GRADUATE TEACHING ASSISTANTS' MATHEMATICAL UNDERSTANDING FOR TEACHING TRIGONOMETRY

by

HEE JUNG KIM

(Under the Direction of Patricia S. Wilson)

This study described the mathematical understanding, exhibited by graduate teaching assistants in a Department of Mathematics (GTA-Ms), that is useful for teaching trigonometry. The following two research questions guided this study: (1) To what extent do GTA-Ms exhibit an understanding of trigonometric concepts when solving and explaining trigonometry problems? (2) What understanding of trigonometry do GTA-Ms use in analyzing and responding to students' mathematical thinking about concepts of trigonometry in hypothetical teaching contexts?

I used the framework of Mathematical Understanding for Secondary Teaching (MUST) developed by the Situations Project of the Mid-Atlantic Center for Mathematics Teaching and Learning (MAC-MTL) at Pennsylvania State University and the Center for Proficiency in Teaching Mathematics (CPTM) at the University of Georgia. This framework was a useful guide for designing task items and a good tool for analysis of the data collected from three task-based interviews with each participant because it helped me systemically organize, categorize, and describe the mathematical understanding that emerged from the participants' mathematical work.

In this study, I considered *fundamental concepts of trigonometry* to mean the basic core concepts that underlie teaching and learning trigonometry. The findings from this study showed that the participants exhibited a mathematical understanding characterized in most of the strands of the MUST framework. Although GTA-Ms exhibited proficiency with advanced mathematical concepts, they showed a lack of conceptual understanding of some fundamental concepts useful for teaching mathematics when solving and explaining trigonometry problems.

Given a hypothetical teaching context describing students' mathematical thinking, the participants tended to use their mathematical understanding to respond in formal ways, such as providing rigid definitions, deductive reasoning, and conventional manipulations of mathematical symbols. In particular, their explanations of both advanced and fundamental concepts were more procedural than conceptual, equation-oriented, and definition-based. The findings from this study suggested that mathematical concepts–fundamental as well as advanced concepts–within courses that GTA-Ms teach should be revisited and conceptually developed as part of their preparation for teaching.

INDEX WORDS: Graduate teaching assistants in mathematics, Professional development, Mathematical understanding for teaching, Trigonometry

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by

HEE JUNG KIM

B.S., Ewha Womans University, South Korea, 1991

M.S., Ewha Womans University, South Korea, 1993

Ph.D., The Pennsylvania State University, 2002

M. Ed., Bowling Green State University, 2007

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HEE JUNG KIM

Major Professor: Patricia S. Wilson

Committee: Denise A. Spangler
AnnaMarie Conner

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
May 2013

DEDICATION

I would like to dedicate this dissertation to my Lord, Christ Jesus.

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CHAPTER 1

INTRODUCTION

Because of today's competitive educational environment, universities are quite interested in improving the quality of their undergraduate education. A particularly important aspect of this goal is the necessity of providing undergraduates learning mathematics with an academically strong, educational experience. Offering a high quality mathematics program is additionally important to a university system because mathematics is seen as a foundational discipline for many other programs in science, business, and even in the arts in this technological age.

One very common trend in major universities is the employment of graduate teaching assistants for introductory undergraduate mathematics courses in the Department of Mathematics. At the time of their employment, graduate teaching assistants bring various backgrounds to the department. In addition, their knowledge, beliefs, and experience developed along the way, while carrying out their instructional roles, appear to be influential in the quality of undergraduate mathematics education. Therefore, it is necessary to pay attention to what they know, learn, experience, believe, and think when they serve as college mathematics instructors. In this study, I focused on what they know and sought to understand the nature of their mathematical understanding for teaching undergraduate mathematics, such as precalculus.

Background

There are some prevailing erroneous beliefs about university instruction deeply rooted in school and society. These beliefs impede the improvement of undergraduate mathematics education (Alsina, 2001; National Research Council [NRC], 1991). One such belief claims that because mathematicians have expertise in the field of mathematics, they are sufficiently qualified to teach mathematics. However, research on K-12 mathematics education (e.g., Begle, 1979; Monk, 1994) shows that knowledge of advanced mathematics does not guarantee effective teaching of mathematics. This belief may result in students suffering from ineffective teaching by experts in mathematics.

Another academic belief is that undergraduate mathematics teaching does not require special training for teaching. Instead, all that is required for the successful teaching of mathematics is on-the-job, accumulated experience, clear presentation, and solid content knowledge (Alsina, 2001). In fact, college mathematics instructors including graduate teaching assistants have limited opportunities to participate in specific training for teaching or to learn about existing mathematics education research. They tend to construct models for teaching primarily based on their prior learning experiences during their own school years, including undergraduate and graduate years, as well as their own teaching experience (Alsina, 2001; NRC, 1991; Speer, Gutmann, & Murphy, 2005). However, researchers (e.g., Bass, 1997; Speer et al., 2005) suggested that professional development for college instructors including graduate teaching assistants in the Department of Mathematics is indeed necessary to improve the quality of undergraduate mathematics education.

A third popular belief argues that pedagogy in undergraduate mathematics education merely consists of the need for a clear and logical presentation of content (Selden & Selden, 1993). Because of this belief, undergraduate mathematics courses have a lecture-mastery paradigm. The predominant perspective for undergraduate mathematics teaching remains that of knowledge transmission or apprenticeship (Alsina, 2001; Prosser & Trigwell, 1999), although Tall (1991) noted, “Current approaches to undergraduate teaching tend to give students the *product of mathematical thought* rather than the *process of mathematical thinking*….A logical presentation may not be appropriate for the cognitive development of the learner" (p. 3).

Departments of Mathematics are one of the largest employers of graduate teaching assistants (GTAs) among U.S. colleges and universities (Henderson, 1997). Graduate teaching assistants in Departments of Mathematics (GTA-Ms) play various roles in undergraduate mathematics education (Speer et al., 2005). Their roles include grading homework and quizzes, solving problems in recitation sessions, and providing tutoring services for mathematics courses (Speer et al., 2005). However, their role becomes more significant in undergraduate mathematics education when they teach undergraduate courses, such as lower-division mathematics courses or content courses for prospective teachers, as instructors of record (Speer et al., 2005). A recent survey reported that about one-half of all GTA-Ms serve as instructors of record and teach 6-14% of the sections at the precollege-level (remedial), introductory-level, and calculus-level undergraduate mathematics courses at U.S. colleges and universities (Belnap & Withers, 2008; Lutzer, Rodi, Kirkman, & Maxwell, 2005). From 2000 to 2005, the percentage of precollege-level sections taught by GTA-Ms increased from 9.5 to 14.6%; the percentage of sections of introductory-level courses (including college algebra, precalculus, mathematics for liberal arts, etc.) taught by GTA-Ms increased from 10.4 to 10.5%; and the percentage of calculus-level sections taught by GTA-Ms increased from 6.4 to 7.6% (Lutzer et al., 2005). In Departments of Mathematics that offer a Ph.D. program in mathematics, approximately 35% of the sections (excluding distance learning) of introductory-level courses (including college algebra and precalculus) were taught by GTA-Ms in 2005 (Belnap & Withers, 2008; Lutzer et al., 2005). These statistics show that not only has the level of involvement of GTA-Ms in undergraduate mathematics education been increasing, but also their potential impact on undergraduate students' learning of mathematics has become greater (Speer et al., 2005).

Rationale

Study of GTA-Ms

In the 1980s, increasing concerns about students' learning of undergraduate mathematics stimulated research on undergraduate mathematics education, which has become active since then (Speer et al., 2005; Speer, Murphy, & Gutmann, 2009). The early focus of research on undergraduate mathematics education was placed primarily on student learning (Hart, 1999; Speer et al., 2005). As researchers became aware of the significant involvement of GTA-Ms in undergraduate mathematics education, they began to pay attention to GTA-Ms "as current and future key players" (Speer et al., 2005, p.76) in the field of undergraduate mathematics education because some GTA-Ms were seen the potential mathematics faculty members of the future (Belnap, 2005). The National Science Foundation [NSF] (1996) report on undergraduate education in Science, Mathematics, Engineering and Technology recommended that college and university governing boards and administrators provide opportunities for graduate students to learn about effective teaching strategies as part of their graduate programs. Resources (e.g., materials, activities, or programs) for the professional development of GTA-Ms became abundantly available in the 1980s and 1990s (Speer et al., 2009). Speer et al. (2009) explained that these resources were developed from "various groups' collective wisdom from practical experience about key issues in learning to teach," (p. 5) but were not based on research about issues related to GTA-Ms and their professional development. Therefore, research-based inquiry on GTA-Ms' learning to teach broke new ground in the late 1990s. At present, however, there is still only a limited body of "groundwork research" (Speer et al., 2005, p. 78) about GTA-Ms as college instructors, compared to a substantial body of research about K-12 teachers (Speer et al., 2005). GTA-Ms' characteristics, such as what they know, think, believe, experience, and need in relation to undergraduate mathematics teaching and learning, remain relatively unknown, although research is in progress (Harris, Froman, & Surles, 2009; Hart, 1999; Speer et al., 2005; Speer et al., 2009; Speer, Smith III, & Horvath, 2010). Lack of research about GTA-Ms as college instructors could restrict our understanding of their teaching practices and professional development for teaching preparation (Speer et al., 2010). Therefore, this study of GTA-Ms' characteristics, in particular, of their mathematical understanding for teaching, was conducted to fill a gap in the current research literature.

Study of GTA-Ms' Mathematical Understanding for Teaching

Teachers' mathematical knowledge has been an important issue in K-12 mathematics education because knowledge and understanding of mathematics to teach is integral to teaching and is influential in students' learning (Fennema & Franke, 1992). The National Council of Teachers of Mathematics [NCTM] (2000) highlighted the importance for teachers' mathematical knowledge by noting, "To be effective, teachers must know and understand deeply the mathematics they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks" (p. 17).

In past decades, researchers (e.g., Begle, 1972; Grossman, Wilson, & Shulman, 1989; Monk, 1994; Monk & King, 1994) attempted to find the relationship between teachers' mathematical knowledge and student performance quantitatively. They used, for example, test scores, the number of undergraduate mathematics courses taken, or grade point averages as indicators of teachers' mathematical knowledge. However, researchers failed to find a consistent correlation between teachers' mathematical knowledge and students' mathematics achievement.

Such failure provided researchers with new insight into the complex aspects of teacher knowledge, which could not be quantified with numbers (Grossman et al., 1989). Researchers began to explore various aspects of teacher knowledge from, for example, its nature, form, organization, and content (Grossman et al., 1989).

Inspired by Shulman and his colleagues' work (Shulman, 1986; Grossman et al., 1989) for categorizing content-related teacher knowledge, researchers (e.g., Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008; Even, 1990; Grossman, 1990; Hill, Ball, & Schilling, 2008; Ma, 1999) proposed theoretical models of the dimensions of teacher knowledge and identified the nature and the effect of teacher knowledge on teaching practice and on student learning. Researchers (e.g., Ball & McDiarmid, 1990; Fennema & Franke, 1992) found that neither prospective nor practicing teachers had good mathematical preparation. They also showed a lack of adequate mathematical knowledge for teaching. Therefore, researchers began to turn their attention to teachers' mathematical preparation in teacher education and in professional development. As a result of continuous active research about this area (e.g., Brown & Borko, 1992; Fennema & Franke, 1992), the body of research on K-12 teacher knowledge became extensive and substantial. Therefore, its findings have served to improve K-12 mathematics education as a foundation not only for K-12 teacher development models and programs but also for research about effects of the programs on teaching practice.

Even though professional development programs for GTA-Ms became widespread, most existing programs including activities and materials were designed by experienced instructors (Speer et al., 2009; Speer et al., 2010). Although the resources might be useful and valuable for novice college instructors, they were not based on GTA-Ms research, such as research about their knowledge or beliefs (Speer et al., 2005; Speer et al., 2010).

GTA-Ms are different from K-12 teachers in the sense that GTA-Ms form a unique group of "teachers," who are young mathematicians teaching undergraduate mathematics with little or limited systemic teaching training (Li, 2009). However, they are similar to K-12 teachers in the sense that GTA-Ms, as college instructors, go through a similar developmental process to become effective teachers (Belnap, 2005; Li, 2009; Speer et al., 2005). Therefore, researchers inspired by K-12 research on mathematics education suggested that research-based K-12 teacher development should be a model for research on GTA-Ms' professional development (Speer et al., 2005; Speer et al., 2009; Speer et al., 2010). Speer and her colleagues (2005) asserted,

> The similarities [of GTA-Ms and K-12 teachers] may point to ways in which the existing research base on K-12 teacher development can be applied to TAs [mathematics teaching assistants]. Differences may help identify areas where additional research is especially needed. In both situations, making use of and building on what is known from research in K-12 teacher development could be an important component of design and implementation of professional development for TAs. (p. 78)

Although findings from research on K-12 teacher knowledge have served as a foundation of K-12 teacher development (e.g., Ball et al., 2001; Cooney, Shealy, & Arvold, 1998), very little empirical research has explored GTA-Ms' knowledge for teaching (Li, 2009; Speer et al., 2005). Kung (2010) said, "Teacher knowledge at the college level remains a largely an unexplored subject, despite the importance for such knowledge to college teaching and the preparation of future teachers at the high school and college level" (p. 143). Only a few recent studies have investigated GTA-Ms' knowledge of student thinking or learning (Kung, 2010; Kung, Speer, & Gucler, 2006; Speer, Strickland, & Johnson, 2005) and knowledge of students' strategies and difficulties (Speer, Strickland, Johnson, & Gucler, 2006).

Stein, Baxter, and Leinhardt (1990) highlighted the importance for research on teacher knowledge of subject matter: "In order to build a solid understanding of how teacher knowledge relates to instructional practice, we need to develop and draw upon detailed, qualitative descriptions of how teachers know, understand, and communicate their subject matters" (p. 640). However, because it was assumed that GTA-Ms have a strong understanding of the mathematics to be taught, their mathematical understanding for teaching has been given less attention than other domains of teacher knowledge:

> *Even fewer*–virtually *none* [italics added], in fact–have studied the knowledge required for teaching by university mathematicians….Moreover, research mathematicians are *unlikely* [italics added] to lack mathematical content knowledge, and as a result, there may be much for the mathematics education research community to learn about the *other* [italics added] kinds of knowledge required for effective reform-minded teaching. (Wagner, Speer, & Rossa, 2007, p. 248)

I found similar statements related to the assumption in the literature about GTA-Ms. For example, "*Little* [italics added] concern has been raised regarding these TAs' depth of understanding of the content knowledge, yet they often experience significant challenges teaching undergraduate students for *other* [italics added] reasons" (Chae, Lim, & Fisher, 2009, p. 246).

However, Ball (2003) asserted that mathematical knowledge for effective teaching is different from that needed by mathematicians. Although most GTA-Ms are mathematicians who are considered as having mathematical expertise, "knowing something for oneself or for communication to an expert colleague is not the same as knowing it for explanation to a student" (Bass, 1997, p. 19). Results of K-12 research have supported such an argument. For example, Begle's (1972) study showed that teacher effectiveness did not directly relate to further advanced mathematics coursework or majoring in mathematics. He highlighted the importance for teachers' understanding of the content they teach:

> Teacher understanding of modern algebra (groups, rings, and fields) has *no* [italics added] significant correlation with student achievement in algebraic computation or in the understanding of ninth grade algebra….However, teacher understanding of the algebra of the real number system *does* [italics added] have significant positive correlation with student achievement in the understanding of ninth grade algebra. (1972, p. 8)

Regarding the result of the Begle's study (1972), Ball et al. (2001) made the following plausible conjecture:

> One explanation might rest with the increasing compression of knowledge that accompanies increasingly advanced mathematical work, a compression that may interfere with the unpacking of content that teachers need to do (e.g., Ball & Bass, 2000; Cohen, in preparation). (p. 442)

Cuoco (2001) suggested that teachers should "develop the habit of 'mining' the topics they teach for substantial mathematics" (p. 170) because knowledge of advanced undergraduate mathematics is not easily connected with the school mathematics they teach. Researchers (e.g., Conference Board of the Mathematical Sciences [CBMS], 2001; Ma, 1999) also suggested that teachers should develop a deeper and more specialized *understanding* of the mathematical concepts they teach, which might not be naturally acquired from learning advanced mathematics.

Many GTA-Ms undergo significant academic training by striving to learn the advanced and specialized mathematical knowledge that they need to be research mathematicians in their degree programs. In contrast, they have relatively little preparation to be college instructors before or while they are teaching (Li, 2009). The content of introductory-level undergraduate mathematics, which most GTA-Ms are assigned to teach, might not be difficult for GTA-Ms. However, the assumption that their understanding of *any* concept of a subject that they teach is sufficient for effectively communicating the content for teaching is not always warranted, even if they "know" the content. Ball et al. (2005) argued, "Knowing mathematics for teaching demands a kind of depth and detail that goes well beyond what is needed to carry out the algorithm reliably" (p. 21). In this study, I used the term *understanding* as more than knowledge and as knowing and using at a deeper level in a dynamic sense. Therefore, this research was conducted to understand the nature of GTA-Ms' mathematical understanding of the subject of undergraduate mathematics they teach, which might provide insight into their teaching practices and, further, their teaching preparation. In particular, this study chose trigonometry for the subject.

Study of Trigonometry

Trigonometry is a complex and interrelated subject in school mathematics (Brown, 2005). Understanding trigonometry is fundamental to understanding topics in mathematics, such as calculus, and other branches of science, such as physics and astronomy. For example, the concept of trigonometry connects to the study of polar coordinates, complex numbers, vectors, rotations, and modeling periodic phenomena within and outside mathematics (Brown, 2005; Fi, 2003; Weber, 2005). Talley (2009) found that college calculus instructors believe that calculus students' prior knowledge of algebra and trigonometry is essential for successful learning of calculus. Although trigonometry is an integral part of mathematics courses, such as precalculus (CBMS, 2001), it is known as a difficult concept for both students and teachers (Brown, 2005; Fi, 2003; Moore, 2010a). As explained by the CBMS (2001),

> Although there is a geometric basis for the subject of trigonometry, right triangle, and periodic-function aspects of that topic have been traditionally taught in a separate high school course and as part of pre-calculus studies. This may be one of reason why prospective high school mathematics teachers often have some technical proficiency in the trigonometry of right triangles when they come to undergraduate studies, but lack deep understanding of the core geometric principles that make trigonometry possible. (p. 132)

Despite its significance in mathematics education, trigonometry did not draw sufficient attention from researchers over the past decades (Akkoc, 2008; Fi, 2003). Therefore, little research has been conducted on teaching and learning of trigonometry; in particular, empirical research on teachers' knowledge of trigonometry is rare (Akkoc, 2008; Fi, 2003). In addition, research on GTA-Ms' mathematical knowledge of the trigonometry they teach was not found.

As reported by a CBMS survey (2005), total enrollment, including distance-learning enrollment, in the undergraduate introductory-level courses teaching trigonometry (e.g., precalculus, college algebra, trigonometry, and combined courses of college algebra and trigonometry) increased from 59 to 70% over the 15 years from 1990 to 2005. The report showed that not only was there a significant increase in students taking those courses over this period, but in 2005, GTA-Ms in doctoral-level mathematics departments taught about 43% of the sections of trigonometry courses (excluding distance learning), 29% of the sections of combined courses, and 40% of the sections of precalculus courses. The statistics also indicated that the instructional responsibility of GTA-Ms for teaching trigonometry to undergraduate students has increased. Therefore, it is important to understand GTA-Ms' knowledge of the trigonometry they teach because their knowledge influences both their teaching ability and their students' learning of the concepts (Ball et al., 2001; Even, 1990; Ma, 1999; McDiarmid, Ball, & Anderson, 1989). Therefore, this research was conducted to understand the nature of GTA-Ms' mathematical understanding of the subject of the trigonometry they teach, in most cases, in a precalculus course.

Purpose of the Study and Research Questions

It may be the case that GTA-Ms' mathematical knowledge of the content that they teach has not been researched because they are regarded as content knowledge "experts." However, I assert that advanced mathematics coursework and research in a specific mathematical domain do not necessarily guarantee that GTA-Ms have an understanding of fundamental undergraduate mathematics that is useful for teaching in a conceptual manner. It is plausible that GTA-Ms' mathematical understanding of the subjects they teach could vary by concepts and could contribute to the variation of their teaching practices. The lack of research on GTA-Ms' mathematical understanding of a subject for teaching is problematic because such research could serve as groundwork for developing research-based effective professional development for their teaching preparation.

The purpose of this study was to explore the nature of GTA-Ms' mathematical understanding of trigonometry for teaching undergraduate students. I conducted this study to answer the following two research questions:

1. To what extent do GTA-Ms exhibit an understanding of trigonometric concepts when solving and explaining trigonometry problems?
2. What understanding of trigonometry do GTA-Ms use in analyzing and responding to students' mathematical thinking about concepts of trigonometry in hypothetical teaching contexts?

Significance of the Study

This study offers two major contributions to research on GTA-Ms' mathematical understanding of teaching trigonometry. First, this study provides an understanding of GTA-Ms' knowledge, proficiencies, and ideas about teaching that could serve as one piece of the groundwork for developing effective research-based teaching development activities, materials, or programs (Speer et al., 2005). Findings from this study may provide professional developers and policy makers with insight into the nature of GTA-Ms' understanding of the mathematics they teach in undergraduate mathematics courses. This study suggests that GTA-Ms' mathematical understanding of fundamental concepts as well as advanced concepts of a subject that they teach should be developed in their teaching preparation.

Second, this study demonstrates the usefulness of the Mathematical Understanding for Secondary Teaching (MUST) framework developed by the Center for Proficiency in Teaching Mathematics and the Mid-Atlantic Center for Mathematics Teaching and Learning.[1] The framework was significant in designing the study and in the analysis of the data. In particular, the framework was useful for organizing and categorizing the participants' mathematical understanding for teaching exhibited and observed in their written and verbal work. Although the framework is still evolving and improving, this study provides an example of how to use it as a lens through which further studies on teachers' knowledge of content for teaching might be examined.

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Definition of Terms

Graduate teaching assistants in the Department of Mathematics. In this study, the term graduate teaching assistants in the Department of Mathematics (GTA-Ms) refers to graduate students seeking a doctoral degree in mathematics, who are employed on a basis of a temporary contract (possibly renewed per semester or academic year) by the Department and assist in teaching or teach, as instructors of record, undergraduate mathematics courses. This term includes those who also hold both research and teaching assistantships, and those who have taught an undergraduate mathematics course including recitation sessions as GTA-Ms, but currently became research assistants.

Teachers and college instructors. In this study, the label *teachers* refers to prospective or practicing teachers at the K-12 level. In contrast, the label *college instructors* refers to those who serve as instructors of record for a course at institutions of higher education. Many GTA-Ms teach undergraduate mathematics as college instructors.

Overview of the Study

Chapter 1 has presented the background and rationale for the study, and explained the purpose of the study including two research questions. This was followed by the significance of the study and definitions of terms. Chapter 2 presents the theoretical foundation and framework for this study. Chapter 3 offers the context of the study and a description of the participants, and details the study's design, including data collection and analysis. Limitations of the study are also discussed. Chapter 4 presents the findings according to the three components of the MUST framework. Chapter 5 discusses the nature of GTA-Ms' mathematical understanding for teaching trigonometry from the findings. Chapter 6 includes a summary and conclusions of the study, the methodological contributions, and implications for future research.

CHAPTER 2

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

Graduate Teaching Assistants

Research on Graduate Teaching Assistants

In the late 1800s, universities began offering graduate teaching assistantships in all fields as a source of graduate financial support to recruit and retain students in graduate programs (DeFranco & McGivney-Burelle, 2001; McGivney-Burelle, DeFranco, Visonhaler, & Santucci, 2001). Originally, graduate teaching assistants (GTAs) were expected to assist professors in teaching lower- and upper-division courses by taking attendance, grading, or preparing class materials (DeFranco & McGivney-Burelle, 2001). As the number of large classes at many institutions increased during the 1960s and 1970s, so did the number of supplemental sessions conducted by GTAs (Nyquist, Abbott, Wulff, & Sprague, 1991). Today, the instructional services by GTAs are integral to undergraduate education (Speer et al., 2005) and a considerable portion of undergraduate courses are taught by GTAs at many institutions (DeFranco & McGivney-Burelle, 2001; Luo, Bellows, & Grady, 2000). For example, GTAs teach more than one-half of the undergraduate courses offered at Yale University (Luo et al., 2000).

As the instructional roles and duties of GTAs have increased, some educational concerns about GTAs for undergraduate education have been raised (DeFranco & McGivney-Burelle, 2001). First, departments generally hire and financially support GTAs based on consideration of their potential as graduate students of their disciplines, not on their teaching competence (Henderson, 1997; Sprague, 1992). As a consequence, some new GTAs begin teaching with little or no prior teaching experience or formal training for teaching, and tend to primarily rely on their prior learning experiences as a source for ideas about teaching (Belnap, 2009; Li, 2009; McGivney-Burelle et al., 2001; Speer et al., 2005). In fact, researchers found that GTAs often lack support and guidance in making the transition from learners to college instructors of their particular academic disciplines (Childs, 2008).

Second, GTAs are most likely assigned to teach introductory-level undergraduate courses. Therefore, they are often regarded as the representatives of the department of the academic discipline. Because undergraduate students typically learn the course content through interactions with GTAs, the students may develop perspectives including values and beliefs about the discipline from their learning experiences with GTAs (Daly, 1992).

Third, many current GTAs are prospective faculty members. Teaching is an important function they will perform at institutions of higher education upon graduation. Golde and Dore (2001) surveyed doctoral students in 11 arts and sciences disciplines including mathematics, from 27 universities with doctoral education. The survey showed that approximately 60 to 90% of the doctoral students surveyed (except for molecular biology and chemistry) had a faculty career in mind. In particular, 75% of the mathematics doctoral students surveyed showed interest in a faculty career after graduation. The survey also showed that 83% of the doctoral students surveyed *across* disciplines, selected a factor of “enjoyment of teaching” as the one positively influencing their decision to pursue a faculty career. Therefore, it is plausible to say that GTAs will bring their teaching experiences and preparation into their future professions, which makes a long-term effect on undergraduate education.

To address such growing concerns, since the 1980s, numerous GTA preparation programs have sprung up and been developed at universities and departments; such programs include GTA orientation as university-wide training and departmental seminars designed to enhance the teaching skills of GTAs (Belnap & Withers, 2008; Childs, 2008). Therefore, over the past decades, research focused on describing methods and strategies for effective GTA preparation or training. For example, researchers suggested GTA workshops or faculty supervision and mentoring through classroom observations and feedback (Bass, 1997; Johnson, 2001). Most professional development programs for GTAs have been designed from experienced instructors' practical knowledge, and provided GTAs with institutional and departmental policies, a laundry list of basic routine instructional duties, and "how-to-do-it" teaching skills, such as communication skills or classroom management (Galvin, 1992; Shannon, Twale, & Moore, 1998).

On the other hand, research investigating the effects of existing GTA teaching training has recently begun (Childs, 2008; Shannon et al., 1998). Researchers found that, despite university efforts for GTA training, GTAs preparation programs do not significantly impact GTAs' teaching practices. In addition, they also found that GTAs perceive that the university provides limited support in helping them prepare to teach and, as a result, tend to depend upon teaching models shaped by their learning experience (Defranco & McGivney-Burelle, 2001; Golde & Dore, 2001; Shannon et al., 1998). Therefore, numerous researchers (Daly, 1992; Saroyan, Amundsen, McAlpine, Weston, Winer, & Gandell, 2004) asserted that discipline-free and skill-based training programs are generally ineffective to improve teaching because teaching always happens in a context that interacts with perspectives, knowledge, and actions. They highlighted that effective workshops for teaching at the postsecondary level require an understanding of the relevance of general pedagogies to a particular discipline in a teaching context. They also suggested that research on GTAs should explore the process of preparing

GTAs in a specific discipline rather than across disciplines because different content domains require distinctive knowledge and methods for teaching.

Research on Graduate Teaching Assistants in the Department of Mathematics

Reports on failure in producing positive effects of GTA's teaching preparation programs (Defranco & McGivney-Burelle, 2001; Golde & Dore, 2001; Shannon et al., 1998) suggested a new direction of research focus in relation to the preparation of graduate teaching assistants' in the Department of Mathematics (GTA-Ms') (Speer et al., 2005). Researchers in undergraduate mathematics education began to pay attention to fundamental aspects involved in GTA-Ms' learning to teach. Influenced by literature on K-12 teacher development, they attempted to identify aspects of their development as college instructors, which might not have been considered or supported by existing teaching training programs (Speer et al., 2009). Because it has been shown at the K-12 level that teacher knowledge is one of critical factors that impact teaching practice and development of teacher knowledge in professional development programs is essential for effective teacher training (Fennema & Franke, 1992), researchers applied and extended ideas of research on K-12 teacher knowledge to research GTA-Ms' knowledge for teaching (Speer et al., 2009). Their goal was to provide research-based guidance for designing new professional development programs as seen in research on K-12 teacher development, and consequently, to help GTA-Ms improve teaching practices. At present, although research on GTA-Ms is still young, it is in progress (Speer et al., 2005; Speer et al., 2009).

Research studies about GTA-Ms' knowledge for teaching are rare (Speer et al., 2005). Because GTA-Ms are studying advanced mathematics, researchers paid more attention to GTA-Ms' knowledge for teaching in relation to student thinking than their mathematical understanding of the content that they teach. Roach, Roberson, Tsay, and Hauk (2010) explained the nature of GTA-Ms' knowledge:

> The knowledge of MTAs [meaning by GTA-Ms] about tackling the highest level of task, "doing math" (Stein et al.'s term), is densely packed and routinely used as implicit knowledge in their own graduate course. That is, helping instructors [GTA-Ms] learn to "unpack" their knowledge and ways of knowing is a different challenge in the professional development of MTAs than for in-service teachers. (p. 11)

Researchers investigated GTA-Ms' knowledge of student thinking for concepts of calculus to identify the nature of their knowledge (Speer, Strickland et al., 2005; Speer et al., 2006). The findings showed that although GTA-Ms exhibited conceptual understanding and procedural fluency with the concepts when they solved the tasks, they had difficulties describing students' strategies and providing explanation about students' difficulties. Speer and colleagues (2005, 2006) noted that this difficulty was an indicator of limited ideas about student thinking. The GTA-Ms could not offer other solution approaches that students might use other than those that the GTA-Ms had generated as optimal solutions. For example, when they were asked for any other possible solutions, they responded, "Well, I always imagine a student would do exactly what I would do" (Speer, Strickland et al., 2005, pp. 3-4). The findings indicated that GTA-Ms' strong understanding of content did not ensure a comparable understanding of student solution strategies and student difficulties. Speer and colleagues (2005, 2006) suggested that GTA-Ms' professional development should be designed so that GTA-Ms' knowledge of student thinking could be developed.

Trigonometry

Overview of the History of Trigonometry

Over 2000 years ago, the Greeks coined the term trigonometry by combining two words *trigonon* (triangle) and *metria* (measurement), which means triangle measurement (Swokowski & Cole, 2009). Trigonometry originally served as a tool to solve practical application problems in the field of astronomy where researchers needed to perfect astronomical calculations. For this reason, circle trigonometry emerged from the astronomic study of the heavens *prior to* the birth of triangle trigonometry (Bressoud, 2010).

Triangle trigonometry was not fully developed until the 11th century, even though its initial development began with the question of how to determine a shadow's length of a vertical stick in the 2nd century (Bressoud, 2010; Scott, 1960). From the mid-16th century, a ratio definition of trigonometric functions emerged and began to be applied to calculation problems to determine the unknown sides of right triangles (Bressoud, 2010). Since the 17th century, trigonometry was no longer considered as just a tool for astronomy, but developed as an independent branch in mathematics. The ratio definition of trigonometric functions in a triangle context became predominant in the 19th century (Bressoud, 2010).

The development of trigonometry was related to the development of angle measurement. A degree measure of an angle was initially defined as the length of an arc subtended by the angle, which depends on the length of the radius of the circle, and later in the late 19th century, as a fractional part of a complete revolution (Bressoud, 2010; Cooke, 1997). Currently, one degree is defined by 1/360th of the full turn corresponding to 360º (Bressoud, 2010). The focus of the early study of trigonometry was placed on the computation of the length of the *chord.* A chord is a line segment formed by connecting the endpoints of an arc. Since then, the chord functions have evolved to trigonometric functions per se (Scott, 1960). For example, if the chord subtends an arc whose length is 2θ along the circle of radius r, half the chord length is rsin(θ) (Bressoud, 2010). Therefore, the term *sine* means half-chord. When the argument of the trigonometric functions shifted from an arc length to a fraction of a complete revolution in the late 19th century, a unit of angular measure called *radians* appeared, although it is uncertain who first coined the term (Bressoud, 2010).

Research on Learning and Teaching of Trigonometry

Trigonometry, as an important high school subject, has served practical and theoretical purposes over the past centuries. Trigonometry is a powerful tool for science and engineering. For example, the fields of navigation and surveying use trigonometry as a tool for calculation, and identifying periodic real-world phenomena, such as heart rhythms or earthquakes, requires trigonometric models. Mathematical topics such as vectors, polar coordinates, or complex numbers, are all related to trigonometric ratios (NCTM, 1989).

Trigonometry has proven an essential component of mathematical knowledge needed to successfully learn calculus (Talley, 2000) and has also proven to be useful for modeling periodic phenomena (NCTM, 1989, 2000). However, findings from research exploring students' understanding of trigonometry showed that the subject is difficult for students to learn (Brown, 2005; Moore, 2010a). Pritchard and Simpson (1999) agreed and explained that trigonometry is "the confluence of a number of streams of mathematical difficulties" (p. 81). In fact, evidence revealed that, in general, students' understanding of trigonometry is incomplete and fragile. For example, students' understanding of the sine and cosine functions, conceptions of angle measure and the radius as a unit for measuring an angle were weak and fragmented (Brown, 2005; Moore, 2010a, 2010b; Moore, LaForest, Kim, 2012).

Bressoud (2010) attributed the students' difficulties to a dichotomy of teaching approaches:

> The study of trigonometry suffers from a basic dichotomy that presents a serious obstacle to many students. On the one hand, we have triangle trigonometry, in which angles are commonly measured in degrees and trigonometric functions are defined as ratio of sides of a right-angle triangle. On the other hand, we have circle trigonometry, in which angles are commonly measured in radians and trigonometric functions are expressed in terms of the coordinates of a point on the unit circle centered at the origin. Faced with two such distinct conceptual approaches to trigonometry, it is any wonder that so many of our students get confused? (p. 107)

Moore (2010a) asserted that students' difficulty in trigonometry may result from difficulty in reasoning necessary for comprehending trigonometry, and also from a lack of other fundamental conceptions, such as radian measurement, that support understanding trigonometry. For example, an understanding of trigonometric functions requires mathematical reasoning without using algebraic formulae or direct numerical manipulation, which might make it seem difficult to students (Weber, 2005). Bressoud (2010) noted the difficult nature of the concept of radians in learning trigonometric functions:

> The degree becomes $1/360^{th}$ of a full turn, forcing practitioners to devise a name for the unit being used when 2π corresponding to a full turn….It is no wonder that students have difficulty comprehending radians. One 360^{th} of a "full turn" makes sense….Few students can conceptualize what one 2πth of a full turn might be. Of course, $1/(2\pi)$ is a fraction that is mathematically meaningful, but it is also conceptually difficult. (pp. 111-112)

Researchers asserted that the disconnection between pictorial images of the trigonometric situation and symbolic manipulation (Pritchard & Simpson, 1999) and a lack of coherence of trigonometric representations between secondary school and college textbooks might also be potential sources of student difficulties in learning trigonometry (Byers, 2010). Pritchard and Simpson (1999) argued that student difficulty in learning trigonometry is a consequence of using unidirectional reasoning patterns to reach solutions. They found that students had difficulty "moving flexibly between images of trigonometric situations and algebraic/numerical symbolism" (p. 81). For example, when asked to find the angle x such that $\cos(x) = 0.24$, a student replied with "0.24 divided by cos," which indicated evidence of misinterpreting $\cos(x)$ as the product of "cos" and x and misapplying the algebraic procedure to this trigonometric situation (p. 85).

Research on student difficulties in learning trigonometry suggested a need to improve teaching methodologies. Several researchers have discussed appropriate pedagogical considerations for effective teaching of trigonometry. Weber (2005) found that a lecture-based traditional teaching method is ineffective for developing an understanding of trigonometric functions. Calzada and Scriano (2006) suggest that teachers should introduce trigonometry by beginning with concrete concepts from algebra and geometry, such as the concepts of area and the Pythagorean Theorem.

Since the mid-19th century, as the instructional focus shifted from circle trigonometry to triangle trigonometry, a common belief that triangle trigonometry should be taught before introducing circle trigonometry has been developed (Bressoud, 2010). Bressoud (2010) and Kendal and Stacey (1998) compared two approaches to introducing trigonometric functions based on the historical development of trigonometry–ratio method and unit circle method. The ratio method describes trigonometric functions as ratios of pairs of sides of a right triangle. The ratio definition of trigonometric functions fits into applications for measurement as well as surveying. The ratio method was generally used to introduce trigonometric functions until the 1960s when another method emerged, the unit circle method. This method defines cosine and sine as the x and y coordinates of a point on a unit circle, and highlights the nature of trigonometric functions as functions of a real variable. The unit circle definition of trigonometric functions fits into applications related to periodic phenomena.

Some researchers discussed the pedagogical pros and cons of each of the methods for effective teaching of trigonometry. Akkoc (2008) and Bressoud (2010) asserted that prior learning of trigonometric functions as ratios might hinder the shift to viewing the argument of trigonometry functions as an arc length and the functions as lengths. They also argued that the concrete and precise meaning of trigonometric functions comes from an understanding of the unit circle and radian measurement. On the other hand, Kendal and Stacey (1998) found the ratio method more effective, resulting in better student performance on trigonometric problems, such as finding an unknown length of a given right triangle. Therefore, Kendal and Stacey (1998) suggested a combined method, where teachers introduce unit circle definitions first and make a connection with ratio definitions in order to adopt the techniques of the ratio method for triangle problems. As the title, *Returning to the beginnings of trigonometry–the circle–has implications for how we teach it*, of Bressoud's article (2010) implies, he suggested restructuring the curriculum of teaching trigonometry based on a historical point of view:

> We would do well to introduce trigonometry by imitating the astronomers who first discovered and explored these functional relationships by seeing them as connecting lengths of arcs and lengths of line segments….If trigonometric functions are first

introduced as lengths of line segments in a circle of radius 1, then they have a concrete meaning….For students who first memorize trigonometric functions as ratios, making the transition to seeing them as lengths much harder. (p. 112)

Research on Teacher Knowledge of Trigonometry

The NCTM (2007) recommended that teachers should develop a conceptual understanding of trigonometry from the geometric view point as well as a procedural understanding so that teachers can in turn apply the concept of trigonometry to solving problems involving calculus. Research findings related to teacher knowledge of trigonometry has been consistent with those of teacher knowledge of other mathematical subjects, such as functions. For example, the CBMS (2001) reported that prospective teachers tend to lack an understanding of the core geometric principles related to trigonometry and often show only procedural fluency in triangle trigonometry. Fi (2003) found that prospective mathematics teachers did not have a good, comprehensive understanding of the radian measure of angles, inverse trigonometric functions, reciprocal functions, periodicity, or co-functions. For example, none of his participants could accurately define a radian angle measure in terms of a ratio of the length of arc and the radius of a given circle.

