Pricing of Adjustable Rate Mortgages Subject to Prepayment and Default Risk

by

Jinbao Tong

(Under the direction of James B. Kau)

Abstract

In this paper, a reduced-form approach is adopted to price Adjustable Rate Mortgages subject to prepayment and default. The reduced-form approach to valuation facilitates the utilization of real life data in asset pricing. A given ARM is priced using the risk-neutral pricing principle, incorporating both default and prepayment risks. In the process, the default and prepayment risks are assumed to be exogenous and empirically estimated. The key feature of the model in this paper is the way the default and prepayment risks are modelled. Going beyond the conventional way of treating prepayment and default risks as deterministic, we assume a particular type of stochastic data generating process which will better capture the dynamic nature of the default and prepayment behaviors. Pricing of a given ARM proceeds in several stages. At stage one, under a Cox proportional hazard framework, the effects on prepayment and default risks of an individual borrower underlying a mortgage are captured in a multiplicative way, while the baseline prepayment and default hazards are allowed to be independent of the idiosyncrasies of the individual borrowers. The effects of the individuals are thus estimated using observed histories on a large number of comparable mortgages. At stage two, the baseline hazards are assumed to come from stochastic data generating processes that can be modelled as CIR-type diffusion processes. The diffusion processes are estimated using a particle filtering technique in a state space framework where the underlying state variables can only be observed indirectly through measurement equations. At stage three, after a two-factor CIR model for the term structure of interest rate is assumed and estimated using a particle filtering technique, we calibrate the default and prepayment hazard processes so that the model would produce a price of a given mortgage that was consistent with what was observed in the marketplace. Finally, we conduct tests on the out-of-sample pricing performance and demonstrate the application of the model in mortgage pricing strategy.

Index words: Asset Pricing, ARM, Mortgage, Reduced-Form Pricing, Particle Filtering, Term Structure of Interest Rate

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Jinbao Tong

B.S., Fudan University, 1985, P.R.China

M.S., Fudan University, 1988, P.R.China

MBA, The University of Georgia, 1998

A Dissertation Submitted to the Graduate Faculty
of The University of Georgia in Partial Fulfillment
of the
Requirements for the Degree

Doctor of Philosophy

Athens, Georgia

2007

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Jinbao Tong

Approved:

Major Professor: James B. Kau

Committee: Donald C. Keenan
Henry J. Munneke
Richard W. Martin

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
May 2007

Acknowledgments

I would like to thank Professor Kau and Professor Keenan for all the valuable support that make this project possible. I am also thankful to my dissertation committee members, Dr. Munneke and Dr. Martin for their constructive comments and valuable advice.

Table of Contents

Chapter 1

Introduction

Many financial obligations that are contractual in nature are exposed to default risk when one party fails to honor a promise to another party. A residential mortgage is an example of financial obligations that face default risk on the part of borrowers. In this case, the borrower might, at some point of time into the contract, fail to continue the payment as required by the mortgage contract.

Essentially, the default risk consists of default timing risk and default payoff risk. The default timing refers to the probability of the default over the time interval from the origination date to any given future date. Default payoff is the payoff to be received in the event of default at any given future date.

There are generally two types of approaches to the valuation of financial instruments subject to default risk. The first, termed "the structural approach" is based upon the option-pricing theory developed by Black and Scholes [26] and was formalized by Merton [149] and extended by Black and Cox [27] and Geske [95]. In this framework, the underlying firm value process is modeled directly. A financial instrument subject to default is valued as a contingent claim on the value of the underlying firm. The default is triggered the first time the value of the underlying firm crosses a default boundary. In one variant of this approach, the default boundary as well as default payoff are determined endogenously.

Kau et al. [129, 131, 124] started out to apply the structural approach to price residential mortgages subject to default risk. A lender, the holder of a mortgage, has a contingent claim on the value of a house subject to default on the part of the borrower. The value of the mortgage is measured based upon two stochastic processes: the value of a house and the short-term interest rate. Default occurs the first time the value of the house falls below the value of the mortgage and the payoff upon default is the difference between the house value and the unpaid mortgage balance. Default in this framework is a predictable stopping time with respect to the reference filtration that represents the information flow available. This means that the random time of default is announced by an increasing sequence of stopping times. Intuitively, given the value of the house is falling down and approaching a threshold, the default is becoming more and more inevitable.

The advantage of the structural approach in general is intuitive: the model of an underlying asset process has a very clear economic interpretation and the event of default is defined clearly in terms of market fundamentals. In addition, the derivation of hedging strategies for defaultable claims is straightforward. However, this approach is very difficult to implement in practice due to the un-observability of the underlying asset process in general. In some cases, it may be inconvenient to obtain information on the individual components of the firm's balance sheet. In other cases, it may be unrealistic to assume that default is equivalent to a situation in which the market value of assets is reduced to or below that of liabilities. For example, default may arise from illiquidity associated with credit restrictions or imperfect information. In general, default could occur well before or after the time at which the market value of assets falls below that of liabilities (Dufffie [73]). Furthermore, as the resulting random time is predictable with respect to the underlying filtration, it is the source of the observed discrepancy between the credit spreads for short maturities predicted by structural models and the market data.

In the case of mortgage, houses usually are not so often traded in the market. As a result, the parameters of the structural model can not be estimated accurately based upon sporadic transaction prices. In addition, the structural approach implicitly assumes that the default is the result only of a financial optimization process and that other non-financial factors do not play a role behind the default behaviors. People derive more than functional utilities (i.e. the need to have a place to live) from buying houses. While A deciding factor in most cases, financial consideration alone does not make a whole picture here. The motivation behind default, therefore, can not be fully explained by financial consideration only.

Unlike the structural approach, the "reduced form" approach treats default as an exogenous, totally unpredictable surprise. The default timing is modeled as the first jump time of an exogenously defined stochastic point process. The jump intensity, which determines the default probability, can be allowed to depend on time and state variables to reflect changes in economic environments (e.g. Lando [138, 139], Madan and Unal [144]). The payoff upon default is also determined exogenously by a recovery rate, which may be deterministic or stochastic (Das and Tufano [62]).

Choice of a specific treatment of recovery in the models often involves a trade-off between computational burden and conceptual appeal. Under RMV (Recovery of market value) assumption adopted by Duffie and Singleton [82], the recovery of the contingent claim upon default is a fraction of pre-default market value of the claim. Duffie and Singleton [82] then showed that the price of a defaultable claim could be expressed as the present value of the promised payoff, treated as if it were default-free, discounted by the default-adjusted short-term interest rate. The appeal of this treatment, even though not so closely reflecting reality as other treatments, is that once the risk-free discount rate is default-adjusted, the conventional term structure models can be used to handle defaultable contingent claims as if they were default-free. The RFV (Recovery of Face Value) assumption studied by Duffie [75] stipulates that bonds of the same issuers, seniority, and face value will have the same recovery at default, regardless of remaining maturity. This assumption is a result of a strict legal interpretation of recovery that is preferred by industry researchers.

In the case of mortgage, in this framework, the default on a mortgage can be the result of either a financial optimization process or a non-financial decision process while under the structural framework, only default that is related to a financial optimization is considered and non-financial motivation is left unaccounted for. One particular advantage is that the default frequency is easily observed over time, which allows us to link the default to some underlying state variables that describe the economic environment where the default occurs.

The objective of this dissertation is to price residential ARMs subject to both default risk and prepayment risk utilizing observed historical data on default and prepayment in a reduced-form framework.

The attributes of ARMs that have major influences on default and prepayment behaviors are first identified through proportional hazard models. Then a stochastic element is introduced into the specification of the baseline hazard function by treating each of stratum-based hazard functions as one random draw from a stochastic data generating process. The two stochastic processes are estimated independently assuming that they are following square-root mean-reverting diffusion processes. The estimation of the processes is achieved using a particle filtering scheme to address the issue of the nonlinear and non-normal nature of the underlying distributions. A two-factor CIR model of term structure of interest rate is estimated and serves as the data generating process upon which the coupon rate of an ARM is built. Finally, calibration is made to estimate risk-adjustment factors needed to convert two processes to risk-neutral processes.

The dissertation is organized as follows. Chapter 2 provides a brief review of the existing literature about valuation of mortgage in general, with special focus on the valuation of ARMs. Chapter 3 discusses the general theoretical foundation of risk-neutral pricing of mortgage subject to default and prepayment. Chapter 4 discusses the technical details about the estimation of a non-linear, non-normal diffusion process using particle filtering method. In chapter 5, a Cox proportional hazard model is employed to study factors that influence the termination of ARMs based on observed historical data on default and prepayment. In Chapter 6 and 7, the particle filtering scheme is applied to the estimation of stochastic processes for prepayment hazard rate, default hazard rate and the term structure of interest rates. Chapter 8 deals with the calibration of prepayment and default hazard rate processes using Monte Carlo simulation, so that the predicted values of a subset of ARMs are consistent with the their corresponding observed values.

Chapter 2

Literature review on valuation of ARM

2.1 The Characteristics of ARMs

Adjustable rate mortgage gained popularity during 1980s, reaching dominant status in 1984, accounting for two-thirds of all mortgages originated at the time. The basic ARM contract has the following features:

- Index: The coupon rate changes at each reset date based on the underlying index to which the rate is tied. Common indexes include one-year constant maturity Treasury yield, one-year LIBOR, the federal Housing Finance Board (FHFB) national average contract interest rate,11th District Federal Home Loan Bank cost of funds (EDCOFI).

- Margin: points added to the reference indices to determine the coupon rate for the next period.

- Coupon rate: on any adjustment date the rate is determined by adding a margin plus the prevailing level of the underlying index, subject to certain caps. It is usually constant and ranges typically from 100 to 300 basis points.

- Teaser rate: it is common for the initial coupon rate to be lower than the fully indexed rate by a teaser rate.

- Annual interest rate cap or payment caps: the interest caps limit the rise (or fall) in the contract rate on any adjustment date and are absolute in the sense that if market rates rise above the cap, the lost interest to the investor is not recoverable. Typical interest rate caps are 1 or 2 percentage point above or below the prior period's coupon rate. Payment caps also directly limit the rise in monthly payments, but the lost interest from a binding cap is usually recoverable through negative amortization. Typical payment caps are 7.5 percent per year with negative amortization limited to 125 percent of the original loan balance.

- Lifetime caps: the lifetime caps set the maximum level of interest rate an ARM can ever reach during its lifetime and typically is 5 percentage points above the initial contract rate.

- Reset frequency: the coupon rate on an ARM contract is adjusted at pre-specified intervals, which usually are every 6 month or one year.

2.2 The Valuation of ARMs

2.2.1 Valuation principle

Several facts cause lots of difficulties in the valuation of an ARM. First of all, the interest rate used to discount the future cash flows is uncertain. (This also applies to fixed rate mortgage.) Secondly, the coupon payments that are tied up with the prevailing interest rate at future time are also uncertain. The coupon rate changes on each adjustment date according to the level of index interest rate at the time. (This is unique to ARM only.) Lastly, the timing of payback of the mortgage is uncertain. Few ARMs have ever been paid strictly according to the pre-determined schedule. The borrower may at some point of time in the future refinance an ARM into a FRM or another ARM; he may be forced to sell the house and use the proceeds to pay back the mortgage and to satisfy the personal obligation; finally, he may pay more than pre-determined amount to speed up the time of paying down the debt. In any event, the mortgage will likely be paid back prematurely by the borrower. The uncertainty in timing of payments affects the value of the mortgage because of the reinvestment risk on the fund paid back prematurely. When the mortgage is prepaid when the interest rate is low, which usually is the case when the reason for the pre-mature payback is refinancing, the lender will have to reinvest the fund at lower rate than before. As a result, a static valuation model such as the discounted cash flow model is not adequate in addressing various forms of uncertainties associated with ARMs valuation.

The contingent claims model, developed by Black and Scholes [26] and Cox, Ingersoll and Ross [57], provides the foundation for the valuation of mortgage under uncertainty of the interest rate and timing of payments. Under contingent claims framework, the value of a mortgage is the expected present value of all the future cash flows with respect to all the possible interest rates and timings of payments. Applications of contingent claims framework in the valuation of ARMs come in different forms depending on the way the uncertainty of timing of the payments is treated.

2.2.2 Types of models

Endogenous Pricing Model

The first type of pricing models assumes that the uncertainty in timing of payments is due to prepayment behavior on the part of the borrower and that the prepayment decision is made purely from financial standpoint, i.e. the borrower will make prepayment whenever he finds it financially optimal. Under this framework, prepayment takes only two forms: one is refinancing, the other is default.

1. Prepayment only - Kau et al. [128] consider the valuation of ARMs subject to rational refinancing only. The type of ARMs considered floats off the one-year default-free pure discount bond yield. In this context, all the uncertainties are associated and determined by the uncertainty of the interest rate. Once the evolution of the interest rate is specified, the valuation of an ARM in a contingent claims framework is equivalent to solving a fundamental partial differential equation subject to certain boundary conditions. Optimal refinancing decision is determined on each payment date by comparing the mortgage balance to the value of the remaining payment stream which is dependent on the path of future interest rates. The PDE is therefore has to be solved backward starting on the next to the last payment date of the mortgage. Because the coupon is dependent on the path of past interest rates and is not available when the PDE is solved backwards, an auxiliary state variable is introduced that holds the value for the coupon rate in the previous period, and the usual backward algorithm can then be applied. Richard Stanton and Nancy Wallace [179] consider the valuation of a particular ARM that is indexed to the Eleventh District Cost of Funds (EDCOFI). The driving factor for uncertainty is again the interest rate, which as done in Kau et al [128], evolves according a square-root mean-reversion diffusion process. Because of special features associated with the index, they model explicitly the relationship between the index and the term structure of interest rate and point out the importance of lags in the index in influencing the value of the ARM. The optimal prepayment strategy is again endogenously determined.

2. Prepayment and default - By introducing another factor that governs the uncertainty of the value of the house underlying the mortgage, Kau et al. [129] are able to consider uncertainties due to both optimal refinancing and

optimal default. Upon default, the lender will receive either payment from mortgage insurance company if the mortgage is insured or proceeds from the selling of the house. In any event, the consequence is similar to a refinancing. The optimal default decision is made on each payment date by comparing the value of the remaining payment stream to the value of the underlying house. As a result, default occurs whenever the value of the underlying house is below the value of the remaining payment stream.

Exogenous Pricing Model

All the endogenous pricing models of ARMs suffer on several fronts. First of all, the predictions from those models are not consistent with the observed prepayment and default frequencies: borrowers seem to refinance more slowly and default more scarcely than predicted (Quigley and Van Order [162]). One possible explanation is the costs associated with refinancing and default. While some of those costs are quantifiable, such as transactions costs relevant to the refinancing and default, other costs are more intangible, such as loss of reputation and deterioration of credit status, emotional attachment to the house etc, and thus are more difficult to quantify.

Secondly, empirical implementation of the endogenous pricing model requires the data about the value of the underlying house when default is to be considered. However, given illiquidity of the housing market, it is nearly impossible for one to observe a series of values associated with one particular house over time. Therefore, endogenous pricing models have serious limitations in real applications.

Second type of pricing models is developed based on the recognition that people are not ruthless in their decision regarding the prepayment. On one hand, they may prepay the mortgages for reasons other than financial considerations even at times when it is not financially optimal to do so, such as relocation due to job changes, divorce etc. On the other hand, they may be reluctant to default even when it is financially optimal to do so because of personal attachment to the house, and concerns about reputation, credit status etc. Instead of considering the prepayment endogenously, this type of models treats prepayment exogenously.

This type of models assumes that at each point in time during the mortgage contract period, the mortgage has a certain probability of termination, conditional on the survival of the mortgage. It incorporates a set of time-varying covariates that are deemed to have influences on the mortgage prepayment and default behaviors that would be considered suboptimal under a pure contingent claim framework.

1. Prepayment only - Jonathan Berk and Richard Roll [19] model the interest rate as a lognormal random walk and specify the prepayment function empirically. Mcconnell and Singh [148] estimate a prepayment function empirically for Freddie Mac ARM-based securities and then incorporate it into a two-factor model of term structure of interest rate to develop an ARM-backed securities valuation model.

2. Prepayment and default - So far no study has incorporated both prepayment and default in the valuation of ARMs. However, lots of studies exist of prepayment and default for fixed rate mortgages. Kau et al.([132]) treat both prepayment and default exogenously for FRMs in a reduced form framework for valuation of mortgages. Deng [65] adopts a competing risk approach in the studying the termination of FRMs.

Chapter 3

Risk-neutral pricing of an ARM subject to prepayment and default

3.1 The General Set-up

Consider the pricing of ARM subject to prepayment and default risk. The termination time of an ARM is not known beforehand and is often conveniently modeled as a stopping time τ. When an ARM is facing both prepayment and default risk, the termination time is either the prepayment time or default time. Assume that the prepayment time and default time are modeled as two stopping times, τ_p and τ_d, then $\tau = min(\tau_p, \tau_d)$

The stopping time can be characterized by their corresponding intensities of arrivals. In a reduced-form framework, the prepayment and default intensities are assumed to be driven by exogenously-determined stochastic baseline intensity processes and the realized prepayment and default hazards are the function of the baseline intensities. Various specifications can be used in principle. However, in this paper, we have adopted a variant of the square-root mean-reverting processes for prepayment and default to ensure non-negativity of the baseline intensities.

$$d\lambda_0^p = \kappa^p[\theta_t^p - \lambda_0^p]dt + \sigma^p \sqrt{\lambda_0^p} dz^p \tag{3.1}$$

$$d\lambda_0^d = \kappa^d[\theta_t^d - \lambda_0^d]dt + \sigma^d \sqrt{\lambda_0^d} dz^d \tag{3.2}$$

$$\lambda^p = e^{\beta_p X_p'(t)} \lambda_0^p \tag{3.3}$$

$$\lambda^d = e^{\beta_d X_d'(t)} \lambda_0^d \tag{3.4}$$

where dz^p and dz^d are assumed to be independent and the θ_t is time-varying function representing the long term mean-reverting level. The $X'(t)s$ represent the vector of exogenous variables that are assumed to have an impact on the termination, and $\beta's$ are the corresponding coefficients. One important assumption here is that the exogenous variables are assumed to influence the hazard in a multiplicative manner.

While the baseline intensities are evolving in continuous time, the actual prepayment and default will only occur at the payment dates. Therefore, the intensity processes here are interpreted as aggregated hazard processes $\Lambda(t) = \sum_{t_i \leq t}(1 - \lambda_i)$

The processes above are regulated by a real probability measure. To obtain a risk-neutral price of an ARM, the processes have to be represented in a risk-neutral probability measure. Assuming that the necessary conditions for the existence of such risk-neutral probability measure are satisfied and that the risk adjustments have the following specifications, we have:

$$dz^{\mathbb{Q}} = dz^{\mathbb{R}} - \upsilon_t dt \tag{3.5}$$

$$d\Lambda^{\mathbb{Q}} = \mu_t d\Lambda^{\mathbb{R}} \tag{3.6}$$

where $dz^{\mathbb{R}}$ refers the real probability measure and $dz^{\mathbb{Q}}$ refers the equivalent martingale measure.

We have made further assumptions in this paper to ensure the tractability of the model. Specifically, we assume that the additive risk adjustment has the following particular form:

$$\upsilon_t = \upsilon\sqrt{\lambda_0(t)}/\sigma \tag{3.7}$$

$$\mu_t = \mu \tag{3.8}$$

Under such specifications, the affine structure of the processes will be preserved in the risk-neutral measure.

3.2 The Valuation Formula

As mentioned above, the termination of an ARM can be characterized by a stopping time $\tau = min(\tau_p, \tau_d)$. The associated risk-neutral hazard process is defined as

$$\Lambda(t) = \Lambda_p(t) + \Lambda_d(t) = \sum_{t_i \le t} (\lambda_i^p + \lambda_i^d) \tag{3.9}$$

According to the risk-neutral pricing principle, the arbitrage-free price of an ARM subject to prepayment and default risk is an expected all discounted future cash-flows with respect to the risk-neutral probability measure. Mathematically, the arbitrage-free value of an ARM can be determined by the following formula:

$$V(t_0) = E^{\mathbb{Q}}[\sum_{i=1}^{360} \mathbb{P}(\tau > t_{i-1}) PV(t_i)] \tag{3.10}$$

$$\mathbb{P}(\tau > t_{i-1}) = \prod_{j=1}^{i-1} (1 - \lambda_j^p - \lambda_j^p) \tag{3.11}$$

$$PV(t_i) = e^{\int_{t_0}^{t_i} \hat{r}(s) ds} CF(t_i) \tag{3.12}$$

$$\hat{r}(s) = (1 - \tau_F) r(s) + l \tag{3.13}$$

$$CF(t_i) = \mathbb{P}(\tau = \tau_d)\, W(t_i) + \mathbb{P}(\tau = \tau_p)\, A(t_i) + \mathbb{P}(\tau > t_i)\, M \tag{3.14}$$

where $CF(t_i)$ is the cash-flow at time t_i, $PV(t_i)$ is the present value of the cash-flow at time t_i, and τ_F and l are the federal tax rate and liquid premium respectively. Conditional on an ARM surviving past time t_{i-1}, the cash-flow at time t_i is one of three payments: the scheduled monthly payment, M; the payment due to prepayment, $A(t_i)$; or the payment due to default, $W(t_i)$. The probability of prepayment, default and continuation are $\mathbb{P}(\tau = \tau_p)$, $\mathbb{P}(\tau = \tau_d)$, and $\mathbb{P}(\tau > t_i)$, respectively.

The instantaneous interest rate is adjusted to account for two factors before it is used in the discounting process. The first factor is that the interest portion of the scheduled payment is tax-deductable at federal level; the second factor is the liquidity premium associated with an ARM.

The specifications for the stochastic processes and the valuation formula discussed here are the theoretical foundation for the empirical study that is covered in the following chapters.

Chapter 4

Estimation of diffusion processes using particle filters

Stochastic differential equations (SDE) have been widely recognized as a very powerful method to model the dynamics of economic and financial variables, such as interest rates, stock prices etc. In many applications, the stochastic differential equations are preferred due to the tractability they offer. However, the actual data are usually observed discretely, either yearly, monthly or daily. As a result, the estimation of the parameters of the SDE can be quite a challenge in many settings where a closed-form solution to the SDE is not available. Naive discretization of diffusion processes can be subject to discretization bias.

A number of methods have been proposed to estimate diffusion parameters. One approach is quasi-maximum likelihood or moment-based estimators based on a Euler discretization of the process [42]. An alternative strategy relies on various forms of approximations to the likelihood function [177, 22, 7]. A third approach is nonparametric methods as proposed by Ait-Sahalia, Y. [5, 6], Jiang [118] and Stanton [178].

