knowledge by assimilation or accommodation. Steffe and Wiegel (1996) refer to assimilation as concerning "the integration of any sort of reality into a structure," where any sort of reality is interpreted to mean reality from an observer's point of view (p. 491). While assimilation accounts for a significant part of the construction of knowledge, without accommodation, one's current ways of operating would never be stretched, allowing for cognitive development. Thus, an accommodation refers to "a modification of conceptual structure in response to a perturbation" (Steffe & Wiegel, 1996, p. 491).

If a teacher's goal is for the child to assimilate, or bring an idea into a child's existing structures, then she would choose a task that would extend strategies or concepts that the child already knew. However, if a teacher's goal is for a child to make an accommodation, then that teacher would choose a task in order to elicit a perturbation.

How does a teacher continue to work productively with a child when they are deliberately causing a child perturbation? Essentially, a teacher must continue to base her decisions on her model of the child's mathematics by choosing tasks that fall within the child's zone of potential construction (ZPC). These decisions include "situations to create, critical questions to ask, and the types of learning to encourage" (Steffe, 1991, p. 177) . Hackenberg (2005) suggests that a child's ZPC is a temporary concept since a child's mathematics is always changing. This concept includes a teacher's conjectures about potential ways in which a child could operate based on the teacher's view of his or her current schema.

Hackenberg (2005) discusses ZPC in terms of a short term ZPC and a long term ZPC. For instance, strategies that a teacher believes the child can construct within the next month would be in a student's short term ZPC, whereas those a teacher believes are within the child's grasp within the year are in the child's long term ZPC. Since the participants' field experience was only eight weeks long, my participants were only dealing with their children's short term ZPC.

Hackenberg (2005) suggests that teachers often realize that problems are within a child's short term ZPC in retrospect. One way a teacher may realize whether a problem was in a child's short term ZPC is if they were able to solve the problem using his or her current ways of operating. If so, the student is able to assimilate the problem. When a teacher wants to choose a task that will potentially create a perturbation - causing an accommodation in a child's current schemes - she would choose a task that she believes to be at the edge of the child's short term ZPC. The goal for these task choices are to "open up new possibilities for students' mathematical thinking" (Hackenberg, 2005, p. 26). A teacher's goal is not necessarily to prompt students to act as she does, but to engender ways of operating that the students have yet to imagine, in hopes of leading them to more sophisticated ways of operating. This is how the concept of ZPC differs from Vygotsky's Zone of Proximal Development or ZPD (Vygotsky, 1986). When teachers are attempting to operate within a child's ZPD, there is a mathematics that she is trying to get the student to know, namely her conception of mathematics (Hackenberg, 2005).

Essentially, for teachers who operate within a constructivist framework, the problems and tasks chosen for a child are based on hypotheses about their interpretations of a child's mathematics, or their model of the child's mathematics. The teacher must attempt to decenter and try to assume the mathematical point of view of her student (Steffe, 1991). As the child engages in the tasks, the teacher's model can be modified to account for her new observations (Steffe, 1991).

How do teachers begin to conjecture about children's mathematics? With a lack of experience, preservice and novice teachers must rely on researchers' experiential models of children's mathematics in order to form living models of their individual students' mathematics.

These living models serve as a basis for their work as a teacher, which is why I have chosen to use the language working model.

The difference between the models that researchers construct of children's mathematics and the models to which I refer in my research questions is that researchers' models attempt to explain and follow the progression of learning of particular content not only for the child with whom they are working but for children more universally. The models of the preservice teachers were based on the mathematics of one child and were not necessarily meant to be applied to other children. Despite this, I do believe that their experience could inform the ways in which they interact with children in the future.

Teacher's Knowledge

There are many facets of knowledge that can contribute to the success of teachers' construction of viable models of children's mathematics and their ability to use those models to make appropriate instructional decisions for their students. In my review of literature I have chosen to focus on two such facets, both of which are part of the Knowledge of Teachers indicated in Simon's MTC above. In the following sections I discuss student learning of particular content and teachers' knowledge of mathematics.

Knowledge of Student Learning of Particular Content

Carpenter, Fennema, Peterson, Chiang, and Loef (1989) found that helping teachers understand research results about children's thinking in particular content areas allowed teachers to identify the strategies that their students would use to solve a problem. These teachers could evaluate their students' knowledge more effectively and use their knowledge of students' mathematics to inform their instructional decisions (Carpenter et al., 1989). There are many different ways that teachers can frame their knowledge of student learning of particular content, specifically with respect to numbers and operations. For instance, Cognitively Guided Instruction (CGI) is a research project that constructed descriptive models of how children progress in whole numbers and operations.

Researchers involved in the CGI project suggest that children can solve problems by using their informal, intuitive strategies before they are ever introduced to formal symbolic notations in mathematics (Carey, Fennema, Carpenter, & Franke, 1995). Thus, the goal of the teacher would be to connect the informal knowledge to the standard mathematical notation. The informal processes that children use to solve problems include actions such as joining, separating, comparing, grouping, and partitioning. The level at which a child is operating is based on the strategies available for them to use. It was found that children begin to solve problems by first "directly modeling" the story in the problem with concrete manipulatives(Carey et al., 1995). Eventually children begin to use counting strategies, progressing from counting all to counting on from the first number, to counting on from the largest number presented in the problem. The next level of operating involves children using related facts to solve unknown facts (Carey et al., 1995).

CGI tends to orient teachers to the idea that children begin school with richer mathematical knowledge than traditional mathematics curricula typically assumes. They also orient a teacher to listening to her students' mathematical ideas and strategies. CGI researchers found that teachers exposed to the CGI framework of children's mathematics showed improvement in students' achievement in problem solving and concepts (Carpenter et al., 1989). However, this is not the framework that was presented to the preservice teachers in my study.

An Experiential Model of Children's Whole Number Development. Von Glasersfeld, Steffe, and Richards (1983) have presented an experiential model to explain children's development of whole number, including the concepts of counting and unit. The purpose of experiential models of children's mathematics of this type is so that adults can use their mathematics in order to more readily understand the child with which they are working. However, Steffe and Wiegel (1996) assert that there is no simple transformation of the mathematics of adults that offers an understanding of the children's mathematics.

Steffe and colleagues (1983) suggest that children learn to count by interacting with an adult. However, what they take as countable items depends on their development within the experiential model that I will describe below. This experiential model is not meant to be taken as a map that must be followed by every individual child. What the researchers present is a trajectory "with more than one possible path toward the final abstraction" (von Glasersfeld & Richards, 1983, p. 21). Each stage of this trajectory is dependent on the "degree of development [the child's] concept of unit has undergone" (p. 21). In other words, whether the child is numerical or not depends on whether they have developed the schema known as "unitizing".

Unitizing. "A unit is that by virtue of which each of the things that exist is called one" (von Glasersfeld & Richards, 1983). In my mind, a unit is the concept of "oneness." Euclid regarded a number as a multitude composed of units or a unit composed of units. Unitizing is an example of Piaget's notion of Reflective Abstraction, which is a higher level of abstraction than simple empirical abstraction. While empirical abstraction is abstraction based on observables, where units are derived from sensory-motor material, or objects to count, reflective abstraction takes the units as material for construction. In other words reflective abstraction requires one to act on their operations, not just on empirical stimuli. Consider the number three. Three could be the result of the act of counting three objects. In this sense, three is the result of empirical abstraction. However, when a child can then act on three as an object or unit by counting on from three, then that child is capable of reflective abstraction. In this sense, “a particular number is produced as a double act of abstraction” (von Glasersfeld & Richards, 1983).

Theory of Counting Types. According to Steffe and colleagues, it is the operation of unitizing that distinguishes a child from being pre-numerical and being numerical. The participants in the course discussed five pre-numerical stages of counting. Much of the literature and discussions in class revolved around learning a way to frame children’s whole number development in terms of hierarchal levels of sophistication ranging from pre-numerical to numerical stages.

Students in level one of this framework are referred to as the perceptual counters and are only able to count objects in their perceptual field. When objects are hidden, they can no longer count the objects because they have not abstracted figurative objects. Often their counting is nonroutine and not necessarily one to one. Figurative counting is the next level of sophistication, students now having the ability to count objects by visualizing them and, thus, they do not require actual objects in their perceptual field in order to count although students often use fingers as substitutions for actual objects. Counting can still be nonroutine and they may not be able to count physical actions such as pointing at this level. Students in the next level of sophistication are referred to as counters of motor items. Motor items are actions such as pointing, nodding, jumping, or clapping that can be taken as items to be counted. These children can also count hidden objects, and fingers can usually be reused because the movement is what is counted not the fingers. Counters of verbal unit items are more specialized counters of motor items. For these students, “the motor acts become arbitrary” (Steffe, Cobb, and Richards, 1983, p. 60). The speaking of the actual number words themselves can be a motor act to be counted. “When, from the child’s point of view, the act of creating a countable item has been reduced to uttering a number word, we call that child a 'counter of verbal unit items'" (p. 60). All of these levels that I have described so far are what Steffe et. al. refer to as prenumerical stages.

The Numerical Child. A child is numerical when they are "aware of the numerical structure designated by a number word is a composite of so many individual units" (Steffe et al., 1983, p. 67). These students can keep track of their counting acts or, in other words, they can count their counts. The first such numerical stage is called initial number sequence (INS). This stage requires the unitizing schema, which refers to the operation of seeing a number such as five as one unit or as five ones. Five becomes the result of counting from one to five, but also means five items. These students have cardinality. They can develop the strategy of counting on, which requires a monitoring of counting acts.

The next numerical stage is what Dr. H. referred to as INS+. In this stage, the child can do all that an INS child can do, but they can switch addends so as to be more efficient when tracking their counts. For instance, when given a story problem that involves the sum four plus nine, in that order, they could begin counting on from the largest number even though that is not the first number presented in the story problem. Another distinction between an INS and an INS+ child is that for INS child the number 80 would be represented as a long string of 80 objects where as an INS+ child may be able to group the objects in convenient ways such as eight groups of ten. INS+ children have two number sequences available, by ones and a grouping of ones called a unit. For instance, they can see fifteen as fifteen ones or three fives. INS children have only one number sequence available.

The third numerical level discussed is the Strategic Additive Reasoning (SAR) stage. These students can partition numbers in convenient ways to solve addition and subtraction problems that they do not already know. They use addition and subtraction to get answers to multiplication problems. They can make connections among addition and subtraction. They can use doubles to solve problems such as 4+5=9 because they know 4+4 is 8 plus one more is 9. It is important to note that SAR children use these strategies on a regular basis, where as INS and INS+ children mostly use the strategy of counting on, counting down, or counting up to in order to solve multiplication, addition, and subtraction problems.

After SAR comes strategic multiplicative reasoning (SMR). SMR students can use multiplication to solve problems on a regular basis. They can also use the distributive property, but would not necessarily be aware of it. For instance, to solve 24 x 2 they may use 20 x 2 is 40 and 4 x 2 is 8, so 24 x 2 is 48. These students can begin constructing powerful fraction schemes.

Application of the Experiential Model. The course that provides the context for my study focuses on children's construction of number, including whole number concepts and fractions. The purpose of the field experience associated with this course is to provide preservice teachers with the opportunity to learn to listen to and make sense of the mathematics that children generate and to consider how children's mathematical thinking informs their instructional decisions (Mewborn, 2000). Since the field component is only eight weeks long the preservice teachers can at best be engaged in a partial teaching experiment.

Steffe and Thomson (2000) suggest that the purpose of using teaching experiments is to experience children's construction of mathematics and reasoning. The goal of a teaching experiment, based on a constructivist philosophy, is for the teacher to come into the understanding of the child. Acknowledging that it is not possible to know exactly what the child is thinking, the teacher's goal then becomes to construct a model of the child's thinking.

While I do not contend that my participants were engaged in a teaching experiment, they were learning to problematize the teaching of children in much the same way. My study involves the way in which preservice teachers in this field experience made sense of children's mathematics. My question is what models do preservice teachers construct of children's mathematics. I also want to know how they use such models to come up with their next task or question.

My study concerns ways in which PSTs interpret the realities of children with respect to their mathematics. Through the interaction, PSTs will determine the *mathematics of children*, which refers to the models that they construct of the mathematical realities of children (Steffe & Thompson, 2000). My hope is that these PSTs will use *mathematics of children* to determine the *mathematics for children* as they are teaching in classes. I want to know how they will use mathematics of children in their planning during the field experience. Steffe states that "we regard the mathematics of students as a legitimate mathematics to the extent we can find rational grounds for what students say and do" (Steffe & Thompson, 2000). The PSTs should be engaged in making sense of what they see students do, including changes in their current ways of operating.

In asking which working models preservice teachers construct of children's mathematics, it is important that I describe what I mean by a working model. Lesh, Doerr, Carmona, and Hjalmarson (2003) claim that "people interpret their experiences using models. These models consist of conceptual systems that are expressed using a variety of interacting media (concrete materials, written symbols, spoken language) for constructing, describing, explaining, manipulating, predicting, or controlling systems that occur in the world" (p. 214). Though it has been noted that constructivism is an epistemology, not a method of teaching (Clements, 2000), Steffe and D'Ambrosio (1995) state that "if the teacher formulates a model of how she makes sense of chidren's mathematical knowledge, including its construction, this would be a constructivist model of teaching" (p. 146). They refer to living models, which "get adjusted and readjusted, or even completely replaced" as more or less viable in working with the living system. I prefer the term working model, which refers to a model that is used as long as it works. In my case, preservice teachers are working to understand the mathematics of the children with whom they are working. The mathematics of the child is the living system - the result of the child's experiential reality. The preservice teachers can never know a child's mathematical reality, but can make a model of the child's mathematics that will enable them to work more effectively with that child. In this sense, they are building what Lesh and Clarke (2000) have described as a model.

As the preservice teacher works with the child they are adjusting and refining their model in order to plan appropriate tasks for that child. In order to prepare the preservice teachers for engaging in their teaching experiment in the field, Dr. H. used the language of making a model of a child's thinking. She described this process as decentering from your own ways of thinking, or cognitive decentering, to try to make a model of someone else. Cognitive decentering refers to putting your own ways of thinking aside to try to think the way someone else does. When we are trying to construct a model of someone else, we are operating under the assumption that our reality is not the reality. The goal of this model building is to use this as a basis of communication.

Knowledge of Mathematics

While the preservice teachers' knowledge of mathematics was not an explicit part of my research questions, many researchers contend that it is a necessary, but perhaps insufficient, factor in a teacher's ability to construct viable models of students' mathematics (Ball, Lubienski, & Mewborn, 2001; Ma, 1999; A. G. Thompson & Thompson, 1996). NCTM (2000) suggests that teachers need to have a deep understanding of the mathematics they are teaching. They further suggest that this knowledge must become usable in their everyday tasks of teaching (NCTM, 2000).

Thompson and Thompson (1994) investigated a middle school teacher engaged in a teaching experiment in which the goal was to teach a student the concept of rate. In this study, they found that, while the teacher possessed a deep understanding of the idea of rate, he did not understand it in a way that made it accessible to anyone whose understanding differed from his. Essentially, this teaching experiment failed because this middle school teacher lacked the ability to de-center from his own ways of knowing in order to adequately make sense of the student's mathematics. Thus, his instruction was not sensitive to the student's ways of operating in that he could only listen to this child evaluatively, with only the voice of the discipline as his guide (D'Ambrosio, 2004). Ball (1997) suggests that while teachers need to be assisted by their knowledge in mathematics in understanding what students know, they do not need to be limited by their understanding of mathematics.

Simon and Schifter (1991) note that the ways in which teachers think about mathematics a key determinant of how they teach. How can teachers learn mathematics in ways that allow them to make it accessible to their students? Research suggests that, through focusing on students' mathematics, teachers can actually strengthen their own mathematical understanding (Borasi & Fonzi, 2002). This suggests that experiences where preservice teachers are engaged in learning how to listen to students and act upon their listening are imperative to help teachers develop content and a spirit of inquiry towards their own practice (Borasi & Fonzi, 2002).

How this framework informs my study

It has been asserted that, while we have predictive frameworks in the area of student learning such as the framework described above, "we are still at the stage of explanatory frameworks in the area of teaching" (Mewborn, 2005, p. 5). With my study, I seek to contribute to a framework that predicts how preservice teachers construct working models of children's mathematics and how these models inform their instructional decisions. I believe that the participants' learning of children's mathematics was a cyclic process. The preservice teachers' construction of the framework from class or the experiential model informed their model of children's mathematics, which then informed their instructional decisions. Since the framework from class was presented just prior to the experience in the field, it is safe to say that each PST was constructing the framework from class simultaneously with her model of her child's mathematics and that both concepts were informing each other. In other words, as the participants were working with the children, this interaction was affecting how they perceived the experiential model on whole number development (the framework) presented in the course. This cyclic process can be situated in the elaborated version of Simon's Mathematical Teaching Cycles pictured above.

How do Teachers Learn These Knowledge and Skills?

Doyle (1990) describes five paradigms in what constitutes effective teaching education programs, specifically with respect to field experiences. These paradigms include the good employee, the junior professor, the fully functioning person, the innovator, and the reflective professional.

Programs operating under the good employee paradigm see the goal of teacher education to be on training preservice teachers to "slip easily into the teacher's role and be skillful in enforcing the rules, in managing classrooms, and in carrying out the standard forms of instruction and evaluation with a minimum amount of supervision" (Doyle, 1990, p. 5). The focus is on the technical skills of teaching that can be developed through field experiences.

While the focus of the good employee paradigm is on the practice of teaching, or craft knowledge, the junior professor paradigm focuses on content knowledge in core disciplines. The assumption tends to be that if preservice teachers increase their academic work, then the quality of their teaching will increase as well.

The assumptions of the fully functioning person paradigm are that knowledge for teaching is personal. Emphasis is placed on development of a teacher's personal style, self efficacy, and clearly defining values.

Institutions operating under the innovator paradigm tend to deemphasize traditional methods of teaching in favor of those based on the latest research. New teachers coming out of these programs tend to fall into "survival mode" neglecting the innovative approaches and cling to "conventional modes of thinking" (p. 6).

The goal of institutions operating under the reflective professional paradigm is for preservice teacher education to "foster reflective capacities of observation, analysis, interpretation, and decision making" (p. 6). In this paradigm, three modes of knowledge are valued, including craft knowledge, personal knowledge, and propositional knowledge, the latter including knowledge and frameworks developed from research. Doyle suggests that these three forms of knowledge should be constructed simultaneously with opportunities for preservice teachers to connect all three forms in field experiences and the university classroom.

CHAPTER 3

METHODS

I conducted a collective case study documenting the field experiences of four preservice elementary teachers who were engaged in a partial teaching experiment as part of their first course on methods of teaching mathematics, specifically focused on children's development of numbers and operations. This provided the context for my study. The instructor's goals for the course made it likely that the participants would learn to listen to and learn from students as well as to learn to act upon their listening.

Description of Course

This mathematics methods course was the first of two mathematics education courses taken by the elementary cohort. The mathematics content foci of the course were numbers and operations, specifically concerning counting, addition, subtraction, number sense, multiplication, division, fractions and decimals. This course concentrated on children's mathematics and included a field experience in which preservice teachers (PSTs) were working one on one with a third grade child and were expected to focus on learning how children learn mathematics.

The instructor of the course, Dr. H., worked very hard to connect the ideas from the class discussion to the preservice teachers' field experiences. In the syllabus she wrote, "you should make an effort to look through the eyes of children, trying to understand how they generate mathematical ideas." She also expected PSTs to learn to act upon their listening in order to design and enact appropriate activities for children. Her goals made the students in this course the perfect participants for my study; however, I still needed to carefully select participants from this class.

Participant Selection

From the 29 preservice teachers in the mathematics methods course, I selected four participants and examined their experiences very closely. I selected a purposeful sample of preservice teachers as is typical in qualitative research methods. Patton (2001) suggests that purposeful sampling is typically necessary when an in-depth understanding is preferred over empirical generalization. I was interested in an in-depth understanding of how preservice teachers developed and used models of children's thinking about numbers and operations and therefore chose each of the four participants because they showed potential for building and using models of children's thinking based on their early assignments and comments during class discussions.

The third grade children, with whom the preservice teachers in the course worked, were chosen by their teachers. The children were from three different teachers' classes. Those chosen particularly to work with my participants were ones that the teachers thought would be willing to explain their reasoning.

Number of Participants

In choosing the number of participants for my study, I considered that I wanted enough participants so that I would see be able to see similarities and differences in the participants' models of children's mathematics. I also wanted a range of possibilities with respect to the ways in which their models informed their instructional decisions. I determined that four participants would offer enough data for themes to emerge and to offer variation in and among those themes, as well as being a manageable number of students on which to focus, allowing me to observe each participant twice.

Selection Criteria

My participants were chosen on the basis of a “buy in” criterion and, in their willingness to articulate their learning and their questions about children’s mathematics and the methods being presented, variation in their mathematical backgrounds and perceived abilities. By “buy in,” criteria I refer to the preservice teachers buying into the goals of the course as presented by Dr. H. Preservice teachers were considered to have a “buy in” criteria if there was evidence that they saw listening to and using children’s mathematics as an important part of what teachers do as opposed to leading children to do mathematics in a predetermined way. Evidence of this disposition and of their ability to verbalize their thinking was based on early class products as well as classroom observations. Their mathematics ability was determined by the instructor’s comments on homework as well as their own account of their mathematics as described in their mathematics autobiography.

Buy In Criteria and Ability to Articulate Thinking

I used many sources to determine whether the preservice teachers were “buying into” the goals presented by Dr. H, especially goals pertaining to my research questions such as learning to think about children’s mathematical learning. I used classroom observations as well as many early assignments to determine evidence for this “buy in”. Early assignments included informal writing and reactions to the readings as well as analyses of counting taken from excerpts of children solving problems. I also used the instructor’s comments and my own classroom observations to further inform my selection.

In order to obtain information about discussions of the literature as the preservice teachers became familiar with the research regarding constructivism and existing experiential models of children's mathematics, I observed most of the methods classes offered during the time frame of the study. These classroom observations aided me in my participant selection and in the development of interview questions for both participants and the instructor of the course.

Their early class assignments included reading and reacting to their framing literature including *Young Children Reinvent Arithmetic* by Kamii and Housman (2000), *Children's Counting Types* by Steffe, von Glasersfeld, Richards, and Cobb (1983), and other readings based on a constructivist epistemology. As I investigated their reactions to the readings and heard the class discussion about these readings, I was looking specifically for students who suggested that they were curious about the ideals in constructivism and the ways in which children learn mathematics. For instance, in class, one of my participants explained that she was beginning to see teaching as "more prompting rather than telling." She further stated that sometimes teachers need to "put students in a position where they are not comfortable." She felt that this was an important goal for teachers, and called it an "art form."

The participants that I chose asked many questions in class, suggesting that they were working hard to try to understand the experiential model given as a framework to think about children's development of whole number. In response to the reading, one of my participants asked whether or not "we hurt kids by not teaching the algorithm," to which another participant that I chose replied, "I don't think any test ever tests a certain method." I felt that their discussion, along with other evidence, showed that the participants I eventually chose were thinking deeply about the readings that they were responsible for in the course.

The counting analyses assignment that influenced my participant selection was based on their reading of *Children's Counting Types* by Steffe and colleagues (Steffe et al., 1983). The instructor selected excerpts from sections of this book which discussed children engaged in tasks presented by a researcher. Students were asked to analyze the counting of three different students and their analyses included many ideas that were relevant to my study. For instance, in the assignment they were asked to state their inferences about the child's mathematical thinking, list the types of counting activities they thought would be appropriate for the child, and provide a rationale for why they selected or designed these activities. I was looking for preservice teachers who, by the instructor's opinions, were working hard to do the type of work involved in this particular assignment and Dr. H's judgments guided my selection of participants.

Variation in Mathematics Background

The first informal writing assignment was a very brief mathematical autobiography in which the preservice teachers were instructed to situate themselves as mathematics learners. I wanted a diverse group in terms of their mathematics backgrounds, thus I used their responses on this assignment, as well as their responses on their mathematical problems, to gauge them as mathematics learners. One participant that I chose described that she had always struggled, and continues to struggle, in mathematics. Two of the participants described “ups and downs” in mathematics and the other participant suggested that she had always done well in mathematics and that mathematics always came easy to her.

I used the instructor's opinions of their mathematics, based on their responses to the problems posed, to verify that I had a fairly diverse group of participants in terms of their mathematical ability. Essentially their views of themselves matched the instructor's opinion of their mathematics background. I chose a diverse group in terms of their mathematics because I wanted to see if their self-proclaimed mathematical ability had any affect on the working models they were able to construct and use of children's mathematics.

In the syllabus, Dr. H claimed that deepening one's own understanding of the way she came to know mathematics is one of the most significant factors in building an understanding of children's mathematics; therefore a major focus of the course was about doing mathematics, generating mathematical conversation, and reflecting on one's own mathematical knowledge. She assigned several mathematical problems over the course of the semester and, although I did not have access to the preservice teachers' solutions to the mathematical tasks posed by the instructor, I did use her comments pertaining to their mathematical abilities in order to determine a range of mathematical abilities among my participants.

Rationale for Participant Selection Criteria

The research questions that continuously guided my methods were what working models preservice teachers constructed of children's mathematics and how these working models informed their instructional decisions. As a group, the preservice teachers chosen met the criteria that I thought would generate good data to help me answer these questions. I needed participants who would be able to articulate their thinking about how their child acted mathematically, their interpretation of these mathematical actions, and how this informed their instructional decisions. In order for these participants to be able to do this work, I thought that it was important for them to see this as a necessary part of learning to be a teacher. This work was not easy and I believe that learning to do this required more motivation than just being part of the expectations of a course. I included the criteria for variation within mathematics backgrounds because Thompson and Thompson (1996) concluded that an inservice teacher failed to get a student in a teaching experiment to understand a topic conceptually due to his lack of mathematical knowledge of teaching. While this teacher seemed to have a lot of formal mathematical knowledge, he did not seem to understand mathematics in a way that allowed him to listen to a student, construct a viable model of her mathematics, and use this model to help her understand it. Thompson & Thompson's conclusion made me curious to see if these preservice teachers' perceived mathematical backgrounds and abilities would have any effect on their ability to construct a viable working model of their child's mathematics and be able to use it to make appropriate instructional decisions.

All four of my participants were very articulate in class about how they were thinking about the readings and discussion from class. They asked questions and made comments that suggested that they each displayed a curiosity and intrigue about the ideas of constructivism and the experiential model presented in class. They varied in terms of the instructor's and their own perceptions of their mathematical skills and background. These traits made them particularly good candidates for participants in my study and I felt very fortunate to have participants with this kind of potential.

Reactions to the Readings and Class Discussion. In class, the students were responsible for reading and responding to many articles and texts that were written from a constructivist perspective. They were also reading articles that described the experiential model that they would eventually use as a framework to guide their work in the field. I was looking for students whose comments in class, along with written reactions to readings, suggested that they were curious about the ideals in constructivism and the ways in which children learn mathematics. In the section above concerning the selection criteria that I have called the PST's buy in and ability to articulate, I have described some of the specific instances of their responses that led to my choice of participants.