Some researchers (e.g., Akkoc, 2008; Topçu, Kertil, Akkoç, Yilma, & Önder, 2006) discussed teachers' concept images in trigonometry, which refer to the total cognitive structure associated with the concept or "the set of all the mental pictures associated in the student's mind with the concept name, together with all the properties characterizing them" (Vinner & Dreyfus, 1989, p. 356). They found that prospective and practicing teachers have weaker concept images of the radian concept than they do of the degree concept in trigonometry. They asserted that this imbalance of the concept images might hinder the teachers from perceiving a radian as a real number and, furthermore, trigonometric functions as functions of real numbers. On the other hand, the teachers with strong concept images of the radian made appropriate conceptual connections among concepts in trigonometry.

Many GTA-Ms, as college instructors, are responsible for undergraduate students' learning of trigonometry when they teach introductory-level mathematics courses. Despite its importance, as of now, there is no previous empirical research on GTA-Ms' mathematical understanding of trigonometry for teaching, although other topics were considered for research on GTA-Ms' knowledge for teaching, such as limits or derivatives (Kung et al., 2006; Speer et al., 2005).

Teacher Knowledge of Subject Matter

Research on Teacher Knowledge

Teacher knowledge is defined as "the total knowledge that a teacher has at his or her disposal at a particular moment...which underlies his or her actions" (Carter, 1990, as cited in Verloop, Driel, & Meijer, 2001, p. 445). It is well known that teacher knowledge is one of the key factors influential in teaching and student learning (Fennema & Franke, 1992). Therefore, it is an important aspect for consideration in professional development, when referring to teacher preparation for prospective teachers and to the development of practicing teachers: "professional development must provide opportunities for professional growth…and motivate them to develop the knowledge, skills, and dispositions they need to teach mathematics well….Professional growth is…marked by change in teachers' knowledge, beliefs, and instructional strategies" (Sowder, 2007, pp. 160-161). Researchers (e.g., Porter, Desimone, Birman, & Yoon, 2001; Sowder, 2007; Verloop et al., 2001) asserted that preparing teachers without addressing teacher knowledge in a meaningful way results in ineffective preparation.

Some of the early research in the 1970s on teaching focused on identifying teaching behaviors and procedures that could effectively promote student learning. These research findings have been employed in teacher education and teacher evaluation for the purpose of teachers' developing effective teaching skills (Fenstermacher & Richardson, 2005; Shulman, 1986). When researchers failed to find a consistent relationship with what teachers knew and what students learned in a quantitative way, they turned their research focus to the nature of teacher knowledge and its roles in teaching (Grossman et al., 1989). Coinciding with the shift in focus of research, the perspective of teaching also shifted from skillful performance or knowledge transmission to "a highly complex, context-specific, interactive activity in which differences across classrooms, schools, and communities" (Cochran-Smith & Lytle, 1990, p. 3).

After a long period of neglect concerning the "content dimension of teaching" (p. 6) in the research community, Shulman (1986) described its absence as the "'missing paradigm' problem" (p. 6) in research on teaching. He asserted that teachers' content knowledge for teaching needs more attention as an important topic of research on teaching and teachers because teaching is knowledge-based professional work that requires content-specific pedagogy. He classified teachers' content knowledge for teaching into three domains: (1) content knowledge, (2) pedagogical content knowledge, and (3) curricular knowledge.

Research on Teacher Knowledge of Subject Matter

Overview. Shulman (1986) defined *content knowledge* as "the amount and organization of knowledge per se in the mind of the teacher" (p. 9). Leinhardt and Smith (1985) refined Shulman's definition as, "Subject matter knowledge includes concepts, algorithmic operations, the connections among different algorithmic procedures, the subset of the number system being drawn upon, the understanding of classes of student errors, and curriculum presentation" (p.

247). Inspired by Shulman's work of identifying teacher knowledge domains over the past decades, numerous researchers (e.g., Ball, 1990; Ball & Feiman-Nemser, 1988; Ball et al., 2005; Brown & Borko, 1992; Even, 1990; Grossman et al., 1989; Leinhardt & Smith, 1985; Ma, 1999) explored teacher knowledge of subject matter and its impact on lesson planning and teaching practices.

Nature of teacher knowledge of subject matter. Many researchers noted that knowledge of subject matter for teaching is different from knowledge of subject matter for the discipline (Bass, 1997; Grossman et al., 1989). Grossman and her colleagues (1989) argued,

> While some of what teachers need to know about their subjects overlaps with the knowledge of scholars of the discipline, teachers also need to understand their subject matter in ways that promote learning. Teachers and scholars have different goals. Scholars create a new knowledge in the discipline. Teachers help students acquire knowledge within a subject area. These differing goals require related but *distinct understandings* [italics added] of the subject matter. (pp. 24–25)

Researchers also showed that knowledge of advanced mathematics was not correlated towards improving student performance (Begel, 1979; Monk, 1994). Because knowledge of advanced undergraduate mathematics is not easily connected with the school mathematics they teach, teachers should develop a deeper and more specialized understanding of the mathematical concepts taught, which might not be naturally acquired from learning advanced mathematics (e.g., CBMS, 2001; Cuoco, 2001; Ma, 1999).

Research (e.g., Ball, 1990; Ma, 1999) found teachers' fragmented knowledge of subject matter, which affected the instructional content, process, and pedagogical decisions—what they teach and how they teach it—and ultimately, student learning. Researchers (Grossman et al., 1989; Brown & Borko, 1992; Shulman, 1986) asserted that teacher knowledge of subject matter is influenced by other domains of teacher knowledge and can transform and grow through the continual process of preparing and teaching over extended periods of time.

Importance for teacher knowledge of subject matter in relation to pedagogical knowledge. Shulman (1986, 1987) coined the term *pedagogical content knowledge* as another domain of teacher knowledge that might be influential in teaching practice and student learning. He defined it as the integration of content and pedagogy, a pedagogical understanding of subject matter, or an understanding of how to represent the content in efficient ways to make the subject comprehensible to students with diverse interests and abilities. It also includes knowledge of student difficulty in learning as well as typical perceptions and misconceptions.

Shulman's conception of pedagogical content knowledge continues to be refined because of its ambiguous characteristics (Marks, 1990). Several researchers (e.g., Bennett & Turner-Bisset, 1993; McEwan & Bull, 1991; Marks, 1990) asserted that Shulman's distinction between subject matter content knowledge and pedagogical content knowledge is not only ambiguous but also impossible to detect because all teacher knowledge is pedagogical in various ways, especially in the context of teaching. In a quantitative study, Neubrand, Seago, Agudelo-Valderrama, DeBlois, and Leikin (2009) found that teacher knowledge of subject matter and pedagogical content knowledge are highly correlated. Moreover, some aspects of pedagogical content knowledge are deeply rooted in teacher knowledge of subject matter, such as explaining mathematical concepts (Marks, 1990).

Impact of teacher knowledge of subject matter on teaching practice. Several researchers investigated how teacher knowledge of subject matter relates to teaching practice. Ball (1988) asserted that how teachers understand and think about subject matter, as well as what they know, are important for teaching. Shulman (1987) found that a teacher with a poor understanding of the content taught in a less flexible and less interactive manner, which implies that, "teaching behavior is bound up with comprehension and transformation of understanding" (p. 18). He argued that a weak knowledge of content is a reason for ineffective teaching. Van Dooren, Verschaffel, and Onghena (2002) found that a lack of prospective teachers' understanding of the content could limit their ability to assess student learning properly because this makes it difficult for them to deal with students' various ideas and methods to reach solutions. Stein et al. (1990) found that a teacher's limited understanding of the content narrowed his or her teaching practice in the following sense: "the lack of provision of groundwork for future learning in this [the given] area, overemphasis of a limited truth, and missed opportunities for fostering meaningful connections between key concepts and representations" (p. 659). Researchers (e.g., Even, 1990; Grossman et al., 1989; Leinhardt & Smith, 1985) asserted that teacher knowledge of subject matter is influential in the ability to critique textbooks, select material or examples, formulate explanations and demonstrations, structure lessons, conduct instruction, and pedagogical decisions, such as asking questions or facilitating activities.

Impact of teacher knowledge of subject matter on teacher education. Research findings on teacher knowledge of subject matter indicated that subject matter preparation of K-12 teachers should be an essential element in teacher education (Ball & McDiarmid, 1990; Ball et al., 2005; NCTM, 2007). Sowder (2007) noted that "developing mathematical content knowledge" (p. 162) is one of several goals of professional development. Because "the knowledge of subject matter that is central to teaching is also knowledge that is central to 'knowing' a discipline" (Grossman et al., 1989, p. 24), it is suggested that teachers should develop proficiency in mathematics, such as conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (NRC, 2001). Ma (1999) suggested that elementary school teachers should achieve a "deep, vast, and thorough" (p. 120) understanding of the mathematical topics beyond simply knowing the content that they teach so that they can use solid basic ideas, make disciplinary connections, have multiple perspectives, and be aware of longitudinal coherence.

Although numerous studies support the importance for teacher knowledge of subject matter in K-12 teacher education, GTA-Ms' professional development programs tend to overlook it and are designed to inform GTA-Ms of their basic routine duties and provide teaching strategies and discuss pedagogical issues. Developing "deep, vast, and thorough" knowledge (Ma, 1999, p. 120) of the content they teach is also essential in their professional development as college instructors.

Knowledge and understanding. Even and Tirosh (2002) noted that "A rather frustrating phenomenon, often described by both researchers and teachers, is that students who are known to have all the knowledge that is needed to solve a problem are unable to employ this knowledge to solve unfamiliar problems" (p. 225). The phenomenon could be similarly observed in research on the area of teacher knowledge of subject matter. For example, one of Ball's participants (1990), a prospective teacher, said, "Like long division–I can *do* it–but I don't know if I could really *teach* it because I don't know if I really *know* it" (p. 449). Ball commented, "Although she can do the mathematics, she may not have the kind of mathematical understandings she will need in order to help students learn" (p. 450). The major research focus in her study was not to determine if the prospective teachers possessed the knowledge to be able to solve the problems themselves but to see the degree to which they possessed knowledge about the meaning of the mathematical concept. The study found that they failed in "unpacking" (p. 454) the meaning of mathematical concepts and showed their “narrow understanding” (p. 457) of the subject. As seen in this example, quite often many researchers use the terms *knowledge* and *understanding* interchangeably in literature on teacher knowledge of subject matter, and what they mean by knowledge and understanding varies greatly.

Therefore, Silverman and Thompson (2005) suggested that teacher knowledge of subject matter should be refined to be more useful. Regarding knowledge, Ma (1999) developed a notion of *profound understanding of mathematics* with “depth, breadth, and thoroughness” (p. 121) because how such knowledge is held by teachers is important for their teaching practice (NCTM, 2007). She claimed that teachers should develop such understanding for effective teaching. McDiarmid et al. (1989) suggested that novice teachers should develop a “flexible, thoughtful, and conceptual understanding” of subject matter to be taught (p. 198), which is necessary for cultivating students’ understanding of subject matter. In particular,

Mason and Spence (1999) discussed types and degrees of knowledge using the term “knowing” instead of “to know” to emphasize its “dynamic, situated, and evolving” aspects (p. 140). They argued that “knowing-that” (“factually”) does not warrant “knowing-how” (“to perform acts”) or “knowing-why” (“having stories to account for phenomena and actions”) when the knowledge is fragmented (p. 137). They discussed the variation of each knowing*:* “knowing-that” ranging from discrete to integrated knowing; “knowing-how” ranging from simple to complex; “knowing-why” ranging from the intuitive to the rigorous (p. 139). They identified a group of the three kinds of knowing as “knowing-about,” and suggested a fourth kind of knowing, which is “knowing-to act,” which refers to “knowledge that enables people to act creatively….[and that] can be used or called upon when required” (p. 136, p.138). With their notion of knowing, they argued that literature on teacher knowledge impacting teaching practices and student learning is discussed "from a static, possessive stand rather than from a dynamic and evolving one" (p. 139).

Pirie and Kieren (1994) referred to *mathematical understanding* as "a constant, consistent organization of one's knowledge structures: a dynamic process, not an acquisition of categories of knowing" (p. 187). They developed a model of eight non-linear, embedded rings for the growth of mathematical understanding from the level of *primitive knowing* to the level of *inventising*: *primitive knowing*, *image making*, *image having*, *property noticing*, *formalizing*, *observing*, *structuring*, and *inventising* (p. 167). Hiebert and Carpenter (1992) defined *understanding* mathematical ideas, procedures, or facts as coherently making a connection between its mental representation and an existing internal network of representations. According to their notion, the frequency and the strength of connection determine the degree of understanding. They described a building process of understanding as follows:

> Networks of mental representations are built gradually as new information is connected to existing networks or as new relationships are constructed between previously disconnected information....Understanding can be rather *limited* [italics added] if only some of the mental representations of potentially related ideas are connected or if the connections are weak. Connections that are *weak* [italics added] and *fragile* [italics added] may be useless in the face of conflicting or nonsupportive situations. Understanding increases as networks grow and as relationships become strengthened with reinforcing experiences and tighter network structuring. (p. 69)

In this dissertation study, my definition of *understanding* followed Hiebert and Carpenter's notion of understanding because meaningful and conceptual mental connections were considered an important feature of understanding in the analysis of data. Based on Hiebert and Carpenter' description, I used the terms of *weak understanding* or *limited understanding* in the analysis.

Theoretical Framework

Developing a theoretical framework for understanding teacher knowledge is indispensable to understand the essence of teaching (Silverman & Thompson, 2008). Frameworks are useful for conceptualizing, identifying, or analyzing teacher knowledge and also applicable for systemic improvement in teacher quality (Hill et al., 2008; Silverman & Thompson, 2008). For decades, extensive research has been conducted for the purpose of establishing useful frameworks for teacher knowledge in the area of mathematics at the K-12 level.

Research on Theoretical Framework of Teacher Knowledge

Shulman (1986) classified teachers' content knowledge within a discipline into three categories: (1) *subject matter content knowledge*, (2) *pedagogical content knowledge*, and (3) *curricular knowledge*. In his notion, subject matter content knowledge includes understanding of the structure, as well as the facts or procedures within a discipline. Ball (1991) elaborated Shulman's dimensions of subject matter content knowledge within the discipline of mathematics. She noted that understanding mathematics means a mixture of "propositional and procedural knowledge *of* mathematics," such as an understanding of the relationships between concepts, and "knowledge *about* mathematics," such as an understanding of the history of mathematics (1991, p. 6).

As an expansion of Shulman's initial conception of teachers' content knowledge, Ball and her colleagues (Hill et al., 2008) developed a theoretical model of the structures of mathematical knowledge for teaching elementary mathematics. They defined *mathematical* *knowledge for teaching* as the mathematical knowledge needed to teach mathematics, which is distinct from mathematical knowledge needed by other professions. In their model, they further divided Shulman's notion of subject matter content knowledge into the three subcategories: (1) *common content knowledge*, (2) *specialized content knowledge*, and (3) *horizon content knowle*dge (p. 403). Figure 1 shows the framework's domains. *Common content knowledge* refers to the mathematical knowledge and skills used in a wide variety of settings other than teaching. In contrast, *specialized content knowledge* is the mathematical knowledge and skills unique to teaching. However, they acknowledged that it can be difficult to discern among the categories due to ambiguity of the definition of each category (Ball et al., 2008). For example, it might be difficult to distinguish specialized content knowledge from common content knowledge or from knowledge of content and students in specific teaching situations.

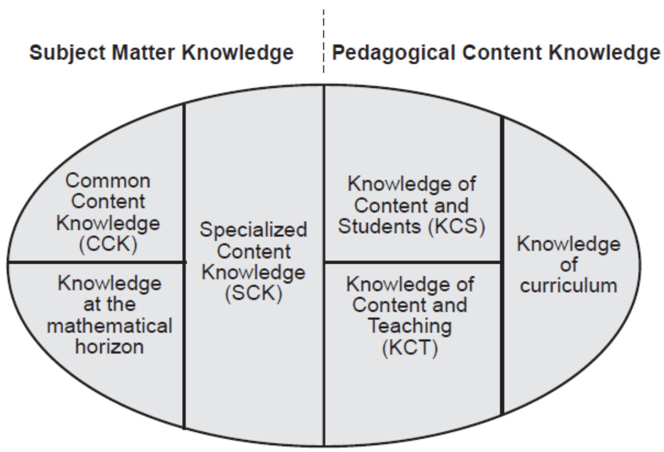


Figure 1. Mathematical knowledge for teaching (Hill et al., 2008, p. 377).

Ma (1999) characterized the mathematical knowledge that elementary teachers should possess for teaching mathematics conceptually. She called such knowledge *profound understanding of fundamental mathematics* (PUFM) (p. 120). By "profound" she meant "deep, vast, and thorough" (p. 120). In the teaching of teachers who possess PUFM, four features can be observed–*connectedness*, *multiple perspectives*, *identifying basic ideas*, and *longitudinal* *coherence* (p. 122). Teachers with PUFM make connections among concepts and procedures, have multiple approaches to problem solving, are knowledgeable about the "simple but powerful concepts and principles of mathematics" (p. 122), and can help students make meaningful connections between what they already know and what they will learn.

Schoenfeld and Kilpatrick's provisional framework (2008) consists of several dimensions that characterize proficiency for teaching mathematics. The domains include: (1) Knowing school mathematics in depth and breadth, (2) knowing students as thinkers, (3) knowing students as learners, (4) crafting and managing learning environments, (4) developing classroom norms an supporting classroom discourse as part of "teaching for understanding," (5) building relationships that support learning, and (6) reflecting on one's practice (p. 322). Ma's notion of PUFM is aligned with the domain, "knowing school mathematics in depth and breadth," because it is described as "broad and connected knowledge of the content at hand, deep knowledge of where the content comes from and where it might lead, an understanding of 'big ideas' or major themes" (p. 327).

Furthermore, PUFM is closely related to the five components of mathematical proficiency identified for successful learning proposed by the National Research Council [NRC] (2001):

- Conceptual understanding: comprehension of mathematical concepts, operations, and relations
- Procedural fluency: skills in carrying out procedures flexibly, accurately, efficiently, and appropriately
- Strategic competence: ability to formulate, represent, and solve mathematical problems

- Adaptive reasoning: capacity for logical thought, reflection, explanation, and justification
- Productive disposition: habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (p. 116)

The components are not disjoint but interrelated as seen in Figure 2. When applying them to a teacher's mathematical proficiency for teaching, a teacher with PUFM can show the proficiency proposed by the framework to help students' successful learn mathematics.

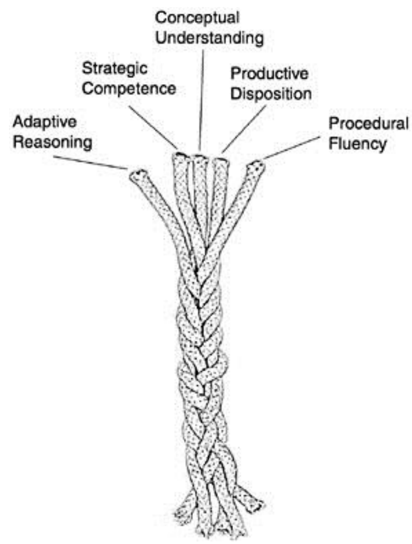


Figure 2. Five strands of mathematical proficiency (NRC, 2001, p. 117).[2]

Even (1990) also claimed that students' meaningful understanding of subject matter relies on teachers' robust knowledge of subject matter. She developed an analytic framework for teacher knowledge of a specific mathematical topic at the secondary level with the following components of teacher knowledge of subject matter: *Essential features*, *basic repertoire*, *knowledge and understanding of a concept*, *different representations*, *alternative ways of approaching*, *the strength of the concept*, and *knowledge about mathematics*. Teachers who

[2] Reproduced with permission from *Adding it up: Helping children learn mathematics*, 2001, by the National Academy of Sciences, Courtesy of the National Academies Press, Washington, D.C.

understand *essential features* of a concept can reason from the definitions of the concept, and correctly distinguish between examples and non-examples. Teachers, who have an understanding of the *basic repertoire*, can show powerful specific examples that illustrate important principles or properties. Teachers with *knowledge and understanding of a concept* can dynamically use both conceptual knowledge and procedural knowledge for problem-solving. Teachers should understand *different representations* of a concept and their relationships because such an understanding makes understanding of the concept “better, deeper, more powerful, and more complete” (p. 524). Teachers should know not only a main approach but also *alternative ways of approaching* a complex concept across the areas of mathematics or other disciplines. Teachers, who have knowledge of *the strength of a concept*, are familiar with unique aspects of the concept and are able to utilize sub-topics or sub-concepts with any other concepts. *Knowledge about mathematics* is a general knowledge about the discipline, and teachers with this knowledge have an understanding of the nature, structure, and development of mathematics.

The components of Even’s framework are related to the five strands of mathematical proficiency (NRC, 2001) as shown in Table 1. To understand essential features and basic repertoire of a concept requires conceptual understanding of the concept. Knowledge and understanding of a concept involve both conceptual understanding and procedural knowledge. Uses of different representations and alternative ways of approaching a concept relate to strategic competence for problem solving. Strength of the concept requires adaptive reasoning. Finally, knowledge about mathematics relates to productive disposition of mathematics.

Table 1

Comparison Between Even's Framework and Mathematical Proficiency Framework

Even's Framework	**Mathematical Proficiency Framework**
Essential Features	Conceptual Understanding
Basic Repertoire	
Knowledge and Understanding of a Concept	Procedural Fluency
Different Representations	Strategic Competence
Alternative Ways of Approaching	
The Strength of the Concept	Adaptive Reasoning
Knowledge about Mathematics	Productive Disposition

Framework for Mathematical Understanding for Secondary Teaching

Background. In the Situations Project of the Mid-Atlantic Center for Mathematics Teaching and Learning (MAC-MTL) at Pennsylvania State University and the Center for Proficiency in Teaching Mathematics (CPTM) at the University of Georgia, researchers developed a framework for Mathematical Understanding for Secondary Teaching (MUST framework) over a period of several years. A purpose for developing this framework was to conceptualize teachers' mathematical knowledge that adequately promotes students' mathematical proficiency at the secondary level. When taking an initial step of developing the framework, researchers in this project were inspired by other researchers' work developing frameworks of teacher knowledge for teaching (e.g., Ball et al. 2008; Even, 1990; NRC, 2001). To reflect a more dynamic characteristic of mathematical knowledge, the designers collected mathematical classroom events that took place in secondary school courses or collegiate courses that prepared secondary teachers of mathematics. The events were often interesting questions posed by students or provocative statements made by teachers. The following is an example of a classroom mathematical event in which a student's question about the concept of an exponent was involved (MAC-MTL & CPTM, 2012, p. 9).

> In an Algebra II class, students had just finished reviewing the rules for exponents. The teacher wrote $x^m \bullet x^n = x^{m+n}$ on the board and asked the students to make a list of values for *m* and *n* that made the statement true. After a few minutes, one asked, “Can we write them all down? I keep thinking of more.”

Selected events were analyzed and organized with specific pieces of mathematics and mathematical thinking, which were critical for helping students build mathematical understanding, were identified and organized. The researchers in this project called each of the mathematical events a “Situation.” In the process of describing the Situations, pertinent aspects of mathematical understanding useful to teachers were constructed and refined.

Over the course of their work, the title of the framework evolved from Mathematical Proficiency for Teaching (MPT) to Mathematical Understanding for Secondary Teaching (MUST) because the mathematics that is useful to secondary teachers is not only about mathematical proficiency (i.e., knowing and doing mathematics), but also about the work of teaching others to become mathematically proficient. Here, understanding is more than knowledge; it includes knowing and using mathematical knowledge, as well as helping others to know and use mathematics.

There are four unique features of the MUST framework (MAC-MTL & CPTM, 2012). First, the framework was developed from actual classroom mathematical events and not previous literature, although previous literature influenced the MUST framework. When over 50 Situations were examined and analyzed, the strands of the framework were determined. Second, the framework reflected the dynamics and situated nature of teacher knowledge which shifts, grows, and deepens in the teaching context. It seems very compatible with Mason and Spence’s (1999) argument that teacher knowledge tends to be discussed “from a static, possessive stand rather than from a dynamic and evolving one" (p. 139). This dynamic feature of the framework also indicates that the development of the framework is still in progress. Third, the framework is specialized for secondary teaching. Therefore, the framework focused on secondary school mathematics and characterized the mathematical understanding useful to secondary mathematics teachers. Fourth, the framework provided a unique perspective of teachers' mathematical understanding for teaching because the development of the framework was based on teaching practices of practicing teachers and student teachers in various teaching contexts.

Components and strands. There are three components within the MUST framework: (1) *Mathematical Proficiency*, (2) *Mathematical Activity*, and (3) *Mathematical Work of Teaching* (MAC-MTL & CPTM, 2012). The components are not completely separable, but quite interrelated and overlapped (Figure 3). Each component consists of several strands (Figure 4).

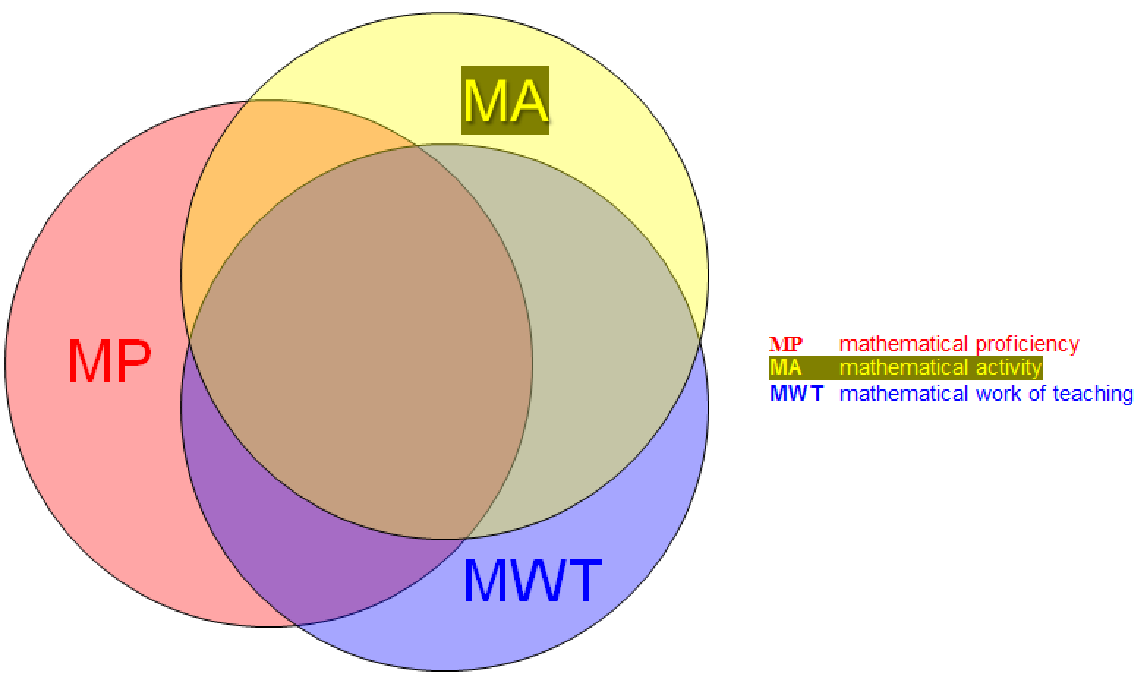


Figure 3. Three components of mathematical understanding for secondary teaching (MAC-MTL & CPTM, 2012, p. 7).

Mathematical Proficiency (MP)

- Conceptual Understanding
- Procedural Fluency
- Strategic Competence
- Adaptive Reasoning
- Productive Disposition
- Historical and Cultural Knowledge

Mathematical Activity (MA)

- Mathematical Noticing
    - Observing Structure of Mathematical Systems
    - Discerning Symbolic Forms
    - Detecting the Form of an Argument
    - Connecting within and outside Mathematics
- Mathematical Reasoning
    - Justifying/Proving
    - Reasoning when Conjecturing and Generalizing
    - Constraining and Extending
- Mathematical Creating
    - Representing
    - Defining
    - Modifying/Transforming/Manipulating
- Integrating Strands of Mathematical Activity

Mathematical Work of Teaching (MWT)

- Analyze Mathematical Ideas
- Access and Understand the Mathematical Thinking of Learners
- Know and Use the Curriculum
- Assess the Mathematical Knowledge of Learners
- Reflect on the Mathematics in One's Practice

Figure 4. The components and strands of the MUST framework (MAC-MTL & CPTM, 2012, p. 5).

Mathematical Proficiency can be thought of as "knowing mathematics." This component consists of the five strands of mathematical proficiency by NRC (2001) with an additional strand, Historical and Cultural Knowledge, which is an understanding of the historical origin and cultural influence on mathematics. This component includes deep and thorough understanding to help students be able to develop students' proficiency in mathematics.

Mathematical Activity can be defined as "doing mathematics." This component of teachers' mathematical understanding is required, especially, when they engage their students in mathematical activities. Teachers, who are proficient in mathematical activities, are more likely to be able to facilitate students' doing and experiencing mathematics. This component consists of the four strands: (1) *Mathematical Noticing*, (2) *Mathematical Reasoning*, (3) *Mathematical Creating*, and (4) *Integrating Strands of Mathematical Activity*. Mathematical Noticing involves observing mathematical structures, discerning symbolic forms, detecting the forms of mathematical arguments, and connecting within and outside mathematics. Mathematical Reasoning includes justifying or proving mathematical arguments in logical ways, reasoning in the context of conjecturing and generalizing, and constraining and extending domains, arguments, or classes of objects. Mathematical Creating is the process of producing new mathematical entities through the mathematical activities of representing, defining, and modifying/transforming/manipulating. Because the three strands are intertwined, an ability to integrate the three strands of Mathematical Activity forms a strand of Mathematical Activity.

Mathematical Work of Teaching can be thought as "teaching mathematics." This component is unique to the mathematics teaching profession. This component consists of the five strands: (1) *Analyze Mathematical Ideas*, (2) *Access and Understand the Mathematical Thinking of Learners*, (3) *Know and Use the Curriculum*, (4) *Assess the Mathematical Knowledge of Learners*, and (5) *Reflect on the Mathematics in One's Practice*.

Use of the framework. The MUST framework focuses on teachers' mathematical understanding of the content at the secondary level. In this study, however, I employed the MUST framework for designing task items and analyzing GTA-Ms' mathematical understanding for teaching trigonometry in a post-secondary mathematics teaching context. I did so for the following reasons. First, most introductory-level undergraduate mathematics topics are parallel to those taught in mathematics courses at the secondary level, such as trigonometry in precalculus. Second, the range of majors for undergraduate students taking introductory-level undergraduate mathematics courses is very wide. Therefore, the mathematics topics taught in these courses are rather general and not specific to any particular major. Third, the MUST framework can be seen as both comprehensive and overarching because it reflected a body of literature and other theoretical frameworks of K-12 teacher knowledge (Wilson & Conner, 2009).

CHAPTER 3

METHODOLOGY

The purpose of this study was to explore and understand the nature of mathematical understanding for teaching trigonometry exhibited by graduate teaching assistants in the Department of Mathematics (GTA-Ms). In this qualitative study, a sequence of task-based interviews was conducted to investigate the research questions: (1) To what extent do GTA-Ms exhibit an understanding of trigonometric concepts when solving and explaining trigonometry problems? (2) What understanding of trigonometry do GTA-Ms use in analyzing and responding to students' mathematical thinking about concepts of trigonometry in hypothetical teaching contexts? Each task-based interview was conducted while each participant explained his/her work for a given mathematical task consisting of several mathematical problems, called *task items*, on trigonometry. This method was useful for this study because "task-based interviews can serve as research instruments for making systemic observations in the psychology of learning mathematics and solving mathematical problems" and "they also can be adapted as assessment tools for *describing the subject's knowledge* [italics added]" (Goldin, 2000, p. 520). In addition, this method was chosen to make the mathematical processes more salient. Goldin (2000) explained:

> In comparison with conventional paper-and-pencil test-based method, task-based interviews make it possible to focus research attention more directly on the *subjects' processes of addressing mathematical tasks* [italics added], rather than just on the patterns of correct and incorrect answers in the results they produce. (p. 520)

This chapter presents the site of research and the participants as well as describes how I designed the data sources, collected data, and analyzed the data.

Site of Research

Department of Mathematics

This study took place at a university in the southern region of the United States. The student population was approximately 26,000 undergraduate and 8,200 graduate students when the study was conducted. The Department of Mathematics at the university offers four degrees in Mathematics: Bachelor of Science (B.S.), Master of Arts (M.A.), Master of Applied Mathematical Science (M.A., M.S.), and Doctor of Philosophy (Ph.D.). There were about 50 graduate students and about 60 full-time faculty members. Approximately 20 graduate students served as teaching assistants (GTA-M) during fall and spring semesters, and approximately 10 GTA-Ms teach during the summer sessions. GTA-Ms were usually appointed to teach MATH1113 (Precalculus) or MATH2200 (Analytic Geometry and Calculus). Only a few GTA-Ms were assigned to teach MATH2250 (Calculus I for Science and Engineering) or MATH2260 (Calculus II for Science and Engineering). Novice GTA-Ms usually teach MATH1113 (Precalculus) during their first semester of teaching and MATH2200 (Analytic Geometry and Calculus) the following semester under departmental supervision. Native English speaking GTA-Ms were expected to teach their own course by fall semester of their second year. And all other GTA-Ms are expected to teach their own course by fall semester of their third year.

According to the teaching guidelines and policy of the department, all new GTA-Ms assigned to teach are required to take MATH7005 (teaching seminar for first-time MATH1113 instructors), MATH9005 (teaching seminar for first-time MATH2200 instructors), and a discipline specific section of GRSC7770 (the graduate teaching seminar). These three seminars are credit courses offered by the department for GTA-Ms' teaching preparation.

Participants

The targeted subjects for this study were graduate students in the Department of Mathematics who had taught the course MATH1113 (Precalculus). I selected this course because it covered trigonometry as one of the major content areas. I considered graduate students who had previously taught the course at the university because I assumed they would feel more comfortable solving and explaining the tasks on trigonometry during the task-based interviews.

When I recruited participants for this study, I sent an invitation email to nine graduate students in the Department of Mathematics. The selection of these nine people was based on my previous experience recruiting GTA-Ms for projects for a qualitative research class. It is difficult for GTA-Ms to find time to participate in any project that moves them beyond the study of pure mathematics. Therefore, for this study, I decided to invite graduate students who I expected might be highly interested in participating in a mathematics education study and thinking about teaching. Five of the nine invitees were the recipients of the *University Outstanding Teaching Assistant Award*. Four of the invited GTA-Ms agreed to participate in the study and signed a consent form; three were the recipients of the teaching award and the fourth was a person who helped me with a previous mathematics education project. Each participant was paid $200 after completing the procedures for this study.

The participants consisted of four GTA-Ms, one female and three males. Their names (pseudonyms) were Gloria, Kyle, Leo, and Micah. Table 2 summarized the participants' academic backgrounds. They had similar academic backgrounds and teaching experiences for the department. The list of courses in the table provides only the courses that the participants took or taught at the university where the study occurred.

Table 2

Participants' Academic Backgrounds

	Gloria	Kyle	Leo	Micah
Education	B.S. in Math	B.S. in Math	B.S. in Molecular Biology	B.S. in Math B.S. in Physics
	M.S. in Math	M.S. in Math	M.A. in Math	M.A. in Math
Status	5th year doctoral student	4th year doctoral student	5th year doctoral student	6th year doctoral student
Recipient of outstanding teaching award	Yes	No	Yes	Yes
Math courses taught	M1113 (Precalculus)	M1113 (Precalculus)	M1113 (Precalculus)	M1113 (Precalculus)
	M2200 (Analytic geometry and calculus)	M2200 (Analytic geometry and calculus)	M2200 (Analytic geometry and calculus)	M2200 (Analytic geometry and calculus)
		M2250 (Calculus I for science and engineering)	M2250 (Calculus I for science and engineering)	
	M5001 (Arithmetic and problem solving)			M5001 (Arithmetic and problem solving)
	M5030 (Geometry and Measurement for Middle School Teachers)			
Number of graduate math courses taken	5	10	19	13
Number of math education courses taken	5	0	2	0
Teaching development	M7005/9005 (Doctoral graduate student teaching seminar)	M7005/9005 (Doctoral graduate student teaching seminar)	M7005/9005 (Doctoral graduate student teaching seminar)	M7005/9005 (Doctoral graduate student teaching seminar)
	GRSC7770 (GTA seminar)	GRSC7770 (GTA seminar)	GRSC7770 (GTA seminar)	GRSC7770 (GTA seminar)

		VIGRE Teaching seminar	EMAT9700 (Teaching observation seminar)	MEFT (Mathematicians Educating Future Teachers)

Data Sources

Designing mathematical tasks was a crucial stage before collecting data because a major portion of the data was collected by observing the participants' mathematical problem-solving process through task-based interviews. Each task consisted of several mathematical problems, called *task items*, on trigonometry. This section explains how the tasks items were designed and how the theoretical framework influenced the design. The participants' written and verbal data were generated during three task-based interviews in three interview sessions. To gain a better understanding of the participants and their teaching and learning experiences, supplementary data were gathered for each participant through (1) a background information sheet, (2) a pre-task interview, and (3) a post-task interview.

Main Data Sources: Tasks

Framework. The task design was guided by the Mathematical Understanding for Secondary Teaching (MUST) framework because I planned to analyze the data through the lens of the framework to explore GTA-M's mathematical understanding for teaching trigonometry. Because the MUST framework consisted of three components, three tasks were considered. Each task item was written to address the components and strands of the framework (see Appendix A). Task I was designed to explore the participants' *mathematical proficiency* in understanding of concepts of trigonometry, Task II was designed to explore their *mathematical activities* when solving and explaining trigonometric problems. Task III was designed to explore how the participants use their mathematical understanding in the hypothetical *mathematical work of teaching* contexts in which they respond to students' ideas or arguments involving misconceptions of trigonometry. Each task consisted of a series of task items on trigonometry. Each item was designed to address specific strands of a component (or components) of the framework. For example, item Task I.1 was aligned with Conceptual Understanding of the first component–Mathematical Proficiency; item Task II.1 was aligned with Mathematical Noticing (Observing structure of mathematical system) and with Mathematical Creating (Defining) in the second component–Mathematical Activity; and item Task III.1 was aligned with Mathematical Noticing (Detecting the form of an argument) in the component, Mathematical Activity and with Analyze Mathematical Ideas in the third component–Mathematical Work of Teaching (see Table 3).

Table 3

Example of Alignment of the Task with the MUST Framework

Task	MUST Framework
I.1	MP: Conceptual Understanding
II.1	MA: Mathematical Noticing: Observing Structure of Mathematical Systems MA: Mathematical Creating: Defining
III.1	MA: Mathematical Noticing: Detecting the Form of an Argument MWT: Analyze Mathematical Ideas

Concepts. It was necessary to select some specific concepts among various concepts of trigonometry to design the tasks. The concepts were selected from the *Faculty Course Outline of Precalculus* of the Department of Mathematics at the university where the research was conducted because the participants had taught precalculus according to the course outline (see Appendix B) in the department. The course outline categorized trigonometry as "elementary trigonometry" or as "advanced trigonometry" depending upon the concepts covered. The course coordinator said that this classification was set so that students could take an examination assessing elementary topics before they took the examination assessing advanced topics. Table 4 shows the contents for each section of chapters 5, 6, and 7 (Swokowski & Cole, 2009) and indicates the level of the section. The department designated sections of 5.1, 5.2, 5.3, and 5.4 of the textbook as “elementary trigonometry,” and sections of 5.5, 5.6, 5.7, 6.2, 6.3, and 6.4 as “advanced trigonometry.” I included the additional sections of 6.6, 7.1, and 7.2 designated as “future study of trigonometry” and “advanced trigonometry.” Table 5 shows a list and frequencies of the trigonometric concepts that appeared in the task items.