For any parametric model, maximum likelihood is the method of choice because of its superior statistical properties compared to others methods. However, the exact likelihood is usually not available in the continuous-time diffusion setting and certain forms of approximations are often used in the applications. When the variables of interest can be observed directly at certain frequency, simulated-based approach [74, 94] can be quite useful because it is applicable to quite general diffusion processes. In many applications, however, the variables of interest are driven by some latent variables which are not directly accessible. This is the case when a two-factor interest rate model has to be constructed. What we can observe in each month are series of yields on Treasury bill with different maturities. The underling factors can only be inferred but not directly observable.

In general, in a class of state space models, there is a state equation that governs the transition of the underlying state variables over time and there is a measurement equation that links the observed variables to the underlying state variables. The estimation of the parameters for a state space model when the transition equation is a continuous-time diffusion equation thus has to consider both the discretization bias and the hidden variable problem. In order to make the estimation feasible in actual applications, we have to also consider the computational time needed for the implementation of the program. In this chapter, a particle filtering technique is adopted in the estimation of the parameters for a general continuous-time diffusion equation with unobserved variables. The following material is based on Michael. K. Pitt [157], Neil Gordon etc [101].

4.1 Likelihood evaluation for state space models

In a state space model, the evolution of the hidden variables is specified via a state (transition) equation

$$\mathtt{x}_t = g(\mathtt{x}_{t-1}, \mathtt{v}_{t-1}|\theta_1), \tag{4.1}$$

where $g : \mathbb{R}^{\mathtt{n}_x} \times \mathbb{R}^{\mathtt{n}_v} \to \mathbb{R}^{\mathtt{n}_x}$ is a possibly non-linear function of the state $\mathtt{x}_{t-1}$,$\{\mathtt{v}_{t-1}, t \in \mathbb{N}\}$ is an i.i.d process noise sequence, N_x, N_v are dimensions of the state and process noise vectors, respectively, and the θ_1 is the parameter vector. The observed variables are related to the hidden variables via an observation (measurement) equation

$$\mathtt{y}_t = f(\mathtt{x}_t, \mathtt{w}_t|\theta_2), \tag{4.2}$$

where $f : \mathbb{R}^{N_x} \times \mathbb{R}^{N_w} \rightarrow \mathbb{R}^{N_y}$ is a possibly non-linear function of the state $\mathtt{x}_t$, $\{\mathtt{w}_t, t \in \mathbb{N}\}$ is an i.i.d measurement noise sequence, and $n_{\mathtt{x}}, n_{\mathtt{w}}$ are dimensions of the measurement and measurement noise vectors, respectively. Note that θ_2 is the parameter vector.

Let $\theta = (\theta_1, \theta_2)$ be the fixed parameter vector and assume that the transition pdf is available and the observation equation can be evaluated explicitly. Given a time series $\{\mathtt{y}_t, t = 1, \ldots, n\}$, the usual maximum likelihood estimation calls for an evaluation of the following log-likelihood

$$\begin{aligned} logL(\theta) &= logf(\mathtt{y}_1, \ldots, \mathtt{y}_n|\theta) \\ &= \sum_{i=1}^{n} logf(\mathtt{y}_t|\theta; \mathcal{F}_{t-1}), \end{aligned} \tag{4.3}$$

where $\mathcal{F}_t = \{\mathtt{y}_1, \ldots, \mathtt{y}_t\}$ are information streams up to time t and $f(\mathtt{y}_t|\theta; \mathcal{F}_{t-1})$ is the transition pdf. Conceptually, this likelihood can be evaluated using the following:

$$f(\mathtt{y}_t|\mathcal{F}_{t-1}) = \int f(\mathtt{y}_t|\mathtt{x}_t) f(\mathtt{x}_t|\mathcal{F}_{t-1}) d\mathtt{x}_t. \tag{4.4}$$

It is clear that the above evaluation hinges on the tractability of the pdf $f(\mathtt{x}_t|\mathcal{F}_{t-1})$. Suppose that the pdf $f(\mathtt{x}_{t-1}|\mathcal{F}_{t-1})$ is available at time t. Then in theory, $f(\mathtt{x}_t|\theta; \mathcal{F}_{t-1})$ can be evaluated recursively using the following set of equations based on the application of the Chapman-Kolmogorov theorem and Bayes' rule,

$$f(\mathtt{x}_t|\theta; \mathcal{F}_{t-1}) = \int f(\mathtt{x}_t|\mathtt{x}_{t-1}) f(\mathtt{x}_{t-1}|\mathcal{F}_{t-1}) d\mathtt{x}_{t-1} \tag{4.5}$$

$$f(\mathtt{x}_t|\theta; \mathcal{F}_t) = \frac{f(\mathtt{y}_t|\mathtt{x}_t) f(\mathtt{x}_t|\mathcal{F}_{t-1})}{f(\mathtt{y}_t|\mathcal{F}_{t-1})}. \tag{4.6}$$

Using (4.5) and (4.6) recurrently, we will be able to evaluate the likelihood function (4.3) for the whole time series. The problem is that in most of cases, (4.5) and (4.6) are not tractable. That is, they can not be determined analytically. Therefore, the feasibility of evaluating the likelihood depends on our ability to approximate the posterior pdf $f(\mathtt{x}_t|\theta; \mathcal{F}_t)$ for any time t.

4.2 Particle filtering algorithms

In several special cases where the transition equation and the measurement equation have special forms that are conducive to an analytic solution, there exits an optimal algorithm that will provide the best estimate of the likelihood using the (4.5) and (4.6). For example, under the following conditions:

- $\mathtt{v}_{t-1}$ and $\mathtt{w}_t$ have Gaussian distributions
- $g(\mathtt{x}_{t-1}, \mathtt{v}_{t-1})$ is known and is a linear function of $\mathtt{x}_{t-1}$ and $\mathtt{v}_{t-1}$
- $f(\mathtt{x}_t, \mathtt{w}_t)$ is known and is a linear function of $\mathtt{x}_t$ and $\mathtt{w}_t$

the Kalman filter has been proved to provide the optimal solution to the problem of estimating the posterior distribution of $f(\mathtt{x}_t|\mathcal{F}_t)$. If the state space is discrete and consists of a finite number of states, then grid-based methods has been shown to provide the optimal recursion of the filtered density, $f(\mathtt{x}_t|\mathcal{F}_t)$.

When those conditions do not hold, approximations are necessary. Three approaches have been developed along this line. The basic idea underlying the extended Kalman filter is to approximate a non-linear function using local linearization technique and to apply the usual Kalman filter to the linearized version of the system. Approximate grid-based methods rely on the decomposing a continuous state space into N 'cells', $\{\mathtt{x}_k^i : i = 1, \ldots, N\}$ and then apply the grid-based method as usual. Instead of approximating the transition function or decomposing the state space, particle filtering methods attempt to represent the posterior density function by a set of random samples with associated weights and to compute estimates based these samples and weights. When the number of samples gets larger, this random sample representation of the distribution will become equivalent to usual functional representation of the same distribution. As a result, any computation involving the pdf can be equivalently done using the random samples.

4.2.1 The foundation of particle filtering technique

Assume the distribution density $p(x)$ from which we want to sample is not directly accessible. However, if $p(x) \sim \pi(x)$ and $\pi(x)$ can be evaluated, then sampling from $p(x)$ can be accomplished via the following two steps. First, draw a sample x^i from an alternative proposal density $q(x)$ for which it is easy to draw a sample and then attach a weight to x^i using the following formula.

$$w^i \propto \frac{\pi(x^i)}{q(x^i)} \tag{4.7}$$

Now, the density $p(x)$ can be approximated by

$$p(x) \approx \sum_{i=1}^{N} w^i \delta(x - x^i) \tag{4.8}$$

This is the principle of importance sampling.

Let $\{\mathtt{x}_{0:t}^i, w_t^i\}_{i=1}^N$ denote a random measure that characterizes a pdf $p(\mathtt{x}_{0:t})$, where $\{\mathtt{x}_{0:t}^i, i = 0, \ldots, N\}$ is a set of supporting points with associated masses $\{w_t^i, i = 1, \ldots, N\}$ and $\mathtt{x}_{0:t} = \{x_j, j = 0, \ldots, t\}$ is the set of all states up to time t. The masses are normalized such that $\sum_i w_t^i = 1$. Then, applying the principle of importance sampling, the density at time t can be approximated as

$$p(\mathtt{x}_{0:t}) \approx \sum_{i=1}^{N} w_t^i \delta(\mathtt{x}_{0:t} - \mathtt{x}_{0:t}^i) \tag{4.9}$$

where the weights are determined using the following:

$$w_t^i \propto \frac{p(\mathtt{x}_{0:t}^i)}{q(\mathtt{x}_{0:t}^i)} \tag{4.10}$$

and $\mathtt{x}_{0:t}^i \sim q(x)$, are random samples from the density that is easy to draw from.

If $q(\mathtt{x}_t|\mathtt{x}_{0:t-1}, \mathtt{y}_{1:t}) = q(\mathtt{x}_t|\mathtt{x}_{t-1}, \mathtt{y}_t)$, then the importance density depends only on $\mathtt{x}_{t-1}$ and $\mathtt{y}_t$. Then, the weight simplifies to

$$w_t^i \propto w_{t-1}^i \frac{p(\mathtt{y}_t|\mathtt{x}_t^i) p(\mathtt{x}_t^i|x_{t-1}^i)}{q(\mathtt{x}_t^i|\mathtt{x}_{t-1}^i, \mathtt{y}_t)} \tag{4.11}$$

and the posterior filtered density $p(\mathtt{x}_t|\mathtt{y}_{1:t})$ can be approximated as

$$p(\mathtt{x}_t|\mathtt{y}_{1:t}) \approx \sum_{i=1}^{N} w_t^i \delta(\mathtt{x}_t - \mathtt{x}_t^i) \tag{4.12}$$

The SIS algorithm consists of recursive propagation of the weights and mass points as each observation is made sequentially. The algorithm is described in the following pseudo-code:

Algorithm 1: Sequential Importance Sampling

- For $i = 1, N$
  - Draw $\mathtt{x}_t^i \sim q(\mathtt{x}_t|\mathtt{x}_{t-1}^i, \mathtt{y}_t)$
  - Attach a weight to $\mathtt{x}_t^i$ according to (4.11)
- End for

In actual applications, however, the SIS algorithm suffers from so-called degeneracy phenomenon where after a few iterations, all but one mass point will have negligible weight. It has been shown this problem is impossible to avoid. One measurement of the severity of the problem can be obtained by

$$N_{eff} = \frac{1}{\sum_{i=1}^{N} (w_t^i)^2} \tag{4.13}$$

where w_t^i is the normalized weight obtained by (4.11). Small N_{eff} indicates severe degeneracy.

One method proposed to reduce the degeneracy problem is to use re-sampling whenever a significant degeneracy is observed, as measured by (4.13). A common criteria is to use re-sample when $N_{eff} < N/3$, where N is the sample size. Re-sampling eliminates mass points that have small weights and concentrates on mass points with large weights. It works by drawing N samples with replacement at the probability of $Pr(\mathtt{x}_t^{i*} = \mathtt{x}_t^j) = w_t^j$. The algorithm is described in the flowing pseudo-code:

Algorithm 2: Re-sampling

- initialize the CDF: $c_1 = 0$
- For $i = 2, N$
  - Construct CDF: $c_i = c_{i-1} + w_t^i$
- End for
- Start at the bottom of the CDF: $i = 1$
- Draw a starting point: $u_1 \sim \mathbb{U}[0, N^{-1}]$
- For j=1,N
  - Move along the CDF: $u_j = u_1 + N^{-1}(j - 1)$
  - Where $u_j > c_i$
    - $i = i + 1$
  - End where
  - Assign sample: $\mathtt{x}_t^{j*} = \mathtt{x}_t^i$
  - Assign weight: $w_t^j = N^{-1}$
- End for

Incorporating the re-sampling technique into the sequential importance sampling produces the following generic particle filter algorithm:

Algorithm 3: Generic Particle Filter

- For $i = 1, N$

  - Draw $\mathtt{x}_t^i \sim q(\mathtt{x}_t|\mathtt{x}_{t-1}^i, \mathtt{y}_t)$
  - Assign a weight, w_t^i, according to (4.11)

- End for
- Calculate the total weights: $w\_sum = \mathtt{SUM}[\{w_t^i\}_{i=1}^N]$
- Normalize weights:$w_t^i = w_t^i / w\_sum$
- Determine the degeneracy according to (4.13)
- If $N_{eff} < N_T$
  - Re-sample using algorithm 2:
    - $[\{\mathtt{x_t^{i*}}, w_t^i\}_{i=1}^N]$=Resample$[\{\mathtt{x}_t^i, w_t^i\}_{i=1}^N]$
- End if

While the re-sampling technique helps to reduce the degeneracy problem, there are several implications of its use in evaluating likelihood. First, as the re-sample process relies on sampling with replacement, the points with large weights will be sampled more often than points with smaller weights. As a result, the i.i.d samples from the discrete approximation to the continuous pdf contains many repeated points and diversity of points is lost. This problem which is known as sample impoverishment, is particularly serious in the case of very small noise where all points will collapse to a single point within a few iterations. Second, the gradient-based maximum likelihood will no longer be possible as the estimated likelihood will be very rough and the estimated gradient will bear no resemblance to the true gradient.

4.2.2 Efficient and smooth likelihood estimation

Given samples from a filtering density at time t, according to (4.4), the likelihood can be estimated as

$$\hat{f}(\mathtt{y}_t|\theta;\mathcal{F}_{t-1}) = \int f(\mathtt{y}_t|\mathtt{x}_t)\left\{\frac{1}{N}\sum_{i=1}^{N} f(\mathtt{x}_t|\mathtt{x}_{t-1}^i)\right\}d\mathtt{x}_t \tag{4.14}$$

The above evaluation still involves an integral that in general can not be evaluated directly. However, the filtering method we discussed previously provides a good foundation for estimating the integral.

Theorem 4.2.1 *Let the importance density be* $q(\mathtt{x}_t) = f(\mathtt{x}_t|\mathtt{x}_{t-1})$ *and assume* $\{\mathtt{x}_t\}_{i=1}^N \sim q(\mathtt{x}_t)$*,* $\{w_t^i\}_{i=1}^N$ *are determined according to (4.11).*

Then the (4.14) can be approximated using the following:

$$\hat{f}(\mathtt{y}_t|\theta;\mathcal{F}_{t-1}) = \frac{1}{N}\sum_{i=1}^{N} w_t^i \tag{4.15}$$

The estimation of the likelihood by (4.15) is, however, not necessarily continuous with respect to the parameter vector θ. The problem lies in the discrete representation of pdf $f(\mathtt{x}_t|\mathcal{F}_t)$ by the probabilities w_t^i proportional to $f(\mathtt{y}_t|\mathtt{x}_t^i)$. This will make the application of the usual gradient-based maximization routine problematic. One technique that will mitigate the magnitude of the roughness is to use the same random seed for each set of parameters. That is, we will use the same random number sequences to evaluate the likelihood under different values of the parameters θ. This will reduce the variance resulting from using different random number sequences.

To ensure the likelihood function continuous w.r.t. to the parameters, the samples from $f(\mathtt{x}_t|\mathcal{F}_t)$ also are required to be smooth as a function of the parameters θ. The basic idea is that instead of sampling from the discrete representation of the distribution, we draw samples from the continuous version of the distribution. The problem then boils down to how to make a continuous version of distribution out of a given discrete representation of a distribution.

Case 1: State variables of one dimension

For state variables with one dimension, Michael, K. Pitt [157] proposed a piecewise linear smoothing scheme. Suppose $\{x^i, \pi^i\}_{i=1}^N$ is the discrete representation of a distribution of a random variable and $\{x^i\}_{i=1}^N$ is sorted in ascending order. Define a region S_i where the associated probability P_i follows:

$$S_i = [x^i, x^{i+1}], \quad i = 1, \ldots, N-1 \tag{4.16}$$

$$P_i = \begin{cases} \frac{1}{2}(\pi_i + \pi_{i+1}), & i = 2, \ldots, N-2 \\ \frac{1}{2}(2\pi_1 + \pi_2), & i = 1 \\ \frac{1}{2}(\pi_{N-1} + 2\pi_N), & i = N-1 \end{cases} \tag{4.17}$$

Within each region S_i, the conditional densities $G(x|i)$ are defined as follows:

$$G(x|i) = \begin{cases} \frac{1}{x^{i+1}-x^i}, & x \in S_i, \quad i = 2, \ldots, N-2 \\ \frac{\pi_1}{2\pi_1+\pi_2}, & x = x^1 \\ (\frac{1}{x^2-x^1})\,\frac{\pi_1+\pi_2}{2\pi_1+\pi_2}, & x \in S_1 \\ \frac{\pi_N}{\pi_{N-1}+2\pi_N}, & x = x^N \\ (\frac{1}{x^N-x^{N-1}})\,\frac{\pi_{N-1}+\pi_N}{\pi_{N-1}+2\pi_N}, & x \in S_{N-1} \end{cases} \tag{4.18}$$

Denoting the continuous cdf by $\tilde{F}(x)$, the discrete cdf by $\hat{F}(x)$ and the true cdf by $F(x)$, it can be proved that as $N \to \infty$, $\tilde{F}(x) \to \hat{F}(x) \to F(x)$ (Smith and Gelfand [176])

To draw samples from the continuous version of the cdf, first we generate N samples of ordered uniform via the following:

$$u \sim \mathbb{U}[0,1], \text{uniform distribution} \tag{4.19}$$

$$u_j = \frac{j-1}{N} + \frac{u}{N}, \text{ where } j = 1, \ldots, N \tag{4.20}$$

Given the sorted uniform samples, the ordered samples from the continuous version of the cdf can be obtained through the following algorithm.

Algorithm 4: Smooth Re-sampling in the univariate case

- Set $s = 0, j = 1$
- For $i = 1, N - 1$
  - $s = s + P_i$
  - While ($u_j \leq s$ and $j \leq N$)
    - r_j=i;
    - $u_j^* = (u_j - (s - P_i))/P_i$
    - $j = j + 1$
  - End while
- End for
- For $j = 1, N$
  - $y_j = (x^{r_j+1} - x^{r_j}) \times u_j^* + x^{r_j}$ for $r_j = 2, \ldots, N - 2$
  - Invert the cdf, for $r_j = 1$ and $r_j = N - 1$
- End for

Case 2: State variables of two dimensions

For state variables with two dimensions, a multivariate kernel smoothing technique as discussed in [175] can be employed to get a continuous pdf. The discussions below are based on Smith [176].

Let K be a probability density function on $\mathbb{R}^n$. Then the two-dimension kernel density estimator is defined as

$$\hat{f}(X) = \frac{1}{nh^2}\sum_{i=1}^{n} K\left\{\frac{1}{h}(X - \boldsymbol{X_i})\right\} \tag{4.21}$$

Common choices of kernel functions include the standard multivariate normal density function

$$K(X) = (2\pi)^{d/2} exp(-\frac{1}{2}X^T X) \tag{4.22}$$

and the multivariate Epanechnikov kernel

$$K_e(X) = \begin{cases} \frac{1}{2}c_d^{-1}(d+2)(1 - X^T X), & \text{if } X^T X < 1 \\ 0, & \text{otherwise} \end{cases} \tag{4.23}$$

where d is the dimension of the X and c_d is the volume of the unit d-dimensional sphere: $c_1 = 2, c_2 = \pi, c_3 = 4\pi/3$.

The key to successful kernel smoothing of an empirical density is the choice of window width h. In general, the bias of the density estimation depends on the window width h: a larger h will reduce the random variance at the expense of introducing large bias into the estimation. That is, it over-smooths the density. A smaller h, on the other hand, will result in larger integrated variance and the density will be less smooth. The optimal window width is the one that will minimize the approximate mean integrated square error. If the distribution underlying the samples is a standard one, then there is a closed-form formula for the determination of the optimal window width. For example, the optimal window width for the smoothing of normally distributed data with unit variance is given by

$$h_{opt} = A(K)n^{-1/(d+4)} \tag{4.24}$$

with the constant

$$A(K) = \left[d\beta\alpha^{-2}\left\{\int(\nabla^2\phi)^2\right\}^{-1}\right]^{1/(d+4)} \tag{4.25}$$

and

$$\alpha = \int t^2 K(t) dt; \qquad \beta = \int K(t)^2 dt$$

In cases of two dimensions, where $d = 2$, the values of $A(K)$ are given as follows:

$$A(K) = \begin{cases} 0.96 & \text{for 2-dimension normal kernel} \\ 1.77 & \text{for 2-dimesion Epanechnikov kernel} \end{cases} \tag{4.26}$$

In the general cases where the underlying densities are unknown, we approximate the densities using the above optimal window width, assuming either Gaussian or Epanechnikov kernel. As sampling from a Gaussian is usually less time-consuming, in the later applications, we will use a Gaussian kernel with a covariance matrix equal to the empirical covariance matrix of the samples. To accommodate the possible multi-modal densities, the actual window width is chosen to be $h = h_{opt}/2$.

Sampling from a kernel-smoothed density can proceed as follows:

Algorithm 5: Smooth Re-sampling in the bivariate case

- Calculate the empirical covariance matrix S
- Compute D such that $DD^T = S$
- Re-sample using algorithm 2:
  - $[\{\mathtt{x_i}^*, w_i\}_{i=1}^N]$=Resample$[\{\mathtt{x}_i, w_i\}_{i=1}^N]$
- For $j = 1, N$
  - Draw $\epsilon^j \sim K$, where K is the kernel density
  - $\mathtt{x}_j^{**} = \mathtt{x}_j + h_{opt}\, D\, \epsilon^j$
- End for

4.3 Estimation of discretely sampled diffusion processes using particle filtering

The application of the particle filtering technique in evaluating a likelihood has two requirements: First, we should be able to sample from a transition density. Secondly, we should be able to evaluate the likelihood of having the actual observations.

In most cases, the diffusion processes to be estimated do not have closed-form transition densities. Instead, we have to rely on certain discretization scheme to approximate the transition density. The common discretization scheme is the Euler scheme, which gives the first order approximation to the underlying stochastic differential equations. Higher order approximations are also available with increased complexities. In the following two applications, we have chosen to use the Euler scheme for its easy implementation without much loss in the precision of the approximation.

The first case is a typical example of state-space models with a diffusion state process where the observation is taken under the influence of measurement errors. The second case is special in that the observation is the result of a random draw from a distribution of a random variable. The measurement errors are indirectly reflected in the randomness of the underlying random variable itself.

