UNDERSTANDING MATHEMATICAL CONCEPTS: THE CASE OF THE LOGARITHMIC FUNCTION

by

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Under the direction of Dr. James W. Wilson

ABSTRACT

The purpose of this study was to describe students' understanding of the logarithmic function, changes in their understanding, and ways of knowing they use to investigate problems involving the logarithmic function. Understanding is defined as a student's beliefs about a mathematical concept. Four categories of evidence (conception, representation, connection, and application) were used to make conjectures about the students' beliefs over three instructional phases (preinstruction, instruction, and postinstruction). Nine interviews were conducted over a two-month period with students (3 female, 1 male) from two college algebra classes at a rural southeastern two-year college. Case studies were developed based on evidence gathered from phenomenological interviews, clinical interviews, participant observation, student constructed maps, and drawings.

The students' understanding contained a central theme: the logarithmic function as a collection of problems to do. Four categories of beliefs (level of difficulty, problem types, tools, character of the function) associated with the theme were identified in all three phases of the study. The static nature of the categories suggests students' understanding of a mathematical concept is influenced by their beliefs about mathematics and understanding. The changes in the content of the beliefs were the result of instruction and the reconstructive nature of memory. A modified theory of understanding using beliefs about mathematics and understanding and four categories of evidence is suggested for further research.

Four ways of knowing (number patterns, successive approximation, More A – More B, and responses to inconsistencies) were used by the students to investigate problems involving the logarithmic function. These ways of knowing are suggested as a starting point for the teaching of logarithmic functions.

INDEX WORDS: Understanding, Beliefs, Representation, The logarithmic function, College algebra

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DEDICATION

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TABLE OF CONTENTS

CHAPTER 1: DEFINING THE PROBLEM AND CRAFTING A SOLUTION PATH

Rationale

"One of the most widely accepted ideas within the mathematics education community is the idea that students should understand mathematics" (Hiebert & Carpenter, 1992, p. 65). This fundamental assumption was the basis for this study. As a teacher, I came to believe that students should understand mathematics. A teacher often assumes a student understands a concept that has been presented and finds, in a subsequent class, the student cannot recall it. This experience with students occurred each term I taught college algebra.

In college algebra the most frustrating concept for me to teach was the logarithmic function. Even those students I thought understood the concept could not remember or use its properties in subsequent courses. This absence of memory motivated me to ask why. Why did my students fail to remember what they had seemed to know so well just a few months earlier? Did they really understand the concept in the first place? As a teacher I had to assume something was wrong. I wanted to know the nature of the problem and how to attack it. This report is the result of a question I posed as a mathematics teacher: Why can't my students remember the definition of, properties of, and how to use the logarithmic function?

I tried to understand my question. I believed if my students understood the logarithmic function they would remember it. If I taught the concept differently, I reasoned, students would understand. So I searched for the perfect curriculum. My search produced a collection of materials based on approaches that included the traditional introduction of the *logarithmic function as the inverse of the exponential function.* Two other approaches also seemed effective: historical (Toumasis, 1993; Katz, 1995) and the logarithmic function as an area (School Mathematics Study Group, 1965). I focused on finding <u>the</u> curriculum that would produce understanding in my students.

After reading and analyzing the historical development of logarithms over the course of several months, I suddenly realized the focus of my research was not the logarithmic function but students' understanding of the logarithmic function. This shift gave my research the focus I had been looking for. My question became what does it mean for students to understand the logarithmic function? Now I needed to be more precise. I turned to the literature for definitions and theories about students' understanding of mathematical concepts.

Several theories of understanding seemed helpful (Hiebert & Carpenter, 1992; Pirie & Kieren, 1992; Sierpinska, 1994; Skemp, 1987). The theories agreed on one point, the location of understanding is in the mind of the individual. Two of the theories (Sierpinska, 1994; Skemp, 1987) explicitly stated the individual can, at times, consciously control his or her understanding. Despite situating the locus of control for understanding within the individual, none of the researchers had asked the individuals what they understood. This omission appeared inconsistent to me. I assumed if understanding was occurring within the individual, he or she could tell the researcher about it. This assumption proved to be a fairly strong one. I realized that students might not share my definition of understanding, but I initially failed to consider that they might not be using their verbally described definitions. Indeed, how someone defines understanding might not involve personal action. But as Bruner (1990) explained, meaning lies somewhere between a person's actions during an experience and the person's explanations of his or her actions. Hence, if I gathered students' definitions of understanding, their reflections on understanding and not understanding mathematical concepts, and saw them in action, I might discover their meaning of the term. Students' meanings and reflections could then help me interpret their actions and the understanding that resulted from them. Using my observations and interpretations I could build descriptions of students' understanding of the logarithmic function. Realizing that I wanted to ask the students about their own understanding did not help me define the term.

All of the theories in the literature seem plausible, but no single theory seemed to explain what it meant for a student to understand a mathematical concept.

When I began thinking about students' understanding of the logarithmic function, my goal was to find a way to teach the concept so students would remember it. As I pursued the goal, I took various paths. I no longer studied curriculum, looking for what might work. Instead, I studied students, their ways of knowing, and their explanations of their ways of knowing in a mathematical context. The purpose of the study was threefold: to develop descriptions of students' understanding of the logarithmic function, of changes in their understanding of the function, and of ways of knowing that they use to investigate problems involving the logarithmic function.

Theoretical Framework

Before I present my definition of understanding, I would like to clarify the assumptions on which the study was based. First, I assumed the goal of mathematics teaching is student understanding. Second, I assumed a student's understanding of a mathematical concept exists in his or her mind. Third, I was aware I could not know precisely what was in a student's mind, but assumed I could infer the workings of the mind from external evidence (Goldin, 1998a; Skemp, 1987). Fourth, I assumed that when students try to solve mathematics problems, they are trying to make sense to and for themselves (self-referencing). Finally, I assumed a student's understanding is qualitatively richer and quantitatively larger than external evidence and ultimately my descriptions can indicate. Hence although my descriptions may not match students' understanding, they provide useful information for those who teach the logarithmic function, design curriculum to be used in the teaching of the logarithmic function, and investigate students' understanding of mathematical concepts.

Understanding

Understanding can change. A student's understanding of a mathematical concept may become either more or less consistent with standard mathematical views of the concept, but the most powerful mediator of understanding is a student's prior knowledge.

"One observation that assumes near axiomatic status in cognitive science is that student's prior knowledge influences what they learn and how they perform" (Hiebert & Carpenter, 1992, p. 80). When a mathematical concept is presented to a student, he or she attempts to make sense of it using prior knowledge of the concept, mathematics, strategies, and available resources. The presentation and what the student thinks he or she is trying to learn influence these attempts. Is it how to do a problem, how to simplify an expression, or what the concept is? Student's attempts at sense making result in a collection of beliefs about the mathematical concept. According to Stavy and Tirosh (2000), such collections have been referred to in mathematics and science education literature as "misconceptions, naïve conceptions, alternative conceptions, intuitive conceptions, and preconceptions" (p. i). I will refer to these privately held beliefs as the student's understanding of a mathematical concept.