Collection of Data

Hays (2004) states that “it is most important that the researcher remember that triangulation requires multiple sources of data and multiple methods in answering each question” (2004, p. 230). By using multiple data sources to understand the PST’s models and how they used their models, I had more confidence in my findings. Data collection consisted of class products, observations and videos of the teaching experiments, interviews with the four participants, and interviews with the instructor of the course.

Each of these artifacts provided me with a unique contribution to answer my research questions. Some of their class products were reflective pieces that showed their current thinking about their children’s mathematics, including their interpretation with respect to the experiential model, which served as their guide for this interpretation. Other sources offered a plan for instruction for each interaction. The observations of the interactions during the field experience allowed me to inquire immediately about the instructional decisions they made, the way that they were questioning, and how they were listening to the children. The videos of these interactions allowed for retrospective data analysis and acted as a source to help me come up with questions for the final interviews.

Class Products

By using a variety of class products, I was able to obtain a richer source of information to help me answer my research questions The course focused on children’s development of numbers and operations. Thus, many of the products were designed to help preservice teachers begin to understand how children “generate mathematical ideas” (Hackenberg, Syllabus, p. 1). The class products that provided helpful sources of data were their initial student interviews, ongoing activity reports, case reports on their school buddy, learning trajectories, and final portfolios. These class products proved to be invaluable sources of data for my study.

Initial Student Interview

The first class product that I used as a source of data was the initial student interview, which consisted of each PST's response to the first session with her child in the field experience and what she learned from said interview. The preservice teachers were given an interview protocol which, based on their school buddies's responses during this session, allowed them to skip problems that they felt would be too easy or too difficult for their buddies. The purpose of the initial interview was for each participant to learn about her school buddy's "strengths and areas of potential development in mathematics" (Hackenberg, Syllabus, p. 3).   Essentially the preservice teachers summarized their interviews, including descriptions of their buddiess solutions to problems that were posed, what they learned about their experiences, and what the experiences implied about instruction. I used this data as a base line of my analysis, which gave me an initial sense of how they began to use the experiential model to construct their own working model for the first time.

Activity Reports

For each session of their field experience, PSTs were responsible for preparing an activity report, part of which had to be filled out in advance while the other part was to be filled out after their session with their school buddy. The part that had to be filled out in advance was essentially a lesson plan in which basic objectives had to be listed and activities described. After each session, the PST remained at the school to comment on the problems or successes the child had and give a brief description of what she planned to do in the next session.

Case Report

The case report on each PST's school buddy was a product where the PST was asked to "describe a pedagogical dilemma" that she encountered during her interaction with her buddy (Hackenberg, Syllabus, p. 4). They were to bring their cases to class to swap with two of their classmates where they were then responsible for providing feedback concerning pedagogical dilemmas found in each case. While I did not analyze their feedback to peers, I observed that some of their instructional decisions were based on the feedback received.

Hypothetical Learning Trajectory

The learning trajectory assignment was an essay in which PSTs had to describe, hypothetically, moving from one mathematical topic to another with a class of students. They were instructed to use mathematical topics that they had discussed in class and to provide a "big picture view of [their] goals and ideas" - a description of how they would know what their students knew in the beginning and ideas for "problematic situations that [they would] pose to different groups of students" so that their hypothetical students would be engaged in "productive mathematical activity" based on their assessment. This particular assignment was not specifically about their field experiences, but they did draw on their interactions with their school buddies in order to think about their hypothetical learning trajectories.

Final Portfolio

The last product analyzed was the final portfolio in which each PST summarized her entire field experience. In this assignment, PSTs were instructed to "reflect on [their] growth over the semester and on the growth of [their school] buddy. They were encouraged to be creative, not just to give an account or "chronological story of [their] semester," but to provide evidence of analysis of their interactions with their school buddies. This particular product was a nice parallel to the student interview assignment since both are reflective and summative pieces, one at the beginning and one at the end. The order in which I analyzed the data, starting with the initial interview and moving to the final portfolio, allowed me to see if there were changes in the participants' thinking about their children's mathematics.

Observations

While their class products did provide rich sources of data, those alone could not give me the data I needed to answer my research questions. Since I was interested in the working models that these preservice teachers constructed and how these models informed their instructional decisions, I needed to study their interactions with their school buddies. Hays (2004) claims, "the interaction of individuals cannot be understood without observations" (p. 229). Thus, I observed the teaching experiments conducted by the preservice teachers with their school buddies and took field notes during the teaching experiments, which included the preservice teacher's progression of activities, the progression of numbers used, some strategies that the child described to answer problems, and questions that the preservice teacher asked during the interaction. These notes were used to quickly generate interview questions immediately following each session.

I videotaped all eight of each participant's field experience sessions for retro analysis of the sessions. I also intended to use these videos during some of my interviews in order to help PSTs recall parts of the interaction about which I wanted to inquire. I actually only used videos in this manner during two interviews.

I was unable to video the first session for each PST because I had yet to have permission from the parents, however, the first sessions were guided by the assigned protocol, and they wrote extensively about this initial session in their student interview assignment. Therefore, I believe that I did not lose a great amount of data on that first day. I took turns observing their interactions with their school buddies and interviewed them immediately following my observation.

All videos were partially transcribed. These partial transcriptions were supplemented by written descriptions of relevant mathematical actions. I began by taking general notes on the videos. While the partial transcriptions included commentary, I found it necessary to transcribe significant portions of the video and, over time, I took detailed notes for each video of each participant.

Interviews

For each participant, I conducted three interviews, each of which were transcribed. Two of the interviews were about specific sessions, most of which immediately followed my observations of their interaction with their school buddies. These interviews were short, typically fifteen minutes. The first two days were consumed by managerial tasks including getting consent forms signed, recruiting help, and setting up equipment. I began the interviews on the third day of their field experience because I had two people operating cameras while I was free to observe a session from beginning to end. Two of my interviews about specific sessions did not immediately following the session. Instead, they were conducted one or two days later using the video to prompt the participants' memories. All interviews with participants addressed their sense of the mathematics of the student with whom they were working during the field experience.

Session Interviews

During the interviews about specific sessions, I posed both questions that were predetermined and many that I formulated during my observation of the interaction; questions such as, “What led you to ask that particular question?” or “To what extent were you thinking about the mathematics of your student when you planned the problems in which to engage him or her?” (See Appendix A). I also asked how they chose their progressions of problems and their progressions of numbers in the problems posed. The data from the interviews helped me to understand the way they enacted their plans for instruction. The data from the session interviews provided another source to confirm or disconfirm data from my observations. I was able to get a sense of their planning from these interviews, for instance, what types of questions and tasks were planned and which were spontaneous. I also learned the sources of their planning, for example, whether the tasks they used were suggestions from the instructor, their peers, or an article from class.

Interview questions that I used for these session interviews included whether they still agreed with their assessment of their children from the initial interview assignment, how they chose their progressions of tasks and activities, including the numbers used during the tasks, and how they decided when to probe their children’s thinking for particular solutions and when not to probe. I asked the participants which questions, tasks, and other decisions occurred spontaneously and which ones were planned in advance. Their answers to these questions helped me get a sense of the extent to which they were thinking about the students’ mathematics as they were planning the problems, tasks, and activities in which to engage their children.

Final Interviews

The final interviews were much longer - typically about an hour. There was a definite protocol for these interviews (see appendix), but I did deviate slightly for each participant depending on how the conversation was going. Many of the questions were determined before I began data collection and were directly related to my research questions. For instance, I knew that I wanted to ask the participants about listening to and learning from children since this was a major goal of the course and a big part of my first research question. I also knew that I wanted to ask them about acting upon their listening in order to design and implement appropriate activities and tasks for their school buddies. I asked them about their working models of their buddies based on the literature from class and wanted to know if they saw their buddies do anything that did not correspond to the literature from class. I also wanted them to reflect on the differences between their mathematics and their buddies' mathematics and to discuss any ways that their buddies surprised them.

Other questions for the final interview were determined as I was collecting data; for instance, I asked about their major goals for their field experience sessions and what they thought they worked on most with their buddies. I asked them to write or describe a problem that they thought would be on the edge of what their buddies could do. I also asked them what their buddies were not doing that they wished they would do. I used the videos and their activity reports to come up with some of the questions for their final interviews, such as why participants continued to work on specific types of problems throughout their meetings. I also showed one of my participants a video from her session in which I was particularly interested in her instructional decisions and then focused the first part of her final interview on this particular session. The last question I posed to the participants was whether they thought they would continue the type of work that they did in the field experience in their future teaching, including making and using working models of children's mathematics.

Instructor Interview

The interview with the instructor concerned her ways of getting the preservice teachers to engage in such a teaching experiment as well as her sense of their success at formulating models of the children with whom they were working. While I had many informal conversations with Dr. H. there was only one formal interview. I used the formal interview to help me formulate my own ideas of what it means to construct a model of a child's mathematics as well as a way to triangulate the data.

Data Analysis

In order to analyze my data, I began with participants' initial student interview assignments. I read these assignments over and over until themes began to emerge based on similarities across participants. Strauss and Corbin (1998) call this a microanalysis of the data. They describe microanalysis as "the detailed line-by-line analysis necessary at the beginning of a study to generate initial categories (with their properties and dimensions) and to suggest relationships among categories; a combination of open and axial coding" (Strauss & Corbin, 1998, p. 57).

After the initial interviews, I analyzed the final portfolios to see if those same themes fit. While those themes were supported, new themes also began to emerge. Then I moved to the interviews where I read the transcripts of the interviews with my themes in mind and categorized the transcripts according to these themes. After each categorization of each piece of data, I summarized the findings for each participant and then again across participants. Eventually some themes were collapsed into subcategories of other themes.

My data analysis was guided by my assumption that children, and all people, construct mathematics and that teachers can and should construct a model of how their students construct mathematics. These hardcore assumptions framed how I was looking at the data. While I was indeed looking for evidence that my participants were doing this kind of work, that is constructing models of their children's mathematics and using those models to inform their instructional decisions, I was open to conflicting evidence as well.

Guided by my research questions, I was looking for instances in their writing where they related their experiences in the field with the framing experiential model from class, as well as for instances where they discussed their instructional decisions that were based on their interpretations. I found many instances of these in their writing for their initial interview assignment and throughout each artifact that I analyzed, coding statements about these ideas as well as other ideas that emerged as themes during my readings and re-readings of the initial interview assignment.

Data Analysis and the Framework

In this section I will connect my methods of data analysis to the other parts of my conceptual framework. This includes three parts: their Mathematical Teaching Cycle (MTC), listening, and acting upon listening (Simon, 1995). While I was not using Simon's framework of MTC at the beginning stages of my analysis, it was helpful in discussing the themes that did emerge.

The preservice teachers seemed to be engaging in the field experience in a way that teacher educators would want professional teachers to engage. Each of the participants was carefully observing her child's mathematical ways of operating, interpreting those ways of operating, making hypotheses about their interpretations, and orienting her instruction to "meet the student where he/she was," as well as to open possibilities for the student to make progress within the levels of the experiential model from class (Hackenberg, personal email). These are the actions and knowledge that Simon identifies in his MTC.

Simon (1997) defines mathematics pedagogy as “intentional interventions” that are “made to promote the construction of powerful mathematical ideas” (p. 58). The PSTs intentionally intervened in order to get their students to progress to the next level of sophistication with respect to their framework from class. Simon (1997) suggests that constructivism makes us aware that students will not likely make the same sense of something that teachers do. Furthermore, he states that it makes little sense to tell each student the same thing since individual students will most likely make sense of a message differently. He suggests that teachers need to understand students’ current knowledge and thinking and to act on this understanding, however, teachers’ interpretations of students’ knowledge is limited at best. Therefore teachers must construct a model that provides them with working knowledge of how to engage an individual student with what she needs. Simon suggests that this work is essential to a model of teaching that is aligned with constructivist perceptions.

He calls his model of teaching mathematics the “Mathematics Teaching Cycle” (MTC) (Simon, 1997) and it is this cycle that I claim that my participants were engaged in during the field experience. My study was focused on certain elements of this cycle, particularly a teacher’s knowledge with respect to her models of students’ knowledge, teacher’s knowledge of student learning of particular content, how this knowledge is developed from her interaction with students, and how the teacher’s knowledge affects her instructional decisions, which Simon calls Hypothetical Learning Trajectory (HLT). The differences between my study and Simon’s MTC is that, rather than focusing on a classroom, each participant was working one on one with a child, therefore interaction with students becomes interaction with one child. Another important difference is that Simon’s model is intended to be a model of how teachers interact and plan for their students in their classroom while my study focuses on how preservice teachers are learning to do this work with one child. Simon's model identifies a teacher's knowledge of student learning of particular content that is most likely gained from years of experience working with children in a particular content area, however my participants developed their knowledge from the experiential model presented in class for guiding their experiences in the field since they lack previous experiences working with children. The use of the experiential model as a guide was an essential element of their work with the children in the field experience and I believe what made their work possible.

As the PSTs were engaged in the MTC, listening was an important piece of constructing their models of their children's mathematics. Initially I was looking through the data for instances of listening as described by Davis (1996) and D'ambrosio (2004), however, the preservice teachers did not explicitly discuss listening in their initial interview assignments, nor in any of the other class products that I coded. However, while listening did not come up as a theme, it can be inferred from the some of the themes, which will be discussed in chapter five.

As teachers were acting on their listening, they were working hard to choose and design tasks that their children would be able to do and that would stretch their mathematics. Steffe refers to this work as operating within a child's Zone of Potential Construction (ZPC). This idea is similar to Vygotsky's well known concept of the Zone of Proximal Development (ZPD), which is typically described as the distance between what a child can do on their own and what they are capable of with a more knowledgeable tutor (Vygotsky, 1986). If the PST had been attempting to work within a child's ZPD, her primary goal would have been to get a child to understand mathematics in the way that she, the more knowledgeable one, understood mathematics, however the instructional goals of the preservice teacher attempting to operate in a child's ZPC are determined from previous work with children (Steffe, 1991). In this case, the preservice teachers had not done previous work with children, rather they were using the work of Steffe and colleagues by operating with the experiential model as their guide. In chapter five, evidence will be presented to prove that the PSTs were attempting to operate within their children's ZPCs.

I initially planned to report my data as the case studies of each participant, which would typically follow the way I analyzed my data, however, as I was writing the draft of the reports of each participant, I noticed that these cases sounded very monotonous. One reason for presenting multiple case studies is that each case may represent a different thematic finding (TESOL, 1996), however the same themes emerged from every participant. Using a thematic organization helped me to elaborate the themes by discussing both the similarities and differences across the participants.

CHAPTER 4

BACKGROUNDS OF PARTICIPANTS

In choosing participants, I wanted variation in mathematics background, disposition, and ability. I used their initial letters to Dr. H., which included their mathematics autobiography, to gauge this variation. All of my participants were females in their early twenties, which is typical of the students in the elementary cohort at the University of Georgia. One participant was Chilean, while the others were Caucasian. Below I describe the background of each participant. In these descriptions I include information from the preservice teachers' letters to the instructor, their mathematics metaphors, information from their initial interview assignments, and a brief description of their sessions during the field experience.

Each piece of data included in the backgrounds of the participants was chosen deliberately. The letters included their feelings about and previous experiences with mathematics as well as their backgrounds in working with children. These were important because they showed a variation in my participants' backgrounds with respect to their mathematics abilities and dispositions.

In order to offer a comprehensive description of their backgrounds with respect to my research, I decided to report their beliefs about mathematics, which required that I adapt an existing instrument. This instrument was e-mailed to the participants approximately a year after the School Buddy Experience. By the time they received the instrument they had completed their student teaching[1]. The mathematics instrument was designed to understand their beliefs about mathematics, mathematical learning, and mathematics teaching (See Appendix F). I did

[1] Student teaching is meant to refer to the participants' last semester field experience as a senior where they were working with a whole class setting not the School Buddy Experience associated with the context of my study.

not originally intend to obtain data concerning their beliefs about mathematics. However, as I was analyzing the data, I began to think it was important to determine their beliefs about mathematics because according to Davis (1996), the ways in which a teacher is able to listen to her students is related to her beliefs about mathematics.

I decided to include my analysis of each participant's initial interview assignment because it provided a baseline for where each participant began using the language from class to describe the mathematical actions of their buddies and their initial constructions of working models of their children's mathematics. I thought that it was important to tell the story of each individual participant's beginning thinking at the first interaction with the child during the field experience as it would allow for a more thorough understanding of each participant in relation to my study. I also thought it would be a good lead in to the thematic organization because it is this piece of data where almost all of the themes in my analysis emerged. The last part of each participant's reported background is a streamlined summary of her work in the field, included so that the reader could better understand the context of this research.

Emma

Letter

Emma described herself as always having been good in mathematics. Before her first mathematics for teachers course she did not have to take mathematics courses at UGA because she scored high enough on the mathematics placement tests. Throughout her school experience, she was always placed in accelerated mathematics courses and always earned A's in mathematics without much effort. In high school she took Advanced Placement Calculus, which was the first time she actually had to study. Although she enjoys working through the steps of a mathematics problem, she does not like to explain on paper what she did to solve the problem.

She admitted that, although she did well in upper level mathematics courses, she felt that she only had a procedural understanding of the material. She wrote,

> I remember from Calculus that I sometimes felt as if I didn't really understand what I was doing or why I was doing it, but rather was simply following the step-by-step directions that the teacher had given me to solve the problem.

Emma had various experiences working with children prior to the semester of the study. She has been volunteering to work with children since middle school. For instance, when she was in middle school she volunteered to work in a tutoring program where she helped elementary children with their homework or read to them. She also worked with children in the nursery at her church and, more recently, volunteered to work in a second grade classroom once a week, helping the children if needed and observing the teacher, attending specifically to her management strategies.

In this particular semester, she stated that she hoped to learn about the ways that children learn mathematics and other subjects, as well as some strategies that children use in order to solve mathematics problems. Eventually she would like to be able to incorporate what she learns about these strategies that children use when she teaches in her own classroom.

Mathematics Instrument

Emma's simile for learning mathematics was that learning mathematics is like building a house; "You have to determine what the students already know mathematically and build on that preexisting knowledge." She indicated that a mathematics teacher is a combination of a coach and a missionary and that "students need to be encouraged and made to feel that they can succeed as a math student." She felt that the purpose of teaching school mathematics is to enable students to function in society as well as to see mathematics as a connected whole. She saw mathematics as

> …a set of rules and truths. For me mathematics is another way of understanding the world in which we live. As people (mathematicians and the like) come to have this understanding, the rules and truths that are mathematics are discovered. The mathematical rules that we teach to our students are those that have been discovered by those who came before us.

Emma seemed to believe that mathematics has always existed in the world and that it is absolutely true. She stated, "Our world is structured in a certain way and mathematics has come about as a means of understanding that structure."

She indicated a belief that, when it came to teaching mathematics, skills and concepts were equally important and Emma gave each equal time during her student teaching experience. She wrote,

> It is very easy to simply follow through with the motions of doing something without fully understanding what exactly is going on. I had several instances during student teaching where I would double check with my students that they were understanding why we added things a certain way or something like that. If they didn't understand, I would explain it again in a different way until they did. I wanted them to have the concept so that they could be confident in their abilities to practice the skills.

Initial Interview Assignment

Description of Emma's Model. Emma's interview assignment showed that she was able to assess Matthew, her buddy, according to the experiential model from class, which framed her thinking about children's mathematics. She often used language from class to discuss the ways in which she observed Matthew operate. She wrote, “So, in a matter of seconds, he switched addends and strategically broke seven into two and five.” She also recognized what he was doing as strategically reasoning: “The use of strategic reasoning continued with the counting-off vs. counting-down problems, the addition and subtraction problems I chose to pose, the place value addition and subtraction problems, and for one of the multiplication problems.”

It seemed clear that Emma was building a model of Matthew’s mathematics from the start. She discussed her expectations of what he would be able to do and what strategies he would use based on a few minutes of posing tasks and observing him solve them. “Matthew added 30+20 to get 50 to start with, which is something I expected him to do based on the strategies he had used earlier.” She was also using inductive reasoning to find patterns in Matthew’s mathematical actions: “ One major trend I noticed was that at no time did Matthew subtract off for any of the problems. He always added up to the bigger number.” She used the patterns she observed to connect to the literature from class. She also saw the propositional knowledge in action. She wrote, “This reinforces a notion we talked about in class, that children prefer using addition to arrive at their answers.” Based on her construction of the framework for whole number development, she described her working model of Matthew in the following way: “Based on what I observed through this initial interaction with Matthew and based on what we have discussed in class, I would say that he is a strategic additive reasoner.” She based her assessment on evidence consistent with what children at the SAR level should be able to do and it may be important to note that her course instructor agreed with this assessment.

Does she see this work as useful? Emma seemed to believe that the process of interviewing children in order to construct a working model for future instruction is useful beyond her field experience. She wrote,

> I think that this interview process is a great way to establish what mathematical level a child is thinking on. It can tell you what areas the child has problems in and whether or not it is safe to move onto more challenging problems and activities. All teachers can benefit from such a process in terms of knowing their student's strengths and weaknesses.

She also acknowledged that the students are helped, which seems like an important point for her to make: "The students can also benefit from it because it can help the teacher maximize each student's mathematical education."

Early Instructional Decisions. Many of the instructional decisions Emma discussed were spontaneous; for instance, she indicated that she skipped problems that she felt were below Mathew's level of operating: "I skipped a few of the problems, believing them to be too easy for him." She also made decisions to adapt certain problems in the interview protocol that she determined would be in his short term ZPC: "To make things a little more challenging, I covered the stacks with my napkin." On occasion, she regretted not posing some of the questions she skipped, which is important to note because there is an indication that she wanted to understand how Matthew was thinking about certain topics, such as division. "I did not pose the M&M's problem and in hindsight I wish I had to see where he stood with division."

On one problem, Emma referred to an instructional decision she made not to correct a mistake that she noticed in Matthew's thinking. She described the problem as,

> "Misha has 34 dollars. How many dollars does she need to have 57 dollars?" Matthew added 30+20 to get 50 to start with, which is something I expected him to do based on the strategies he had used earlier. Then he added 7+4 to get eleven and added the 20 and the eleven to get a final answer of 31, which is incorrect.

Then she discussed what he was doing and what may have led to his mistake:

> I believe, to start with, Matthew knew he needed to add on to the 34 until he got to 57 which is why he said 30+20=50, but then I think he got so wrapped up in the idea of adding that he confused his own thinking without realizing it.

She wasn't satisfied with her speculation of why he may have made a mistake and wanted to explore further how Matthew might think about this type of problem, which was missing addend with regrouping:

> In the future, I would like to give Matthew some more problems like this one to see if it is something about the wording that led to his mistake because his reasoning for the problem that followed, involving the rock collection, was correct.

Types of questions that elicit children's mathematical thinking. Emma discussed a few instances when her questions helped her determine how Matthew was thinking. These questions were meant to prod Matthew to use another method besides the traditional algorithm.

> For the place value addition problem involving adding 28 girls and 35 boys playing on a playground, Matthew used the addition algorithm to get his answer. When I asked him if there was another way he could do it, he failed to see any reason why he should find another way when he had already determined the answer.

In class, the preservice teachers discussed that if a child uses the traditional algorithm to solve multi-digit addition and subtraction problems involving regrouping, it doesn't help determine the level on which the child is operating according to the framework. I suspect this is why Emma continued to ask Matthew to come up with other methods to solve a problem. Again she indicated, "He used the algorithm correctly, but I did ask him to use other ways besides the algorithm for the rest of the problems." It is important to note that, even though she only mentioned two questions in the initial interview assignment, this does not mean that she did not ask more, only that she may not have written down every probing question that she posed.

Indications of Surprise. Emma definitely indicated moments of being surprised during her initial interaction with Matthew. She stated, “I was floored by how quickly Matthew went through the activities I posed to him.” She was also surprised by what Matthew was able to do mathematically. “My surprise at his mathematical abilities was immediate as I posed the first counting-on problem.” While I would expect a preservice teacher interacting with a child in this experience for the first time to be surprised, Emma’s writing about her interview experience indicated that she overwhelmingly surprised. “To be perfectly honest, I was shocked at how well he did. I was not expecting him to be as advanced as he was.”

Field Experience

Since Emma determined on the initial interview assignment that Matthew was operating at the strategic additive reasoner level of whole number operations, she continued to work with Matthew on problems and tasks that might elicit strategic multiplicative reasoning. Throughout the field experience, she worked with getting Matthew to track the number of stacks and total number of Unifix cubes within those stacks of five or more cubes. She also worked with arrays. In working with the arrays she would first show Matthew the array and then hide it, asking Matthew how many rows, columns, and total dots he saw. She worked with Cartesian products, including two level and three levels. These problems involved combinations such as how many outfits can you make if you have 4 different pairs of shorts and three different shirts. She presented him with word problems involving measurement and partitive division and, toward the end of the experience, she began exploring what Matthew was capable of with fractions, especially with sharing candy bars, naming fractional pieces, and comparing fractions.

Maria

Letter

Maria was of Chilean descent. She wrote, "I would describe myself tentatively as a thorough, yet unenthusiastic mathematics student." She, like Emma, was in an accelerated mathematics program from middle school. She took both AP Calculus AB and AP Statistics and became disenfranchised with mathematics after getting a score of 4 on the AP Statistics Exam. She wrote, "I vowed to never take another math course." She claimed that she thought that her Mathematics for Elementary Teachers course, Math 5001, was too easy, but found it useful when she started tutoring young children. She discussed that this course enabled her to explain things fully and break the content down so that a child could understand.

Maria wrote about her experience working with children in almost every elementary grade level. She, like both of the other participants, volunteered to work with children when she was in high school, particularly in helping them to learn Spanish. While in college, she worked with an organization called Oasis, tutoring Hispanic children in various subjects. She also worked one-on-one with a first grader in reading. More recently she spent 100 hours observing a fourth and fifth grade classroom at a local elementary school.

From her comments concerning what she wanted to learn this particular semester, Maria seemed to view teaching as explaining, claiming that she wanted to learn to "explain complex ideas to younger minds." She wanted to be able to "get through" to students by being able to synthesize ideas so that her wording was at the appropriate level. Her other desire is best captured in her statement, "I would like to regain my original love for mathematics so that I may teach it with the enthusiasm necessary for the development and growth of my students' love of math."

Mathematics Instrument

Like Emma, Maria indicated a belief that learning mathematics is like building a house, and suggested, “you must first create a solid foundation to build knowledge upon because, without that base knowledge, you cannot make connections and learn more difficult concepts.” She discussed that a mathematics teacher is like a gardener and that “He/she plants the seeds of knowledge, fertilizes the plant with coaching, direction, and guidance, and watches the plant grow over time.” For Maria, the purpose of school mathematics is to enable students to be able to

> …see mathematics as a connected whole. I would love to say that the purpose of school mathematics is to enable students to function in society, but before entering that society, the students must learn the concepts and excel at them in school. Once they have mastered them in the school environment, they can apply them in society; thus, the students see how mathematics will continuously be relevant or important to know. It is also wonderful to watch students realize that math skills build upon each other and are connected.