Case 1: A two-factor CIR term structure model of the interest rate

Under a two-factor CIR model, the instantaneous interest rate is assumed to be determined by two unobserved independent factors, each evolving according to a diffusion process:

$$r_t = x_t^1 + x_t^2 \tag{4.27}$$

$$dx_t^1 = \kappa_1(\theta_1 - x_t^1)dt + \sigma_1\sqrt{x_t^1}dw_t^1 \tag{4.28}$$

$$dx_t^2 = \kappa_2(\theta_2 - x_t^2)dt + \sigma_2\sqrt{x_t^2}dw_t^2 \tag{4.29}$$

The above stochastic differential equations have closed-form transition densities:

$$p(x_t^i, t; x_s^i, s) = c_i e^{-u_i - v_i}(v_i u_i)^{q_i/2} I_q(2\sqrt{u_i v_i}), \quad i = 1, 2. \tag{4.30}$$

where $\Delta = t - s > 0$, $c_i = 2\kappa_i/[\sigma_i^2(1 - e^{-\kappa_i \Delta})]$, $u_i = c_i x_s^i e^{-\kappa_i \Delta}, v_i = c_i x_t^i$, $q_i = 2\kappa_i\theta_i/\sigma_i^2 - 1$, and $I_q(.)$ is the modified Bessel function of the first kind of order q. Let $y_t^i = 2c_i x_t^i$, then $y_t|y_s$ is distributed in a noncentral chi-squared manner with $4\kappa_i\theta_i/\sigma_i^2$ degrees of freedom and noncentrality parameter $y_s^i e^{-\kappa_i \Delta}$. To draw samples from these transition densities, two methods as mentioned in Chen and Scott [45] can be used.

The first method is based on the following property of the distribution:

$$\hat{\chi}_v^2(\delta) = \hat{\chi}_1^2(\delta) + \chi_{v-1}^2$$
$$\hat{\chi}_1^2(\delta) = (Z + \sqrt{\delta})^2$$

where $\hat{\chi}_v^2(\delta)$ is a noncentral χ^2 distribution with v degrees of freedom and noncentrality parameter δ, χ_{v-1}^2 is a central χ^2 distribution with $v - 1$ degree of freedom, and Z is a standard normal. The samples from the distribution can then be drawn by letting $v = 4\kappa_i\theta_i/\sigma_i^2$ and $\delta = 2c_i x_s^i e^{-\kappa \Delta}$. This method cannot be used when the degrees of freedom is less than one.

The second method is based on the observation that the non-central χ^2 distribution can be expressed as a mixture of central χ^2 distribution with degrees of freedom proportional to random variates from a Poisson distribution. Specifically, to draw a sample from the distribution, we first draw j from a Poisson distribution with mean $\mu = c_i x_s^i e^{-\kappa_i \Delta}$ and calculate the degrees of freedom of the central χ^2 as $df = 2q_i + 2 + 2j$. Then draw a sample from the central χ^2 distribution with df degrees of freedom. This method can be used even when the degrees of freedom is less than one.

The specification of the underlying factors yields a closed-form pricing formula for a pure discount bond. At each point of time t, a series of yields $\{y_t^i\}_{i=1}^4$ with different maturities are observed and they are assumed to deviate from the theoretical yields by some observation errors.

$$y_t^i = -\frac{1}{\tau_i} log[P_t^i(x_t^1, x_t^2, \tau_i)] + \epsilon_i, \quad i = 1, \ldots, 4 \tag{4.31}$$

where P_t^i is the price of a pure discount bond with maturity $\{\tau_i\} = \{3/12, 6/12, 5, 10\}$ at time t and is given by the following formulae:

$$\begin{aligned} P_t &= A_1(t,\tau)A_2(t,\tau)e^{-B_1(t,\tau)x_t^1 - B_2(t,\tau)x_t^2} \\ A_i(t,\tau) &= \left[\frac{2\gamma_i e^{\frac{1}{2}(\kappa_i+\lambda_i-\gamma_i)\tau}}{2\gamma_i e^{-\gamma_i\tau} + (\kappa_i + \lambda_i + \gamma_i)(1 - e^{-\gamma_i\tau})}\right]^{2\kappa_i\theta_i/\sigma_i^2} \\ B_i(t,\tau) &= \frac{2(1 - e^{-\gamma_i\tau})}{2\gamma_i e^{-\gamma_i\tau} + (\kappa_i + \lambda_i + \gamma_i)(1 - e^{-\gamma_i\tau})} \\ \gamma_i &= \sqrt{(\kappa_i + \lambda_i)^2 + 2\sigma_i^2}, \quad i = 1, 2 \end{aligned}$$

The observation errors are assumed to be normally distributed with zero mean and constant variance: $\epsilon_i \sim N(0, \sigma^2)$

Case 2: Stochastic hazard processes for prepayment and default

Assume that the baseline hazard rates for prepayment and default in the context of mortgages termination can be described by two square root diffusion processes:

$$\lambda_0^p(t) = x_t^1, \quad \lambda_0^d(t) = x_t^2 \tag{4.32}$$

$$dx_t^1 = k_1[\theta_1(t) - x_t^1]dt + \sigma_1\sqrt{x_t^1}dw_t^1 \tag{4.33}$$

$$dx_t^2 = k_2[\theta_2(t) - x_t^2]dt + \sigma_2\sqrt{x_t^2}dw_t^2 \tag{4.34}$$

The individual prepayment and default hazard rates are assumed to be related to the baseline hazard rates in a multiplicative manner:

$$\lambda_t^p = \lambda_0^p e^{X_p'\beta_p}, \quad \lambda_t^d = \lambda_0^d e^{X_d'\beta_d} \tag{4.35}$$

where X_p and X_d are sets of mortgage specific variables deemed to reflect the mortgage-specific impact on top of the baseline hazard rates while β_p and β_d are parameters associated with corresponding variables.

At each point of time, we can observe the number of mortgages that are prepaid or defaulted. The observed numbers are assumed to follow Poisson distribution with means equal to the hazard rates.

$$Pr(N_p = k) = e^{-\lambda_t^p} \frac{(\lambda_t^p)^k}{k\,!} \tag{4.36}$$

$$Pr(N_d = k) = e^{-\lambda_t^d} \frac{(\lambda_t^d)^k}{k\,!} \tag{4.37}$$

where N_p and N_d are the number of the prepayments and defaults respectively.

There are no closed-form transition densities in this case because of the time-varying mean reverting levels. However, utilizing a discretization scheme, we can get an approximation to the underlying transition densities. Specifically, given $X(s)$, to get $X(t)$, we first partition the interval $[s, t]$ into N subintervals $[s_j, s_{j+1}]$, where

$$s_j = s + \Delta * j, \quad \Delta = \frac{t-s}{N}, \quad j = 0, \ldots, N$$

$$u_0^i = x^i(s), \quad u_j^i = x^i(s_j), \quad i = 1, 2$$

Then apply the Euler scheme to each sub-interval $[s_j, s_{j+1}]$,

$$u_{j+1}^i = u_j^i + \kappa_i[\theta_i(s_j) - u_j^i]\Delta + \sigma_i \sqrt{u_j^i \Delta}\, z_j^i \tag{4.38}$$

$$z_j^i \sim N(0,1), \text{ i.i.d.} \quad j = 0, \ldots, N-1 \tag{4.39}$$

$$x^i(t) = u_N^i \tag{4.40}$$

Chapter 5

A proportional hazard model of ARM termination

5.1 Background

As specified in the previous chapter, we have assumed that there are two underlying stochastic processes which determine the baseline prepayment and default hazards at any time. The idiosyncrasy of any given ARM is assumed to have a multiplicative impact on top of the baseline hazards. This chapter aims to identify and estimate those idiosyncrasies with respect to an ARM, using a proportional hazard model.

Several considerations that are particular to this paper make the application of proportional hazard models differ from their usual use. First, a different implicit assumption about the underlying baseline hazards is made in the process. In the usual application of proportional hazard models, the baseline hazard function is left unspecified. That is, no parametric structure is imposed upon it. Instead, the baseline hazards are estimated non-parametrically from the data and are used in later predication. Nonetheless, the baseline hazards are implicitly assumed to be deterministic. The implicit assumption made in this paper, instead, is that the underlying baseline hazards are governed by stochastic processes. As a result, the resultant baseline hazards are stochastic instead of deterministic. The seemingly inconsistency between the two assumptions about the baseline hazards are reconciled through the use of stratification in the models. Specifically, the data are stratified by the time of origination measured in quarters. All mortgages with same vintage are then assumed to share the same realization of the baseline hazards. The baseline hazards for mortgages with different vintages are assumed to the result of different realizations of the same processes over time.

Another important consideration is that we intend to use the result here to facilitate pricing of an ARM using only loan specific information. As a result, we seek to extract only the impact that is due to the idiosyncrasy of an ARM. In other words, anything exogenous to the loan will not be considered in the model even though the conventional wisdom may suggest that they are important in predicating termination behaviors of an ARM. Examples include home sales, macro-economic indicators other than the interest rates, etc.

Usually, the observation window for prepayment and default is a month. That is, the number of events is grouped into an interval of a month. As a result, there are often lots of ties in a month, which violates one of the fundamental assumptions underlying a Cox proportional hazard model. The Cox framework is built on a continuous process assuming no tie. Therefore, we employed a variant of a standard Cox model, specifying hazards of termination as discrete processes instead of continuous processes. It has been shown in Prentice and Kalbfleisch [120] that the conventional Breslow estimator of covariate effects is still consistent, though the standard errors have to be adjusted.

5.2 Data

The data used in the analysis are from Bank of America and include various types of ARMs originated between 1980s and 2000s. There are ARMs that differ in the indices used and hybrid ARMs that act like fixed rate mortgages for certain years and become ARMs thereafter. Since the emphasis here is to study the termination behavior of a typical ARM, only three types of ARMs are included in the study.

They differ only in the re-set frequencies and the reference indices used to determine the coupon rates.

One type of ARM, labeled CD, uses the 6-month CD rate as the benchmark index and is reset semi-annually. The second type of ARM, labeled TRA, uses the 6-month average of the 1-year CMT rate as its benchmark index and is also reset semi-annually. The third type of ARM, labeled TRS, uses the 1-year CMT spot rate as its benchmark index and is reset annually.

There are 203,016 loans in total for the study. Table 5.1a provides an overview of the basic characteristics of the data universe. As the table shows, the distribution of three types of ARMs is 44.8% for CD type, 33.0% for TRA type and 22.2% for TRS type. Other than the re-set frequencies, these three types of ARMs are pretty similar in key loan attributes, such as LTV, margin, ceiling, original coupon rates, etc. For this reason, they are treated as homogenous in the model and only the differences in the re-set frequency are isolated, using a dummy variable.

Table 5.1b compares the basic loan characteristics between prepaid vs non-prepaid loans and defaulted vs non-defaulted loans. Prepaid loans tend to have large loan amounts and lower points, while defaulted loans tend to have small loan amounts and higher points. These preliminary observations are consistent with common wisdoms. That is, larger loans benefit more from prepayment and higher points discourage prepayment. If the loan size is a proxy for the financial status, people with small loans are more vulnerable to any shock, particularly to an interest rate shock, and thus are more likely to default.

Table 5.2a and Table 5.2b report the distributions of prepayment and default by years of origination, termination and observation. Out of 203,016 loans, roughly 90% of them are observed to be prepaid sometime during the period, while 1.2% of them eventually defaulted. In the study by Kau et al. [132], 37.7% of 917,703 fixed rate mortgages are observed to be prepaid sometime during their lives while 0.5%

are observed to be defaulted. The percentages of prepayment and default for ARMs is more than double those for FRMs.

5.3 Model Specification

Lots of studies have been done on the causes of ARM termination (Roll et al.[16], Sanyal [168], Ambrose et al. [10], Davis [63]). Basically, the causes fall into four categories:

1. Natural housing turnover
2. Refinancing
3. Defaults
4. Curtailment

People terminate the ARMs prematurely when they have to sell their houses either because of job relocation or court judgement in a divorce settlement etc. By self-selection, ARM borrowers are more mobile than fixed-rate mortgage borrowers. There are two types of refinancing facing ARM mortgagors: refinancing into another ARM or refinancing into an FRM. The majority of ARM borrowers choose to refinance into fixed-rate mortgages. One explanation for a relative higher ARM default rate is the payment shock that ARM mortgagors experience when ARM coupons are adjusted upward. Curtailment occurs when an ARM is partially prepaid. Previous studies indicate that housing turnover and refinancing are the major drivers of termination, while defaults and curtailments play less important roles.

It is a well-known fact that ARMs are more sensitive to the level of interest rates. Therefore, we have paid particular attention to capturing the impact from the level of interest rates in the model. Specifically, the relative spread between the coupon rate and the 1-year Treasury bill yield is used to measure the impact of refinancing into another ARM. The relative spread between the coupon rate and the 10-year Treasury bond yield is used to measure the impact of refinancing into an FRM. The relative spread is defined as (yield/coupon - 1). To capture the nonlinearity of the impacts, polynomials of the relative spreads up to order 4 are used in the model.

A low 1-year Treasury bill yield relative to current coupon rate of an ARM will induce the mortgagor to refinance into another ARM with a lower coupon rate. As a result, the prepayment hazard will increase; A narrowed spread between the 10-year Treasury bond yield and current coupon rate will provide an incentive for the mortgagor to lock into a fixed rate by refinancing into a FRM, resulting a spike in prepayment. The relative spreads are lagged by two-month in the model to reflect the time necessary to complete the refinancing process.

Other things being equal, the mortgagors will benefit more from refinancing on a larger mortgage balance. The inflation adjusted mortgage balance is used in the model as a proxy for the refinancing benefit.

While prepayment may not necessarily hurt the lenders, financially-savvy mortgagors often benefit from prepayment at the expense of the lenders. To protect themselves from prepayment risk, the lenders have come up some schemes to mitigate and/or control the prepayment risk. One scheme is the provision of prepayment penalty. The other is the points charged up-front. Both schemes have the effect of increasing the costs of refinancing and decreasing the gains. It is quite natural to see prepayment hazards for ARMs with such attachments compressed a bit compared to ARMs without such attachments.

Other variables that are considered in the model include rate-cap, ceiling, LTV, and margin. Rate-cap is a ratio of the original coupon rate to the annual cap. If the original coupon rate is small relative to the cap, the magnitude of upward adjustment will be large. Therefore, it will pay for the mortgagor to refinance either into a FRM or into another ARM. Likewise, a higher ceiling will impose greater upward adjustment risk on the mortgagor, making prepayment more beneficial, increasing prepayment hazards. A higher LTV will make default more valuable and prepayment more costly because of the well-known substitution effect between prepayment and default, lowering the prepayment hazard. Other things being equal, the more frequently the coupon rate is re-set, the greater the interest rate risk to the mortgagor and the more beneficial the prepayment. Thus, semi-annually adjusted ARMs tend to have higher prepayment hazards than do annually-adjusted ARMs.

The major determinants for default are LTV and payment shock. To capture the nonlinearity of the impact of LTV on the default, a quadratic function of LTV is used in the model. The proxy for measurement of a payment shock is the relative spread between the current coupon rate and the 1-year treasury bill rate, the benchmark index that the coupon rate is pegged upon. A widened relative spread indicates potential large payment shock on the reset date, increasing the probability of default.

5.4 Results

The estimation of the models is implemented in SAS using proc PHREG routine. Table 5.3 summarizes the estimated coefficients for the variables just discussed. [1] Table 5.3a and Table 5.3b summarizes effects of covariates on the prepayment and default hazard rate respectively. The estimates of the prepayment model for the most part are consistent with previous discussion. In particular, loan balance, LTV, points and prepayment penalty all have correct signs and are significant. As for impact of interest rates, the narrowed relative spreads all increase prepayment hazards. As expected, LTV and relative spread between coupon rate and 1-year Treasury bill rate have correct signs and are significant. Overall, the results are consistent with those in previous studies.

[1] The adjusted standard errors are reported in the parentheses next to the normal standard errors.

Look at Figure 5.1a that shows the Kaplan-Meier non-parametric estimate of prepayment hazard. The monthly prepayment hazard makes two big jumps in the 12th and 36th months, corresponding to the 1st and the 3rd reset dates. It then slowly creeps upward and peaks in the 120th month. After the 10th year, the monthly hazard declines steadily in the later part of maturity. Towards the end of an ARM's life, there is a burst of prepayment activities due to the nuisance effect: the mortgagors pay out all the remaining mortgage balance because of its relative insignificance in saving compared to hassle of making several payments.

Figure 5.1b indicates that 50% of ARMs will be prepaid by the 45th month and another 25% will be prepaid in the next 30 months. By the 108th month, only 10% of ARMs remain active. Even though the ARMs under study all have 30-year maturities, they actually look more like a 10-year loan because less than 5% of them are around after the 10th year.

Figure 5.2a shows that the monthly default hazard increases at an increasing speed and reaches the peak in about 72th month. It then deceases from the 6th year through 10th year. The monthly default hazard becomes very erratic beyond the 10th year with lots of zeros accompanied by lots of big spikes. As less than 5% of ARMs will ever survive beyond the 10th year, by which time the payment shock periods have already passed, one possible explanation for the erratic behavior of the hazard is that in the later part of an ARM's life, the unobserved heterogeneity of the mortgagors plays a dominant role. When the sample base is small, the point estimate of the hazard can be very sensitive to outliers. By outliers we mean those ARMs whose behaviors are not adequately captured by the factors considered in the model. It may well be that some unobserved factors that are left out of the consideration in the model are responsible for the seemingly erratic behavior. Given the limitation of the data, we have no way to know for sure what those factors are.

The Kaplan-Meier estimate is a non-parametric estimate that treats each incidence of prepayment or default equally. That is, any prepayment or default event will be counted as one event regardless of the difference in the underlying loans. To adjust for the impact of loan characteristics, we also calculated the predicated prepayment and default hazard based on the variables valued at the mean at each point of time. Figure 5.3a indicates that the adjustment for prepayment hazard causes the whole hazard curve to shift downward a little. [2] As a result, the Figure 5.3b shows that the overall survival rate increases a little. It seems that the prepayment hazard rate is not influenced as much by loan's characteristics as by the general economic environment. [3]

The impact of the loan characteristics on the default hazard is more noticeable. Figure 5.4b shows that the Kaplan-Meier estimate has overestimated the cumulative default rate by 4 percentage point. Figure 5.4a indicates that default hazard is driven more by the idiosyncrasies of the mortgagors than by the general economic environment as represented by the baseline hazard.

5.5 Stratification Test

Notice that we have estimated two proportional hazard models based on quarterly stratification. The same models could be estimated without any stratification or based on annually or semi-annually stratification. As mentioned earlier in this

[2] In Figure 5.3a and Figure 5.4a, the dot line represents the simple Kaplan-Meier estimates of the hazard rates treating each event equally. That is, without taking into consideration each loan's characteristics. Each prepayment(default) is counted as one incidence whether it is a high LTV loan or low LTV or large loan amount or small loan amount. The solid line represents the estimates of the hazard rates taking loan's attributes into account. Basically the incidence of each prepayment(default) is weighted by the loan's attributes based on the estimated coefficients from the Cox proportional hazard models.

[3] In Figure 5.3b and Figure 5.4b, the thick line represents the cumulative hazard rate based on simple Kaplan-Meier estimates of the hazard rates while the thin line represents the estimates of the same hazard rates taking into account the loan's characteristics.

chapter, stratification scheme is employed here to convert static baseline hazards to stochastic ones so that they can be later modeled as diffusion processes. One implicit assumption here is that there exists stratum-specific effects. To test the validity of this assumption, a clustering statistic developed by Ridder and Tunali [165] is used that utilizes the information from both the stratified partial likelihood estimation and the unstratified partial likelihood estimation. The null hypothesis is that there is no strata-specific effect. The test statistic takes the following form

$$C = (\hat{\beta}_s - \hat{\beta}_u)' V(\hat{\beta}_s - \hat{\beta}_u)^{-1} (\hat{\beta}_s - \hat{\beta}_u) \tag{5.1}$$

where $\hat{\beta}_s$ and $\hat{\beta}_u$ are vectors of coefficient estimates from a stratified model and an unstratified model respectively. The test statistic C has a chi-square distribution with p degree of freedom, where p is the number of covariates in the model. Table 5.4 reports the test results and indicates that the null hypothesis is rejected in favor of stratification.

Under the regularity conditions specified in Anderson and Gill [12], $\hat{\beta}_s - \hat{\beta}_u$ is asymptotically normal with variance given by

$$V(\hat{\beta}_s - \hat{\beta}_u) = V(\hat{\beta}_s) - V(\hat{\beta}_u) \tag{5.2}$$

where $V(\hat{\beta}_s)$ and $V(\hat{\beta}_u)$ are inverted Hessians for the respective log-likelihoods evaluated at their maximum.

Up to this point, we have just adopted the usual way of modeling the termination of ARMs and the results indicated that the ARMs under current study conform to the general pattern revealed in similar studies completed by other researchers. As indicated in previous discussion, the purpose of this section is to isolate the loan idiosyncratic impact from the baseline hazard. Once the loan specific impact has been identified, the dynamics of the residual baseline hazards can then be attributed to the dynamics of economic environment, which can be modeled via some stochastic processes. And that will the subject of the next chapter.