Table 4

The List of the Sections of Trigonometry in the Textbook

	Chapter 5 The trigonometric functions	Chapter 6 Analytic trigonometry	Chapter 7 Applications of trigonometry
Section 1	5.1 Angles	6.1 Verifying trigonometric identities	7.1 The law of sines
Section 2	5.2 Trigonometric functions of angles	6.2 Trigonometric equations	7.2 The law of cosines
Section 3	5.3 Trigonometric functions of real numbers	6.3 The addition and subtraction formulas	
Section 4	5.4 Values of the trigonometric functions	6.4 Multiple-angle formula	
Section 5	5.5 Trigonometric graphs	6.5 Product-to-sum and sum-to-product formulas	
Section 6	5.6 Additional trigonometric graphs	6.6 The inverse trigonometric functions	

Table 5

The Concepts of Trigonometry Involved in the Tasks

Concepts	**Task I (8 items)**	**Task II (7 items)**	**Task III (10 items)**
Angle measure/Radians/Degree	I.1, I.2, I.3	II.1, II.2	III.1, III.4, III.5
Circumference formula	I.2		III.5
Arc length formula	I.4	II.1, II.2	III.1, III.4
Sine/Cosine functions	I.5, I.6, I.7	II.2, II.3, II.4, II.5, II.6	III.2, III.8, III.9, III.10
Units/Unit conversion	I.6	II.1	III.1, III.4
Unit circle trigonometry and right triangle trigonometry	I.8	II.3	III.10
Inverse trigonometric functions		II.2	III.6, III.8
Composite functions		II.4, II.5	III.2, III.6, III.8
Periodic functions/Periods		II.5	III.7
Addition formula for sine/cosine functions		II.7	III.9
Unit circle			III.3
Co-function formula			III.9
Modeling of trigonometric functions		II.6	

Sources for the task items. The task items for the three tasks were mathematical problems on trigonometry. These task items were mainly designed by selecting some problems from the book *Pathways to Calculus: A Problem Solving Approach* (Carlson & Oehrtman, 2009), developed by the Project *Pathways*. Some other task items were also designed from class materials used in a mathematics education class for prospective secondary teachers of mathematics.[3] Permission to use the textbook problems and materials from class to design the task items was granted by Dr. Marilyn P. Carlson at Arizona State University and Dr. Kevin C. Moore at the University of Georgia, respectively. In addition, I also referred to research papers on mathematics education and internet book chapters on trigonometry. For example, a research

[3] Materials were prepared by Dr. Kevin C. Moore who used *Pathways to Calculus: A Problem Solving Approach* for his class.

paper (Thompson, Carlson, & Silverman, 2007, p. 417) provided me with the idea of designing a hypothetical situation between a college instructor and two students for Task III.2. A modeling problem of a sine function in Task II.6 was extracted from an internet book chapter on trigonometry (www.aw-bc.com/scp/lial_hornsby.../LIALMC06_0321227638.pdf, p. 566).

The webpage (https://www.rationalreasoning.net/products.php) for the online text of the book *Pathways to Calculus: A Problem Solving Approach* (Carlson & Oehrtman, 2009) described itself as offering "Research based educational products, supporting teachers and educating students." The overview of the book from the same website stated,

> This text was designed to develop students' conceptual knowledge, problem solving abilities and skills that are foundational for success in calculus….Teacher support materials include cognitively scaffolded worksheets (with detailed teacher notes) that are designed to keep students' minds active in making critical connections for understanding the course's key ideas.

To achieve these goals, the book consisted of various research-based conceptual precalculus problems to promote students' mathematical problem solving. In particular, Module 7 (entitled *Angle Measure and Introduction to Trigonometric Functions in the Context of the Unit Circle*) of the book contained conceptual problems related to the selected trigonometric concepts for this study. Therefore, some problems in Module 7 were selected or modified as task items for this study.

Formats. After collecting some trigonometric problems from several sources, such as textbooks and papers, I sorted them by guessing a reasonable time to complete each task. I designed Task I to consist of eight task items, Task II seven task items, and Task III ten task items. The format for the items of each task was determined by considering the components of the framework. To investigate the participants' mathematical proficiency or mathematical activity, the task items for Tasks I and II were open-ended mathematical problems of trigonometric concepts to explore the participants' problem-solving process. For example, Task II.4 asked the participants to find a function to represent the relationship between the given mathematical quantities in a real-world situation.

After finishing Tasks I and II, respectively, the participants were asked to rate the difficulty level and importance for each task item. An intention to design ratings was to collect data about their use of mathematical understanding to analyze each item from the perspective of teaching and student learning. On the first page of Tasks I and II, the following instructions about rating the difficulty level and importance for each task item were provided:

> After you finish explaining each task item, please classify the level and rate the importance for each task item with reasons.
>
> - Please classify each item as easy (**E**), medium difficulty (**M**), or difficult (**D**) for *you* (as an instructor) and for *your students*, explaining the reason for your classification.
> - Please rate (**1**: **less** important; **2**: important; **3**: **most** important) the importance for each item for *you* (as an instructor) to know and for *your students* to know, explaining the reason for your rating.

Table 6 shows an example of the rating table that was provided on the last page of Tasks I and II respectively.

Table 6

Tables for Rating the Difficulty Level and Importance for a Task Item

Level	E	M	D	Reason
For you				
For students				

Importance	1	2	3	Reason
For you				
For students				

For Task III, there were two kinds of task items: (1) seven task items (Task III.1–III.6 and III.10) involving hypothetical teaching situations between a college instructor and students and (2) three task items (Task III.7, III.8, and III.9) about "test" problems. For both kinds of items, the participants were supposed to respond to several subsequent questions related to the situations or to the given test problems. The hypothetical teaching situations in Task III.1 through III.6 and III.10 were created for the purpose of observing how well the participants could identify the students' misconceptions or misunderstandings and how well they could help their students understand mathematical concepts key to the situations. This design was based on Biza, Nardi, and Zachariades' (2007) suggestion that tasks involving situation-specific contexts should be useful tools to explore teacher knowledge:

> By asking the teacher to engage with a specific (fictional yet plausible) student response that is characterised by a subtle mathematical error we can explore not only whether the teacher can identify the error but probe into its causes and grasp the didactical opportunity it offers (and the fruitful cognitive conflict it has the potential to generate). (p. 303)

Types of questions. To collect rich data on the participants' mathematical understanding for teaching trigonometry, I used several types of questions identified by Zazkis and Hazzan (1999): *performance questions*, *construction tasks*, *"give an example" tasks*, *reflection questions*, and *"twist" questions*. Most of the items for Tasks I and II could be identified as *performance questions*. For example, the question for Task I.3, "Describe how to use the arc length and circumference of the circle displayed below to determine how many of the "mystery" angle measure units mark off (or cut off) any circle's circumference," was an example of a *performance question* that requests an explanation of "how the answer was found, why an action or procedure was chosen and how a decision was reached" (Zazkis & Hazzan, 1999, p. 431). The question for Task II.4, “Define a function that relates the measure of the angle (in radians) swept out by the fan blade as a function of time elapsed,” was an example of a *construction task* that required “building mathematical objects which satisfy certain properties” (Zazkis & Hazzan, 1999, p. 431). One of the subsequent questions for Task III.1, “What questions or examples would you ask or use to help students better understand the mathematical ideas/concepts involved in this situation?” was an example of a “*give an example*” *question*, which could be useful to gain an insight about the participant’s understanding of a situation from the examples s/he generated (Zazkis & Hazzan, 1999). Some questions in the Task III items involving hypothetical teaching situations could be identified as *reflection questions* because they were designed for the participant “to distance himself/herself from the personal performance by responding to someone else’s ideas” and to “shift the focus to the reason for the [someone else’s] solution, rather than to the solution itself” (Zazkis & Hazzan, 1999, p. 434). For example, one of the subsequent questions in Task III.4, “What might be possible sources of his/her conception?” was an example of a *reflection question*. Zazkis and Hazzan (1999) identified *“twist” questions* as a type of question that “presents a variation on a familiar situation” (p. 432). For example, the question for Task III.10, “How would you respond to each student’s question?” was a twist question because the question was given in a situation in which students asked how the mnemonic device “SOA-CAH-TOA” could be used in a general context, such as for obtuse angles, not just in the right triangle context.

Task-based interviews. According to Goldin (2000),

> Structured, task-based interviews for the study of mathematical behavior involve minimally a subject (the problem solver) and an interviewer (the clinician), interacting in

> relation to one or more tasks (questions, problems, or activities) introduced to the subject by the clinician in a preplanned way. The latter component justifies the term *task-based* italics added], so that the subjects' interactions are not merely with the interviewers, but with the task environments. (p. 519)

In this study, the participants had three task-based interviews conducted individually in the same fashion as follows. In the beginning of a task-based interview, task material was presented to the participants as an "interview script" (Goldin, 2000, p. 518). The scripts had several pages, with a task item written and blank space for the participants' mathematical work. The task items for each task-based interview are listed in Appendix C. The interview started with me reading the instructions on the first page of the interview script. I asked them to act as college instructors and consider me as one of their students who came to ask the questions in the task items during their office hours. I also encouraged the participants to "'think out loud' during problem solving….Thus, the interview contingencies were almost unstructured, but important structures were imposed in the choices of structured tasks and the thinking-aloud procedure" (Goldin, 2000, p. 521).

During the interviews, my intervention was minimal as Goldin described above. If possible, I attempted not to interrupt while they were thinking out loud. If necessary, I asked additional "why" and "how" questions, such as "How or why did you do (find, say, etc.) that?" to help them articulate their ways of thinking or clarify their explanations. I asked such questions at the moment they made a short pause in the middle or end of their explanation. During the task-based interviews, my major roles, as an interviewer, included recording the participants' verbal mathematical work through audiotaping; observing their mathematical work, such as explaining, writing, or drawing; asking them follow-up questions, when necessary; and collecting their written work on the interview script.

Supplementary Data Sources

Background information sheet. A background information sheet (see Appendix D) was utilized for collecting the participants' demographic and educational background data, including a personal history of teaching and training experience for teaching undergraduate mathematics.

Pre-task interview. Before conducting task-based interviews, the participants had a pre-task interview (see Appendix E), a 30-minute interview about their backgrounds (e.g., education, TA experiences) and their thoughts about learning and teaching trigonometry. For example, they were asked to answer a question such as "Tell me five things related to trigonometry that you want your students to remember even after finishing the course."

Post-task interview. After completing the third task-based interview, the participants completed a 15-minute post-task interview (see Appendix F) designed to provide them with an opportunity to reflect on the experiences that they had during this study. For example, they were asked to answer a question, such as "What do you think about the task items?"

Data Collection

The data collection was completed during the second and third weeks of November 2011. Each meeting took approximately two hours, which was longer than I expected. Table 7 shows the meeting schedule with the participants and Table 8 shows the data collection for each interview session.

Table 7

Meeting Schedule with the Participants (November)

Sun	Mon	Tues	Wed	Thu	Fri
6 Kyle I	7	8 Gloria I	9 Kyle II	10 Leo I	11 Leo II Gloria II
13 Micah I	14 Gloria III	15 Micah II	16	17 Leo III	18 Micah III Kyle III

Table 8

Interview Sessions for Data Collection

Interview Session	**The Participants did…**
Session I	Background information sheet
	Pre-task interview
	Task-based interview with Task I
Session II	Task-based interview with Task II
Session III	Task-based interview with Task III
	Post-task interview

To collect data, each participant had three interview sessions. During the first interview session, each participant completed a background information sheet and a pre-task interview for collecting data about their teaching and learning backgrounds. It took approximately 40 minutes, as planned. After completing the pre-task interview, the participants achieved the first task-based interview by responding to the written task items for Task I. Every time they finished a task item, they were asked to rate the level of difficulty and importance for each task item. During the second session, they completed another task-based interview with Task II. Finally, during the last session, they concluded the third task-based interview with Task III. During the three task-based interviews, the participants showed similar reactions to the interviews. They appeared comfortable when explaining their mathematical work and were willing to spend considerable time to compete each task, although it took longer than I anticipated. Table 9 shows a comparison of the planned time periods and actual duration of time periods for each participant's completion of each task- based interview. This demonstrates their persistence in figuring out each task item. After finishing the third task-based interview, they completed a short post-task interview to reflect on their participation in this study. This interview took approximately 15 minutes. All the interviews were audio taped and transcribed.

Table 9

Elapsed Time Periods for Working on Task Items

	Plan	Gloria	Kyle	Leo	Micah
Task I	1 hour	2 hours 17 min	2 hours 4 min	1 hour 18 min	2 hours 3 min
Task II	1 hour 20 min	4 hours 13 min	2 hours 29min	1 hour 48 min	2 hours 12 min
Task III	1 hour 20 min	2 hours 7 min	2 hours 16 min	1 hour 45 min	2 hours 41 min
Total	3 hours 40 min	8 hours 37 min	6 hours 49min	4 hours 51 min	6 hours 56 min

Data Analysis

According to Glesne (2006),

> Data analysis involves organizing what you have seen, heard, and read so that you can make sense of what you have learned. Working with the data, you describe, create explanations, pose hypotheses, develop theories, and link your story to other stories. To do so, you must categorize, synthesize, search for patterns, and interpret the data you have collected….The art of data transformation is in combining the more mundane organizational tasks with insight and thoughtful interpretations. (p. 147, p. 154)

The MUST framework was utilized as an analysis tool to categorize and organize the data collected. This section describes the approach I employed to the data analysis to answer the research questions. There were three key stages in the analysis process: (1) Sorting and reducing data *by task item* for each participant, (2) Reorganizing the data *by strand* across the participants, and (3) Finding patterns/themes and interpreting the data.

The First Stage: Sorting and Reducing Data by Task Item for Each Participant

During this initial stage, I scrutinized the transcripts of the three task-based interviews for each participant using the lens of the strands of the three components of the MUST framework. I sorted the data by identifying specific pieces of data that corresponded to each strand. This sorting work was processed by scrutinizing the written and verbal data derived from each participant's responses to each task item.

To categorize and organize the data, key ideas of the description for each strand of each component were important. For example, the participants' comments about formal definitions, illustration of properties (Essential Features of Even's framework), or use of examples (Basic Repertoire of Even's framework) were counted as a good indicator for showing their Conceptual Understanding of the component–Mathematical Proficiency.

A lens of the framework helped me closely examine the raw data and discover more underlying strands than I originally expected. For example, Task I.1 was designed to see the strand, Conceptual Understanding, for Mathematical Proficiency. However, I found that Kyle's raw data showed two more strands– Procedural Fluency of Mathematical Proficiency and Know the Curriculum of Mathematical Work of Teaching. Figure 5 is part of my note that shows how Kyle's raw data from Task I.1 were categorized and organized according to the framework. I summarized what I learned from the raw data with selected quotations to exemplify the participants' explanations. I displayed the summary data with labeling, for example, K.I.1 (i.e., Kyle's Task I.1 item), and with the relevant strand and the component. Additionally, I included raw data as evidence, when necessary. I quoted what the participants said from the raw data and included the task item number along with the participant's pseudonym for further reference.

- **Kyle**
  - **Task I.1**

    K.I.1
    <MP: Conceptual Understanding-Essential Features>
    His meaning of one degree measure is one of the 360 equal parts along the circumference. On the other hand, to explain the meaning of the radian measure he converted the given 2.3 radians to the number of degrees. He thinks, "Degrees are nicer to think about because they are more intuitive…you could easily…see how big that angle is" (Task I.1, Kyle).

    <MP: Procedural Fluency>
    He fluently converted 2.3 radians to the number of degrees using a conversion factor from the fact that π radians are the same as 180 degrees. In the computation of 2.3 rad × 180º/π rad, he used the unit cancellation by saying, "the radians cancelling out here we are going from radians to degrees" (Task I.1, Kyle).

    <MWT: Know the Curriculum>
    He thinks that knowing radians is important because it is used to define trigonometric functions on the real numbers.

Figure 5. Example of data organization for an interview.

At the end of this stage, I tabulated the strands observed from each participant's raw data (see Appendix G). The tables were useful to compare the strands that I expected, assisting me to see the strands that I actually observed. The tables also allow comparison of the strands by task item across all the participants. For example, Table 10 shows the expected and observed strands from the participant's responses to Task I.1 item.

Table 10

Expected Strands and Observed Strands in Task I.1 Across the Participants

Task	Expected Strands	Observed Strands
GI.1	MP: Conceptual Understanding	MP: Conceptual Understanding (Essential Features) Procedural Fluency Historical Knowledge
		MWT: Know the Curriculum
KI.1	MP: Conceptual Understanding	MP: Conceptual Understanding (Essential Features) Procedural Fluency Historical Knowledge
		MWT: Know the Curriculum
LI.1	MP: Conceptual Understanding	MP: Conceptual Understanding (Basic Repertoire) Historical Knowledge
MI.1	MP: Conceptual Understanding	MP: Conceptual Understanding (Essential Features)
		MWT: Know the Curriculum

The Second Stage: Reorganizing Data by Strand Across the Participants

The classification completed for *each participant* during the first stage was reorganized with respect to *each strand* for the framework across the participants during the second stage. With respect to a strand of a component, I arranged the data pieces together to form "data clumps" (Glesne, 2006, p. 152) across the task items and the participants. The data labeling completed during the first stage was useful for identifying and organizing the data relevant to each strand. The data corresponding to a strand were displayed by task item, which made it easier to compare the data across the participants. Figure 6 shows an example of how the data were reorganized from the organization completed during the first stage. It appears like a data clump consisting of all data corresponding to a specific strand, saying, Access and Understand the Mathematical Thinking of Learners, of Mathematical Work of Teaching, across the task items and the participants.

- **Mathematical Work of Teaching (MWT)**
    - **Access and Understand the Mathematical Thinking of Learners**

GIII.1
< MWT: Access and understand the mathematical thinking of learners>
She thinks that the student's argument is related to the idea of the definition of the radian measure in some sense that the radian measure can be taken as the arc length given by an angle of a circle with radius 1. She attempted to understand what the student was doing. She focused on what the student was possibly doing right rather than what he/she was doing wrong.

L.III.1
<**MWT: Access and understand the mathematical thinking of learners>**
He said that the student's argument is not "generally accepted…so this is not a general mathematical convention" (Task III.1, Leo). Hence, he said that the argument could be mathematically right, if it is corrected to say that the arc length on a circle of radius 1 associated to the angle is 1.7 inches.

GIII.2
<MWT: Access and understand the mathematical thinking of learners>
She used the typical cosine graph to show the outputs of the cosine function are real numbers, not angles to correct student A because he/she misinterpreted the output of the cosine as degrees, and supported the argument of student B who argued that the result must be a real number. But, she explained that it is possible to convert the given real number into an angle in degrees.

GIII.3
<MWT: Access and understand the mathematical thinking of learners>
She said that student A is partially right and should be more careful about dealing with 1 unit. She would explain to student A how to make a unit circle out of 2.8 feet with a radius. She thinks that the student's difficulty in understanding 1 unit of a unit circle came from not placing a unit on the drawing of a unit circle, in general.

M.III.3
<MWT: Access and understand the mathematical thinking of learners>
He thinks that it would be difficult for students to deal with the units concretely and understand that the unit you choose does not matter in the concept of a unit circle. For example, making a circle 2.8 feet a unit circle and making your units 2.8 feet could probably confuse students. On the other hand, if unit feet are chosen, the circle is not a unit feet circle.

M.III.6
<MWT: Access and understand the mathematical thinking of learners>
He thinks that the student did not understand the rules of function algebra such as function composition. So, showing a counterexample using the same function may not be helpful. He would provide an example to distinguish between the function composition and function multiplication from the standard functions: "If f of x is x^2 and g of x is x^3, then this is that [the composition] which is x to the 6^{th}. But this [the multiplication] is x to the 5^{th}" (Task III.6, Micah).

M.III.7
<MWT: Access and understand the mathematical thinking of learners>
Students might easily forget the condition of the minimality of the period regarding the length, if they understand a period only as some section on which the function repeated itself.

GIII.9
<MWT: Access and understand the mathematical thinking of learners>
For students' better understanding, she could give students an arbitrary point at the angle, t, on the unit circle without providing specific coordinates and ask them to locate the point at the angle, π/2 – t, to determine where they would become confused this problem.

M.III.9
<MWT: Access and understand the mathematical thinking of learners>
When P of t is the constant point (4/5, 3/5), he thinks that students easily make a mistake for P of π/2 – t as (π/2 – 4/5, π/2 – 3/5), which may come from a lack of understanding of parametric equations and function composition.

Figure 6. Example of a data clump corresponding to a strand.

For the strand, Conceptual Understanding, of Mathematical Proficiency, additional work had to be completed at the second stage of data analysis. Because data about the participants' conceptual understanding formed a relatively large data clump, it was important to refine my categorization based on concepts. In other words, the data regarding each strand were reorganized according to the concepts I considered when I designed the task items. For this additional work, the chart (see Table 5) of the concepts embedded in the task items was useful for arranging and further organizing the data. Figure 7 shows an example of the organization of the data about Conceptual Understanding of the concept “unit circle trigonometry and right triangle trigonometry.”

- **Mathematical Proficiency (MP)**
    - **Conceptual Understanding**
        - **Unit circle trig and right triangle trig**

G.I.8 Unit circle trig and right triangle trig
<MP: Conceptual Understanding>
<MP: Historical Knowledge>
For the problem I.8 asking about historical development of unit circle and right triangle trigonometry, as well as connection of trigonometry with inside and outside of mathematics, she said that she has no ideas of history, but thinks that unit circle trigonometry is a generalization of right triangle trigonometry “by allowing side lengths to be negative and by relating the obtuse angles to its unique reference triangle.” (Task I.8, Gloria)

K.II.3 Sine/cosine function; Unit circle trig and right triangle trig
<**MP: Conceptual Understanding**>
He connected right triangle trigonometry with unit circle trigonometry by thinking of the length of the hypotenuse as the radius of a circle. He used the *x* coordinate of the point which is a vertex of the given right triangle to find the value of the cosine function because he said that the definition is the *x* coordinate. He corrected himself that the *x* coordinate is the radius times the cosine.
He said that the circle is not the unit circle because the radius is 5. He said, “Let me take that circle and shrink it down by a factor of 5. I’ll get the unit circle. Nothing has changed in the picture just smaller now. Everything has been divided by 5… this is 1 this [the *x* coordinate] is 4/5 this [the *y* coordinate] is 3/5 and so the point that we’re looking at is 4/5, 3/5 now you can see that these number are what we have found to be the cosine or the sine” (Task II.3, Kyle).
He confirmed the definition of the cosine and sine function on the unit circle using the right triangle with radius 1, the length *x* for the base, the length *y* for height. Without making reference to triangle he had no idea how to define the sine and cosine functions with unit circle trigonometry.

L.II.3 Sine/cosine function; Unit circle trig and right triangle trig
<**MP: Conceptual Understanding**>
<MA: Mathematical Noticing: Observing structure of Mathematical Systems>
When asked to find the cosine value in unit circle trigonometry, he rescaled down the size of the given right triangle by dividing each side by 5 and obtained a similar triangle. He placed the right triangle inside the unit circle using the hypotenuse as the radius of 1. The vertex *A* for the right triangle became a point on the circle and the coordinates are (4/5, 3/5). He picked the *x* coordinate as the cosine of θ.

M.II.3 Sine/cosine function; Unit circle trig and right triangle trig
<MP: Conceptual Understanding>
<MA: Mathematical Noticing: Observing Structure of Mathematical Systems>
He saw the connection between right triangle trigonometry and unit circle trigonometry. He overlapped the given right triangle and a circle with radius 5 which matches with the length of hypotenuse of the triangle. The vertex *A* of the triangle became a point (4, 3) on the circle and the vertex *C* became the origin. He found the cosine value from the *x* coordinates of *A* which is 5 cosine θ.

Figure 7. Example of a data clump corresponding to a strand and a concept.

The Third Stage: Finding Patterns/Themes and Interpretation

During this last stage of data analysis, I synthesized, analyzed, and searched for patterns and themes within each organized and classified data clump corresponding to each strand. I frequently returned to the raw data (both written and verbal data) to understand the original context. This stage was the analysis phase for me to “think with” my data, “reflecting upon what” I “have learned, making new connections and gaining new insights, and imagining how the final write-up” would “appear” (Glesne, 2006, p. 154). Therefore, I was able to begin writing the findings from the data and interpretation in this stage, which will be discussed in the next two chapters.

Limitations

This section identifies some of the limitations of the methodology utilized in this study. First, this study used only one major method to collect data, individual task-based interviews. The data for the second research question were collected from the participants' responses to hypothetical teaching situations without observing the participants' actual classroom teaching. Due to this limitation, I could not find any evidence of their Reflection on Mathematics in One's Practice of the third component–Mathematical Work of Teaching.

Second, Task III was designed to explore how the participants thought about and responded to students' difficulties and misunderstandings. The hypothetical situations in Task III consisted of short dialogues between one or two students and a college instructor. The short lengths of the situation descriptions for the task items limited the participants' responses to their mathematical understanding related to the situations, instead of their thinking about student thinking. Their responses also might have been led by the sub-questions for the task items, such as "How would you describe mathematical concepts key to the situation?" Due to this limitation, although Task III was designed to reveal the strands of the third component–Mathematical Work of Teaching, data from Task III provided more information about the other two components of the MUST framework.

Third, the vocabulary "radians" misused in four task items unintentionally. For example, the first question for Task I.6 contained the term "radians," which was intended to mean "radius lengths" or "radii." The question was stated as follows:

> What is the general form of (x rad, y rad) as the ordered pair *in radians*, of any point on a circle of radius r kilometers that forms an arc length of s kilometers as illustrated on the diagram? Why?

In this question, (*x* rad, *y* rad) should be corrected to (*x* radii, *y* radii) because *x* and *y* represent the values for distance. Because when a *radius length* is used as a unit of angle measure, the unit is referred to as a *radian* (Carlson & Oehrtman, 2009), I mistakenly assumed that the two terms could be used interchangeably. Although they are conceptually related, these terms should be mathematically distinguished because the term “radians” is used only for angles. This terminological error confused the participants, who claimed that a radian measure is only for measuring an angle not distance.

Fourth, although it was not planned, three out of the four participants in this study were recipients of the *University Outstanding Teaching Assistant Award*. This participant group in this study might be considered *special* because such GTA-Ms are generally perceived as “good” college instructors by students, peers, and professors.

CHAPTER 4

FINDINGS

The purpose of this chapter is to report the findings focusing on the participants' mathematical understanding for teaching trigonometry. The data were organized and coded using three components and several strands of the Mathematical Understanding of Secondary Teaching (MUST) framework. Therefore, this chapter presents the findings according to the three components of the framework–Mathematical Proficiency, Mathematical Activity, and Mathematical Work of Teaching (MAC-MTL & CPTM, 2012).

Mathematical Proficiency

Mathematical Proficiency can be thought of as "knowing mathematics." This includes teachers' deep and thorough understanding of mathematics, which helps students develop proficiency in mathematics (MAC-MTL & CPTM, 2012).

Because it was not easy to have direct access to people's mathematical understanding and to organize it explicitly, I referred to Even's framework (1990) for teacher knowledge of subject matter along with the strands of Mathematical Proficiency in the MUST framework (see Chapter 2). Even's framework provided insight and helped me recognize and describe Mathematical Proficiency from the data. However, I used the titles of the strands from the MUST framework to organize the findings.

Conceptual Understanding

Conceptual understanding refers to comprehension of mathematical concepts, operations, and relations (NRC, 2001). The data about the participants' conceptual understanding were collected through three task-based interviews. In this section, I organize and present the findings of conceptual understanding that the participants exhibited *by concept*.

Angle measures. Traditionally, the meaning of an angle is not discussed in trigonometry class because it is assumed that students already know it. However, Leo and Micah mentioned the meaning of an angle. Leo briefly described an angle as a place "where two lines intersect" (Interview 1, Task I.1) and drew an angle using two rays. Micah mathematically articulated an angle saying, "An angle θ whose vertex is at the circle's center cuts off s [the arc length] inches of the circle's circumference as the terminal side of the angle opens in a counterclockwise direction from the initial side," and drew an angle in a circle using two rays (Interview 1, Task I.4).

When the participants were asked to discuss the *meaning* of 10 degrees and 2.3 radians, they interpreted the question in terms of *how big* is the openness of the angles. They had no problem figuring out how big an angle of 10 degrees was when placed within a circle. To determine the openness of an angle of 2.3 radians, Kyle converted radians to degrees, Gloria and Leo approximated the values of $\pi/2$ and π, respectively, and Micah used the fraction of $2.3/2\pi$ of the circumference (Figure 8). None of the participants, except for Micah, discussed the degree or radian angle measure as a fractional part of the circumference of a circle that has its center at the vertex of the angle in Task I. However, Gloria described the degree measure as, "A degree is…a ratio of something….One degree is $1/360^{th}$ of a circle" in Task III (Interview 3, Task III.2).

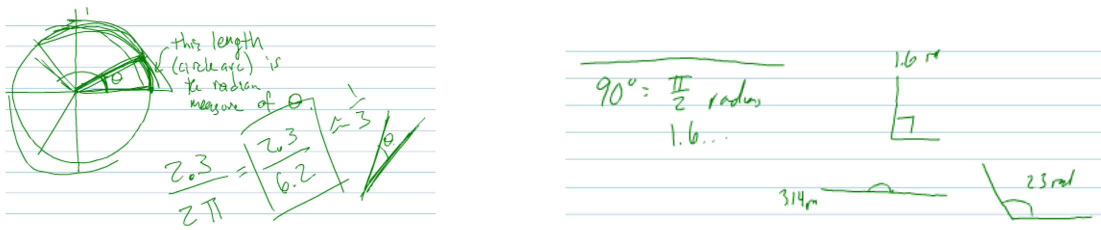


Figure 8. Micah's and Leo's explanation of an angle of 2.3 radians.

Unit circles. The participants showed a dual conception of the unit circle. One conception was a common concept of the unit circle. They perceived the radius as a unit of measurement and claimed that every circle is a unit circle. For example, Gloria explained what a unit is in the definition of the unit circle saying, "It is completely up to every individual as to what a unit is….A unit is not a unique object….What is one unit? It's precisely this distance (indicating the radius length of a circle she drew)" (Interview 3, Task III.3). She identified the circle with a radius of 2.8 feet as a unit circle by defining the radius of 2.8 feet as one unit. Interestingly, she drew the unit circle separately from the original circle, although she claimed that they are the same (Figure 9). When I asked why she drew them separately, she answered, "Because I want to make a point that this is my one unit but this right here is 2.8 feet" (Interview 3, Task III.3).

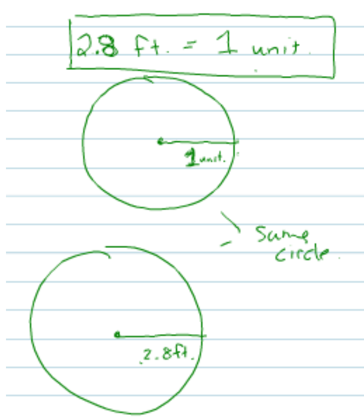


Figure 9. Gloria's unit circle as a separate circle.

Another conception was that a unit circle should be accompanied by a specific unit, such as a unit of an inch. For example, Kyle claimed that any circle is a unit circle *with* a choice of unit of measure and said, "A unit circle is a pair of a circle and a unit of measurement….Every circle can be made into a unit circle by choosing the appropriate unit of measurement" (Interview 3, Task III.3). For the circle of a radius of 2.8 feet, he argued that it could or could not be a unit circle depending upon the choice of the unit of measure. In a similar sense, Micah mentioned that it did not make sense to ask whether a circle is a unit circle without specifying a unit saying, "A circle of radius 12 inches is not a unit inch circle and is a unit circle of the unit of 12 inches" (Interview 3, Task III.3). Kyle and Micah also discussed rescaling the size of a circle to have a unit circle and drew a unit circle by changing the length of the radius from the original circle (Figure 10).

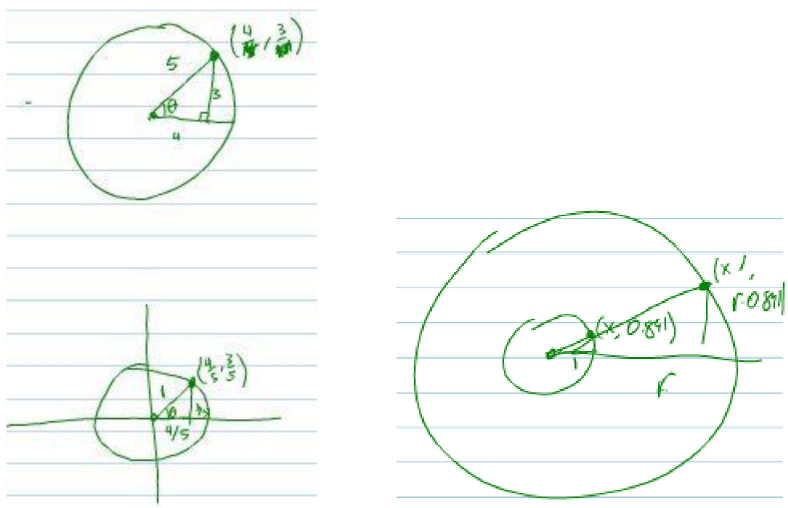


Figure 10. Kyle's and Micah's drawings of unit circles as rescaled from the original circle.

Sine or cosine functions. The participants knew that the inputs of sine or cosine functions are angles in radians or real numbers and that the outputs are real numbers. When they were asked about the interpretation of the input and output of the sine function sin(1.1) ≈ 0.891 in a given Ferris wheel context (Figure 11), they easily responded that the input of 1.1 is the angle measure in radians from the 3 o'clock position. Kyle interpreted the meaning of the angle of 1.1 radians as how wide it opens, just as he did in Task I.1. To figure it out, he converted it to the degree measure using the conversion factor of 180º/π, which showed his consistent reasoning while working on radians.

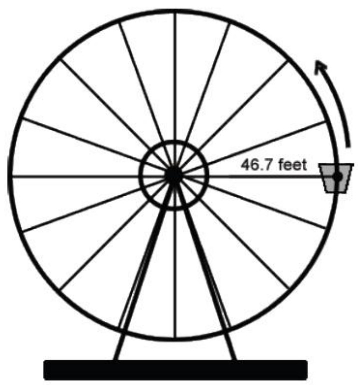
To interpret the output of 0.891, Gloria represented the *y*-coordinate of a point on a circle as the sine function $y = r\sin\theta$ and demonstrated that 0.891 is the unitless *ratio* of the height above the 3 o'clock position to the radius of the Ferris wheel, by explaining:

> The input value is the angle measurement in radians….So, it's the angle measurement from the 3 o'clock position through a measurement through an angle 1.1 radians….It [0.891] is the scale factor…. *y* is equal to r times sine of θ, *y* over *r* would be sine of θ. So, maybe the best way to say…is .0.819 is the ratio of the height above the 3 o'clock position with the Ferris wheel radius….This [1.1] is a real number….I have a hard time. I don't like thinking about the output of…a sine function having a unit…It [1.1] is *unitless* [italics added]…. I *never* [italics added] use radians as a unit. (Interview 1, Task I.7)

Although she considered the output as the ratio of the vertical height with respect to the radius, there was no evidence of her perception of the output as one being measured in radius lengths.

Kyle explained that multiplying the number 0.891 by the radius length is the actual height of the bucket above the horizontal diameter of the Ferris wheel. Micah clarified the unit of the output by saying, "This number [0.891] is really unitless….The unit lives there (indicating the radius length of 46.7 feet)" when he wrote "Height = 46.7(ft)·(0.891)" (Interview 1, Task I.7). Leo said, "1.1 corresponds to an angle…If there are no units, always assume it's radians" (Interview 1, Task I.1). However, Leo made a minor mistake in connecting the context and the given sine function. Although he interpreted 0.891 as a vertical distance above the bucket when it started to move, he answered 0.891 "feet," because he thought that the unit of the output value of the given sine function depends upon the given unit of the radius.

John is sitting in a bucket of a Ferris wheel. He is exactly 46.7 feet from the center and is at the 3 o'clock position as the Ferris wheel starts turning.



I.7.1 What does the **input** value of the sine function $\sin(1.1) \approx 0.891$ represent in this context?
I.7.2 What does the **output** value of the sine function $\sin(1.1) \approx 0.891$ represent in this context?

Figure 11. Task I.7.

In another problem regarding the input and output of trigonometric functions, I identified evidence of the participants' struggles. When they had a given value of $\cos(\sin(35^\circ)) \approx 48.125^\circ$ in Task III.2, they expressed unfamiliarity with the symbolic notation of the output in degrees. For example, Gloria said, "I don't think that I like this problem at all because…this is bad notation…this whole expression…would be outputting a real number. But a degree isn't a real number" (Interview 3, Task III.2). They argued that the output of the cosine function should be a real number because they said that the cosine function is a real-valued function. For example, Leo disagreed with the idea of treating a length as an angle in degrees because he said, "No matter what, it [the cosine function] is always going to spit out a real number. It's never going to give you a degree. It's always going to give you a real number. So that's sort of a confusing question" (Interview 3, Task III.2). He represented the cosine or sine value of an angle as the length of the base or height of a right triangle with a hypotenuse of one. Additionally, he argued that the unit of the cosine or sine value might be in any linear measure but could not possibly be in degrees (Figure 12).

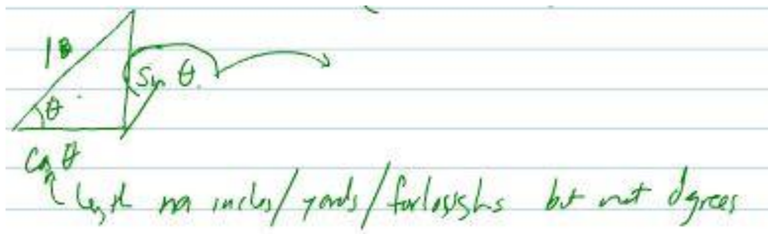


Figure 12. Leo's representation of the cosine and sine values.

Inverse trigonometric functions. The participants were very familiar with the definitions, properties, and graphs of inverse trigonometric functions. They had no problem discerning the mathematical symbol for the inverse function compared to the reciprocal function. They clearly stated the restricted domain between -π/2 and π/2 of the sine function for the existence of the inverse sine function and drew its graph by flipping the sine function with respect to the line $y = x$. They knew that the range of the sine inverse is the domain of the sine. Kyle, for example, explained it,

> When we do this reflection across the *x* and *y* axis, what we are really doing is we are switching *x* coordinate for *y* coordinate on each point. And so, our axes are going to be switched as well when we do that. I'm just assuming some facts about inverse functions, like here, the domain of the inverse is a range of the function. (Interview 3, Task III.8)

He added that for the existence of the inverse sine function the sine function should be one-to-one, meaning that the function does not repeat itself and that the restricted domain could vary as long as it is as large as possible with no repetitive outputs.

Micah described the same idea in a different way, saying "Where are you going to cut its domain?....In some sense, it doesn't really matter where you cut its domain as long as you cut its domain somewhere…it passes the horizontal line test" (Interview 3, Task III.8). Leo articulated the formal definition of the inverse function: "It exists if *f* inverse composed with *f* of *x* gives me back *x* and *f* of *f* inverse of *x* also gives me back *x*" (Interview 3, Task III.6).