A Definition of Understanding

A student's understanding of a mathematical concept is his or her collection of privately held beliefs about the concept. This definition does not imply that a student with a collection of beliefs about a concept understands it. Certainly a person who believes the logarithmic function is a number such as e or π does not understand the logarithmic function. Instead, having beliefs implies that a student has *an understanding* of the concept. I draw a distinction between a declaration that a student *understands* and that he or she has *an understanding.* Mathematics teaching is meant to encourage the growth of a system of beliefs within the student that are consistent with culturally accepted beliefs. A student is said to *understand* a mathematical concept when, based on an analysis of available evidence, the system of beliefs attributed to the student is consistent with culturally accepted beliefs about the concept. It is evidence of consistency that is used to decide whether or not a student *understands* a mathematical concept. My purpose is not to answer such a question, but to analyze the available evidence and describe the collections of beliefs I attribute to the student. I will call this collection of beliefs *the student's understanding* of the logarithmic function. From the descriptions of the students and their beliefs each reader can judge for him- or herself if the students portrayed understood the logarithmic function.

A collection of beliefs might seem a rather odd definition of understanding, but is less so if we reflect on how we behave when a new mathematical concept is presented to us. We quickly attempt to give meaning (Sfard, 2000) to a concept by applying our existing beliefs about mathematics and our knowledge of mathematical concepts. Reports of novice-expert studies (Chi, Feltovich, & Glaser, 1981; Glaser, 1984) describe differences in what the two groups view as important. If beliefs are different, understanding will be different. Schoenfeld (1988) found students' beliefs about geometry were largely the result of their experiences with the subject. For example, they believed most geometry proofs could be done in a very few minutes. The students' experience became the basis for the theory they acted on. If they could not do a proof in a few minutes, they gave up. A student's understanding of a mathematical concept is much the same: his or her collection of beliefs about a concept is used to decide when, if, and how a concept is used. Thus it is students' beliefs about mathematics and specific beliefs about concepts that govern their learning and form their understanding of concepts.

Categories of Evidence

I will base my inferences about students' beliefs on four categories of evidence: conception, representation, connection, and application. A *conception* is a student's communicated feelings and ideas about the concept. A *representation* is a symbol the student uses to communicate the concept. A *connection* is a relationship between representations. An *application* is a use of the concept to solve a problem. After defining and giving an example of each of these categories of evidence, I will explain why they are indications of students' theories about a mathematical concept.

Conception.

A student's conception of a mathematical concept is his or her communicated feelings and ideas about the concept. For example, a student may describe the logarithmic function as a collection of letters or "frustrating." These are expressions of the student's conception of the function. Conceptions may be the result of various factors including, but not limited to, a student's goals for his or her mathematical activity.

A student's conception of a mathematical concept can affect his or her future attempts to learn more about or apply the concept (Sierpinska, 1994; Skemp, 1987). It is human nature to attempt to categorize objects we perceive. Mathematical objects are no exception to this rule. When a student sees mathematical notation for a function, he or she will try to make sense of it based on past experiences with mathematics. Research on students' classifications of function illustrates this point. If a student believes all functions are linear, when faced with a coordinate axes on which several points are plotted and asked to draw as many function as possible through the points, what will the student draw? Lines. A student's conception affects how the concept is applied. Hence his or her conception is evidence of his or her understanding of the concept.

Representation.

A student's representation of a mathematical concept consists of symbols the student uses to think about the concept and communicate it to others. In the study, I focused on four modes of representation: written, pictorial, tabular, and oral. Briefly, a *written* representation is a collection of letters and numerals, a *pictorial* representation consists of an image, a *tabular* representation is a compilation of numerical data in a table, and an *oral* representation is spoken. A student is likely to use a combination of these four modes when thinking or communicating about a concept.

Written representations are notations students use to think about and communicate a mathematical concept in writing. The written representations discussed in this report are *names*, *notations*, *maxims*, and *descriptions*. *Names* are terms that refer to mathematical objects, procedures, and collections of the two. One example of a name is the term *base*. *Notations* are definitions, properties, and examples of mathematical concepts written using mathematical symbols: $\log_2 1 = 0$ and $\log_a a = 1$ are examples of notation. *Maxims* are short statements meant to serve as mathematical rules or guides.

One example is the common: logarithms are exponents. *Descriptions* are explanations of procedures, outcomes of procedures, mathematical objects, and relationships. "A function is a collection of letters and numbers" is one example of a description.

Pictorial representations are images students use to think about and communicate a mathematical concept visually. An example of a pictorial representation often used in the exploration of the logarithmic function is the graph of $y = \log_2 x$, as shown in Figure 1:

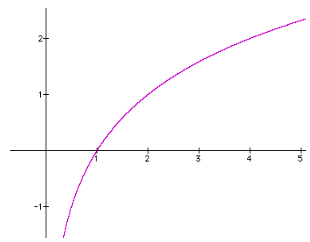


Figure 1. Graph of $y = \log_2 x$.

Tabular representations are tables of numerical data students use to think about and communicate a mathematical concept. An example of a tabular representation of $y = \log_2 x$ is shown below:

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2
$\log_2 x$	-2	-1	0	1

Oral representations are spoken words and expressions students use to talk about a mathematical concept. As with written representations, oral representations are names, notations, maxims, and descriptions. The definitions for these terms are the same except they are spoken not written. An example of an oral representation is: "Log of one is zero."

Representations play a role in all mathematical communication. They are used to convey an approximation of our thinking. The theories of understanding cited in this report (Hiebert & Carpenter, 1992; Pirie & Kieren, 1989, 1994a, 1994b; Sierpinska, 1994; Skemp, 1987) use representation. Although the understanding of a mathematical concept exists in the mind of the individual, how a student uses symbols to represent the concept is evidence of his or her beliefs about the concept. For example, if a student attempts to approximate $\log_3 2$ by graphing the function, this is evidence of the student's belief that a logarithm is associated to the graph of the logarithmic function. On the other hand if the student uses his or her calculator and the change of base formulas, then we might conjecture the student sees a logarithm as an algebraic computation. Students' uses of representations are indications of their understanding of a mathematical concept.

Connection.

If a student *translates* a representation from one mode to another or *transforms* a representation to another in the same mode (Lesh, Post, & Behr, 1987), I will say he or she has *connected* the two representations. For example, if a student can identify the graph of a logarithmic function (see Figure 2), he or she has translated a representation in the pictorial

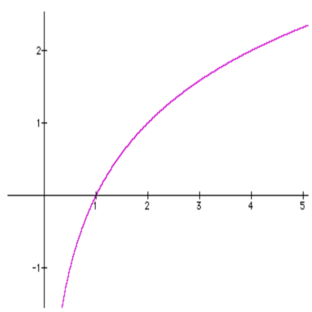


Figure 2. Example of a question designed to investigate connections.

mode to one in the written mode. The student has connected the graph, a pictorial representation, and its algebraic expression, a written representation. If a student rewrites the written representation $y = \log_2 x$ as $y = \dfrac{\log x}{\log 2}$, he or she has also translated the representation. Hence, there is a connection within the written mode.

According to Hiebert and Carpenter (1992), "the degree of understanding is determined by the number and strength of connections" (p. 67). The connections referred to are internal ones between representations, but Hiebert and Carpenter also note that evidence of a connection can be observed when a student relates two or more external representations. That connections are evidence of understanding is not a new idea, as Hiebert and Carpenter explain: "It is a theme that runs through some of the classic works within mathematics education literature" (p. 67). The connections I have defined and described are limited to external ones, but they provide evidence of students' beliefs about a mathematical concept.

Application.

An *application* of a mathematical concept is the use of the concept to help solve a problem. If a student uses a mathematical concept to solve a problem, he or she has linked the problem to the concept. This link indicates an understanding of how the concept can be used. For example, a student's ability to find $\log_3 8$, given $\log_3 2$ is evidence of the student's *beliefs* about the logarithmic function.