Table 5.1a

Descriptive Statistics For Entire Sample

Variable	N	Mean	Median	Std Dev.	Minimum	Maximum
LTV	203,016	74.01%	78.00%	16.68%	0.49%	124.30%
OriginalAmount	203,016	$223,275	$171,000	$188,386	$5,000	$2,850,000
Orig_Rate	203,016	5.43%	5.38%	1.36%	2.75%	18.38%
Points	203,016	0.02%	0.00%	1.07%	-7.00%	4.50%
Margin	203,016	2.67%	2.50%	0.90%	0.03%	10.00%
Ceiling	203,016	11.50%	11.75%	1.18%	3.88%	28.00%

Descriptive Statistics By Type of ARM

Product=CD

Variable	N	Mean	Median	Std Dev.	Minimum	Maximum
LTV	91,112	71.67%	75.00%	17.42%	4.00%	124.13%
OriginalAmount	91,112	$194,766	$162,000	$138,641	$12,000	$2,815,000
Orig_Rate	91,112	4.75%	4.38%	1.16%	2.75%	10.00%
Points	91,112	0.28%	0.00%	1.00%	-6.00%	3.50%
Margin	91,112	2.61%	2.50%	1.01%	0.25%	10.00%
Ceiling	91,112	11.14%	10.88%	1.18%	4.25%	22.50%

Product=TRA

Variable	N	Mean	Median	Std Dev.	Minimum	Maximum
LTV	67,078	76.97%	80.00%	16.02%	5.00%	124.30%
OriginalAmount	67,078	$236,210	$172,000	$207,515	$5,000	$2,850,000
Orig_Rate	67,078	5.91%	5.75%	1.02%	3.13%	10.63%
Points	67,078	-0.29%	0.00%	1.13%	-7.00%	4.50%
Margin	67,078	2.68%	2.63%	0.83%	0.03%	10.00%
Ceiling	67,078	11.72%	11.88%	0.84%	7.63%	21.63%

Product=TRS

Variable	N	Mean	Median	Std Dev.	Minimum	Maximum
LTV	44,826	74.34%	78.00%	15.33%	0.49%	123.07%
OriginalAmount	44,826	$261,865	$203,000	$232,020	$11,700	$2,800,000
Orig_Rate	44,826	6.09%	5.75%	1.54%	3.00%	18.38%
Points	44,826	-0.04%	0.00%	0.97%	-5.00%	4.50%
Margin	44,826	2.79%	2.75%	0.75%	0.03%	10.00%
Ceiling	44,826	11.89%	11.88%	1.38%	3.88%	28.00%

Table 5.1b

Descriptive Statistics By Prepayment

Not Prepaid Loans

Variable	N	Mean	Median	Std Dev.	Minimum	Maximum
LTV	20,670	75.12%	80.00%	17.22%	0.53%	122.66%
OriginalAmount	20,670	$167,268	$130,000	$157,014	$10,450	$2,661,100
Orig_Rate	20,670	5.78%	5.38%	1.73%	2.88%	17.25%
Points	20,670	0.10%	0.00%	1.09%	-6.00%	3.50%
Margin	20,670	2.78%	2.50%	1.33%	0.25%	10.00%
Ceiling	20,670	11.74%	11.88%	1.43%	6.25%	21.50%

Prepaid Loans

Variable	N	Mean	Median	Std Dev.	Minimum	Maximum
LTV	182,346	73.89%	78.00%	16.61%	0.49%	124.30%
OriginalAmount	182,346	$229,623	$177,600	$190,583	$5,000	$2,850,000
Orig_Rate	182,346	5.39%	5.38%	1.31%	2.75%	18.38%
Points	182,346	0.01%	0.00%	1.06%	-7.00%	4.50%
Margin	182,346	2.66%	2.50%	0.84%	0.03%	10.00%
Ceiling	182,346	11.47%	11.63%	1.14%	3.88%	28.00%

Descriptive Statistics By Default

Not Defaulted Loans

Variable	N	Mean	Median	Std Dev.	Minimum	Maximum
LTV	200,540	73.87%	78.00%	16.70%	0.49%	124.30%
OriginalAmount	200,540	$224,125	$172,000	$189,040	$5,000	$2,850,000
Orig_Rate	200,540	5.42%	5.38%	1.36%	2.75%	18.38%
Points	200,540	0.02%	0.00%	1.07%	-7.00%	4.50%
Margin	200,540	2.67%	2.50%	0.90%	0.03%	10.00%
Ceiling	200,540	11.49%	11.75%	1.18%	3.88%	28.00%

Defaulted Loans

Variable	N	Mean	Median	Std Dev.	Minimum	Maximum
LTV	2,476	85.51%	90.00%	9.39%	28.00%	122.66%
OriginalAmount	2,476	$154,403	$135,000	$103,522	$10,450	$1,435,000
Orig_Rate	2,476	5.73%	5.50%	1.45%	3.00%	14.63%
Points	2,476	0.22%	0.00%	1.15%	-6.00%	3.50%
Margin	2,476	2.71%	2.63%	0.84%	1.25%	10.00%
Ceiling	2,476	11.91%	11.88%	1.18%	8.50%	19.63%

Table 5.2a

Loan Origination,Default and Prepayment by Year of Origination

Year	# Originated	# Prepaid	# Defaulted	% Prepaid	% Default
1977	1	1	-	100.00	-
1978	2	1	-	50.00	-
1979	6	5	-	83.33	-
1980	9	3	-	33.33	-
1981	11	4	1	36.36	9.09
1982	25	13	-	52.00	-
1983	181	88	3	48.62	1.66
1984	927	523	13	56.42	1.40
1985	909	525	18	57.76	1.98
1986	464	299	11	64.44	2.37
1987	484	291	8	60.12	1.65
1988	457	305	6	66.74	1.31
1989	304	202	6	66.45	1.97
1990	3,659	3,111	78	85.02	2.13
1991	17,584	15,786	324	89.77	1.84
1992	33,330	29,074	787	87.23	2.36
1993	39,262	35,977	355	91.63	0.90
1994	42,911	38,674	484	90.13	1.13
1995	13,753	12,303	238	89.46	1.73
1996	13,912	13,230	85	95.10	0.61
1997	11,663	11,168	26	95.76	0.22
1998	13,930	13,055	13	93.72	0.09
1999	5,767	5,049	10	87.55	0.17
2000	3,272	2,489	9	76.07	0.28
2001	186	163	1	87.63	0.54
2002	7	7	-	100.00	-
Total	203,016	182,346	2,476	89.82	1.22

Loan Termination,Default and Prepayment by Year of Termination

Year	# Terminated	# Prepaid	# Defaulted	% Prepaid	% Default
1992	1	1	0	100.00	-
1993	4,130	4125	4	99.88	0.10
1994	10,513	10451	23	99.41	0.22
1995	12,055	11957	78	99.19	0.65
1996	17,844	14366	211	80.51	1.18
1997	26,651	26010	611	97.59	2.29
1998	42,119	39346	566	93.42	1.34
1999	23,951	23375	523	97.60	2.18
2000	12,582	12354	214	98.19	1.70
2001	18,616	18264	110	98.11	0.59
2002	11,506	11266	68	97.91	0.59
2003	7,884	7815	43	99.12	0.55
2004	15,164	3016	25	19.89	0.16
Total	203,016	182,346	2,476	89.82	1.22

Table 5.2b

Active Loans Observed, Default and Prepayment by Year of Observation

Beg_Y

Year	# Observed	# Prepaid	# Defaulted	% Prepaid	% Default
1989	13	10	0	76.92	-
1990	3,390	2920	72	86.14	2.12
1991	17,325	15606	324	90.08	1.87
1992	32,786	28659	779	87.41	2.38
1993	39,108	35899	348	91.79	0.89
1994	41,814	37897	452	90.63	1.08
1995	12,888	11648	212	90.38	1.64
1996	13,174	12677	61	96.23	0.46
1997	11,688	11227	21	96.06	0.18
1998	13,889	13021	13	93.75	0.09
1999	13,464	10113	184	75.11	1.37
2000	3,274	2491	9	76.08	0.27
2001	196	171	1	87.24	0.51
2002	7	7	0	100.00	-
Total	203,016	182,346	2,476	89.82	1.22

Active Loans Observed, Default and Prepayment by Year of Observation

End_Y

Year	# Observed	# Prepaid	# Defaulted	% Prepaid	% Default
1993	5,216	5,212	4	99.92	0.08
1994	9,943	9,884	25	99.41	0.25
1995	12,867	11,435	76	88.87	0.59
1996	18,037	15,833	276	87.78	1.53
1997	27,553	26,942	577	97.78	2.09
1998	41,790	39,014	572	93.36	1.37
1999	22,663	22,113	501	97.57	2.21
2000	12,752	12,528	212	98.24	1.66
2001	19,097	18,750	105	98.18	0.55
2002	10,787	10,541	68	97.72	0.63
2003	7,409	7,334	37	98.99	0.50
2004	14,902	2,760	23	18.52	0.15
Total	203,016	182,346	2,476	89.82	1.22

Table 5.3
Coefficients Estimates for Proportional Hazard Model with Competing Risks of Default and Prepayment (Stratified)

Prepayment Model

Variable	DF	Parameter Estimate	Standard Error		Chi-Square	Pr>ChiSq	Hazard Ratio
ln_begbal	1	3.9103E-01	0.0038	(0.0037)	10,828.48	<.0001	1.48
LTV	1	-3.1400E-03	0.0001	(0.0001)	467.19	<.0001	1.00
Points	1	-9.0180E-02	0.0031	(0.0031)	824.52	<.0001	0.91
Margin	1	-6.0670E-02	0.0027	(0.0026)	504.38	<.0001	0.94
Ceiling	1	8.3780E-02	0.0058	(0.0057)	208.02	<.0001	1.09
Rate_cap	1	-9.2190E-02	0.0048	(0.0047)	365.37	<.0001	0.91
i_Penalty	1	-5.1134E-01	0.0068	(0.0066)	5,699.07	<.0001	0.60
spd_short1	1	-1.7000E-02	0.0014	(0.0013)	156.57	<.0001	0.98
spd_short2	1	-3.2720E-04	0.0000	(0.0000)	172.20	<.0001	1.00
spd_short3	1	1.0214E-07	0.0000	(0.0000)	0.40	0.5258	1.00
spd_short4	1	1.3569E-08	0.0000	(0.0000)	108.90	<.0001	1.00
spd_long1	1	-1.3530E-02	0.0010	(0.0009)	195.90	<.0001	0.99
spd_long2	1	2.4070E-04	0.0000	(0.0000)	422.21	<.0001	1.00
spd_long3	1	-1.6765E-06	0.0000	(0.0000)	78.15	<.0001	1.00
spd_long4	1	3.7640E-09	0.0000	(0.0000)	22.53	<.0001	1.00
i_AdjFrq	1	-1.8720E-01	0.0149	(0.0141)	158.36	<.0001	0.83

Default Model

Variable	DF	Parameter Estimate	Standard Error		Chi-Square	Pr>ChiSq	Hazard Ratio
ln_begbal	1	-1.7701E-01	0.0401	(0.0393)	19.49	<.0001	0.84
LTV	1	1.1490E-01	0.0217	(0.0212)	28.15	<.0001	1.12
LTV_sq	1	-3.0950E-04	0.0001	(0.0001)	5.25	0.0219	1.00
Points	1	-4.4310E-02	0.0288	(0.0282)	2.37	0.1234	0.96
Margin	1	-1.3678E-01	0.0208	(0.0204)	43.12	<.0001	0.87
Ceiling	1	7.1390E-02	0.0458	(0.0449)	2.43	0.1193	1.07
Rate_cap	1	1.4996E-01	0.0380	(0.0372)	15.57	<.0001	1.16
i_Penalty	1	-1.6478E-01	0.0663	(0.0650)	6.18	0.013	0.85
spd_short1	1	-1.3700E-02	0.0037	(0.0036)	14.00	0.0002	0.99
spd_short2	1	3.5270E-04	0.0000	(0.0000)	133.70	<.0001	1.00
i_AdjFrq	1	6.8978E-01	0.1211	(0.1187)	32.43	<.0001	1.99

Note:

spd_short1:	(t1y_L2/cpn_lag2-1)*100
spd_long1:	(t10y_L2/cpn_lag2-1)*100
ln_begbal	log(begbal/cmphi)
Rate_cap:	Orig_rate/year_cap
i_Penalty:	1, if prepay penalty applies; 0, o.w.
i_AdjFrq:	1, if reset annually; 0, o.w.

Figure 5.0a Effects of Covariates on Prepayment Hazard

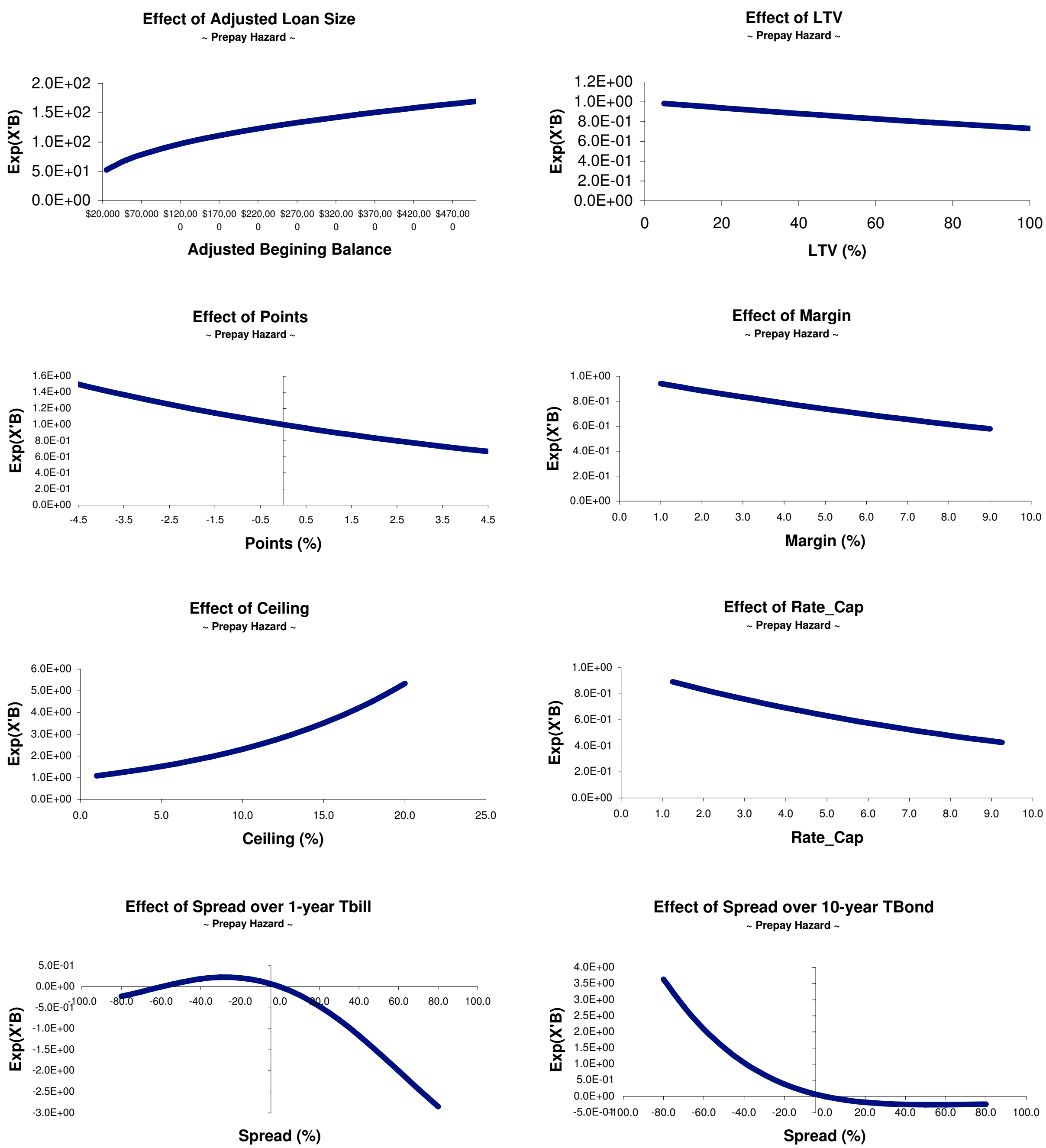
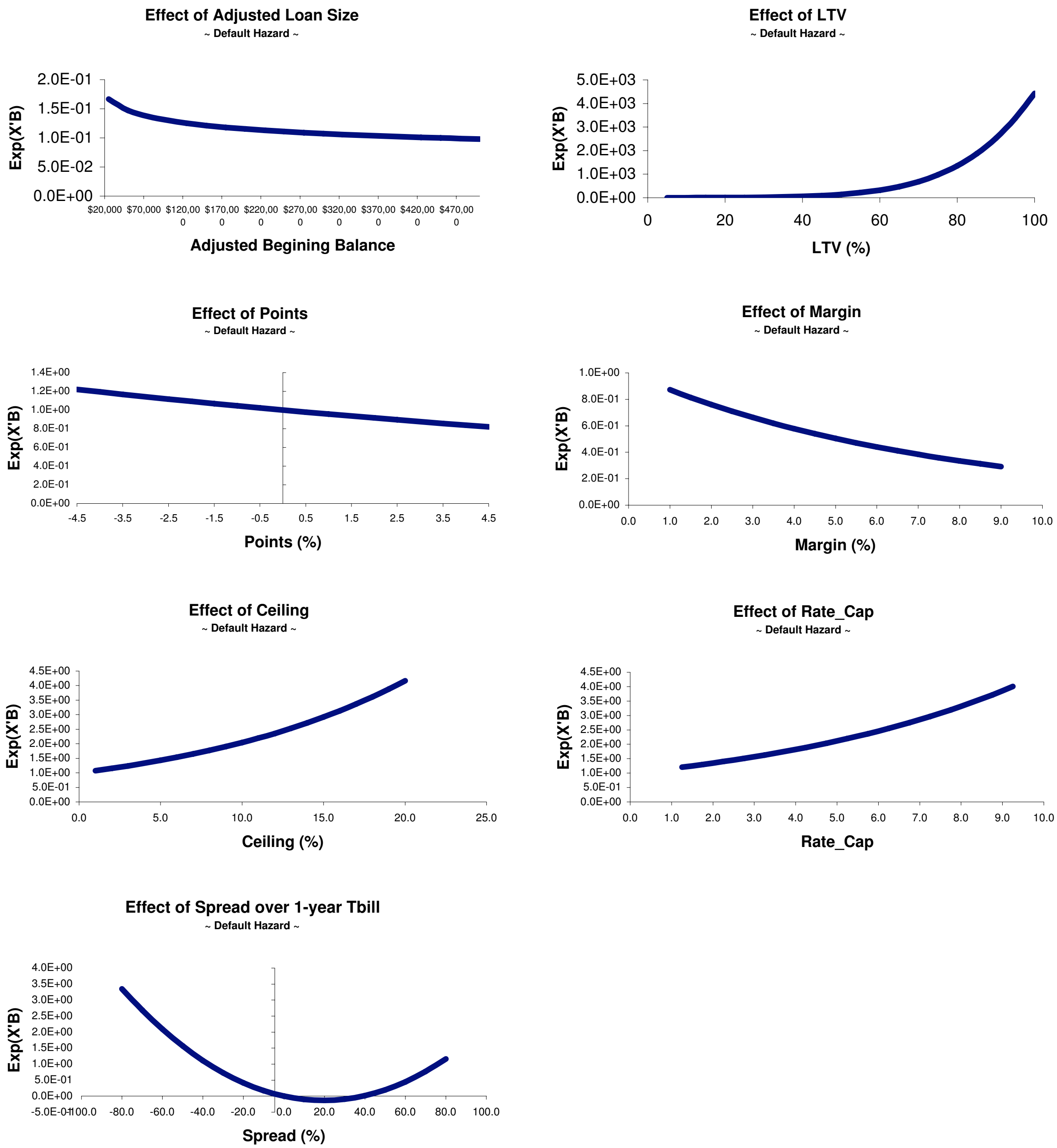


Figure 5.0b Effects of Covariates on Default Hazard



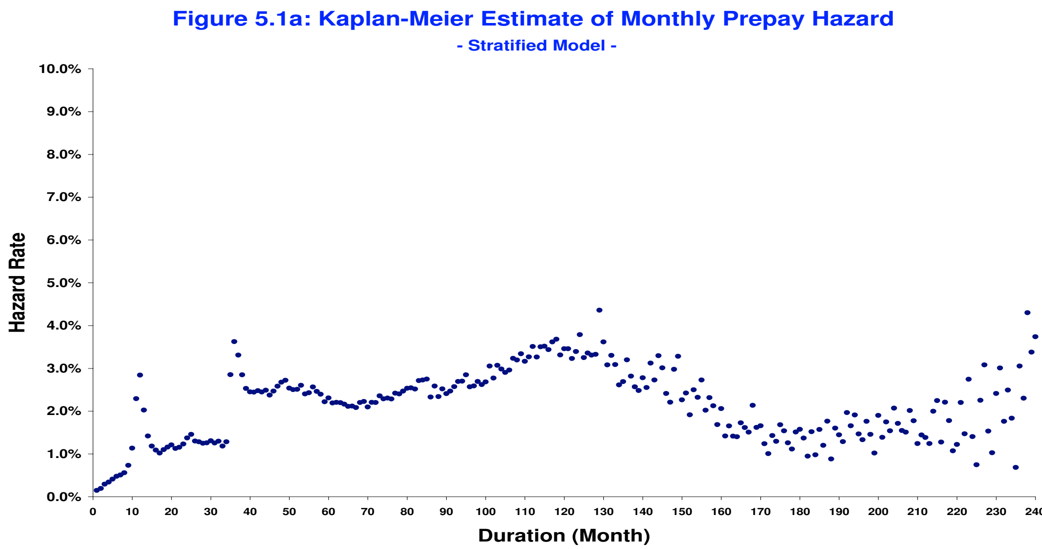


Figure 5.1a: Kaplan-Meier Estimate of Monthly Prepay Hazard
- Stratified Model -
Hazard Rate
Duration (Month)
10.0%
9.0%
8.0%
7.0%
6.0%
5.0%
4.0%
3.0%
2.0%
1.0%
0.0%
0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200 210 220 230 240

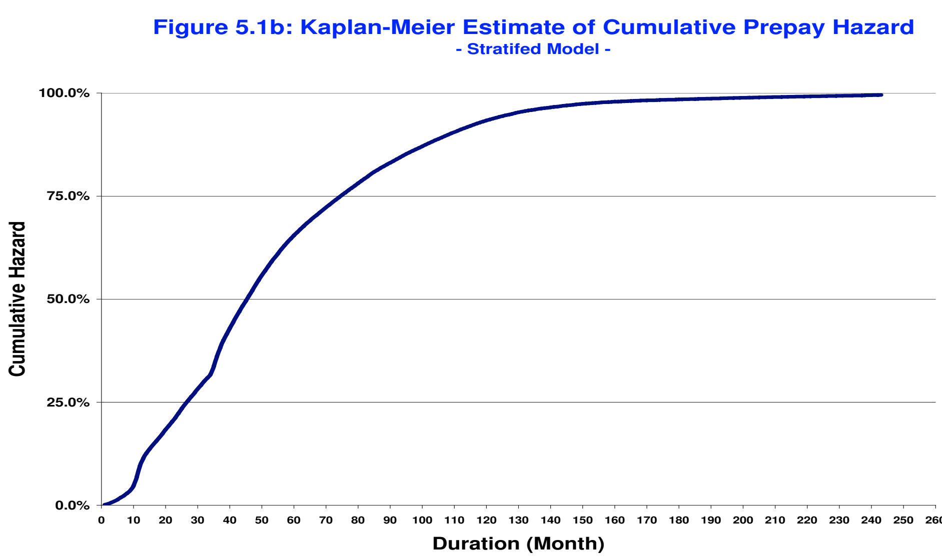


Figure 5.1b: Kaplan-Meier Estimate of Cumulative Prepay Hazard
- Stratifed Model -
Cumulative Hazard
Duration (Month)
100.0%
75.0%
50.0%
25.0%
0.0%
0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200 210 220 230 240 250 260

Figure 5.2a: Kaplan-Meier Estimate of Monthly Default Hazard
- Stratified Model -