Periodic functions/Periods. The participants had no problem presenting the formal definition of a periodic function. They recognized that the sine function $y = \sin(x^2)$ is not periodic because they used the definition of a period and found no constant *c* satisfying $f(x) = f(x + c)$ for all *x*. Gloria said that many variables were involved in finding the constant c, which made it impossible to find c for all *x*. When asked if 2π is the period because of $\sin(x^2 + 2\pi) = \sin(x^2)$, she responded that 2π could not be a period because 2π did not satisfy the condition of the definition of a period. All the participants, except for Kyle, successfully sketched a rough graph of $y = \sin(x^2)$ by plotting several *x*-intercepts using the square root function and considering the maximum and the minimum values. Their graphs oscillated between 1 and -1 and oscillated faster as *x* increased. Kyle also had a similar idea and drew a graph oscillating. However, he did not pay attention to the critical idea of the extreme values of the function and behaviors of the *x*-intercepts from the argument of x^2. Therefore, his graph was not right because it illustrated that the *x*-intercepts were getting farther away and the maximum and minimum values varied (Figure 13).

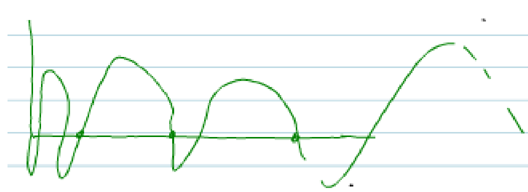


Figure 13. Kyle's graph of $y = \sin(x^2)$.

Unit circle trigonometry and right triangle trigonometry. The participants had no trouble connecting right triangle trigonometry with unit circle trigonometry by thinking of the length of the hypotenuse as the radius of a circle, the length of the base as x, and the height as y. They noticed that the definitions in the unit circle context and in the right triangle context were compatible. They fluently represented any point on a circle as a pair of the cosine and sine functions ($r\cos\theta$, $r\sin\theta$) and used the representation in various problem contexts (Figure 14).

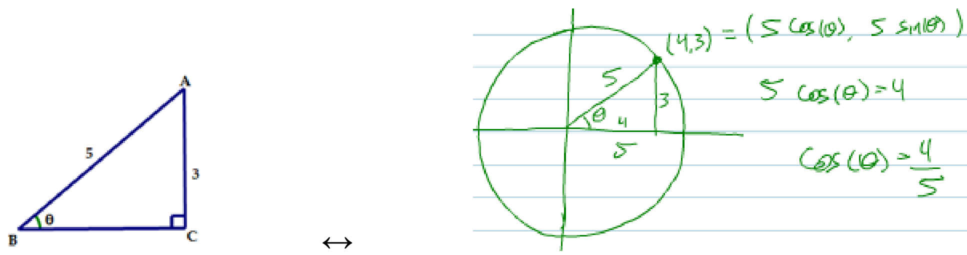


Figure 14. Micah's connection of the right triangle to a circle context.

Co-functions. The participants understood the sine and cosine functions as co-functions of each other. Given a point $P(t)$ on the unit circle corresponding to the angle t between 0 to 2π, they had no difficulty finding the coordinates for the point $P(\pi/2 - t)$ by using the addition and subtraction formula for sine and cosine. In addition, they explained that $\pi/2 - t$ is the complementary angle of t, hence the coordinates for $P(\pi/2 - t)$ could be found by using the co-function formula. They drew right triangles in the unit circle to explain the relationship between the co-function pair of sine and cosine. Leo, for example, explained it this way:

> Another fancier way to do…is to know that the cosine of an angle is actually equal to the sine of the compliment of that angle. That's where the name comes from….If I want to know the cosine of $\pi/2 - t$,…that's the compliment of t, that has to be the sine. (Interview 3, Task III.9)

Micah, however, misunderstood the function $P(t) = (4/5, 3/5)$ as a constant parametric description of the circle and failed to remember that the description consisted of the cosine and sine functions of t on the unit circle for an angle t. Therefore, he responded in an incorrect way:

> P of t is the constant point (4/5, 3/5), so in other words, this description of how I'm running around the circle says I'm just standing here forever. So where am I standing at $\pi/2 - t$, well, I never moved so I'm still standing there. So P of $\pi/2 - t$ is (4/5, 3/5). P of anything is (4/5, 3/5), in other words, this is constant in t. (Interview 3, Task III.9)

Procedural Fluency

Procedural fluency is a skill in carrying out mathematical procedures flexibly, accurately, efficiently, and appropriately (NRC, 2001). The participants showed procedural fluency in algebraic computations for solving task items in trigonometry, in most cases, with meaningful explanations. For example, they had no problem converting one angle measure to another angle measure using a conversion factor. In the conversion process, they utilized a traditional method of unit cancellation. For example, to convert 2.3 radians into degree measure, Kyle explained, "The radians are cancelling out. Here we are going from radians to degrees" (Interview 1, Task I.1) in the computation of "2.3 rad × 180º/π rad." Leo asserted that unit cancellation is important for calculation and said, "If the units cancel out correctly, then…you are doing things in the right direction and it's not upside down….The units guide you of which way I should write this way or that way….I always try to use these things because I think it makes it less tricky for them [students]" (Interview 2, Task II.1).

They were good at using the arc length formula, circumference formula, and equivalence of ratios of parts to the whole to solve problems involving angles. Their fluent algebraic work involved not only the Pythagorean Theorem and the mnemonic device "SOA-CAH-TOA" in a right triangle context but also a general symbolic representation ($r\cos\theta$, $r\sin\theta$) of a point on a circle in a circle context. Their algebraic manipulations for modeling trigonometric functions from a real world application problem were also skillful. They did not show any difficulty in algorithmic work with inverse functions and their restricted domains.

Strategic Competence

Strategic competence refers to proficiency in formulating, representing, and solving problems using mathematical strategies (NRC, 2001). It is a skill that concerns more than just "knowing how" because it involves creativity and flexibility. The participants were good at dealing with unknown variables and formulating or representing problems mathematically. Their problem-solving strategies were similar with slight differences depending upon task items. For example, when formulating Task I.3 (Figure 15), they all used either part to the whole ratios or ratios of angles to arc lengths. For this problem, no participant set up and explained equations using the concept of an angle as a subtended arc or as a fraction of a circle's circumference. For example, $(s/C)\cdot x = 7$ or $(7/x)\cdot C = s$, where s is the length of the arc subtended by the angle, C the circumference, and x the total number of the "mystery" units.

A student measured the angle displayed below and determined that its measure was 7. However, he did not label the units in which he measured the angle. The unit he measured in is not grads, radians, or degrees. Describe how to use the arc length and circumference of the circle displayed below to determine how many of the "mystery" angle measure units mark off any circle's circumference.

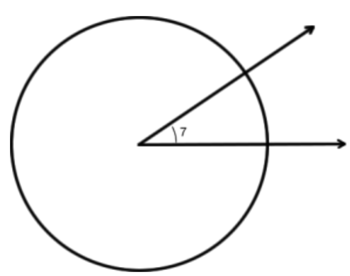
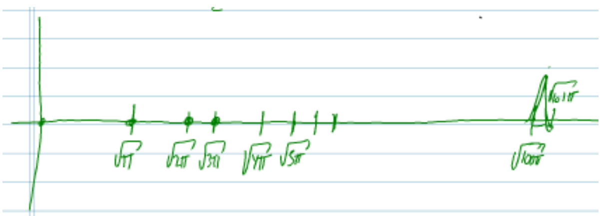


Figure 15. Task I.3.

The participants also showed similar strategies to represent functions graphically or symbolically. To sketch a graph, they all plotted special points, such as the outputs of some special angles, the *x*-intercepts, and/or the extreme values, and connected the points smoothly (e.g., Figure 16).



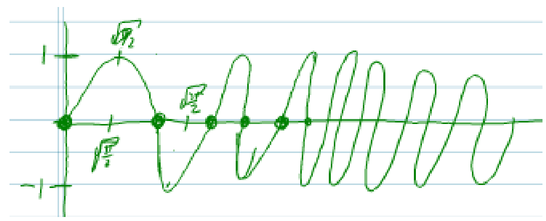


Figure 16. Leo's sketch of the graph of $y = \sin(x^2)$.

When the participants had item Task III.2 (Figure 17), they all agreed with student B and struggled with the assumption that the instructor's representation of the cosine function as an angle in degrees was true because they thought that the cosine value should be a real number, not an angle.

Instructor (You): The answer for computing cos(sin(35º)) is 48.125º.
Student A: I put my calculator in the degree mode and then sin(35) produced 0.5736 and cos(0.5736) produced 0.9999. The answer is 0.9999º.
Student B: I think that the answer 48.125º could be wrong because the value for cos(sin(35º)) must be a real number.

- How would you compute cos(sin(35º))?
- How would you help the students derive 48.125º?
- How would you describe mathematical concepts key to the situation?
- What might be possible sources for their error or conception?
- What questions or examples would you ask or use to help students better understand the mathematical ideas/concepts involved in this situation?

Figure 17. Task III.2.

Despite their discomfort, they patiently attempted to figure out the item. They generated a strategy in which they attempted to associate a real number with an angle. For example, Gloria explained her strategy as follows (Figure 18):

Student B says that 48.125 degrees could be wrong because the [cosine] value should be a real number. I actually agree with the student on that….That [48.125 degrees] just seems sort of weird to me because a cosine does spit out a real number….Any real number, such as .840 [= cos(sin(35º))] can be thought of as a radian because you can take your circle of radius one and you could trace out what's an arc length if you will be of .840 and that would be your corresponding angle. So to this arc length of .840 you would get an angle measurement, which you could then measure in terms of degrees. (Interview 2, Task III.2)

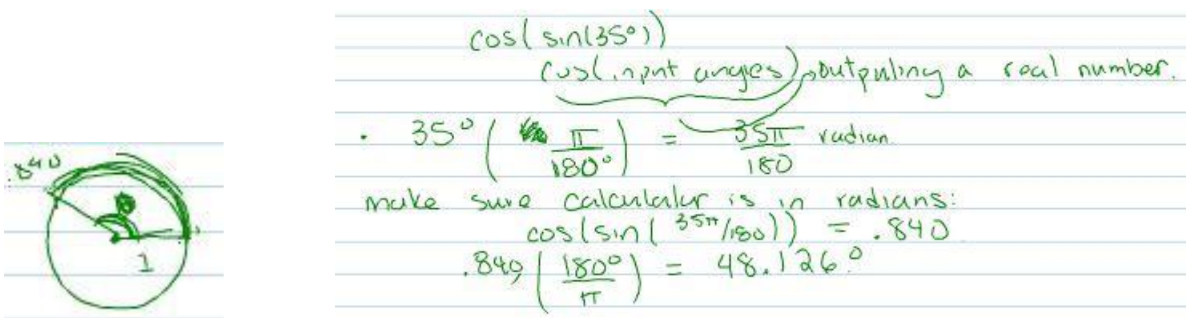


Figure 18. Gloria's strategy.

She claimed that the association between the real number and the angle does not mean that they are the same but that there is a one-to-one correspondence. For the item, she identified the real number of .840 as an angle of .840 radians and converted it to an angle in degrees using the conversion factor of 180º/π, which was the same way Kyle approached the item. Although Micah also generated and used the same strategy to figure out the item, he was not satisfied with the approach.

You are thinking of the map that goes from the numbers to the angles. And then I take the degree measurement of that corresponding angle….To me that is just a very convoluted thing to do. I don't understand why one would do that….Certainly the only thing that this could possibly mean is you get some number you now think of that number [cos(sin(35º))] as a radian measurement and I convert that radian measurement to degrees and….I should get that angle [48.125º]. But to me it is very unnatural to think of the output of the cosine function to be an angle. That's very unnatural to me….In fact I think it's safe to say that until this moment I have never thought of the output of the cosine as an angle. (Interview 2, Task III.2)

Adaptive Reasoning

Adaptive reasoning refers to capacity for logical thought, reflection, explanation, and justification including intuitive, deductive, and inductive reasoning based on patterns or analogy (NRC, 2001). This proficiency helps a person place facts, concepts, procedures, and solution methods together in a meaningful way (NRC, 2001).

Even when the participants misunderstood task questions, they attempted to reason accordingly. For example, when they were asked to "interpret" the circumference formula $C = 2\pi r$, they attempted to figure out "where it came from." Kyle and Micah, for example, guessed the origin of the number π as half of the circumference of the unit circle. And then they used the similarity of circles and generalized their definition to the circumference formula for any circle of the radius of r using r as an expanding or shrinking factor.

Although the participants showed no difficulty sketching trigonometric functions, their reasoning about the smoothness and concavity of the curve made sense but was intuitive. For example, to graph a sine function Gloria plotted some points, connected them smoothly, and explained that the height of an object in a circular motion was changing as a continuous motion. She reasoned about the concavity of the sine curve in a sense of rate of change as follows:

I would have…to sort of measure this distance [the height] and then as you get closer and closer to the top you can see that the distance is not changing as much like this distance is really close to that distance and so that implies that as you're really close up here you are not growing all that fast. So you would have a certain concavity. When you are down here though that angle measurement from here to here whatever that is, that's going to make a bigger difference in the growth from here to here maybe. And just kind of show them that by looking at that in try to plot it you can see you can almost kind of see how the concavity has to go by looking at sort of the speeds and the changes of the distances…. We look at the rate of change of distances to get concavity. (Interview 1, Task I.5)

Leo and Micah justified their informal arguments using contradiction. Leo mentioned that talking about concavity requires knowledge of derivatives, and that motions in the real world are nice and smooth most of time, except for some cases. He said, "It [the curve] has to slowly come up here," because *otherwise* [italics added] at the maximum value "that would be a sharp turn….It [the bug] is not like it gets to here and suddenly jumps back down that way" (Interview 1, Task I.5). Micah asserted,

> This bug's motion is completely smooth. I never sort of stop and then drastically sort of speed up or slow down….Whereas, if maybe you were worried that this thing was concave up…if I get there right and now I've got to start decreasing again all of a sudden…I'm going to get some sort of weird point if I try to do that, so maybe intuitively it makes more sense for this function to be nice and smooth. (Interview 1, Task I.5)

The participants' reasoning about the unit circle was similar. Although they perceived the radius as a unit of measure for the concept of the unit circle, they also talked about a unit circle depending upon the choice of a unit. Therefore, I could observe that they drew the unit circle as a separate or a different circle from a given circle (see Figures 9 and 10). In most cases, they regarded the radius length as a scaling factor to shrink or enlarge the original circle to the unit circle, keeping the linear unit of the radius of the original circle. They reasoned about the concept of the unit circle in a similar sense of proportionate properties of similar triangles. For example, Gloria stated that the unit circle is “a scaled-down version” (Interview 2, Task II.3)–a smaller or bigger circle of radius “one,” using a scaling factor of 1/r for the given circle with radius r.

I observed reasoning by analogy in the participants’ work. One of the examples was Gloria’s analogy of “a rubber stamp” to explain the concept of the period of the function. She illustrated that each chunk of stamp should be connected, have the shortest distance, and not overlap as stamping along.

Productive Disposition

Productive disposition refers to a positive attitude toward and beliefs about mathematics including self-efficacy (NRC, 2001). Through task-based interviews, I found that the participants seemed to already have “habits of mind” (Cuoco, Goldenberg, & Mark, 1996) as mathematicians. For example, after Kyle finished a task item on a proof, he said that proving was fun for him. They also put forth a great deal of effort to think about, do, and explain the task items. For example, for one of the task items in Task I, Gloria misunderstood what the task item asked and became lost. She attempted a couple of ways but failed to figure it out. After completing all other items for Task I, she wanted to return to the task item and patiently try it again from a fresh perspective. She finally figured it out when she realized that her confusion came from her misunderstanding of the question and that the item was not that complicated. She spent about 20 minutes on the task item in total. This occurred not only to her, but also to the other participants. Table 9 in Chapter 3 showed the amount of time the participants spent completing each task, which also demonstrates their persistence in figuring out each task item.

They did not mind taking a great deal of time and were willing to finish the tasks, although the length of time elapsed was longer than the participants and anticipated originally.

I observed that Leo tended to show more non-negotiable attitudes toward mathematical facts that he believes to be correct than the other participants. This is evident in his working time, which was relatively shorter than the other participants' (see Table 9). For example, in Task III.2 (Figure 17) all the participants wondered how the cosine value could be expressed as an angle in degrees, which they did not agree. Different from Gloria, Kyle, and Micah who attempted to think about a way to figure it out, Leo refused to consider the cosine value as a degree measure, making the following argument:

> I don't think this [48.125º] is the answer….This doesn't make sense to me….Cosine is a length….It's sort of where the fuzzy line between length units switching to degrees. And how that sort of fuzzy line when we say things like the cosine of the sine of 35 degrees, well, this is a length but now we are suddenly treating it like it's an angle measure. So we are kind of fuzzying the line between length and angle measure. (Interview 3, Task III.2)

Historical and Cultural Knowledge

The MUST framework added historical and cultural knowledge to the five strands of mathematical proficiency identified in *Adding it Up* (NRC, 2001) because understanding the origins and conventions in mathematics can foster an understanding of mathematical ideas in a more conceptual way. The participants frankly said that they did not know the history of trigonometry well and that their knowledge of the history of trigonometry was limited. Therefore, they attempted to guess the history of trigonometry when asked about the historical background of trigonometry. For example, Gloria guessed that the origin of degree measurement might be related to a nautical term without further explanation. Although the origin of the degree as a unit of angles is unknown, it is known that ancient astronomers used a degree when concerning the ecliptic path of the sun (Scott, 1960). Kyle said that astronomy might provide the historical background for trigonometry. Interestingly, they all guessed that trigonometry involved only angles and triangles in the beginning, hence triangle trigonometry developed first followed by circle trigonometry, which is not true in the history of trigonometry.

The participants were able to articulate the mathematical conventions regarding angles. They mentioned that the conventional direction of angle rotation is counterclockwise and the absence of the unit for an angle conventionally means the radian measure. With this understanding, they could recognize, for example, $\sin(3) \neq \sin(3^{\circ})$ because "3" in sin(3) denotes the radian angle. Gloria asserted that mathematical convention makes mathematics work in a coherent way. Through the data, I could not observe any evidence of their knowledge of trigonometry from a cultural perspective.

Mathematical Activity

Mathematical Activity can be thought of as the process of "doing mathematics," which involves teachers' mathematical actions to notice, reason, create, or integrate mathematical ideas (MAC-MTL & CPTM, 2012). This component consists of three strands–Mathematical Noticing, Mathematical Reasoning, and Mathematical Creating. Because these strands are intertwined, adding another strand–Integrating Strands of Mathematical Activity–was unavoidable.

Mathematical Noticing

Mathematical noticing involves recognizing similarities and differences in mathematical structures, mathematical symbolic conventions, argumentation, and connections of mathematical entities in varied mathematical areas or other fields (MAC-MTL & CPTM, 2012).

Observing structure of mathematical systems. Teachers need to recognize the structural similarities or differences in mathematical systems, which helps them flexibly deal with mathematical objects in varied systems (MAC-MTL & CPTM, 2012). The participants noticed the connection between right triangle trigonometry and unit circle trigonometry by associating the hypotenuse of a right triangle with the radius of a circle. However, they did not perceive the length of the hypotenuse of the triangle as a unit of measure so that they could consider the circle as a unit circle. When asked to find a cosine value in Task II.3, they rescaled the length of the hypotenuse to one for "their" unit circles and other lengths by using the same scaling factor (e.g., Figure 10). Their understanding of the connection of the two structures–right triangle trigonometry and unit circle trigonometry–was also observed in their responses to Task III.2 (Figure 17). For this task item, they all disagreed with the instructor and agreed with student B. Their responses to the instructor's solution, 48.125º, were similar. They claimed that the output of the cosine function should be a real number, not an angle in degrees. For example, Gloria said that writing 48.125º in degrees for the output value of the composite function was "bad notation" (Interview 3, Task III.2). Leo was concerned about "fuzzying the line between length and angle measure" (Interview 3, Task III.2). Micah mentioned that it would be "very dangerous to blur the distinction between numbers and angles" (Interview 3, Task III.2).

Although they identified the hypotenuse of a right triangle with the radius of a circle to make a connection between right triangle trigonometry and unit circle trigonometry, I could not find evidence that they interpreted the cosine value known as the ratio of one length (adjacent to a given angle) over the hypotenuse as a proportional length to the circle's radius or considered the unit of the cosine value to be in radius length. Such interpretation makes it possible to find a corresponding angle in degrees to an arc whose length is a fraction of the circle's circumference in the following sense: cos(32.864º) = 0.834 means that cos(32.864º) is a length of 0.834 of the circle's radius. Because the circle's radius is $1/2\pi$ of the circle's circumference, cos(32.864º) corresponds to an arc whose portion is $0.834/2\pi$ of the circle's circumference. If an angle of one degree is defined as an arc whose length is 1/360th of the circle's circumference, cos(32.864º) corresponds to an angle of $(0.834/2\pi)\cdot 360^\circ \approx 48.125^\circ$ (Thompson et al., 2007). Bressoud (2010) expressed the radius of a circle in degrees, "If the circumference is 360º, then the radius should be $360^\circ/2\pi \approx 57.3^\circ$" (pp. 109-110). Because cos(32.864º) has a unit of radius lengths, a length of 0.834 of the circle's radius, $\cos(32.864^\circ) \approx 0.834\cdot 57.3^\circ$ provides as an angle in degrees.

Discerning symbolic forms. Teachers need to be aware of symbolic forms so that they can identify and explain the mathematical meanings behind the symbolic rules (MAC-MTL & CPTM, 2012). The participants were good at dealing with symbolic representations, such as the representation of an arbitrary point on a circle as ($r\cos\theta$, $r\sin\theta$). They also noticed the symbolic differences between the reciprocal of a number and the inverse of a function. For example, Gloria and Leo pointed out that the notational idea of the exponent to -1 in between "number mathematics" and "function mathematics" should be distinctively interpreted as the reciprocal of a number and the inverse of a function respectively. Kyle mentioned that the superscript -1 notation for inverse trigonometric functions is ambiguous and confusing because it can be interpreted as either its reciprocal or its inverse function. He suggested that, to indicate an inverse trigonometric function, it could be appropriate to use the terminology "arc" instead of using the superscript -1, which might be a way to avoid students' confusion. Micah said that it is "a matter of convention" for cosine to the -1 of x to "mean the function inverse not the inverse in the range" (Interview 3, Task III.6). He emphasized the importance for notation in mathematics saying, "I would say…notation is everything" (Interview 3, Task III.6).

Regarding rules related to symbols, the participants were aware of students' common mistakes. For example, Leo responded that cancelling "cos" in $\cos\left(\frac{1}{\cos x}\right)$ is "an illegal cancellation" and "we cannot do this against the rules" (Interview 3, Task III.6). Gloria remarked that $\cos\left(\frac{1}{\cos x}\right) = 1$ might come from the confusion between the composition and multiplication of functions. Kyle was aware that students easily take constants out of trigonometric functions, such as $\sin(2x) = 2\sin x$.

Detecting the form of an argument. Teachers need to notice the form of mathematical arguments generated by textbooks or students so that they can identify a missing or repeated portion of the arguments (MAC-MTL & CPTM, 2012). I expected to see this strand from the items of Task III in which some of students' arguments in hypothetical teaching situations were stated. However, due to the short length of the descriptions of the situations in the task items, the participants focused on explaining more about mathematics related to the situations rather than analyzing the students' arguments.

Interestingly, the participants attempted to avoid directly judging the students' arguments in the task items and to understand their thinking even when the arguments were not mathematically correct. For example, when responding to Task III.1 (Figure 19), they all noticed that the student's argument came from his/her misconception of the arc length formula.

Instructor (You): Can an angle's openness be measured with a linear unit of measure, such as in inches?
Student: Yes.
Instructor (You): Why?
Student: For example, the measure of an angle is 1.7 "inches" as an angle measure when the arc length is 1.7 inches on the circle of radius 1 inch.

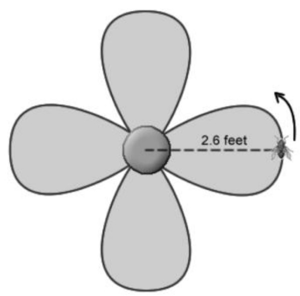
- How would you describe the mathematical concepts key to the situation?
- What might be possible sources of his or her conception?
- How would you correct the student's answer, if necessary?
- What questions or examples would you ask or use to help students better understand the mathematical ideas/concepts involved in this situation?

Figure 19. Task III.1.

However, they did not agree that the student's argument was completely invalid. Rather, Gloria attempted to support the student's argument. She reasoned that the student's thinking could imply generalizing a concept of the radian measure and "theoretically…a well-defined angle measure…this method would consistently measure angles in an appropriate way" (Interview 3, Task III.1). She demonstrated that, as the radius changes, she could rescale the radius down by a scaling factor onto a circle with a one-inch radius. Leo mentioned that it would be okay for him as a mathematician to redefine an angle as an object "assigned a unit length of 1.7 inches if the arc length…associated to the angle is 1.7 inches long on a circle of radius 1," although the student's argument was not "generally accepted…so this is not a general mathematical convention" (Interview 3, Task III.1). Micah agreed with the student's argument and claimed that an angle in a linear measurement could be well defined as long as a unit of length is given because he defined an angle by a point on the unit circle. Their descriptions might be partially correct as long as the angle was represented in the coordination of a 1.7-inch arc length *with* a one-inch radius and a new measurement of the arc length had to be taken when the radius of the circle changed. However, no participant argued that it is *impossible* to measure an angle's openness in a linear measurement because the angle measure unit should convey the fractional part of a circle's circumference and be consistent regardless of the size of the circle (Carlson & Oehrtman, 2009).

In this study the terminology "radians" was incorrectly used in some task items in place of the terminology "radius lengths" or "radii." The participants' responses to the unplanned misuse of the term were categorized into their mathematical noticing of the form of an argument because their first reaction was to say that the task items had something wrong mathematically. The first question for Task I.5 was an example (Figure 20).

Imagine a bug sitting on the end of a blade of a fan as the blade revolves in a counter-clockwise direction. The bug is exactly 2.6 feet from the center of the fan and is at the 3 o'clock position as the blade begins to turn.



I.5.1 Sketch a graph of a function *f* over the input interval from 0 to 2π to illustrate how the bug's vertical distance (in radians) above the horizontal diameter co-varies with the measure of the angle swept out the bug's fan blade (in radians). Justify the shape of the graph.
I.5.2 Determine symbolic representations of the function *f* in part **I.5.1**.

Figure 20. An example task item containing an incorrect use of the term "radians."

At first, the participants attempted to figure out the shape of a graph representing the relationship between the angle and the vertical distance without paying attention of the designated unit of the vertical distance, which was supposed to be "in radius lengths" not "in radians." When asked about the unit of the height, they noticed the given unit in the first question. They did not agree with "distance in radians" and considered it weird in mathematics. For example, Kyle responded, "Well, I don't know. I can't see what it would mean to measure this distance in radians….That's what I did here except for the fact that there is… a *wrong* [italics added] unit” (Interview 1, Task I.5). When Leo noticed the wrong terminology, which was not mathematically acceptable to him, he argued:

> You *can’t* [italics added] talk about vertical distance in radians because radian is not a distance. So, you can only talk about…it would be bug’s vertical distance in feet. To talk about it in radians is like saying something along the lines of what time is it in inches….I would say this problem is worded *incorrectly* [italics added] and…that *mathematically* [italics added] this doesn’t make sense…. That just seems *strange* to me….as a math person….That is kind of *confusing* [italics added]. (Interview 1, Task I.5)

Connecting within and outside mathematics. Teachers need to be able to flexibly connect mathematical entities within varied mathematical areas. In addition, they need to seek and notice mathematical applications in fields other than mathematics (MAC-MTL & CPTM, 2012).

Few of the participants mentioned connections within or outside mathematics when they were completing the task items. Gloria offered a few descriptions in regard to this strand. She noticed that the unit conversion could be used in other fields, such as chemistry for calculations involving moles or atoms and made the following comments: “How different trig functions behave as compositions” can be used “for applied math a lot…with waves, and so these are the compositions of graphs like these are the ones that come up a lot, I think, in nature” (Interview 2, Task II.5).

Mathematical Reasoning

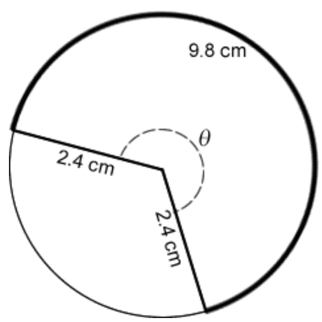
Mathematical reasoning involves reasoning in the context of justifying/proving, conjecturing/generalizing, and constraining/extending mathematical arguments.

Justifying/Proving. Teachers need to be proficient in justifying and proving mathematical arguments or claims formally or informally in logical ways. They should be aware of the assumptions underlying arguments and familiar with various approaches to justification or proof (MAC-MTL & CPTM, 2012).

The participants showed the step-by-step logical procedures necessary to reach solutions or conclusions through the task items because they seemed to be well trained in their major field, mathematics. However, their main approach to their explanations was algebraic. For example, in Task II.1 (Figure 21) they set up proportions and performed algebraic work by placing units and unit cancellation in computations to justify the unit for their final answer (e.g., Figure 22).

The *grad* is a unit of angle measure that is sometimes used in France, where every circle's circumference is 400 grads.

II.1.1 Determine θ in grads in the following figure.



II.1.2 If a circle has a radius 7.1 inches, what is the arc length in inches of the angle of 3 grads?
II.1.3 How many radians are equivalent to 10.2 grads?
II.1.4 Name your own unit of angle measure and define how many of these units mark off the circumference of a circle so that you can create a protractor to measure any angle in your unit. Describe the meaning of an angle of 18.2 (name of your unit).
II.1.5 Define a function that converts a number of grads to a number of your unit. Explain the meaning of the formula.

Figure 21. Task II.1.

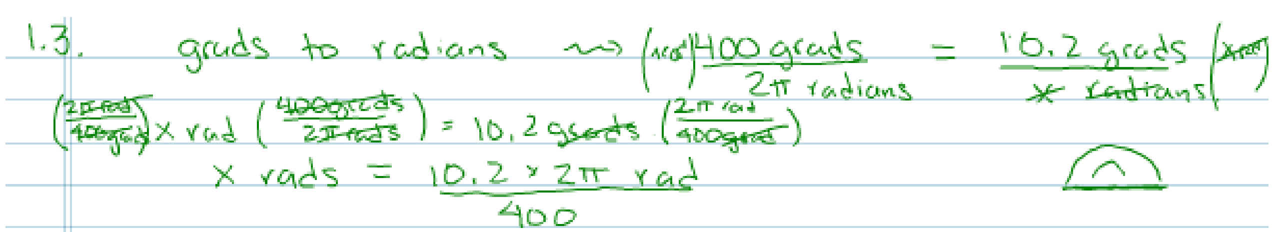


Figure 22. Gloria's work using proportions and unit cancellation.

In some task items the participants reasoned by analogy. They used various kinds of analogies in an effort to make their explanations clearer and simpler. For example, Leo explained the meaning of the unit conversion factor of 400 grads/2π radians (or 2π radians/400 grads) as a comparison or an identity by using an analogy of units of money. He provided an example of 100 pennies to a dollar and 5 nickels to a dollar because both 2π radians and 400 grads indicate a full circle. Gloria used an analogy of decimals versus percentages to explain radians versus degrees in computation and explained that multiplying by a conversion factor is similar to multiplying by 100 or 1/100. She said that although percentages are easy to conceptually understand, the decimal form is required when solving a problem, such as an interest rate problem. She explained that as percentage does not play with real numbers, degrees do not play with the real numbers nicely, but the radians do. She also explained that the ratio of the circumference to the diameter being constant is similar to the ratio of the perimeter of squares to the side length being constant. She also used an analogy of a "stamp" to explain the concept of the period of the function as noted earlier. Micah explained the angle function $\theta(t) = (3/15)\cdot t$ with respect to time t using an analogy of the distance formula, which states that the distance equals rate times time. He said, "It was this same kind of idea. If I have some measurement divided by seconds and I multiply it by time, then I get the amount of measurement it went" (Interview 2, Task II.4). Kyle explained with an analogy of the semi-upper parts of two circles to justify that the number of radius lengths that the circumference has does not depend upon the radius length although the length of the leftover part is longer for a bigger circle. He demonstrated that although the arc length of the bigger circle is longer than that of the smaller one, the fraction of the circumference, 1/2, is the same for both circles. He also used the case of two similar triangles whose lengths might change while their proportions remained the same.

The participants also demonstrated awareness of assumptions of problems. For example, when a rate of 3 radians every 15 seconds was given in a task item, Gloria and Kyle made sure that it should be constant. Gloria mentioned that an angle in her work was assumed measured in a counterclockwise direction when the direction of the angle was not shown in the given figure.

When the participants were asked to geometrically prove the trigonometric identity $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ in Task III.7, their common strategy was to draw several auxiliary lines to form right triangles labeled with the necessary variables in the figure. Gloria and Kyle remarked that finding the "right" right triangles and seeing their connections was the key to its proof of the task item.

Reasoning when conjecturing and generalizing. Teachers need to be able not only to construct conjectures but also to test them. Generalizing requires reasoning about some mathematical properties from one class of mathematical entities to another. Creating counter-instances could be a way to reason about the application of conjectures or to generalize to the domain that is extended (MAC-MTL & CPTM, 2012).

When the participants were asked to sketch the graph of the function $y = \sin(x^2)$, they conjectured that the function might be non-periodic because the argument of x^2 is non-linear, and they justified it by the definition of a periodic function. Their algebraic work was based on the definition, and they could not find a constant c such that $\sin(x^2) = \sin(x + c)^2$ for all x because c turned out to depend upon the choice of x. After justifying that the function was not periodic, they also conjectured that the graph would still oscillate between 1 and -1 and repeat more quickly as the input values increase. Using the square root functions, their conjecture was justified by plotting several x-intercepts, which became closer as x increased. They explained that the graph became tighter because the augment x^2 grew faster as x increased.

The participants attempted to use the simplest possible examples or counterexamples when the students' thinking appeared overgeneralized and, hence, mathematically incorrect. For example, Leo said that he often provided students with counterexamples using the special angles to correct the students' incorrect work involving overgeneralization. For example, he used the pair $x = \pi/2$ and $y = \pi/2$ for a student work "$\sin(x + y) = \sin x + \sin y$" to justify why the linearity of the sine function does not generally work and the value of $x = \pi/4$ for "$\cos(1/(\cos(x)) = 1/x$" to justify why the cancellation of "cos" does not make sense.

For Task III.2 (Figure 17), the participants struggled with the cosine value in degrees. To figure it out, they used a strategy of identifying the real value with an angle in radians and converting it to an angle in degrees. To justify their strategy, they made a conjecture that a real number could correspond to an angle in radians and attempted to prove it. Each participant showed slightly different reasoning processes to generalize the relationship between real numbers and angles in radians. For example, Gloria proposed a generalized correspondence between a real number x and an angle θ in radians whose arc length has the length of the real number x on the unit circle (Figure 23):

> If you were to give me any real number and call it *x*, I know what *x* is supposed to represent as soon as I have a number line. So, as soon as I label out one, that determines what my *x* is, so maybe here is my *x* here. Again, I can imagine taking a string and placing it and marking off the length of *x*, and I could take my circle. I want this circle to have a length of one measured from the real number line that I drew, and I could draw an arc length. I could put this dot here and, then, traced it out on the circle until I reached the length of *x*. So, as soon as I do that, I traced out an angle and there is a correspondence between this arc length which I'm calling *x*, and this angle θ….They are not the same, but

there is a one-to-one correspondence…it's precisely that one-to-one correspondence that allows us to describe angles as real numbers. (Interview 3, Task III.2)

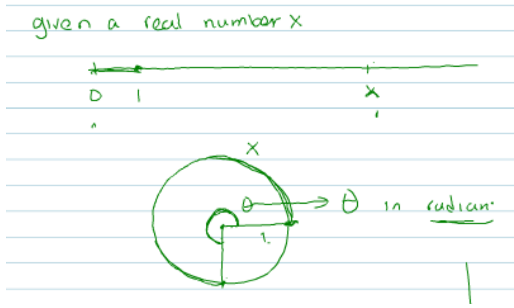


Figure 23. Gloria's depiction of a correspondence of a real number to an angle.

She did not rigorously explain how the correspondence could be one-to-one or what happens when x is negative or large. Although she asserted that a real number x and the corresponding angle θ are not the same, she identified real numbers with angles in radians without justification for solving and explaining Task III.2.

Kyle's attempt to generalize the relationship between real numbers and angles was similar to Gloria's. The only difference was that he considered a circle of radius r, instead of the unit circle. Although he mentioned that "dividing by r" was necessary, he did not discuss how "dividing by r" was applied when solving Task III.2. He identified the cosine value with an angle in radians as Gloria did (Figure 24).

I'm identifying in some way an angle being measured in radians with a real number….Because for radians, that is what we do…to define the measure of an angle in radians is exactly to look at the arc length. So, when I say that this angle is π/2 radians, I'm identifying this angle with this real number, which is the length of that arc. So, this is π/2. I would say every real number represents an angle measured in radians. And, how do you know given a real number what is that angle? Well, you would take…any circle with any radius that you want, and…you would take your real number and see what arc length it cuts off, and I guess divide by r, and that's what we would call this angle. So, to every real number, you can associate an angle in radians with the same measure…. I'm not even mentioning the trig functions when I say when I make this definition. You take your real number and any circle, and you go along a certain distance, and you look at that angle, and that gives you the correspondence between the real numbers and the angles. I haven't mentioned sine or cosine or anything. That is the correspondence. (Interview 3, Task III.2)

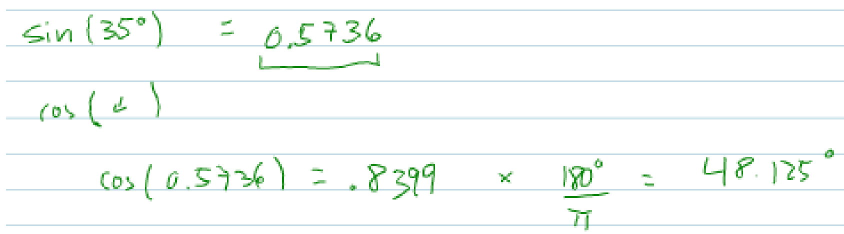


Figure 24. Kyle's work identifying the real number with an angle in radians.

Micah's unique view that the other participants did not mention was that he thought of angles as points on the unit circle. He argued, "You have things like $e^{i\theta}$ and you know it's certainly why I think of angles as points on the complex plane cause, then you can make sense out of this angle being a number" (Interview 3, Task III.1). He described that what it means for an angle to be a number is the pre-image under the map $t \rightarrow e^{it}$ from the real line to the unit circle (Figure 25).

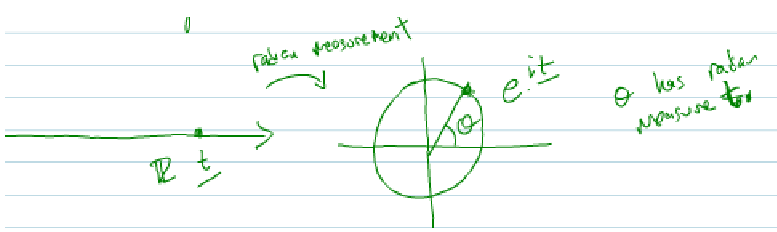


Figure 25. Micah's mapping.

His construction and justification for a mapping to make a connection between real numbers and angles in radians was reasonably close to the idea of construction of the *wrapping function* (see Figure 30) shown in Chapter 5:

> What does it mean for θ to equal a number?...What is an angle?...I would say a point on the unit circle in the complex plane….There is a well-defined map from the real lines to the unit circle. And what do I mean by measuring an angle by a number, I mean, an angle is a point on the unit circle….Angle I am measuring by this number is this image under this map…where the $e^{i\theta}$ is. (Interview 3, Task III.1)
>
> I think if I was asked to be formal about what it means to measure an angle in radians, I would say it's that map [$t \to e^{it}$]. This map is what I mean by measuring angles in radians. So, it's not the identity…..That's a t and then I would say that this angle…θ has radian measure *t*…. that angle is a point on the circle. I'm just calling it by many names, and the set of names that I'm calling it as the pre-image of this point on the circle on this map. And so, I have a bunch of these dots, and I can call this angle by any one of those numbers. (Interview 3, Task III.2)

Although the participants attempted to construct a mapping that represented a relationship between a real number and an angle in radians, their justification was not mathematically rigorous; in other words, they did not discuss how their generalization could work and make sense for *any* real number or for *any* angle in radians. On the other hand, Leo continued to argue that the radian measurement is unitless and considered it a real number without justification, hence he did not talk about the correspondence as did the other participants.