Maria views mathematics as

> a personal experience. There will always be problems that arise, but there is no single, correct solution or method to solve that problem. Thus, math becomes a personal experience because one chooses what skills to apply to solve the problem. There are many ways to arrive at the same end, and if we all have a variety of tools and abilities from which to select, we can use creative freedom to approach the situation.

When it comes to teaching mathematics, she believed that equal time should be spent on concepts and skills. In reference to her student teaching experience, she stated

> Although I would like to say that I concentrated equally on skills and concepts during student teaching, it was more like I spent an equal amount of time on different topics and emphasized either skills or concepts. For example, I first taught two-digit long-division, focusing on skills, not concepts. Although I attempted to explain the concept, the students found it easier to simply memorize the steps to complete the problem. On the other hand, I spent a great deal of time discussing concepts in geometry. I believe that the topic under discussion has a great impact on whether you emphasize skills or concepts.

She believed all knowledge, including mathematics, is "dynamic and constantly changing." She stated that

> There may be ideas that are currently accepted as true, but there are also scientists and mathematicians and researchers that are discovering new bits of information or exceptions to rules. We cannot disregard that uncertainty does exist in mathematics and life in general, but we must also embrace the knowledge we currently possess and utilize it to the best of our abilities until new information is given to us.

Initial Interview Assignment

Description of Maria's Model. Maria recognized and named mathematical actions discussed in class and it was clear that she was thinking about how Ansley's mathematical actions fit into the framework she learned in class. She also discussed her excitement of seeing what she read about come to life through the actions of her buddy: "This sign of strategic reasoning excited me, but I did not want to make any conclusions prematurely." While she noted seeing actions that were consistent with strategic additive reasoners, she did not immediately classify her buddy, Ansley, in that category. She wrote about recognizing Ansley counting off versus counting down because Ansley started counting back past 19, the subtrahend.

> I thought it was interesting how she had accidentally counted off, but because she did not understand that by doing so the answer was one number below the final number she counted on her fingers, she thought she had done the problem incorrectly. However, because Ansley was so certain that the response was 16, she was able to solve the problem in a different way, by counting down.

Dr. H. suggested that Ansley may have been counting down to start with. When describing Ansley's number sequence, Maria judged a problem's degree of difficulty for Ansley by the speed at which she responded: "This showed me that it is easier for Ansley to find the number word after a particular number than to find the number word before that number, especially when the counting back involves crossing a century number."

Maria's questions allowed her to recognize strategies that Ansley was using to add and subtract whole numbers. She wrote, "I quickly asked Ansley how she had solved the problem, and she said, "I knew that 28 minus 8 was 20, minus another 1 is 19. So the answer is 19." She described Ansley's solution as, "Ansley had once again used strategic reasoning to break up the subtraction problem into two steps that made the calculation easier for her!" Many times during her interview assignment, she described observing Ansley use strategic additive reasoning. She also recognized the significance of Ansley being able to count on from the largest number, confirming that Ansley was above the INS level: "she quickly switched numbers—a confirming sign of an INS Plus child—to count up four from 9 to get the correct response, 13." She knew that Ansley was numerical, but further reflection was required before she determined which numerical category to place Ansley: "During the experience, I was able to observe Ansley enough to definitely categorize her as a numerical child. However, with regards to INS, INS

Plus, and strategic additive reasoner, I became a little unsure of her classification." She described all of the strategies that Ansley used that were consistent with those of a SAR:

> Ansley showed clear evidence of using strategic reasoning for addition and subtraction problems throughout the interview, using such strategies as partitioning numbers, compensating, counting by tens for addition, using common sums, and so on. Ansley was also able to construct and use place value for addition problems, only displaying difficulty with subtraction problems."

Because Ansley could not consistently use place value in addition and subtraction problems, Maria was hesitant to classify her as a SAR, drawing from the framework that a student is not a SAR unless place value is usable in problem solving: "Due to her problems with place value, I think I would classify her as above the INS Plus level, but not quite yet at the strategic additive reasoning level." Maria indicated that, while she classified Ansley as INS Plus, Ansley may have been on the verge of being a SAR.

Does she see this work as useful? Maria anticipated her work with her child because she wanted to put to use the theory she was learning in the classroom. Referring to the methods course prior to the field experience, she wrote, "I felt like I was learning all this useful information, but I was not able to utilize it in a classroom or see it for myself."

Like Emma, Maria saw the value in interviewing Ansley for the purposes of planning for their interactions during the field experience:

> After observing and interacting with Ansley, I believe that our time together was very useful for the planning of my future activity reports. It is essential for a teacher to know the placement of all of his/her students in order to create activities that will not bore them or overwhelm them, designing extensions in case the activity is too simple or alternate

> activities that will help them practice if it is too hard. By interviewing Ansley and utilizing all the knowledge and insights I gained during the process, I will now be able to best cater to her needs, thus helping her grow as a mathematics student and thrive in the classroom.

Maria wrote about using what she learned from interviewing Ansley in order to plan tasks for her during this field experience and, like Emma, Maria acknowledged that interviewing children in order to formulate a working model of their mathematics was useful beyond her field experience. She indicated that constructing a working model, which she called "knowing the placement" of students, was essential in order to plan appropriate tasks for students. One idea that Maria mentioned that the other participants did not was that she believed that she was helping Ansley do better in her third grade math class and helping her become a better mathematics learner overall.

Early Instructional Decisions. Many of Maria's instructional decisions were spontaneously determined based on Ansley's degree of struggle with previous tasks.

> Because Ansley had had some difficulty with the second problem, I posed the third question to see if she would be more successful. Since she easily solved number nine, I proceeded to pose the missing subtrahend problem because, although it was harder, I thought Ansley would be able to reason her way through it. This was by far the most difficult problem for Ansley and the one that caused her the most frustration; therefore, I decided to skip number 26 which would require similar reasoning. Because this seemed so easy for Ansley, I told her to pretend I was subtracting 3 ten rods from the 63 and to predict how many there would be after that.

Maria knew from class that she could not determine the level at which Ansley was operating if she only observed her using the traditional algorithm to solve multi-digit addition or subtraction, however, in order to keep Ansley engaged, Maria allowed her to use it from time to time. She wrote, "I finally allowed her to solve the problem by using the subtraction algorithm." I do not think that Maria wanted her to use the algorithm, but she allowed it so that Ansley would continue to act mathematically rather than shut down in confusion.

Other instructional decisions were made because Maria had to improvise due to a lack of materials: "Because we had run low on Unifix cubes, I asked her to imagine that I was adding one more ten rod to the group and predict how many there would be then." Immediately, Ansley said 53. I repeated this one more time, and Ansley said 63.

One instructional decision Maria made was not about this interview; rather it was for planning for later sessions in the field experience where she noted that, in order to solve a partitive division problem, Ansley distributed counters one by one until she was out and then recounted to be sure the division was fair. Maria indicated that Ansley had not met her expectations based on the working model she had determined from observing Ansley work previous tasks. Since Ansley was not able to solve these problems using strategic reasoning, she noted "that this would be something we could work on together and continued."

Types of questions that elicit children's mathematical thinking. Maria described her goals for the interview as she was talking to Ansley. Maria wrote, "I told Ansley that I was more interested in how she arrived at her answers than in if she answered all the questions correctly". In order to construct Ansley's mathematics, Maria asked many probing questions, and, on many occasions, Maria asked Ansley how she determined her answers: "Rather than telling her she was incorrect, I asked her to tell me how she had solved the problem." At times, Maria could observe that Ansley used her fingers to help her solve a problem, however, she was never satisfied with that observation. She wanted to know precisely how Ansley used her fingers, therefore she was in pursuit of a working model of Ansley's mathematics.

> Soon after presenting this problem, Ansley told me the answer was 16. I had noticed that she moved her fingers a little while calculating the response, so I asked her how she had solved the problem. Ansley said that she had just counted backward. I probed a little further and asked her to tell me exactly what she had done in her head. She lifted one finger and said 19, lifted a second finger and said 18, and lifted a third finger and said 17.

Through her questioning Maria learned that Ansley was able to reason strategically with some addition problems: "I quickly asked Ansley how she had solved the problem, and she said, 'I knew that 28 minus 8 was 20, minus another 1 is 19. So the answer is 19.'" Maria also often asked Ansley to try to solve a problem in a different way, especially if she observed Ansley use the traditional algorithm. She encouraged Ansley to use more mental strategies rather than having to write down her work. Much of her suggestions to try not to use paper were to discourage Ansley's use of the traditional algorithm in favor of strategic reasoning to make mental computation more fluent.

There were times that it seemed like Maria was not asking questions simply to probe Ansley's mathematics. Rather, many of her questions seemed to be pushing Ansley's thinking to another level. After Maria noticed that Ansley was capable of splitting numbers and making nice numbers, she was not satisfied when Ansley resorted to counting all by ones in the division and multiplication section. Thus, Maria continued to prompt Ansley to get her to try other methods with these types of problems.

> I asked her if there was another way she could tell me how many cubes there were without counting by 1s, but she could not count by 4s without monitoring her counting on her fingers which was essentially counting by 1s again.

Indications of Surprise. Maria wrote, "Ansley surprised me from the beginning." She noted many times that Ansley's mathematical actions did not match her expectations. Each surprising incident is a significant point that caused Maria to change or refine her working model. On one occasion Maria used an incident of surprise to determine plans for her future sessions with Ansley:

> Although Ansley had not done as well as I had expected in this section, she was able to form and keep track of groups and total numbers of counters, so I made a note that this would be something we could work on together and continued.

Maria indicated intrigue that Ansley used a counting strategy that had been discussed in class to solve a problem. She wrote, "I thought it was interesting how she had accidentally counted off." Maria claimed that Ansley used this method accidentally because she had a look of confusion as she counted "19, 18, 17," when her answer had been 16. Because Ansley appeared to have difficulty with this problem, Maria posed the next related problem and, when Ansley answered this problem with ease, Maria indicated another moment of surprise.

> What happened next really surprised me. I asked Ansley to put 28 counters into her cup. After she did so, I removed 9 counters and told her I had done so. Upon asking Ansley how many counters were left in the cup, she quickly responded with 19. I was shocked at how quickly she had solved what I had thought would be a much more difficult problem for her.

Maria described how Ansley used strategic reasoning to solve a missing addend problem. With her description, she indicated another moment of surprise:

> I told her that Susan had 12 gummy bears and gave some to her friend, and now she had 7 gummy bears. Then, I asked her how many gummy bears Susan had given to her friend. Ansley immediately said 5. When I asked her how she solved the problem, she said, "I knew 12 minus 6 was 6, minus 1 more is 5." It seemed so easy for her to use this type of strategic reasoning. I was very impressed.

After Ansley got an answer of 14 when solving an addition problem involving 7+6, Maria asked Ansley how she arrived at her answer. Maria quoted Ansley's response as, "I knew that 7 plus 7 was 14, but I had to subtract 1, so I got…wait a minute! I messed up. The answer is 13!" Once again Maria seemed surprised but this time she was surprised at Ansley's ability to self-correct..

Maria also seemed intrigued by how Ansley resorted back to counting strategies to solve similar types of problems; however, she interpreted Ansley's use of different methods to indicate that Ansley possessed an understanding of the mathematical concepts:

> I thought it was interesting that she used the same method to solve different levels of missing addend problems and a comparing problem; this proved to me that she truly understood the math process involved in each situation even though the circumstances and the wording were different.

For another missing addend problem involving the whole of 21 and a part of 16, Maria described their interaction. She indicated that Ansley was confused and wanted to write it down. While Maria encouraged her to try it another way, she finally allowed Ansley to use the standard subtraction algorithm. After she was able to successfully arrive at an answer of five, Maria once again asked Ansley to try it another way. This time, Ansley counted up to solve the problem.

Maria indicated surprise and interpreted Ansley's response in the following way: "This was interesting to me because it showed me that Ansley seemed to trust the algorithms over her own reasoning when dealing with missing addend problems."

Maria was surprised at Ansley's response to a partitive division problem:

> I told Ansley that there were 20 students that were riding to the park in cars but that only four students could fit in each car. After asking her how many cars they would need, Ansley asked me if she could use pencil and paper. What happened next was incredibly fascinating. Ansley drew a car and put 4 tally marks in it, drew another car and put 4 tally marks in it (at this point, she said she knew 4 plus 4 was 8), drew another car (here, she counted up by 1s using tally marks to get to 12), drew another car (once again she counted up by 1s using tally marks), and drew one more car (finally, she reached 20 by counting up by 1s). Ansley had kept track of her groups of 4 tally marks by drawing them inside cars, thus making it easier for her to count the groups at the end and get the answer, 5.

When determining how many tens were between two numbers, Ansley explained that she was imagining a number line. Maria wrote, "…when I asked how many tens were between 35 and 55, 22 and 42, and 27 and 67, she explained to me that she was imagining the beginning numbers on a number line and moving along the number line by tens until arriving at the final numbers." About Ansley's response Maria wrote, "I thought this was a very different way of keeping track of the tens."

In this assignment, Maria indicated many moments of surprise during her first experience trying to understand children's mathematics. She wrote, Overall, my experience with Ansley with the counting problems and activities proved very intriguing. I was pleasantly surprised. What I had feared would be a long session of hard work had turned out to be a fun exploration of mathematics for the both of us.

Field Experience

In her initial interview assignment, Maria determined, with some uncertainty, that Ansley was operating at the Initial Number Sequence Plus level of whole number development. With this in mind, she spent some time working toward strategic additive reasoning by working with Ansley on addition and subtraction with double digit whole numbers. As she worked on computation, Maria prompted Ansley to use place value and other forms of strategic reasoning by working with base ten blocks. Maria also worked on multiplication and division with Ansley, continuously having her track stacks and cubes with stacks of size 3, 4, 5, and 6, sometimes hidden and sometimes not. Maria also worked on partitive and measurement division by having Ansley make equal groups from a quantity of cubes. She wanted her to determine the missing factor, as either the missing number of groups or the missing number of cubes in the groups. She also had her find multiples of four on a handout on one occasion and multiples of six on another occasion. This handout included a triangle with dots partitioned into distinct regions. Maria instructed Ansley to enclose a number of dots that would be a multiple of four. This activity immediately followed counting by fours by tracking stacks of four. Maria then explained that the multiples of four were the numbers that you just said when you were counting by fours.

Other activities that Maria worked on toward the end of the experience involved arrays and pairing problems. Maria used a book called “The Grapes of Math” to introduce the idea of arrays. She also showed Ansley a 4x6 array of dots and then hid it, asking her how many rows, how many columns and how many dots. She followed this activity by asking Ansley to draw as many different arrays of 24 dots as she could by using distinct arrangements other than the 4 rows and 6 columns. Her goals in working with multiplication included making the connection between addition, grouping, and multiplication and discovering how multiplication can be used to solve pairing problems. One issue that Maria had working against her during the field experience was that Ansley was absent three out of the eight sessions. This made it difficult to consistently refine her working model and make appropriate instructional decisions; however, I think that Maria did a more than adequate job in the field.

Megan

Letter

Unlike Emma, Megan claimed that mathematics had always been her absolute worst subject. She described that her lack of mathematical ability has made her dislike the subject, stating that she has always had to struggle to achieve Bs. She attributed her only positive experience in mathematics to a "nice" teacher who provided a great deal of individual help and encouragement.

Of the four participants, Megan had the most experience working with children before this semester. She, like Emma, volunteered to work with children before college. In her senior year of high school, she described her experience in a "Teacher Cadet" program in which she spent three hours a day in a first grade classroom. She worked with these children daily, both one on one and in small groups. She stated, " I made games, bulletin boards, and worksheets, etc. to help the students learn." She also worked at a summer camp with underprivileged children for four summers. She stated that this experience was what made her want to be a teacher. More recently, she spent 60 hours observing different elementary classrooms.

Megan's comments about what she hoped to learn during the course suggested that she saw teaching as explaining. She indicated that, during the class, she hoped to learn how to explain concepts in mathematics to children, in multiple ways in order to accommodate children with different learning styles. She realized that not all children would understand problems in the ways that feel most natural to her.

Mathematics Instrument

Megan described her simile for learning mathematics in two ways: how she used to view mathematics and how she currently thinks of mathematics. She indicated that she used to think that learning mathematics was like cooking with a recipe "because as long as you did all the steps in the correct order you would get the correct answer," but now believes that learning mathematics is like building a house because "you have to start from the foundation, but after that there are no set rules. Everything is up to the designer's interpretation." She felt that a mathematics teacher is like a coach, stating that "He [the teacher] teaches the play and then observes as the players carry it out. He watches as players create new plays and he encourages them to use their talent to the best of their ability." She believes that the purpose of school mathematics is to enable students to function in society, reason mathematically, do well on standardized tests, solve mathematics problems, function (or excel) in college mathematics courses, see mathematics as a connected whole, and see the beauty in mathematics. She stated, "it's a little bit of all of these."

While Megan stated that she gave equal time to both skills and concepts when she taught math to a pre-kindergarten class during her student teaching, she felt that overall, skills should typically be emphasized over concepts. She stated, "I felt like the [pre-k] students needed an even foundation to start from." She believed that if students are given the skills necessary, they can use those skills to solve problems. She viewed mathematics as a personal experience because "everyone experiences math in a different way, and the same math experiences mean different things to different people." She believed that, while mathematics may be incomplete because we "are learning new things every day about mathematics," people continue to use it to benefit society.

Initial Interview Assignment

Description of Megan's Model. Megan determined that her child was INS+, "I assessed that Grace was numerical because she was able to count on," and provided evidence suggesting reasons for thinking Grace was numerical, using phrases she learned in class such as counting on, reasoning strategically, and counting on from first. She indicated moments of intrigue while putting to use and testing the theory learned in class. For Megan, I believe this is where her newly found propositional knowledge (research and theory) met craft knowledge (teaching skill) in that she felt she was seeing the propositional knowledge in action She stated, "This next question was my favorite part of the interview because it gave me some insight into Grace's ability to reason strategically." Even though she noticed Grace reason strategically, she didn't assume that this meant that Grace was in the level of SAR. She indicated why she placed her child in one level and not the next level of sophistication, stating, "Another implication she is INS+ is that although she did reason strategically at least once in our interview, she didn't reason strategically as much as she could have—as much as a strategic additive reasoner would have." Megan was constructing a working model of Grace's mathematics that would enable her to create and or choose tasks that would be appropriate for Grace's level of reasoning. According to D'Ambrosio (2004) this is the purpose of constructing such a model of students' mathematics.

Megan indicated that she used her initial interview with Grace to determine a learning goal for her work during her field experience. She wrote, "I'm hoping she [Grace] will begin reasoning strategically."

Does She See This Work as Useful? Megan saw this interview process as a valuable part of planning her sessions in this field experience and, according to her writing on this assignment, she also saw the interview process as useful beyond this particular field experience. Megan wrote,

> I think interviewing has a lot of implications for instruction. Had I not interviewed Grace, I would not be able to pose the correct level of problems to reinforce what she has already learned, nor would I be able to pose the correct level of problems to challenge her.

There is an indication in the statement above that teachers generally need to do this work in order to pose an appropriate level of problems for their students. This suggests that she saw listening to children for planning as useful beyond her particular field experience.

Early Instructional Decisions.   One of my research questions is how the preservice teachers used their working model to make instructional decisions. Megan saw that the purpose of conducting this initial interview was to assess Grace's mathematics in order to plan appropriate instruction, choose and create tasks for Megan to solve, and make spontaneous instructional decisions. Making spontaneous instructional decisions includes, but is not limited to, abandoning problems that may be too challenging for the child, asking a question that may make a problem more challenging for a child, or changing the numbers in a problem to create the appropriate challenge. This may be termed working in the zone of potential construction (Steffe & Thompson, 2000) for the child. There is some evidence that Megan was making both spontaneous and long-term instructional decisions as she was conducting this interview. One example of an interaction where she decided it was best to abandon a problem rather than to pursue Grace's thinking in the area is described below.

> Q: What number comes after 13? 100? 375? 422? 600?
>
> Grace knew the number words in order backwards and forward until about 600. When I asked her what came after 600 she said "650." Then when I asked her what came after 601 (thinking that might help her) she said "300." It was all downhill after that. I tried to go back down to smaller numbers, but she was obviously confused or shaken up somehow because when I asked her what came after 100 (a question she answered easily before), she said "230." I aborted the question.

There is a small amount of evidence that would suggest Megan was constructing learning goals during the initial interview that were based on her working model of Grace. Megan wrote, "I'm hoping she will begin reasoning strategically." This suggests that Megan's future problems and tasks for Grace would be chosen with the concept of helping her learn to reason strategically in mind, but she didn't refer to any specific plans or strategies that she wanted to pursue with Grace.

Types of questions that elicit children's mathematical thinking. In order to construct a working model of a child's mathematics, one must become proficient at asking good probing questions. While the participants were experiencing their first times trying to understand a child's mathematical ways of thinking, they did ask probing questions. Megan included many of the questions she asked Grace in an attempt to build a model of her mathematics. After posing a question on the interview protocol, Megan observed Grace count on her fingers in order to arrive at an answer. Some novices may have been satisfied to say that the child counted on her fingers to get the answer, but Megan decided to probe further in order to determine how Grace thinks about mathematics:

> "After she arrived at the correct answer, I asked her how she got her answer. She explained that she counted up from 8. I asked her how many times she counted up, and she said, "7, because you told me that there were 7 counters to start with, and then I added 8 more." "Why did you start with 8?" I asked."

It was clear that Megan was making a significant effort to build a model of Grace's mathematics. She asked a series of questions on one problem to understand the operations that Grace was using to arrive at an answer. On another question, Megan described Grace's actions using language from class:

> Grace put 19 counters in the cup. She then took out 3 to give to me. After she put the counters down, she counted backwards on her fingers uttering "18, 17, 16" while putting up one finger per number word. She told me "16 counters are left in the cup."

Again, Megan was not satisfied just to recognize the actions Grace was taking. She asked her "why she started counting backwards at 18 instead of 19." Megan wrote up Grace's response as, "she said, 'You don't have to say 19 because you already have 19—you just need to take 3 off; that's why I counted backwards.'" She then described this response in terms of the literature and determined that Grace's actions on this problem displayed evidence of Grace's ability to count down. Other questions she consistently asked Grace throughout the interview were whether or not Grace could solve a problem in a different way, how Grace came to her answers, and how Grace knew that her method would work.

One interaction that I thought showed Megan's skill at asking probing questions to construct a working model of Grace's mathematics is described below:

Q: Marcus has some toy dinosaurs. He goes to the store and buys 3 more toy dinosaurs, and then he has 9 dinosaurs all together. How many dinosaurs did Marcus have to start with?

This one was a little bit harder for Grace. She used the counters. She begins by putting out 9 counters and then separating 3 with her hand. She then counts the remaining counters and gets the answer “6.” “How did you get that?” I asked-. Grace says, “Well, I knew he had 9 at the end, and he got 3 from the store so I moved those and counted the rest and that told me there were 6 to start with.” I asked her if there was another way she could have done the problem, and she said “I could have not used the counters.” “How could you have done it without using the counters?” “I could have started with 9 and counted back 3?” She said in a questionable tone. “Why don’t you try that?” I asked. So, she utters “8, 7, 6…6,” she says. “Did it work?” I asked. “Yup,” Grace says. After some more questioning, Grace goes on to explain that her counting back from 9 is the same as when she used the counters and “hid” 3 with her hand.

From the interaction that Megan described above, I believe that she was working hard to understand Grace’s mathematics by asking a series of questions for one problem in order to see what Grace was capable of and to see how Grace fit with the literature from class. Considering that this was Megan’s first time working with a child in this capacity, she showed surprising skill at asking probing questions and describing the child’s actions using the language from class in order to analyze the child’s responses by using the framing experiential model.

Field Experience

Megan spent practically every moment of her field experience trying to prompt Grace to use strategic additive reasoning. Specifically, she was trying to get Grace to use doubles. In an interview, Megan mentioned that the reason she began to prompt her to use doubles was that she noticed that Grace used that strategy once in the initial interview assessment. Megan developed card games designed to elicit strategic additive reasoning. One such example is a card game that uses Phase Ten cards and cards with an equal sign, subtraction symbols, and addition symbols. In this game, Megan placed an equal sign and a number card, such as 11. Grace was instructed to create a math sentence that equaled eleven. Initially, Megan's directions were very open but, as Megan discovered that Grace intended to use many plus ones to get the desired result, she changed the rules of the game so that she could elicit strategic additive reasoning, eventually telling Grace that she could only use the one card once. As the game progressed, it seemed that Grace's goals and Megan's goals were different; Grace wanted to make the sentence as long as possible, while Megan wanted Grace to use number sentences such as 3+3+1=7. Over and over she tried to prompt Grace to use doubles in her sentence, and to get what she wanted, Megan eventually changed the rules again. She told Grace to use exactly three cards. While Megan never quite got the types of responses she was trying to prompt, it was a good game to get Grace to partition numbers in various ways.

On another occasion, Megan used a game that she called Number Sentence Go Fish. In this game, the rules were similar to the rules of Go Fish, however instead of asking for a five, you had to ask for a 2+2+1 or any other way to partition 5. The response of the other person then had to be, "no, I do not have a 5". Megan also posed many problems that were clearly selected to elicit the use of doubles strategies, including join, comparing, and take away. Some problems were result unknown, change unknown, and start unknown. The numbers in the problems and the sequences of problem were deliberately chosen to prompt strategic additive reasoning. This will be more evident in the thematic discussion to follow.

Samantha

Letter

Samantha described herself as having a negative attitude toward mathematics. She attributed her negativity to choosing to take courses that were too difficult for her in high school and to “sub-par teachers and teachers that were absent from the classroom for extended periods of time, thus hindering the learning process.” In high school, she took gifted mathematics courses and then AP Statistics her senior year. She described her first mathematics course at UGA, Mathematics of Decision Making (Math 1060), as fairly easy, mentioning that she only enjoyed parts of it. She, like Maria, referred to the first course that she had to take as an elementary preservice teacher as being very beneficial. She suggested that, while challenging at first, it helped her better understand the reasoning behind the ways in which we solve mathematical problems. She admitted that she did not “comprehend the true reasoning as to why a certain method worked” in elementary school. She, like Emma, described simply being able to work the procedure in the same manner as the teacher and typically getting the right answer.

Like all other participants, she volunteered to work with children while she was in high school. She worked in a second grade classroom her senior year, which she described as enjoyable. She also taught Sunday school to four and five year old children for two years. More recently, she interned with a first grade classroom at a local elementary school for several months.

This particular semester, she described looking forward to working one-on-one with her School Buddy. She wanted to “see the learning process in mathematics first-hand.” She, like Emma, wanted to be able to incorporate what she learned in this particular course in her future classroom. By reflecting on her own mathematical learning, she hoped to help her students avoid some of the frustrations that she experienced with mathematics.

Mathematics Instrument

Although she offered no reason, Samantha indicated a belief that learning mathematics is like working a jigsaw puzzle, and that a mathematics teacher is like a coach. She believes the purpose of school mathematics should be to enable students to see mathematics as a connected whole and, when it comes to teaching mathematics, she believes that concepts should be emphasized over skills. She views mathematics as a changing body of knowledge that is constructed socially. Further she suggested that mathematics might be incomplete, dynamic, fallible, and not universally or objectively true. The reason she gave for her view of mathematics was that, "People are constantly providing their own interpretations of mathematical discoveries. Research continues (and will continue forever) in this field because it will always be ever-changing and incomplete."