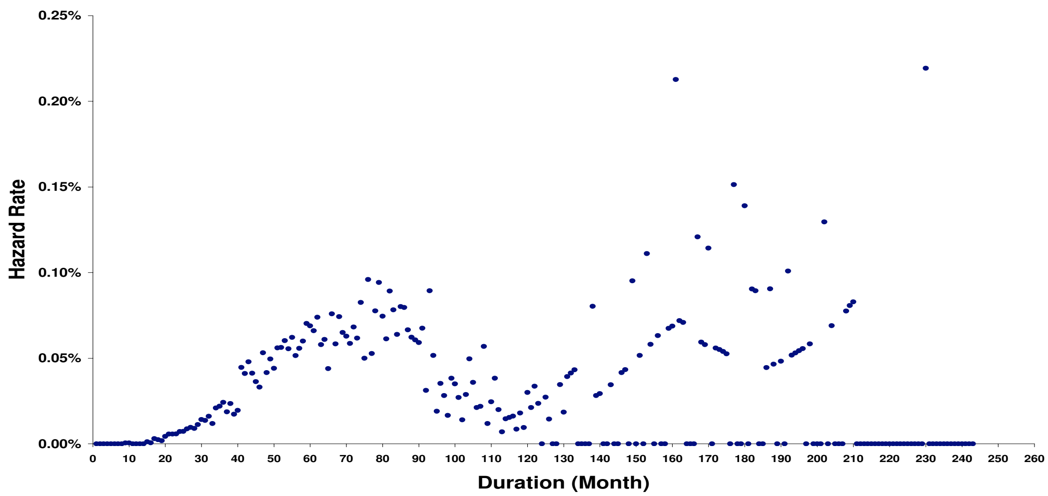
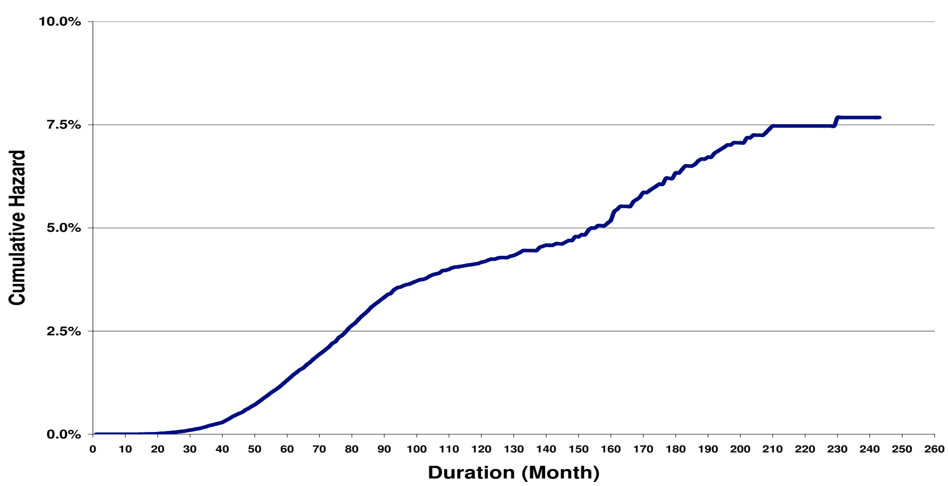


Figure 5.2b: Kaplan-Meier Estimate of Cumulative Default Hazard
- Stratified Model -



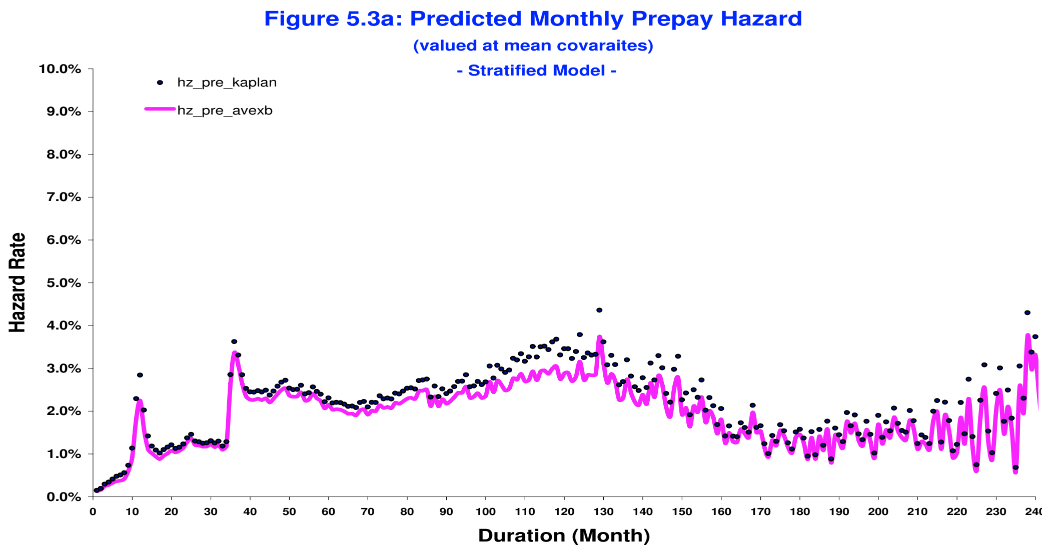


Figure 5.3a: Predicted Monthly Prepay Hazard
(valued at mean covaraites)
- Stratified Model -
hz_pre_kaplan
hz_pre_avexb
Hazard Rate
Duration (Month)
10.0%
9.0%
8.0%
7.0%
6.0%
5.0%
4.0%
3.0%
2.0%
1.0%
0.0%
0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200 210 220 230 240

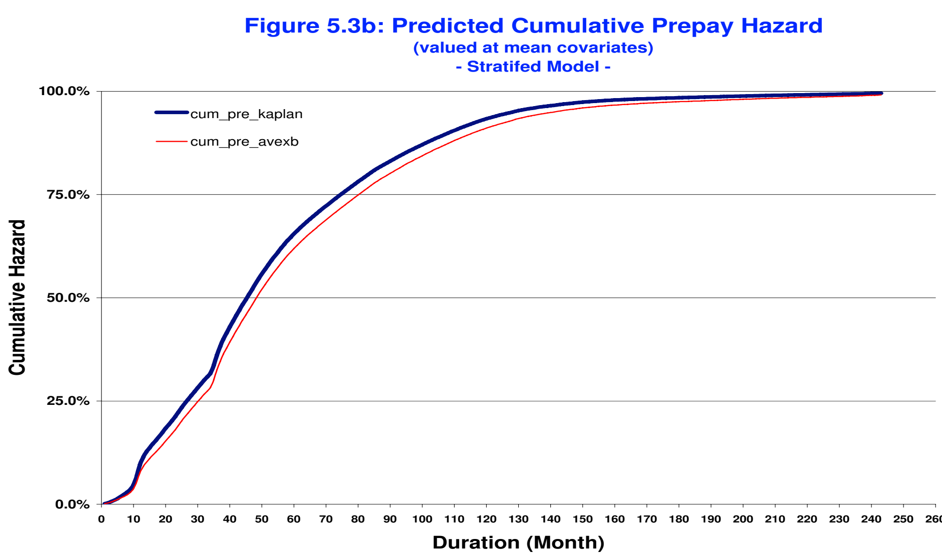


Figure 5.3b: Predicted Cumulative Prepay Hazard
(valued at mean covariates)
- Stratifed Model -
cum_pre_kaplan
cum_pre_avexb
Cumulative Hazard
Duration (Month)
100.0%
75.0%
50.0%
25.0%
0.0%
0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200 210 220 230 240 250 260

Figure 5.4a: Predicted Default Hazard

(valued at mean covariates)

- Stratified Model -

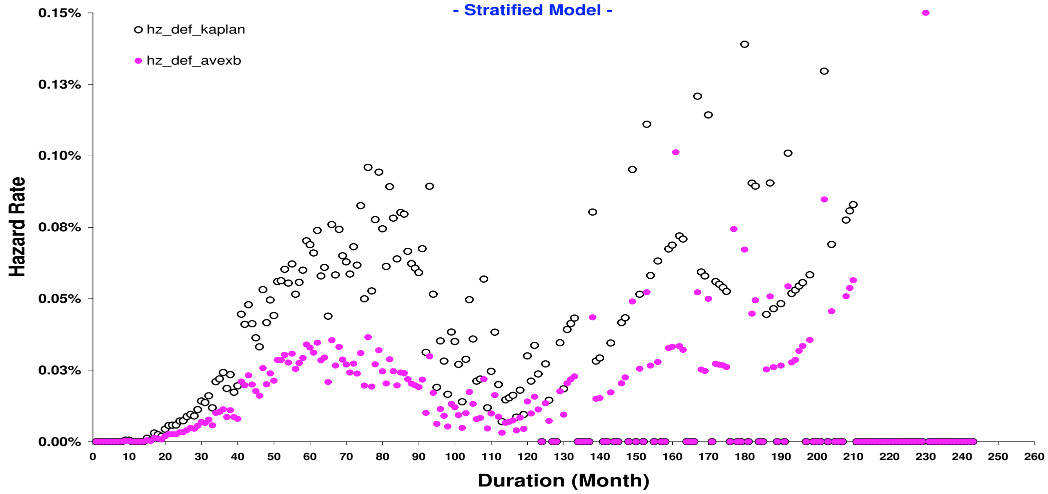


Figure 5.4b: Predicted Cumulative Default Hazard

(valued at mean covariates)

- Stratified Model -

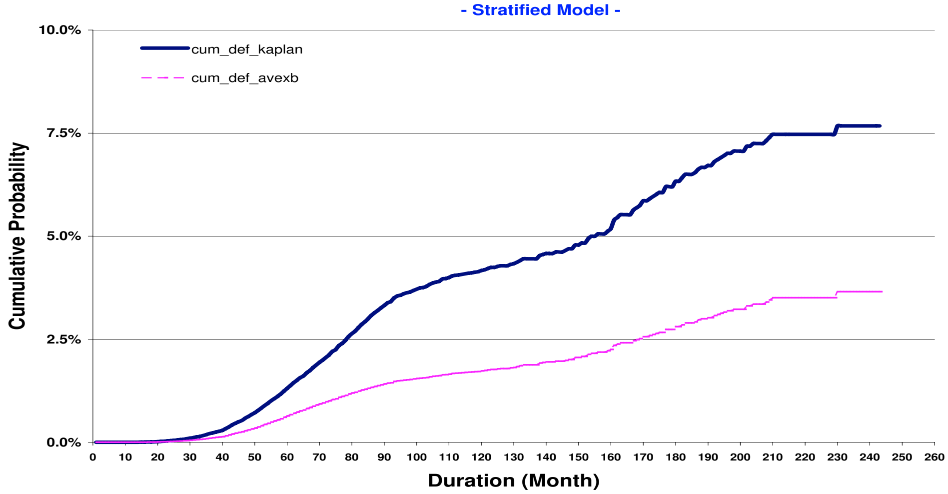


Table 5.4
Test for the Presence of Stratification
Sample Size: N=203,016
Degree of Freedom: p=26

Stratification	**C-Statistic**	**P-Value**
Quarterly vs None	2552.17	0.0000
Quarterly vs Annually	646.04	0.0000
Quarterly vs Semi-annually	211.48	0.0000

Note:
- The C-statistic is a clustering statistic that has a chi-square distribution with p degree of freedom under the null hypothesis.

- The null hypothesis is that there is no stratum-specific effect.

Chapter 6

Estimation of diffusion processes for prepayment and default hazard

A Cox proportional hazard model is often used to model the termination of mortgages whether it is a FRM or ARM. The advantage of the model is its ability to allow us to study the impact of loan idiosyncrasy on the termination of a loan, without imposing an explicit assumption on the baseline function. The baseline function is determined by the data under study non-parametrically. Most of the previous studies also use non-parametric estimates of the baseline function in predications of the termination hazard. The implicit assumption those researchers makes, though, is that the baseline function is deterministic. That is, the baseline hazard rate is always the same for a given duration regardless the general economic environment. This implicit assumption is probably the result of a belief that all the economic impacts are supposed to be captured by the covariates, a very strong assumption unlikely to be true in the reality. As a result, the resultant prediction is static and not conducive to the risk-neutral pricing of loans.

Kau et al. [132] make the first attempt to reconcile the inconsistency between the assumption of a Cox proportional model and the economic reality. They propose a scheme that incorporates the dynamic nature of the general environment into a baseline hazard function. The advantage of a dynamic baseline hazard function is its flexibility to capture the residual impact left by the covariate terms. One side-effect of the scheme, though, is the possibility of model mis-specification due to the assumptions made explicitly on the stochastic processes. In this chapter, we will discuss the details of implementing the scheme proposed by Kau et al. [132] in studying the dynamics of baseline hazards for ARMs.

6.1 The State-space Models of Baseline Prepayment and Default

Generally speaking, in a state-space framework, state variables that are inaccessible directly evolve through time according to certain stochastic laws that have representations as stochastic diffusion processes. At each point of time, the state variables can be measured or observed indirectly via a measurement mechanism. In the case of ARMs, two state-space models can be constructed as follows.

First, the state variables are baseline prepayment and default hazards. We have assumed that their dynamics can be well represented by two independent square-root diffusion processes with mean reversion. One reason for this particular specification is the non-negativity of a square-root diffusion process, reflecting the fact that the baseline hazard is non-negative at any time.

$$\lambda_0^p(t) = x_t^1 \quad \lambda_0^d(t) = x_t^2 \tag{6.1}$$

$$dx_t^1 = k_1[\theta_1(t) - x_t^1]dt + \sigma_1\sqrt{x_t^1}dw_t^1 \tag{6.2}$$

$$dx_t^2 = k_2[\theta_2(t) - x_t^2]dt + \sigma_2\sqrt{x_t^2}dw_t^2 \tag{6.3}$$

where $\theta_1(t)$ and $\theta_2(t)$ are time-varying mean reverting level functions for baseline prepayment and default hazards respectively. In the discussion that follows, we will elaborate on the specifications of these two functions because they are critical in state-space modeling.

At any given point of time, we cannot observe the baseline prepayment and default hazards but only the fact that a number of ARMs have prepaid or defaulted. We further assume that the prepayment and default hazards are related to the corresponding baseline hazards in a multiplicative manner.

$$\lambda_t^p = \lambda_0^p(t)\, e^{X_p'(t)\beta_p}, \quad \lambda_t^d = \lambda_0^d(t)\, e^{X_d'(t)\beta_d} \tag{6.4}$$

where X_p and X_d are sets of mortgage specific variables deemed to reflect the mortgage specific impact on top of the baseline hazard rates, and β_p and β_d are parameters associated with corresponding variables.

The number of prepayments or defaults can then be approximated using Poisson distributions with means equal to the hazard rates. Assuming that each individual mortgage is terminated independently, the number of prepayments or defaults at any given time is then binomially distributed with the probability equal to the hazard rate. When the probability of an event occurring is very small relative to the number of trials, the Poisson distribution will provide a very good approximation to the binomial distribution, so

$$Pr(N_p = k) = e^{-\lambda_t^p} \frac{(\lambda_t^p)^k}{k\,!} \tag{6.5}$$

$$Pr(N_d = k) = e^{-\lambda_t^d} \frac{(\lambda_t^d)^k}{k\,!} \tag{6.6}$$

where N_p and N_d are the number of prepayment and default respectively.

A complete state-space model of baseline prepayment and default hazards is as follows:

$$
\begin{aligned}
&\text{The State (Transition) Equation:}\\
&dx_t^1 = \kappa_1[\theta_1(t) - x_t^1]dt + \sigma_1\sqrt{x_t^1}dw_t^1\\
&dx_t^2 = \kappa_2[\theta_2(t) - x_t^2]dt + \sigma_2\sqrt{x_t^2}dw_t^2\\
&\lambda_0^p(t) = x_t^1, \quad \lambda_0^d(t) = x_t^2\\
&\text{The Observation (Measurement) Equation:}\\
&Pr(N_p = k) = e^{-\lambda_t^p}\frac{(\lambda_t^p)^k}{k\,!}\\
&Pr(N_d = k) = e^{-\lambda_t^d}\frac{(\lambda_t^d)^k}{k\,!}\\
&\lambda_t^p = \lambda_0^p(t)\, e^{X_p'(t)\beta_p}, \quad \lambda_t^d = \lambda_0^d(t)\, e^{X_d'(t)\beta_d}
\end{aligned}
$$

where dw_t^1 and dw_t^2 are independent diffusion processes.

6.2 Implementation Scheme

In the previous chapter, a Cox proportional model was utilized with stratification based on the origination quarter of an ARM. According to the scheme proposed by Kau et al. [132], ARMs in the same stratum are assumed to share one realization of baseline hazards from the underlying diffusion processes. ARMs in different strata are assumed to follow different realizations of the baseline hazards from the same diffusion processes. At any given time t, the observed number of prepayments and defaults within a given stratum is related to one realization of the underlying baseline hazards. Thus K different strata are related to K different underlying baseline hazards at time t. The log-likelihood of observing $N_p(t,k)$ prepayments and $N_d(t,k)$ defaults at time t among stratum k can be represented as $log f_p(N_{t,k}^p)$ and $log f_d(N_{t,k}^d)$ respectively. The total log-likelihood of observing different numbers of prepayments and default during the life of ARMs can then be formalized as the following:

$$log f_p = \sum_{k=1}^{K} \sum_{t=1}^{T_k} log f_p(N_{t,k}^p) \tag{6.7}$$

$$log f_d = \sum_{k=1}^{K} \sum_{t=1}^{T_k} log f_d(N_{t,k}^d) \tag{6.8}$$

where T_k is the max duration in months for stratum k.

The ARMs in this study are pooled into 73 strata by month of origination. [1] For each stratum, the number of prepayments and defaults at time t and the corresponding covariate terms are defined as follows.

$$N_{t,k}^p \triangleq \sum_{i=1}^{M(t,k)} \mathbb{I}_p(i|t,k) \tag{6.9}$$

$$N_{t,k}^d \triangleq \sum_{i=1}^{M(t,k)} \mathbb{I}_d(i|t,k) \tag{6.10}$$

$$e^{X'_p(t,k)\beta_p} \triangleq \sum_{i=1}^{M(t,k)} e^{X'_p(i|t,k)\beta_p} \tag{6.11}$$

$$e^{X'_d(t,k)\beta_d} \triangleq \sum_{i=1}^{M(t,k)} e^{X'_d(i|t,k)\beta_d} \tag{6.12}$$

where $\mathbb{I}_p, \mathbb{I}_d$ are indicator functions of whether there is a prepayment or default and $X_p, X_d, \beta_p, \beta_d$ are from previous Cox proportional hazard models.

Essentially, the original loan-based data is transformed into panel-style data where the cross sectional dimension is the strata and the time series dimension is the duration of the loans. The transformation produced $7,273$ panel-style observations out of $203,016$ ARMs.

[1] There are total 93 strata in the original data. 20 strata, most of which are those loans originated in early 80's, do not have many observations, fewer than 10 or so. Therefore, the hazard rates can be greatly distorted by one or two prepaid or defaulted loans, which caused problems in the estimation of the processes. As a result, they are not used in the estimation.

The mean-reverting functions in the state-space models are not yet specified because they have to be developed based on the investigation of the empirical estimates of the baseline hazards. In the process of transforming the original data into panel-style data, we estimated times series of the empirical baseline hazards for prepayment and default respectively for the whole duration of ARMs. The estimates at time t are basically the average baseline hazards with respect to all strata that are still active at time t. The estimation method is based on the idea elaborated before. That is, the underlying baseline hazard at time t will be randomly distributed according to some probability law and the empirical estimate of the baseline hazard based on one stratum corresponds to one observation of the baseline hazard. The mean baseline hazard estimated at time t can then be used as an approximation of the mean-reverting level at that time. By fitting to these time series, we identified the specification of the mean-reverting level functions for prepayment and default respectively. Figure 6.1 and Figure 6.2 show the empirical estimates of the baseline hazard for prepayment and default respectively. We then developed two function specifications best fitted to the empirical curves: a polynomial of $7th$ order for the baseline prepayment hazard and an un-normalized gamma function for the baseline default hazard. The empirical baseline hazards for prepayment and default are scaled upward by a factor of 10^3 and 10^8 respectively to avoid numerical problems. The coefficients will serve as the initial values in the optimization process:

$$\theta_p(t) = \sum_{i=0}^{7} \alpha_i t^i \tag{6.13}$$

$$\theta_d(t) = \frac{\beta}{\Gamma(\rho) r^\rho} t^{\rho-1} e^{-t/r} \tag{6.14}$$

The set of parameters to be optimized are:

Prepayment hazard:	$\{k_1, \sigma_1, \alpha_0, \ldots, \alpha_7\}$
Default hazard:	$\{k_2, \sigma_2, \beta, \rho, r\}$

With the models for baseline prepayment and default hazards fully specified, the estimation technique that has been discussed in chapter 4 can be used to estimate the coefficients of the diffusion processes by maximizing the log-likelihood of 6.7 and 6.8. The number of intermediate steps used in the optimization is 30 for both prepayment and default. The number of particles used to simulate the distribution is 1000 with the number of re-samples equal to 1300.[2]

6.3 Results and Specification Test

The estimated coefficients are listed in Table 6.1 and Table 6.2. By standard measures, all the estimates are significant.

Table 6.3 and Table 6.4 provide comparisons between the specification and alternative ones. All the specifications differ only in the specification of the trend function. For the prepayment process, we have experimented with constant trends, polynomial trends of different orders, gamma trends, etc. Given the apparent spikes around the 1st year and the 3rd year, we also tried a hybrid function that consists of a 3rd order polynomial plus two normal spike functions. The two spike functions were fixed at the 1st and 3rd years respectively with two parameters associated with each spike

[2]The particle filter technique is computationally intensive. Trade-off has to be made between more accurate estimation of parameters and feasible computing time. Normally, for short time series, a large number of particles (40,000 to 50,000) will be needed in the estimation of the likelihood. In our case, the time series is long (up to 240 months for some stratum) and we found there was no major difference in the estimated likelihood using 1000 particles vs 2500 particles. The estimated likelihoods usually have same first 3 to 4 digits to the left of the decimal and only differ on the last couple of digits to the right of the decimal.

function. One parameter measured the variance and the other measured the height of the spike.

For the default process, we have also experimented with constant trends, polynomial trends of different orders, chi-square and gamma trends etc.

Figure 6.5 and Figure 6.6 provide an overview of MLE estimates of trend functions relative to their stratum-based empirical counterparts. Clearly, there is lots of volatility in both processes across strata. The baseline default hazard process exhibits a clearer pattern than does the prepayment process. A gamma trend function captured well the overall profile across strata and the mean-reversion is relatively strong.

The baseline prepayment hazard process is more volatile and the mean-reverting tendency is weak. None of the trend specifications did a decent job in capturing the overall pattern, which remains murky for the most part.

The determination of the final specification for the trend functions is based on the AIC (Akaike Information Criterion) and SBC (Bayesian Information Criterion), two usual criteria for model selection.