Constraining and extending. Mathematical domains, arguments or classes of objects can be constrained or extended for new mathematics. Teachers need to be aware of the consequences of the effects of constraining or extending mathematical assumptions, ideas, concepts, or properties (MAC-MTL & CPTM, 2012).

The participants exhibited a good understanding of the inverse trigonometric functions and, in particular, their restricted domains and ranges. They explained the one-to-one function as the condition for the sine function to have its inverse mathematically. Leo described the one-to-one function as follows:

> I hit it exactly once. I don't repeat myself….I can sort of limit myself to and on this little portion on the sine graph. It's exactly everything is one-to-one. For every *y* value, there is one *x* value; for every *x* value, there is one *y* value….So we can talk about…one of the rules for sine inverse. (Interview 3, Task III.8)

Micah and Kyle remarked that the sine function is not injective on its domain and that the domain should be cut, although it does not matter where it is cut as long as it passes the horizontal line test; the mathematical convention is to restrict the domain to $-\pi/2$ to $\pi/2$ for its inverse sine function. Kyle described a one-to-one function as a function that "does not repeat itself." At first, he said that the interval for a full period of the graph makes the sine invertible, but he recognized his mistake and corrected himself. Gloria also clearly stated the restricted domain and the range of the inverse sine function and briefly sketched its graph by flipping the sine function with respect to the line $y = x$. Kyle illustrated the meaning of the reflection of the graph across the line $y = x$ as follows.

> What we are really doing is we are switching the *x* coordinate for *y* coordinate on each point…assuming some facts about inverse functions….The domain of the inverse is a range of the function. So, the domain of arcsine is the range of sine. It's between -1 and 1. So my *x* values are between -1 and 1. And, similarly, the range of the sine inverse is

the domain of sine. But we have restricted the domain to make it invertible. And, the domain that I chose here it was $-\pi/2$ to $\pi/2$. So, that's my domain. Alright. That's the graph of the arcsine. (Interview 3, Task III.8)

Mathematical Creating

There are several creative activities in doing mathematics such as representing, defining, modifying, transforming, or manipulating, through which teachers can construct new mathematical entities.

Representing. Creating representations involves selecting a useful form that can convey important strands of mathematical entities with consideration of mathematical structures, constraints, and properties. It is useful to create examples, non-examples, and counterexamples (MAC-MTL & CPTM, 2012). The participants were fluent in dealing with symbolic representations. For example, to formulate the relationship between a number of grads and a number of a new unit called mats, Micah started with different symbols (θ grads and φ mats) for variables and solved an equation of two proportions (θ grads/400 grads = φ mats/100 mats) to obtain a conversion function ($\varphi = \theta/4$) from grads to mats. His symbolic representation of the function was simple and clear.

For Task II.7, Gloria and Kyle labeled various lengths in symbolic forms, figured out the relationships among symbolic expressions and proved the trigonometric identity $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ (e.g., Figure 26).

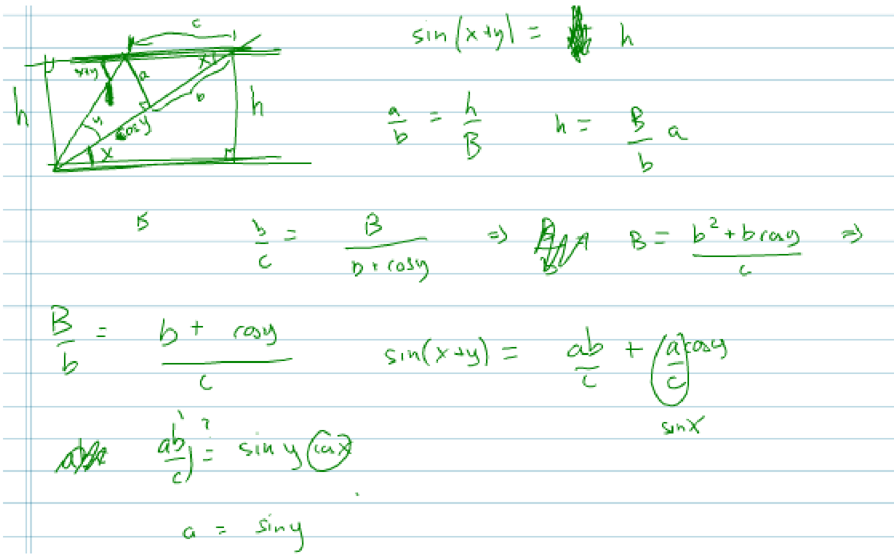
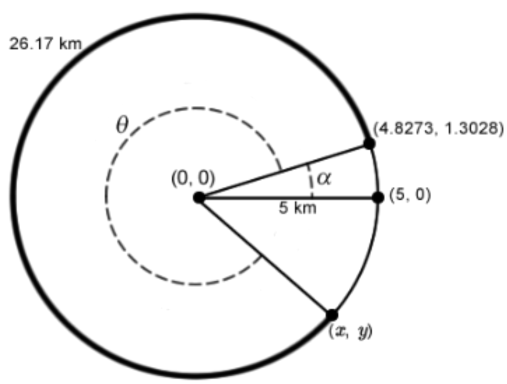


Figure 26. Kyle's manipulation of symbolic expressions.

The participants had no difficulty in representing any point on a circle in terms of trigonometric functions. For example, they used the general symbolic representation (x, y) = ($r$cosθ, $r$sinθ) of a point on a circle with radius r and angle θ to figure out the angles and the coordinates of the given point (x, y) in Task II.2 (Figure 27).

Given the following circle and undetermined angle measures of α and θ radians, answer the following questions.



II.2.1 What is the value of θ in radians, the measure of the angle indicated in the figure above?
II.2.2 How many kilometers did an object sweep out a counterclockwise angle beginning from the position (5, 0) along the circle and ending at the position (4.8273, 1.3028)?
II.2.3 Determine values for the coordinate point (x km, y km) as the ordered pair in kilometers.

Figure 27. Task II.2.

Defining. Teachers need to be able to appeal to definitions to solve problems and reason from definitions (MAC-MTL & CPTM, 2012). The participants were good at reasoning from definitions. They accurately described formal definitions and attempted to convey what a formal definition meant. They tended to refer to formal definitions when they began explaining task items. For example, when Leo was asked to identify key concepts in a given hypothetical situation in Task III, he articulated the formal definition of an inverse function as follows: "If it exists such that *f* inverse composed with *f* of *x* gives me back *x* and *f* of *f* inverse of *x* also gives me back *x*" (Interview 3, Task III.6).

On the other hand, Micah's approach was somewhat opposite that of Leo's for the same task item. He did not start his solution with the formal definition of period. Instead, he attempted to explain its meaning when testing each graph to see if it was periodic.

> I would say start somewhere at the left most point of the bold section, and you want to just trace through the bold section till you get back to where you started. And, then, what you want to ask yourself is, am I about to do the exact same thing as I just did? And, if that is true, then it's a period. And, if it's false, then it's not a period. So, let's test this one. I am going to start here. I'm going to go until I get back to where I started. Now, am I about to do exactly the same thing? Well, yes. So, A is periodic. (Interview 3, Task III.7)

He also used the term "pattern repetition" for the period and briefly described its meaning as "the smallest number that you can add to the argument so that you get the same thing" (Interview 2, Task II.5). After testing each graph, he introduced the formal definition of period as "the smallest number p so that *f* of *x* + *p* is *f* of *x* for every *x*" (Interview 2, Task II.5).

For the definition of an angle in a linear unit of measure generated by a student in Task III.1 (Figure 19), the participants' responses were similar. No one disagreed with the student's definition. For example, Leo pointed out that the definition is neither conventional nor generally accepted in mathematics because the angle measure depends upon the size of the circle. And he suggested revising the definition to make it understandable. He showed partial agreement with the student's definition.

> You can't really measure this angle, can't really be measured in inches, only the arc length can be measured in inches. So, if we assume the radius is one, then that sort of makes sense. But…you would want to correct to say the arc length on a circle of radius one associated to your angle is 1.7 inches. This would be more mathematically correct….This statement [the student's definition] is not exactly true because…for different size circles we have different lengths of things like that. So…if we want to define…an angle can be assigned a unit length of 1.7 inches if the arc length…associated to the angle is 1.7 inches long on a circle of radius one….If you want to define it this way and lock it in this way then as a mathematician, I'm completely ok with that. However, this is not generally accepted….So, this is not a general mathematical convention, but if you want to define this, then we can do that. (Interview 3, Task III.1)

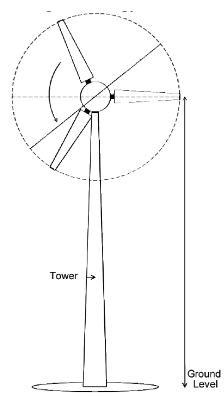
The other three participants agreed with the student's definition in the same manner that Leo described it.

The participants showed a dual concept of the meaning of "one" in the definition of a unit circle, a circle with radius one. For them, it could be either any radius length or a unit length in linear measurement, such as an inch. They provided inconsistent answers in their responses to a question asking whether a given circle is a unit circle. For example, although Gloria agreed that every circle is a unit circle by using the given radius as a unit of measure, she also said that the unit should be redefined each time when considering a unit circle. Micah's dual conception was shown in the following description.

> It doesn't make sense to ask whether or not something is a unit circle without specifying what you are talking about. And if they are supposed to be implying by the fact that these two units were being measured in feet, then what they mean is, the circle of radius 2.8 feet is not a feet unit circle. That is a true statement….But, to say that it's not a unit circle, to me, is a stronger statement than there is no unit for which this is a unit circle. And, that's a false statement. That's a unit circle in the unit 2.8 feet. (Interview 3, Task III.3)

Modifying/transforming/manipulating. Symbolic manipulation is an example of transformations that convey particular mathematical ideas (MAC-MTL & CPTM, 2012). The participants showed fluency in symbolic manipulation regardless of minor computational errors, in particular when they represented a real world application problem. I observed that they exhibited various symbolic manipulations in their work for Task II.4 (Figure 28).

Wind turbines, or windmills, are currently used in an attempt to produce green energy. The wind turbine rotates at a rate of 3 radians every 15 seconds. Let *h* be the height (measured in meters) between the horizontal diameter of the turbine and the ground, so the tip of the fan is *h* meters away from the ground when it is at the 3 o'clock position. Let *r* be the turbine's radius (in meters).



II.4.1 Define a function that relates the measure of the angle (in radians) swept out by the fan blade as a function of time elapsed.
II.4.2 Define a function *f* that represents the distance of the fan blade's tip (in meters) of the windmill above the ground as a function of the number of seconds that have elapsed since the fan started rotating from the 3 o'clock position.
II.4.3 Define a function *g* that represents the distance of the fan blade's tip (in radians) of the windmill above the ground as a function of the number of seconds that have elapsed since the fan started rotating from the 6 o'clock position.

Figure 28. Task II.4.

For example, Gloria started the item by labeling unknowns in the picture before finding their relationship as a function. She represented "one second corresponds to $1/5^{th}$ of a radian" as an angle function $R(x) = (1/5)x$ and $\theta = (1/5)t$, saying, "you input the time and it outputs the radian" (Interview 2, Task II.4). When representing the vertical distance of the fan blade's tip of the windmill above the ground from the 3 o'clock position, she used the sine function and composed the sine function with the angle function to obtain a function representing the distance function with respect to the time. She drew the right triangle to represent and formulate the function starting from the 6 o'clock position (Figure 29).

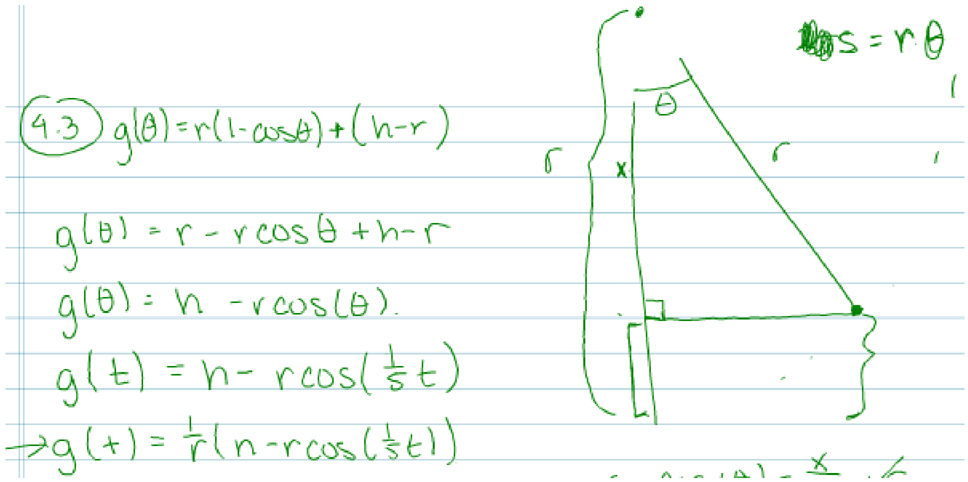


Figure 29. Gloria’s symbolic representation.

Leo used a symbol, T_6, to indicate the starting time at the 6 o’clock position distinct from the conventional 3 o’clock position. He formulated the equation $f(T_6) = h + r\sin(T_6/5) = h - r$ from the function $f(t) = h + r\sin(t/5)$, where the starting time, T_6, was negative $5\pi/2$, saying “this is our new time element” (Interview 2, Task II.4). He used a different function notation (*f* tilde of *t*) because it was slightly modified from *f* of *t*. In the modification of the function *f*, he hesitated as to whether he should use $(t + (-5\pi/2))$ or $(t - (-5\pi/2))$, but finally selected $t + (-5\pi/2)$ from the initial condition that the height should be $h - r$ at time $t = 0$ at the 6 o’clock position.

Integrating Strands of Mathematical Activity

Mathematical modeling requires bringing a real world situation into a formal mathematical system, and in the modeling process, teachers can use all strands of mathematical activity (MAC-MTL & CPTM, 2012).

The participants’ fluency in modeling was observed when they solved a problem on modeling temperatures for Task II. They were familiar with the formal mathematical terms related to a translation of trigonometric functions and fluently explained the meaning for each constant in the function $y = a\sin[b(x - d)] + c$. In addition, they described the effects of the constants on the standard sine function when they found the values of the constants. For example, Micah found the constant *c* by using a line of symmetry and explained, "That is telling us we are shifting this graph up" (Interview 2, Task II.6). He described the meaning of *a* by the using the term "amplitude" and also explained a way to find it saying, "How much does the temperature swing?...It's measured from…the center line,…from the average temperature. What's the swing? We need to know what this line of symmetry is" (Interview 2, Task II.6). He drew the standard sine function to explain the meanings for each unknown constant by comparing it to the given graph. He said that one of the most confusing aspects of the graph translations is the constants *b* and *d* in the argument. He explained that *b* causes the period to expand and contract, hence makes the graph oscillate faster or slower and *d* shifts the graph left or right relative to the original sine function. He mistakenly thought that *b* was the period of the function but later corrected himself using algebraic work from $bx = 2\pi$.

Mathematical Work of Teaching

Mathematical Work of Teaching can be thought of as the "knowing of teaching mathematics" based on teachers' understanding of mathematical content and activities. The goal of teachers' mathematical work of teaching is to facilitate students' development of "knowing" and "doing" mathematics so that students can improve their mathematical understanding (MAC-MTL & CPTM, 2012).

Analyze Mathematical Ideas

Teachers need to be able to analyze complex and condensed mathematical objects and ideas by investigating them and pulling them apart using content knowledge and doing mathematical activities (MAC-MTL & CPTM, 2012). When the participants analyzed students' mathematics occurring in hypothetical situations, they were good at pointing out key concepts, errors, difficult features for students, possible sources for misconception, or related examples. For example, Gloria explained that a possible source for a student's mistake shown in his or her work of $\cos(\sec(x)) = \cos\left(\frac{1}{\cos x}\right) = 1$ in Task III was confusion of the distinction between "number math" and "function math" saying,

> This is something that a lot of students actually do struggle with because they are sort of like number math, so to speak, and function math. So for instance if I were to do 2^{-1} that means 1/2, that means take the reciprocal. In function math f^{-1} means the inverse of the function *f*. (Interview 3, Task III.6)

In addition, she pointed out that the difference between the composition (*f* of *g*) and multiplication (*f* times *g*) would also be difficult for students and provided an example using non-trigonometric functions such as $f(x) = x^2$, $g(x) = x^{1/3} + 1$ to show the difference between the results of the two function operations.

Access and Understand the Mathematical Thinking of Learners

Teachers need to be able to understand what students understand and learn by accessing their work. They need to know how to interpret students' mathematically incorrect explanations and how to help them correct these explanations and guide them to essential mathematical concepts. This proficiency requires a comprehensive understanding of mathematical concepts, reasoning processes, conventions and terminology (MAC-MTL & CPTM, 2012).

Although the participants could not have access to actual students in the hypothetical teaching situations I posed, the participants viewed the students' definitions or arguments carefully through a mathematical lens and expressed their mathematical ideas appropriately in mathematical language. For example, as a response to a student's argument about an angle measure in linear measurement, Leo pointed out that it is not "generally accepted…so this is not a general mathematical convention" (Interview 3, Task III.1) and suggested that it should be redefined so that the argument could be mathematically correctly.

In the situation that involved the unit circle in Task III, Gloria and Micah noted that it would be difficult for students to deal with the units concretely and to understand that the chosen unit does not matter in the concept of a unit circle. They perceived that the students' difficulty in understanding "one unit" of a unit circle might be a result of instructors' not including a unit on drawings of a unit circle, in general.

For the student who argued that sec *x* is the inverse function of cos(*x*) because sec(*x*) = $\cos^{-1}(x)$ and cos(sec(*x*)) = $\cos\left(\frac{1}{\cos x}\right)$ = 1, Micah thought that the student did not understand the basic rules of function algebra, such as function compositions, and said that showing a counterexample using the same function might not be helpful for the student. He provided an example to make a distinction between the function composition and function multiplication using simple polynomial functions: "If *f* of *x* is x^2 and *g* of *x* is x^3, then this is that [the composition] which is *x* to the 6^{th}. But this [the multiplication] is *x* to the 5^{th}" (Interview 3, Task III.6).

Know and Use the Curriculum

Teachers need to be equipped with knowledge to make the curriculum meaningful, connected, and useful so that they can help students achieve curricular goals. They should be able to identify foundational and prerequisite concepts that enhance and support current and future learning (MAC-MTL & CPTM, 2012).

Because I did not observe the participants' teaching practice, it was impossible to collect data about their knowledge of how to use the curriculum. Instead, from the activity for rating the task items in Tasks I and II, I observed their understanding of the curriculum or curricular goals.

The participants showed similar curricular knowledge of angles. They believed that an understanding of how angles are measured is fundamental in trigonometry and that students should be able to work fluently with angles in both measurements–degrees and radians. When Gloria rated the level of difficulty and the importance for Task I.1, she highlighted the usefulness and importance for students to know the angle measures.

> For the students I would say it [Task I.1] is medium [difficulty], maybe even difficult because the conversion between degrees and radians or the conversion/difference between degrees and radians isn't natural, it's not an easy concept for them….You must know both ways of thinking about angles and for the students the same thing, especially, because both conventions exist and are used in different contexts. If they are ever going to have any hope of actually working fluently with angles in either an algebraic context you would probably need radians or in a geometric you probably want to use degrees. So knowing both is important. And going between the two is also important.

In addition, they agreed that understanding radian measure is important because it is used to define trigonometric functions on the real numbers. They also said that a radian measure is useful for calculus because it works a lot better than degrees in calculus. Despite its importance, Leo thought that the concept of radians could confuse many students because it was mathematically "made up out of nowhere" (Interview 1, Task I.1).

Regarding the arc length formula, they thought that deriving the formula could provide motivation for learning the definition of radians because the usual content order in curricula or in precalculus textbooks is that the arc length formula is followed by the radian measure. Despite this, they agreed that students might have trouble reasoning why the arc length formula is true.

Therefore, they believed that knowing how to use the formula for computations is more important than knowing the theory at the level of precalculus.

All the participants valued word problems in real world contexts for trigonometry because they thought that these problems might provide students a physical interpretation of the functions and require them to use intuition, critical thinking, and knowledge. They agreed that solving real world application problems in trigonometry would be a good way to make students think about what mathematics really means in some concrete cases. In addition, Gloria demonstrated that graph transformations are essential to precalculus students, and a modeling trigonometric problem is "a good way to test their understanding of graph transformations with respect to different families of graphs" (Interview 2, Task II.6).

The participants understood the subject of trigonometry in connection with calculus and advanced mathematics in a general sense. Gloria said that trigonometry could be used in calculus, differential topology, geometry, and analysis. Micah commented that trigonometric polynomials and trigonometric expansions are used in advanced mathematics, such as Fourier analysis. Leo listed detailed examples of applications of trigonometry:

> Trigonometry is especially helpful…in further mathematics when you are doing calculus….There are a lot of things you want to figure out like how fast balloons are rising and…how fast like a plane is moving….You could use trigonometry a lot of times…for surveying, for example, if you want to know how tall a mountain is….And they use it…when a cop pulls you over for speeding they actually use lasers and it figures out how fast the angle went there. (Interview 1, Task I.8)

Kyle and Micah mentioned that trigonometry is useful in integral calculus because, for example, trigonometric substitution is a useful method for integral computations.

The participants showed some understanding of how right triangle trigonometry and unit circle trigonometry are connected in related task items, such as Task II.3, and discussed their importance. Micah asserted that understanding the relationship between right triangle trigonometry and unit circle trigonometry is “the heart of trigonometry” and “all of trigonometry is based on these two ideas” (Interview 2, Task II.3). In addition, he emphasized its importance for teaching: “I think it would be hard to teach trigonometry well without understanding, you know, this relationship between triangles and circles” (Interview 2, Task II.3). Gloria described the relationship as follows:

> Right triangle trigonometry just measures the ratios of sides of triangles. But right triangles can’t have obtuse angles trying to generalize sine cosine tangent etc. So, trying to generalize sine cosine and tangent to obtuse angles leads to the idea of reference triangles or reference angles which lead to reference triangles. So the unit circle is really just a generalization of right triangle trig by allowing side lengths to be negative and by relating the obtuse angle to its unique reference triangle. (Interview 1, Task I.8)

Leo observed that “SOH-CAH-TOA” (i.e., Sine: Opposite/Hypotenuse-Cosine: Adjacent/Hypotenuse-Tangent: Opposite/Adjacent) works only for acute angles, and can be generalized to any angle using the concept of reference angles and knowing the change of the signs for the values of trigonometric functions. He mentioned that the mnemonic “All-Students-Take-Calculus and this says that All…is positive, Sine is positive, Tangent is positive, Cosine is positive” (Interview 3, Task III.10). Gloria explained that “All-Students-Take-Calculus” helps students remember how to associate the negative signs with the four quadrants. She remarked that unit circle trigonometry enables students to find the value of trigonometric functions for *any* angle, such as cos(π/2), using the representation (cosine of the angle, sine of the angle) of a point on a unit circle.

The participants believed that knowledge of the history of mathematics is valuable for teachers because talking about historical background might motivate students to learn and increase their interest and enjoyment of the subject. However, they did not think that it is essential for students to know the history of mathematics to learn trigonometry. In fact, they said that they do not talk about the historical aspects of mathematics in their classes unless students ask about it.

Assess the Mathematical Knowledge of Learners

Assessing students' understanding includes a teacher's proficiency to evaluate not only what students understand and learn but also how they use and connect essential mathematical ideas. Teachers also need to be attentive to common student errors and identify students' stages in the learning trajectory to determine students' mathematical progress (MAC-MTL & CPTM, 2012).

The participants noticed and discussed several misconceptions that students had in hypothetical teaching situations. For example, Leo talked about an illegal cancellation–cancelling "cos" in $\cos\left(\frac{1}{\cos x}\right) = 1$ and said that it happens remarkably often in students' work. He suggested that, sometimes, it is more convincing for students to use the example that they generate and then to explain why it does not work. He also noted that students may easily attempt to cancel the sine and the sine inverse in the expression $\sin\left(\frac{1}{2}\sin^{-1}\left(\frac{\pi}{2}\right)\right)$ by saying, "The reason you can't so…is you don't get an answer this way, which means you must have done something wrong somewhere" (Interview 3, Task III.8). He did not explain the mathematical thinking behind the students' work in either of the examples. In contrast, Kyle explained that factoring 1/2 out and the cancellation of the sine and the sine inverse in $\sin\left(\frac{1}{2}\sin^{-1}\left(\frac{\pi}{2}\right)\right)$ might occur in students' work because many students tend to consider composition and multiplication of functions as the same thing.

The participants also commented on specific difficulties that students had in the process of doing mathematics. For example, Micah remarked that students usually have a difficult time placing labels or unknowns on figures when solving word problems and sketching or interpreting functions, especially composed functions, such as $y = \sin(x^2)$ would be difficult or abstract for students. Gloria recognized that many students struggle with making a distinction between number mathematics and function mathematics; for example, they confuse the reciprocal of a number and the inverse of a function due to confusing the superscript notation of -1. Kyle stated that this is why students easily think of the inverse sine of π/2 as 1 over sine of π/2.

The participants also understood and identified difficulty in mathematics learning in a general sense. Leo said, "Students often freeze when they don't see how to do the whole problem at the beginning" (Interview 2, Task II.6). Gloria also explained that even if the concept of a problem is not that difficult for students as a standard problem, when they do not know "how to get started…they might have a hard time….But once they see it they understand it" (Interview 3, Task III.9).

To correct students' misconceptions or errors, the participants identified two major approaches–asking questions or providing counterexamples. For example, in the situation where a student argued that an angle could be measured in linear measurement, Gloria and Kyle chose to respond by asking the student a question to help him or her think about how the definition that the student generated works in the case of a radius other than 1. She said that she could ask, "If this is your circle of radius 5… how would you determine this angle here?" (Interview 3, Task III.1). Leo drew two circles with radius one and two at the same center and showed that the angle should not be measured in inches because it changes depending upon the size of the circle as shown in the drawing. In the task item, Micah did not want to correct the student's argument. Instead of responding to the argument, he attempted to demonstrate his understanding of what the student meant by measuring an angle.

Reflect on the Mathematics in One's Practice

Teachers need to reflect on, analyze, and assess their teaching practice using a mathematical lens to improve their teaching for better student learning (MAC-MTL & CPTM, 2012). In this study I used only hypothetical teaching situations in Task III and did not observe the participants' teaching practices in the classroom. There was not an authentic way to gather data for this strand.

CHAPTER 5

DISCUSSION

In this study about mathematical understanding of graduate teaching assistants for teaching trigonometry in the Department of Mathematics (GTA-Ms), I examined the mathematical proficiency that they exhibited when responding to various mathematical questions and to students' mathematical thinking in hypothetical teaching situations involving trigonometry. This study was guided by the following research questions:

1. To what extent do GTA-Ms exhibit an understanding of trigonometric concepts when solving and explaining trigonometry problems?

2. What understanding of trigonometry do GTA-Ms use in analyzing and responding to students' mathematical thinking about concepts of trigonometry in hypothetical teaching contexts?

This chapter includes responses to the two research questions, based on my analysis and interpretation of the findings.

Response to the First Research Question

Ma (1999) described *fundamental* mathematics as mathematics that is elementary, foundational, and primary: "It is elementary because it is the beginning of mathematics learning. It is primary because it contains the rudiments of more advanced mathematical concepts. It is foundational because it provides a foundation for students' further mathematics learning" (p. 124). She highlighted teaching with a "profound understanding of fundamental mathematics (PUFM)," because teachers with PUFM reinforce "simple but powerful" basic ideas, make connections among concepts, and have multiple perspectives and longitudinal coherence (p. 120, p. 122).

According to the *Faculty Course Outline of Precalculus* of the Department of Mathematics (see Appendix B), the concepts of trigonometry in the textbook belong to one of two kinds of trigonometry–"elementary trigonometry" or "advanced trigonometry." Borrowing Ma's perception of elementary mathematics as fundamental mathematics (1999), I interpret *fundamental concepts* as not only the concepts of elementary trigonometry described in the course outline, but also as *basic core mathematical concepts* for teaching and learning trigonometry.

In this study, the participants' approaches to explaining task items reflected their conceptual understandings of a variety of concepts in trigonometry. This study found that the four participants' explanations and responses to task items were similar to each other's, differing only slightly depending upon the task item. As described in the summary of the findings, they showed a good understanding of the selected concepts of trigonometry, which should be no surprise to many people who think of GTA-Ms as mathematical content "experts" and "qualified" mathematics instructors.

However, this study provided evidence that although the participants showed great mathematical fluency with advanced concepts, they struggled with fundamental concepts of trigonometry. In particular, the findings identified certain issues concerning their understanding of some fundamental concepts, which is consistent with the results of previous research on student learning and teaching of concepts of trigonometry. In this section, I will discuss the nature of GTA-Ms' mathematical understanding for teaching trigonometry in terms of their proficiency in advanced concepts and fundamental concepts as well as challenges they had with some fundamental concepts. I will show examples exhibiting the participants' proficiency as well as areas where they seemed less knowledgeable.

Advanced Concepts of Trigonometry

This section discusses the participants' proficiency with some advanced concepts of trigonometry, which include graphing and modeling trigonometric functions, inverse trigonometric functions, and co-functions.

Graphs of trigonometric functions. The participants had no difficulty graphing the trigonometric functions. When asked to sketch a graph of the composition function $y = \sin(x^2)$, they used a common strategy of observing the graphical behavior of how the graph hits the x axis (which are the x-intercepts) from the argument x^2 and the extreme values of the sine function.

Modeling trigonometric functions. The participants showed considerable fluency in dealing with trigonometry functions for modeling real world situations. When they were asked to determine unknown constants in the sine function $y = a\sin[b(x - d)] + c$ to model the average monthly temperature in a city, they fluently found the constants and explained the meaning and the effect of each constant on the standard sine function graph. They fluently used formal mathematical terminology, such as *amplitude* for the variable a in the function, related to a translation of trigonometric functions.

Inverse trigonometric functions. The participants displayed a strong understanding of inverse trigonometric functions. They were familiar with the definitions and properties of the inverse trigonometric functions, and conceptually explained the graphical features of the inverse trigonometric functions as the reflection of the original function about the line $y = x$.

They showed no evidence of confusion with inverse symbols such as $\cos^{-1}x$ or with reciprocals of trigonometric functions such as sec *x*. In addition, they easily distinguished between multiplication and the composition of trigonometric functions. They even noticed that students easily made the mistake of using the linearity property for trigonometric functions.

Co-functions. The participants had a good understanding of the relationship between sine and cosine functions as co-functions of each other. To find the coordinates of the point $P(\pi/2 - t)$ for a given point $P(t)$ on the unit circle that corresponds to the angle t, they flexibly used not only the addition and subtraction formulas for sine and cosine but also the co-function formula.

These findings demonstrate the participants' considerable proficiency in the advanced concepts of trigonometry, which contrasted with Fi's (2003) findings about pre-service secondary mathematics teachers' knowledge of trigonometry. His participants showed a weak, segmented understanding of these concepts of trigonometry.

Fundamental Concepts of Trigonometry

Brown (2005) observed students' difficulties with the fundamental concepts of trigonometry, which involved the understanding of the three representations of sine and cosine as ratios of sides of a reference triangle, coordinates of a point on the unit circle, and the directed horizontal and vertical distance in unit circle trigonometry, and also of the relationship among these. However, this study revealed that the participants showed considerable proficiency in dealing with these representations of sine and cosine and successfully solved related task items. However, this finding does not mean that they showed the same level of proficiency in all the fundamental concepts of trigonometry.

Hiebert and Carpenter (1992) defined *understanding* mathematical ideas, procedures, or facts as coherently making a connection between its mental representation and an existing internal network of representations. When focusing on the feature of meaningful, conceptual connections in their theoretical notion of *understanding*, I found that the participants showed a weak understanding of some *other* fundamental concepts of trigonometry. Therefore, this section will focus more on discussing the challenges that the participants encountered as they worked with some fundamental concepts, such as interpreting formulas, angle measures, inputs and outputs of trigonometric functions, and unit circles.

Interpreting formulas. In this study, the participants had difficulty conceptually interpreting or explaining the *meaning* of some formulas. The first example was the circumference formula. When asked to interpret the circumference formula, no one meaningfully interpreted it in a way that all circles have a circumference length of 2π radius lengths (for $C = 2\pi r$) or that $2\pi \approx 6.28$ is the number of radius lengths that mark off the circle's circumference (for $C/r = 2\pi$). Instead of *interpreting* the circumference formula, they attempted to describe the symbolic representation or explain the origin or history of the real number π as the ratio of the circumference to the diameter. Gloria explained the division (C/r) as "the ratio of the circumference over a radius" (Interview 1, Task I.2). She understood this interpretation question as a why-question and responded to it that the formula is "organic" in a sense that "it came from nature" and could be "observed from lots of examples" (Interview 1, Task I.2). She rated its interpretation as medium-difficult because the formula is "organic" and "the reasoning isn't as clear maybe to define" (Interview 1, Task I.2). Kyle interpreted $C = 2\pi r$ as, "This formula is telling you how to find the length of going around any circle if you are given…the radius" (Interview 1, Task I.2). Micah described the meaning of the formula from the definition of π:

> I think the definition of π is…half the circumference of the unit circle….So, the fact that the circumference of the unit circle is 2π is really just the definition of π. And then what this formula says is that if I scale the circle's radius then I'm scaling the circumference by the same amount….I would say this [problem] is difficult for me…because it's not clear what "interpret" means. (Interview 1, Task I.2)

Another example of the participants' difficulty interpreting formulas involved the arc length formula. When asked the meaning of the formula, no one conceptually explained it in terms of the multiplicative relationship among the three quantities as follows:

- The formula $\theta = s/r$ describes an angle measure θ (in radians) as representing the number of radius lengths r needed to make up the arc length s of that circle's circumference.
- The formula $s = r\theta$ conveys that a specific length of any arc subtended by the rays of an angle θ can be determined by the radius length times as long as the angle measure θ.

Gloria responded to the question of what $\theta = s/r$ means: "The…length of the arc is given by angle θ divided by…or the proportion…the arc length over the radius" (Interview 1, Task I.4), which was consistent with her response when asked the question about what *C*/*r* meant. Because the arc length formula was related to the definition of the radian measurement, a weak ability to interpret the meaning of the arc length formula was connected to a weak conceptual understanding of radians, in particular, perceiving the length of the radius as the unit of measure. Kyle did not show a conceptual interpretation of the arc length formula either. He responded,

> That *s* is just the product of *r* times θ [$s = r\cdot\theta$]….With the radius 1, the definition of the radians tells us that this length (indicating the arc length corresponding to the angle θ) from here to there is θ. To obtain this picture (indicating a circle with radius of *r*), I am stretching everything by a factor of *r*. The radius gets stretched by *r*, and, therefore, this

length gets stretched by r. So, s…is going to be r times this length, which is θ. So, that's where that comes from. (Interview 1, Task I.4)

Micah repeated the same thing without a meaningful explanation when asked the meaning of the division $\theta = s/r$: "The radian measure of an angle is the ratio of the arc length to the radius. So, in other words, what do you mean by θ? Well, you mean s over r, that means s equals $r\theta$" (Interview 1, Task I.4). Although all of the participants argued that the arc length formula was related to the definition of the radians, they did not relate the argument to justify why radians are unitless. It is noticeable that only Micah briefly mentioned that the radian measure is unitless because he said that it is defined as a ratio s/r of the arc length to the radius length, both have the same units. The other participants used the radian measurement being unitless as a fact without sufficient explanations.

Their weak ability to interpret the formulas (the circumference formula and the arc length formula) seemed related to their limited understanding of the radius length as a unit for measuring the angle in the ratios ($2\pi = C/r$, $\theta = s/r$). Simon and Blume's study (1994) on prospective teachers' understanding of the concept of a ratio showed that prospective elementary teachers had difficulty identifying a ratio as a measure of an attribute or as a representation of a quantitative relationship. For example, the prospective teachers were unlikely to perceive the ratio of height to base, such as 3:2, of a box as indicating the height is 1.5 times the length of the base (p. 195). Simon and Blume (1994) further argued that because "ratio-as-measure" (which refers to "identifying a ratio as the appropriate measure of a given attribute" (p. 184)) "involves the expression of a quantitative relationship between two quantities as ratio" (p. 191), the ability to explain a meaning of the ratio as representation of the relationship requires an understanding of "what it means for a mathematical expression to represent a physical relationship" in a more general sense (p.191). Their notion of “ratio-as-measure” indicates that the participants’ limited ability to interpret the ratios ($2\pi = C/r$, $\theta = s/r$) might be caused by a weak understanding of the multiplicative relationship between the two quantities.

Angle measures. A unit of measurement for the openness of an angle (or arc length) should convey the fractional part of a circle’s circumference cut off by the angle’s two rays, keeping the vertex of the angle located at the circle’s center because the measurement does not depend upon the circle’s size (Carlson & Oehrtman, 2009). An angle in *degrees* means the length of an arc subtended by the angle measured in units of $1/360^{th}$ of a circle’s circumference. Researchers (e.g., Carlson & Oehrtman, 2009; Moore, 2010a, 2010b; Thompson et al., 2007) have asserted that it is more convenient and beneficial to measure angles using a radius length as a unit of angle measure than degrees because a radian measure describes a multiplicative comparison between an arc length and the radius length. In addition, Thompson et al. (2007) argued that a unit of an angle in radius lengths allows inputs and outputs of trigonometric functions to be measured in the same unit. Because 2π radius lengths measure every circle’s circumference, an *angle measure of one radius length* cuts off an arc that is the fractional part ($1/2\pi$) of the circle’s circumference. Therefore, it makes sense to use this radius length as a unit of angle measure, which is commonly referred to as a *radian* (Carlson & Oehrtman, 2009). Thompson et al. (2007) found that “using a circle's radius as a unit of length for measuring arcs” along with "thinking of arc length as angle measure” was beneficial for teachers to “build a coherent system of meanings” of angles and trigonometric functions (pp. 420-421).

The participants did notice the importance for the concept of angle measurement in trigonometry, especially the radian measure. For example, Micah pointed out, “I think that understanding how we measure angles is the fundamental thing in trigonometry….It’s very important for instructors to know this…because I think a lot of students have trouble with radians" (Interview 1, Task I.1). In particular, he emphasized the importance for understanding the radian measure for future study, such as calculus, saying,

> There is no geometric meaning for degrees, whereas a radian has a very specific geometric meaning. It's…measuring angles by lengths on the unit circle. And I think this is particularly important to remember it [radian] because when you get to calculus and you start doing things with trigonometry in calculus,…sine and cosine…are in radians. And…that [radian] is…the more natural way to measure angles. So, I think this is particularly important if you are going to go onto calculus so you really should…forget degrees. (Pre-interview)

However, the participants showed a lack of conceptual understanding of angle measures. Their understanding of the nature of angle measures as the fractional part of the circumference of any circle centered at the vertex was not exhibited in their work on the task items. Their responses to the meaning of radian measures were to explain how wide it was by converting it to degree measures using the conversion factor. For Task III.1 about angle measurement, the participants partially agreed with the student's wrong argument that an angle can be measured in a linear measure, such as 1.7 inches. In fact, if a linear unit, such as inches, is used to measure an angle, the measurement will depend upon the size of the circle centered at the vertex of the angle and, hence, does not convey the fractional part of the circle's circumference regardless of the size of the circle in which the angle is embedded (Carlson & Oehrtman, 2009). However, the participants suggested that the student would be correct if the angle is represented by the coordination of a 1.7-inch arc length with a one-inch radius. No one argued that the student's definition is impossible based on the nature of angle measures.