Initial Interview Assignment

Description of Samantha's Model. Samantha determined very early in the interview process that the child she was working with was beyond the INS stage: "Immediately I was able to figure out that Isaac was indeed numerical because he was able to count on." After observing more, she noticed that he could do more than just count on. "With Isaac's answer to question eight, I was able to figure out that he knows to switch addends (so as to count on from a larger number in an addition problem.)" As she probed more into his thought process, she began to consider the possibility that Isaac may be a strategic additive reasoner; however she didn't determine this based on the first time she noticed him strategically reason, feeling that she would need more evidence: "When I asked Isaac about how he subtracted the 7 from the 12, he said that he subtracted 2 from 12 and then 5 from that…This answer indicated to me that perhaps Isaac was a strategic reasoner, but I would still have to dig deeper."

When she completed her interview and reflected on what she just observed, she determined where Isaac fit into the experiential model that they had learned in class: "Based on my interview with Isaac and what we have learned so far in our class discussions and reading, I would probably classify him at the INS-plus level." She also decided that while he was at the INS-Plus level, he was on the edge of being in the next level of sophistication. She concluded, "I think he is already well on his way to the strategic reasoner level." From the feedback on this assignment, it seemed that the professor agreed with her assessment. Samantha described her understanding of a student in the INS-plus level as, "A child at the INS-plus level can count on" and further provided evidence that Isaac fit that description. She also noted the efficiency with which children in this stage of development work.

> INS-plus children can also switch addends when counting on so they are counting on from larger numbers. … Another characteristic of INS-plus children is that they are able to count up to and count down. Over and over, Isaac used the strategy of counting up (problems 7&8). On problems 3 and 4, I was able to hear him explain how he counted back when counting down. INS-plus children have two number sequences to use, which aids them in these beginning multiplication activities. Isaac was very capable of keeping track of the stacks of 4-rods and 3-rods. INS-plus children struggle with strategic reasoning. Isaac did not strategically reason often. Isaac did strategically reason in problems 10 and 13, but I do not believe reasoning strategically is something that is completely natural for Isaac yet. One more characteristic to note about INS-plus children is that place value is not usable in their thinking. Isaac could track the number of tens

> and ones, but he could not *use* this knowledge to solve problems 24 through 26, which dealt with addition and subtraction problems that were two digits. I would say with the way Isaac is beginning to strategically reason like he did in problems 10 and 13, he is definitely on his way to advancing to the "strategic reasoner" level soon.

How useful is this practice? There was an indication that Samantha believed that interviewing Isaac in order to formulate a model of his thinking in mathematics was useful for her, but her writing doesn't go beyond usefulness for her in this particular field experience. For instance, there is no indication that she believed that this would be a beneficial practice for teachers in their classrooms in general; however, this does not mean that she did not believe this. Samantha wrote, "Now that I have this basic sketch of where Isaac is thinking mathematically, I hope to be able to plan and create meaningful sessions where I can assist him in discovering new and meaningful ways of thinking about mathematics."

Early Instructional Decisions. Samantha used her preliminary model of Isaac to plan her instruction for him during the field experience, based on her idea that he was close to becoming a strategic additive reasoner. She indicated that:

> In future sessions, I hope to work with Isaac more on place value, so it can become usable to him in solving problems. I also hope to work with some more addition and subtraction problems to see if we can formulate some more strategies in solving problems, like reaching a benchmark number, making doubles, etc. so Isaac would solve these problems more like a strategic reasoner.

She also wanted to help him form some multiplication and division strategies, which could indicate that she thought he could even become a strategic multiplicative reasoner, a stage beyond SAR.

Some of her instructional decisions were not based on her model of Isaac's mathematics; rather they were about exploring his thoughts on a particular concept in order to more precisely formulate her model of Isaac: "I would like to explore multiplication and division (*especially*, division) more with Isaac." She discussed running out of time before really getting into the fraction problems; however, she did find out that he could partition a candy bar to share with three people, although he could not decide what to call each piece. She wanted to find out more about Isaac's thinking in fractions, stating:

> Next time we meet, I hope to delve back into this topic [referring to fractions] so I can better understand Isaac's thinking in this area. As I said earlier, I would really like to go back and explore his knowledge of fractions

Other instructional decisions were subtler; for instance, she determined that one particular problem would be easy for Isaac based on his thinking on previous problems. She indicated that she knew before he began working on the place value problems, where she was adding and taking away ten rods, that he would not struggle with tracking the number of cubes and the number of rods. Samantha stated, " I had figured this problem would not be difficult for him because he had figured out the 4-rod problems so quickly."

Types of questions that elicit children's mathematical thinking. Samantha was rarely satisfied with just an answer, also wanting to know how Isaac arrived at his answers. To find out how he was thinking so that she could construct a more accurate model, she often asked probing questions. For instance, she wrote "When he came up with the answer '15' I asked him how he solved it and he said by thinking in his head '9, 10, 11, etc.'" She described Isaac's counting by threes in the following way, "The 3's sequence was fluent at first, but once Isaac got past 15, he started pausing for a few seconds." While she recognized that he could count by threes, however slowly, she decided to ask him how he was thinking about this sequence: “When I asked him how he was figuring out what came next, he said at a certain point that he was just thinking of it by adding on 1+1+1.”

In her description about how Isaac solved a missing subtrahend problem, she wrote, “In problem 10, Isaac knew that he would subtract 7 from 12 to figure out how many gummy bears Susan had given to her friend”. However, she was not satisfied with knowing that he subtracted to solve the problem, so she probed further: “When I asked Isaac about how he subtracted the 7 from the 12, he said that he subtracted 2 from 12 and then 5 from that.” This process helped her formulate her model for Isaac. “This answer indicated to me that perhaps Isaac was a strategic reasoner, but I would still have to dig deeper”. Even though she noticed that instead of using counting strategies to solve this problem, he used strategic reasoning, she did not determine right away that he was in the SAR level, indicating that she took this model building very seriously. She knew more would have to be determined in order to place Isaac within her interpretation of the framework.

In this initial interview assignment, Samantha was not always explicit about the questions she asked Isaac to elicit his mathematical schema. From this description, it seems clear that she asked Isaac how he arrived at an answer of five for a particular division problem, but the actual question is never stated:

> When I asked him about the division problem in question 18, it took Isaac a moment, but then he answered “5.” From what I understand, Isaac said they had done a class project that involved buying animals at a certain price each. He had said that he figured out how to do this problem because he had remembered that to buy 4 dogs, it was $20. Isaac had also remembered that each dog was $5. He said that was how he figured out they needed

> 5 cars. In retrospect, I feel like I should have probed him more on this problem or continued on with the harder problem involving the 42 children.

I believe that the statement above is an indication of Samantha's desire to get better at asking probing questions in order to construct her model of Isaac's mathematics. She was also reflecting about her experience in order to think about her instruction for future sessions.

The following exchange, described by Samantha, offers a look into her questioning techniques:

> Coming to an answer on this problem took Isaac longer than he had spent on any other problem so far. He asked me in the middle of his thinking, "Each person gets less than 10, right?" I said yes, and I asked him how he knew that. (He did not answer me until I asked him again after he had finished solving the problem.) Isaac said that he had started by giving each person in his head 6 M&Ms. He said he added 6+6+6 up in his head, but he did not get 18, so he thought his answer was wrong. When he continued and tried giving each child 7 and added 7+7+7, he got more than 18. He said he then went back and tried 6 again and it worked this time. When I questioned him again about how he knew that each person would get less than 10, he figured out that if each person got more than 10, then we would need 20 for 2 people and even more for 3.

Through her questioning techniques, Samantha was able to obtain a lot of information about the way Isaac was thinking mathematically that she would not have known otherwise, however, she was not always successful at figuring out what he was doing. There are a few examples from the interview assignment that point to Samantha trying to make sense of Isaac's ways of operating, however unsuccessful. Samantha wrote, "I tried a couple of times to figure out what he was thinking, but for the sake of time and the continued confusion, I decided to move on." This statement also points to an early instructional decision of abandoning a problem in order to keep him motivated and to deal with a perceived lack of time. On another problem, she described not being able to figure out how Isaac was thinking about the solution and again makes the decision to abandon and move on:

> Based on my observations, he appeared to be genuinely perplexed by this problem and was rambling various numbers. I was unable to follow his train of thought as to how he was formulating his solution. I decided to move on again to the rock collection problem. At first, Isaac was trying to reason strategically. He started off by saying, "I rounded 70 and 26. 70 rounds to 70, and 26 rounds to 30." Isaac subtracted and got 40, but then always added 20 and then 4 more. I am not sure why Isaac thought he had to add the 20 onto 40. Perhaps he wanted to add the 4 because he had added 4 earlier to round the 26 to 30, so he felt like he had to add 4 again.

The last example was more about her determining if Isaac knew the typical language for fractions. She saw that he could partition a candy bar to share with two people, but probed as to whether he could name each piece. Isaac was successful in dividing the candy bar between two people but, "When [she] asked him what we call each piece he answered, '[Uttered] Oooo…hmm.'" Again, because of time, she abandoned this one, but suggested she wanted to further explore his knowledge of fractions.

Indications of Surprise. Samantha indicated having many instances of surprise about the ways in which Isaac was operating mathematically. These surprises possibly indicate a conflict with her current working model and what she was observing, which could cause her to make adjustments in her model. Her first statement suggested to me that Isaac might be operating in a way that was right above what she thought might be possible for a third grader to do in mathematics. She stated, “Right away I was very impressed with the way that Isaac was thinking about counting and number.” These anomalies may serve as disconfirming evidence that lead to an accommodation in her model of children’s mathematics. Another indication of being impressed is where she writes, “When moving on into the multiplication and division word problems, I was also very impressed with Isaac’s mathematical thinking,” for instance, “I was really intrigued by the way he solved the M&Ms problem.”

An action she described as interesting concerns Isaac’s visual way of operating as he verified a solution to a problem. She wrote, “One thing I found interesting is that he said he checked his answer by looking at a wall made up of glass squares. He said he counted 6 squares and then added 7 more squares and got 13 squares.” In this particular description of Isaac’s strategy, she was noting the ways Isaac could and did operate when confronted with join, result unknown problems. The next indication of surprise suggested that she may have been observing limitations regarding teaching the standard algorithm as the way to operate with multi-digit addition and subtraction:

> When continuing on to question 11, Isaac’s explanation of how he solved the problem surprised me. He said that he “just knew” that 13-5 was 8. When I asked him how he knew that, I found it surprising that he was trying to apply the standard algorithm to this problem. Although this did not make a lot of sense to me, I found it interesting that he felt like he had to use the standard algorithm on a problem where it is of no use.

Early on, she observed Isaac using counting strategies that would suggest he was operating at the INS or INS+ level of whole number development. Thus, when he showed evidence of strategic reasoning, she noted being surprised. This caused her to think maybe he could be a SAR child, but she hesitated to make that assumption and noted that more evidence would be required than using it on one problem.

> He used some strategic reasoning on problem 13. He said that he thought of the problem as 10-3=7. Isaac knew that 9 is one less, so the answer is 6. I was surprised he was able to reason this so easily.

She was surprised at how Isaac was determining how many tens came between two off-decade numbers. She wrote,

> To figure out how many tens between 22 and 42 he used the same strategy. "22 to 30 is one ten and 30 to 40 is another ten. I know it is not 3 tens because I am not going all the way up to 50. I am stopping at 42." I found it surprising that he was considering 35 to 40 as "one ten."

All of these indications of surprise served as opportunities for her to refine or change her model of Isaac's mathematics. These moments of interest or intrigue were of particular significance as I considered her model and how it was evolving through working with Isaac.

Field Experience

As stated in Samantha's initial interview assignment, her goal was to get Isaac to be a strategic additive reasoner. Therefore, like Megan, she spent the majority of the field experience working to get Isaac to strategically reason. She used games, word problems, and tracking stacks and cubes in order to reach her objective; however she did not create the games that she used. She used one of Megan's games involving using number cards to partition a given sum. The other game she used was one described in the literature from class called "Going Bananas" (Kamii & Housman, 2000). In this game, two dice are rolled, one black and one white. The black die had numbers 1, 2, 2, 3, 3, and 4 on each of the faces, while the faces of the white die were labeled with numbers 4, 5, 6, 7, 8, 9. Her buddy was to roll both dice, add them together, and place cubes on a hundreds grid that equaled the sum, until he filled up the grid. Along the way, she would ask questions such as how many more cubes do they needed to have 10, 20, 50, and 100. She also asked Isaac how many cubes they would need to subtract, or how many bananas they would need to eat, to have ten bananas. She would probe Isaac along the way by asking him how he came to a particular sum. On an activity report, Samantha stated that her goal in using this game was to "get him to manipulate the numbers to help him feel more comfortable with using these strategies." By "these strategies", she was referring to using doubles, making benchmark numbers such as ten and one hundred, and counting off decade by tens.

Unlike Megan, Samantha was working more with multiplication and division and had Isaac tracking hidden stacks and cubes on a regular basis. The sizes of the stacks were three, four, six, and ten. She also had Isaac working with arrays in order to help him learn his multiplication facts. Her goal was to work with rows, columns, position, and total dots. In order to continue working with strategic additive reasoning, she often had him track stacks to count by elevens, twelves, twenties, and even thirties, sometimes starting at zero, sometimes not. With these stacks, she had Isaac count both forwards and backwards, taking stacks away. Following these counting sessions, she would transition into word problems involving adding or subtracting with an addend or subtrahend of eleven, twelve, twenty, twenty-one, or thirty to see if he could transfer his counting into counting strategies for solving problems.

CHAPTER 5

FINDINGS

D'Ambrosio (2004) suggests that what makes teaching from a constructivist epistemology unique is that teachers work to build a model of each of their students' mathematics. My research set out to describe working models that preservice teachers construct of children's mathematics and to understand how they use their models to make instructional decisions. Through my investigation I was able to determine that my participants were able to construct and use a working model of their child's mathematics. I also discuss other findings that came out of my research, including the preservice teachers' evolving definition of teaching and their learning of mathematics.

Participants' Working Models of Their Children's Mathematics

In determining the working models that the preservice teachers constructed of children's mathematics, first the implicit question should be whether or not they did construct a working model of their child's mathematics. I believe that the preservice teachers were engaged in making a working model of their child's mathematics and that their working models encompassed three important elements: description, analysis, and application. In class they were exposed to this language of making models and were instructed to do so in their initial interview assignment. Therefore, it may not come as a total surprise that, in every written description, the preservice teachers described the mathematical actions of their buddies and determined their buddies' levels of whole number development based on the literature they had read in class. They provided evidence for such placement and received confirming responses from the instructor, Dr. H., on their initial assessments.

In class, the participants were presented with an article that discussed an experiential model to explain children's whole number development which they then used to practice analyzing the mathematics of children in videos and written excerpts. This experiential model provided each participant with a framework with which to construct her own working model of her child's mathematics in the field experience. I believe that the participants' working models of the children's mathematics, influenced by the framework from class, informed their knowledge of children's mathematics from a more global perspective. In the final portfolio Maria stated, "Most importantly, however, I learned how to think like a child which will be incredibly helpful when I am attempting to design activities for my own classroom in the future."

Participants' Description

Initially I made a distinction between the participants' descriptions of the mathematical actions of the children and their constructions of working models of the children's mathematics. I believed that description was a necessary component for model building, but it alone is was not the working model. Rather, it was how the preservice teachers analyzed their descriptions that qualified as construction of a working model. Since describing is an essential first step in the process of constructing a working model, I eventually decided to merge description under the working model category as a subcategory rather than keeping it a separate category. I therefore decided to report the participants' descriptions and analyses as their working models of the children's mathematics.

Using Language from Class

The preservice teachers had been immersed in the language of whole number schema such as counting on, counting off, disembedding, unitizing, etc. They were using this language fluently to describe the mathematical actions in which the children engaged. All four of my participants were able to use the language from class to discuss the ways in which their buddies were operating mathematically. One typical example of this type of description is from Samantha's final portfolio. She stated, "He had the ability to count-up and down-to. He was switching addends, which was another indication he was at least at the INS+ level." She used her description in order to analyze Isaac's mathematical actions with respect to the experiential model from class.

> … place value was not quite "usable" in his thinking. He was very capable of tracking tens and ones, but when applied to addition and subtraction problems, he could not use his knowledge about place value to solve. When explaining how he solved addition and subtraction problems, he often used the standard algorithm for addition (or subtraction) as the method of solving.

Through her observation, she deduced that Isaac was not quite at the SAR level and thus was at the level of INS Plus.

Using the language from class in their descriptions was a necessary first step in the teachers' constructions of working models of their children's mathematics with respect to the framework from class. Subsequently, they used their descriptions of their children's mathematics in order to provide evidence for their analyses.

Using Their Own Language

Although they were comfortably using the language from class, they did not always use that language to describe what their children were doing. However, their other ways were equally as valid as descriptions of the children's mathematics. For instance, on the final portfolio, Megan used her own language to describe Grace's mathematical actions with respect to their work with multiplication. Megan also described the questions she used in order to change Grace's mathematical actions to her desired actions. I called this line of questioning prompting. Megan stated,

> While Grace and I worked very little on multiplication, she certainly learned a great deal about multiplication in the few lessons we had. The first time I introduced multiplication to Grace I posed this problem: "I have 3 pieces of candy in one hand and 3 pieces in the other hand. How many pieces do I have in all?" Grace used the counters and made 2 groups of 3. She concluded there was 6 pieces of candy in all. Then I asked her to make a multiplication problem, and she wrote 3x3= 9. "But you just told me there were only 6 pieces of candy," I said. "Well, now it's multiplication, so you need more candy." No matter what problem I posed or how I worded it, Grace would always get the correct addition sentence and write a different, incorrect multiplication sentence.

In the quote above, Megan is highlighting one of the concepts with which Grace struggled during the field experience. This was a source of frustration for Megan. Each time Megan reminded Grace that her multiplication statement was resulting in a larger value than her solution to the problem, Grace continued to be unperturbed because to her, "multiplication makes bigger." However, Megan was eventually able to settle this by repeatedly asking Grace what the first number in a multiplication sentence represented and what the second number represented in the context of a story problem. Megan began to see a pattern in the way that Grace understood multiplication contextually. For Grace, the first number represented the number of objects each group and the second number represented the number of groups. For instance, for the symbolic expression, 4x3, Grace would continuously refer to this as 4 three times, which for Grace seemed to suggest that she had four objects in three groups. In the quote below, Megan is describing Grace's mathematical actions on a later multiplication problem, after she thought Grace had resolved the multiplication-always-makes-bigger-than-addition issue. On the final portfolio Megan wrote,

> Every so often Grace would slip back into her previous ways of thinking about multiplication... The problem …is, "If there are 3 birds sitting on a wire, how many feet are on the wire? Can you write a multiplication sentence?" At first, Grace wrote "3x3=9." I pointed to the first 3 and said, "What does this number represent?" She said, "There are 3 birds." Okay, well what about this 3?" "The number of feet---wait! Birds don't have 3 feet!" She then proceeded to write the correct multiplication sentence, 3x2=6, and I asked her to draw a picture of the 3 birds sitting on a wire.

Through her questioning, Megan was able to create the kind of perturbation that would lead to a natural change in Grace's schema involving multiplication and she was able to document her description of Grace's actions on these problems in a way that would lead to the type of instruction to make an accommodation in Grace's current multiplication schema possible.

Through describing their children's mathematical actions in their own words, the preservice teachers were able to develop appropriate problems and questioning techniques. These instructional decisions lead to each participant's buddy's construction of productive mathematical schema, such as the kinds of thought Megan described about Grace.

Participants' Analysis

Each teacher was able to describe what she saw her buddy do mathematically, but, she was also able to go deeper than just description, taking that description and assessing the level at which her buddy was operating, according to the framework from class, and then making inferences about the types of problems her child could solve and the approaches the child might take to solve the problems. It is this type of assessing and inferring that I chose to call analysis.

An interesting similarity among the working models of all the participants is that each participant felt that her buddy was very close to the next level of whole number development. Samantha indicated on many occasions, from the very first interview assignment to at least the fourth session in the field, she felt like Isaac was toward the end of being INS Plus to being at the beginning of the SAR stage. In the interview immediately following the session on October 26, she indicated that she felt like he was even closer to SAR. She determined this because on the "break through" day, Isaac used one of the additive strategies without her prompting him to do so. She stated,

> I think he is very close to strategic. He might be very much on the cuff of it. Last week was kind of a really big break through. Because like before I would probe him about using doubles and making ten and he would never do it. And then last week all of the sudden he just started doing it. So um then my questions and concerns are… I had mentioned that I was concerned whether I should just move on to something else like multiplication or keep working with that. And she [Dr. H.] said that I should just keep working on it so it gets solidified in his head, and just work with higher numbers. So that's what I decided to do. I don't think he's completely there, but I think he's pretty much getting to that SAR level.

Similarly Megan felt that Grace was on the verge of being a Strategic Additive Reasoner. In her final portfolio she wrote,

> Although Grace improved tremendously, I am not convinced that she was a strategic additive reasoner by the end of our last meeting. Grace was capable of reasoning strategically for most problems. However, her strategic reasoning did not always come from her own logical necessity as much as it came from me probing [prompting] her to

> "do the problem a different way." I think Grace is on the verge of being a strategic additive reasoner, and I can see how much she has grown mathematically from our first lesson together, so I feel confident that she will be at the SAR level soon.

Emma also thought that Matthew was on the border between two levels of whole number development. However, unlike the rest of the participants, she felt that her buddy was operating at a higher level than INS+, indicating her opinion that he was very close to being at the end of SAR or the beginning of SMR. In the interview immediately following the session on September 2 she stated that

> Although he likes to strategically reason all the time, I think we're getting into the multiplicative reasoning because he's doubling and he's quadrupling and he's tracking things so well when he does stuff like that I am starting to suspect that maybe he's starting to get up there, but I don't think that he really realizes what he's doing. He's just trying to figure out the best way to solve his problems.

In the same interview she stated:

> I think he's on the brink of moving beyond strategic additive reasoning. If he you know… He still doesn't remember certain things. Like he doesn't have his multiplication facts memorized, which kind of slows that process down I think a little bit. But he's right there on the brink. He's somewhere in the gray area I think of figuring all that out.

Emma thought that the only thing standing between Matthew and multiplicative reasoning was his memorizing of the multiplication tables. She seemed to believe that, if he could only achieve that, he would be on his way. Even though she described Matthew using a particular strategy on a regular basis, which could be interpreted as multiplicative, she was not convinced that he was at the SMR level of whole number development. She stated, "He has his tens [multiplication facts memorized]. I always think it's neat. Sometimes he pretends nine is ten and then subtracts. …But since he doesn't have them [multiplication facts] memorized, that's sort of slowing his process down and he's only a third grader." In class, it was discussed that the aspect, which essentially defines strategic multiplicative reasoning, was the use of the distributive property. The strategy that she described Matthew using was for multiplication problems such as 8x9. She discussed that for this problem, he would solve 10x8 and then subtract 8. Actually, this is a demonstration of the distributive property, which Emma did not seem to recognize. For instance, if we use a series of equations to follow Matthew's train of thought, we could write,

$$8 \times 9 = 8 \times (10-1) = 8 \times 10 - 8 \times 1 = 80 - 8 = 72.$$

Essentially, Matthew was proficiently using the distributive property in this strategy by distributing the eight onto the 10 and onto the one. It could be that she did not feel like he was using strategic multiplicative reasoning on different types of multiplication problems since he only used it with nines multiplication tables because nine was so close to ten.

After the initial interview, Maria, like the Samantha and Megan, thought that her buddy was not quite at the SAR level and was at the end of INS Plus. On the final portfolio Maria wrote,

> Well when we first started, I felt like she was beyond INS plus, but not quite a strategic additive reasoner. Because like for strategic additive reasoning, that's when you can keep track of tens and like use place value in the activities that I was just talking about [that involved computation with two digit numbers] and she wasn't able to do that at first.

Notice that, in the quote above, Maria is supplying evidence for her analysis of Ansley in her working model. She remembered from class that students operating at the SAR level can use place value in computation and noticed that Ansley could not do this, thus placing her in the level just below SAR. However, unlike the other three participants, Maria felt as though Ansley had progressed to the next level during their time together. She felt like this progression was a result of their interactions, even though there were only five sessions due to Ansley's absences. In the final interview Maria stated,

> …so we worked on that [strategic additive reasoning] and I really think that she got it at the end. I mean there might still be some things that she needs to work on that she's not particularly that strong in, but I mean she improved a lot over the time that we worked together. So I guess at the end, probably, I mean like I would say she was at the beginning of the strategic additive reasoning stage because I wouldn't have to prompt her [to use strategic additive reasoning] so much like I did at the beginning.

Simon (1997) acknowledges that teachers' knowledge of student learning pertaining to particular content is crucial to their overall knowledge and should be used to inform their instructional decisions as part of their hypothetical learning trajectory. The PSTs seemed to be well-versed in the language from class, allowing them to analyze their buddy's mathematics with respect to the experiential model that guided their thinking about children's development of whole numbers. Each participant was clearly using her construction of the experiential model to directly inform her model of her child's mathematics which indicates that, for my study, Simon's expanded model of MTC should be adapted to include an arrow from the teacher's knowledge of student learning of particular content to the teacher's model of her student's mathematics. While it could be speculated that this should be a bidirectional arrow, I have no evidence that the PSTs' understanding of this experiential model was directly informed by their interaction with the child.

Instructional Decisions

Simon (1997) suggested that a teacher's initial construction of a hypothetical learning trajectory happens before instruction in the classroom during the process in which the teacher develops a plan for what will happen in her class. "However, as the teacher interacts with the students, the teacher and students collectively constitute an experience" (Simon, 1997, p. 78). Consequently, the experience would not be the same as the one that the teacher planned. The interaction with students leads to a modification in her initial ideas and knowledge as she makes sense of what is happening and what has happened in the classroom. In my research, each preservice teacher was working with a child rather than a classroom of children; however, the essence of Simon's idea of a learning trajectory was maintained except that the plan for what would happen during the interactions occurred after the first meeting with the child. Thus, rather than beginning with a plan for the initial goals, the cycle of experience began with an initial interview with a buddy, which served as the basis of each preservice teacher's goals for her student's learning.

As discussed in the review of literature, Simon's Hypothetical Learning Trajectory has three main components: the goals for student learning, the plan for student learning activities, and hypotheses of the student learning process. The PSTs' goals and activities for student learning are what I have classified as the applications of their working models of each of their children's mathematics whereas their hypotheses of the student learning process is a part of their working models that I described above.