The ability to apply a mathematical concept in an unfamiliar situation is probably the most widely used test of understanding. According to Brown, Bransford, Ferrera, and Champion (1983) "We are reluctant to say that someone has learned elementary physics or mathematics if they can solve only the problems they have practiced in class" (p. 143). If a student can apply a concept to a novel problem situation, he or she understands something about both the problem and the concept. What a student knows can be extracted from his or her actions. If a student can transform the function $f(x) = \frac{1}{x}$ into linear form, I would conjecture that he or she knows more about the logarithmic function than a student who cannot. The application is evidence of the student's beliefs about the function.

Changes in Understanding

When the term *understanding* is used in colloquial speech, it generally means a static quantity, but teachers of mathematics know that a student's understanding of a mathematical concept changes over time. Students study and develop their beliefs in and out of class, and hence how their understanding changes is not always clear. Pirie and

Kieren (1994b) have hypothesized that these changes are occur in a process involving various levels of abstraction, indicating the development of understanding is orderly. They are careful to point out that their theory is based on conjectures of an observer. To an observer the student's actions appear in sequence or like a process, but the experience for the student is often very different. He or she may experience understanding as changes in their beliefs based on their experiences during various activities. For the student the understanding is not a process.

I do not claim that understanding is a process. In my own attempts at understanding I impose a structure on my activities to make the experience of learning feel less chaotic, but it does indeed feel chaotic (Halmos, 1985; Poincaré, 1946). The most I can claim about a student's understanding of a mathematical concept is that it changes. In this study, changes in students' understanding of a concept were seen as changes in their beliefs about the concept. The second purpose of the study was to identify changes in students' understanding of the logarithmic function.

Ways of Knowing

When a student tries to solve a problem, he or she does not always approach the problem the way I would. I have often been surprised at the approaches a student takes. For example, given the sequence 1, 2, 4, 8, … some students explain the action in the geometric sequence as multiplying each term by two to get the next term and coordinated (Smith & Confrey, 1994) this action with adding 1 in the arithmetic sequence 0, 1, 2, 3…. Describing the relationship between the two sequences as a map of multiplication by 2 to addition by 1 allows the students to predict terms in the arithmetic sequence that correspond to nonnegative integer powers of 2, but is not flexible enough to allow the student to make predictions about arithmetic corresponding with terms such as $\sqrt{2}$. I will call students approaches *ways of knowing*. Hence, students' ways of knowing are defined as operations and strategies they use to investigate problems they are asked to solve.

The students' ways of knowing can be used as the basis for future understanding, but they can also be constraining (Skemp, 1987). If a student sees the relationship between the two sequences as a map where multiplication by 2 in one sequence corresponds to adding 1 in the other, he or she may have an extremely difficult time reversing the relationship to find what $\frac{1}{2}$, inserted in the geometric sequence, maps to.

Despite the constraints a student's way of knowing may create, they are sources of meaning. As sources of meaning, the constraints have the potential to be what Sierpinska (1994) calls the *basis* (chapter 2) for the student's understanding. For me a student's ways of knowing provides insight into how a student's understanding of the logarithmic function can grow.

Research Questions

The primary purpose of this study was to describe students' understanding of the logarithmic function. Hence, the first two questions were about understanding. The first was about understanding the logarithmic function, and the second was about changes in understanding. The secondary purpose was to identify ways of knowing used by the students that could be used as a basis for growth of understanding. Hence, the third question was posed to look beyond what students understand and toward what they might be able to understand.

1. What is students' understanding of the logarithmic function?
2. What changes occur in students' understanding of the logarithmic function during the instructional process?
3. What ways of knowing do students use to investigate problems that include a representation of the logarithmic function?

CHAPTER 2: DISCUSSION OF RELEVANT LITERATURE

The purpose of this chapter is to summarize and discuss my interpretation of an conversation among educational researchers about understanding. The participants in the conversation were selected because of their influence on my view of understanding, the design of the study, and the data analysis.

In this chapter I outline four theories of mathematical understanding proposed in the last 25 years. Each of the theories was developed from a different perspective and uses a different definition of understanding. However, as I will show, these theories have common elements I used to investigate and describe students' understanding.

In the data analysis phase of the study, it was useful to examine the effects of students' beliefs about mathematics and understanding on their understanding of the logarithmic function. Some literature illustrating the connection between students' actions and their beliefs is discussed.

In the rationale for this study I noted my own dismay at the students' failure to remember either the definition of the logarithmic function, the basic properties of the function, or the graph. In addition, during the collection of data, students themselves remarked about either being able to remember or not remember. Hence, research on the connection between remembering and understanding is discussed.

Finally, the historical development of the concept of logarithms sheds some light on how the tasks in the study were developed. A historical development of the concept of logarithm is presented, and Smith and Confrey's (1994) work on both the exponential function and the logarithmic function is discussed.

Theories of Understanding

Skemp's Theory of Understanding

In 1976, Richard Skemp marked the beginning of the study of understanding in mathematics education research. His classic article entitled "Relational and Instrumental

Understanding" sought to define and describe these two types of understanding and to explain why so many teachers felt instrumental understanding was a type of understanding. Skemp credited Stieg Mellin-Olsen with the coining and definition of the terms. According to Skemp, *relational understanding* is "knowing what to do and why," whereas *instrumental understanding* is "rules without reasons" (p. 152). The emphasis on knowing what and why in Skemp's article gives one the impression he associates understanding with the type of knowing it produces (Sierpinska, 1990). A bit more reading reveals an expansion and revision of Skemp's categories of understanding.

Following the publication of Skemp's (1976) article in *Mathematics Teaching*, debate about the definitions and categories of understanding Skemp described ensued (Backhouse, 1978; Buxton, 1978; Byers & Herscovics, 1977; Tall, 1978). This discussion prompted Skemp to revise his definitions of and to include a new type of understanding he called *formal understanding.* Skemp (1987) elaborated on these new definitions attributed to Byers and Herscovics:

> *Instrumental understanding* is the ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works.
>
> *Relational understanding* is the ability to deduce specific rules or procedures from more general mathematical relationships.
>
> *Formal understanding* ...is the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning. (p. 166)

The language of "knowing" found in Skemp's (1976) original work regarding instrumental and relational understanding is replaced in this excerpt with "ability." Hence, for Skemp, understanding is linked to abilities. The question remains, how does one acquire these abilities?

In his book the *Psychology of Learning Mathematics*, Skemp writes "*To understand something means to assimilate it into an appropriate schema*" (p. 29). We can unravel this sentence a bit if we know what *assimilate* and *schema* mean. By schema, Skemp means a group of connected concepts, each of which has been formed by abstracting invariant properties from sensory motor input or from other concepts. The concepts are connected by relations or transformations. An example of how a schema works is given by Skemp:

> When we see some particular car, we automatically recognize it as a member of the class of private cars. But this class-concept is linked by our mental schemas with a vast number of other concepts, which are available to help us behave adaptively with respect to the many different situations in which a car can form a part. Suppose the car is for sale. Then all our motoring experience is brought to bear, reviews of performance may be recalled, questions to be asked (m.p.g.?) present themselves. (p. 24)

This characterization does not mean schemas are used only when we have had some previous experience with a situation: they are also used in problem situations with which we have no experience. For example, if one has never solved a logarithmic equation, but has solved linear equations, various techniques and information about solving linear equations might come to mind as one tried to solve the logarithmic equation. According to Skemp, “The more schemas we have available, the better our chance of coping with the unexpected” (p. 24).