The findings that the participants showed a lack of conceptual understanding of angle measures confirmed findings from other studies on angle measures. For example, research (e.g., Akkoc, 2008) has shown that a weak understanding of angle measures leads students and teachers to difficulties in learning and teaching trigonometry. Moore (2010) identified precalculus students' multiple incoherent meanings of angle measures. Some researchers (e.g., Akkoc, 2008; Topçu et al., 2006) found that both pre-service and in-service teachers have weak concept images of the concept of radian measurement. Fi (2003) found that although pre-service teachers were good at converting between degrees and radians, they did not show a deep understanding of what a radian measure means because they were not able to describe it as a ratio of the arc length to the radius length.

Inputs and outputs of trigonometric functions. Fi (2003) examined pre-service teachers' abilities to figure out the effect of transformations on the domain and the range of trigonometric functions. His study assumed that the trigonometric functions are real-valued functions. He found that his participants showed a strong understanding of the domain but a lack of understanding of the range of trigonometric functions.

In contrast with Fi's findings (2003), the participants in this study had no difficulty dealing with either the domain or the range of trigonometric functions as real valued functions. For example, when asked the meaning of the input and output values of the sine function in the Ferris wheel context, the participants easily figured out that the input of 1.1 corresponds to an angle in radians and the output of 0.891 corresponds to a length or height and claimed the units of the input and output are unitless.

In connection with the concept of radian measurement, however, they showed a limited understanding of inputs and outputs of trigonometric functions, which confirms the findings from other research (e.g., Moore, 2010a, 2010b; Thompson et al., 2007). Although Gloria considered the output as the ratio of the vertical height with respect to the radius, there was no evidence of her perception of the output as the one measured in radius lengths. The participants' limited understanding of the input and output of trigonometric functions evidently was reflected in their responses to $\cos(\sin(35^\circ)) \approx 48.125^\circ$ in Task III.2 (Figure 17). This task item required a coherent understanding of a measure of an angle as a measurement of the length of arc subtended by the angle, as a fraction of the circumference, or as the number of radii, and the inputs and outputs of a trigonometric function as a measurement in units of a circle's radius. The participants' understanding of the radian as a "unitless" measurement and of the inputs and outputs of sine or cosine functions as "real numbers" was not meaningfully connected with an understanding of a unit of radius length. Because their limited understanding of the angle measure inhibited them from seeing the radius lengths as units for the input and output values of sine and cosine, they struggled with the related task items.

Unit circles. The concept of the unit circle as a geometric object is fundamental in trigonometry. It can be used also as a major organizing tool in trigonometry to generate, for example, the trigonometric values of arguments in the quadrants (Fi, 2003). A unit circle is defined as a circle of radius "one." A key idea in this definition is to understand what a unit is. The unit circle has a radius equal to one unit, hence "one unit" is equal to the radius of the unit circle regardless of the measurement of the length of the radius in any measure. However, the participants in this study exhibited a weak understanding of the connection between the concept of the unit circle and a unit of radius lengths. Their explanations for a unit circle were inconsistent due to their limited understanding of the meaning of "one." For example, Kyle demonstrated the concept of the unit circle as follows.

It’s kind of a *paradox* [italics added].…If you use one unit of measurement like feet then you would say this [a circle with radius of 12 inches] is a unit circle. But…the student is confused because if you say 12 inches, that doesn’t have *radius one* [italics added]. It has *radius 12* [italics added]. So, they are confused about what a unit circle means because they are missing the fact that you have to keep track of the unit of measurement that you are using.…With one unit of measurement it is a unit circle and with another one it is not. (Interview 3, Task III.3)

In fact, he expressed his confusion about the concept by calling it a “paradox” and was unable to clearly explain what “one” in the definition meant. The other participants also showed a similar limited understanding. Gloria responded:

Interviewer (I): Then, is it fair to say that every circle is a unit circle?

Gloria: Sure. But then you would be redefining the unit every time.…What does it mean to have one unit? It would be precisely one would equal the radius. Or, one unit would always equal the radius measurement of the circle, or your favorite measurement of your choice to use. (Interview 3, Task III.3)

The participants’ work on Task II.3 was another piece of evidence of their limited understanding of the unit circle. To determine the value of cos(θ) in unit circle trigonometry (Figure 30), the participants used a scaling factor of 1/5. Kyle said,

This [the given circle in the figure] is *not* [italics added] the unit circle; obviously, this is a circle of radius 5. But let me take that circle and *shrink* [italics added] it down by a factor of 5. I’ll get the unit circle; nothing has changed in the picture, just smaller now. Everything has been divided by 5. (Interview 2, Task II.3)

Although they fluently used the hypotenuse and the radius interchangeably when asked the connection between unit circle trigonometry and right triangle trigonometry, no one determined cos(θ) in unit circle trigonometry in relation to right triangle trigonometry in a conceptual way such that cos(θ) is "the percentage of the radius length made by the length of the side adjacent to the origin in the embedded right triangle" (Thompson et al., 2007, p. 417); therefore, cos(θ) equals 4/5.

II.3.1 Determine cos(θ) (without determining the value of θ) in right triangle trigonometry.
II.3.2 Determine cos(θ) (without determining the value of θ) in unit circle trigonometry.

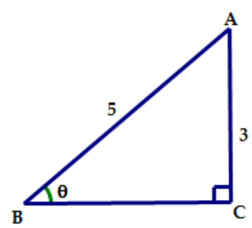


Figure 30. Task II.3.

The participants drew separate circles to represent the unit circle by scaling the radius down or up to one (see Figure 10). Even when Gloria regarded the given radius length as one unit, she drew a separate circle for a unit circle (see Figure 9). These findings were consistent with the recent findings from other studies showing that both students and teachers showed limited understandings of concepts of the radians and the unit circle (e.g., Akkoc, 2008; Moore, 2009, 2010a, 2010b; Moore, LaForest, & Kim, 2012).

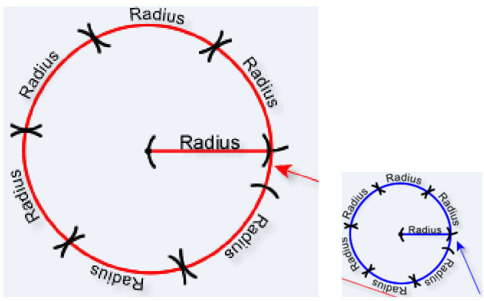
Response to the Second Research Question

When the participants responded to students' thinking in the hypothetical teaching contexts, they exhibited some similar tendencies. For example, they tended to use more procedural fluency than conceptual understanding. Their explanations were equation-oriented and definition-based. They were very good at justifying by manipulating symbolic representations for their logical explanations when they responded to students' arguments. A related example was their frequent use of unit cancellation. As another way of justifying their explanations, they reasoned from formal definitions of concepts. In this section, I will discuss the nature of GTA-Ms' mathematical understanding for teaching trigonometry in terms of their tendencies when responding to students' thinking in hypothetical teaching situations.

Mathematical Proficiency: Procedural Fluency

The participants' challenge with the interpretation of the circumference formula was shown in Task III.5 (Figure 31). Kyle said, "This is not easy to explain for me" in the beginning of his explanation (Interview 3, Task III.5). The challenge seemed to be related to their tendency to explain in a procedural way using symbolic representations rather than in a conceptual way because they showed a strong ability to manipulate mathematical symbols through the task items.

Instructor (You): Tell me about the relationship between the size of a circle and the length of a radius of any circle.
Student A: I think that the larger a circle is, the more radius lengths the circumference has.
Student B: I agree with him/her because the leftover portion on the circumference (where the arrows are pointing in the diagrams below) is longer for a larger circle than the one for a smaller circle.



- How would you describe mathematical concepts key to the situation?
- What might be possible sources of his/her conception?
- How would you correct the students' answers, if necessary?
- What questions or examples would you ask or use to help students better understand the mathematical ideas/concepts involved in this situation?

Figure 31. Task III.5.

For example, to correct the students' answers, Gloria explained using the equalities $L/\mathrm{R} = 2\pi - 6 = l/r$, where L is the length of the leftover of the bigger circle with radius R and l is for the smaller circle with radius r. Leo also used the equalities $C_L/r_L = 2\pi = C_s/r_s$, where C_L is the circumference of the bigger circle with radius r_L and C_s is the circumference for the smaller circle with radius r_s.

Another piece of evidence of the participants' procedural approaches was found in their explanation of periods of a sine function. In a modeling problem for a sine function, the participants almost perfectly described what each constant meant, how it was evaluated, and how each constant affected and transformed the standard sine graph. However, when asked *why* the coefficient b of x in the function $y = a\sin[b(x - d)] + c$ could be obtained by 2π over the period of 12 of the function, they responded in multiple ways which were algebraic rather than conceptual. For example, Gloria and Leo demonstrated that the period should be $2\pi/b$ because $2\pi/b$ makes 2π appear inside the sine function. Gloria explained in detail with a slightly modified sine function $y = A\sin[Bx + C]$:

> The period is given by $2\pi/B$…because if you were to…write this as a function of x, plug in the value of $2\pi/B$, you get A sine of B times $2\pi/B + C$, and you get A sine the B is cancelled $2\pi + C$…sine of 0 is equal to a sine of B times $0 + C$, so you get A sine of C sine is a periodic function and…we go from C to $C + 2\pi$ exactly when we plug in $2\pi/B$. (Interview 2, Task II.6)

Micah responded in an algorithmic way:

> How do you figure this out?…Sine of x has period 2π and the sine of $2x$ has a period of π…because when b was 1, it was 2π, and when b was 2, it was π. So that [the period] is 2π divided by b. (Interview 2, Task II.6)

The participants did not connect the coefficient of *x* (the slope) with the concept of a rate of change for a meaningful explanation. Therefore, no one conceptually explained the relationship of *b* with the period using the concept of the rate of change in such a way that the coefficient *b* of *x* represents the argument change by 2π when the input changes by 12, which gives a rate of change of $2\pi/12$.

Mathematical Reasoning: Justifying

When the participants were asked to help the students obtain the value 48.125º in a hypothetical situation in Task III.2, they did not like the expression $\cos(\sin(35^\circ)) \approx 48.125^\circ$ because they argued that the outputs of trigonometric functions should be real numbers with no unit. However, they reluctantly chose a way to consider the outputs of the sine and cosine functions as angles in radians and converted to angles in degrees. To justify their method, they conjectured and attempted to construct a mapping possibly representing the relationship between real numbers and angles in radians. Although their construction of the mapping was not complete, their mapping was close to what *the wrapping function* (which "wraps" a real number line with origin at (1, 0) around the unit circle) says (Figure 30). It is known that the wrapping function allows "mathematicians to define trigonometric functions with domain and range consisting of real numbers using the radian measure" (Akkoc, 2008, p. 858).

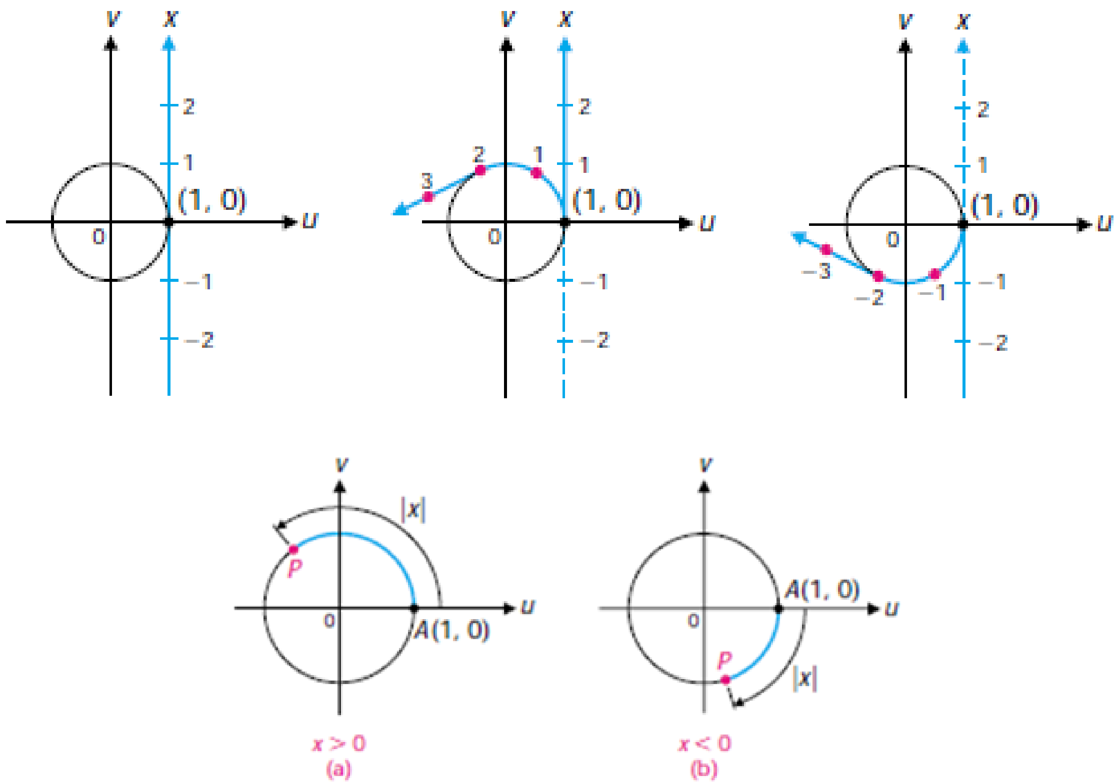


Figure 32. The wrapping function (http://www.mhhe.com/math/precalc/barnettpc5/graphics/barnett05pcfg/ch05/others/bpc5_ch05-01.pdf).

For the wrapping function to make sense, there are three assumptions to be considered: (1) a point on the unit circle should be considered as an angle, (2) an angle measure should be defined in terms of the length of arc formed by the distance from (1, 0) to a point on the unit circle, and (3) the radius of the unit circle should be used as the unit for measuring the arc length. Therefore, any real number x possibly corresponds to an angle represented by a point P on the unit circle whose arc length is $|x|$ units (Figure 30). Here, $|x|$ units would be the distance equal to $|x|$ times the radius length of the unit circle because the unit circle has a radius equal to one unit.

The concept of the wrapping function makes it possible to think that the angles in radians can be considered real numbers as a number of radius lengths, and any real number can be associated with an angle in radians on the unit circle. Because there was no evidence showing that the participants exhibited such an understanding in this study, it was not surprising that the participants could not properly construct the wrapping function to explain the relationship between real numbers and angles in radians measured by radius lengths, although they frequently mentioned angles in radians as real numbers. This finding concerning their limited understanding of inputs and outputs of trigonometric functions was consistent with Thompson et al.'s finding (2007) showing that teachers did not develop a coherent understanding of the concepts of angle measure and sine and cosine functions.

Mathematical Creating: Defining

Definitions of mathematical concepts are known to play a crucial role in mathematics. Typically, advanced mathematics courses are presented in the form of a sequence of *definition-theorem-proof*, which might influence one's thinking about the way mathematics is taught (Vinner, 1991). The participants, as mathematicians, articulated and showed a considerable proficiency in dealing with definitions in their explanations. When they encountered students, who had misconception in hypothetical teaching situations, they tended to emphasize the definition and often started explanations with formal definitions.

For example, for Task II.5, the participants highlighted the formal definition of a periodic function. Gloria's response to a follow-up question was a typical example.

> **Interviewer (I):** If a student asks you that it [$y = \sin(x^2)$] could be a periodic function because sine x squared plus 2π equals sine x squared, then how would you respond?
>
> **Gloria:** The answer is no because *the definition of periodic* [italics added] means that there is a constant c so that f of x is equal to f of x plus c for all x. So that means that you can literally take this graph and shift it….With our function, I'm going to use a functional notation here. What is f of x plus c? Well, that's sine of $x + c$ quantity squared because of

the way of the composition of functions work, so that is equal to sine of $x^2 + 2xc + c^2$. (Interview 2, Task II.5)

When I asked a similar question to Leo, he also started with the definition, as Gloria did.

Interviewer (I): This is an additional question. If a student asks, without looking at the graph, whether this [$y = \sin(x^2)$] is periodic or not, then how will you answer?

Leo: Well, I would say, well, periodic *by definition* [italics added] means that if we have to look up *what that means* [italics added], what it means is that there is a number I'll call it p such that *f* of *x* + p equals *f* of *x* for all *x*. (Interview 2, Task II.2)

In another problem on periods in Task III, he explained the definition and then its graphical meaning.

The first thing [italics added] that we would talk about is what is a period of a function….If *f* has a period, then *f* of *t* + *p* equals *f* of *t* for all *t*…this is an algebraic thing. But, pictorially… the graph can be covered by exact copies of the period by laying them next to one another….without overlap. (Interview 3, Task III.7)

When asked what questions or examples would be helpful to a student who was confused between the reciprocal and the inverse of a trigonometric function, Kyle responded,

For instance, let's do one example what is here is…*f* of *x* equal to *x*….What is the inverse of that function? Well, if you apply the same reasoning [as the student did], here you would say that the inverse is one over *x*. Now let's ask that question, are these two functions inverses of each other? Well, what's *the definition of being an inverse*? [italics added] That means that when you compose them you get *x* in either order. So if I do *f* of *g* of *x* or *g* of *f* of *x* if these truly are inverse functions then in both cases I'm going to get *x*. Let's see. So *f* of g of *x* would be *f* of one over *x*. But *f* doesn't do anything to its input so this is just one over *x*. And particularly this is not *x*. So these are not inverse functions. (Interview 3, Task III.6)

Even when Kyle reflected on how he performed in the task items in the post-task interview, he attributed his challenges to doing and constructing definitions carelessly:

> I now am not comfortable with how good I am at explaining things. Even though I understand them, I cannot explain them sometimes because I just had never thought of very carefully *doing definitions* [italics added] and *constructing all the functions out of those definitions*. (Final reflection interview)

However, Micah had a different opinion about the role of definitions in teaching, saying,

> Would I ever tell a student [struggling with concepts] that they…actually don't know a definition for angle? No, I don't think I would ever say that to a student because that only is going to confuse them. It's only going to make them doubt that they know what they are doing. (Interview 3, Task III.3)

When he explained a solution of the task item on periods, he did not begin with a formal definition of the period. Instead, he explained the meaning in terms of graphs as "A period isn't just some section on which the function repeated itself, it's the smallest section of which the function will always repeat itself" (Interview 3, Task III. 7) to test if each graph is periodic. He introduced the formal mathematical definition of the period at the end to confirm the features of the period.

Keiser (2000) argued that in many cases definitions alone cannot answer students' questions. She explained, "All definitions, by their precise, exclusive nature, may *limit* [italics added] exploration and prohibit questioning because they offer an authoritative 'last word'" (p. 510). She observed that sharing and challenging classmates' ideas and images of a given concept helped her students construct a broader and more complete concept image that they could use in other mathematical contexts. She suggested that students should develop definitions by getting actively involved in the defining process. Therefore, GTA-Ms should learn that providing students with formal definitions might not always be the best way to help them and think about how they could form their class as "an intellectual community" (p. 511) in which students are not passive recipients of the existing definitions, but have ownership of learning and are able to construct and refine their understandings in the defining process.

At this point, it is noteworthy that the tendency that the participants showed regarding the use of definitions in teaching was likely influenced by the "cultural climate in which mathematicians work" (Speer et al., 2005, p. 78). Speer et al. called this phenomenon "the [GTA-Ms'] issue of enculturation" (p. 78), for example, a conflict of teaching styles (e.g., teacher-centered and student-centered). They explained a plausible origin of the issue as follows:

> Pressures to become part of existing culture are strong. Even TAs who arrive in graduate school with substantial concern for undergraduate education and strong motivation to teach may find that holding on to those ideals is incompatible with success as defined by their departments, their faculty mentors, and the discipline as a whole. (p. 78)

Therefore, they suggested that "*proper* [italics added] support and enculturation" are necessary for GTA-Ms' development of "good intentions and practice" in their work of teaching (p. 78).

Mathematical Creating: Manipulating

The participants showed fluency in manipulating mathematical symbols and units, which seemed to support the frequent use of logical explanation for students or their responses to students' thinking in hypothetical situations. One of their common ways to manipulate symbols was to use unit cancellation. Unit cancellation (or dimensional analysis) is commonly used to convert one measure in the given unit to an equivalent measure in the desired unit in mathematics, engineering, and physics (Moore, LaForest, & Kim, 2012). The participants showed considerable procedural fluency in converting from one unit of angle measure to another using a conversion factor. Their dominant approach to conversion was setting up a proportion and solving it by unit cancellation to justify the resulting unit in a solution. For example, Figure 21 showed Gloria's work using a proportion and unit cancellation to convert 10.2 grads to radian measures for Task II.3. However, I did not observe any participants who responded to this problem in a conceptual way without unit cancellation. An alternative, more conceptual, method is that the angle measure of 10.2 grads is 10.2/400 of a circle's circumference in length, and because a circumference has 2π radius lengths (or radians), 10.2 grads is (10.2/400) times as long as 2π radians (i.e., $(10.2/400)\cdot 2\pi$), which obviously is in radians without using unit cancellation.

The participants claimed that unit cancellation is important for calculation. For example, Leo said,

> This unit thing really helps a lot….If the units cancel out correctly then…you are doing things in the right direction and it's not upside down…the units guide you of which way I should write this way or that way…I always try to use these things because I think it makes it less tricky for them [students]… .It's a good thing. But this trusty trick [unit cancellation] right here is what I use all the time. (Interview 2, Task II.1)

Micah highlighted placing units in each step of computation to justify the unit of the final answer by saying,

> Let's write inches just to make sure we are keeping track of units…and be explicit about all of our units… Notice that those grads will cancel and what I end up with is inches and

> that matches with the fact that we were trying to compute a length. (Interview 2, Task II.1)

However, different from the participants' positive opinions about unit cancellation for teaching, some researchers (e.g., Reed, 2006; Thompson, 1994) found that although unit cancellation might be useful to check procedures after an equation has been constructed (as the participants did in this study), unit cancellation is of little help to improve students' quantitative understanding. In fact, Thompson (1994) pointed out that unit cancellation is a mechanical device that does not require students' conceptual understanding of the procedure, as his critique below makes clear:

> We should condemn dimensional analysis, at least when proposed as "arithmetic of units," and hope that it is banned from mathematics education. Its aim is to help students "get more answers," and it amounts to a formalistic substitute for comprehension. (p. 226)

He argued that teachers need to understand that unit cancellation might hinder students' conceptual understanding of the unit conversions and lead to only mechanical computations. Therefore, they need opportunities to think about teaching unit conversion in conceptual ways.

CHAPTER 6

SUMMARY AND CONCLUSIONS

This chapter consists of three sections. The first section overviews this study and includes the purpose of the study, the research questions, research methodology for collecting and analyzing data, and a brief summary of the findings. In the second section, I present conclusions of this study. In the last section, I make some recommendations based on the findings of this study for future research on graduate teaching assistants in the Department of Mathematics and for their professional development.

Summary

Graduate teaching assistants in the Department of Mathematics (GTA-Ms) play significant instructional roles in undergraduate mathematics education (Speer et al., 2005). The purpose of this study was to explore their mathematical understanding for teaching trigonometry because it is known that how teachers understand the mathematical content they teach is integral to teaching and is influential in students' learning (Fennema & Franke, 1992). I conducted this study to answer the following two research questions:

1. To what extent do GTA-Ms exhibit an understanding of trigonometric concepts when solving and explaining trigonometry problems?

2. What understanding of trigonometry do GTA-Ms use in analyzing and responding to students' mathematical thinking about concepts of trigonometry in hypothetical teaching contexts?

The participants in this study were four GTA-Ms who have taught trigonometry as college instructors in the Department of Mathematics at the university where they were pursuing their PhD in mathematics. The data were collected from three task-based interviews with each participant. Each task-based interview was conducted while each participant showed and explained his/her work for given mathematical tasks consisting of several mathematical problems, called *task items*, on trigonometry. This method was useful because it exposed the mathematical understanding that GTA-Ms exhibited in their written and verbal mathematical work while they solved and explained the task items.

The data collected from task-based interviews were analyzed using the Mathematical Understanding for Secondary Teaching (MUST) framework, which characterizes teachers' mathematical understanding useful for teaching mathematics. The findings in this study showed that the participants were proficient in understanding and using advanced concepts of trigonometry, but struggled with some of the fundamental concepts. All participants exhibited an understanding of trigonometry described in the three overarching components of the framework.

The first component–Mathematical Proficiency–is based on the idea that teachers with a deep and thorough understanding of mathematical content are better able to help students develop mathematical proficiency in the process of learning mathematics. It includes Conceptual Understanding, Procedural Fluency, Strategic Competence, Adaptive Reasoning, Productive Reasoning, and Historical and Cultural Knowledge (MAC-MTL & CPTM, 2012).

The participants showed a good understanding of most of the selected concepts for trigonometry. They computed accurately and satisfactorily when solving the task items, which showed their procedural fluency in algebraic and algorithmic work. In addition, they were able to clearly explain why and how the procedures worked. They were good at formulating and representing problems mathematically. They showed similar problem-solving strategies with slight differences depending upon the task items. They used adaptive reasoning in their logical explanations for justifications or proofs of the task items. They even corrected themselves when they found errors or mistakes. There was no doubt from their efforts and persistence in completing task items that they, as mathematicians, had a productive disposition toward mathematics. However, they did not exhibit much historical or cultural knowledge of trigonometry, although they claimed that knowledge of the history of mathematics is valuable for teaching. When they were asked about the history of unit circle trigonometry and right triangle trigonometry, they all guessed that triangle trigonometry developed first and circle trigonometry came afterward, which was not historically true. Their thinking seemed to reflect the order of content in current textbooks of trigonometry in which triangle trigonometry is commonly followed by unit circle trigonometry. Historically, circle trigonometry emerged for astronomical calculations prior to the birth of triangle trigonometry, not fully developed until the 11^{th} century (Bressoud, 2010). Polya (1981) suggested that an understanding of the historical development could help teachers gain an insight into structuring a curriculum.

The second component–Mathematical Activity, can be thought of as the process of "doing mathematics," which involves teachers' mathematical actions to notice, reason, create, or integrate mathematical ideas. The three strands of Mathematical Activity–Mathematical Noticing, Mathematical Reasoning, and Mathematical Creating–contain several sub-strands that focus on teachers' mathematical actions taken, when dealing with mathematical objects (MAC-MTL & CPTM, 2012).

The participants noticed symbolic differences in different mathematical systems. They easily distinguished the notation of the raised -1in the system of numbers from that of the system of functions. They also differentiated between the multiplication and composition of two trigonometric functions. They made a connection between right triangle trigonometry and unit circle trigonometry by identifying the hypotenuse of a right triangle as the radius of a circle. However, I did not observe evidence that this connection led to their perception of the length of the hypotenuse of the triangle as a unit of measure for the sine and cosine values. Through the task items, they showed step-by-step logical procedures for their work, conjectures, and generalization. To justify their arguments, they used analogies, examples, and counterexamples. They showed a tendency to explain concepts more algebraically than conceptually, for example, to determine the coefficient of *x* in the argument of a sine function. They were good at formulating and representing mathematical objects, such as the symbolic and graphic representations of functions or verbal representations using mathematical terminologies. They were familiar with and often mentioned the abbreviation "SOH-CAH-TOA," which stands for Sine equals Opposite over Hypotenuse, Cosine equals Adjacent over Hypotenuse, and Tangent equals Opposite over Adjacent, as the definition of the trigonometric functions of an acute angle of a right triangle. In fact, the abbreviation was introduced in the official textbook that the participants used for teaching in the department as follows: "A mnemonic device for remembering the top row in the definition is SOH CAH TOA, where SOH is an abbreviation for <u>S</u>in(θ) = <u>O</u>pp/<u>H</u>yp, and so forth" (Swokowski & Cole, 2009, p. 372). They accurately stated formal definitions and often reasoned from them. The value that they placed on mathematical definitions was implied by the fact that they often referred to them as they began to explain task. The participants' fluency in modeling by modifying, transforming, and manipulating a sine function was evident when they solved an application trigonometric problem in a real world setting.

The third component–Mathematical Work of Teaching, is teachers' proficiency in mathematical teaching based on their robust understanding of mathematical content and mathematical actions. Teachers should have this proficiency so that they can enhance students' mathematical understanding (MAC-MTL & CPTM, 2012).

Although the participants had to deal with students' mathematical understanding occurring in written hypothetical teaching situations during the task-based interviews, they did exhibit the key strands of Mathematical Work of Teaching in the framework. They were good at analyzing students' mathematical ideas or arguments, and understood how to interpret students' misconceptions and errors. Specifically, they recognized students' possible difficulties or errors. To correct students' misconceptions or errors, the participants used two major approaches–asking questions and providing counterexamples. They agreed that trigonometric word problems in a real world setting might provide students with a physical interpretation of the functions and would necessitate intuition, critical thinking, and knowledge. The participants recognized how trigonometry was connected with other subjects in mathematics, such as calculus and advanced mathematics

Although the participants in this study showed a good understanding of selected concepts on trigonometry that they teach and exhibited all the strands of the MUST framework, they struggled with concepts foundational to learning trigonometry, such as the radian measure and the unit circle. For example, their weak understanding of radius length as a unit of measure created difficulty for them to explain the meaning of "one" in the definition of the unit circle and interpreting the circumference formula.

When the participants used their mathematical understanding in responding to hypothetical student thinking, the findings showed that GTA-Ms tended to use formal definitions, justify arguments using logical, deductive statements, and illustrate ideas by manipulating mathematical symbols. In particular, their explanations tended more procedural than conceptual, and more equation-oriented and definition-based when explaining concepts–not only advanced ones but also fundamental ones–and when responding to students' mathematical thinking that involved misconceptions. For example, they tended to start explaining with a formal definition and use unit cancellation when converting units or justifying the unit of the final answer.

Conclusions

The GTA-Ms in this study were proficient in solving trigonometric problems and using formal approaches to explain trigonometric concepts. Three of the four participants were university outstanding teaching award recipients, who were recognized as "good" instructors by professors, students, and peers. They were thoughtful about their teaching. However, they also struggled with fundamental concepts that may have interfered with their ability to move beyond formal explanations in their approach to address students' misconceptions. For example, the GTA-Ms valued unit cancellation, but Thompson (1994) argued that teachers need to understand that unit cancellation might hinder students' conceptual understanding of the unit conversions and lead to only mechanical computations. In many cases, Keiser (2000) found that definitions alone cannot answer students' questions because providing definitions might limit students' mathematical exploration. She suggested that teachers should help students become engaged in the defining process to construct more complete concept images, instead of providing formal definitions.

Both mathematics teachers and mathematicians need substantial mathematics and mathematical proficiency (NRC, 2001). However, their purposes for this need are different (Ball, 2003; Bass, 1997; Ferrini-Mundi & Findell, 2001; Grossman et al., 1989). Mathematics teachers need mathematical knowledge to use it "to make curriculum decisions, plan lessons, understand their students' work," and facilitate and monitor students' mathematical learning (CBMS, 2001, p. 39). In contrast, mathematicians need mathematical knowledge to pursue a deeper and more professional level of mathematics. They continue to explore and investigate mathematical structures of numbers and spaces, as well as the dynamics of core mathematics; they view mathematics not only as a tool to use for problem-solving but also as a dynamic object that can be explored, created, and justified (Bass, 1997).

Researchers also noted that knowledge of advanced undergraduate mathematics is not easily connected with the school mathematics that teachers teach, and suggested that teachers should develop a deeper and more *specialized* understanding of the mathematical concepts taught, which might *not* be naturally acquired from learning advanced mathematics (e.g., CBMS, 2001; Cuoco, 2001; Ma, 1999). The findings from this study confirmed researchers' arguments that knowing advanced mathematics does not always mean knowing mathematics for teaching (Ball, 2003; Bass, 1997; Ferrini-Mundi & Findell, 2001; Grossman et al., 1989).

There is a Chinese proverb, "To give a student a cup of water, a teacher should have a bucket of water." People have various interpretations of this saying, but I interpret it from this perspective: To teach knowledge of a concept to a student, a teacher should have knowledge far beyond the knowledge that the student will learn. Therefore, the teacher should have a broad, deep, and thorough understanding of the concepts they teach (Ma, 1999).

GTA-Ms, as college instructors, play significant roles and have considerable responsibilities in undergraduate mathematics education. Because being a content expert in a field does not always imply having content expertise for teaching, this study suggests that, for quality teaching, GTA-Ms should develop a deep mathematical understanding of the subjects of undergraduate mathematics they teach.

Recommendations

Based on the findings of this study, I offer recommendations in two areas. I make recommendations for the kinds of studies that would advance research on GTA-Ms, and for methodological approaches. I also make recommendations for planning and implementing professional development for GTA-Ms.

Recommendations for Future Research

This section presents some recommendations for future research on GTA-Ms' mathematical understanding for teaching in terms of selecting of topics and participants and using the MUST framework.

Selection of topics and participants. In this study, I chose the subject of trigonometry because it is rare to find empirical studies on teachers' understanding of trigonometry (Akkoc, 2008; Fi, 2003). Furthermore, studies on GTA-Ms' mathematical understanding of trigonometry for teaching are absent. Currently, GTA-Ms are assigned to teach a variety of undergraduate courses, and, in particular, the percentage of sections of introductory-level courses (including college algebra, precalculus, mathematics for liberal arts, etc.) taught by GTA-Ms has been increasing as enrollments continue to increase (Lutzer et al., 2005; Speer et al., 2005). Because this study found that the participants showed less proficiency in some fundamental concepts than they did in advanced concepts, a future study should explore their mathematical understanding of topics in the areas of introductory-level undergraduate mathematics, such as college algebra, for teaching.

In this study, three of the participants were recipients of the outstanding teaching assistant award. Although they were "good" college instructors officially acknowledged by students, peers, and professors, this study found evidence of their struggles with some fundamental concepts in explaining task items on trigonometry. However, to explore the nature of "general" GTA-Ms' mathematical understanding for teaching, a future study should take the number and backgrounds of GTA-Ms into consideration, when recruiting. The number should be more than four and their backgrounds should be more diverse including international or novice GTA-Ms.

Methodological approaches. I offer some recommendations regarding research methodology for future research on GTA-Ms' mathematical understanding for teaching. First, facilitating focus group discussions with mathematical tasks could be an avenue to observe how GTA-Ms might mathematically react to other GTA-Ms' explanations or reasoning processes while sharing mathematical solutions, strategies, and approaches in a focus group setting. Data from such focus group discussions could reinforce data collected from task-based interviews with mathematical tasks because more mathematical understanding for teaching might be exhibited in a multi-directional setting than in a solving-and-explaining setting.

Second, it would be helpful to conduct task-based interviews with mathematical tasks involving hypothetical teaching situations for a future study. To prompt GTA-Ms to deeply think about mathematics in response to students' mathematical thinking in the hypothetical context, descriptions of teaching situations in tasks should be elaborated with well-posed open sub-questions. Data from class observation of GTA-Ms' actual classroom teaching practices and interactions with their students might supplement or enforce data collected from participants' responses to hypothetical teaching situations and make it easier to describe their mathematical understanding for teaching within the third component–Mathematical Work of Teaching–in the

MUST framework, when using the MUST framework. Visiting classes covering selected concepts for a study and observing review sessions for tests would be useful to collect data about how they use their mathematical understanding to introduce the concepts, pose questions, and respond to students' questions related to the concepts.

Third, semi-structured interviews with GTA-Ms before and after lessons or grading tests could provide useful data about the participants' mathematical understanding in relation to student thinking. In particular, interviews *before* they grade students' mathematical work could help to investigate how they use their mathematical understanding to anticipate and access students' mathematical thinking or misconceptions; interviews *after* they grade students' mathematical work could help to how they use their mathematical understanding to analyze and assess students' mathematical thinking or misconceptions. In addition, rubrics for grading tests that participants construct as well as their feedback on students' work could be also useful data to investigate their mathematical understanding that can be exhibited in relation to students' learning and thinking.

Fourth, the MUST framework is a useful guide for designing task items and a good tool for data analysis for a future study on GTA-Ms' mathematical understanding for teaching, although there is room for the framework to evolve and improve. In this study, the framework helped designing tasks, each of which addresses designated components and strands in the framework. However, data analysis showed that *more* components or strands were found in each task than I expected to observe as well as the components or strands that I anticipated. The tables for comparisons of the strands that I expected to observe and the strands that I actually observed in all the task items are listed in Appendix G. In this study, the framework also facilitated systematic organization and categorization of the mathematical understandings that were exhibited from the participants' responses to mathematical task items during task-based interviews.

Some observations can be made about the framework after using it for this study. For the first component–Mathematical Proficiency, I also considered using Even's framework for teacher knowledge of subject matter because both referred to mathematical content knowledge; in particular, Even's Essential Features and Basic Repertoire provided me with insight into Conceptual Understanding in the MUST framework. Table 11 compares the strands of Mathematical Proficiency in the MUST framework and the aspects of Even's framework.

Table 11

Comparison Between Mathematical Proficiency in the MUST Framework and Even's Framework

<table>
<tr><th>Mathematical Proficiency in the MUST framework</th><th>Even's Framework</th></tr>
<tr><td rowspan="3">Conceptual Understanding</td><td>Essential Features</td></tr>
<tr><td>Basic Repertoire</td></tr>
<tr><td rowspan="2">Knowledge and Understanding of a Concept</td></tr>
<tr><td>Procedural Fluency</td></tr>
<tr><td rowspan="2">Strategic Competence</td><td>Different Representations</td></tr>
<tr><td>Alternative Ways of Approaching</td></tr>
<tr><td>Adaptive Reasoning</td><td>The Strength of the Concept</td></tr>
<tr><td>Productive Disposition</td><td rowspan="2">Knowledge about Mathematics</td></tr>
<tr><td>Historical and Cultural Knowledge</td></tr>
</table>

Because the strands of Mathematical Proficiency are interrelated and intertwined, some data were categorized into more than one strand simultaneously. For example, data showing a participant's Strategic Competence also exhibited his or her Conceptual Understanding and Procedural Fluency because problem-solving requires both kinds of knowledge. It was not easy to make a distinction between Adaptive Reasoning in the first component–Mathematical Proficiency–and Mathematical Reasoning in the second component–Mathematical Activity.

Therefore, data about reasoning were categorized under both strands during data analysis. The strand Productive Disposition was observed throughout the task-based interviews. Therefore, it is suggested that data about this strand should be collected deliberately through the process of collecting data in future studies rather than by asking a few task items.

Among the strands for the second component–Mathematical Activity, Mathematics Noticing was not easy for me to observe because it occurs internally. For example, Discerning Symbolic Forms is recognizing symbolic forms to identify potential symbolic rules with those forms, which is distinguished from symbolic manipulation for Mathematics Creating. Therefore, it could not be easy to make a clear-cut distinction between the participants' discerning symbolic forms and manipulating mathematical symbols that were exhibited in their mathematical work.

The third component–Mathematical Work of Teaching–distinguishes between mathematics teachers and mathematicians. In this study, the participants' responses to the hypothetical teaching situations in a written form provided limited data about their mathematical understanding for teaching within the third component. Therefore, this component could be more useful for analyzing data from GTA-Ms' planning, teaching, and reflecting on lessons.