According to Simon (1997), teachers' models of students' knowledge and teachers' knowledge of student learning regarding particular content are two essential elements of teachers' knowledge that should inform their hypothetical learning trajectory. My research question concerns how a preservice teacher's working model of her child's mathematics informs her instructional decisions. We know that the working model of each child's mathematics was informed by the experiential model of whole number development from class, which the preservice teachers used as a guide for the field experience and acts as, what I assume to be, the basis of their knowledge of student learning of particular content, whole numbers and operations. Unless a preservice teacher could work with her model, it would not be appropriate to classify her model as a working model. By working with the model I mean planning appropriate tasks, goals and other instructional decisions based on her model of the child's mathematics. I consider the instructional decisions based on her model of the child's mathematics to be the application part of her working model. Thus, the question is could the participants use their models to inform their instructional decisions? If they did, how did their models inform their instructional decisions?

Preservice Teachers' Initial Goals for Student Learning

It may seem rather early to have a working model of someone's mathematics based on one interaction. Perhaps it seems even less likely that a novice would be able to use their model to determine types of problems that would be in a child's zone of potential construction. However each participant, to varying degrees, determined some direction for work with their child based on the first interaction and constructed a working model of their child during the initial interview.

Initial Goals Based on their Working Model

Each participant wrote about instructional decisions on their initial interview assignment to some extent. Many of these decisions were spontaneously made in order to accommodate her buddy's level of operating mathematically. Making spontaneous instructional decisions include, but are not limited to, abandoning problems that may be too challenging for the child, asking a question that may make a problem more challenging for a child, or changing the numbers in a problem to create the appropriate challenge (Steffe, 1991). This may be termed working in the zone of potential construction (Steffe & Thompson, 2000) for the child.

Some of the participants also made instructional decisions, based on their working models, in order to plan instruction for future sessions while other participants' decisions indicated that they wanted the opportunity to explore how their buddies thought about a particular topic. These decisions were often based on some instance that they found intriguing but lacked the time to explore during the initial interviews.

Early Spontaneous Decisions Based On Model

For Megan, Emma, and Maria most of their early instructional decisions were spontaneous. Emma and Maria skipped problems that they thought were too difficult for their buddies or abandoned problems if they thought their buddies were on the brink of giving up. Emma's spontaneous decisions were often made by thinking that the problems were not difficult enough for Matthew. For instance, Emma skipped problems that she felt were below Matthew's level of operating. She also made decisions to adapt problems in the interview protocol to make them more challenging for Matthew. On one problem, Emma refers to a mistake that Matthew made, which she decided not to correct.

Samantha only has one indication of an instructional decision that was made spontaneously, however, there was no indication that she used this spontaneous decision as Megan, Emma, and Maria had—to skip or modify a problem. She determined that one particular problem would be easy for Isaac based on his thinking on previous problems. She indicated that she knew before he began working on the place value problems, where she was adding and taking away ten rods, that he would not struggle with tracking the number of cubes and the number of rods. Samantha stated, "I had figured this problem would not be difficult for him because he had figured out the 4-rod problems so quickly." Her statement suggested that she was making instructional decisions based on her working model of Isaac's mathematics because she seemed to be anticipating Isaac's success on a particular type of problem based on her limited experiences with him up to this point.

Early Planning for Future Sessions Based on Model

Many of Megan, Maria, and Samantha's early instructional decisions on the initial interview assignment were based on getting their buddies to the next level of sophistication, which for all of them was the SAR level. However, only Samantha made very specific plans about the types of strategies she wanted Isaac to develop. There is no indication on the initial interview assignment that Emma made instructional decisions for future sessions based on her working model of Matthew's mathematics.

Only one of Maria's decisions was aimed at their future work together. Her goal for future sessions was to get Ansley to become a SAR, noting that, since Ansley was not able to strategically solve the problems she posed, this would provide an area for their future work together. Maria wrote, "…so I made a note that this would be something we could work on together and continued."

There is little evidence that Megan was constructing long term learning goals based on her working model of Grace, but she did indicate that she wanted Grace to begin reasoning strategically. She wrote, "I'm hoping she will begin reasoning strategically." Her statement suggests that future problems and tasks for Grace will be chosen to help her learn to reason strategically, but she did not refer to any specific plans.

Unlike Megan and Maria, Samantha wrote about specific strategies that she wanted Isaac to be able to use in solving problems, clearly using her preliminary working model of Isaac to plan her instruction for him for future field sessions. These decisions were based on her idea that he was close to becoming a SAR. She stated,

> In future sessions, I hope to work with Isaac more on place value, so it can become usable to him in solving problems. I also hope to work with some more addition and subtraction problems to see if we can formulate some more strategies in solving problems, like reaching a benchmark number, making doubles, etc. so Isaac would solve these problems more like a strategic reasoner.

She also wanted to help him form some multiplication and division strategies which could indicate that she thought he could even become a strategic multiplicative reasoner, the stage beyond SAR. She stated, “I think together we could formulate some strategies about multiplication and division.”

By the end of the field experience, Megan could point to some initial goals on which she based her sessions with Grace. On the final portfolio, Megan stated,

> During the Interview, Grace answered most of the questions by counting on or down with her fingers. Again, since I saw part of myself in Grace as a math student, it was important to me that she begin reasoning strategically to answer addition and subtraction problems instead of counting on or down on her fingers. My initial goal was to have Grace become a strategic additive reasoner.

It will become clear as I discuss Megan’s planning of learning activities that she did have this goal in mind from her initial interview with Grace.

Early Planning for Future Sessions Based on Exploring

Not all plans for future sessions were based on the participants' working models of their child's mathematics. Samantha and Emma were the only two who made plans for future sessions based on wanting to explore their buddies' thinking in some mathematical area. For Emma, one instructional decision was based on a perceived mistake that she observed Matthew make and she noted that she wanted to see if it was a fluke or if it was, in fact, a "flaw in Matthew's thinking" for this particular type of problem. Emma implied that she also wanted to explore the concept of division with Matthew and, similarly, Samantha expressed wanting to explore Isaac's thinking in particular topics such as multiplication, division, and fractions.

Some of Samantha's instructional decisions were not necessarily based on her model of Isaac's mathematics, rather they were about exploring his thoughts on a particular concept in order to more precisely formulate her model of Isaac. She wrote, "I would like to explore multiplication and division (*especially*, division) more with Isaac." She discussed running out of time before really getting into the fraction problems; however, she did find out that he could partition a candy bar to share with three people even though he could not remember what to call each piece. She wanted to find out more about Isaac's thinking in fractions. She wrote,

> Next time we meet, I hope to delve back into this topic [referring to fractions] so I can better understand Isaac's thinking in this area…As I said earlier, I would really like to go back and explore his knowledge of fractions.

Emma indicated regret in not posing a problem that involved partitive division. This is important to note because there was an indication that she wanted to understand how Matthew was thinking about certain topics, such as division. She stated, "I did not pose the M&M's

problem, and in hindsight I wish I had to see where he stood with division." Another indication that Emma was making an instructional decision to explore Matthew's mathematical thinking was indicated by the following discussion where Emma wrote about discovering a potential error in her buddy's reasoning. She interpreted his mistake by saying, "I think he got so wrapped up in the idea of adding that he confused his own thinking without realizing it." However, she was not satisfied in her speculation of why he may have made a mistake and indicated a desire to further explore how Matthew thought about that particular type of problem, a missing addend task that involved regrouping. With respect to Matthew's mistake, she wrote,

> In the future, I would like to give Matthew some more problems like this one to see if it is something about the wording that led to his mistake because his reasoning for the problem that followed, involving the rock collection, was correct.

The participants' instructional decisions featured on their initial interview assignment were limited to skipping problems on the protocol and making major instructional goals for future sessions. Only Samantha indicated wanting to work toward specific strategies with her buddy during future sessions, but even she did not refer to specific problems or types of problems that she wanted to pose in future sessions. However, in later sessions, they did have to plan the activities, many of which were based on their working models of their children's mathematics.

Preservice Teachers' Plans for Student Learning Activities

After each preservice teacher developed learning goals for her buddy, informed by her working model of her buddy's mathematics, each created or chose activities she felt were appropriate for meeting those goals. Megan developed many games in response to her goals for Grace's learning. One such game involved the use of Phase Ten cards and cards with pluses, minuses, and an equal sign. The Phase Ten cards included four of each of the numbers from one to twelve. In this game, Megan intended for Grace to use her knowledge of doubles in order to write number sentences. Megan placed an equal sign and a sum, such as seven, on the table and instructed Grace to use three or more cards to make the math sentence for a given sum. A warm up activity for this game was for Grace to label all of the doubles on a chart with all of the one digit sums from 1+0=1 to 9+9=18 in columns. The doubles appeared on a diagonal in this chart. The purpose of this beginning activity was to orient Grace to focus on the doubles so that she would know that they were available for use during the game.

As Megan's hypothetical learning trajectory was enacted during the session, the interaction lead to a modification in her plan; Megan and Grace "collectively constituted an experience" (p. 78), an occurrence in agreement with Simon (1997). Megan's rules for the game were adapted as Grace's mathematics became more evident. For instance, as Megan noticed that Grace's goal was to make her sentence as long as possible, often using 1+1+1…, she implemented a rule that Grace could only use a one card once in her number sentence. There were other such adaptations of the rules of the game as the session continued.

These instructional decisions were significant to my research questions because they provided evidence that Megan was using her working model of Grace's mathematics in order to design the learning activity. On the initial interview assignment, Megan determined that Grace was at the INS Plus level of whole number development. She indicated that her goal for her sessions in the field would be to push Grace to the strategic additive reasoning level, thus, her specific purpose in constructing this game was to get Grace to use doubles to reasoning strategically. Many of her learning activities and the learning activities of Maria and Samantha were designed or chosen with this purpose in mind. In fact, Samantha used Megan's game in her work with Isaac. Similarly, Emma's learning activities were chosen in order to get Matthew from the SAR stage to the SMR stage.

Besides their goals and learning activities, other instructional decisions included asking questions, some of which came out of their work spontaneously while others were built into their planning. These questions were both based on their working model and used to inform and refine their working model. Since these questions were such an important part of their work, I decided to talk about separately.

Questions: Probing, Prodding, Prompting

Teaching for understanding requires teachers to learn to ask questions that will allow them to understand their students' mathematics (Malave, Howlett, & Collins, 1993). I found that the preservice teachers asked their buddies many questions during the field experience and that the questions they asked could be categorized as three different types: probing, prodding, and prompting. Probing involves questions to figure out what their child was thinking, while prodding questions intended to spur the child into mathematical action, and prompting questions attempted to elicit a specific response or strategy to a problem or task posed. I describe in greater detail these three types of questions below.

The category of questions could be considered a subcategory of instructional decisions since the PST had to decide what the questions would be, when to ask, and when not to ask. However, since the questions themselves point to the type of listening in which the PST engaged, I decided to keep this category separate from the other instructional decisions, such as choosing problems or tasks and deciding what types of tasks their buddy would be able to solve. Initially this category was called questions to probe the child's thinking but I eventually broadened the category to include questions or suggestions to prompt or prod a child into mathematical action.

Note that I make a distinction between prompting and prodding, which is discussed in the subcategories below. Although some of their questions were planned in advanced, I believe that most of their questions were formulated during the interaction with their buddies.

Participants continuously asked many questions during the interactions with their buddies. Davis (1997) suggested that we can infer listening from the types of questions teachers ask. Below, I show evidence that the participants' questions can be placed in three different categories: probing, prodding and prompting. However, the type of listening in which the participants were engaged was not necessarily determined by the type of questions they were asking as they often asked all three types within the same interaction. What seemed to determine the type of listening in which they were engaged had more to do with the motivation for their questions rather than the type of questions they were asking.

Probing

According to the Merriam-Webster dictionary (2007), probing is considered to be a penetrating or critical investigation to search into and explore very thoroughly. Many of the planned and spontaneous instructional decisions included asking their buddies questions to elicit responses from the children that would help them create a working model of the child. In their written descriptions of the interviews, each participant noted questions that they asked her child beyond the questions in the protocol. Such questions included asking how the child arrived at a particular solution, if the child could solve a problem another way, and why the child used a particular strategy. The question of how the child arrived at a particular solution is considered an example of a probing question, while asking the child to solve a problem in another way is an example of a prodding question. The participants' use of probing questions was a theme that was relevant to my research question because it was an essential part of model building or making sense of a child's mathematics.

All participants used probing questions throughout the experience. Questions that were used repeatedly by all participants were various versions of "Can you tell me how you did that?" Probing questions were used to allow the preservice teachers to construct a more viable models of their children's mathematics.

Megan showed a great amount of skill in asking Grace probing questions even from their first interaction, clearly showing great effort in attempting to build a model of Grace's mathematics. She asked a series of questions on one problem to understand the operations that Grace was using to arrive at an answer. Megan was never satisfied just to recognize the actions Grace was taking, continuously asked her how she solved a problem. She asked Grace questions such as "why she started counting backwards at 18 instead of 19." This question is situated in the excerpt described by Megan in the initial interview assignment.

> Grace put 19 counters in the cup. She then took out 3 to give to me. After she put the counters down, she counted backwards on her fingers uttering "18, 17, 16" while putting up one finger per number word. She told me "16 counters are left in the cup." I asked her, why she started counting backwards at 18 instead of 19. 'She said, "You don't have to say 19 because you already have 19—you just need to take 3 off; that's why I counted backwards."

It is important to note that Megan determined that Grace's actions on this problem displayed evidence of her ability to count down, which she knew to be consistent with INS children. It was through her questioning that she was able to use the literature from class to construct an initial working model of Grace's mathematics.

On another problem from the initial interview assignment, Megan was able to determine that Grace had not yet constructed the commutative property of multiplication. Through her questioning, Megan found out that children do not automatically know that the order can be changed in multiplication to arrive at the same solution every time. On her initial interview assignment, Megan wrote,

> Q: #17 A pack of gum has 5 pieces. You have 3 packs of gum. How many pieces of gum do you have in all?
>
> For this problem, Grace counted aloud, “5, 10, 15…15.” “Why did you count by 5’s?” I asked. “Well, I knew there were 3 packs with 5 pieces of gum so I counted 5 (3) times,” Grace said. I asked her if the answer would be different if she had 5 packs of gum, each with 3 pieces. She didn’t know off the top of her head, but she counted “3, 6, 9, 12, 13, 14, 15” to get the right answer. I asked her why she counted by 3’s until she got to 12 but then started counting by ones after that. She said she just changed her mind.

Like Megan, Samantha was rarely satisfied with just an answer. She also wanted to know how her buddy, Isaac, arrived at his answers. Specific questions she asked him included how he figured out what came next, how he knew that each person would get less than 10, and how he solved a particular problem. Like Megan, she was not satisfied with knowing that he subtracted to solve a problem, so she probed further when she asked Isaac about how he subtracted the 7 from the 12. This process of asking probing questions to help her determine her model of Isaac’s mathematics helped her decide that perhaps Isaac was a strategic reasoner, but she knew she would still have to dig deeper.

A difference between Megan and Samantha is that, on her initial interview assignment, Samantha was not always explicit about the questions that she asked Isaac to elicit his mathematical schema. From her description, it seemed clear that she asked Isaac how he arrived at an answer for a particular division problem, but the actual question was rarely stated. Another difference was that Samantha indicated regret for not probing more. She wrote, "In retrospect, I feel like I should have probed him more on this problem or continued on with the harder problem involving the 42 children."

Through her questioning techniques, Samantha was able to obtain a lot of information about the way Isaac was thinking mathematically that she would not have known otherwise; however, she was not always successful at figuring out what he was doing. There were a few examples from the initial interview assignment that pointed to Samantha trying to make sense of Isaac's ways of operating; however she was unsuccessful. Samantha wrote, "I tried a couple of times to figure out what he was thinking, but for the sake of time and the continued confusion, I decided to move on." This statement also pointed to an early instructional decision of abandoning a problem in order to keep her buddy motivated and to deal with a perceived lack of time.

Unlike Megan, Emma and Maria wrote very little with respect to the specific questions they asked during their first interaction with their buddies on their initial interview assignment. I found only one instance where Emma referred to a specific question on her initial interview assignment to help her determine how Matthew was thinking mathematically. Like Megan and Samantha, most of Maria's questions were asking her buddy how she solved the tasks that she posed.

The participants constantly used probing questions suggesting that each preservice teacher was trying to learn and interpret the thinking of her buddy's mathematics. This suggests that each participant was attempting to listen interpretively.

Prodding

Prodding can be defined as inciting into action or to urging someone on (Merriam-Webster, 2007) and is distinguished from prompting in that it is just getting someone to act, not necessarily cueing them to do or say something specific. This particular type of question or teacher move was often designed to keep the participant's buddy acting mathematically rather than shutting down completely. Teacher moves such as these include, but are not limited to, abandoning a complex problem for a simpler, but related, problem. Another example of a prodding question is asking the child to solve a problem in a different way, as long as it does not seem like the preservice teacher is trying to elicit a particular strategy. Prodding questions were also used to help the participants construct a more viable model of their child's mathematics.

On the initial interview assignment, and throughout the field experience, Emma, Maria, and Samantha often asked their buddies to use another strategy besides the traditional algorithms. By traditional algorithms they were referring to aligning the multi-digit numbers vertically, on paper or in their head, and adding or subtracting starting with the ones, regrouping as necessary. They asked their buddies to use a different method because they knew that, were the child to use a traditional algorithm, they would learn very little information about their child's mathematics and they would not be able to analyze their child's mathematics with respect to the framework from class.

Maria, Megan, and Samantha also asked their buddies to try using strategies other than their comfortable counting strategies. Megan consistently asked Grace other prodding questions throughout the initial interview asking her to explain how she knew her method would work. This was prodding her to justify her mathematical actions, which is an important part of doing mathematics.

I refer to these questions as prodding because the preservice teachers were not trying to incite a particular strategy; they were simply trying to get their children to use strategies that would allow them to construct more viable working models of their children's mathematics, thus using prodding questions was to stretch their children's mathematics, but in no particular direction. They were trying to see what other directions their buddies might take a problem beyond a first attempt.

A typical way that Emma used prodding questions to engage Mathew in mathematical tasks is described below.

> Emma: Alright we're going to do some of the pairing problems again today. How many outfits can we create if we have 3 hats, 4 shirts and 5 pants?
>
> Matthew: 4 shirts?
>
> Emma: 4 shirts.
>
> Matthew: [draws a picture of 3 hats, 4 shirts, and…]
>
> Emma: 5 pairs of pants.
>
> Matthew: [draws 5 pairs of pants. He draws lines from one shirt to a pair of pants and a hat. Then from that same shirt, he draws lines to another pair of pants. Then to all the pants from that same shirt. Then he points with his pencil. Then writes something I can't see.] 60
>
> Emma: 60! Wow! How did you get that?
>
> Matthew: You can get any of these hats with that shirt and any of these pants with that shirt. [He added 15, 4 times.]
>
> Emma: So you figured out how many hats for the shirts and then the pants with the shirts? How else could you solve it without using addition? Do you have any other strategies?

Another prodding example that Emma used was on a pairing problem involving the number of ways to take a trip.

> Emma: There are two roads connecting Arnoldsville and Winterville, and there are two roads connecting Winterville and Bishop. How many different ways are there to travel from Arnoldsville to Bishop. Do you want to try it with just…well let's try it with just one road? [Two roads and one road] Let's see. Then we'll try it with two roads.
>
> Matthew: [inaudible]
>
> Emma: It's a round trip.
>
> Matthew: I think 4.
>
> Emma: Okay, show me your tracks.
>
> Matthew: [Shows his tracks by tracing them with a pencil.]
>
> Emma: Okay, now let's try it with the two roads from Arnoldsville to Winterville.
>
> Emma: This has been a little difficult to track. So we're going to label the roads. Maybe you could keep a list as you go along the different routes.
>
> Matthew: [He begins tracing different routes and listing out the roads by using his labels.]
>
> 11
>
> Emma: okay

This was the second time that she presented Matthew with the road problem. Emma kept Matthew from shutting down the way he did during the first task through use of her prodding questions. The purpose of her prodding was to put the problem more within his reach to see if it would give him any insight into the original problem posed, which was a nice strategy. Her suggestion to label could be considered a prompt, however, even if it was a prompt, the purpose for her suggestion was to provoke him into mathematical action rather than to incite a particular strategy. During the session interview following this interaction, she discussed her instructional decisions while engaging Matthew in this task. In the interview following this session, Emma stated,

> He kind of…I don't remember what he said but something can we just do it with one road and I think that was sort of him telling me well I'm not sure what we're doing with the two roads. Okay. What is it going to look like if we just do that? And I think that with the doing of one road that really made something click that I'm going to have to come back on this road so I'm going to have to take four different roads for one route. It's going to take four roads to complete one route and that got him realizing that this is a round trip, and I think that really… Cause there was sort of a break through with that problem once I got back to the two roads he realized that he was going to have to come back different ways…and something clicked I don't exactly know how to describe it but that sort of helped get past the eight roads because before when I had done the problem with him he came up with eight roads and sixteen and he never came up with the right answer but I was happy that he made that connection.

From her discussion, Emma was obviously more concerned about prodding him to consider the process of the round trip rather than prompting a correct procedure or solution. Her abandoning the original problem for a simpler problem was her attempt to operate within Matthew's zone of potential construction while building up to the original task.

Prompting

To prompt can be described as to move to action, as to incite or assist one to act or recite often by suggesting or saying the next words of something forgotten or imperfectly learned**.** A synonym of prompt is to cue or to serve as the inciting cause of (Merriam-Webster, 2007). In the context of this study, to prompt keeps this meaning and can be recognized by the preservice teachers trying to get their buddies to say something or to do something specific, such as to use a particular strategy.

Prompting was a questioning technique that the participants used throughout the field experience. From early comments in class it seemed like they were questioning the teaching-by-telling model. For instance, in class one participant claimed that she was beginning to see teaching as more prompting, however, the line between prompting and telling was often blurred. In the first interview Megan stated

> It's hard [laughing]. I got so frustrated because I was trying to explain it to her the best way I knew how and I knew that I wasn't doing…I wasn't explaining it to her the way she was understanding it, but it was hard for me to explain it to her without actually telling her 3+3 is 6 and if you add one more it equals 7. You know? And you can do all of them like this. **…** and it was hard for me to explain around it without telling her. Because I don't want to tell her you know this is the easy way to do it or this is the better way to do it you know. I want her to figure it out on her own. But it's hard to explain around it I guess.

In trying to get Isaac to strategically reason with tens, Samantha asked,

> Samantha: Is there a way to solve it without the blocks and do 35 + 28 in your head? Could you maybe keep one of those numbers in your head and count by tens? Okay so you started with 28. How many tens are in thirty?
>
> Isaac: 3
>
> Samantha: So how many tens do you need to add to twenty-eight?

The purpose of this line of questioning was to get the child to use a particular strategy. In a sense, Samantha was asking questions in order to tell Isaac how to act in order to push him toward strategic additive reasoning. This was not a strategy that he had used before and it was unlikely that he would begin using it spontaneously without some prompting. Her purpose for this line of questioning was to stretch his mathematics, not to see how he is currently thinking mathematically.

Maria prompted Ansley to see a different way to count a group of arrays. She read parts of a book called the *Grapes of Math*. This book had poems along with objects grouped strategically on the pages. One page included a group of five arrays with seven objects in each array. After Ansley struggled to count the objects by sevens, Maria stated, “Okay, can I do something so that maybe there’s a different way to solve it?” Maria rotated the book and reminded Ansley of the poem part that said “don’t just count the ones you see, consider the missing ones too”, which prompted Ansley to count by fives (Tang, 2001). Maria then stated,

> Do you understand how counting 5 rows with 7 in them is the same as counting 7 rows with 5 in them? Looking at it this way [as she rotated book back around so that the rows had 7 in them] made it harder. That was good though.

The purpose of Maria’s prompts was to get Ansley to see an easier strategy than the one she was struggling to use. Her summarizing prompt pointed to the idea of commutativity in multiplication, pointing out that seven groups of five is the same as five groups of seven, but counting by fives is easier.

Both Samantha and Megan used prompting questions to get their buddies to use doubles to reason strategically with one digit addition. For instance, to begin their second session, both Samantha and Megan went over doubles with their buddies to get them started thinking about doubles. They drilled doubles such as 2+2, 3+3, 4+4…9+9. After which, they presented their buddies "near doubles" addition tasks. Samantha began with 4+5.

> Samantha: Now um using the doubles that we just talked about. Can you figure out how to add 4+5? (Prompting)
>
> Isaac: 9
>
> Samantha: Did you use that with how you regularly solved it? How did you figure that out? (Probing)
>
> Isaac: You said 4 and 5. First I put four in my mind and then I went 4 5 6 7 8 9.
>
> Samantha: Oh okay can you try using the doubles that we talked about to solve 4+5? Remember when we did 4+4 and 5+5? Can you use the doubles to solve 4+5? (Prompting)
>
> Isaac: 4+4 is 8.
>
> Samantha: uh huh and
>
> Isaac: 5+5 is 10. There is one before and one behind.
>
> Samantha: One before and one behind so if I gave you the problem 6+7, What double could you use to solve that?
>
> Isaac: I could do
>
> Samantha: Could you use a double and then add something to that? (Prompting)
>
> Isaac: 7+7.
>
> Samantha: So if it's 6+7 what do we need to do to get from 7+7 to 6+7? (Prompting)
>
> Isaac: Take away one.
>
> Samantha: Good job! Now using the doubles again, do you think you could solve 7+8? (Prompting)

Isaac: 8+8.

Samantha: Okay since it's 7+8, what do we need to do? (Prompting)

Isaac: Take one away.

Samantha: Take one away okay can you try um 5+6, but this time instead of subtracting can you use adding to solve it instead of subtracting? (Prompting)

Isaac: 11

Samantha: Okay what did you do to solve it? (Probing)

Isaac: 6+6 and then I did 6+5.

Samantha: Can you maybe use the doubles and add to get the answer so if I asked you to solve 7+8 what double could we use to add instead of subtract. (Prompting)

Isaac: We could use 9+9 is 18 then subtract three and then no no. You could add 6 and 12 and then add one more.

Samantha: 7+8 Could it be possible that we could use the double of seven to figure this one out? 7+8 could you use the double of 7 to figure that out? (Prompting)

Isaac: Cause you could use 7+7 is 14 then try different numbers to figure that out and then add one.???

Samantha: So seven plus eight you could do 7+7 so how many more is 8 than 7? One more. So if 7+7 is 14 what would 7+8 be? (Prompting)

Isaac: 15

Even though this excerpt included two probing questions, I included it in the prompting section because Samantha was really asking Isaac how he figured the problem out to be sure he was using the doubles strategy that she was trying to prompt. Megan went through a similar series of prompting questions with Grace to prompt the doubles strategy. Both Megan and Samantha chose to begin prompting the doubles strategy because each noticed during the initial interview that her buddy was proficient with doubles. They also noticed that their buddies both used this strategy once, but not on a regular basis. Megan's and Samantha's goal was to make this strategy readily available to her buddy. Both lines of prompting questions relate to the doubles strategy based on the working models of each child's mathematics. This was evidence that both were using their working models of their children's mathematics to make instructional decisions, particularly to determine what strategies to prompt. On October 12 during the first session interview with Megan she stated,

> Megan: Sometimes she will [use strategic reasoning], but it's just very very seldom. And it's only with certain problems, and I can't remember one off the top of my head, but she did on one problem say well I know that 5+5 is 10 and then I add one more and that makes eleven.
>
> Interviewer: So she used doubles the first time?
>
> Megan: Yeah she did, and that's part of my basis for this.
>
> Interviewer: Okay.
>
> Megan: Because she…there's plenty of problems where she could've used doubles, and she only used it on that one problem. So I was thinking well maybe if I just work with her a little bit more then it will come to her fairly quickly.