As Skemp points out his definition of understanding is not based on finding *the* appropriate schema, but *an* appropriate schema. This distinction explains why students may think they understand a concept when they do not. Suppose for example a student thinks the notation $f(x)$ means $f \cdot x$. The student may believe he or she understands the notation. It is assimilated into his or her schema for multiplication and will be detrimental to his or her understanding of the concept of function. The student can *reconstruct* a schema if he or she encounters situations for which his or her existing schemas are not adequate. Skemp notes that this process is not easy or comfortable, because of the strength of existing schema. “If situations are then encountered for which they are not adequate, this stability of the schemas becomes an obstacle to adaptability” (p. 27).

Instrumental understanding is understanding. The problem is that to use such understanding the student must be able to identify the problem type and associate it with a solution procedure. Unfortunately, there are many types of problems a particular mathematical concept can be used to solve and memorizing all of them would be both difficult and inefficient. Nevertheless, many students memorize procedures and problem types, some with the encouragement of their teachers. Skemp notes the connection between the procedures and problem types is likely to deteriorate rather quickly, leaving the student unable to match the problem with the concept. Hence, instrumental understanding fails to have two qualities of relational understanding: adaptability and integration. A student who attempts to understand relationally will try to link a new concept with other concepts he or she has developed and then reflect on the similarities and differences between the new concept and those previously understood. The student then has resources to draw on when he or she gets stuck in a problem.

Logical understanding, according to Skemp, is what allows a student to communicate mathematically and be understood by others. Although a student might solve a problem correctly and understand it, that is no guarantee he or she could prove formally that the sequence of actions is based on a series of logical inferences used in mathematical proof. The following example illustrates how a student may have instrumental and relational understanding, but not logical understanding. A student may be able to find $f(4)$ given $f(x) = \log_2 x$, by proceeding as follows (example adapted from Skemp, 1987, p. 170):

$$f(x) = \log_2 x = \log_2 4 = 2.$$

When asked why he or she wrote this expression the student might respond, “To find any range value that corresponds with a given domain value for a given function *f*, one simply evaluates the function at the given domain value.” This explanation indicates that the student has relational understanding, but what he or she has written indicates a flaw in logic, namely, that the range of the function, $f(x)$, could equal a particular range value, $f(4)$. Although Skemp (1987) was able, with the help of his colleagues, to propose the construct of logical understanding, he was, as he put it, “working in his own frontier zone” when it came to describing “what kind of schema are involved in” (p. 171) this kind of understanding. His hypothesis was that schema involved were built from concepts consisting of classes of statements connected by logical implications. Hence, Skemp and his colleagues saw formal logical mathematical argumentation as part of understanding.

Skemp’s view on understanding expressed in his 1976 article is a very brief introduction to his theory of learning mathematics. Knowing how and knowing what are by-products of learning, or schema building, whereas understanding is part of the schema building process.

Pirie and Kieren’s Theory of Understanding

Pirie and Kieren’s (1994b) theory of understanding (Pirie-Kieren theory) is based on their belief that “mathematical understanding is a process, grounded within a person, within a topic, within a particular environment” (p. 39). The theory was developed in response to what Pirie (1988) saw as the inadequacies of category theories like Skemp’s for explaining children’s understanding of mathematical concepts. Pirie (1988) explained how a student, Katie, who had in previous interviews constructed an understanding of the division of a fraction by a fraction using pie diagrams, in a later interview could not give a pictorial representation for her symbolic action. Katie was able to divide fractions effortlessly using the standard algorithm but could not recall her previous way of operating with fractions. In the final interview, Katie illustrated how she divided fractions by inserting shaded pie pieces for the number symbols. Any researcher who observed this demonstration would conclude Katie did not understand the division of fractions but was simply following an algorithm. However, the interviewer having seen

Katie's actions in previous interviews, asked her if 50 divided by $\frac{1}{3}$ would be bigger or smaller than 50. According to Pirie (1988), Katie's instant response that the answer would be bigger was evidence she had an understanding of the division of fractions. The question was what understanding did she have? Into which of Skemp's categories did Katie's understanding fit? Pirie noted that extreme care should be used in labeling students' understanding. Obviously the researcher's experience with Katie allowed her to have an insider's perspective on changes in Katie's understanding. This context altered the meaning the researcher gave to Katie's actions.

Shortly after the appearance of Pirie's (1988) article, Pirie and Kieren (1989) published "A Recursive Theory of Mathematical Understanding." In this article, they describe a theory of understanding and illustrate how the theory can be used to explain a student's understanding of a mathematical concept. Their theory is one of *transcendent recursion*. It is transcendent in that each level of knowing, while compatible with prior levels, transcends those levels in sophistication. It is recursive in that the structure of the understanding at one level is similar to the structure of the understanding at another, and one level of understanding can call into action a previous understanding. For example, if a conflict occurs at a current level of understanding, the student has access to previous ways of knowing that can be used to help resolve the conflict. The result of this perspective is the following definition of understanding:

> Mathematical understanding can be characterized as leveled but non-linear. It is a recursive phenomenon and recursion is seen to occur when thinking moves between levels of sophistication.… Indeed each level of understanding is contained within succeeding levels. Any particular level is dependent on the forms and processes within and further, is constrained by those without. (Pirie & Kieren, 1989, p. 8)

In addition to a definition of understanding, Pirie and Kieren (1994a) developed a pictorial representation of their theory highlighting eight levels in the process of the growth of understanding (see Figure 3).

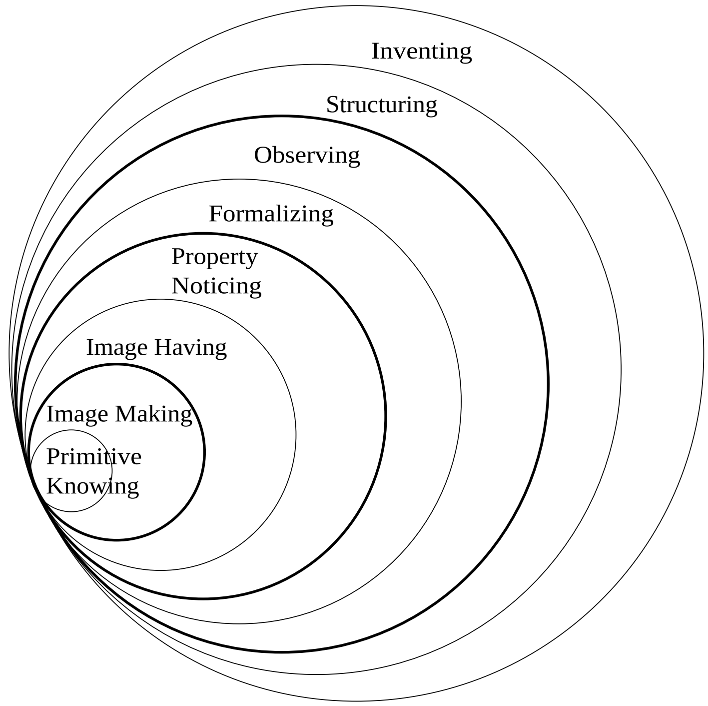


Figure 3. Pirie and Kieren's pictorial representation of understanding (1994a, p. 167).