Recommendations for Professional Development

The findings from this study, describing the nature of GTA-Ms' mathematical understanding of trigonometric concepts for teaching, could serve as a baseline or groundwork for designing professional development. Those designing professional development for GTA-Ms could provide better preparation of GTA-Ms as college instructors if they included a focus on fundamental concepts, attention to developing conceptual understanding, and tasks involving hypothetical teaching situations. Based on the findings from this study, I offer the following recommendations.

First, professional development programs should enhance and deepen GTA-Ms' mathematical understanding of *fundamental concepts* of the subject they teach. Although the participants demonstrated a strong understanding of the advanced concepts of trigonometry, the findings showed that their understandings of some fundamental concepts were limited and not meaningfully connected. When considering that an understanding of fundamental concepts they teach might not naturally be constructed or acquired from studying advanced mathematics, it is essential that professional development programs should help GTA-Ms develop fundamental concepts of the topics in the subject area that they teach.

Second, GTA-Ms should acquire a conceptual understanding of the mathematics that they teach which will enable them to teach more than mathematical procedures. Professional development should involve the GTA-Ms in developing teaching strategies that go beyond demonstrating procedures. In this study, despite proficiency in advanced concepts, the participants' approaches to explaining task items tended to be procedural rather than conceptual. One example was their frequent use of unit cancellation. In the conversion process, they often utilized a traditional method of unit cancellation to justify the unit for their final answer (e.g., Figure 22). Leo, for example, claimed that "unit cancellation is important for calculation.…I *always* [italics added] try to use these things because I think it makes it less tricky for them [students]" (Interview 2, Task II.1). However, researchers (e.g., Reed, 2006; Thompson, 1994) argued that teachers need to be cautious to use unit cancellation because algorithmic computations of unit cancellation might hinder students from developing conceptual understanding of unit conversion.

The participants in this study agreed with the importance for GTA-Ms' acquiring conceptual understanding for teaching. The following remark from Gloria reflects her belief that teaching requires more than possession of knowledge of the content taught.

> I think if you are teaching anything, you have to have a *deep knowledge* [italics added] of the material….You have to be willing to expect unexpected questions and when you have unexpected questions…you need to rely on a really *thorough understanding* [italics added] of the material to give a justified answer….So, you need to have an understanding of something to justify it. (Final reflection interview)

Therefore, one of foci in GTA-Ms' teaching preparation should be placed on developing their conceptual understanding.

Third, professional development programs should use task items similar to those used in this study. In this study, task items–both those involving teaching situations and conceptual task items–used during task-based interviews prompted the participants' mathematical thinking and had the potentials to broaden and deepen their mathematical understanding.

Although the participants in this study were not exposed to actual teaching situations, the items involving hypothetical teaching situations prompted the participants to think about mathematics in relation to students' mathematical thinking. Biza et al. (2007) explained the importance of such tasks:

> The tasks offer an opportunity to explore and develop teachers' sensitivity to student difficulty and needs…as well as an ability to provide adequate…feedback to students. Particularly by asking the teacher to engage with a specific (fictional yet plausible) student response that is characterized by a subtle mathematical error we can explore not

only whether the teacher can identify the error but probe into its causes and grasp the didactical opportunity it offers. (p. 303)

The participants in this study also mentioned that they had an opportunity to think about what they had not thought about by solving and explaining some conceptual task items. For example, when Kyle talked about the unit circle, he asked himself, “What is this ‘one’ [in the definition of the unit circle]? What unit is that ‘one’ measured in? I’ve never thought of that” (Interview 1, Task I.1). In the final reflection interview after completing the task-based interviews, he added, “Most of these [task items] were very…creative questions. I think that really *made me think* [italics added]….Many of these questions I had never thought about…were just unexpected to me. They were not standard questions” (Final reflection interview). Micah also noted, “I’ve learned that trig questions can be much more difficult than they are in our course. And I think it would be great to get to a place where these were the kinds of questions that we were asking” (Final reflection interview).

This study also found that the participants improved their explanations through doing the task-based interviews. For Task I, for example, Gloria explained the degree measure by dividing the circumference by 360 tick marks and counting tick marks along the circumference, which was not a conceptual interpretation of degree measures in terms of the fractional amount of a circle’s circumference. However, for Task II, she described the degree measure in this way: “A degree is…the angle swept out by going $1/360^{th}$ of a circle’s circumference,” (Interview 2, Task II.4), which was more conceptual. She seemed to make self-improvement as she struggled with the meaning of angle measure, while working on the task items. Leo also showed some improvement in his interpretation of the circumference formula because he provided a better and conceptual explanation of it for Task III than he did for Task I. For Task I.2, his interpretation of the circumference formula was the following:

> The meaning of this [the circumference formula] is saying that…if you want to know how far this distance [the circumference] is,…we could take a piece of string and measure around here, but we can't do that very easily. So, we are going to have this formula instead, which always…tells us…this is always true for any circle in the whole world, we know the distance around here is $2\pi r$. (Interview 1, Task I.2)

However, when he was forced to think about the meaning of the division ($C/r = 2\pi$) in Task III.5, he explained it conceptually as follows:

> **Interviewer (I):** Related to this division, could you explain how we could have six marks off here?
>
> **Leo:** This idea of division is…that…I'm going to split this into a number of pieces equal length pieces….This piece is one and this is *a set length* [italics added]. And this is another piece of length and this is another piece of length and I just go around I count and so it is like 1, 2, 3, 4, 5, 6 and then I have this little bit extra which is the remainder….What I'm doing is I'm actually starting with *a set length which is the radius* [italics added] and then I'm seeing how many times can I make it around the whole circle….It's always six and a little bit more. And no matter what, it's always…the same. (Interview 3, Task III.5)

His description showed that he seemed to perceive the radius as a unit of measure and 2π ("6 and a little more") as the number of radius lengths measuring the length of the circumference of any circle.

Although this study investigated the trigonometric understanding of talented GTA-Ms, the findings offer points to be considered in designing professional development for GTA-Ms preparing to teach other topics in mathematics. GTA-Ms, who understand fundamental concepts, value conceptual learning as well as procedural learning, and have studied hypothetical teaching situations, are likely to develop the deep mathematical understanding needed to enhance the learning of the undergraduate students they teach.

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APPENDIX A

CONCEPTS AND MUST FRAMEWORK FOR EACH TASK ITEM

Task Items	Concepts	Expected Strands
	Task I	
I.1	Angle measure/Radians/Degree	MP: Conceptual Understanding
I.2	Angle measure/Radians/Degree Circumference formula	MP: Conceptual Understanding
I.3	Angle measure/Radians/Degree	MP: Procedural Fluency Adaptive Reasoning
I.4	Arc length formula	MP: Conceptual Understanding
I.5	Sine/cosine functions	MP: Conceptual Understanding
I.6	Sine/cosine functions Units/unit conversion	MP: Conceptual Understanding
I.7	Sine/cosine functions	MP: Conceptual Understanding
1.8	Unit circle trig and right triangle trig	MP: Historical and Cultural Knowledge
	Task II	
II.1	Angle measure/Radians/Degree Arc length formula Units/unit conversion	MA: Mathematical Noticing – Observing Structure of Mathematical Systems Mathematical Creating – Defining
II.2	Angle measure/Radians/Degree Arc length formula Sine/cosine functions Inverse trig functions	MP: Procedural Fluency Strategic Competency MA: Mathematical Noticing – Discerning Symbolic Forms
II.3	Sine/cosine functions Unit circle trig and right triangle trig	MA: Mathematical Noticing – Observing Structure of Mathematical Systems – Connecting within and outside Math
II.4	Sine/cosine functions Composite functions	MA: Mathematical Creating – Representing – Modifying/Transforming/Manipulating

II.5	Sine/cosine functions Composite functions Periodic functions/Period	MP: Adaptive Reasoning MA: Mathematical Reasoning – Constraining and Extending – Conjecturing and Generalizing
II.6	Sine/cosine functions Modeling trigonometric functions	MA: Integrating Strands of Mathematical Activity
II.7	Addition formula for sine functions	MA: Mathematical Reasoning – Justifying/Proving
	Task III	
III.1	Angle measure/Radian/Degree Units/unit conversion Arc length formula	MA: Mathematical Noticing – Detecting the Form of an Argument MWT: Analyze Mathematical Ideas
III.2	Sine/cosine functions Composite functions	MA: Mathematical Noticing – Detecting the Form of an Argument – Observing Structure of Mathematical Systems MWT: Analyze Mathematical Ideas Assess the Mathematical Knowledge of Learners Know and Use the Curriculum
III.3	Unit circle	MA: Mathematical Noticing – Detecting the Form of an Argument MWT: Know and Use the Curriculum Assess the Mathematical Knowledge of Learners
III.4	Angle measure/Radian/Degree Units/unit conversion Arc length formula	MA: Mathematical Noticing – Detecting the Form of an Argument MWT: Assess the Mathematical Knowledge of Learners
III.5	Angle measure/Radian/Degree Circumference formula	MA: Mathematical Noticing – Detecting the Form of an Argument MWT: Assess the Mathematical Knowledge of Learners
III.6	Inverse trig functions Composite functions	MA: Mathematical Noticing – Detecting the Form of an Argument – Discerning Symbolic Forms

		MWT: Assess the Mathematical Knowledge of Learners
III.7	Periodic functions/Period	MWT: Assess the Mathematical Knowledge of Learners
III.8	Inverse trig functions Composite functions Sine/cosine functions	MA: Mathematical Noticing – Discerning Symbolic Forms MWT: Assess the Mathematical Knowledge of Learners
III.9	Sine/cosine functions Co-function formula Addition formula for sine/cosine functions	MA: Mathematical Noticing – Discerning Symbolic Forms MWT: Assess the Mathematical Knowledge of Learners
III.10	Sine/cosine functions Unit circle trig and right triangle trig	MWT: Know and Use the Curriculum

APPENDIX B

FACULTY COURSE OUTLINE OF MATH1113 (PRECALCULUS)

Text: Swokowski-Cole, Precalculus: Functions and Graphs, 12th ed., Cengage Publishing
Remark: there is the regular text from Cengage and our custom-printed text which has fewer chapters and it is cheaper (comparing new to new) to students. 11th edition is also acceptable.

Unit 1: The Cartesian plane, interval notation, midpoint and distance formula, circles (including complete the square), graphs of equations, their intercepts and symmetry tests; lines and linear models, the definition of function, identifying functions, computing function values, function domains and ranges, difference quotients, linear functions, modeling functions (find geometry figures on back endpaper of text, PLUS they should know a box and rectangles of course)
Sections 2.1, 2.2, 2.3, 2.4
The text HW 2.2 doesn't have many symmetry tests—do that in WebAssign

Unit 2: Graphs of functions, even/odd functions, shifts, reflections, or stretching/compressing of graphs; (opt: greatest integer function and) absolute value functions (complete list/pix of fns we expect students to be familiar with on p. 790-791, appendix I), quadratic functions (general and standard forms, completing the square to get to standard form), extreme values of quadratics (some word problems here), operations on functions, modeling and interpreting function models, one-to-one functions and their inverses (graphically and symbolically) Sections (1.4), 2.5, 2.6, 2.7, 4.1

Unit 3: Exponential and Logarithmic functions and applications. Definitions, domain/range and graphs (including shifts and reflections), the number e, using exponential models, definition of log functions, exponential and logarithm properties, modeling with exponential and logarithmic functions, including business models, and solving equations involving exponentials and logarithms. (We don't do the logistic curve or limit-type questions) NOTE: we do ONLY $-type applications first, then after 4.6 we look at additional models and see the same principles at work. Sections 4.2, 4.3, 4.4, 4.5, 4.6

Unit 4: Elementary trigonometry: Angle measure using degrees and radians, arc length and sector area (no angular motion), right triangle trigonometry and extension to arbitrary angles (using $P(x,y)$ on the terminal side), reciprocal and Pythagorean identities, trigonometric functions of real numbers (a brief mention of the unit circle), graphs of the six trigonometric functions and domain/range, computation of trigonometric functions of arbitrary special angles via reference angles (no mention of that word on assignments). Some applications using trigonometric functions. **Sections 5.1, 5.2, 5.3, 5.4**

Unit 5: Advanced trigonometry: Analyze and modeling functions of the form $y = A\sin(bx+c)$ (or cosine). Applications involving right triangle trigonometry, depression/elevation, triangle area (this is an addition here because it is covered in LOS/LOC section, but we do early). Solving trigonometric equations (special angle answers), addition and double angle formulas for sine and cosine. **Sections 5.5,(5.6),5.7, 6.2, 6.3, 6.4**

Additional study before the final exam: Inverse trigonometric functions, the graphs, domains/ranges, properties, computations, and uses for solving trigonometric equations (arctan(5) type answers), Laws of Sines and Cosines and their applications (bearings covered here). **Sections 6.6, 7.1, 7.2**

Addendum: Intensive Precalculus sections
Additional content covered from sections 1.1-1.4 of the text: real numbers and interval notation, definition of and working with absolute value, order of operations and laws of exponents, operations involving exponents: radicals and fractional exponents, definition of polynomials, factoring polynomials, solving linear, quadratic, and rational equations. Most of these concepts are introduced as needed within the context of the content studied in the individual units of the precalculus course.

Testing Information: Tests are taken outside of class time, in Room 222 or 324. Students sign up on the 1113 webpage. See the 1113 webpage for a general description of test protocol: calculators, ID, scratch, honor code, no phones/hats, etc. Each test is 12 questions in 75 min. The final exam is 24 questions in 165 minutes. Coach your students to use WebAssign effectively during a test: 1. you can minimize the timer if it is worrisome 2. Submit each part of each question separately, since you can repair errors and minimize your penalty 3. WebAssign is CASE SenSiTivE, and can't read your misspellings correctly (*x* for *p*, for example), but you the teacher can override such an error later (should you so choose). 4. Never use the Back Button.

Test 1 content items: Working with distance and midpoint formulas, finding/using equations of circles, completing the square to find centers/radii, intercepts of graphs, observing symmetry (just observing, not computational yet), linear equations from information, parallel/perpendicular lines, finding/comparing/using slopes, finding/using intercepts of lines, horizontal/vertical lines, linear models, function computations: graphical, tabular, numeric like $f(2)$ and symbolic $f(1/x)$, difference quotients, domains of functions, building a functional model. Special notes: quadratic equations at this point lend themselves to solving without the quadratic formula (although you can of course), and some problems involve multiple-step algebra, as we are preparing students for calculus. There are 3 word problems, one linear, one experiential, one unseen by students (so watch what you show them–please stick to the hw and wq problems). The content is in order, except all word problems are at the end of test. Note that kids tend to memorize problem solution types rather than methods and this can be a problem, as they will memorize the answer to a cylinder problem and see a cylinder with new information (volume fixed instead of area, for example). This is a good thing to point out, as "there are many questions you can ask about this cylinder/cone/box, so don't expect that the image is only attached to the problem you drew this time."

From the testing coordinator:
Some testing protocol. In this case, “we” means my proctors.

1. Students will need to remember their username/password. A lot of these kids save their login information on their computers and thus do not type these information regularly. They will not be able to reset their passwords because email (or any email server for that matter) is blocked. If they come in not knowing their password, I send them away.

2. We accept the following forms of photo identification: student ID, state licensed drivers license, federally issued passports. Pretty much anything that has a photo identification.

3. We will not be allowing any handheld calculators of any kind. If your student does not want to use the computerized vti-83, then they can make do without a calculator.

4. We will not be providing syntax help on WebAssign or calculator tips during the exam. Please stress this fact to your students that they need to be practicing both WebAssign inputs and ti-83/ti-84 keystrokes.

5. Like their webquizzes, students get multiple tries on their test. However, there is no third try for tests. I've had a few students in the past asking about this. Remember its full credit for the first submission, 75% credit on the second submission.

6. Students will need to regularly submit their work. Any un-submitted work that is lost for any reason (computer freezing, power outages, etc.) is unrecoverable.

Makeup Exam protocol:
1. It is up to you to decide whether a student should be allowed to take his/her test outside of the two days that they are scheduled to do so. This normally means that they provide you with a written note. Students can take their test after their scheduled test days, but not before.

2. If you deem that a student has valid reasons for taking a makeup test, I will need a notification from you. The easiest way to do this is by email. Something along the lines of: “Please allow John Doe to make up Test n.” Your student will then need to contact me as soon as possible so that we can schedule his/her makeup test.

3. Makeups are generally given the four school days immediately after Group 2's second day. I will let you know the hours I'm available to administer these exams as the semester progresses.

4. I'll repeat this again later, but any student that misses Test 5 will have to take it AFTER Thanksgiving Break. Rules 1, 2, and 3 still apply; we will not be administering exams before their scheduled days. Students in group 2 that book their flights early will have to suck it up. They will also need your permission and it will be your call whether or not skipping town early is a valid reason for missing their test.

APPENDIX C

TASKS USED FOR THE TASK-BASED INTERVIEWS

Task I

Date: ______________________ Name__________________________

Instruction

- Task I consists of 8 problems.
- Suppose each question were in a textbook and you as an instructor needed to explain it to students. Please tell me how you would approach explaining an ideal answer to a student.
- After you finish explaining each problem, please classify the level and rate the importance for each task item with reasons.
  - Please classify each item as easy (**E**), medium difficulty (**M**), or difficult (**D**) for *you* (as an instructor) and for *your students*, explaining the reason for your classification.
  - Please rate (**1**: **less** important; **2**: important; **3**: **most** important) the importance for each item for *you* (as an instructor) to know and for *your students* to know, explaining the reason for your rating.

I.1

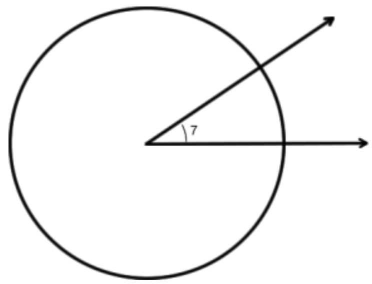
I.1.1 What does it mean for an angle to have a measure of 10 degrees?
I.1.2 What does it mean for an angle to have a measure of 2.3 radians?

I.2

Interpret the circumference formula $C = 2\pi r$, where r is the radius length of any circle.

I.3

A student measured the angle displayed below and determined that its measure was 7. However, he did not label the units in which he measured the angle. The unit he measured in is not grads, radians, or degrees. Describe how to use the arc length and circumference of the circle displayed below to determine how many of the "mystery" angle measure units mark off (or cut off) any circle's circumference.

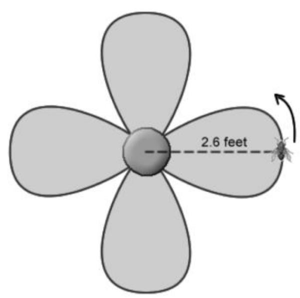


I.4

A circle has a radius of r inches. An angle θ whose vertex is at the circle's center cuts off s inches of the circle's circumference as the terminal side of the angle opens in a counter clockwise direction from the initial side of the angle.
Write a formula that conveys the relationship between the radius length *r* (in inches), the angle measure θ (in radians), and the arc length *s* (in inches).
Explain the meaning of the formulation of the relationship between three quantities.

I.5

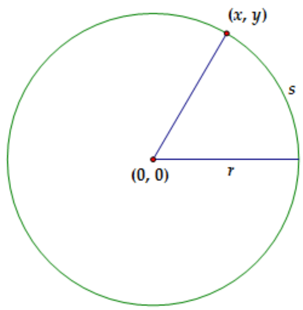
Imagine a bug sitting on the end of a blade of a fan as the blade revolves in a counter-clockwise direction. The bug is exactly 2.6 feet from the center of the fan and is at the 3 o'clock position as the blade begins to turn.



I.5.1 Sketch a graph of a function *f* over the input interval from 0 to 2π to illustrate how the bug's vertical distance (in radians) *above* the horizontal diameter co-varies with the measure of the angle swept out the bug's fan blade (in radians).[4] Justify the shape of the graph.
I.5.2 Determine symbolic representations of the function *f* in part **I.5.1**.

[4] This wording should be changed to *in radius lengths* or *in radii* to clarify the intent.

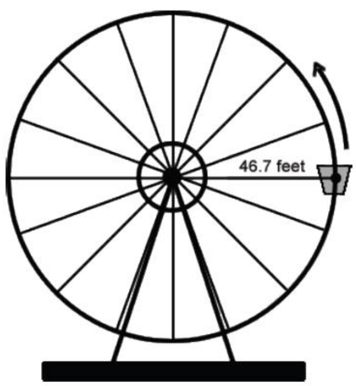
I.6



I.6.1 What is the general form of (x rad, y rad) as the ordered pair in radians,[5] of any point on a circle of radius r kilometers that forms an arc length of s kilometers as illustrated on the diagram? Why?
I.6.2 What is the general form of (x km, y km) as the ordered pair in kilometers, of any point on a circle of radius r kilometers that forms an arc length of s kilometers as illustrated on the diagram? Why?

I.7

John is sitting in a bucket of a Ferris wheel. He is exactly 46.7 feet from the center and is at the 3 o'clock position as the Ferris wheel starts turning.



I.7.1 What does the input value of the sine function $\sin(1.1) \approx 0.891$ represent in this context?
I.7.2 What does the output value of the sine function $\sin(1.1) \approx 0.891$ represent in this context?

I.8

I.8.1 Tell me about the historical development of unit circle trigonometry and right triangle trigonometry.
I.8.2 Tell me about how trigonometry connects with areas within and outside mathematics.

[5] This wording should be changed to *in radius lengths* or *in radii* to clarify the intent.

Please rate the level and the importance for each problem with reasons.

I.1

Level	E	M	D	Reason
For you				
For students				

Importance	1	2	3	Reason
For you				
For students				

I.2

Level	E	M	D	Reason
For you				
For students				

Importance	1	2	3	Reason
For you				
For students				

I.3

Level	E	M	D	Reason
For you				
For students				

Importance	1	2	3	Reason
For you				
For students				

I.4

Level	E	M	D	Reason
For you				
For students				

Importance	1	2	3	Reason
For you				
For students				

I.5

Level	E	M	D	Reason
For you				
For students				

Importance	1	2	3	Reason
For you				
For students				

I.6

Level	E	M	D	Reason
For you				
For students				

Importance	1	2	3	Reason
For you				
For students				

I.7

Level	E	M	D	Reason
For you				
For students				

Importance	1	2	3	Reason
For you				
For students				

I.8

Level	E	M	D	Reason
For you				
For students				

Importance	1	2	3	Reason
For you				
For students				

Task II

Date: ____________________ Name________________________

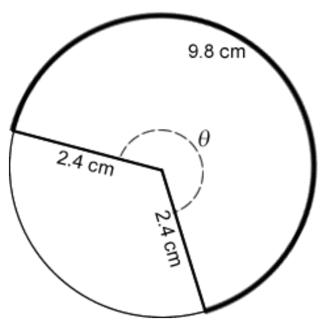
Instruction

- Task II consists of 7 problems.
- Suppose each question were in a textbook and you as an instructor needed to explain it to students. Please tell me how you would approach explaining an ideal answer to a student.
- After you finish explaining each task item, please classify the level and rate the importance for each task item with reasons.
  - Please classify each item as easy (**E**), medium difficulty (**M**), or difficult (**D**) for *you* (as an instructor) and for *your students*, explaining the reason for your classification.
  - Please rate (**1**: **less** important; **2**: important; **3**: **most** important) the importance for each item for *you* (as an instructor) to know and for *your students* to know, explaining the reason for your rating.

II.1

The *grad* is a unit of angle measure that is sometimes used in France, where every circle's circumference is 400 grads.

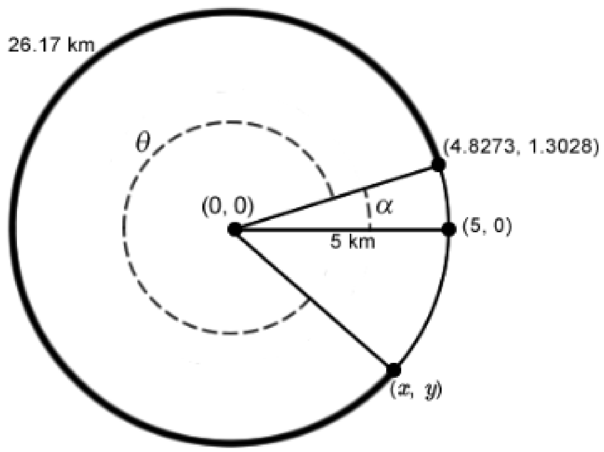
II.1.1 Determine θ in grads in the following figure.



II.1.2 If a circle has a radius 7.1 inches, what is the arc length in inches of the angle of 3 grads?
II.1.3 How many radians are equivalent to 10.2 grads?
II.1.4 Name your own unit of angle measure and define how many of these units mark off the circumference of a circle so that you can create a protractor to measure any angle in your unit. Describe the meaning of an angle of 18.2 (name of your unit).
II.1.5 Define a function that converts a number of grads to a number of your unit. Explain the meaning of the formula.

II.2

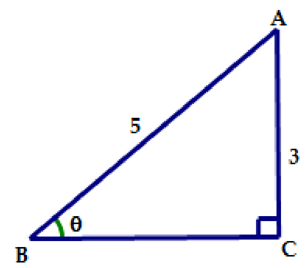
Given the following circle and undetermined angle measures of α and θ radians, answer the following questions.



II.2.1 What is the value of θ in radians, the measure of the angle indicated in the figure above?
II.2.2 How many kilometers did an object sweep out a counterclockwise angle beginning from the position (5, 0) along the circle and ending at the position (4.8273, 1.3028)?
II.2.3 Determine values for the coordinate point (*x* km, *y* km) as the ordered pair in kilometers.
II.2.4 Determine values for the coordinate point (*x* rad, *y* rad) as the ordered pair in radians.[6]

II.3

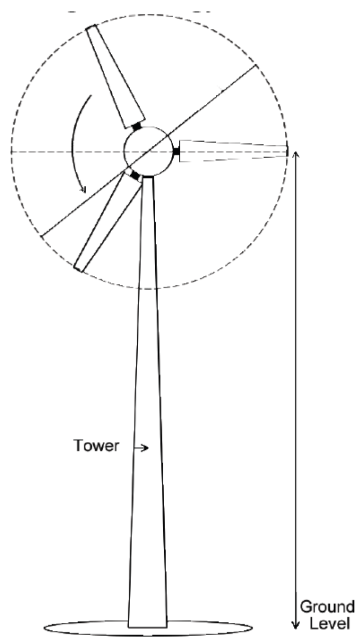
II.3.1 Determine cos(θ) (without determining the value of θ) in right triangle trigonometry.
II.3.2 Determine cos(θ) (without determining the value of θ) in unit circle trigonometry.



[6] This wording should be changed to *in radius lengths* or *in radii* to clarify the intent.

II.4

Windmills are currently used in an attempt to produce green energy. The windmill rotates at a rate of 3 radians every 15 seconds. Let *h* be the height (in meters) between the horizontal diameter of the windmill and the ground, so the tip of the fan is h meters away from the ground when it is the 3 o'clock position. Let *r* be the turbine's radius (in meters).



II.4.1 Define a function that relates the measure of the angle (in radians) swept out by the fan blade as a function of time elapsed.
II.4.2 Define a function *f* that represents the distance of the fan blade's tip (in meters) of the windmill above the ground as a function of the number of seconds that have elapsed since the fan started rotating from the 3 o'clock position.
II.4.3 Define a function *g* that represents the distance of the fan blade's tip (in radians)[7] of the windmill above the ground as a function of the number of seconds that have elapsed since the fan started rotating from the 6 o'clock position.

II.5

Sketch the graph of the function $y = \sin(x^2)$ for $x \geq 0$.
Hint: You may use the following line of reasoning. The sine function has a period of 2π, which means that it goes through a full cycle whenever its argument varies by 2π.
For example, $\sin(3x + 5)$ will repeat whenever the argument $3x + 5$ varies by 2π. So $\sin(3x + 5)$ will repeat whenever x varies by $2\pi/3$.

(Thompson, Carlson, & Silverman, 2007, p. 420)

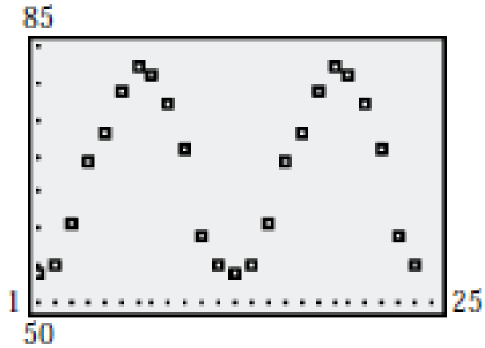
[7] This wording should be changed to *in radius lengths* or *in radii* to clarify the intent.

II.6

The maximum average monthly temperature in New Orleans is 82°F and the minimum is 54°F. The table shows the average monthly temperatures. The scatter diagram for a 2-year interval in the figure below strongly suggests that the temperatures can be modeled with a sine curve.

Month	°F	Month	°F
Jan	54	July	82
Feb	55	Aug	81
Mar	61	Sept	77
Apr	69	Oct	71
May	73	Nov	59
June	79	Dec	55

Source: Miller, A., J. Thompson, and R. Peterson, *Elements of Meteorology, 4th Edition*, Charles E. Merrill Publishing Co., 1983.



To model the average monthly temperature in New Orleans, determine a function of the form

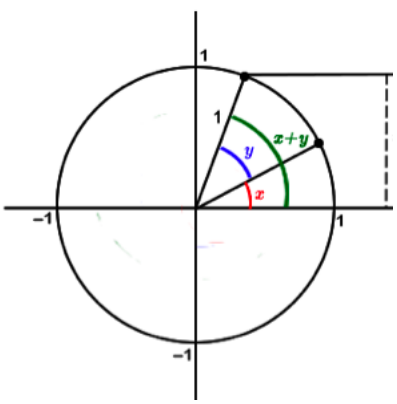
$$f(x) = a\sin[b(x - d)] + c$$

where a, b, c, and d are constants, and x represents the month, with January corresponding to $x = 1$.

(www.aw-bc.com/scp/lial_hornsby.../LIALMC06_0321227638.pdf, p. 566)

II.7

Consider the following image of the unit circle with angles of measure x, y, and $x + y$.



Prove the trigonometric identity $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ geometrically.

Please rate the level and the importance for each problem with reasons.

II.1

Level	E	M	D	Reason
For you				
For students				

Importance	1	2	3	Reason
For you				
For students				

II.2

Level	E	M	D	Reason
For you				
For students				

Importance	1	2	3	Reason
For you				
For students				

I.3

Level	E	M	D	Reason
For you				
For students				

Importance	1	2	3	Reason
For you				
For students				

II.4

Level	E	M	D	Reason
For you				
For students				

Importance	1	2	3	Reason
For you				
For students				

II.5

Level	E	M	D	Reason
For you				
For students				

Importance	1	2	3	Reason
For you				
For students				

II.6

Level	E	M	D	Reason
For you				
For students				

Importance	1	2	3	Reason
For you				
For students				

II.7

Level	E	M	D	Reason
For you				
For students				

Importance	1	2	3	Reason
For you				
For students				

Task III

Date: ______________________ Name__________________________

Instruction

- Task III consists of 10 problems.

- Assume that some situations (III.1 – III.6, III.10) between you and your student(s) happened in your class or during office hours. In III.7, III.8, and III.9, test questions were given to your students. Please respond to questions for each task problem.

III.1

Instructor (You): Can an angle's openness be measured with a linear unit of measure such as in inches?
Student: Yes.
Instructor (You): Why?
Student: For example, the measure of an angle is 1.7 "inches" as an angle measure when the arc length is 1.7 inches on the circle of radius 1 inch.

- How would you describe mathematical concepts key to the situation?
- What might be possible sources of his or her conception?
- How would you correct the student's answer, if necessary?
- What questions or examples would you ask or use to help students better understand the mathematical ideas/concepts involved in this situation?

III.2

Instructor (You): The answer for computing cos(sin(35º)) is 48.125º.
Student A: I put my calculator in the degree mode and then sin(35) produced 0.5736 and cos(0.5736) produced 0.9999. The answer is 0.9999º.
Student B: I think that the answer 48.125º could be wrong because the value for cos(sin(35º)) must be a real number.

- How would you compute cos(sin(35º))?
- How would you help the students derive 48.125º?
- How would you describe mathematical concepts key to the situation?
- What might be possible sources for their error or conception?
- What questions or examples would you ask or use to help students better understand the mathematical ideas/concepts involved in this situation?

(Thompson, Carlson, & Silverman, 2007)

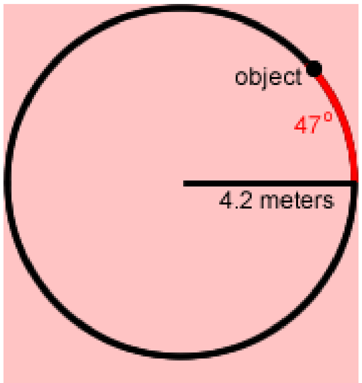
III.3

Instructor (You): Can anyone give an example of a unit circle?
Student A: I think that a circle with radius of 1 foot is a unit circle, but a circle with radius of 2.8 feet is not a unit circle.
Student B: I agree. But I am confused about whether a circle with radius of 12 inches is a unit circle or not because 12 inches is 1 foot.

- How would you describe mathematical concepts key to the situation?
- What might be possible sources of their conception?
- How would you correct the student's answer, if necessary?
- What questions or examples would you ask or use to help students better understand the mathematical ideas/concepts involved in this situation?

III.4

Instructor (You): How many meters does the object travel when it sweeps out 47 degrees on the circular path?

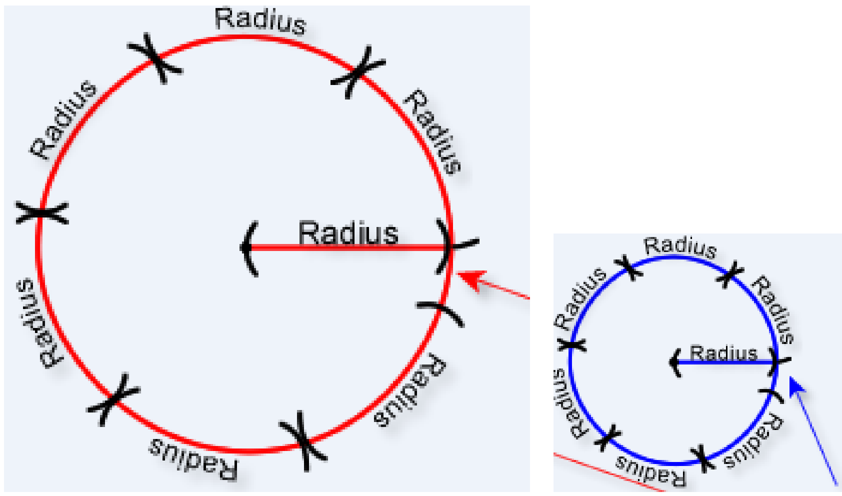


Student: That's easy. Using the formula $s = r \cdot \theta$, I can have the arc length $s = (4.2) \cdot 47 = 197.4$ meters.

- How would you describe mathematical concepts key to the situation?
- What might be possible sources of his/her conception?
- How would you correct the student's answer, if necessary?
- What questions or examples would you ask or use to help students better understand the mathematical ideas/concepts involved in this situation?

III.5

Instructor (You): Tell me about the relationship between the size of a circle and the length of a radius of any circle.
Student A: I think that the larger a circle is, the more radius lengths the circumference has.
Student B: I agree with him. As you can see in the diagrams below, the leftover portion on the circumference where the arrows are pointing is longer for a larger circle than the one for a smaller circle.



- How would you describe mathematical concepts key to the situation?
- What might be possible sources of their conception?
- How would you correct the students' answers, if necessary?
- What questions or examples would you ask or use to help students better understand the mathematical ideas/concepts involved in this situation?

III.6

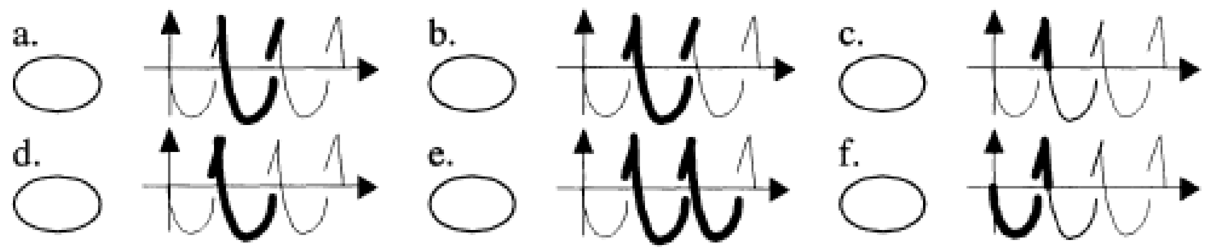
Instructor (You): Tell me about the inverse function of the cosine function.
Student: I think that sec *x* is the inverse function of cos(*x*) because $\sec(x) = \cos^{-1}(x)$ and

$$\cos(\sec(x)) = \cos\left(\frac{1}{\cos x}\right) = 1.$$

- How would you describe mathematical concepts key to the situation?
- What might be possible sources of his or her conception?
- How would you correct the student's answer, if necessary?
- What questions or examples would you ask or use to help students better understand the mathematical ideas/concepts involved in this situation?

III.7

Test Question: Below is a graphical representation of a periodic function, in a certain domain. In each drawing a part of the graph is bold. Write 'Yes' next to a drawing in which you think the bolded part is a period of the function and 'No' next to a drawing in which you think the bolded part is not a period of the function.



- How would you explain a solution to students?
- How would you describe mathematical concepts key to this question?
- What might be plausible causes of students' incorrect choices?
- What questions or examples would you ask or use to help students better understand the mathematical ideas/concepts involved in this test question?

(Shama, 1998)

III.8

Test Question: Find the exact value of the expression if it is defined.

$$\sin\left(\frac{1}{2}\sin^{-1}\left(\frac{\pi}{2}\right)\right)$$

- How would you explain a solution to students?
- How would you describe mathematical concepts key to this question?
- Any possible students' incorrect solutions?
- What might be plausible causes of the incorrect solutions?
- What questions or examples would you ask or use to help students better understand the mathematical ideas/concepts involved in this test question?

III.9

Test Question: Let $P(t)$ be the point on the unit circle that corresponds to t for $[0, 2\pi)$.

If $P(t) = (4/5, 3/5)$, find $P\left(\frac{\pi}{2} - t\right)$.

- How would you explain a solution to students?
- How would you describe mathematical concepts key to this question?
- Any possible students' correct solutions that are different from the way you solve?
- Any possible students' incorrect solutions?
- What might be plausible causes of the incorrect solutions?
- What questions or examples would you ask or use to help students better understand the mathematical ideas/concepts involved in this test question?

III.10

Student A: A book introduced SOH (Sine equals Opposite over Hypotenuse)-CAH (Cosine equals Adjacent over Hypotenuse)-TOA (Tangent equals Opposite over Adjacent) as a way to memorize the sine, cosine, and tangent values using ratios of the lengths and hypotenuse of a right triangle. But I am just wondering how I can find the value of $\cos(\pi/2)$ using it.
Student B: I also got the same question. In addition, what about the values of the trigonometric functions of any other angles than acute angles?

- How would you respond to each student's question?

(Thompson, Carlson, & Silverman, 2007)

APPENDIX D

BACKGROUND INFORMATION SHEET

Instructions: Information provided on this sheet remains strictly confidential. If you feel uncomfortable answering any of the questions, then you may leave them blank. Thank you for your time.

- Name:_________________________________
- Email:_________________________________
- Phone number: __________________________
- Are you a full time graduate student in the Department of Mathematics at UGA? Yes__ No__
- Degree that you are pursuing in mathematics at UGA: ________________________
- You are currently TA___ or RA____ or both____ this semester.
- Number of years as a graduate student in the program:___________________
- Number of semesters as a teaching assistant in the department:___________

- Mathematics course(s) you are teaching or have taught at UGA

Course name	Year/semester

- Have you ever taught trigonometry *prior to* coming to UGA? Yes_____ No______

  If yes, please provide the details below.