A standard measure of goodness of fit for count data is the Pearson statistic (see Cameron and Trivedi [39], $p151 \sim 152$).

$$P = \sum_{i=1}^{n} \frac{(y_i - \hat{\mu}_i)^2}{\hat{\omega}_i} \tag{6.15}$$

$$E \sum_{i=1}^{n} \frac{(y_i - \mu_i)^2}{\omega_i} = n \tag{6.16}$$

where $\hat{\mu_i}$ and $\hat{\omega_i}$ are estimates of μ_i and ω_i. If the mean and variance are correctly specified, then Eq (6.16) will hold. The P is then compared to $(n - k)$, where k is a degree of freedom correction term. For a Poisson model, $\omega_i = \mu_i$. The judgement rule is that $P > n - k$ indicates over-dispersion while $P < n - k$ indicates under-dispersion. Over-dispersion means the true variance exceeds the mean.

Several specification tests are conducted and reported in Figure 6.3 and Figure 6.4. The Pearson statistics for both models indicate an evidence of under-dispersion, where the true variance smaller than the mean. Because the Pearson statistic is based on an assumption that is only true in a special case, not assumed to apply here, its diagnostic power of model mis-specification is somewhat limited.

To formally test whether the model specifications are correct, we also have conducted the Conditional Moment test as discussed in Cameron and Trivedi ([39], p $47 \sim 51$). The results indicate that the default model cannot be rejected while the prepayment model is rejected. Rejection is often interpreted as indication of model misspecification, although it is not always immediately apparent in what direction the model is misspecified. One possible cause of the rejection of the prepayment model is that the baseline prepayment hazard apparently has some jumps in early periods as indicated in the empirical estimate of the baseline prepayment hazard. As a result, a diffusion process that is only capable of representing small and continuous movement cannot capture such jump dynamics. While a jump-diffusion processes have been used in the some theoretical research, the application of the process using real data is still in its infancy. The major hurdle is the difficulty in estimating the jump-related parameters. The current study is meant to explore ways of incorporating observable prepayment and default information into a risk-neutral pricing framework. The major consideration here is the trade-off between the tractability of the model and correct specification.

Figure 6.1: Estimated Mean Baseline Prepay Hazard
(Scale: 10E+3)

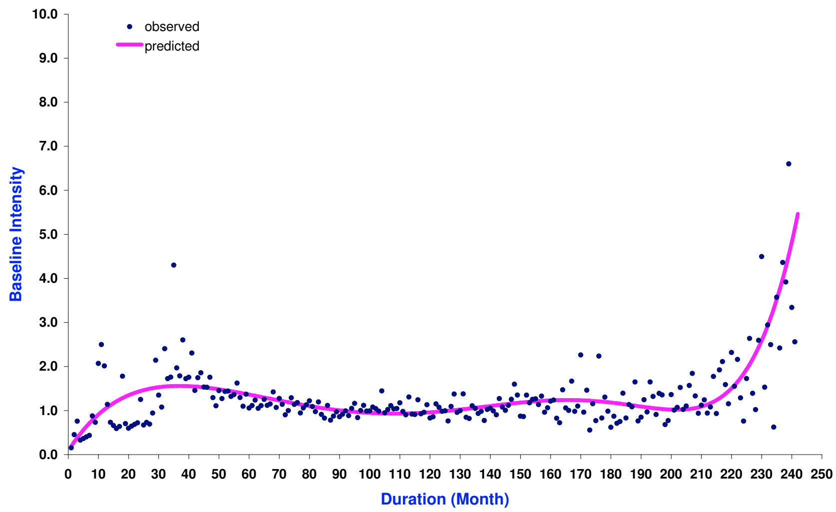


Figure 6.2: Estimated Mean Baseline Default Hazard
(Scale: 10E+8)

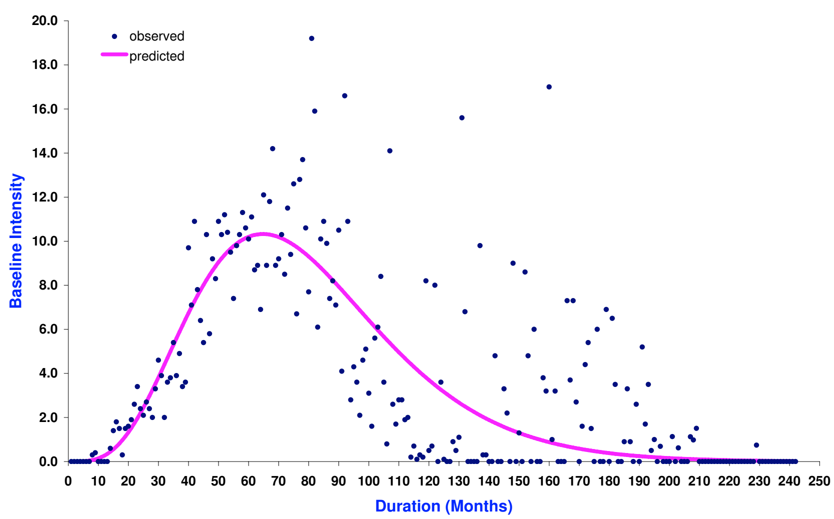


Table 6.1: Maximum Loglikelihood Estimation the Diffusion Process
- the Baseline Prepayment Hazard -

Sample Size: N=7,273
Numver of Mass Particles: M=1000
Numver of Resample Particles: R=1300
Discretization Scheme: Euler
Monthly Time Subintervals: 30
Scale factor: 10^3

Coefficients	Estimate	Std Err
k_1	0.1270	0.0301
σ_1	1.4719	0.0057
α_0	-0.4571	0.2812
α_1	18.6780	3.9179
α_2	-28.1040	11.0260
α_3	58.5880	19.4920
α_4	-81.9270	29.4970
α_5	59.5290	20.7340
α_6	-22.0140	9.2148
α_7	3.5197	2.2630
Log-Likelihood	-19,610.58	

Table 6.2: Maximum Loglikelihood Estimation the Diffusion Process
- the Baseline Default Hazard -

Sample Size: N=7,273
Numver of Mass Particles: M=1000
Numver of Resample Particles: R=1300
Discretization Scheme: Euler
Monthly Time Subintervals: 30
Scale factor: 10^8

Coefficients	Estimate	Std Err
k_2	0.4736	0.0874
σ_2	3.6314	0.2221
β	51.1110	8.2747
ρ	4.6544	1.1046
r	1.1266	0.3246
Log-Likelihood	-3,364.33	

Figure 6.3: Specification Test for Prepayment Hazard Process

Prepayment Model
Predicted vs Actual Probabilities

Counts	Actual	Predicted	|diff|
0	24.8%	24.1%	0.0076
1-10	29.6%	30.4%	0.0072
11-20	11.5%	11.8%	0.0030
21-30	8.1%	8.2%	0.0008
31-40	6.0%	6.2%	0.0016
41-50	4.5%	3.8%	0.0068
51-60	3.5%	2.9%	0.0062
61-70	2.7%	2.2%	0.0046
71-80	1.8%	1.7%	0.0006
81-90	1.2%	1.4%	0.0013
91-100	0.9%	1.1%	0.0020
>100	5.3%	6.3%	0.0097
	100.0%	100.0%	
Pearson_statistic:		6,262	
Degrees of Freedom:		7,190	

Conditional Moment Test
Degrees of Freedom: Q=12

Covariance Used	Test Statistic	P-Value
OPG	299.06	<0.05

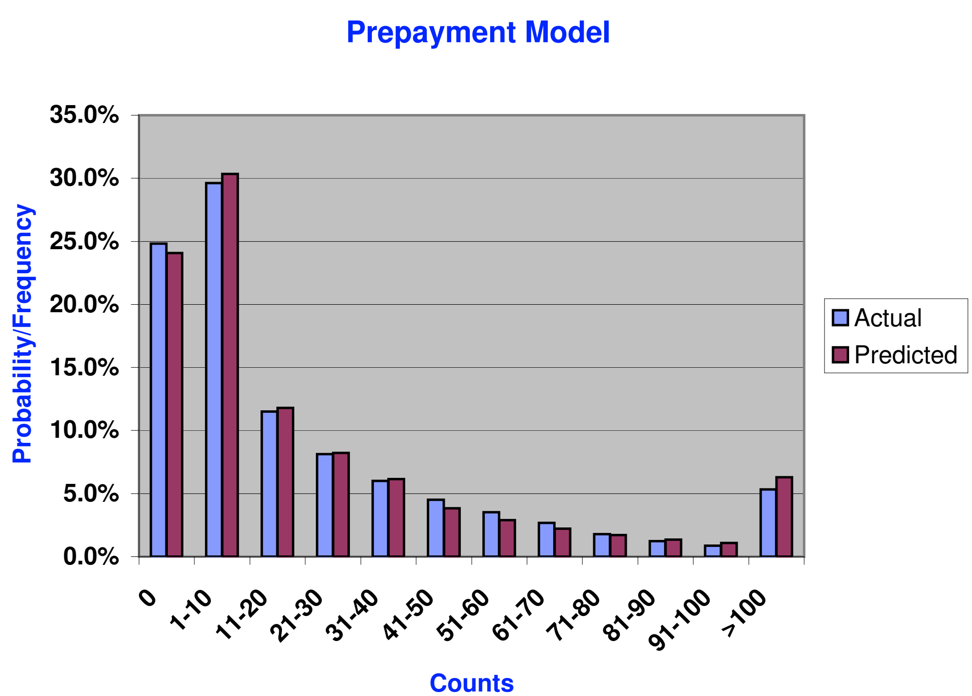


Figure 6.4: Specification Test for Default Hazard Process

Default Model
Predicted vs Actual Probabilities

Counts	Actual	Predicted	|diff|
0	82.9%	82.3%	0.0062
1	9.1%	9.9%	0.0071
2	3.7%	3.8%	0.0014
3	1.8%	1.9%	0.0004
4	1.2%	1.0%	0.0018
5	0.6%	0.5%	0.0010
6	0.4%	0.3%	0.0005
7	0.1%	0.2%	0.0002
8	0.1%	0.1%	0.0001
9	0.0%	0.1%	0.0002
>=10	0.0%	0.1%	0.0002
	100.0%	100.0%	
Pearson_statistic=		5,946	
Degrees of Freedom:		7,195	

Conditional Moment Test
Degrees of Freedom: Q=11

Covariance Used	Test Statistic	P-Value
OPG	10.96	>0.05

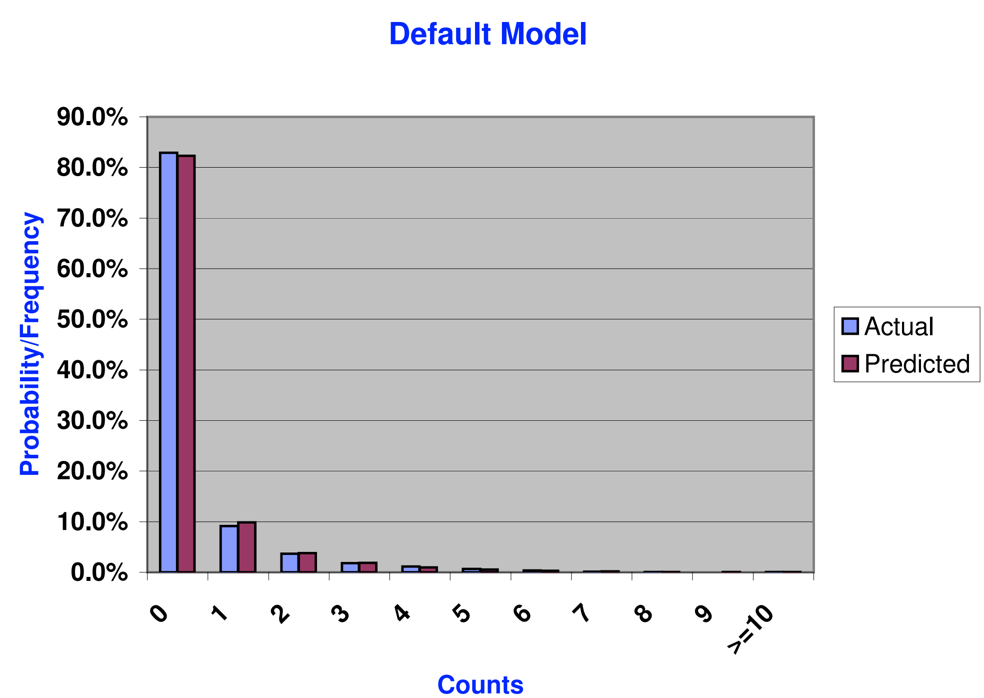


Table 6.3: Comparison MLEs of Alternative Model Specifications
- the Baseline Prepayment Hazard -

Sample Size: N=7,273
Numver of Mass Particles: M=1000
Numver of Resample Particles: R=1300
Discretization Scheme: Euler
Monthly Time Subintervals: 30
Scale factor: 10^3

Coefficients	Constant	Polynomial (5th Order)	Polynomial (7th Order)	Polynomial (3rd Order) & Normal Spikes	Gamma
k_1	0.0146	0.2029	0.1270	0.3839	0.1531
σ_1	1.4874	1.4918	1.4719	1.4821	1.4895
α_0	9.7933	-0.3666	-0.4571	-0.2919	
α_1		9.5550	18.6780	7.6790	
α_2		-1.0312	-28.1040	-7.1967	
α_3		-2.2401	58.5880	2.5453	
α_4		-2.5013	-81.9270		
α_5		1.8914	59.5290		
α_6			-22.0140		
α_7			3.5197		
β					195.7900
ρ					3.2725
r					6.0145
α_{11}				0.0205	
σ_{11}				0.2452	
α_{12}				8.5908	
σ_{12}				0.1868	
Log-Likelihood	-20,660.85	-20,506.17	-19,610.58	-20,600.46	-20,598.55
K	3	8	10	10	5
N	7273	7273	7273	7273	7273
AIC	41,327.70	41,028.35	39,241.15	41,220.92	41,207.10
SBC	41,348.38	41,083.48	39,310.07	41,289.84	41,241.56

Table 6.4: Comparison MLEs of Alternative Model Specifications
- the Baseline Default Hazard -

Sample Size: N=7,273
Numver of Mass Particles: M=1000
Numver of Resample Particles: R=1300
Discretization Scheme: Euler
Monthly Time Subintervals: 30
Scale factor: 10^8

Coefficients	Constant	Polynomial (5th Order)	Chi-Square	Gamma
k_1	0.3758	0.1816	0.5133	0.4736
σ_1	3.8168	3.7759	3.9043	3.6314
α_0	3.6304	-1.7805		
α_1		37.1960		
α_2		-3.4583		
α_3		-17.5780		
α_4		-12.9140		
α_5		10.0240		
α_6				
α_7				
β			65.4550	51.1110
ρ			6.6217	4.6544
r				1.1266
α_{11}				
σ_{11}				
α_{12}				
σ_{12}				
Log-Likelihood	-3,419.28	-3,420.25	-3,381.55	-3,364.33
K	3	8	4	5
N	7273	7273	7273	7273
AIC	6,844.55	6,856.50	6,771.09	6,738.66
SBC	6,865.23	6,911.64	6,798.66	6,773.12

Figure 6.5 The Trend for Prepayment Hazard Process
(Particle Filter Estimate vs Stratified Empirical Estimate)

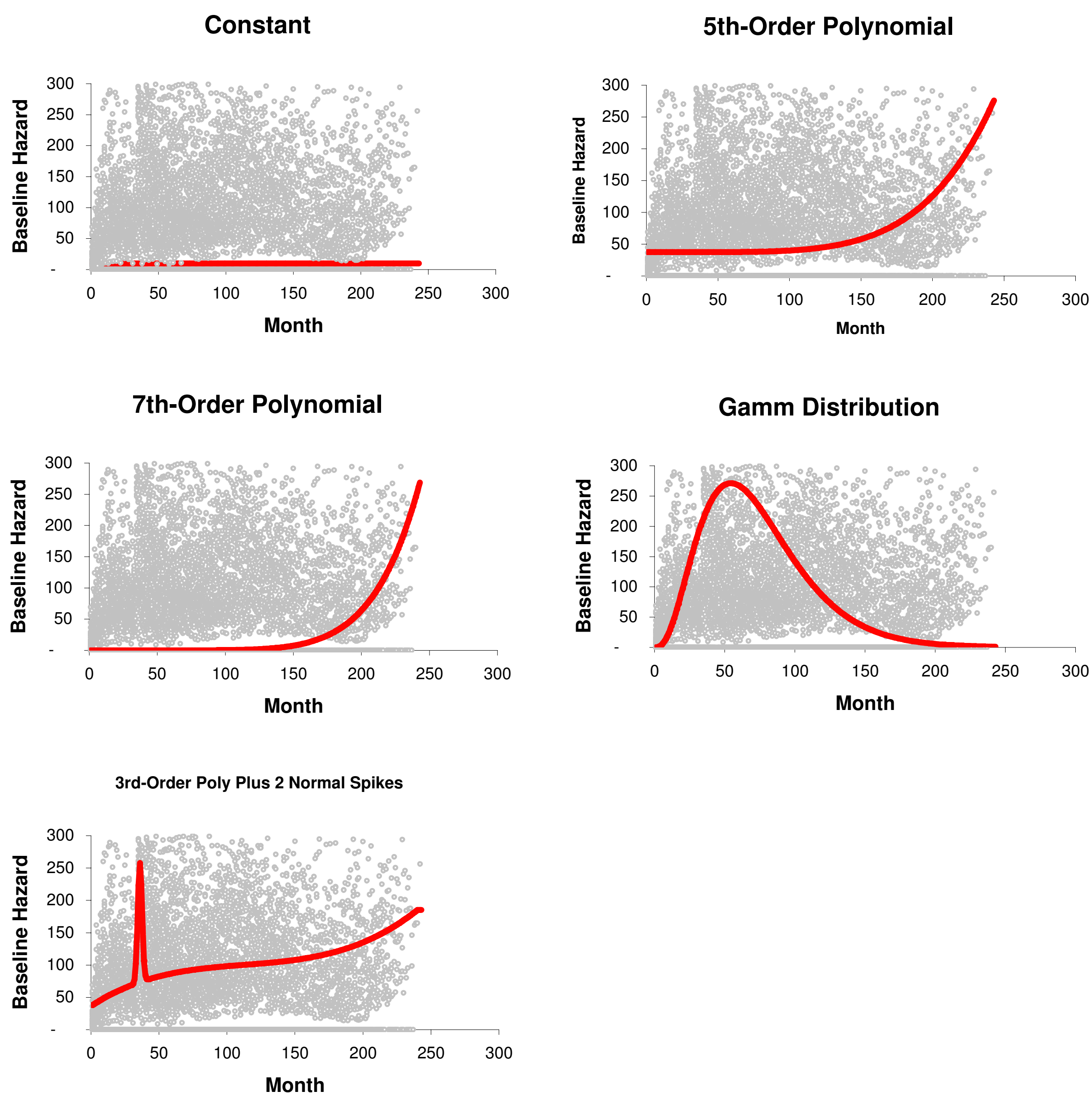
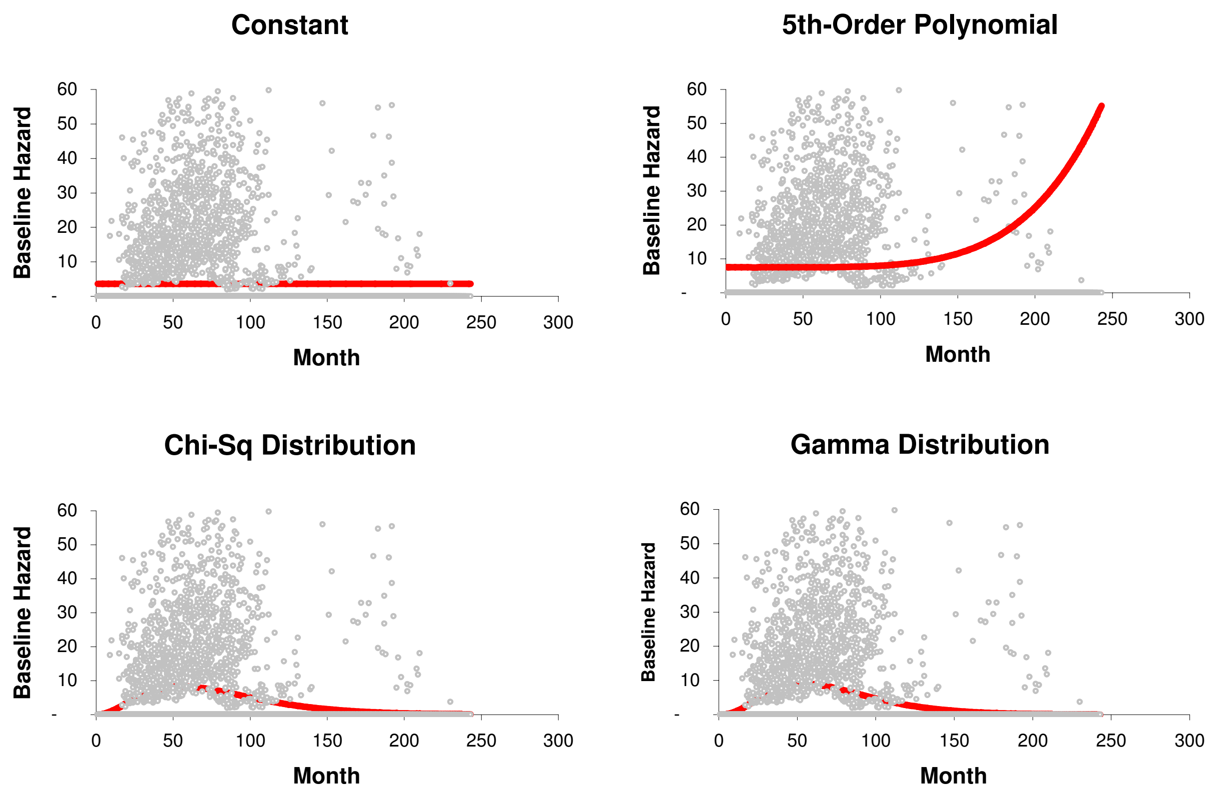


Figure 6.6 The Trend for Default Hazard Process
(Particle Filter Estimate vs Stratified Empirical Estimate)



Chapter 7

Estimation of a two-factor CIR affine term structure model of interest rate

The coupon rate for an ARM with annual reset is adjusted every year based on a specified reference index such as the 1-year Treasury bill rate. As a result, an ARM is path-dependent on the interest rate process. The valuation of the cash-flows for an ARM also calls for the specification of the instantaneous interest rate.

Previous research (Chen and Scott [45], [46], [47]) indicate that at least two factors are needed for a model to adequately capture the dynamics of interest rates. In this chapter, we will specify and estimate a two-factor CIR-type interest rate model using a state-space framework. As the focus of this research is not on the interest rate model per-se, we will not elaborate on the characteristics of our model as compared to other similar models other than present the estimation methodology and the major results.