Although the representation is static, the intention is that it be used as a tool for mapping an individual's changes in understanding of a mathematical concept over time. When such a map is completed, according to Pirie and Kieren (1992), it represents the student's process of understanding. In general, the inner levels of understanding leading up to formalizing are context dependent. The particular problems the student does and actions he or she takes will enable and constrain the properties he or she abstracts from them. Formalizing marks the beginning of reflections on mental objects free of the contexts from which they were derived and also marks the development and proof of theorems regarding these mental objects. Detailed definitions of each of the levels can be found in Pirie and Kieren (1992).

There is a major problem with the Pirie-Kieran theory. It is unclear how it was developed. There is no evidence in published articles (Kieren & Pirie, 1991; Pirie, 1988; Pirie & Kieren, 1989, 1992, 1994a, 1994b) that the model was adapted over time using the results of investigations with students. The theory is used to describe students' understanding, but there has been no critique of it by the authors. Unless a theory of understanding is tested and modified using data collected from students who perform both standard and nonstandard tasks, it is an opinion rather than a theory.

Sierpinska's Theory of Understanding

Sierpinska's (1994) theory is based on understanding an "act of grasping meaning" (1990, p. 27). She calls these *act of understanding.* Each act of understanding is composed of four components: the understanding subject, the object of understanding, the basis of understanding, and mental operations that link the object with the basis.

The understanding subject that Sierpinska (1994) refers to is not the psychological subject one teacher has in his or her class, but rather is the epistemic subject referred to by Beth and Piaget (1961). This subject is a compilation of all subjects who have grasped various meanings of mathematical concepts. For example, the epistemic subject developing the concept of real numbers would encounter difficulty with both the concept of zero and the negative integers. Historically in the development of the real numbers, both of these concepts were met with resistance, hence the epistemic subject would experience these concepts as obstacles. Called epistemological obstacles or "barriers to changes in frame of mind" (Sierpinska, 1994, p. 121), they become opportunities for the occurrence of an act of understanding. Sierpinska provides the following justification for her focus on the epistemic subject:

> If we want to speak about understanding of some mathematical topic in normative terms this notion of sujet épistémique comes in handy. To be exact, it is not the way 'a certain concrete Gauss' has developed his understanding between one work and another that will give us some guidance as to what acts of understanding have to be experienced or what

> epistemological obstacles have to be overcome in today's students. We have to know how a notion has developed over large periods of time, and in what conditions (questions, problems, paradoxes) were the great breakthroughs in this development brought about. This, and not historical facts about exactly who did what and when, can be instructive in designing our teaching and facilitating understanding processes in our students. (p. 40)

Hence, though observations of a single subject and his or her acts of understanding may help an individual teacher working with a concrete student, the construction of an epistemic subject can help any teacher working with any student.

Despite the inclusion by Sierpinska of the understanding subject, this subject has none of the human features of the students in our classrooms. The students we teach bring to their study of mathematics obstacles in mathematics, mathematical thinking, and obstacles in their beliefs and goals. They often believe they understand what was taught despite a mismatch with instructional goals set by the teacher. According to Skemp (1987), this belief of the student can shut the door to further examination of a mathematical concept. A student who feels he or she has failed to understand can be motivated to conduct further investigation whereas a student who feels he or she understands often terminates his or her investigation. Hence an investigation of understanding should include both the researcher's interpretation of understanding and the student's.

The second, third, and fourth components (object, basis, and mental operations) of Sierpinska's (1994) theory of understanding provide a picture of how a student attempts to construct meaning. These three components work together to produce an act of understanding. First, there is a mental operation of identification: "Identification is the main operation involved in acts of understanding ... acts that consist in a re-organization of the field of consciousness so that some objects" (p. 57) that were in the background are now in the foreground. Other mental operations identified and defined by Sierpinska are discrimination, generalization, and synthesis. Discrimination is the identification of differences between objects. Seeing an object as a particular case of a situation is defined as generalization. Synthesis is the "search for a common link" (p. 60) between generalizations.

Once an object is identified, the subject searches for a basis for his or her understanding. How can the object be given meaning? Objects are connected using mental operations. The existing object is called the basis for understanding the new object. In terms of the questions at hand, how does a student's understanding of the logarithmic function change? In Sierpinska's terms, if a student is able to identify the logarithmic function as an object to understand and can find a basis for it, he or she will understand. We must ask: What bases can there be for understanding the logarithmic function? To answer this question we must look at students in action.

Hiebert and Carpenter's Theory of Understanding

Hiebert and Carpenter (1992) propose a cognitive science perspective on students' understanding of mathematics. They conjecture the existence of networks of internal representations. The "number and strength" (p. 67) of the connections between representations is used as a measure of the degree of understanding. Hence a student who has an internal representation for the logarithmic function connected to the definition of function and to the graph of the function will have a stronger understanding than a student who has simply heard of the function. The theory is based on three assumptions: "Knowledge is represented internally and these internal representations are structural" (p. 66). There is a relationship between internal representations and external ones. And internal representations are connected. Hiebert and Carpenter further explain that internal representations and connections can be inferred from analyzing a student's external representations and connections.

An example of the connection between internal and external representation is provided by Lawler (1981). He presented his daughter, Merriam, with three tasks he viewed as involving the same computation, adding 75 and 26. First, Lawler presented the problem orally, then in terms of money (75 cents and 26 cents), and finally on paper. In each case, Merriam used a different method of computation. Context (external representation) made a difference in Merriam's thinking. In terms of Hiebert and Carpenter's definition of understanding, the external representations were related to Merriam's mental (internal) ones.

What then are these mental representations and connections? Although mathematics educators do not know how a student is representing mathematical concepts internally or the nature of these representations, according to Hiebert and Carpenter, the student's solution to a problem is influenced by external representations (physical materials, pictures, symbols, etc.) in the problem. Problems solved both in and out of school affect the internal representations and help form networks. Hiebert and Carpenter contend these mental representations are needed to "think about mathematical ideas" (p. 66).

Hiebert and Carpenter (1992) propose two metaphors for these networks of representations. First, networks are structured like vertical hierarchies. Representations are details of other more overarching representation. Hence if a student has a mental representation of function, in terms of a vertical hierarchy, an associated representation would be a linear function. Second networks are structured like webs. Representations of information form nodes connected to other nodes. Connections, according to Hiebert and Carpenter, are formed in one of two ways: by noting similarities and differences, and by inclusion. A new idea is compared with other ideas already represented mentally. Once similarities and differences are cataloged, a student can connect his or her mental representation of the idea to existing structures.

Hiebert and Carpenter (1992) explain the growth of understanding in terms of adjoining and reorganizing existing networks. Adjoining may occur when a student becomes aware of a mathematical idea for the first time. In an attempt to make sense of the idea the student searches for connections to existing mental representations. One result of this process is the connection of new ideas to unrelated mental representations.

For example, consider the addition of logarithms: log 4 + log 5. A student might connect this representation to his or her knowledge of the distributive property. This connection will result in the following calculation: log 4 + log 5 = log 9. Hence the idea is adjoined, but the connection is not useful. This connection can be modified through a process that Hiebert and Carpenter call reorganization. Reorganization can occur when a student reflects on his or her thinking and is aware of an inconsistency. For example, if a student subsequently sees log 4 + log 5 = log 20, he or she may have cause for reorganization. The new information is not consistent with current mental representations for adding logarithms.

Noting the importance placed on the communication and understanding of mathematics in both school and society, Hiebert and Carpenter (1992) explain how written symbols can be understood by students. If a symbol is to carry some meaning for a student, it "must be represented internally as a mathematical object" (p. 72). Hence in order for a student, to understand log 4, for example, he or she must represent the notation as mathematical object rather than a collection of symbols.