Course name	Year/semester	School/College/Institution

- Have you ever received any formal training (such as taking education courses) prior to teaching precalculus? Yes_____ No_____
  If yes, please provide the details below.

Course/Program name	Year/Semester	School/College/Institution

- Please check (√) mathematics graduate courses you are taking and have taken at UGA (The list is cited from http://www.math.uga.edu/graduate/gradcourses.html.)

Foundations

( ) 7900 Foundations for Graduate Mathematics An intensive review of techniques and material essential for graduate study in mathematics, including background in calculus and linear algebra. Emphasis is on small group study and presentations. Topics include proofs, induction, the metric Observing Structure of the reals, the Bolzano-Weierstrass theorem, and the diagonalization theorem.

Algebra/Group Theory

( ) 6000 Modern Algebra and Geometry I An introduction to the ideas and constructs of abstract aglebra, emphasizing geometric motivation and applications. Beginning with a careful study of integers, modular arithmetic, the Euclidean algorithm, the course moves on to fields, isometries of the complex plain, polynomials, splitting fields, rings, homomorphisms, field extensions and compass and straightedge constructions.

( ) 6010 Modern Algebra and Geometry II More advanced abstract algebraic Observing Structure s and concepts, such as groups, symmetry, group actions, counting principles, symmetry groups of the regular polyhedra, Burnside's Theorem, isometries of R3 , Galois theory, affine and projective geometry.

( ) 6050 Advanced Linear Algebra Orthogonal and unitary groups, spectral theorem; infinite-dimensional vector spaces; Jordan and rational conconical forms and applications.

( ) 6080 Advanced Algebra A course in linear algebra, grouops, rings, and modules, intermediate in level between MATH 6010 and MATH 8000. Topics include the finite-dimensional spectral theorem, group actions, classification of finitely generated modules over principal ideal domains, and canonical forms of linear operators.

( ) 8000 Algebra A course in groups, fields and rings, designed to prepare the student for the algebra prelims. Some topics covered include the Sylow theorems, solvable and simple groups, Galois theory, finite fields, Noetherian rings and modules.

( ) 8010 Representation Theory of Finite Groups Irreducible and indecomposable representations, Schur's Lemma, Maschke's theorem, the Wedderburn Observing Structure theorem, characters and orthogonality relations, induced representations and Frobenius reciprocity, central characters and central idempotents, Burnside's theorem, Frobenius normal p-complement theorem.

( ) 8020 Commutative Algebra Localization and completion, Nakayama's lemma, Dedekind domains, Hilbert's basis theorem, Hilbert's Nullstellensatz, Krull dimension, depth and Cohen-Macaulay rings, regular local rings.

( ) 8030 Topics in Algebra This course will present topics in abstract algebra at the level of current research.

( ) 8080 Lie Algebras Nilpotent and solvable Lie algebras, Observing Structure and classification of semisimple Lie algebras, roots, weights, finite-dimensional representations

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Analysis

( ) 6100 Real Analysis Metric spaces and continuity; differentiable and integrable functions of one variable; sequences and series of functions.

( ) 6110 The Lebesgue Integral and Applications The Lebesgue integral, with applications to Fourier analysis and probability.

( ) 6120 Multivariable Analysis The continuation of MATH 4100 to the multivariable setting: the derivative as a linear map, inverse and implicit function theorems, change of variables in multiple integrals; manifolds, differential forms, and the generalized Stokes' Theorem.

( ) 6150 Complex Variables Differential and integral calculus of functions of a complex variable, with applications. Topics include the Cauchy integral formula, power series and Laurent series, and the residue theorem.

( ) 8100 Real Analysis I Measure and integration theory with relevant examples from Lebesgue integration, Hilbert spaces (only with regard to L2), L2 spaces and the related Riesz representation theorem. Hahn, Jordan and Lebesgue decomposition theorems, Radon-Nikodym Theorem and Fubini's Theorem.

( ) 8110 Real Analysis II Topics including: Haar Integral, change of variable formula, Hahn-Banach theorem for Hilbert spaces, Banach spaces and Fourier theory (series, transform, Gelfand-Fourier homomorphism).

( ) 8150 Complex Variables I The Cauchy-Riemann Equations, linear fractional transformations and elementary conformal mappings, Cauchy's theorems and its consequences including: Morera's theorem, Taylor and Laurent expansions, maximum principle, residue theorem, argument principle, residue theorem, argument principle, Rouche's theorem and Liouville's theorem.

( ) 8160 Complex Variables II Topics including Riemann Mapping Theorem, elliptic functions, Mittag-Leffler and Weierstrass Theorems, analytic continuation and Riemann surfaces.

( ) 8170 Functional Analysis I Introduction to Hilbert spaces and Banach spaces, spectral theory, topological vector spaces, comvexity and its consequences including the Krein-Milman theorem.

( ) 8180 Functional Analysis II Introduction to operator theory, spectral theorem for normal operators, distribution theory, the Schwartz spaces, topics from C*-algebras and von Neumann algebras.

( ) 8190 Lie Groups Classical groups, exponential map, Poincare-Birkhoff-Witt Theorem, homogeneous spaces, adjoint representation, covering groups, compact groups, Peter-Weyl Theorem, Weyl character formula.

Applied Mathematics and Differential Equations

( ) 6700 Qualitative Ordinary Differential Equations Transform methods, linear and nonlinear systems of ordinary differential equations, stability, and chaos.

( ) 6720 Introduction to Partial Differential Equations The basic partial differential equations of mathematical physics: Laplace's equation, the wave equation, and the heat equation. Separation of variables and Fourier series techniques are featured.

( ) 6780 Mathematical Biology Mathematical models in the biological sciences: compartmental flow models, dynamic system models, discrete and continuous models, deterministic and stochastic models.

( ) 8700 Applied Mathematics: Applications in Industry Mathematical modeling of some real-world industrial problems. Topics will be selected from a list which includes air quality modeling, crystal precipitation, electron beam lithography, image processing, photographic film development, production planning in manufacturing, and optimal control of chemical reactions.

( ) 8710 Applied Mathematics: Variational Methods/Perturbation Theory Calculus of variations, Euler-Lagrange equations, Hamilton's principle, approximate methods, eigenvalue problems, asymptotic expansions, method of steepest descent, method of stationary phase,

perturbation of eigenvalues, nonlinear eigenvalue problems, oscillations and periodic solutions, Hopf bifurcation, singular perturbation theory, applications.
( ) 8740 Ordinary Differential Equations Solutions of initial value problems: existence, uniqueness, and dependence on parameters, differential inequalities, maximal and minimal solutions, continuation of solutions, linear systems, self-adjoint eigenvalue problems, Floquet Theory.
( ) 8750 Introduction to Dynamical Systems Continuous dynamical systems, trajectories, periodic orbits, invariant sets, Observing Structure of alpha and omega limit sets, applications to two-dimensional autonomous systems of ODE's, Poincare-Bendixson Theorem, discrete dynamical systems, infinite dimensional spaces, semi-dynamical systems, functional differential equations.
( ) 8770 Partial Differential Equations Classification of second order linear partial differential equations, modern treatment of characteristics, function spaces, Sobolev spaces, Fourier transform of generalized functions, generalized and classical solutions, initial and boundary value problems, eigenvalue problems.

Algebraic Geometry
( ) 6300 Introduction to Algebraic Curves Polynomials and resultants, projective spaces. The focus is on plane algebraic curves: intersection, Bezout's theorem, linear systems, rational curves, singularities, blowing up.
( ) 8300 Introduction to Algebraic Geometry An invitation to algebraic through a study of examples. Affine and projective varieties, regular and rational maps, Nullstellensatz. Veronese and Segre varieties, Grassmannians, algebraic groups, quadrics. Smoothness and tangent spaces, singularities and tangent cones.
( ) 8310 Geometry of Schemes The language of Grothendieck's theory of schemes. Topics include the spectrum of a ring, "gluing" spectra to form schemes, products, quasi-coherent sheaves of ideals, and the functor of points.
( ) 8320 Algebraic Curves The theory of curves, including linear series and the Riemann Roch theorem. Either the algebraic (variety), arithmetic (function field), or analytic (Riemann surface) aspect of the subject may be emphasized in different years.
( ) 8330 Topics in Algebraic Geometry Advanced topics such as algebraic surfaces, or cohomology and sheaves.

Topology/Geometry
( ) 6200 Point Set Topology Topological spaces, continuity; connectedness, compactness; separation axioms and Tietze extension theorem; function spaces.
( ) 6220 Differential Topology Manifolds in Euclidean space: fundamental ideas of transversality, homotopy, and intersection theory; differential forms, Stokes' Theorem, deRham cohomology, and degree theory.
( ) 6250 Differential Geometry An introduction to the geometry of curves and surfaces in Euclidean space: Frenet formulas for curves, notions of curvature for surfaces; Gauss-Bonnet Theorem; discussion of non-Euclidean geometries.
( ) 8200 Algebraic Topology The fundamental group, van Kampen's theorem, and covering spaces. Introduction to homology: simplicial, singular, and cellular. Applications.
( ) 8210 Topology of Manifolds Poincar duality, deRham's theorem, topics from differential topology.
( ) 8220 Homotopy Theory Topics in homotopy theory, including homotopy groups, CW complexes, and fibrations.
( ) 8230 Topics in Topology and Geometry Advanced topics in topology and/or differential geometry leading to and including research level material.
( ) 8250 Differential Geometry I Differentiable manifolds, vector bundles, tensors, flows, and Frobenius' theorem. Introduction to Riemannian geometry.
( ) 8260 Differential Geometry II Riemannian geometry: connections, curvature, first and second variation; geometry of submanifolds. Gauss-Bonnet theorem. Additional topics, such as characteristic classes, complex manifolds, integral geometry.

Number Theory
( ) 6400 Number Theory Euler's theorem, public key cryptology, pseudoprimes, multiplicative functions, primitive roots, quadratic reciprocity, continued fractions, sums of two squares and Gaussian integers.
( ) 6450 Cryptology and Computational Number Theory Recognizing prime numbers, factoring composite numbers, finite fields, elliptic curves, discrete logarithms, private key cryptology, key exchange systems, signature authentication, public key cryptology.
( ) 8400 Algebraic/Analytic Number Theory I The core material of algebraic number theory: number fields, rings of integers, discriminants, ideal class groups, Dirichlet's unit theorem, splitting of primes; p-adic fields, Hensel's lemma, adeles and ideles, the strong approximation theorem.
( ) 8410 Algebraic/Analytic Number Theory II A continuation of Algebraic and Analytic Number Theory I, introducing analytic methods: the Riemann Zeta function, its analytic continuation and functional equation, the Prime number theorem; sieves, the Bombieri-Vinogradov theorem, the Chebotarev density theorem.
( ) 8430 Topics in Arithmetic Geometry Topics in Algebraic number theory and Arithmetic geometry, such as class field theory, Iwasawa theory, elliptic curves, complex multiplication, cohomology theories, Arakelov theory, diophantine geometry, automorphic forms, L-functions, representation theory.
( ) 8440 Topics in Combinatorial/Analytic Number Theory Topics in combinatorial and analytic number theory, such as sieve methods, probabilistic models of prime numbers, the distribution of arithmetic functions, the circle method, additive number theory, transcendence methods.
( ) 8450 Topics in Algorithmic Number Theory Topics in computational number theory and algebraic geometry, such as factoring and primality testing, cryptography and coding theory, algorithms in number theory and arithmetic geometry.

Numerical Analysis
( ) 6500 Numerical Analysis I Methods for finding approximate numerical solutions to a variety of mathematical problems, featuring careful error analysis. A mathematical software package will be used to implement iterative techniques for nonlinear equations, polynomial interpolation, integration, and problems in linear algebra such as matrix inversion, eigenvalues and eigenvectors.
( ) 6510 Numerical Analysis II Numerical solutions of ordinary and partial differential equations, higher-dimensional Newton's method, and splines.

( ) 8500 Advanced Numerical Analysis I Numerical solution of nonlinear equations in one and several variables, numerical methods for constrained and unconstrained optimization, numerical solution of linear systems, numerical methods for computing eigenvalues and eigenvectors, numerical solution of linear least squares problems, computer applications for applied problems.
( ) 8510 Advanced Numerical Analysis II Polynomial and spline interpolation and approximation theory, numerical integration methods, numerical solution of ordinary differential equations, computer applications for applied problems.
( ) 8520 Advanced Numerical Analysis III Finite difference and finite element methods for elliptic, parabolic, and hyperbolic partial differential equations convergence and stability of those methods, numerical algorithms for the implementation of those methods.
( ) 8550 Special Topics in Numerical Analysis Special topics in numerical analysis, including iterative methods for large linear systems, computer aided geometric design, multivariate splines, numerical solutions for pde's, numerical quadrature and cubature, numerical optimization, wavelet analysis for numerical imaging. In any semester, one of the above topics will be covered.

Probability, Stochastic Processes and Combinatorics
( ) 6600 Probability Discrete and continuous random variables, expectation, independence and conditional probability; binomial, Bernoulli, normal, and Poisson distributions; law of large numbers and central limit theorem.
( ) 6630 Mathematical Analysis of Computer Algorithms Discrete algorithms (number-theoretic, graph-theoretic, combinatorial, and algebraic) with an emphasis on techniques for their mathematical analysis.
( ) 6670 Combinatorics Basic counting principles: permutations, combinations, probability, occupancy problems, and binomial coefficients. More sophisticated methods include generating functions, recurrence relations, inclusion/exclusion principle, and the pigeonhole principle. Additional topics include asymptotic enumeration, Polya counting theory, combinatorial designs, coding theory, and combinatorial optimization.
( ) 6690 Graph Theory Elementary theory of graphs and digraphs. Topics include connectivity, reconstruction, trees, Euler's problem, hamiltonicity, network flows, planarity, node and edge colorings, tournaments, matchings, and extremal graphs. A number of algorithms and applications are included.
( ) 8600 Probability Probability spaces, random variables, distributions, expectation and higher moments, conditional probability and expectation, convergence of sequences and series of random variables, strong and weak laws of large numbers, characteristic functions, infinitely divisible distributions, weak convergence of measures, central limit theorems.
( ) 8620 Stochastic Processes Conditional expectation, Markov processes, martingales and convergence theorems, stationary processes, introduction to stochastic integration.
( ) 8630 Stochastic Analysis Conditional expectation, Brownian motion, semimartingales, stochastic calculus, stochastic differential equations, stochastic control, stochastic filtering.

Education
( ) 7040 Basic Ideas of Calculus I Survey of one-variable calculus in preparation for teaching calculus at the secondary level: combines review of basic techniques with careful study of underlying concepts. This is MATH 2400H for graduate students in Mathematics Education.
( ) 7050 Basic Ideas of Calculus II A continuation of Basic Ideas of Calculus I focusing on functions of several variables. This is MATH 2410H for graduate students in Mathematics Education.
( ) Other (Please list them.) ____

APPENDIX E

THE PRE-TASK INTERVIEW PROTOCOL

1. Tell me about your experience in learning trigonometry during your K-12 and/or college years.

2. Tell me about your experience with graduate teaching assistants during your college years, if applicable.

3. What roles of graduate teaching assistants are important for teaching undergraduate mathematics?

4. Tell me about your experience of the first semester as a teaching assistant in the department.

5. Think of a time when you had difficulties/challenges in teaching trigonometry to undergraduates and tell me about that. How did you resolve such problems?

6. Tell me what you think graduate teaching assistants need to know to teach trigonometry.

7. How did the teaching seminar influence your teaching of trigonometry?

8. How did teaching trigonometry influence your ideas about teaching trigonometry?

9. In what ways might you improve your teaching of trigonometry to undergraduates?

10. Why do you think undergraduate students should learn trigonometry in precalculus?

11. What are the most important ideas to understand in trigonometry?

12. Tell me five things related to trigonometry that you want your students to remember even after finishing the course.

APPENDIX F

THE POST-TASK INTERVIEW PROTOCOL

1. Tell me about your experience of participating in this research project.

2. What do you think about the task items?

3. What project activities did you like? Why?

4. What project activities did you not enjoy? Why?

5. What do you think about your knowledge of trigonometry for teaching?

6. What do you think about the relationship between your knowledge of trigonometry and your teaching?

7. What did you learn from the project activities?

8. What are your comments about this research project? Is there anything else that would be helpful for me to know?

APPENDIX G

COMPARISONS OF EXPECTED STRANDS AND OBSERVED STRANDS

Task I

Gloria

Task	Expected Strands	Observed Strands
GI.1	MP: Conceptual Understanding	MP: Conceptual Understanding (Essential Features) Procedural Fluency Historical Knowledge
		MWT: Know the Curriculum
GI.2	MP: Conceptual Understanding	MP: Conceptual Understanding (Basic Repertoire) Historical Knowledge
		MWT: Know the Curriculum
GI.3	MP: Procedural Fluency Adaptive Reasoning	MP: Procedural Fluency
		MWT: Know the Curriculum
GI.4	MP: Conceptual Understanding	MP: Conceptual Understanding (Basic Repertoire) Adaptive Reasoning
		MWT: Know the Curriculum
GI.5	MP: Conceptual Understanding	MP: Adaptive Reasoning (Strength of the Concept)
GI.6	MP: Conceptual Understanding	MP: Strategic Competence (Different Representation) Positive Disposition
GI.7	MP: Conceptual Understanding	MP: Conceptual Understanding (Essential Features)
GI.8	MP: Historical and Cultural Knowledge	MP: Conceptual Understanding Historical Knowledge
		MWT: Know the Curriculum

Kyle

Task	Expected Strands	Observed Strands
KI.1	MP: Conceptual Understanding	MP: Conceptual Understanding (Essential Features) Procedural Fluency Historical Knowledge
		MWT: Know the Curriculum
KI.2	MP: Conceptual Understanding	MP: Conceptual Understanding (Essential Features) Adaptive Reasoning Historical Knowledge
		MWT: Know the Curriculum
KI.3	MP: Procedural Fluency Adaptive Reasoning	MP: Procedural Fluency Adaptive Reasoning
KI.4	MP: Conceptual Understanding	MP: Conceptual Understanding (Essential Features)
		MWT: Assess the Mathematical Knowledge of Learners
KI.5	MP: Conceptual Understanding	MP: Conceptual Understanding (Essential Features) Adaptive Reasoning Strategic Competence (Different Representations)
		MWT: Know the Curriculum
KI.6	MP: Conceptual Understanding	MP: Procedural Fluency Adaptive Reasoning
KI.7	MP: Conceptual Understanding	MP: Conceptual Understanding (Essential Features) Adaptive Reasoning
KI.8	MP: Historical and Cultural Knowledge	MP: Historical Knowledge
		MWT: Know the Curriculum

Leo

Task	Expected Strands	Observed Strands
LI.1	MP: Conceptual Understanding	MP: Conceptual Understanding (Basic Repertoire) Historical Knowledge
LI.2	MP: Conceptual Understanding	MP: Conceptual Understanding (Basic Repertoire) Historical Knowledge
		MWT: Know the Curriculum
LI.3	MP: Procedural Fluency Adaptive Reasoning	MP: Strategic Competence (Alternative Ways of Approaching) Historical Knowledge
LI.4	MP: Conceptual Understanding	MP: Conceptual Understanding Strategic Competence (Alternative Ways of Approaching) Adaptive Reasoning Historical Knowledge
		MWT: Know the Curriculum
LI.5	MP: Conceptual Understanding	MP: Strategic Competence Adaptive Reasoning
		MWT: Know the Curriculum
LI.6	MP: Conceptual Understanding	MP: Procedural Fluency
LI.7	MP: Conceptual Understanding	MP: Conceptual Understanding (Essential Features) Historical Knowledge
		MWT: Know the Curriculum
LI.8	MP: Historical and Cultural Knowledge	MP: Historical Knowledge
		MWT: Know the Curriculum

Micah

Task	Expected Strands	Observed Strands
MI.1	MP: Conceptual Understanding	MP: Conceptual Understanding (Essential Features)
		MWT: Know the Curriculum
MI.2	MP: Conceptual Understanding	MP: Adaptive Reasoning Historical Knowledge
MI.3	MP: Procedural Fluency MP: Adaptive Reasoning	MP: Procedural Fluency
		MWT: Know the Curriculum
MI.4	MP: Conceptual Understanding	MP: Conceptual Understanding (Essential Features) Historical Knowledge
MI.5	MP: Conceptual Understanding	MP: Conceptual Understanding (Essential Features) Strategic Competence
		MP: Adaptive Reasoning (Strength of the Concept)
		MWT: Know the Curriculum
MI.6	MP: Conceptual Understanding	MP: Conceptual Understanding (Essential Features) Procedural Fluency
MI.7	MP: Conceptual Understanding	MP: Conceptual Understanding (Essential Features) Adaptive Reasoning
MI.8	MP: Historical and Cultural Knowledge	MP: Historical Knowledge
		MWT: Know the Curriculum

Task II

Gloria

Task	Expected Strands	Observed Strands
GII.1	MA: Mathematical Noticing: Observing Structure of Mathematical Systems MA: Mathematical Creating: Defining	MA: Mathematics Noticing: Connect outside Math Mathematical Reasoning: Justifying Extending Generalizing Mathematical Creating: Defining MWT: Know the Curriculum
GII.2	MP: Procedural Fluency MP: Strategic Competency MA: Mathematical Noticing: Discerning Symbolic Forms	MP: Historical knowledge Productive Disposition MA: Mathematical Noticing: Discerning Symbolic Forms Connecting within Math Mathematical Reasoning: Justifying Constraining Mathematical Creating: Representing Defining
GII.3	MA: Mathematical Noticing: Observing Structure of Mathematical Systems MA: Mathematical Noticing: Connecting within and outside mathematics	MA: Mathematical Reasoning: Justifying
GII.4	MA: Mathematical Creating: Representing MA: Modifying/Transforming/Manipulating	MP: Conceptual Understanding Adaptive Reasoning Historical Knowledge Productive Disposition MA: Mathematical Reasoning: Justifying Mathematical Creating: Representing Manipulating MWT: Analyze Mathematical Ideas
GII.5	MP: Adaptive Reasoning MA: Mathematical Reasoning: Conjecturing and Generalizing MA: Mathematical Reasoning: Constraining and Extending	MP: Conceptual Understanding (Essential Features) MA: Mathematical Noticing: Connecting within and outside Math Mathematical Reasoning: Justifying/Proving MWT: Analyze Mathematical Ideas
GII.6	MA: Integrating Strands of Mathematical Activity	MP: Procedural Fluency MA: Mathematical Reasoning: Justifying Mathematical Creating: Manipulating Transforming Integrating Strands of Mathematical Activity MWT: Know the Curriculum
GII.7	MA: Mathematical Reasoning: Justifying/proving	MA: Mathematical Reasoning: Justifying/Proving

Kyle

Task	Expected Strands	Observed Strands
KII.1	MA: Mathematical Noticing: Observing Structure of Mathematical Systems MA: Mathematical Creating: Defining	MP: Conceptual Understanding (Essential Features) Procedural Fluency
		MA: Mathematical Noticing: Observing Structure of Mathematical Systems MA: Mathematical Reasoning: Justifying Generalizing
KII.2	MP: Procedural Fluency MP: Strategic Competency MA: Mathematical Noticing: Discerning Symbolic Forms	MP: Conceptual Understanding Procedural Fluency Adaptive Reasoning
		MA: Mathematical Noticing: Discerning Symbolic Forms Mathematical Reasoning: Constraining
KII.3	MA: Mathematical Noticing: Observing Structure of Mathematical Systems MA: Mathematical Noticing: Connecting within and outside mathematics	MP: Conceptual Understanding Procedural Fluency
KII.4	MA: Mathematical Creating: Representing MA: Modifying/Transforming/Manipulating	MP: Procedural Fluency
		MA: Mathematical Reasoning: Justifying Mathematical Creating: Manipulating
		MWT: Know the Curriculum
KII.5	MP: Adaptive Reasoning MA: Mathematical Reasoning: Conjecturing and Generalizing MA: Mathematical Reasoning: Constraining and Extending	MP: Conceptual Understanding
		MA: Mathematical Reasoning: Justifying Conjecturing
KII.6	MA: Integrating Strands of Mathematical Activity	MP: Procedural Fluency Strategic Competence (Knowledge and Understanding of Concept) Adaptive Reasoning-Strength of the Concept
		MA: Mathematical Creating: Manipulating Integrating Strands of Mathematical Activity
KII.7	MA: Mathematical Reasoning: Justifying/proving	MP: Productive Disposition
		MA: Mathematical Reasoning: Proving Mathematical Creating: Manipulating

Leo

Task	Expected Strands	Observed Strands
LII.1	MA: Mathematical Noticing: Observing Structure of Mathematical Systems MA: Mathematical Creating: Defining	MP: Procedural Fluency
		MA: Mathematical Reasoning: Justifying
LII.2	MP: Procedural Fluency MP: Strategic Competency MA: Mathematical Noticing: Discerning Symbolic Forms	MA: Mathematical Reasoning: Constraining Mathematical Creating: Representing
		MP: Conceptual Understanding Procedural Fluency Strategic Competence
LII.3	MA: Mathematical Noticing: Observing Structure of Mathematical Systems MA: Mathematical Noticing: Connecting within and outside mathematics	MP: Conceptual Understanding Procedural Fluency
		MA: Mathematical Noticing: Observing Structure of Mathematical Systems
		MWT: Know the Curriculum
LII.4	MA: Mathematical Creating: Representing MA: Modifying/Transforming/Manipulating	MA: Mathematical Reasoning: Justifying Mathematical Creating: Representing Modifying/Manipulating
LII.5	MP: Adaptive Reasoning MA: Mathematical Reasoning: Conjecturing and Generalizing MA: Mathematical Reasoning: Constraining and Extending	MP: Conceptual Understanding (Essential Features)
		MA: Mathematical Reasoning: Justifying Extending
LII.6	MA: Integrating Strands of Mathematical Activity	MP: Procedural Fluency Strategic Competence (Knowledge and Understanding of Concept)
		MA: Mathematical Reasoning: Justifying Mathematical Creating: Manipulating Integrating Strands of Mathematical Activity
LII.7	MA: Mathematical Reasoning: Justifying/proving	MA: Mathematical Noticing: Discerning Symbolic Forms Mathematical Reasoning: Justifying/Proving Mathematical Creating: Representing

Micah

Task	Expected Strands	Observed Strands
MII.1	MA: Mathematical Noticing: Observing Structure of Mathematical Systems MA: Mathematical Creating: Defining	MP: Conceptual Understanding Procedural Fluency
		MA: Mathematical Reasoning: Justifying
		MWT: Know the Curriculum
MII.2	MP: Procedural Fluency MP: Strategic Competency MA: Mathematical Noticing: Discerning Symbolic Forms	MP: Conceptual Understanding Procedural Fluency Strategic Competence Adaptive Reasoning
		MWT: Mathematical Reasoning: Constraining
MII.3	MA: Mathematical Noticing: Observing Structure of Mathematical Systems MA: Mathematical Noticing: Connecting within and outside mathematics	MP: Conceptual Understanding Procedural Fluency
		MA: Mathematical Noticing: Observing Structure of Mathematical Systems
		MWT: Know the Curriculum
MII.4	MA: Mathematical Creating: Representing MA: Mathematical Creating: Modifying/Transforming/Manipulating	MP: Strategic Competence Adaptive Reasoning
		MA: Mathematical Reasoning: Justifying Mathematical Creating: Manipulating
		MWT: Assess the Mathematical Knowledge of Learners
MII.5	MP: Adaptive Reasoning MA: Mathematical Reasoning: Conjecturing and Generalizing MA: Mathematical Reasoning: Constraining and Extending	MA: Mathematical Noticing: Connecting within and outside Mathematics Mathematical Reasoning: Justifying/Proving
		MWT: Analyze Mathematical Ideas
MII.6	MA: Integrating Strands of Mathematical Activity	MP: Procedural Fluency
		MA: Mathematical Creating: Modifying/Transforming/Manipulating Integrating Strands of Mathematical Activity
		MWT: Assess the Mathematical Knowledge of Learners
MII.7	MA: Mathematical Reasoning: Justifying/proving	MA: Mathematical Reasoning: Proving

Task III

Gloria

Task	Expected Strands	Observed Strands
GIII.1	MA: Mathematical Noticing: Detecting the form of an argument MWT: Analyze Mathematical Ideas	MP: Conceptual Understanding (Essential Features)
		MA: Mathematical Noticing: Detecting the Form of Argument Mathematical Creating: Defining
		MWT: Access and Understanding the Mathematical Thinking of Learners Assess the Mathematical Knowledge of Learners
GIII.2	MA: Mathematical Noticing: Observing Structure of Mathematical Systems MA: Mathematical Noticing: Detecting the form of an argument MWT: Analyze Mathematical Ideas MWT: Assess the Mathematical Knowledge of Learners MWT: Know and Use the Curriculum	MP: Procedural Fluency Strategic Competence (Knowledge and Understanding of Concept) Adaptive Reasoning
		MA: Mathematical Noticing: Observing Structure of Mathematical Systems Mathematical Reasoning: Justifying/Proving
		MWT: Access and Understand the Mathematical Thinking of Learners
GIII.3	MA: Mathematical Noticing: Detecting the form of an argument MWT: Assess the Mathematical Knowledge of Learners MWT: Know and Use the Curriculum	MP: Conceptual Understanding (Essential Features) Conceptual Understanding (Basic Repertoire)
		MA: Mathematical Creating: Defining
		MWT: Access and Understand the Mathematical Thinking of Learners Know the Curriculum
GIII.4	MA: Mathematical Noticing: Detecting the form of an argument MWT: Assess the Mathematical Knowledge of Learners	MA: Mathematical Reasoning: Justifying/Proving
		MWT: Analyze Mathematical Ideas Assess the Mathematical Knowledge of Learners
GIII.5	MA: Mathematical Noticing: Detecting the form of an argument MWT: Assess the Mathematical Knowledge of Learners	MWT: Analyze Mathematical Ideas Assess the Mathematical Knowledge of Learners

GIII.6	MA: Mathematical Noticing: Detecting the form of an argument MWT: Assess the Mathematical Knowledge of Learners	MP: Conceptual Understanding-Essential Features
		MA: Mathematical Noticing: Discerning Symbolic Forms Mathematical Creating: Defining
		MWT: Analyze Mathematical Ideas Assess the Mathematical Knowledge of Learners
GIII.7	MWT: Assess the Mathematical Knowledge of Learners	MP: Conceptual Understanding (Basic Repertoire)
		MA: Mathematical Reasoning: Justifying/Proving
		MWT: Analyze Mathematical Ideas Assess the Mathematical Knowledge of Learners
GIII.8	MA: Mathematical Noticing: Discerning Symbolic Forms MWT: Assess the Mathematical Knowledge of Learners	MP: Conceptual Understanding (Essential Features) Procedural Fluency
		MA: Mathematical Reasoning: Constraining
		MWT: Analyze Mathematical Ideas Assess the Mathematical Knowledge of Learners
GIII.9	MA: Mathematical Noticing: Discerning Symbolic Forms MWT: Assess the Mathematical Knowledge of Learners	MP: Strategic Competence
		MWT: Access and Understand the Mathematical Thinking of Learners Assess the Mathematical Knowledge of Learners
GIII.10	MWT: Know and Use the Curriculum	MWT: Analyze Mathematical Ideas Know the Curriculum

Kyle

Task	Expected Strands	Observed Strands
KIII.1	MA: Mathematical Noticing: Detecting the form of an argument MWT: Analyze Mathematical Ideas	MP: Conceptual Understanding (Essential Features) Conceptual Understanding (Basic Repertoire) Adaptive Reasoning MA: Mathematical Noticing: Detecting the Form of Argument Mathematical Creating: Defining MWT: Analyze Mathematical Ideas
KIII.2	MA: Mathematical Noticing: Observing Structure of Mathematical Systems MA: Mathematical Noticing: Detecting the form of an argument MWT: Analyze Mathematical Ideas MWT: Assess the Mathematical Knowledge of Learners MWT: Know and Use the Curriculum	MP: Conceptual Understanding (Essential Features) Procedural Fluency MA: Mathematical Noticing: Observing Structure of Mathematical Systems Mathematical Noticing: Detecting the Form of Argument
KIII.3	MA: Mathematical Noticing: Detecting the form of an argument MWT: Assess the Mathematical Knowledge of Learners MWT: Know and Use the Curriculum	MP: Conceptual Understanding (Essential Features) Conceptual Understanding (Basic Repertoire) Adaptive Reasoning MWT: Assess the Mathematical Knowledge of Learners
KIII.4	MA: Mathematical Noticing: Detecting the form of an argument MWT: Assess the Mathematical Knowledge of Learners	MA: Mathematical Reasoning: Justifying MWT: Analyze Mathematical Ideas Assess the Mathematical Knowledge of Learners
KIII.5	MA: Mathematical Noticing: Detecting the form of an argument MWT: Assess the Mathematical Knowledge of Learners	MP: Adaptive Reasoning MA: Mathematical Reasoning: Justifying

KIII.6	MA: Mathematical Noticing: Detecting the form of an argument MWT: Assess the Mathematical Knowledge of Learners	MP: Conceptual Understanding (Basic Repertoire)
		MA: Mathematical Noticing: Discerning Symbolic Forms Mathematical Reasoning: Justifying
		MWT: Assess the Mathematical Knowledge of Learners
KIII.7	MWT: Assess the Mathematical Knowledge of Learners	MP: Conceptual Understanding (Essential Features) Conceptual Understanding (Basic Repertoire)
		MA: Mathematical Creating: Defining
KIII.8	MA: Mathematical Noticing: Discerning Symbolic Forms MWT: Assess the Mathematical Knowledge of Learners	MP: Conceptual Understanding (Essential Features)
		MA: Mathematical Noticing: Discerning Symbolic Forms Mathematical Reasoning: Justifying Constraining
		MWT: Assess the Mathematical Knowledge of Learners
KIII.9	MA: Mathematical Noticing: Discerning Symbolic Forms MWT: Assess the Mathematical Knowledge of Learners	MA: Mathematical Creating: Manipulating
		MWT: Know the Curriculum
KIII.10	MWT: Know and Use the Curriculum	MP: Historical knowledge
		MWT: Know the Curriculum

Leo

Task	Expected Strands	Observed Strands
LIII.1	MA: Mathematical Noticing: Detecting the form of an argument MWT: Analyze Mathematical Ideas	MP: Conceptual Understanding (Essential Features) MWT: Analyze Mathematical Ideas Assess the Mathematical Knowledge of Learners Access and Understand the Mathematical Thinking of Learners
LIII.2	MA: Mathematical Noticing: Observing Structure of Mathematical Systems MA: Mathematical Noticing: Detecting the form of an argument MWT: Analyze Mathematical Ideas MWT: Assess the Mathematical Knowledge of Learners MWT: Know and Use the Curriculum	MP: Conceptual Understanding (Essential Features) Adaptive Reasoning MA: Mathematical Noticing: Observing Structure of Mathematical Systems
LIII.3	MA: Mathematical Noticing: Detecting the form of an argument MWT: Assess the Mathematical Knowledge of Learners MWT: Know and Use the Curriculum	MP: Conceptual Understanding (Essential Features) Conceptual Understanding (Basic Repertoire) MWT: Analyze Mathematical Ideas
LIII.4	MA: Mathematical Noticing: Detecting the form of an argument MWT: Assess the Mathematical Knowledge of Learners	MA: Mathematical Reasoning: Justifying MWT: Analyze Mathematical Ideas
LIII.5	MA: Mathematical Noticing: Detecting the form of an argument MWT: Assess the Mathematical Knowledge of Learners	MP: Conceptual Understanding (Essential Features) Historical Knowledge MA: Mathematical Reasoning: Justifying

LIII.6	MA: Mathematical Noticing: Detecting the form of an argument MWT: Assess the Mathematical Knowledge of Learners	MP: Conceptual Understanding (Essential Features)
		MA: Mathematical Noticing: Discerning Symbolic Forms Mathematical Creating: Defining Mathematical Reasoning: Justifying
		MWT: Know the Curriculum
LIII.7	MWT: Assess the Mathematical Knowledge of Learners	MP: Conceptual Understanding (Essential Features) Conceptual Understanding (Basic Repertoire)
		MA: Mathematical Creating: Defining
LIII.8	MA: Mathematical Noticing: Discerning Symbolic Forms MWT: Assess the Mathematical Knowledge of Learners	MA: Mathematical Noticing: Discerning Symbolic Forms Mathematical Reasoning: Justifying Constraining Mathematical Creating: Manipulating
		MWT: Know the Curriculum
LIII.9	MA: Mathematical Noticing: Discerning Symbolic Forms MWT: Assess the Mathematical Knowledge of Learners	MP: Conceptual Understanding (Essential Features)
		MA: Mathematical Creating: Manipulating
		MWT: Know the Curriculum
LIII.10	MWT: Know and Use the Curriculum	MWT: Know the Curriculum

Micah

Task	Expected Strands	Observed Strands
MIII.1	MA: Mathematical Noticing: Detecting the form of an argument MWT: Analyze Mathematical Ideas	MP: Conceptual Understanding (Essential Features) Adaptive Reasoning MA: Mathematical Creating: Defining MWT: Analyze Mathematical Ideas Know the Curriculum
MIII.2	MA: Mathematical Noticing: Observing Structure of Mathematical Systems MA: Mathematical Noticing: Detecting the form of an argument MWT: Analyze Mathematical Ideas MWT: Assess the Mathematical Knowledge of Learners MWT: Know and Use the Curriculum	MP: Conceptual Understanding (Essential Features) Procedural Fluency Adaptive Reasoning MA: Mathematical Noticing: Observing Structure of Mathematical Systems MWT: Analyze Mathematical Ideas
MIII.3	MA: Mathematical Noticing: Detecting the form of an argument MWT: Assess the Mathematical Knowledge of Learners MWT: Know and Use the Curriculum	MP: Conceptual Understanding (Essential Features) Adaptive Reasoning MWT: Analyze Mathematical Ideas Access and Understand the Mathematical Thinking of Learners
MIII.4	MA: Mathematical Noticing: Detecting the form of an argument MWT: Assess the Mathematical Knowledge of Learners	MP: Conceptual Understanding (Essential Features) Procedural Fluency MA: Mathematical Reasoning: Justifying MWT: Analyze Mathematical Ideas
MIII.5	MA: Mathematical Noticing: Detecting the form of an argument MWT: Assess the Mathematical Knowledge of Learners	MP: Conceptual Understanding (Essential Features) Adaptive Reasoning
MIII.6	MA: Mathematical Noticing: Detecting the form of an argument MWT: Assess the Mathematical Knowledge of Learners	MA: Mathematical Noticing: Discerning Symbolic Forms MWT: Access and Understand the Mathematical Thinking of Learners

MIII.7	MWT: Assess the Mathematical Knowledge of Learners	MP: Conceptual Understanding (Essential Features)
		MA: Mathematical Creating: Defining
		MWT: Access and Understand the Mathematical Thinking of Learners
MIII.8	MA: Mathematical Noticing: Discerning Symbolic Forms	MP: Conceptual Understanding (Basic Repertoire)
		MA: Mathematical Reasoning: Constraining
	MWT: Assess the Mathematical Knowledge of Learners	MWT: Know the Curriculum
MIII.9	MA: Mathematical Noticing: Discerning Symbolic Forms	MA: Mathematical Noticing: Discerning Symbolic Forms
	MWT: Assess the Mathematical Knowledge of Learners	MWT: Access and Understand the Mathematical Thinking of Learners
MIII.10	MWT: Know and Use the Curriculum	MWT: Know the Curriculum