Prompting questions were definitely instructional decisions made by these participants. Many of Samantha's instructional decisions that she discussed in the final interview were based on her working model of Isaac, some made by listening to him solve problems and latching onto something he said or did. For instance, she decided to prompt rounding strategies because she heard Isaac say something about rounding when he was describing how he solved an addition or subtraction task. After talking to the instructor, she decided to use numbers that would be easy for him to round. In an interview, Samantha stated,

> The week before he had done a lot of like rounding strategies…This is a suggestion of Dr. H. I would have a number and to probe like kind of a rounding strategy move. I would do like a problem like sixty-three minus nine and move up to the next one sixty-three minus nineteen. So it's still like basically doing the same process, but it's still having to probe [prompt] his thinking a bit and work on rounding that way.

Her last statement indicates her desire for these strategies to become internalized so that Isaac would use them without her having to prompt them.

Probing, Prodding, and Prompting Working Together

It is important not just to extract the types of questions the PSTs asked separately, but to see how these types of questions were used together during the interactions. Below I use examples from the initial interview assignment and from the sessions during the field experience to show how the different types of questions were used and what they imply about the type of listening in which the participants were engaged.

Below is Megan's account on the initial interview assignment of a typical interaction between Grace and herself. The question she posed to Grace was "Marcus has some toy dinosaurs. He goes to the store and buys 3 more toy dinosaurs, and then he has 9 dinosaurs all together. How many dinosaurs did Marcus have to start with?" Megan describes,

> This one was a little bit harder for Grace. She used the counters. She begins by putting out 9 counters and then separating 3 with her hand. She then counts the remaining counters and gets the answer "6." "How did you get that?" I asked-. Grace says, "Well, I knew he had 9 at the end, and he got 3 from the store so I moved those and counted the

rest and that told me there were 6 to start with." I asked her if there was another way she could have done the problem, and she said "I could have not used the counters." **"**How could you have done it without using the counters?**"** "I could have started with 9 and counted back 3?" She said in a questionable tone. "Why don't you try that?" I asked. So, she utters "8, 7, 6…6," she says. "Did it work?" I asked. "Yup," Grace says. After some more questioning, Grace goes on to explain that her counting back from 9 is the same as when she used the counters and "hid" 3 with her hand.

This excerpt included both prodding and probing questions. Although Megan could see the actions that Grace took to arrive at an answer, she was not satisfied and chose to investigate further by asking Grace how she got her answer. After obtaining information about how Grace solved this problem, she used prodding questions to draw out a different way to solve the problem. I called these questions "prodding questions" because she was not trying to elicit a particular strategy. Megan was not only prodding Grace to state another possible strategy, she was able to get Grace to try out her strategy and determine whether it worked to produce the same result. She was able to successfully get Grace to connect two different strategies.

In another instance, Megan used a series of different types of questions to successfully engage Grace in solving a particular task and to learn about Grace's mathematics. Below is the excerpt of the interaction between Megan and Grace, along with the question categories placed in parentheses.

Megan: How many tens are between 35 and 55?

Grace: 20

Megan: There's 20 what? (Prodding/Probing)

Grace: Tens

Megan: There's 20 tens between 35 and 55?

Grace: One ten.

Megan: One ten?

Grace: Um humm

Megan: Okay. Can you tell me how you know that? (Probing)

Grace: Because I subtracted 55 and 35.

Megan: Okay. Is there a different way to do it maybe? (Prodding)

Grace: The number line?

Megan: How would you do it using the number line? (Prodding)

Grace: Make a number line and go from 35 to 55.

Megan: Oh yeah how many spaces would you have to go? (Prompting)

Grace: 20

Megan: Okay and if I were jumping ten spaces at a time, how many times would I have to jump? (Prompting)

Grace: One I mean two.

Megan: Two? How do you know that? (Probing)

Grace: Because two ten sticks is 20.

Megan: Two ten sticks is 20 so if I want to know how many tens there are between 35 and 55, how many?

Grace: Between 35 and 55?

Megan: Um humm how many tens?

Grace: Two

Megan: Two! Very good! Alright! Let's try a different one.

In the excerpt above, Megan used probing, prodding, and prompting questions. She was open to whatever strategies Grace chose to use to engage in the task. Even though she did use prompting questions, her prompts were based on the strategy that Grace chose to use, the number line, which suggests that Megan was adapting her questions in response to how she was interpreting Grace's mathematical actions, leading me to claim that, in this interaction, Megan was at least listening interpretively. D'Ambrosio (2004) claims that teachers engaged in interpretive listening attempt to make sense of a child through using the voice of the child rather than the voice of the discipline, which is what is required of teachers listening hermeneutically. She claims that teachers listening hermeneutically continuously renegotiate the voice of the discipline in order to give reason to the child. In this type of listening both the student and teacher are learners. She suggests that hermeneutic listening is the type of listening required of constructivist teachers since it is used to construct a model of a student's mathematical understanding in a way that guides her teaching. In contrast, Davis (1996) claims that constructivist teachers listen interpretively rather than hermeneutically. He argued that interpretive listening is premised on conceptions of learning in which the learner is autonomous and isolated. The teacher is actively trying to interpret a child's utterances and actions. Hermeneutic listening is more negotiatory since it is described as "a participation in the unfolding of possibilities through collective action" (Davis, 1996, p. 53).

The probing and prodding questions were used in order to help Megan construct a more viable model of her child's mathematics, however, her use of prompting questions was often based on the working model of her child's mathematics and was determined spontaneously in response to Grace's mathematical actions. Since Megan was basing her teaching actions on her model of Grace's mathematics, I claim that she may have been listening hermeneutically, but I would have to probe further to make this claim with more certainty.

Rethinking Definition of Teaching Mathematics

From their final interviews and their portfolios, it appears that the participants were rethinking their definition of teaching mathematics in four very important ways. They seemed to be letting go of teaching as telling students how to think and how to act and were beginning to see theory as an important part of teaching, departing from teaching as telling and moving toward teaching as posing appropriate tasks for students. They were also becoming aware of teachers as lifelong learners.

Troubling Teaching as Telling

The first way in which the preservice teachers were beginning to change their conceptions of teaching mathematics was that they were deserting their thinking that teaching mathematics meant telling students how to act. In her final portfolio, Maria described her view of teaching as it emerged through her work with her buddy. She stated,

> As I began working with my School Buddy, Ansley, I experienced my first awakening—children do not learn mathematics through eloquent explanations, but through logical necessity and their own reasoning. Thus, my goal changed from being able to explain concepts to children to being able to pose the "right" problems to evoke the desired learning.

Megan also discussed changes in her conception of what it means to be a teacher. In the final interview she stated,

> I like to talk and I guess my innate nature as a teacher or wanting to be a teacher makes it very difficult for me to let someone learn on their own. I really want to teach them and

> you know… By teach I mean my idea of teach, which is tell them how to do it—step by step by step. And so this whole idea of like letting them do it own their own… Even though now I realize how important that is and I see how I was not allowed to do that in my own learning when I was a child… I see how important that is to her [Grace]. I definitely want to do that, but I do think it's difficult just because I like to be in there and I like to teach and I like to be the one that's showing somebody oh if you do it this way and get their reaction … but I think that's just human nature so it has been kind of difficult.

Megan seemed to be pointing to the difficult task of letting go of the conception of teaching as telling. While she acknowledged that she thought, in her own mathematical background, she was potentially harmed by not being allowed to construct mathematics out of her own logical necessity, she still feels drawn to teach by telling, the method by which she was taught.

Samantha also referred to a change in her goals with respect to learning to teach mathematics. On the final portfolio she wrote, "Once I knew we would be studying math education from the lens of a constructivist, my initial goal (of learning to rely on and clearly explain the algorithms) was put aside." She discussed a new learning goal that she developed during her work in the course. She stated, " In general, I wanted to learn more about the notion of silence discussed in our classroom. My goal was to learn to appreciate silence in our interactions, because that is when deep thinking is likely happening." Samantha pointed to an idea that is often overlooked or avoided in the classroom. Often teachers try to fill the silence with their thoughts because silence is often seen as awkward. However, Samantha acknowledged that the silence could be the sound of students thinking. This is not likely something she had thought about prior to her work in this course.

Teaching Requires Theory

I suspect that the need for theory in the practice of teaching was a new concept for the participants. Three of the four participants explicitly refer to seeing the field experience as an opportunity to see theory in action. On the initial interview, Maria wrote, "I felt like I was learning all this useful information, but I was not able to utilize it in a classroom or see it for myself." In this quote, Maria is pointing out that, while she intuitively felt that what she was learning may be useful, she could not fully appreciate it until she saw it come to life through the work with her buddy, Ansley.

Doyle (1990) calls knowledge of research and theory "propositional knowledge" and knowledge of the skills of teaching "craft knowledge." He discussed how both of these forms of knowledge, learned separately, are insufficient and addresses the need for preservice teacher education to include opportunities for propositional knowledge and craft knowledge to be developed simultaneously. The participants seemed to convey a feeling that they were seeing the propositional knowledge in action through their field experience. In the final portfolio Emma wrote,

> This was an opportunity to see what such a transition looked like. I got really excited when I made this realization because I knew that this was a unique opportunity to make observations about these initial stages of strategic multiplicative reasoning.

Emma was referring to a very specific stage of reasoning that she learned as part of the theory from class. She clearly appreciated the unique opportunity that she was given to connect the theory from class to her experiences in the field, making it possible for her to personalize the parts of the theory that she could connect to her buddy, and thus building propositional knowledge into her craft knowledge.

Posing Appropriate Tasks

The participants began to see the task of teaching to include posing appropriate problems for children and all participants indicated that learning to listen to children was an essential part of teaching mathematics. In the final interview, Emma said

> If we can't listen to them, if we don't understand what their thinking we could give them stuff that isn't appropriate for where they are mathematically. We could shoot too far or not give enough um to …challenge them… It's not easy but it's important.

All participants mentioned that their initial interview aided them in choosing and planning appropriate activities for their buddies in the field experience. In referring to how she was personally planning to use the insight she gained on her initial interview with Isaac, Samantha wrote

> Now that I have this basic sketch of where Isaac is thinking mathematically, I hope to be able to plan and create meaningful sessions where I can assist him in discovering new and meaningful ways of thinking about mathematics.

Some participants even went beyond their personal experiences to indicate that the process could be helpful for other teachers and for their future classroom students. On the initial interview, Emma and Maria suggested that they were thinking more broadly. Emma states, "All teachers can benefit from such a process in terms of knowing their student's strengths and weaknesses." She further states that "the students can also benefit from it because it can help the teacher maximize each student's mathematical education." Through their comments, it can be stated that all participants seemed to be striving to choose and design tasks that were in their child's Zone of Potential Construction.

Teaching as Continued Learning

Each participant discussed learning more from their buddy than her buddy learned from the experience. Emma stated,

> In terms of any learning my buddy may have done, I cannot out and out say he did learn anything. I don't know if he did. I wasn't really setting out to teach him anything. In fact, I was the one doing the learning.

On the final portfolio Maria wrote, “At the beginning of this semester, I expected to learn how to be a teacher, but really I learned how to be the student to my student which is an essential part of teaching as well.” All of the participants saw being a learner as part of their task as a teacher in the field. Maria highlights that this is not just part of the field experience, but that teachers in general need to continue to learn from their students.

All participants were able to use their working model of their child’s mathematics in order to construct instructional goals and plan appropriate activities for their child. Of course, while they seemed to feel that listening to and learning from children in order to design appropriate tasks is important for them and for all teachers in general, they still had questions and concerns regarding this work. During the final interview, Samantha expressed concerns about working with children who are not at the same level as Isaac. When asked if she will continue this work with her own classes, she responded with,

> I feel like I’m not very knowledgeable about children who are more advanced than Isaac because I put all my concentration into working with a child at his level so I don’t feel comfortable … working with someone who maybe is above where he’s at… I would love to use this approach because I think it’s very helpful and I think that having them construct their own way is the best from what I’ve seen, but I would need to get more

> information and learn more about it to feel really comfortable with it. I don't think one semester is enough.

All of the participants felt that the type of work they did with one child in their field experience would be very difficult to reproduce with a whole class. In the next statement, I think that Samantha articulated beautifully what most of the participants expressed as concerns about teaching from a constructivist perspective with a whole class. She stated,

> Incorporating it [constructivist approaches] in a whole class context is still kind of overwhelming to me because there's going to be kids at so many different levels. And just trying to remember where every child is at … is just at this point overwhelming … so just like imagining that is kind of daunting.

Emma had many questions that she wrote about in her final portfolio. She wrote,

> Some specific issues that remain open to me are, when does one know when a child has made a complete transition to the next mathematical thinking level and is no longer in the gray area between levels? How will I know when a child has a complete understanding of a brand new topic? If certain algorithms serve to disrupt a child's natural way of thinking then why are they still being taught?

All of the participants' remaining questions suggested that they were thinking deeply about their emerging definitions of learning and teaching and they all expressed concerns about teaching traditional algorithms. In the final interview Maria strongly stated, "algorithms, I wish the teachers would just like stay away!" When asked if she planned to stay away from teaching the traditional algorithms, she stated,

> I don't know. I kind of feel like I'll probably end up doing both, just because I know that other teachers will probably be using the algorithms and it might hurt them in the future that I didn't just introduce it to them.

Maria's conflicting statements concerning the teaching of traditional algorithms suggests that, while she is open to children constructing mathematics "through logical necessity and their own reasoning," she still feels tied to the structures that she experienced as school mathematics.

Even though the participants still have many questions regarding the ideas of constructivism from which the course was framed, their questions and concerns indicate a desire to continue to learn how to teach by listening to children and posing appropriate tasks. In the final portfolio, Emma wrote,

> I want to know how to adjust my own mathematical thinking so that I can understand where children are coming from in order to better teach them. There is so much to know and learn about children's mathematical thinking that my list barely scratches the surface.

Her statement suggested that she sees children's mathematics as a legitimate mathematics and that she is willing to renegotiate her own understanding of the discipline in order to learn how children think about mathematics, which indicates her desire to learn to listen hermeneutically.

Megan beautifully articulated the participants' descriptions of teaching involving continuation of learning. In her final portfolio, she stated,

> While I know that I still have a lot to learn about children's mathematics and the way children learn, this semester has greatly increased my mathematical knowledge and strengths for teaching elementary math. There are so many components that create a good math teacher, and after reflecting on my work and seeing how much I have learned,

I have concluded that the single most important thing I have learned this semester is: There is always more to learn.

The Participants' Mathematics

I chose participants with variation in their mathematics background. In this section I will discuss how this variation did not interfere with building their models or using their models. I will also discuss a second finding that was not necessarily tied to my research questions; I found that each of the participants reported that she learned mathematics through her work with her buddy. As Simon (1997) noted in his MTC framework, a teacher's interaction with students often informs her mathematical knowledge. Even though the context of my research was provided by a mathematics methods course rather than a mathematics content course, the preservice teachers were learning mathematics, specifically the mathematics that they were teaching. Another finding that was related to their mathematics was that each of the participants compared her mathematics to her buddy's mathematics and, while there were strong similarities among the way the participants talked about their mathematics, there were also some minor differences.

Variation in Mathematics Background

As reported in previous sections, there was a variation in the participants' mathematics backgrounds. Megan described her previous experiences in mathematics as a struggle while Emma reported that mathematics had always come easy to her. Maria and Samantha both discussed having a love-hate relationship with the subject.

I found no differences in the way the participants were able to interpret the experiential model from class to analyze the children's mathematics. They each described her child's mathematics using language from class and their own language. They were also able to use their working model to inform their instructional decisions. Megan was the most creative in the use of her model to make instructional decisions since she created games in order to get her buddy to strategically reason. Samantha was the only participant to list specific strategies that she wanted to foster in her buddy on the initial interview assignment. On this assignment, the other participants discussed wanting to get their children to the next level of whole number development but didn't suggest specific strategies to work on within that level. While Emma claimed mathematics came easily to her, I did not see evidence that indicated it was easier for her to build or use a model.

These observations lead me to believe that the preservice teachers' perceived mathematical background had little or no affect on the working models that they were able to construct or the way that they were able to use those models to inform their instructional decisions. In determining whether a stronger mathematics background would allow one to construct a more viable model of a child's mathematics I would have to say that the results are inconclusive.

Similarities Among Participants' Mathematics

When discussing learning to reason strategically, the participants all expressed having always been tied to the traditional algorithms when solving addition and subtraction problems with multi-digits. In the final interview Samantha reiterated her attachment to the standard algorithms. However, she also suggested that she was slowly trying to detach herself now that other strategies are available to her. On the final portfolio she stated,

> Over the semester, my own mathematical learning has evolved. A lot of my learning goals were addressed. But, my greatest learning of the semester coincided with what I was working with on Isaac—reasoning strategically. During elementary school, I was

only taught to use the standard algorithm for addition and subtraction. I cannot remember exploring other strategies for solving addition and subtraction problems. Although I know my doubles and how to make 10, I was not using strategies like these to solve math problems. Before this semester, if someone would have asked me to solve a two-digit addition or subtraction problem, I would have immediately lined up the numbers and executed the standard algorithm. It would not have come naturally to me to strategically think about the problem.

My work with Isaac this semester has really opened my eyes to how I can use various strategies and apply them to addition and subtraction word problems. The work we did with counting off-decade by 10s was especially useful to me. Working with Isaac really got my brain working to figure out how many 10s between two numbers, and apply that strategy to problems I encounter. Now I find myself thinking about two-digit and three-digit addition and subtraction problems strategically in my mind, and I have actually found these methods to be quicker than if I had lined up the numbers in my head. Although my mathematical learning sounds simplistic, it was actually a significant jump in my thinking and it is something that I will continue to build upon.

Differences Among the Participants' Mathematics

The differences among the participants' discussions of their own mathematics concerned their relationships to mathematics. For instance, Megan did not like or understand math and thought she never would. She stated,

> Until this class, I have always thought of math as a subject that I would never understand. I learned just as much, if not more, about myself as I did about my Buddy, or children's learning in general during this course. Since I have always struggled in math, once a

> formula that produced the correct answer to a problem was introduced to me, i.e., the standard algorithm, I ignored any other method; my mathematical creativity was smothered by algorithms, formulas, and shortcuts. Therefore, my most important mathematical learning of the semester was reasoning strategically. I was taught the standard algorithm at a very early age, and I cannot remember ever being encouraged (or allowed for that matter) to answer any type of mathematics problem a different way other than using the standard algorithm. Until this semester, if I was asked to answer the problem "25+24," I would have pictured the standard algorithm in my head and added "5+4" and "2+2" to get "49." However, after "not being allowed" to use the standard algorithm in EMAT [methods course] , I was forced to find an alternative way to answer problems such as, "25+24." I quickly learned to build on previous knowledge and reason strategically to answer any type of problem.

Unlike Megan, at one time Maria claimed to have loved mathematics. While she had formerly lost her love for math, her work in this course revived it. On her final portfolio, Maria wrote,

> I particularly enjoyed the move away from algorithms to the reliance on strategic reasoning because it brought math to life again. Every problem was a challenge that I could not merely dodge by using a formula or algorithm, but that I had to confront and truly reflect on in order to understand thoroughly.

Emma, who reported having always been good at mathematics, had much to say about her mathematical experiences. She felt that it was difficult for her to explain concepts in mathematics because, as an adult, she did not remember how she came to know them. She stated,

> Ever since I was little, math has come fairly easily to me. I was always quick to understand the concept the teacher was teaching and got A's without trying. My mom would drill me

> with addition, subtraction, multiplication, and division flashcards, thus making me very proficient in those areas. As I got into middle and high school, I was repeatedly placed in higher-level math courses until I finally reached A.P. Calculus my Senior year. Throughout those years, I started noticing a trend in my mathematical learning. I would pick up the concepts and strategies very easily, so easily in fact that I hardly ever had to study. But, I had a hard time retaining the knowledge and remembering it later down the road unless someone refreshed my memory. It was not until A.P. Calculus that I really had to put forth a major effort to do well. I had to develop a much deeper understanding of the material than I ever had before, which resulted in the most enriching mathematical experience I have ever had (and a 5 on the A.P. test).

Through Emma's reflection about her mathematical learning as she was growing up, she determined that, rather than understanding mathematics, she knew it rather procedurally, however, she reported wanting to be a teacher that helps children understand mathematics more conceptually. On her final portfolio she stated,

> As I think back on my math in middle school and high school, I have come up with a theory. During those years, it is my belief that my mathematical understanding was not as good as I would like to think it was, but rather I simply knew how to memorize formulas and figure out what kind of problem required which formula. I think part of this is because formulas were all that seemed to be taught and we were not always taught the meaning or the reasoning behind those formulas. I believe my teachers took it for granted that we "higher level" students would automatically understand and they would simply show us what steps to use. That is not the kind of math teacher I want to be.

Emma reported that her most difficult task in mathematics was to explain. She felt that her mathematics was so formal and compact that it was hard for her to think about how she came to know it. On the final portfolio she stated,

> I think that my most important mathematical learning involved realizing how difficult it can be to explain oneself, especially as adults. I think that our mathematical thinking becomes so automatic that we feel we can't give an explanation. We don't know why we come up with our solutions to problems. We don't always remember the reasoning behind our strategies. They have become so engrained in us that, in a way, we have lost their meaning.

Comparing their mathematics to their children's mathematics

Three of the participants compared their mathematics to their buddies' mathematics. Megan saw her mathematical self in her buddy, Grace, but also noted a difference between herself and her buddy. Her comparisons involve affective aspects of doing mathematics. On her final portfolio Megan stated,

> I see a part of myself in Grace as a math student. She is easily frustrated when probed to "think outside the box" or explain her reasoning. However, unlike me, she is very confident in her answers- even when she is wrong.

Maria's comparison between her and her buddy's mathematics was also from an affective perspective. On her final portfolio Maria stated,

> More than her ability to solve these problems and complete the activities I designed, Ansley impressed me with her dedication and diligence… The traits I have deemed both positive and negative in my own personality I find quite pleasing in my student. The main difference between Ansley and me, however, lies in her ability to enjoy the process and not stress out like me.

Like Megan and Maria, Emma compared her and her buddy's mathematics, however, unlike them, she saw her mathematics as very different from her buddy's. Her comments focused more on differences in her strategies and her buddy's. On her final portfolio, she stated,

> I noticed through working with my buddy that he didn't have a set way of solving problems all the time. He had certain tendencies, but he never used just one strategy, but rather, he would use what worked for him that particular day or instance. When I compare that to my own mathematical thinking, which to me is very automatic, structured, and one dimensional, this is a very profound idea. It makes me realize that I am going to have tons of ideas to work off of and build on to when I become a teacher.

Emma was referring to the flexibility with which Matthew seemed to be working, a flexibility that she had lost in what she perceived as the algorithmic way she operated in mathematics.

Summary of Findings in Participants' Mathematics

Through their comments, it seemed that the participants experienced powerful changes in their conceptions of mathematics as a result of their work in the field. Through constructing children's mathematics, they changed their constructions of mathematics by moving away from relying on traditional algorithms and moving towards reasoning strategically. They also seemed to be changing their dispositions towards mathematics. Megan, who thought she would never understand or like mathematics, discussed her move to strategic reasoning; Maria's love for mathematics was revived, and Emma determined that just knowing formulas and when to use them is insufficient for a true understanding of the subject.

CHAPTER 6

SUMMARY AND DISCUSSION

Steffe (1991) emphasizes that one of the key goals that serves to guide the actions of mathematics teachers choosing to operate under a constructivist epistemology is to learn the mathematics of students in order to inform the way teachers communicate with students and view the mathematics for students. Steffe and Wiegel (1996) assert that the most basic element of a constructivist teaching model would include close listening to students and that this listening could yield a working model of students' mathematics; however, they caution that close listening alone is not sufficient. They propose that teachers also need to learn to act on their listening in order to bring out and extend the mathematics of students.

Summary

The purpose of my study was to investigate the working models that four elementary preservice teachers constructed of children's mathematics. I also wanted to know how their models informed their instructional decisions. These four elementary preservice teachers were in a mathematics methods course with a mathematics specific field experience in fall 2005. This course provided the necessary context for my study in that it made it likely that the preservice teachers would attempt to do this work. In the field experience, each preservice teacher was working one-on-one with a third grade child called their school buddy. My data included videos of these interactions, interviews, and written assignments for the university class that were based on their interactions with their buddies. The data were analyzed using qualitative case study methods, including micro-analysis (Strauss & Corbin, 1998).

I found that each preservice teacher constructed a working model of her child's mathematics by describing the mathematical actions of her child and then analyzing her descriptions based on her interpretation of the experiential model explained in class. This experiential model informed the participants' knowledge of content-specific student learning discussed in the Mathematics Teaching Cycle offered by Simon (1995) as a framework for constructivist teaching. I also found that each PST could work with the model she constructed, thus making it a working model. Their models informed many of their instructional decisions, including the tasks and problems that they designed or chose based on their learning goals. These learning goals were based on the preservice teachers' working models of their buddies' mathematics. Some of their instructional decisions were based on their desire to explore their buddies' mathematics in some particular content area, which they used to further inform their models of their buddies' mathematics. Upon analyzing the data, I found that their interactions in the field resembled the MTC presented by Simon (1997) even though Simon's MTC was concerned with inservice teachers in a classroom environment.

I found that each of the participants asked many questions during her interaction with her child and I determined that these questions served three different purposes, including informing the PST's model of each child's mathematics, motivating each child to continue to act mathematically, and motivating each child to extend his or her mathematics. The first type of question that I identified was used to probe each child's mathematics in order to inform and refine the model of the child's mathematics. These types of questions were similar to two different types of questions Driscoll (1999) referred to including prompting mathematical reflection and clarifying questions. Other questions that the preservice teachers asked served to prod their children into mathematical action and were similar to two different categories of questions determined by Driscoll: orienting questions and managing questions. A third type of question that the participants asked was intended to elicit a particular strategy that they felt was in their children's ZPCs. I called these questions prompting questions. These were similar to the category that Driscoll called eliciting algebraic thinking, the difference being that participants were not trying to get their buddies to use algebraic strategies, rather, they were trying to prompt strategies for additive or multiplicative reasoning.

The questions that the participants asked provided evidence for classifying the type of listening in which they may have been engaged. However, it was not the type of question each participant was asking, as Davis (1997) suggested, rather it was the way in which she used the questions to attend to her child's mathematics, as D'Ambrosio (2004) suggested. Similar to my study, D'Ambrosio (2004) discussed a PST responding and attending to one child. However, in her study, the PST was responding to a child through the use of e-mail. Whether in a whole class situation or one-on-one D'Ambrosio contends that it is unlikely that a preservice teacher can listen to students interpretively and even less likely that they can engage in hermeneutic listening when interacting with students. However, I found that the participants in my study were learning to listen to students, constructing models of their students' mathematics, and using their models to inform their instructional decisions. Even when the PSTs were using questions to prompt particular strategies, the strategies that they chose to prompt were based on their working models of their buddies' mathematics and on the experiential model from class and not solely on their previous self-identified algorithmic understanding of mathematics. D'Ambosio suggests that a teacher is listening hermeneutically when she can renegotiate her own understanding of mathematics while interacting with her students; thus I contend that the PSTs were engaging in both interpretive and hermeneutic listening.