Common Elements in the Four Theories of Understanding

The four theories of understanding have five elements in common: obstacles to understanding, modification for efficiency or to overcome obstacles, basis for understanding, mental representation, and connections. Although the language and perspectives of the researchers' theories differ, each of the theories makes use of these five elements.

Obstacles to Understanding and Modification

Each theory of understanding contains the ideas of obstacle and of modifications in the face of obstacles. Skemp does not use the term *obstacle*; instead, he notes that a student may encounter a situation for which his or her schemas are not adequate. In this situation, "this stability of the schemas become an obstacle to adaptability" (p. 27) and the schemas must be reconstructed (modified) "before the new situation can be understood" (p. 27). Naturally there is no guarantee a student will successfully reconstruct his or her schema. Skemp notes that if an effort at reconstruction fails, then "the new experience can no longer be successfully interpreted and adaptive behavior breaks down — the individual can not cope" (p. 27).

An obstacle in the Pirie-Kieren theory is simply a problem that cannot be solved. "When faced with a problem or question at any level, which is not immediately solvable, one needs to fold back to an inner level in order to extend one's current, inadequate understanding" (1994a, p. 173). If the student cannot solve the problem, it is an obstacle to the growth of understanding. *Folding back* is the terminology Pirie and Kieren use to illustrate how students behave when they encounter an obstacle. Students return to inner levels of understanding to generate information and new ways of operating that will help them overcome the obstacles. This return to and modification of inner levels of understanding results in growth of understanding.

Epistemological obstacles are a major feature of understanding in Sierpinska's (1994) theory. Obstacles to the historical development of mathematical ideas indicate mathematical concepts that might be obstacles for students. Obstacles are overcome by what Sierpinska calls reorganizations. "Every next stage starts with a reorganization, at another level, of ways of understanding constructed at the previous stage, the understandings of the early stages become integrated into those of the highest levels" (p. 122). These reorganizations result in modification of the student's *beliefs* about a mathematical concept.

In the Hiebert-Carpenter (1992) theory of understanding, a network of mental representations grows as new and varied problem situations are presented. Growth will be inhibited if the problem types and contexts are of a very limited nature. For example, if students only encounter the logarithmic function as a rule for solving exponential equations, they will have difficulty finding log 8 given log 2. A student's limited network of mental representations is an obstacle to solving novel problems. Changes in understanding occur as the networks grow and connections are strengthened, or as networks are modified.

> The construction of new relationships may force a reconfiguration of affected networks. The reorganizations may be local or widespread and dramatic, reverberating across numerous related networks. Reorganizations are manifested both as new insights, local or global, and as temporary confusions. Ultimately, understanding increases as the reorganizations yield more richly connected, cohesive networks. (p. 69)

Hence, the obstacles in the Hiebert-Carpenter theory are the limited experiences of the students and the modification is the change in existing networks of mental representations.

Basis for Understanding

The third common element in the four theories of understanding is the basis for understanding. A student logically associates what he or she is presented in class with other concepts. These associated concepts are what Sierpinska (1994) calls the basis for the student's understanding of the presented concept.

For Skemp the basis of understanding is existing schema. For example, if a student understands the exponential function and inverse functions, these two concepts could be the basis for understanding the logarithmic function. The logarithmic function could be seen as a special case of an inverse function and be assimilated into the inverse function schema and connected to the exponential function schema.

In the Pirie-Kieren theory, *primitive knowing* is the basis for understanding. Primitive knowing is the understanding the student uses to build his or her understanding of a new concept. In the example in the last paragraph, a student's understanding of the exponential function and inverse functions could serve as primitive knowing for the student's understanding of the logarithmic function. Hence in the Pirie-Kieren theory, the basis for understanding a new mathematical concept is a previous understanding.

The basis for understanding in the Hiebert-Carpenter theory of understanding is the existing network of mental representations and connections. For example, if an external representation of the logarithmic function such as the graph of $y = \log x$ is compared with the graph of $y = 10^x$, the two functions can be identified as inverse functions based on the symmetry of the graphs about the line $y = x$. Hence the basis for understanding the graph of the logarithmic function becomes the graph of the exponential function.

Mental Representations and Connections

The final common elements of the four theories of understanding are mental representations and connections. These two elements play a particularly important role in the development of understanding in each of the theories.

According to Skemp (1987), a mental representation of common properties abstracted from experiences, either mental or physical, is referred to as a concept. Hence in a schema it is the concept that is the mental representation. If we want to be able to use our experiences in the future, they cannot be represented exactly as they have occurred. Instead, they are examined in search of regularities. These regularities can then be adapted to new situations we encounter. Concepts must be connected to form schemas. Connections for Skemp take the form of *relations* and *transformations*. A relation is a common idea connecting two concepts. For example, if one considers the following pairs of functions: $f(x) = 2^x$, $f^{-1}(x) = \log_2 x$; $g(x) = 3^x$, $g^{-1}(x) = \log_3 x$; $h(x) = 5^x$, $h^{-1}(x) = \log_5 x$, the relation between the two would be "is the inverse function of." A transformation "arises from our ability to 'turn one idea into another' by doing something to it" (p. 23). For example,
$8 = 2^3 \rightarrow \log_2 8 = 3$ is a transformation. Both relations and transformations help form connections between existing concepts and new concepts.

In the Pirie-Kieren theory, when a student is at the level of image making, the images may be either external or internal, but when the image having level is attained, the images are internal. The general quality of the images made has been abstracted. These images are one form of mental representations included in the Pirie-Kieren theory. At subsequent levels, operations are performed that abstract qualities from mental images and generalize them. The generalizations are also represented internally.

Both the understanding of concepts and levels of knowing are connected in the Pirie-Kieren theory. Connections between understandings of concepts can be seen in the "fractal like quality"(Pirie & Kieren, 1994a, p. 172) of a theory of a student's understanding. "Inspection of any particular primitive knowing will reveal the layers of inner knowings" (p. 172). Connections within a particular concept are formed as commonalties and are abstracted from the results of mental and physical action. For example, suppose a student has constructed the graph of $y = \log x$ and the graph of $y = \ln x$ on his or her calculator. The student may abstract an image of an increasing function from these two *made* images. This abstraction, although incorrect, is a *noticed* property of these two graphs.

Sierpinska also features mental representations and connections in her theory of understanding. She describes mental representation as a possible basis for understanding and as one source of obstacles. For example, consider the abstraction made by the hypothetical student in the previous paragraph. Based on two graphs, he or she abstracted the idea that the logarithmic function is increasing. This abstraction may prove to be an obstacle when the student tries to determine the limit of $y = \log_{1/2} x$ as x approaches 0 If a mental representation does form the basis of a student's understanding of a mathematical concept, according to Sierpinska, it is connected to the object of understanding by mental operations. The Hiebert-Carpenter definition and theory of understanding is built on representations, internal and external, and connections between representations.

Conclusion

Obstacles and modification, basis, and representations and connections were used in various ways in this study. First, idea of the existence of obstacles and modification was used in the development of the tasks and during the interviews. In particular epistemological obstacles in the historical development of the logarithmic function were considered as I developed the tasks. During the interviews, I pressed students when they considered their own thinking. For example, I always encouraged the students to justify their answers. On many occasions, attempts to justify their responses lead students to modifications their original approaches.