7.1 A State-space model of the interest rate with two independent factors

Under a two-factor CIR model, the instantaneous interest rate is assumed to be determined by two unobserved independent factors, each evolving according to a diffusion process.

The State (Transition) Equation

$$r_t = x_t^1 + x_t^2 \tag{7.1}$$

$$dx_t^1 = \kappa_1(\theta_1 - x_t^1)dt + \sigma_1\sqrt{x_t^1}dw_t^1 \tag{7.2}$$

$$dx_t^2 = \kappa_2(\theta_2 - x_t^2)dt + \sigma_2\sqrt{x_t^2}dw_t^2 \tag{7.3}$$

The specification of the underlying factors yields a closed-form pricing formula for a pure discount bond. At each point of time t, a series of yields $\{y_t^i\}_{i=1}^N$ with different maturities are observed and they are assumed to deviate from the theoretical yields by some observation errors.

The Observation (Measurement) Equation

$$y_t^i = -\frac{1}{\tau_i}log[P_t(\tau_i, x_t^1, x_t^2)] + \epsilon_i, \quad i = 1, \ldots, N \tag{7.4}$$

$$P_t(\tau, x_t^1, x_t^2) = A_1(t,\tau)A_2(t,\tau)e^{-B_1(t,\tau)x_t^1 - B_2(t,\tau)x_t^2} \tag{7.5}$$

$$A_i(t,\tau) = \left[\frac{2\gamma_i e^{\frac{1}{2}(\kappa_i+\lambda_i-\gamma_i)\tau}}{2\gamma_i e^{-\gamma_i\tau} + (\kappa_i+\lambda_i+\gamma_i)(1-e^{-\gamma_i\tau})}\right]^{2\kappa_i\theta_i/\sigma_i^2} \tag{7.6}$$

$$B_i(t,\tau) = \frac{2(1-e^{-\gamma_i\tau})}{2\gamma_i e^{-\gamma_i\tau} + (\kappa_i+\lambda_i+\gamma_i)(1-e^{-\gamma_i\tau})} \tag{7.7}$$

$$\gamma_i = \sqrt{(\kappa_i+\lambda_i)^2 + 2\sigma_i^2}, \quad i = 1, 2 \tag{7.8}$$

where $\{\lambda_i\}$ are risk premia for the underlying factors and ϵ_i are observation errors, normally distributed with zero mean and constant variance: $\epsilon_i \sim N(0, \sigma^2)$.

With the state-space model fully specified, the particle filter technique for the bivariate case as discussed in chapter 4 can be used to evaluate the likelihood and estimate the parameters.

7.2 Data and Results

The data used for the estimation of a two-factor CIR model are obtained using the Bliss-Nelson-Siegel method. Using an extensive data set on Treasury securities, Bliss

[28] demonstrated that this method provided best overall results compared to other traditional methods of obtaining rates of zero-coupon discount bonds such as the unsmoothed Fama-Bliss method, the smoothed Fama-Bliss method, the McCulloch cubic spline method, and the Fisher method et al.

Four yield series of different maturities are used in the estimation: 3-month,6-month,1-year and 10-year yields, covering the period of 1990:01 to 2000:12, the same period during which ARMs under study are observed. In the proportional hazard models of ARMs termination, we have identified 1-year and 10-year rate as the major determinants for refinancing into ARM and FRM respectively. We therefore put more weights on the model's ability to capture the dynamics of the yields in these two maturities. On the other hand, a good fit of the model to the short end of the term structure of the interest rate is necessary for discounting cash-flows of an ARM in deriving its risk-neutral value.

Figure 7.1 provides an overview of the yields movement for 3-month, 1-year and 10-year maturities.

The estimates of the model coefficients are presented in Table 7.1. The standard interpretation in the literature is that one factor represents a "general level" of interest rates that is closely related to the yield with the longest maturity and behaves almost like a random walk. This factor is identified here as the factor with κ_2 and σ_2. It has weak mean reversion ($\kappa_2 = 0.014$) and low volatility ($\sigma_2 = 0.0595$).

The second factor is related to the spread between the yields with the shortest and longest maturity. It is identified here as the one with κ_1 and σ_1. It has strong mean reversion ($\kappa_1 = 0.6615$) and a large volatility ($\sigma_1 = 0.1055$).

The mean reversion parameter under the risk-neutral measure is positive for factor one, $\kappa_1 + \lambda_1$, but negative for factor two, $\kappa_2 + \lambda_2$. This does not pose any problem. As Jamsshidian [112] shows, in the context of one-factor CIR model, if $\kappa + \lambda < 0$, the forward rate volatility curve is upward sloping.

Overall, the results are consistent with previous work, both in terms of magnitude and in terms of interpretation. In addition, by utilizing multiple maturities in the yield curve, the current model is able to capture the dynamics on both the short, middle and long ends of the term structure and provides a good evaluation environment for pricing ARMs.

Figure 7.1: Observed Monthly Treasury Rates

- 1990 to 2000 -

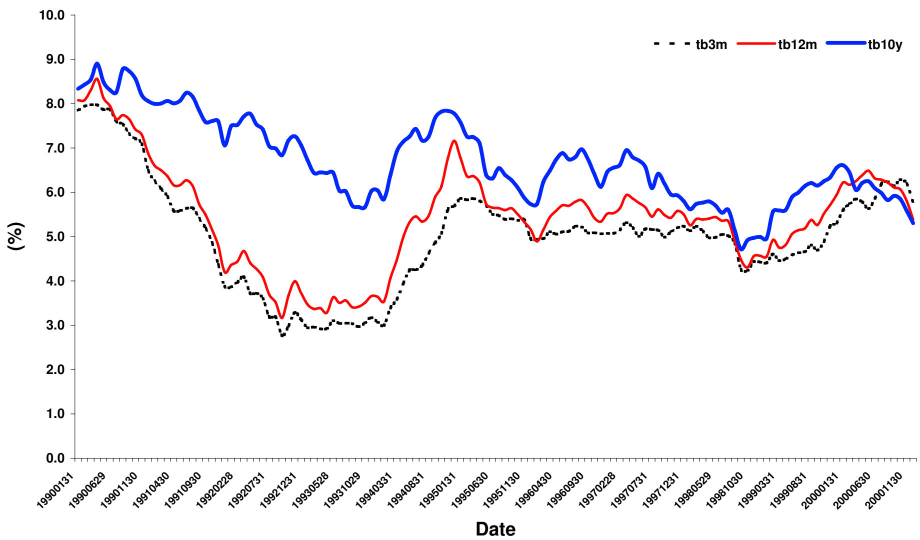


Table 7.1

Maximum Likelihood Estimation of a Two-factor Model of Interest Rate
Observation Period: January 1990 - December 2000
Sample size: 132
Maturities: 3-motnh, 6-moth, 1-year and 10-year
Number of Mass Particles: 20,000
Number of Mass Particles: 24,000
Number of Subintervals Per Month: 30

Coefficient	Estimate	Std Err	Coefficient	Estimate	Std Err
$\kappa 1$	0.6615	0.0141	$\kappa 2$	0.0140	0.0065
$\theta 1$	0.0417	0.0281	$\theta 2$	0.0179	0.0094
$\sigma 1$	0.1055	0.0760	$\sigma 2$	0.0595	0.0231
$\lambda 1$	-0.0608	0.0353	$\lambda 2$	-0.0540	0.0105
σ	0.0012	0.0032			
Likelihood	2,451.22				

Chapter 8

Calibration of prepayment and default hazard processes

8.1 Arbitrage Free Condition

In a simplified world where transaction costs are neglected, to avoid arbitrage, an ARM has to be constructed in such a way that its value at origination has to satisfy the following condition.

$$V(t_0) = L(1-\delta) \tag{8.1}$$

where $V(t_0)$ is the value of the mortgage, L is the loan amount, and δ represents the points charged upfront. In the real world, the value of a mortgage can deviate a little bit from the above condition as a result of transaction costs associated with the arbitrage.

Give a specification of an ARM, its risk-neutral value can be computed using the valuation formula, as discussed in chapter 3.

$$V(t_0) = E^{\mathbb{Q}}[\sum_{i=1}^{360} \mathbb{P}(\tau > t_{i-1}) PV(t_i)] \tag{8.2}$$

$$\mathbb{P}(\tau > t_{i-1}) = \prod_{j=1}^{i-1} (1 - \lambda_j^p - \lambda_j^p) \tag{8.3}$$

$$PV(t_i) = e^{\int_{t_0}^{t_i} \hat{r}(s)ds} CF(t_i) \tag{8.4}$$

$$\hat{r}(s) = (1-\tau_F)r(s) + l \tag{8.5}$$

$$CF(t_i) = \mathbb{P}\{\tau = \tau_d\}\, W(t_i) + \mathbb{P}\{\tau = \tau_p\}\, A(t_i) + \mathbb{P}\{\tau > t_i\}\, M(t_i) \tag{8.6}$$

where $CF(t_i)$ is the cash-flow at time t_i, $PV(t_i)$ is the present value of the cash-flow at time t_i, and τ_F and l are the federal tax rate and liquid premium respectively.

Eq (8.2) indicates that the valuation of an ARM is the expected conditional present values of cash streams with respect to a risk-neutral probability measure. The random variable whose expectation is to be computed is a function of two underlying random variables: τ, the stopping time, and r, the instantaneous interest rate. The joint distribution of the stopping time and the instantaneous interest rate does not have closed-form representation and thus is very difficult to evaluate directly.

Monte Carlo simulation is widely used in computing expectation of random variables whose distribution are not tractable, that is, whose expectation is not easy to compute numerically. In current case, we use Monte Carlo methods to simulate the stopping time and the instantaneous rate separately and compute the expectation based on the simulated realization of the two random variables.

8.2 Implementation Scheme

8.2.1 Simulating the stopping time of an ARM

As we mentioned in the previous chapter 3, the stopping time of an ARM is the function of a stopping time for prepayment and a stopping time for default. $\tau = min(\tau_p, \tau_d)$. Therefore, we can simulate the stopping time for an ARM by simulating two stopping times: one for prepayment the other for default.

The stopping time for prepayment and default are determined completely by their respective hazard processes. In chapter 6, we have specified and estimated the baseline prepayment and baseline default processes based on the historical information about prepayments and defaults for $203,016$ ARMs. The resultant representation of the processes are therefore described in the real probability measure. By making explicit assumptions about the multiplicative and additive risk adjustments, we can get the representation of the two processes in the risk-neutral probability measure.

Hazard Diffusion Processes in Real Probability Measure

$$\lambda^p = e^{\beta_p X_p'(t)} \lambda_0^p \tag{8.7}$$

$$\lambda^d = e^{\beta_d X_d'(t)} \lambda_0^d \tag{8.8}$$

$$d\lambda_0^p = \kappa^p [\theta_t^p - \lambda_0^p] dt + \sigma^p \sqrt{\lambda_0^p} dz_P^p \tag{8.9}$$

$$d\lambda_0^d = \kappa^d [\theta_t^d - \lambda_0^d] dt + \sigma^d \sqrt{\lambda_0^d} dz_P^d \tag{8.10}$$

Hazard Diffusion Processes in Risk-neutral Probability Measure

$$\lambda^p = e^{\beta_p X_p'(t)} \lambda_0^p \tag{8.11}$$

$$\lambda^d = e^{\beta_d X_d'(t)} \lambda_0^d \tag{8.12}$$

$$d\lambda_0^p = \mu^p [\kappa^p \theta_t^p - (\kappa^p + \nu^p) \lambda_0^p] dt + \mu^p \sigma^p \sqrt{\lambda_0^p} dz_Q^p \tag{8.13}$$

$$d\lambda_0^d = \mu^d [\kappa^d \theta_t^d - (\kappa + \nu^d) \lambda_0^d] dt + \mu^d \sigma^d \sqrt{\lambda_0^d} dz_Q^d \tag{8.14}$$

where μ_p, μ_d are multiplicative risk adjustment factors and ν_p, ν_d are additive risk adjustment factors.

With the stochastic processes fully specified in the risk-neutral measure, using the Euler discretization scheme, we can simulate the evolution of the prepayment and default hazards and calculate the conditional probabilities of prepayment and default at any time t. The starting values for both processes are set to values that are close to zero.

8.2.2 Simulating the instantaneous interest rate

The simulation of the instantaneous interest rate is done via the diffusion process we have estimated in chapter 7. Since the estimation of a two-factor CIR interest rate model was based on market data, the process is represented in a risk-neutral measure and no further transformation is needed here.

Diffusion Processes for the Instantaneous Interest Rate

$$r = x_1 + x_2 \tag{8.15}$$

$$dx_1 = [\kappa_1\theta_1 - (\kappa_1 + \lambda_1)x_1]dt + \sigma_1\sqrt{x_1}dz_Q^1 \tag{8.16}$$

$$dx_2 = [\kappa_2\theta_2 - (\kappa_2 + \lambda_2)x_1]dt + \sigma_2\sqrt{x_2}dz_Q^2 \tag{8.17}$$

The discretization scheme is the Euler scheme. The starting values of the two latent factors at time t are the values that generate the best predicated yields for 4 maturities: 3-month, 6-month, 1-year and 1-year against the actual yields as observed at time t.

8.2.3 Simulating the cash flows

The cash flow at any given month can be any of three types of payments: $M(t)$, the scheduled payment, $A(t)$, the payment from prepayment, $W(t)$, the recovery payment from default.

The scheduled payment $M(t)$ is the monthly payment calculated using the on-going coupon rate c_t, the on-going outstanding mortgage balance L_t and the remaining term to maturity n_t is.

$$M(t) = L_t \frac{c_t/12}{1 - \frac{1}{(1+c_t/12)^{n_t}}} \tag{8.18}$$

The payoff from a prepayment consists of the outstanding mortgage balance at time t:

$$A(t) = L_t \tag{8.19}$$

Because the coupon rate is re-set every year against the reference index, the 1-year Treasury bill rate, the on-going outstanding mortgage balance L_t has to be calculated sequentially using the time-varying coupon rate for each month.

The recovery payment upon default depends first on whether the loan is insured. It is assumed here that all loans with $LTV > 80\%$ have insurance. It also depends on the assumption about the basis of recovery. One assumption known as "recovery of face value" stipulates that the recovery is a function of the outstanding mortgage balance at the time of default. Assume that the loss rate ω is a stochastic variable that is proportional to the on-going loan-to-value ratio and independent of the other stochastic state variables, then the expected recovery payment $W(t)$ can be proved to have the following representation (see Schönbucher [170])

$$\omega = \kappa L_t / H_0 \tag{8.20}$$

$$\begin{aligned} W(t) &= (\phi + 1 - E[\omega]) \\ &= (1 + \phi - \kappa^e L_t / H_0) L_t \end{aligned} \tag{8.21}$$

where ϕ is the percent insured, κ^e is the expected value of a random variable κ, H_0 is the original house value and the expectation operator is taken with respect to the risk-neutral measure.

The expected cash-flow at time t $CF(t)$ can then be simulated as follows.

$$CF(t_i) = \mathbb{P}\{\tau = \tau_p\}\, A(t_i) + \mathbb{P}\{\tau = \tau_d\}\, W(t_i) + \mathbb{P}\{\tau > t_i\}\, M(t_i) \tag{8.22}$$

$$\mathbb{P}\{\tau = \tau_p\} = \lambda^p \tag{8.23}$$

$$\mathbb{P}\{\tau = \tau_d\} = \lambda^d \tag{8.24}$$

$$\mathbb{P}\{\tau > t_i\} = 1 - \lambda^p - \lambda^d \tag{8.25}$$

According to Eq(8.2), the risk-neutral value of an ARM can be approximated as the average value over all the simulated stopping times, instantaneous interest rates and cash-flows. 1000 simulations have been used with 30 time increments per month over a 30-year life span of an ARM.

In the estimation of prepayment and default processes in real measure, we have utilized all three different types of ARMs. Here we focus only on TRS type of ARM that is reset annually and uses one year CMT as reference index. The procedure can be easily applied to other types of ARMs.

Similar to Kau et al. [132], we have assumed that the tax rate is 28% at the Federal level and 4.32% at the state level. The parameters to be optimized then include $\mu_p, \nu_p, \mu_d, \nu_d, l, \kappa^e$, where l is a liquid premium and κ^e is the expected loss rate on default. These parameters will be calibrated through the arbitrage free condition (8.1)

Specifically, the calibration process amounts to a multi-variable minimization problem:

$$min \sum_{n=1}^{N} [V_n(t_0|\Theta)/L_n - (1 - \delta_n)]^2 \tag{8.26}$$

$$\Theta = \{\mu_p, \nu_p, \mu_d, \nu_d, l, \kappa^e\}$$

In the actual calibration, we have used the root mean square error (RMSE) as the objective function and tried to find a set of parameters that minimizes the objective function. One implicit assumption associated with using a simple RMSE in the calibration of a model is that the parameters of interest can be identified in this way. This assumption is often violated in the practice, due to either data problems or the misspecification of a model. One usual solution to the problem is to impose additional restrictions on the parameters or to introduce certain penalty functions in the objective function using prior knowledge.

It turned out that naive calibration of prepayment and default processes using RMSE as defined above will not work. Through investigation, we found that the objective function was a monotonic function of some parameters associated with the prepayment and default processes.

In particular, as we decrease the additive prepayment risk premium ν^p into negative territory, the RMSE will decrease forever, albeit gradually. Likewise, as we increase the multiplicative prepayment risk premium μ^p, the RMSE will also decrease forever. Furthermore we found that there was a substitution relationship between the additive prepayment risk premium ν^p and the multiplicative prepayment risk premium μ^p. A given RMSE can be obtained through a high additive risk premium (in absolute sense) and low multiplicative risk premium, or a low additive risk premium (in absolute sense) and high multiplicative risk premium. Another key insight we got from the investigation is that for the same RMSEs, the value decomposition can be quite different and some of the them definitely are not of interest to us.

The implication is that there will be no optimal solution to the calibration of prepayment and default processes using simple RMSE without imposing some restriction on the underlying parameters.[1]

Therefore, we have adopted a two-stage calibration scheme with certain explicit constrains on the parameters to be calibrated. Specifically, at stage one, we first calibrate the prepayment risk parameters using loans with very low $LTV \leq 40\%$ such that the default impact will be trivial. In addition, we only use loans having points. At stage two, we then calibrate the default risk parameters using only loans with high $LTV \geq 90\%$, holding the prepayment risk parameters fixed. Table 8.1 provide an overview of the loans used in each of the two stages.

We further impose the conditions such that both prepayment and default processes will remain mean-reverting in the risk-neutral measure. To further narrow the search for the set of parameters that will render not only small RMSE but also meaningful value decomposition of a given loan, we also check the value decomposition associated with each set of parameters returned from the optimization algorithm. [2]

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[1]Given additive and multiplicative risk premium for prepayment and default processes, we found the RMSE often attains its minimum around a liquid premium equal to 90 bp.

[2]The procedure employed here differs from conventional way of optimization and is more like a hybrid of grid search and gradient-based optimization because certain constraints can not be built into the search process explicitly. The procedure differs from a typical grid search is that the search direction is still determined by gradient. Without such constraints,

Jarrow, Lando, and Yu [114] have formally demonstrated that, with sufficient diversification, the multiplicative risk parameter for default should approach one. The specification of our model and the rich information afforded by the data provide an opportunity for us to test their hypothesis.

8.3 Results

The results in Table 8.2 indicate that the multiplicative default risk parameter is not far from one. While this result is very much dependent on the specification of the model, at least we can conclude that, in the current setting, it is necessary to only consider the additive default risk premium and not the multiplicative default risk, which is likely driven more by idiosyncrasies of the borrowers and can be diversified away with a large pool.

The multiplicative prepayment risk parameter,[3] however, is greater than one, indicating that its risk can not be diversified away by any means.

Both additive risk adjustments are negative, indicating the lenders' aversion to both prepayment and default risk components.

8.4 Pricing Performance

To test the pricing performance, we use another set of 100 loans that were not used in the calibration and randomly selected from the pool of ARMs with annual reset, indexed off the one-year CMT rate, to estimate their values and compare

we found the search can be led to wrong direction based on the gradient alone. This may has something to do with approximation errors of the gradient. See appendix C for the detailed description of the process.

[3] The RMSE in the first stage is 2.53% and the mean error is 2.178%

them against the true value.[4] Table 8.3 shows the profiles of the loans and Table 8.4 provides the overall pricing statistics.

Overall, the performance is satisfactory as far as the RMSE (root mean square error) is concerned. [5]

Our model affords us a unique opportunity to decompose the value into three parts: value due to scheduled payment, value due to prepayment and value due to default. For this purpose, we have selected three typical loans with negative, zero and positive points and computed the value decomposition for each of them. The results are presented in Table 8.5.

Across the board, the prepayment portion accounts for more than 80% of the total value while the default portion accounts for less than 1%.

Finally we have investigated the value of prepayment and default options. The results are presented in Table 8.5. The value of combined option is less than the sum of the value of prepay option and the value of default option because exercising one option will preclude exercising the other option.

8.5 Alternative Calibration Implementation Scheme

As we stated earlier, there are problems using simple RMSE as the objective function and the gradient-based optimization technique to calibrate the prepayment and default processes in risk-neutral measure. We got over these problems by imposing certain restrictions on the parameters, some of which are implemented in an ad-hoc,

[4]The selection criteria used include LTV between 80% to 85%, originated after 1995, with no prepayment penalty.

[5]One reason for out-of-sample pricing error being smaller than the in-sample pricing error is that the out-of-sample is more normal in terms of LTV ratio ($80\% \leq LTV \leq 85\%$), compared to the sample used in calibration ($LTV \geq 90\%$). Using loans with higher than normal LTV makes it easier to calibrate the default process because the objective function is then more responsive to the change in the default process.

non-systematic way. To formalize the process and check the robustness of the parameters calibrated for the resultant out-of-sample pricing performance, we approach the same problem using a different route, one that is based on re-parametrization of the two processes and a grid search optimization technique.

The re-parametrization allows us to look beyond current model specification which may be the cause of the problem due to mis-specification. A grid search technique will avoid errors introduced in the estimation of gradient of the objective function. While the estimation accuracy of the parameters using a grid search may be compromised, it will guarantee that the resultant parameters will be in the vicinity of the true optimal values if they do exist.

The prepayment and default processes still have the same specifications in the real measure as before.

Hazard Diffusion Processes in Real Probability Measure

$$\lambda^p = e^{\beta_p X_p'(t)} \lambda_0^p \tag{8.27}$$

$$\lambda^d = e^{\beta_d X_d'(t)} \lambda_0^d \tag{8.28}$$

$$d\lambda_0^p = \kappa^p [\theta_t^p - \lambda_0^p] dt + \sigma^p \sqrt{\lambda_0^p} dz_P^p \tag{8.29}$$

$$d\lambda_0^d = \kappa^d [\theta_t^d - \lambda_0^d] dt + \sigma^d \sqrt{\lambda_0^d} dz_P^d \tag{8.30}$$

The specifications of the two processes in risk-neutral measure are generalized to be the following.