There were other significant findings that were not directly related to my questions. I found that the participants learned mathematics for themselves as they learned the mathematics of children, all of them referring to becoming less reliant on algorithms and more reliant on strategic reasoning. Some participants were also changing their disposition towards mathematics and were comparing their own mathematics to their buddies'. This finding is consistent with Steffe's (1991) claim that "personal mathematical knowledge can be contrasted with the mathematical knowledge of others; and this comparison is the likeliest way to make progress" (p. 191).

I found that the each PST was redefining what teaching mathematics meant to her. They were moving away from defining teaching as telling and attempting to clearly explain mathematical ideas. They seemed to be moving toward a definition of teaching mathematics that suggested that teaching was more about listening and posing appropriate tasks for students. The participants also indicated that they were seeing the value of theory and research in informing their actions as teachers. They noticed that continued learning, learning about their particular students, the mathematics of students more generally, and mathematics for students, was an imperative part of the work of teachers.

I believe that this finding would not have been possible without the work of the instructor, Dr. H. in closely tying the propositional knowledge from class to their developing craft knowledge in the field. Another part of the context that resulted in the PSTs redefining their view of what it means to teach mathematics was the way this particular mathematics specific field experience was designed. It was set up so that PSTs would have the opportunity to work one-on-one with one child with no pressures to cover curricula objectives. In the "School Buddy" field experience management concerns were essentially eliminated and PSTs were allowed to focus on the mathematics of the child. This has implications for the education of elementary teachers.

Implications for Teacher Education

Ball (1997) suggests that, while we cannot know precisely what a student is thinking, we cannot give up the quest to try to figure it out. She suggests that teachers can use their past experiences with similar students and experiential models developed by researchers in order to guide their interpretations of their students' mathematics. She further advocates that successful methods of teaching depend on teachers having the opportunity to study children and engage in experiments while probing and interpreting children's mathematical ideas. My study investigated one such opportunity for preservice teachers, however, it was clear that learning to do this work was a difficult task.

In the final portfolio Samantha stated,

> Adopting the constructivist philosophy was a struggle for me because it was the first time I had seen it used to teach math. It is natural to think that the way you learned something in school is the best way to teach someone else.

Samantha's personal struggle in identifying with a constructivist orientation to teaching mathematics was consistent with Simon's (1997) claim that, "many teachers have developed their models of teaching in the context of thousands of hours as students in traditional classrooms" rather than being based on explicit models of learning (p. 57). As a result, teachers' expectations, values, suppositions, and assumptions about learning and teaching of mathematics tend to be based solely on a lecture/demonstration model in which teaching is predominantly telling and showing. This suggests that mathematics education reform is not going to happen naturally and that teachers and preservice teachers need opportunities to listen to and make sense of students' mathematics. They also need to be aware of experiential models that help them to make sense of students' mathematics in a manner that informs their instructional decisions.

NCTM has been advocating for reform since the eighties. They assert that teachers need to listen to their students, observe their mathematical actions, and make sense of their listening and observations (A. G. Thompson, 1989). They also suggest that teachers need to base their instructional decisions on their sense of their students' mathematics. Simon's Mathematics Teaching Cycle is aligned with the ideas of reform advocated by NCTM. Since I found that the process in which the PSTs were engaged was closely aligned with Simon's MTC I chose to explicitly incorporate my findings into Simon's diagram of the MTC.

Simon's original MTC involved teachers working in a classroom environment with students. My adaptation is meant to explicitly address the differences between and my study, which involved preservice teachers working one-on-one with a child. Many of the boxes referring to the knowledge of preservice teachers remained unchanged. For instance, Simon's diagram included a box for knowledge of mathematical activities and representations. The adapted version for PSTs includes this same box, but their growing knowledge of mathematical activities and representations was greatly influenced by the suggestions of the instructor, the articles assigned by the instructor and the discussions from class that were facilitated by the instructor. I suspect that inservice teachers learn about representations and activities for mathematics very differently from the way that these preservice teachers did. In fact, all boxes representing the different facets of the PSTs' knowledge in this diagram were heavily influenced by the methods course associated with their field experience, which was taught by Dr. H.

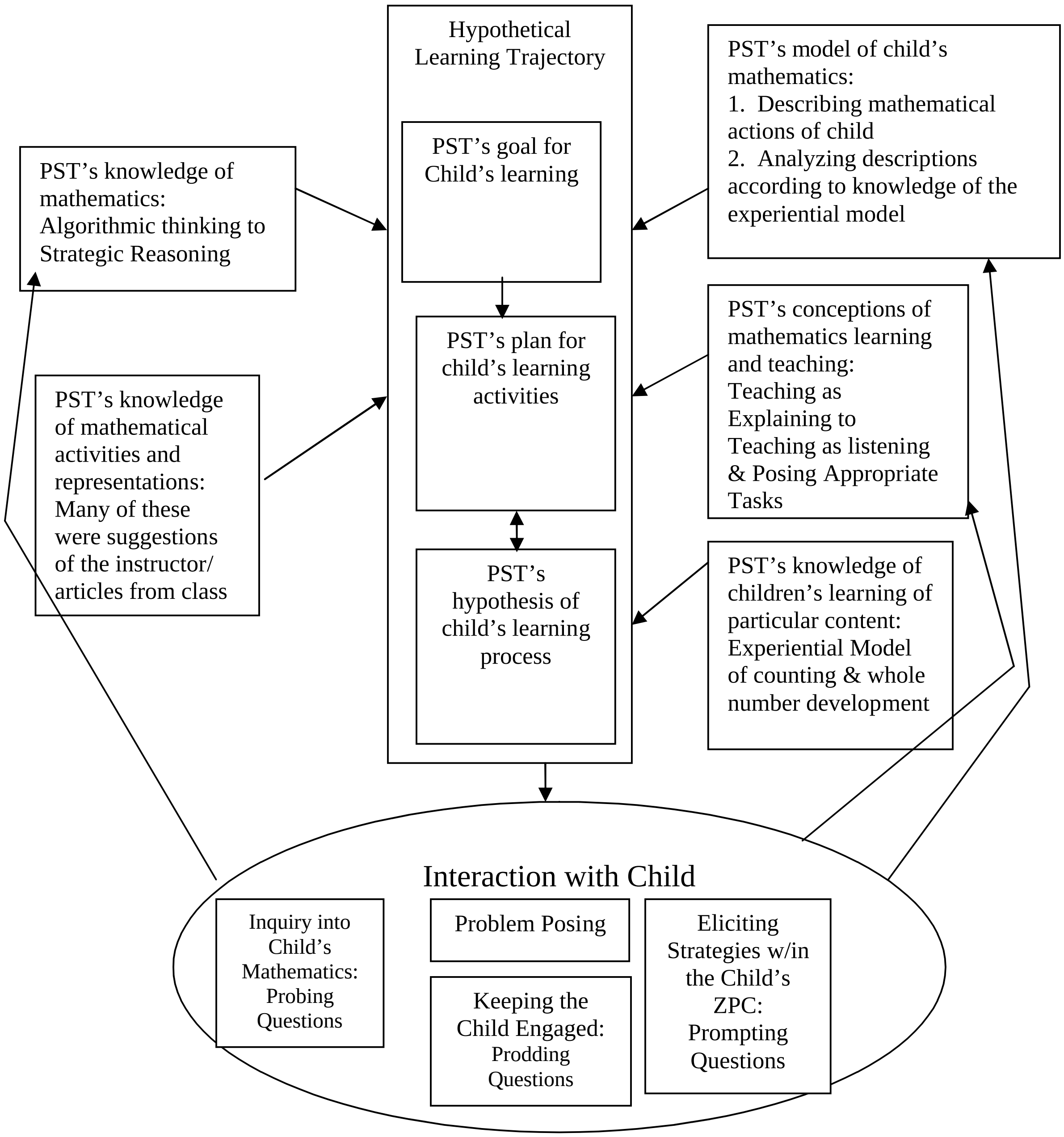

Figure 3: Simon's Extended MTC Adapted for PSTs

In the Simon's diagram the bubble labeled interaction with students has been replaced with the interaction with a child. I have also included the categories of questions that emerged from my study, probing, prompting, and prodding, explicitly into this bubble. These question categories merged effortlessly into Simon's MTC. Like the teachers addressed in Simon's model, the PSTs were inquiring into their buddies' mathematics. Since the purpose of the PSTs' probing questions was to learn about their child's mathematics, I included these questions in that particular aspect of the MTC for preserivce teachers working one on one with a child. In the adapted version of Simon's MTC I changed the box originally titled Facilitation of Discourse to Keeping the Child Engaged and Interactive Constitution of Classroom Practices to Eliciting Strategies within the Child's ZPC.

The Hypothetical Learning Trajectory in Simon's MTC was not altered greatly except that rather than working with students, the preservice teachers were working with a child. However, their goals for their children's learning came out of their initial interviews with their buddies and their knowledge of children's learning of particular content which was presented to them in the methods class. Like Simon's MTC, the PSTs' plan for learning activities came out of their learning goals and their knowledge of mathematical activities and representations.

Ball suggests that a study is needed to look at how different contexts affect teachers' learning to listen to and interpret students (1997). My study looked at one such experience and, in this case, each preservice teacher was focused on hearing and interpreting the mathematical actions of one student. Perhaps this was a necessary step since doing this work in a class of twenty students seems daunting (Ball, 1997). While my study did not set out to look at the effectiveness of this context in helping teachers learn to listen to and learn from students and act upon their listening to enact appropriate instructional decisions, I believe that there is something special about the mathematics-specific field experience in this methods course that directed PSTs to listen to children, learn from their listening, and act on their learning.

Ball (1997) suggests that teaching has traditionally been an isolated endeavor and that teachers need to be involved in creating a professional discourse and sense of community around the task of learning to listen to and learn from students. She further asserts that this kind of work can build and extend teachers' knowledge and resources that could assist them in determining what students know and how to look at students' work. Thus, field experiences for preservice teachers and professional development for inservice teachers need to focus on making sense of students' mathematics and using this sense to inform instructional decisions.

Doyle (1990) suggests that teacher education is most productive when it allows teachers to construct knowledge by "fostering reflective capacities of observation, analysis, interpretation, and decision making" (p. 6). I found that my participants engaged in all of these reflective practices; They each posed problems to a child, observed the child's mathematical actions, interpreted those actions, and then made instructional decisions based on their interpretations. This mathematics specific field experience was a valuable experience where teachers were learning to use children's mathematics as a basis for instructional decisions. They were engaged in learning the task that Ball (1997) suggests is an essential task of teachers.

Many field experiences have been designed so that preservice teachers learn to "slip easily into the teacher's role" (Doyle, 1990, p. 5). In these field experiences, preservice teachers typically start out observing a teacher, and then slowly take on the roles of the teacher. These "apprenticeship experiences" serve to socialize a preservice teacher into the current practices of teaching (Doyle, 1990, p. 5). Traditional field experiences tend to perpetuate the traditional forms of teaching where mathematics is seen as something teachers teach from a text book, "a priori to mathematics learning and disembodied from learning" (Steffe, 1991, p. 188). Mathematics for children is too often something that curriculum developers make decisions about, typically without a notion of the mathematics of children. Thus, field experiences must be carefully constructed in order to give teachers opportunities to focus on the mathematics of students and should give preservice teachers the opportunities to redefine their views of teaching mathematics.

Questions Raised Leading to Further Research

In addition to answering my research questions, my analysis led to the generation of more questions. The questions that still remain regard the participants' surprises, listening, and adaptations to whole class settings. These questions are discussed in the sections to follow. The questions suggest implications for further research.

Surprises

I would argue that every preservice teacher going into a mathematics specific field experience has some expectation of what a child in that age group should and should not be able to do in mathematics. It is always interesting to me when they are surprised by what they see a child do. I believe that surprises like these are essential to constructing a working model of a child and may be a crucial part of the perturbation required in learning children's mathematics. No matter how versed we are in existing constructivist models of children's mathematics, we should always be looking for anomalies. I believe this to be a crucial element of mathematics education research. My participants indicated that they were surprised many times during their interactions with their children. For instance, Emma indicated surprise at the flexibility with which Matthew engaged in problem solving. On the final portfolio she further indicated, "At every session, he did something new that astounded me."

While I did have evidence that each participant was often surprised by her buddy's mathematical actions, I do not know how the surprises influenced the working model of each child's mathematics. Some questions that I have concerning their continued indication of surprise are:

1. How did their surprises affect their model?
2. How did their model affect their surprises?
3. Did they readjust their model in response to their surprises?
4. Did their surprises cause perturbations in their current model which led to accommodations in their model?
5. Could they simply assimilate their surprises into their current model?

I think that these are important questions to explore, but my current study is limited in that it did not help me answer these questions. Perhaps future research could focus more explicitly on these questions.

Listening

Although I've made claims about the type of listening in which the participants were engaged, I think more research needs to be done in terms of how teachers' questions relate to their listening. Davis (1996) claims that, unless teachers are asking hermeneutic questions, they cannot possibly be engaged in hermeneutic listening. By hermeneutic questions, he refers to questions for which the questioner doesn't have an answer in mind. My participants did ask questions to genuinely seek information from their buddies; however other questions, such as prompting questions, were asked in order to stretch their buddies' mathematics in a manner in which they believed to be consistent with strategies in their buddies' ZPC.

My claim that they were prompting particular methods and yet listening hermeneutically is based on D'Ambrosio's (2004) suggestion that listening hermeneutically is the type of listening required in constructivist teaching. She claims that constructivist teaching requires the teacher to construct models of their students' mathematics, using those models as a basis for instruction and that teachers who are able to listen hermeneutically can renegotiate their own mathematics and not be limited by their own understanding of the discipline. Davis (1996) suggests that there is no value in asking a question for which the questioner has an answer in mind or that is intended as an end to learning. I believe this assumption needs to be questioned in order to justify why a teacher would determine learning goals for her students when she never intended to question them with those learning goals in mind.

Adaptations to Whole Class Setting

All participants displayed evidence of reluctance in trying to use a constructivist approach with a whole class setting. For instance, on the final portfolio, Samantha stated:

> Something else I really struggled with over the semester was how to incorporate this less-traditional constructivist approach into a whole-class teaching situation. Thinking about how to apply this approach to a whole-class situation was overwhelming when looking at how much thought and energy was put into just one student (my School Buddy). I struggled with exactly how to use this approach in my future classroom.

Often, studies around constructing models of students' mathematics are done in a teaching experiment, which is typically a context in which a teacher researcher is working with one student. However, Steffe (1991) states that "choosing to work in these laboratory conditions should not be construed to mean that I view constructivism as being restricted in its implications to teaching individual children" (p. 179). Since it is not clear how the current findings extend to a whole class setting, one possible focus for future study would be to investigate how teachers might construct working models of students and use those working models in a whole class setting.

Conclusions

I set out to study the working models of preservice teachers and how their models informed their instructional decisions. The PSTs of this study were able to construct and use their working models of their buddies' mathematics. They asked different types of questions: probing, the purpose of which was to inquire about their buddies' mathematics; prodding, for which the purpose was to keep their buddies acting mathematically; and prompting, for which the purpose was to elicit particular strategies that the PSTs thought were in their buddies' ZPCs.

From these findings I concluded that the PSTs were listening both interpretively and hermeneutically. This finding contradicts previous claims that PSTs are not likely to engage in these types of listening (D'Ambrosio, 2004). These findings also contradict Davis's contention that only hermeneutic questions, which are questions for which the teacher doesn't have particular responses in mind, are associated with hermeneutic listening. I claim that even though the PSTs asked questions to prompt particular strategies, they were still engaged in hermeneutic listening because the strategies were chosen on the basis of their working models of their children's mathematics. This claim is consistent with D'Ambrosio's description of hermeneutic listening as the type of listening in which constructivist teaching is aligned.

One of the most significant findings of my study is that through this mathematics specific field experience, the PST not only learned mathematics, but they were able to redefine their notion of what it means to teach mathematics in a way that is consistent with what NCTM endorses as reform oriented teaching. The PSTs' development through this field experience is an example of a qualitative reorganization of their understanding (Goldsmith & Shifter, 1997). Goldsmith and Shifter (1997) describe a qualitative reorganization of understanding as the learner's construction of increasingly complex cognitive structures, "resulting in the capacity for more complex thought and action" (p. 22). They suggest that teacher education should be set up so that reorganizations of understanding are possible, where PSTs can reorganize their thoughts in ways that allow them to become better mathematicians as well as redefine teaching mathematics so that it is less about "transmissibility of knowledge" and more about helping students build on what they currently understand (p. 23). In my study the PSTs' reorganization of understanding can be represented by Maria's description on her final portfolio:

> I had never tried to actually introduce and teach any mathematical concepts before, and I must admit I was afraid that I would not be able to do so. However, as I began working with Ansley, I started to understand that the explanations would not be coming from my mouth; rather they would be created and comprehended within Ansley's mind. My job would be to create the appropriate environment for learning and to pose problems that would lead to Ansley's construction of knowledge. I would be a guide instead of a dictator. I would be student and teacher to the child.

REFERENCES

Ball, D. L. (1997). What do students know? Facing challenges of distance, context, and desire in trying to hear children. In B. J. Biddle (Ed.), *International Handbook of Teachers and Teaching* (Vol. I, pp. 769-818). Netherlands: Kluwer Academic Publishers.

Ball, D. L., Lubienski, S., & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (Fourth ed., pp. 433-456). New York: Macmillan.

Borasi, R., & Fonzi, J. (2002). Analyzing students' thinking. In *Professional development that supports school mathematics reform* (Vol. 3, pp. 53-66). Arlington, VA: National Science Foundation.

Carey, D. A., Fennema, E., Carpenter, T., P., & Franke, M. L. (1995). Equity and mathematics education. In W. G. Secada, E. Fennema & L. B. Adajian (Eds.), *New directions for equity in mathematics education* (pp. 93-125). Cambridge, MA: Cambridge University Press.

Carpenter, T., P., Fennema, E., Peterson, P. L., Chiang, C.-P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*(1), 499-531.

Cobb, P., & Macclain, K. (2001). An approach for supporting teachers' learning in social context. In F. L. L. T. J. Cooney (Ed.), *Making Sense of Mathematics Teacher Education* (pp. 207-231). Boston: Kluwer Academic.

D'Ambrosio, B. S. (1995). Highlighting the humanistic dimensions of mathematicsactivity through classroom discourse. *The Mathematics Teacher*, *88*(9), 770-772.

D'Ambrosio, B. S. (2004). Preparing teachers to teach mathematics within a constructivist framework: The importance of listening to children. In *AMTE Monograph 1: The Work of Mathematics Teacher Educators* (pp. 135-150).

D'Ambrosio, B. S., & Campos, T. M. M. (1992). Pre-service teachers' representations of children's understanding of mathematical concepts: conflicts and conflict resolution. *Educational Studies in Mathematics, 23*(3), 213-230.

Davis, B. (1996). *Teaching mathematics: Toward a sound alternative* (Vol. 7). New York: Garland Publishing, Inc.

Davis, B. (1997). Listening for differences: An evolving conception of mathematics teaching. *Journal for Research in Mathematics Education, 28*, 355-376.

Doyle, W. (1990). Themes in teacher education research. In W. R. Houston (Ed.), *Handbook of research on teacher education* (pp. 3-24). New York: Macmillan.

Driscoll, M. (1999). *Fostering algebraic thinking: A guide for teachers grades 6-10*. Portsmouth, NH: Heinemann.

Franke, M. L., Carpenter, T., Levi, L., & Fennema, E. (2001). Capturing teachers' generative change: a follow-up study of professional development in mathematics. *American Educational Research Journal, 38*(3), 653-690.

Goldsmith, L. T., & Shifter, D. (1997). Understanding teachers in transition: characteristics of a model for the development of mathematics teaching. In E. Fennema & B. S. Nelson (Eds.), *Mathematics Teachers in Transition* (pp. 19-54). Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Hackenberg, A. J. (2005). *Construction of Algebraic Reasoning and Mathematical Caring Relations.* Unpublished Dissertation, University of Georgia, Athens, Georgia.

Hays, P. A. (2004). Case study research. In K. deMarrais & S. D. Lapan (Eds.), *Foundations for research: Methods of inquiry in education and the social sciences* (pp. 432). Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Kamii, C., & Housman, L. B. (2000). *Young children reinvent arithmetic: Implications of Piaget's theory* (Second ed.). New York: Teachers Colleage Press.

Lesh, R., & Clarke, D. (2000). Formulating operational definitions of desired outcomes of instruction in mathematics and science education. In A. Kelly & R. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 113-149). Mahway, NJ: Lawrence Erlbaum Associates.

Ma, L. (1999). *Knowing and teaching elementary mathematics*. Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Malave, A., Howlett, B., & Collins, P. (1993). Cultivating new visions. In D. Schifter & C. T. Fosnot (Eds.), *Reconstructing mathematics education: Stories of teachers meeting the challenge of reform*. New York: Teachers College Press.

Mewborn, D. S. (1999). Reflective thinking among preservice elementary mathematics teachers. *Journal for Research in Mathematics Education, 30*(3), 316-341.

Mewborn, D. S. (2000). Meaningful integration of mathematics methods instruction and field experience. *Action in Teacher Education, 21*(4), 50-59.

Mewborn, D. S. (2005). *Frameworks for research in mathematics teacher education.* Paper presented at the Twenty-Seventh Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Blacksburg, VA.

NCTM. (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Paley, V. G. (1986). On listening to what the children say. *Harvard Educational Review, 56*(2), 122-131.

Patton, P. Q. (2001). Designing qualitative studies. In *Qualitative research and evaluation methods* (3rd ed., pp. 207-254). London: Sage Publications.

Pellegrino, J. W. (2002). *Knowing what students know*. Retrieved December 2006, 2006, from http://www.issues.org/19.2/pellegrino.htm

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Simon, M. A. (1997). Developing new models of mathematics teaching: An imperative for research on mathematics teacher development. In E. Fennema & B. S. Nelson (Eds.), *Mathematics teachers in transition* (pp. 55-86). Mahwah, New Jersey: Lawrence Erlbaum Associates, Publishers.

Simon, M. A., & Schifter, D. (1991). Towards a constructivist perspective: An intervention study of mathematics teacher development. *Educational Studies in Mathematics, 22*(4), 309-331.

Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning, 6*(2), 91-104.

Steffe, L. P. (1988, October, 1988). *Children's construction of meaning for arithmetical words: A curriculum problem.* Paper presented at the Conference on Implicit and Explicit Knowledge in Science and Mathematics, Tel Aviv University, Tel Aviv, Isreal.

Steffe, L. P. (1991). The constructivist teaching experiment: Illustrations and implications. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (Vol. 7). Dordrecht: Kluwer Academic Publishers.

Steffe, L. P., & D'Ambrosio, B. S. (1995). Toward a working model of constructivist teaching: A reaction to Simon. *Journal for Research in Mathematics Education, 26*(2), 146-159.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: underlying principles and essential elements. In A. Kelly & R. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 267-306). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application*. New York, New York: Praeger Special Studies.

Steffe, L. P., & Wiegel, H. G. (1996). On the nature of a model of mathematical learning. In L. P. N. Steffe, P. (Ed.), *Theories of mathematical learning* (pp. 477-498). Mahwah, NJ: Lawrence Erlbaum Associates.

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Thousands Oaks, CA: Sage Publications.

Tang, G. (2001). *The grapes of math: Mind stretching math riddles*. New York: Scholastic Inc.

TESOL. (1996). *Qualitative research: Case study guidelines*. Retrieved August 8, 2007, from http://www.tesol.org/s_tesol/sec_document.asp?CID=476&DID=2153&rcss=print&print=yes

Thompson, A. G. (1989). Assessing students' learning to inform teaching: The message in NCTM's evaluation standards. *Arithmetic Teacher, 37*(4), 22-26.

Thompson, A. G., & Thompson, P. W. (1996). Talking about rates conceptually, part II: mathematical knowledge for teaching. *Journal for Research in Mathematics Education, 27*(1), 2-24.

Thompson, P. W., & Thompson, A. G. (1994). Talking about rates conceptually, part I: a teacher's struggle. *Journal for Research in Mathematics Education, 25*(3), 279-303.

von Glasersfeld, E. (1990). Environment and communication. In L. P. Steffe & T. Wood (Eds.), *Transforming children's mathematics education: International perspectives* (pp. 30-38). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

von Glasersfeld, E., & Richards, J. (1983). The creation of units as a prerequisite for number: A philosophical review. In L. P. Steffe, E. von Glasersfeld, J. Richards & P. Cobb (Eds.), *Counting types: philosophy, theory, and application*. New York: Praeger Special Studies.

von Glasersfeld, E., & Steffe, L. P. (1991). Conceptual models in educational research and practice. *The Journal of Educational Thought, 25*(2), 91-103.

von Glasersfeld, E., Steffe, L. P., & Richards, J. (1983). An analysis of counting and what is counted. In L. P. Steffe, E. v. Glasersfeld, J. Richards & P. Cobb (Eds.), *Children's counting types: Philosophy, theory, and application* (pp. 21-44). New York, New York: Praeger Publishers.

Vygotsky, L. S. (1986). *Thought and language* (A. Kozulin, Trans.). Cambridge, Massachusetts: Massachusetts Institute of Technology.

APPENDIX A

SESSION INTERVIEW

For this part of the interview, I may play back parts of the video of the participant working with their “School Buddy,” pointing to particular moments in the session that I observed.

1. What led you to ask that particular question?

2. To what extent were you thinking about the mathematics of your student when you planned the problems and tasks in which to engage him or her?

3. How did you choose your progression of numbers?

4. On the initial interview assignment, you determined that your buddy was at the ____level of whole number development. Do you still agree with your initial assessment?

APPENDIX B

FINAL INTERVIEW

Pre-determined Questions:

1. One of the goals of the course, EMAT 3400 is to listen to and learn from children. How have you enacted this goal at School?

    Probes (Has this been a difficult process for you? How important do you find this goal to be for teachers? Why do you think it's important?)

2. A second goal of the course is to act upon your listening in order to design and enact appropriate activities for children. In what ways have you acted upon your listening in order to design and implement tasks for your School Buddy?

3. Based on the models of children's mathematics that you've read about and discussed in class, what is your sense of the mathematics of the student with whom you are working during the field experience?

4. Have you seen your "School Buddy" do anything that does not correspond with the literature from class?

5. Describe differences in the approaches you see your "School Buddy" take to solve the tasks you give and the ways you might approach the task as an adult?

6. In what ways have you been surprised by your "School Buddy?"

Questions developed later:

1. What did you work on the most with your School Buddy?
2. Can you come up with a problem that would be right on the edge of what your School Buddy is able to do?
    a. Why do you think this?
3. What is your School Buddy not doing that you wish (s)he would do?
4. Do you think you will continue this type of work?
    a. I.e. making and using working models of children's mathematics.