Second, I used the idea of the basis of understanding during the data analysis. During the data analysis I often asked myself what the student's basis was for understanding the logarithmic function. Thinking the evidence through using this construct clarified my findings. Third, representations and connections were essential elements of my own theory because they play a role in each of these theories and because of the role representations have played in my own mathematical thinking.

Beliefs

The release of the statistics reported in the Third National Assessment of Educational Progress (NAEP) motivated mathematics educators to study the existence of a connection between students' performance and their beliefs about mathematics. According to Carpenter, Lindquist, Matthews, and Silver (1983), the NAEP data illustrated that "students felt very strongly that mathematics always gives a rule to follow to solve problems" (p. 656). These findings prompted researchers to study the connection between students' beliefs and their behavior. The results of such studies (Kloosterman, 1991; Schoenfeld, 1989,1992; Underhill, 1988) indicated there was a relationship between students' beliefs about mathematics and how they thought about and did mathematics.

Affect is a term used by mathematics educators as a catch-all for emotions, attitudes, and beliefs (McLeod, 1988). McLeod defines affect as "all of the feelings that seem to be related to mathematics learning" (p. 135). Although there seems to be consensus that belief is an affective factor, in the mathematics education community there is little agreement as to the word's definition (Pehkonen & Furinghetti, 2001). Cobb (1986) defined beliefs as "assumptions about the nature of reality that underlie goal-oriented activity" (p. 4). This definition is the most useful one in terms of my approach to the research questions and my discussion of the findings. I chose this definition, first because it does not use the term *understanding* to define belief and second because it was consistent with my assumptions about students' understanding: it occurs within the individual, student's own opinions of their understanding are important, and it may be inconsistent with correct mathematics.

Using Cobb's definition, I took views that students expressed about what it meant to understand a mathematical concept as evidence of their beliefs about mathematics and understanding. Students' described mathematics as a collection of rules and procedures they must learn to achieve their performance goals and understanding. This view of mathematics among students and teachers has been well documented and shown to influence action (Dossey, 1992; Schoenfeld, 1989).

Although links between beliefs, understanding, and learning have been explored by some researchers (Schoenfeld, 1989; Szydlik, 2000), those studying understanding have focused their energy on describing the cognitive factors involved in its development. The theories proposed fail to address students' about mathematics beliefs, which Schoenfeld (1992) noted "shape their behavior in ways that have extraordinarily powerful (and often negative) consequences" (p. 359). Each of the theories discussed in this chapter, assumes understanding occurs within the mind of the individual. It is not clear how an individual's beliefs about mathematics and understanding might influence his or her understanding.

The results of this study illustrate the importance of beliefs in the development of students' understanding of mathematical concepts. Much like those who have studied preservice teachers' beliefs in attempts to explain their practice (Cooney, 1997; Dossey, 1992) I have found beliefs to be a tool to make sense of students' understanding of mathematical concepts.

Representations

Since Janvier's (1987) summary of research and theories on the role of representation in the learning of mathematics, debate has continued regarding how the term might be defined, how internal and external representations are related, and what can be learned from the study of representations (Goldin, 1998b). Some mathematics education researchers (Goldin, 1998a; Goldin & Kaput, 1996) claim that representations are a system we use to learn mathematics. In this study, representations provide evidence of students' understanding.

Sfard (2000) hypothesized the symbol referent pair is dynamic, the use of one influences how the other is seen. For example, the use of the representation log 3, and how and when it is used influence what a student sees as the referent for the representation. Often when a new concept is introduced to students, it is through representations. The student then strives to become proficient with the representation and its syntax. Although that sort of proficiency is not all the understanding mathematics educators hope for, syntactic proficiency is an understanding. External representations may enable a student to develop internal representations of the concept, and they may constrain his or her understanding of the concept.

Remembering and Understanding

A major revolution in experimental psychology was born with Bartlett's (1932) research on remembering. This seminal work illustrated the reconstructive nature of remembering. Since then, some psychologists who choose mathematics as a context in which to study memory have ignored this function. The result of this oversight has been a focus on students' errors as "bugs" (Brown & Burton, 1978). and the popular term used today *misconceptions*. Much of what is labeled misconceptions in mathematics could simply be attributable to the reconstructive function of memory. One good example of this function of memory are overgeneralizations (Byers & Erlwanger, 1985) made by students studying the notation of a new concept such as the sum property of logarithms. The students remember this property as $\log_b A + \log_b B = \log_b (A + B)$, an overgeneralization of the distributive law. Much of what students are asked to do in college algebra is based on remembering rather than understanding.

In 1985, Byers and Erlwanger called for more research on the role of remembering in understanding. They noted that very little work in mathematics education had focused on how remembering impacted students' understanding. In particular, they pointed out distortions in students' thinking. They called these changes in memory *transformations*. One such transformation is reported to have occurred in the daughter of one of the authors. "Her school topic in arithmetic was "carrying" and she was doing beautifully — until the Christmas break. After the break she started to produce consistently wrong answers" (p. 275). Her wrong answers were produced because the child was 'carrying' the ones and recording the tens. Another example of transformation was the example given by Pirie (1988) of Katie, who after learning the algorithm for dividing fractions, could no longer produce drawings that represented division of fractions. Byers and Erlwanger called for research that might explain these transformations and how and when they occur. Neither research focusing on retention nor research on organization, according to Byers and Erlwanger, had been able to accomplish this task.

Both the idea that memory is reconstructive, as suggested by Bartlett, and that remembering can result in errors, as suggested by Byers and Erlwanger, are important in this report. A brief glimpse at the literature on learning mathematical concepts will produce a long list of studies highlighting students' errors (e.g. Arcavi, Bruckheimer, & Ben-Zwi, 1987; Fischbein, Jehiam, & Cohen, 1995; and Schmittau, 1988). Despite the identification of these errors, little attempt has been made to find or inquire how students' understanding either becomes "distorted" (Byers & Erlwanger, 1985) or fails to meet cultural standards for validity.

The Logarithmic Function

Although we know that the development of a concept in the mind of an individual need not follow its historical development, there is much to be gained from knowledge of the historical development of a mathematical concept. In particular, in the study of mathematical understanding, knowledge of historical development of mathematical ideas gives us another lens through which to view a student's actions. Commonalties that occur in the way a student's understanding of a mathematical concept develops and the way it developed historically are, according to Sierpinska (1994) citing Piaget and Garcia

(1989) and Skarga (1989), attributable to commonalties in mechanisms of development and to preservation of historical meanings of terminology. In the present attempt to investigate students' understanding of the logarithmic function, I based several of the tasks on the historical development of the function.

My study of the historical development was motivated, in part, by the work of Smith and Confrey (1994). They outline the historical development of the concept of logarithms and note its consistency with students' actions (Confrey, 1991; Confrey & Smith, 1994; Confrey & Smith, 1995). These consistencies were observed during teaching interviews designed to investigate how students learn about the exponential function.

According to historians of mathematics (Boyer, 1991; Cajori, 1913; Eves, 1983; Katz, 1995), Napier developed logarithms to simplify the difficult job of multiplying large numbers, an admirable aim in the time before calculators. In honor of the tercentenary of Napier's invention, a volume (Knott, 1915) was published containing articles that discussed the invention and how it had been used in the intervening 300 years. Among the articles is one written by Lord Moulton (1915), who hypothesized three stages in the invention of logarithms. In the first stage, Napier identified the correspondence between a geometric and arithmetic sequence as a probable starting point for a solution. Second, he introduced "a geometrical representation of the original arithmetical operations " (p. 13). Third, Napier realized that if pairs of terms in the geometric sequence had the same ratio, then the corresponding terms in the arithmetic sequence were equal distances apart. Illustrations of each of the stages follow. Modern notations and an increasing geometric sequence are used to simplify the treatment. This approach was introduced by Victor Katz (1995).