Hazard Diffusion Processes in Risk-neutral Probability Measure

$$\lambda^p = e^{\beta_p X_p'(t)} \lambda_*^p \tag{8.31}$$

$$\lambda^d = e^{\beta_d X_d'(t)} \lambda_*^d \tag{8.32}$$

$$log(\lambda_*^p) = a_0^p + a_1^p log(\lambda_0^p) \tag{8.33}$$

$$log(\lambda_*^d) = a_0^d + a_1^d log(\lambda_0^d) \tag{8.34}$$

$$d\lambda_0^p = [(\kappa^p \theta_t^p - d_0^p \sigma^p) - (\kappa^p + d_1^p \sigma^p)\lambda_0^p]dt + \sigma^p \sqrt{\lambda_0^p} dz_Q^p \tag{8.35}$$

$$d\lambda_0^d = [(\kappa^d \theta_t^d - d_0^d \sigma^d) - (\kappa^d + d_1^d \sigma^d)\lambda_0^d]dt + \sigma^d \sqrt{\lambda_0^d} dz_Q^d \tag{8.36}$$

Compared to previous specifications, there are two noticeable features with the current generalized version. The first feature is that the market price of risk follows the specification proposed by Jun Pan etc [152]. This extended version will allow the risk premium to change sign over time while the sign of risk premium in the standard version remains fixed. The second feature, which is due to Antje Berndt etc [20], is that the multiplicative risk premium is not necessarily a constant function but can be a function of the baseline hazard rates depending on the coefficient of a_1^p, a_1^d.

The previous specifications are nested in the generalized version with $d_0^p = 0, a_1^p = 1, d_0^d = 0, a_1^d = 1$ and $\mu^p = e^{a_0^p}, \nu^p = d_1^p \sigma^p, \mu^d = e^{a_0^d}, \nu^d = d_1^d \sigma^d$.

Adding liquid premium, $liqp$ and loss rate on default, $loss$, the set of parameters to be optimized is $d_0^p, d_1^p, a_0^p, a_1^p, d_0^d, d_1^d, a_0^d, a_1^d, loss, liqp$.

The optimization of the parameters using a grid search proceeds in two stages. At stage one, we examine how the objective function changes in response to a change in one parameter. The aim is to understand the overall behavior of the objective function so that some of the insights can help us in determining the search direction and feasible domain for each parameter. The search stride is set large enough so that the trend of the objective function can be clearly assessed. In determining the feasibility of a value for a parameter, we consider not only the objective function but also the implication on decomposition of the loans. Values that may result in trivial loan decomposition are considered not feasible for a parameter and are excluded from the search domain.

At stage two, we narrow the search domain for each parameter and use a relatively small stride. In the process, we first look for a set of parameters that will produce the smallest RMSE. When two RMSEs only differ in non-significant digits, we then look for smallest average error (in an absolute sense). Along the way, we also check the implication for stationarity of the processes.[6] Other things being equal or close, we choose a stationary process over a non-stationary one. Within the same stationary process, we choose a small multiplicative risk premium over a large one.

The grid search optimization is not very efficient when a large number of parameters are involved. Therefore, at stage two, we calibrate the parameters for prepayment process first using a set of loans that have very low LTV ($\leq 40\%$) and high points, rendering the probability of any default trivial. Since the default process is no longer at work, we can focus on the calibration of the prepayment process only. Once the prepayment process is calibrated, we then use another set of loans with high LTV ($\geq 90\%$) to calibrate the default process. The value of a mortgage with high LTV is more responsive to the default process than a mortgage with modest or low LTV, which will make it easier for us to find the optimal parameters for the default process. After both processes have been calibrated, we use a third set of loans with normal LTVs (between 80% to 85%) for out-of-sample pricing performance evaluation.

[6] In current context, a process is considered to be stationary if it is mean-reverting.

The results are presented in Table 8.2a. [7] For easy comparison, the parameters have been converted to their equivalency using the previous notation.

Compared to results in Table 8.2, the alternative calibration scheme produced a set of optimal parameters with similar magnitude. Conclusions that are consistent with previous findings can also be drawn. The multiplicative default risk parameter is not far from one, indicating the idiosyncrasies of the default event can be diversified away. The multiplicative risk premium for prepayment is greater than one, indicating that the event has more to do with overall economic environment and less to do with the idiosyncrasies of the borrowers. [8] When mortgages are affected by common factors, a diversification strategy will no longer work.

Table 8.4a reports the out-of-sample pricing performance using the optimal parameters obtained from the alternative calibration scheme. Notice that we have used the very same set of loans for performance testing as is used previously, so that the testing results are comparable. The pricing performance is about as good as the previous one. [9]

Table 8.5a reports the decomposition of value for three loans selected from the set of loans used in pricing performance test. They differ mainly in points charged upfront. The first loan has negative points, the second loan has no points, and the third loan has positive points. They are the same loans used in the previous value decompositions and thus the decomposition of the values are comparable to one other.

[7] To facilitate comparison between two calibration schemes, we have used the same set of loans for both calibrations whenever applicable. The RMSE in the first stage is 1.976% and the mean error is 1.589%

[8] These findings are also consistent with results in Chapter 5 that accounting for the idiosyncrasies of the borrowers in estimating the hazard rate does not yield much improvement over a simple Kaplan-Meier estimate treating everyone equal.

[9] The set of loans used in performance test are mainly those originated during 1998. We also tested the pricing performance on another two set of loans. One consists of loans originated between 1995 to 1996. The RMSE is 1.2246%, with mean error of 0.05%. Another consists of loans originated in 2000. The RMSE is 0.8467% with mean error of −0.33%

In all cases, prepayment accounts for more than 80% of the total value while the default accounts for less than 1% of the total value. The decompositions, while different from previous results, have similar magnitude as before so similar and consistent conclusions can still be drawn. The same point can be made regarding the values of the options.

If the focus is on explanation rather than prediction and trading, then we have demonstrated here that our calibration process is quite robust and can serve this purpose reasonably well.

8.6 Application

After the calibration is completed, we can use the model to analyze how the value of a given loan will change with respect to the major components of the loan. This type of analysis will help lenders to customize their loan offering to different people while maximizing the value of the loan.

We select one loan from the sample and vary the values for major loan components such as coupon rate, LTV, margin, ceiling and points. Using the original setting as the base case, for each of those components except for LTV, we change the values by ± 100 basis points from the base and then estimate the corresponding values using the model.

The detailed results of the analysis are summarized in Table 8.6. The analysis shows that the two most important value drivers for a loan are the margin and coupon rate. LTV comes next as an important driver of mortgage.

8.7 Conclusions

ARMs are more vulnerable to prepayment and default risk than FRMs, as indicated by their higher prepayment and default frequency. Understanding the dynamics of prepayment and default is therefore very important to lenders of ARMs, and to a larger extent, the general investors who have exposure to ARMs. Due to their complexity, the existing research on prepayment and default for ARMs has adopted a static, simplified approach to the problem. The work we have completed here is an attempt to improve our understanding on how prepayment and default affect the value of an ARM and on how the risk premiums are factored into the total value. Given that the market for ARMs is highly illiquid compared to FRMs, this understanding will better facilitate risk management on the part of lenders.

Utilizing the rich historical information about the prepayment and default from a substantial number of ARMs, we have built a unified model that takes into consideration loan idiosyncrasies, general economic environment and interest rate uncertainty. The approach lends itself to lots of future practical applications since the information that is used in the model is readily available and observable. Our effort is consistent with the current trend in the academic community that more and more academic researches should have a real world orientation.

However, the work we have just completed is far from perfect. For simplicity, we have assumed independence between prepayment and default when we estimated stochastic processes for prepayment and default. This independence is not consistent with the competing nature of the risks. We made an attempt to incorporate explicit correlation between prepayment and default processes using particle filter technique without much success. The major hurdle is that estimating the likelihood of a joint distribution using particle filter technique often encounters numerical problems whenever the covariance matrix becomes singular. The particle filter technique is very sensitive to outliers and is not suitable for non-diffusion processes. Some of the prepayments and defaults obviously cannot be accounted for using diffusion processes.

As far as valuation is concerned, prepayment is clearly the dominant factor, while default is less an issue. Given the limited information available to us, it is very difficult to come up with a satisfactory specification for prepayment. The current specification of the model fails to capture apparent prepayment jumps observed during the first couple of years of a typical ARM. Adding a spike function to the mean-reverting function did not help a lot as long as we continued using diffusion process for prepayment. The very notion of mean-reverting for prepayment may be questionable in the first place.

Another important factor that is not accounted for in our study is the borrowers' idiosyncrasies, due to lack of relevant information. For that matter, no single model can ever capture those behaviors that may not be explained using conventional assumption of a rational economic agent.

Table 8.1
Loans Used in Two-stage Calibration

Loan Profile for Calibrating Prepyament Risk Parameters

	Original Loan Amount	Initial Coupon Rate	LTV	Margin	Ceiling	Points
Ave	$145,505	5.87	30.88	3.57	12.29	1.70
Median	$91,500	5.63	32.00	2.75	12.50	1.69
Max	$1,000,000	8.88	40.00	10.00	14.13	2.50
Min	$25,000	3.75	12.00	1.75	9.88	1.50

Loan Profile for Calibrating Default Risk Parameters

	Original Loan Amount	Initial Coupon Rate	LTV	Margin	Ceiling	Points
Ave	$126,571	4.92	94.64	3.20	11.11	0.99
Median	$130,565	5.00	95.00	2.75	10.88	1.00
Max	$298,000	6.63	100.00	10.00	12.50	2.00
Min	$19,250	3.25	91.00	2.13	9.88	-2.00

Table 8.2
Monte Carlo Calibration Results
Number of Loans Used: 100
Number of Simulations: 1000
Disretization Scheme: Euler
Monthly Time Intervals: 30

Coefficient	Estimate
μ_p	2.2341
υ_p	-0.1091
μ_d	1.1096
υ_d	-0.1225
Loss Rate	38.7%
Liquidity	0.93%
RMSE(%)	1.5107

Table 8.3
Loans Used for Pricing Performance Test

	Original Loan Amount	Initial Coupon Rate	LTV	Margin	Ceiling	Points
Ave	$462,411	5.74	80.27	2.63	11.02	-0.70
Median	$396,400	5.75	80.00	2.50	10.88	-1.00
Max	$2,800,000	6.75	85.00	3.00	11.88	1.25
Min	$91,200	4.88	80.00	2.13	9.88	-1.25

Table 8.4
Pricing Performance Statistics

RMSE	0.77%
Mean Pricing Error	-0.22%
STD of Pricing Error	0.74%

Table 8.5
Value Decomposition

Loan Profile

	1		2		3	
Original Loan Amount	$127,500		$255,000		$271,920	
Initial Coupon Rate	5.63		5.75		4.88	
LTV(%)	84.00		80.00		80.00	
Margin	2.75		2.50		2.50	
Ceiling	10.88		10.88		10.88	
Points(%)	-0.38		-		1.25	
Actual Value	$127,978		$255,000		$268,521	
Estimated Value	$127,944		$256,494		$271,018	
Relative Error	-0.03%		0.59%		0.93%	

Decomposition

From scheduled payment	$24,175	*18.9%*	$44,220	*17.2%*	$47,744	*17.6%*
From prepayment	$102,850	*80.4%*	$211,314	*82.4%*	$222,355	*82.0%*
From default	$918	*0.7%*	$960	*0.4%*	$919	*0.3%*
Total	$127,943		$256,494		$271,018	

Value of Options

No Risk	$134,201	$266,591	$281,054
Prepay Risk Only	$128,164	$256,908	$271,412
Default Risk Only	$131,977	$263,426	$278,159
Both Risks	$127,944	$256,494	$271,018
Value of Prepay Option	$6,037	$9,683	$9,642
Value of Default Option	$2,224	$3,165	$2,895
Value of Combined Option	$6,257	$10,097	$10,036

Table 8.1a
Loans Used in Two-stage Calibration

Loan Profile for Calibrating Prepyament Risk Parameters

	Original Loan Amount	Initila Coupon Rate	LTV	Margin	Ceiling	Points
Ave	$145,505	5.87	30.88	3.57	12.29	1.70
Median	$91,500	5.63	32.00	2.75	12.50	1.69
Max	$1,000,000	8.88	40.00	10.00	14.13	2.50
Min	$25,000	3.75	12.00	1.75	9.88	1.50

Loan Profile for Calibrating Default Risk Parameters

	Original Loan Amount	Initila Coupon Rate	LTV	Margin	Ceiling	Points
Ave	$126,571	4.92	94.64	3.20	11.11	0.99
Median	$130,565	5.00	95.00	2.75	10.88	1.00
Max	$298,000	6.63	100.00	10.00	12.50	2.00
Min	$19,250	3.25	91.00	2.13	9.88	-2.00

Table 8.2a
Monte Carlo Calibration Results
Number of Loans Used: 100
Number of Simulations: 1000
Disretization Scheme: Euler
Monthly Time Intervals: 30

Coefficient	Estimate	Coefficient	Estimate
d0_p	0.0000		
d1_p	-0.0660	υ_p	-0.0972
a0_p	1.2000	μ_p	3.3201
a1_p	1.0000		
d0_d	0.0000		
d1_d	-0.0330	υ_d	-0.1209
a0_d	0.2500	μ_d	1.2840
a1_d	1.0000		
Loss Rate	40.0%		40.0%
Liquidity	0.92%		0.92%
RMSE(%)	1.4055		1.4055

Table 8.3a
Loans Used for Pricing Performance Test

	Original Loan Amount	Initial Coupon Rate	LTV	Margin	Ceiling	Points
Ave	$462,411	5.74	80.27	2.63	11.02	-0.70
Median	$396,400	5.75	80.00	2.50	10.88	-1.00
Max	$2,800,000	6.75	85.00	3.00	11.88	1.25
Min	$91,200	4.88	80.00	2.13	9.88	-1.25

Table 8.4a
Pricing Performance Statistics

RMSE	0.87%
Mean Pricing Error	-0.50%
STD of Pricing Error	0.72%

Table 8.5a
Value Decomposition

Loan Profile

	1	2	3
Original Loan Amount	$127,500	$255,000	$271,920
Initial Coupon Rate	5.63	5.75	4.88
LTV(%)	84.00	80.00	80.00
Margin	2.75	2.50	2.50
Ceiling	10.88	10.88	10.88
Points(%)	-0.38	-	1.25
Actual Value	$127,978	$255,000	$268,521
Estimated Value	$127,699	$256,117	$270,648
Relative Error	-0.22%	0.44%	0.79%

Decomposition

From scheduled payment	$21,131	*16.5%*	$38,692	*15.1%*	$41,766	*15.4%*
From prepayment	$105,804	*82.9%*	$216,640	*84.6%*	$228,126	*84.3%*
From default	$765	*0.6%*	$785	*0.3%*	$756	*0.3%*
Total	$127,699		$256,117		$270,648	

Value of Options

No Risk	$134,720	$267,363	$281,976
Prepay Risk Only	$127,889	$256,472	$270,988
Default Risk Only	$132,146	$263,753	$278,663
Both Risks	$127,699	$256,117	$270,648
Value of Prepay Option	$6,831	$10,891	$10,988
Value of Default Option	$2,574	$3,610	$3,313
Value of Combined Option	$7,021	$11,246	$11,328

Table 8.6 Application

~ Sensitity Analysis of Mortgage Value ~

Loan Profile

Origination Year	**1999**
Origination Month	**5**
Loan Size	**$114,300**
Coupon Rate (%)	**5.25**
LTV (%)	**84.04**
Teaser (%)	**1.94**
Annual Cap (%)	**2**
Margin (%)	**2.5**
Ceiling (%)	**10.875**
Points (%)	**2.125**
Acutal Value	**$111,871**
Estimated Value	**$113,567**
Pricing Error	**1.52%**

Sensitivity Analysis

Loan Components		Value Decomposition ($)				Value Decomposition (%)		
	Value	Payment	Prepay	Default	PCHG	Payment	Prepay	Default
Coupon Rate (%)								
4.25	$112,030	$25,569	$85,796	$665	-1.35%	22.8%	76.6%	0.6%
5.25	$113,567	$26,345	$86,467	$755	0.00%	23.2%	76.1%	0.7%
6.25	$114,770	$27,160	$86,761	$849	1.06%	23.7%	75.6%	0.7%
LTV (%)								
74	$113,615	$26,145	$87,121	$349	0.04%	23.0%	76.7%	0.3%
84.4	$113,567	$26,345	$86,467	$755	0.00%	23.2%	76.1%	0.7%
94	$113,505	$26,504	$85,542	$1,460	-0.05%	23.4%	75.4%	1.3%
Margin (%)								
1.5	$111,730	$26,467	$84,558	$704	-1.62%	23.7%	75.7%	0.6%
2.5	$113,567	$26,345	$86,467	$755	0.00%	23.2%	76.1%	0.7%
3.5	$114,882	$26,406	$87,682	$794	1.16%	23.0%	76.3%	0.7%
Ceiling (%)								
9.875	$113,592	$27,050	$85,806	$736	0.02%	23.8%	75.5%	0.6%
10.875	$113,567	$26,345	$86,467	$755	0.00%	23.2%	76.1%	0.7%
11.875	$113,543	$25,660	$87,110	$773	-0.02%	22.6%	76.7%	0.7%
Points (%)								
1.125	$113,547	$25,612	$87,185	$750	-0.02%	22.6%	76.8%	0.7%
2.125	$113,567	$26,345	$86,467	$755	0.00%	23.2%	76.1%	0.7%
3.125	$113,587	$27,100	$85,728	$759	0.02%	23.9%	75.5%	0.7%

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Appendix A

Proof of theorem (4.2.1)

$$
\begin{aligned}
\hat{f}(\mathtt{y}_t|\theta;\mathcal{F}_{t-1}) &= \int f(\mathtt{y}_t|\mathtt{x}_t)\frac{1}{N}\{\sum_{i=1}^{N} f(\mathtt{x}_t|\mathtt{x}_{t-1}^i)d\mathtt{x}_t\} \\
&= \frac{1}{N}\sum_{i=1}^{N}\{\int f(\mathtt{y}_t|\mathtt{x}_t)f(\mathtt{x}_t|\mathtt{x}_{t-1}^i)d\mathtt{x}_t\} \\
&= \frac{1}{N}\sum_{i=1}^{N}\{\int w_t^i f(\mathtt{x}_t|\mathtt{x}_{t-1}^i)d\mathtt{x}_t\} \\
&= \frac{1}{N}\sum_{i=1}^{N} w_t^i \qquad \mathtt{QED}
\end{aligned}
$$

Appendix B

Algorithm: Maximum Likelihood Estimation Using Particle Filter

Module 1: Evaluate the Likelihood of Total Observations $f(\theta)$

- Convert the parameter vector θ into parameters as they exists in the models
- Set the random seed
- For $i = 1, N$ (filtering through the time series stratum: $\{Y_{t_j}^i\}$)
  - Initialize the state vector X_{t_0}
  - For $j = 1, N_i$ (filtering through the observation within the strata)
    - Calculate the likelihood $logf_{ij}$ and estimate state vector X_{t_j} for $[t_{j-1}, t_j]$
      - Evaluate the likelihood of a single period: $logf_{ij} = f(X_{t_j-1}, X_{t_j}, Y_{t_j}^i)$
  - Set $X_{t_j-1} = X_{t_j}$
  - End for (j loop)
- End for (i loop)
- Return $f(\theta) = \sum_{i=1}^{N} \sum_{j=1}^{N_i} logf_{ij}$

Module 2: Evaluate the Likelihood of a Single Period $f(X_{t-1}, X_t, Y_t)$

- For $i = 1, M$ (the number of particles)
  - Propagate through the transition equation: $\mathtt{x}_t^i \sim p(X_t|\mathtt{x}_{t-1}^i)$
  - Attache a weight to each particle based on the observation equation: $w_i \sim f(Y_t|\mathtt{x}_t^i)$
- End for (i loop)
- Calculate the mean of the weights: $\bar{w} = \frac{1}{M}\sum_{i=1}^{M} w_i$
- If $\bar{w} \leq$ vsmall (vsmall is a machine dependent parameter)
  - $logf_t = log(vsamll)$
  - Issue a warning: all weights are too samll
  - $X_t = X_{t-1}$ or re-initialize the state vector
  - Return $f(X_{t-1}, X_t, Y_t) = logf_t$
- Else
  - $logf_t = log(\bar{w})$
- End if
- Normalize $w_i^* = \frac{w_i}{\sum_{i=1}^{M} w_i}$
- For $i = 1, M$
  - Smooth re-sampling:$\{\mathtt{x*}_t^i\}_{i=1}^M = \mathtt{RESAMPLE}[\mathtt{x}_t^i, w_i^*]_{i=1}^M$

$$\begin{cases} \text{Do piecewise-smoothing re-sampling} & \text{if univariate} \\ \text{Do Normal-Kernel-smoothing re-sampling} & \text{if bivariate} \end{cases}$$

- End for
- $X_t = \{\mathtt{x*}_t^i\}_{i=1}^M$
- Return $f(X_{t-1}, X_t, Y_t) = logf$

Appendix C

Calibration Process

The calibration proceeds in three steps:

Step 1: using low LTV loans and high points, we first calibrate the prepayment process and liquidity premium. Based on intermediate output from the minimization routine, we can see how the routine goes about searching for the optimal set of values. To minimize the pricing error, without any further constraints, the routine tends to increase the prepayment speed first by increasing the values for additive risk premium and multiplicative risk premium. After several iteration, the additive risk premium reaches a point that the risk-neutral process is no longer mean-reverting in the risk-neutral measure. While the objective function still decreases, the pace of decreasing is very slow and the decomposition of the loan approaches more and more to a trivial case where most payments come from prepayment. Since there is no easy way to explicitly incorporate all those constraints of economic significance into the minimization process, we have adopted an trial and error approach to stop the minimization routine for the prepayment process.

Step 2: given the parameters for prepayment process from step one, we then calibrate the default process using loans with high LTVs, loss rate and liquidity premium, applying the same principle as we do at step one. If we find that a given set of parameters for the prepayment process can not lead to a meaningful convergence in step two, we go back to step one and select another set of parameters. We find that a proper selection of parameters for the prepayment process will determine a proper convergence for the default process.

Step 3: We test the pricing performance using loans with normal LTVs and points. If the pricing performance is not within reasonable range or the decomposition of the values do not have any economic significance, we go back to step one and start the process all over again. To avoid the minimization routine to search in the invalid domain of the parameters, parameters for speed of adjustment, diffusion,loss rate, liquidity premium are log-transformed to ensure positivity of the parameters prior to being used in the routine.