APPENDIX C

INSTRUCTOR INTERVIEW

1. I notice that your goals for your class include getting the preservice teachers to listen to and understand the mathematics of children. Why do you consider this an important goal?

2. How did you choose the articles that you had your students read?

3. How would you describe the framework for your course?

4. How did you prepare the preservice teachers to engage in the teaching experiments at the School?

5. What is your sense of how well your students are doing at formulating models of the children with whom they are working? (What about my four participants?)

6. Another goal that you list in your syllabus is that preservice teachers will use their models to design and implement instruction at The Elementary School. What is your sense of how your students as a class are using their models at The Elementary School? (What about my four participants?)

7. How would you describe the ways in which you use the models you form as you're working with children to inform your instruction with those children?

8. As an instructor, how do you assess your preservice teachers' assignments if their interpretation of what they see children do differs from your own views?

9. What kind of feedback do you offer to help guide the preservice teachers in their effort to construct models of children's mathematics and to design tasks correlated to these models?

APPENDIX D

COURSE SYLLABUS

EMAT 3400: Children's Mathematical Learning Fall 2005, MW, 8:00-9:55 am, rm 102

Instructor: Dr. H. Office:

Overview

This course is the first of two mathematics education classes you will take in the Early Childhood Program. In this course, you will have classes at UGA and a field experience at The Elementary School—you will meet with one elementary school student for out-of-class activities. During our class meetings at UGA and our activities at The Elementary School we will focus our work on *learning how children learn mathematics*. Although we will talk about classroom ideas and teaching practices, *this course concentrates on children's mathematics*.

Throughout this course, you are expected to combine our discussions in class and the experience at The Elementary School to rethink your own knowledge of mathematics, as well as the learning and teaching of mathematics during the first years of schooling. In particular, you should make an effort to *look through the eyes of children, trying to understand how they generate mathematical ideas.*

We will discuss questions such as: How do we learn mathematics? How do children solve problems in mathematics? How do children construct mathematical concepts? We will address these questions mostly in the context of learning number and reasoning with numbers: counting, addition, subtraction, number sense, multiplication, division, and fractions. As time permits, we will also investigate decimals, percents, ratios, and proportions.

Goals

The goals of this class are for you to:

1. Learn to listen to and learn from children. This goal includes coming to see how capable children are of learning mathematics and solving problems; respecting children's mathematical thinking even when you do not understand it; and allowing what you learn from children to influence your own mathematical thinking and your teaching.
2. Act upon your listening in order to design and enact appropriate activities for children. "This goal includes developing an understanding of how children make changes in their mathematical thinking; engaging in interactions with children that focus on conceptual understanding rather than rote memorization; and increasing your understanding of children's mathematical thinking in contrast with adult mathematical thinking.
3. Work actively to develop greater awareness of your own mathematical thinking. This goal includes rethinking your conceptions about the nature of mathematics; coming to understand aspects of your own mathematical thinking that you take for granted; and developing a habit of searching for foundational ideas in your own mathematical activity.
4. Participate in discussions about a wide range of current issues in mathematics education and research on children's mathematical learning. This goal includes co-creating an environment that is safe for students (both elementary school children and your classmates) to share and justify their thinking, ask questions, makes conjectures, and take risks. This goal also involves thinking about ways to design your future classroom to facilitate mathematical learning for all students.

*****Some of the most important "methods" you have as a teacher of mathematics are**
(a) your own evolving, creative mathematical thinking; and
(b) your evolving understanding of your *students'* mathematical thinking and how it can change through engaging students in productive mathematical activity.***

Required Texts

Course packet from Bel-Jean Copy/Print Center, 163 East Broad Street, 548-3648. ($14.70)

Kami:. C. &. Housman, L. B. (2000). *Young children reinvent arithmetic* (2nd ed.). New York: Teachers College Press. (Available at the UGA Bookstore or on-line. ISBN #0-8077-3904-9. Amazon.com lists the price new at $23.95 and used starting from $19.99).

Optional Text

Van de Walle, J. A. (2001). *Elementary and middle school mathematics: Teaching developmentally* (4th ed.). New York: Longman.

Attendance

Attendance and participation are essential in this class, both for you to learn and so that others may benefit from your input. Attendance is expected because most of class time will be spent on group discussions and activities. The ideas and concepts we work on cannot easily be built up through class notes. You are responsible for all announcements made in class even if you are not there. It is important that you arrive promptly (especially when we are at The Elementary School). Absences and tardiness will affect your grade.

Assignments

I will try to make the purpose of each assignment clear. If you have questions about the purpose of the assignment or what is expected of you, please ask. Late assignments will be assessed a penalty of one grade level. You are expected to demonstrate correct use of the English language with regard to grammar, punctuation, and spelling. I do grade on technical writing skills as well as content. Please proofread your work before turning it in to me. If you have weaknesses in the area of grammar, punctuation, or spelling, find someone who will proofread your work for you before you turn it in to me. It is expected that you will do your writing assignments (i.e., not mathematical problems) on a word processor. Any exceptions must be cleared with me in advance. Assignments that are not typed will be returned without a grade. **I would prefer that you turn in most writing assignments (i.e., not most problem** sets) **via the electronic course folder as I will explain in class, or send them to me as an email attachment. Label each assignment with your last name and the assignment number. For example, to turn in the counting analysis assignment, I would name the file "DRH3."**

University policies

All university policies with regard to withdrawals, early final exams, academic honesty, etc. will be strictly followed. It is your responsibility to be familiar with these policies.

ASSIGNMENT DESCRIPTIONS

1. MATHEMATICAL PROBLEMS

One of your greatest assets in understanding children's mathematics is understanding and deepening your own mathematical thinking, as well as your awareness about your mathematical thinking. Therefore, part of this course is about doing mathematics, generating mathematical conversation, and reflecting on your own mathematical knowledge. We will work on several mathematical problems over the course of the semester. **You do not need to type these assignments.** Please complete them legibly on paper. In some cases, you will turn in electronic files depending on the mathematical area we are exploring.

2. READINGS

For discussion in class, I will ask you to read articles from the course packet, from the required text, and from supplementary texts. In many cases I will ask you to do some informal writing in association with preparing a reading. In a few cases, I will ask you to write a more formal, polished commentary on or analysis of an article either in preparation for class or after class discussion. **All writing in response to readings should be typed.**

3. COUNTING ANALYSIS

Based on our class discussions and readings on how children count, analyze the counting activity of three different students (as provided in class). Your analysis should include what you can infer about the students' mathematical thinking, what kinds of counting activities you think would be appropriate for the child, and a rationale for why you have selected or designed these activities.

4. STUDENT INTERVIEW

On the first day at The Elementary School (probably Wednesday, September 28), you will interview your student to learn about her or his strengths and areas of potential development in mathematics. The purpose of this assignment is to provide you an opportunity to reflect on what you learn from the interview. Write a summary of the interview you conducted. The review should contain the following information:

General Information

- Your name
- The name, age, and grade of the student you interviewed
- The teacher's name
- Any pertinent information about the child you would like to mention

Your analysis

- Include all of the mathematical problems you posed and a brief summary of the child's response. Say more than 'The child solved the problem correctly." Explain how the child solved the problem or what the child said to indicate that she or he could not solve the problem. Some children will not be able to explain how they solved a problem. If this happens, simply indicate this in your summary. Note any behaviors you see the child exhibiting such as counting on fingers or moving lips.
- What did you learn from this experience? Did anything surprise you?
- What, if any, implications does interviewing have for instruction?

Note: Avoid evaluative statements about the child, such as, "she was really smart" or "he seemed slow." You do not know enough about the child to make such statements, and besides, those statements do not provide any information. Instead, provide details, such as, "When I asked her how many marbles she has if she started with 8 and her friend gave her 9 more, she solved it by saying '8 and 8 is 16, and one more is 17.' I thought that was neat because I would not have expected a child to do that," or "I asked him this question and he just looked at me. I asked him if I should repeat the question, and he said 'no.' I did not know how else to reach him."

5. CASE ON SCHOOL BUDDY

Describe a pedagogical dilemma you encountered while interacting mathematically with your School Buddy, and write it in a format similar to the cases we have read in class. The case should be approximately 2 pages long and should provide readers with enough detail so that they feel that they have personally experienced your dilemma. Use a pseudonym for the child's name. Your case should be "open" (i.e., not resolved). Bring *three* copies of your case to class on the due date. You will turn one copy in to me and will give the other *two* copies to peers who will read your case and provide feedback.

6. FEEDBACK ON PEER CASES

You will receive two cases from peers on which you will provide feedback. For each case, provide a 1-2 page reaction to the case, giving suggestions for ways to resolve any problems or dilemmas in the case. Remember to address the data in the case rather than telling your own story in your feedback. Bring two copies of your feedback to class on —one for your peer and one for me. (You may write comments directly on the case you were given. However, you .should also type a 1 -2 page summary of your comments, ideas, and suggestions.)

7. LEARNING TRAJECTORY

Write a learning trajectory in essay form about moving from one mathematical topic to another with students. Use topics that we have worked on this semester so far. As we will discuss in class, formulating a learning trajectory can be the basis for writing many lesson plans, but a learning trajectory is not a plan for a single lesson —instead, it's a big picture view of your goals and ideas. In your trajectory, describe how you will assess informally what your students know when you start and describe ideas for problematic situations that you will pose to different groups of students, based on your assessments, so that they might engage in productive mathematical activity. Be sure to justify your choices of problematic situations based on what mathematical thinking you are working to bring forth in your students.

I will ask for a draft of your learning trajectory, give you feedback, and then ask you to revise it into a final draft. I encourage you to get feedback from your peers on your drafts and revisions.

8. FINAL PORTFOLIO

"The purpose of this assignment is to give you a chance to reflect on your growth over the semester and on the growth of your School Buddy. You may be as creative as you wish in designing your portfolio. However, remember that I am much more interested in the *substance* of what you have to say than the format in which you package it. So put most of your effort into the content of the portfolio. Your portfolio should show evidence of reflection and analysis on the semester. Do not simply create a "scrapbook" in which you tell a chronological story of your semester. I *will give some guidelines for this assignment later in the semester.*

9. ACTIVITY REPORTS

For each session at The Elementary School you will need to prepare an activity report. Your activity report should consist of a description of the general objecti ve(s) of the lesson, any activities that you used, any problems or successes the child had, and a brief description of what you plan to do in the next session. These activities should be described in enough detail that the classroom teacher can figure out what you did. Put most of your emphasis (both in effort and in writing) on analyzing the child's understanding of the concepts you were addressing. A sample activity report will be distributed to the class. **Your activity reports will be copied and given to the classroom teachers so that they can keep track of what their students are doing.** 1 will also make comments on your activity reports to provide you with suggestions for future sessions. In order for these comments to be useful to you in planning your next teaching session, I need to be able to read them and give you feedback before your next teaching session. Therefore, activity reports **must** be turned in to me at The Elementary School before you leave the school.

Please remember that you are a beginning teacher education student and that the comments on your activity report should be appropriate to your level of expertise. You are not qualified to label or diagnose a child as "LD," "BD," "dyslexic," "hyperactive," "gifted," etc. Do not use such labels or other judgmental words in your activity report. Please remember that you are seeing this child for a very short period of time, which amounts to only a fraction of the time that child spends in school. You are also seeing the child in a highly specialized context, so the behavior (social or academic) the child exhibits with you may not be typical of her or his behavior during the rest of the school day. If you notice that your child is having some difficulty (or success) during your sessions, please describe the child's actions as carefully as possible WITHOUT using labels such as the ones listed above. For example, rather than writing "Joshua appears to be dyslexic," write "Joshua consistently writes his 3's and 7's backwards. He generally writes his other numbers correctly. Sometimes he also makes his J's backward when he writes his name." Your goal is to accurately describe what the child is doing, not to diagnose any learning or behavior problems the child may have. It is appropriate for you to include questions on your activity report such as, "Should I correct Joshua every time he writes a letter or numeral backwards?"

At no time should your activity report make any judgmental comments about the child's classroom teacher. It is not your place to question the teacher's methods, curriculum, assignments, or comments about a child. You will observe very little of the classroom teacher's practices, so you will not be in a position to comment on them.

Grading

In general, I grade with rubrics, although I will use a point system for the final exam. The assignments listed above have this approximate weight in your final grade:

Assignment	Approximate Weight (percentage)
1. Mathematical Problems	10
2. Writings on Readings	10
3. Counting Analysis	10
4. Student Interview	5
5. & 6. Case on School Buddy and Feedback on Peer Cases	10
7. Learning Trajectory, initial and revised drafts	10
8. Final Portfolio	15
9. Activity Reports	No grade
Professionalism	5
Final Exam (Monday, 12/12, 8:00-1 1:00 am)	25
Total	100

Your grade for Professionalism will be based on your punctuality and preparedness for class and The Elementary School teaching sessions, your class participation (which includes both your contributions and your reactions to the contributions of others), your response to constructive feedback in the classroom and at The Elementary School, and your demonstration of a professional demeanor (dress, language, attitude) toward others (professors, assistants, classroom teachers, peers, children).

APPENDIX E

PARTICIPANTS' INITIAL INTERVIEW PROTOCOL

Student Interview Assignment

To remember during the interview:

- Set a friendly tone
- Pay close attention to child (How she/he seems to be feeling - what she/he may be thinking - what she/he seems to be doing)
- Take *mental* notes on both child and yourself—you can jot written notes too, but don't let that distract from interacting with the child
- Consider wait time (You may be uncomfortable with silence, but the child may not be)
- Don't assume child wants question repeated - ask or offer to repeat the question (You can also ask the child to tell you what problem they are trying to solve)

Materials you will need to bring:

- paper and pencils-for yourself to jot, and for the child to draw or write
- 1-2 large plastic cups (that you cannot see through) and a paper napkin or cloth
- Rectangular candy bars on paper (cut these apart) and sheets of paper with pictures of string or licorice (see last four pages); these all go with #27-29.
- You will be able to get about 60 unifix cubes from the cart at The Elementary School. Pre-make about 5 stacks of 4 cubes (4-rods) to use with #15.

Basics to get started:

- Set child at ease: Ask the child's name and introduce yourself.
- Explain that this is an assignment for your class. Tell the child that o This will not be graded. o Some questions will be easy, and some might be hard—that's OK. Just do the best you can. o I am interested in how you are thinking, not how many correct answers you get. Throughout our time together, I will be asking "How did you get this?" or saying "Huh —show me how you did that."
- **Be sure to probe but also accept child's responses—try to communicate a sense that the child is doing fine even if you are surprised or shocked by her/his responses.**

Do not feel you must pose every question. If a child struggles with some questions, it would make sense to skip more difficult questions. I have made some notes to that effect on many of the questions, but you will have to make some assessments as you go along.

Sample Interview Questions

Counting:

1) *(Goal: assess counting on—is the child numerical?)*
Ask the child to put some number (small, say 7) of counters into their cup. Now tell the child you are giving her/him 8 more counters, and put the counters (not one-by-one—do it as a handful) into the cup. How many counters do they have now? Ask how she/he knows —how did she/he figure it out. "Did you count?" If the child counted on her/his fingers but seem "ashamed" to show you. encourage the child that it's okay —You'd love for her/him to show you exactly what she/he did, how she/he counted. *Note: If the child does not count on, do another similar problem with smaller numbers. If the child still does not count on, pose some more problems with one collection, part of the collection covered with a paper napkin, to try to assess the child's level of counting. It is unlikely that 3rd graders will be pre-numerical, but it is possible.*

2) *(Goal: assess counting off v. counting down)*
Ask the child to put some number (a little larger, like 19) of counters into her/his cup. Now say, something like: "okay, now you give me three counters from your cup." How many counters do you have now? (Don't let the child just count the remaining counters.) Again, try to elicit what she/he did.
Note: If the child can't solve this problem, ask a similar problem but have her/him give you only I counter from the cup.

3) *(Goal: assess counting off or down when going over a decade number)*
If the child solved #2, ask the child to put a larger number like 28 counters in the cup. Then you take out a relatively large number, like 9 counters. Ask the child how many counters they have now. As always, ask her/him to tell or show what she/he did, unless it is obvious to you.
Pay attention to whether the child has trouble counting back past 20.

4) *(Goal: assess counting off v. counting down with a missing addend problem)*
Ask the child to put some number (like 21) of counters into her/his cup. Now say: okay, I'm taking out some counters (take some out, like 5, but don't tell them how many). Tell her/him that there are 16 counters left in the cup. How many did I take out?
Note: Again, pay attention to whether the child counts on or off or down to in order to solve the problem, or whether the child uses strategic reasoning.

5) *(Goal: assess their number word sequence)*
Can you tell me what number comes after 12? What about after 19? After 27? After 73? After 100? Can you tell me what number comes *before* 59? What number comes before 40? Do you know what number comes before 200? What about after 200? How high can you count? Ask what number comes after and before the highest number they say. *Note: Adjust numbers from my suggestions—go high or low depending on your sense of the child's thinking at this point. Some 3rd graders know the number word sequence up to WOO!*

6) *(Goal: assess their number word sequence when skip counting)* Can you count by 2s? 3s? 5s? 10s? 100s? (Have the child show you. See how far he/she can go. You can also ask the child to count backwards by 10s or 100s, if it seems appropriate.)

Addition and Subtraction Problems:
In all of the following, try to assess whether the child is limited to counting on (INS), or can reason .strategically, or is in between (INS+). If the child counts all or counts on to solve the problem, you can ask her/him if she/he can solve the problem another way. You may want to adjust numbers up or down in some problems, depending on your assessment of the child's thinking so far. If the child seems stumped, remind him/her that counters are available to use.

7) Karen has 7 grapes. Her mom gave her 6 more grapes. How many does she have now?

8) Eric has 4 toy cars. He got 9 more toy cars on his birthday. How many toy cars does Eric have now?
Note: Pay attention to whether the child switches to start with the larger added even though the smaller addend was said first. Note that #7 and 8 are both "two collection addition problems," according to Steffe & colleagues.

9) Todd has 14 toy cars. He gives 5 toy cars to Julie. How many toy cars does Todd have left?
Note: Pay attention to whether the child counts off or counts down. Kamii calls this a separating problem.

10) Susan has 12 gummy bears. She gave some to her friend and now Susan has 7. How many did she give to her friend?
Note: This problem is a harder variation on the separating problem above, sometimes called a "missing subtrahend problem."

\ 1) Mary has 5 marbles but she needs to have 13 to play a game. How many more marbles does she need?
Note: Many people call this a missing addend problem; Kamii calls it an equalizing problem.

12) Mike has 9 marbles and Lara has 4 marbles. How many more marbles does Mike have than Lara?
Note: Kamii calls this a comparing problem.

13) Marcus has some toy dinosaurs. He goes to the store and buys 3 more toy dinosaurs, and then he has 9 toy dinosaurs. How many toy dinosaurs did Marcus have to start with? *Note: This is a (harder) variation of a missing addend problem. Skip it if you think it will be too hard.*

14) Beth has some apples. She gives 3 apples to Katie and then she has 4 apples left. How many apples did Beth have before she gave 3 apples to Katie?
Note: This is a harder variation of a missing subtrahend problem. Skip it if you think it will be too hard.

Problems that involve the Beginnings of Multiplication and/or Division
Note that all of these problems can be solved with addition or subtraction or partitioning of numbers. We have not discussed multiplication and division in detail, so just be curious and observe what happens! If you feel your child is up to it, you can pose harder versions of these problems as noted.

15) *(Goal: Assess whether the child can track groups of items)*
Ask the child to make two stacks of 4 unifix cubes each (4-rods). "How many cubes in those two stacks?" Put down another 4-rod in front of the child. "How many cubes in all of those stacks? How many stacks?" Keep going, asking the child to say the number of stacks and the total number of cubes as you continue to put down 4-rods.
Note: Observe whether the child counts by Is, 4s, or reasons strategically. If the child is counting by Is, ask if she/he can do it another way. If you think this activity is easy for the child, try putting down 2 4-rods at once or taking away a 4-rod (or 2 four-rods!).

16) *(Goal: Assess whether the child can make groups of items from a collection)* Give the child 9 counters. Ask the child how many stacks of three cubes she/he can make. If the child is successful, put the three stacks and three more loose cubes under the napkin and tell the child you are doing so. Ask the child if she/he can find out how many stacks of three cubes she/he can make *without looking]*

NOTE 1: If she/he *can't* figure that out, let her/him look. Then try this: Start with 6 cubes and ask how many stacks of two cubes she/he can make. Then put 8 cubes under the napkin and ask how many stacks of two she/he can make. Continue with stacks of two in similarly to what is described below in NOTE 2.

NOTE 2: If she/he *can* figure out the original problem, slide three more cubes under the napkin. See if the child can track how many cubes are under the napkin and how many stacks of three. Repeat. If the child finds this easy, slide *six* more cubes under the napkin, and again ask the child how many cubes and how many stacks of three cubes. If things are going well, go up to thirty cubes under the napkin. Then take away three cubes and ask how many cubes and how many stacks of three cubes.
Note: If the above seems too easy for the child, use something harder, like stacks of 6. In general, stacks of 2 and 5 are the easiest, then 3 and 4, then 6, 7, 8, 9.

17) A pack of gum has 5 pieces. You have 3 packs of gum. How many pieces of gum do you have in all?
Offer the counters or cubes as something the child might use if she/he is stumped. Harder version: A package of pencils contains 4 pencils. How many pencils are there in 7 packages?

18) 20 children in a 3rd grade class are riding to the park in cars. Each car has room for only 4 children. How many cars do they need to drive all 20 children to the park?

Offer the counters as something the child might use if the child is stumped. Harder version: 42 children are forming teams to play games on the playground. Each team has 6 people on it. How many teams can they make with all 42 children?

19) There are 18 M&Ms in a dish on the table. Three children want to share the M&M's fairly. How many M&Ms does each child get?
Offer the counters as something the child might use if the child is stumped. You probably won't need a harder version for this problem, as it is usually quite challenging for most elementary school children.

Place Value:
20) Ask the child to make some ten-rods with you (make a total of five). Put out four ten-rod and three loose cubes, and have the child check to make sure they each have ten cubes (e.g., visually they can usually check). Say something like: "OK, so there are ten cubes in each rod; can you tell me how many cubes there are altogether counting all the cubes?" (Sweep hand over entire collection of rods and cubes.)

Put 1 more 10-rod down: "Watch what I do. I'm putting ten more here. Now how many are there?"
Note: If this seems too easy, ask the child to predict how many more will there be if you put down 1 more ten-rod. You can also take away ten-rods and see if the child can go backwards.

21) Monday Anna played Nintendo for 20 minutes before school and 10 minutes in the evening. How many minutes did she play Nintendo on Monday? *Note: Does the child count on by Is or by IDs? This is key.*

22) How many tens are there in 32?
Note: If child doesn 't know, suggest using the counters or cubes to help. If the child still doesn 't know, skip the remaining questions in this section. Instead, put down 2 cubes. Ask the child "How many cubes?" Put down a ten-rod and ask the child: What's 10 more than 2? The child may have to count by Is to figure it out. See if the child can count by 10s to figure it out. Keep going as in #15 and, if things go well, eventually start taking 1 ten-rod away at a time. If the child does know, try other numbers: 75, 98, 120, 100.

23) How many tens are there between 30 and 40? How many tens between 20 and 50? Between 35 and 55? How did you know that!? How many tens between 22 and 42? Between 27 and 67? How do you know?

24) There were 28 girls and 35 boys on the playground at recess. How many children were there on the playground at recess?

25) Misha has 34 dollars. How many dollars does she need to earn to have 57 dollars?

26) Andrea had 70 rocks in her rock collection. When she brought her collection to school, she lost 26 rocks. How many rocks does she have in her collection now?

Fractions *(if you don't get to this, don't worry!)*

27)

Imagine that this is a picture of a candy bar. Can you make a mark to show me how to share this bar with two people so that each person gets a fair share —the same amount? After the child draws a line somewhere, ask the child if she/he is sure that it's fair. If the child wants to adjust the mark, give a new bar. When the child is satisfied, point to one out of the two pieces and ask what the child would call this piece.

Repeat with a new bar for 3 people, then 4 people. If successful, try 5 people.

28) Here is a picture of my string. Your string is three times longer than mine. Can you draw your string? Show me how you know your string is three times longer than mine.

29) Here is a picture of my licorice. My licorice is two times longer than yours. Can you draw your licorice? Show me how you know my licorice is two times longer than yours. *Note: Probably the child will interpret this problem to be the same as in #28—i.e., by drawing a line twice as long as the one shown. If so, don't worry. Probe a little, stressing that MY licorice is TWO times longer than YOUR licorice. See if the child thinks she/he has solved the problem. Then let it go.*

SEE SYLLABUS (pp. 3-4) **FOR INFORMATION ON STUDENT INTERVIEW ASSIGNMENT WRITE-UP.** The assignment is #4 on the list and should be emailed to me **by** 2 **pm on Monday, October 3rd** (please name the document "Iastname4" and please write no more than 7 pages double-spaced.) *Do not worry if you did not get to all the problems listed on this interview guide.* Just focus on what you did work on with the child. I have requested that you avoid evaluative statements about the child (see p. 4). However, in your response to the question "What did you learn from this experience?" (see p. 3), make some comments about your current assessment of the child's numerical level based on your observations and interactions. If aren't sure, make some comments about why you aren't sure.

APPENDIX F

MATHEMATICS INSTRUMENT

Questions for Student Teaching Reflection

1. Consider the following similes. Choose the response that best describes your thoughts and explain.

   a. Learning mathematics is like:

   Working on an assembly line
   Cooking with a recipe
   Working a jigsaw puzzle
   Building a house
   Working as an apprentice
   Other ________________
   Watching a movie
   Picking fruit from a tree
   Conducting an experiment
   Creating a clay sculpture
   Working on a corporate project team

   b. A mathematics teacher is like:

   News broadcaster
   Doctor
   Gardener
   Missionary
   Other ___________
   Entertainer
   Orchestra conductor
   Coach
   Social worker

2. The purpose of school mathematics is to enable students to be able to:

   Function in society
   Reason mathematically
   Do well on standardized tests
   Solve mathematics problems
   Function (or excel) in college mathematics courses
   See mathematics as a connected whole
   See the beauty in mathematics

3. Which of the following best describes your view of mathematics? Explain.

   A set of rules and truths
   An unquestioned body of useful knowledge
   A body of structured, pure knowledge
   A personal experience
   A changing body of knowledge that is constructed socially

Consider the following continuum. As far as mathematical importance, put an X on the continuum where you believe the importance or emphasis should be in mathematics teaching. Then put an * where you placed importance or emphasis during student teaching. Compare. Explain.

|____|
skills   concepts

4. Two students were having a discussion in the hall. With whom do you most agree? Explain.

   Student A: Mathematical knowledge might be incomplete, dynamic, fallible, and not universally or objectively true. Mathematics is constructed by man, and is based on certain assumptions of truth.

   Student B: Mathematics is out there. It has always existed. It is absolutely true. Mathematics was, is, and shall be. It was here since the beginning of time, and man has discovered it and used it for our benefit.