If we juxtapose an arithmetic and geometric sequence, we can see how Napier could have thought of the idea of changing difficult multiplication into simpler addition.

Arithmetic sequence: 0, 1, 2, 3, 4, 5, ...

Geometric sequence: 1, 2, 4, 8, 16, 32, ...

In the geometric sequence, the product of 4 and 8 can be found by adding the corresponding terms in the arithmetic sequence and finding the number in the geometric sequence corresponding to the number 5, namely 32. Hence, multiplication is easily converted to addition. However, one can also see what Napier's challenge must have been by looking at the terms in both sequences. Neither sequence is dense. Napier needed to be able to multiply any given numbers, not just integer powers of some base. Even the selection of a base near 1 for the geometric sequence could not produce the density Napier desired. Napier's dissatisfaction with this method led to what Lord Moulton identified as Stage 2 in the process of invention.

Napier converted his arithmetic problem to a geometric one by considering points moving on lines as in Figure 4 (Katz, 1995). The point *P* on the upper line is moving

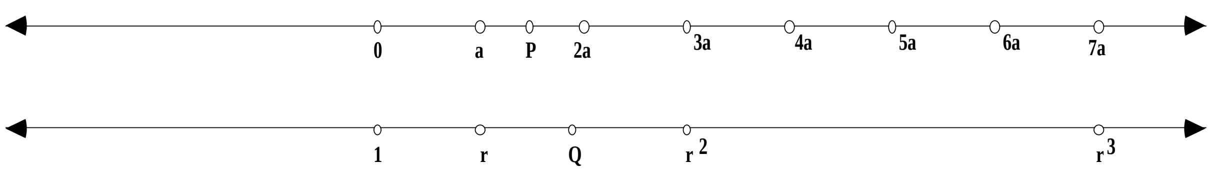


Figure 4. Illustration of Napier's representation of moving points.

arithmetically. *P* moves with constant velocity and covers each interval in the same amount of time. *Q* begins with the same velocity as *P* and covers the distance between points in the same amount of time that *P* moves between points. This requirement produces an increasing velocity on each of the segments [1, r], [r, r^2], [r^2, r^3], and so on. According to Katz, Napier's geometric representation allowed him to think of point Q as moving with a smoothly increasing velocity.

Using his geometric representation, Napier considered any two intervals [α,β] and [γ,σ] on the lower line such that $\frac{\beta}{\alpha} = \frac{\gamma}{\sigma}$ (Katz, 1995). He then noticed when the forgoing relationship held, the time it took *Q* to cover [α, β] was the same as the time it took to cover [γ, σ]. Since the measure of time on the upper and lower lines was the same, the points on the upper line being called the logarithms of those on the lower, and since *P* moved with constant velocity, then log β - log α = log σ - log γ. This correspondence marked what Lord Moulton called the third stage of the invention of logarithms.

Mathematical Ideas Used by Napier

The treatment presented here is simplified: however, it illustrates several very important mathematical ideas. First, it illustrates that multiplication in the geometric sequence corresponds to addition in the arithmetic one. This idea has been identified by Confrey and Smith (1994, 1995) as problematic for students. Citing Rizzuti (1991), they conjecture that the correspondence definition of function attributed to Dirichlet is based on a more primitive concept called *covariation* found in earlier definitions of function. Covariation is defined by Confrey and Smith (1995) as "being able to move operationally from y_m to y_{m+1} coordinating with movement from x_m to x_{m+1}" (p. 137).

Second, Confrey and Smith (1994) identified the idea of a multiplicative unit and a multiplicative rate as primitive actions of students. In interviews with students, Confrey and Smith presented the students with a table of values meant to represent the division of a cell over time. The time, starting at zero, was given in integral values, while the number of cells was given in powers of 9. Students identified multiplication by 9 as the action for moving down the table and division by 9 as the action for moving up the table. According to Confrey and Smith, the students also identified various powers of 9 as a constant ratio between terms. For example, they found the ratio between successive terms $\left(\frac{y_m}{y_{m-1}}\right)$ was to be 9 regardless of the position of y_m in the table. Similarly, $\frac{y_m}{y_{m-2}}$ was identified as 81, again regardless of the position of y_m. Hence, the primitive actions of students were consistent with stages that have been identified as important (Katz, 1995; Moulton, 1915) to the invention of logarithms.

Third, the summary of the invention of logarithms indicates the vital role that representation played in Napier's conjectured solution to the density problem in the coordination of the actions of two sequences. This representation along with his ingenious idea of two points moving along these lines allowed Napier to see how the terms in the sequences could be related using of the positions of the points.

These three important stages in the invention of logarithms provide a lens for examining and analyzing students' actions on tasks using representations of the logarithmic function. In addition, the historical development of the logarithmic function may a guide for how students can make sense of both the function and its modern applications.

Conclusion

Understanding is described in the literature as assimilation, a network, an act, and a process. Although each of the characterizations is different, all of them were developed to explain and predict students' behavior. In addition, representations and connections are elements of each theory. My definition of understanding also incorporates representations and connections. The literature on representation cited in this chapter helped me realize the student's external representations and connections could be used as evidence of their beliefs. Hence I noted that the representations I am looking for and at in this study are external. From these and the student's conception and application of a mathematical concept, I can hypothesize students' beliefs.

The link between beliefs and student behavior as expressed by Schoenfeld (1989, 1992) was useful in explaining what I found in this study. Students' beliefs impact their understanding in very specific ways and the existing literature helped me identify the impact.

A common view held by the students in this study was that remembering was either the same as understanding or a part of understanding. Hence, an examination of how remembering and understanding are related was called for. Bartlett's (1932) view of memory as reconstructive provides a useful perspective for the analysis of the students' actions during problem solving and will be revisited in chapter 6.

The historical development of the logarithmic function discussed in this chapter helped me develop tasks to investigate students' understanding and acted as a lens through which to view students' actions. In addition, the historical development highlighted the importance of representation in the creation of the logarithmic function. Finally the literature on the logarithmic and exponential functions alerted me to various mental actions to look for as I observed the students solving problems. In particular, I proposed coordination of actions between two sequences as a path students might take as they generate understanding of the logarithmic function.

CHAPTER 3: METHODOLOGY

The goal of this chapter is to describe the study, so others who wish to critique or duplicate it may do so easily (Rachlin, 1981). The description will include discussion of techniques, instruments, participant selection, and procedures.

Research Techniques

The research questions and the theoretical framework for the study should suggest appropriate data-gathering techniques for the study. Regarding understanding as existing within an individual suggests evidence of an individual's understanding should be gathered through interaction with the individual. The interpretation of these interactions is classified as qualitative analysis. According to Truran and Truran (1998), "Qualitative analysis interprets spoken or written language, and sometimes other forms of communication, such as drawings or body language" (p. 61). This method is ideally suited for the investigation of understanding.

Techniques and Rationales

Five techniques (phenomenological interviews, clinical interviews, mapping, drawings, and participant observation) were used to gather data regarding students' understanding of the logarithmic function. A description and rationale for each of the techniques is given in this section.

Phenomenological interview. In this study, students' understanding, changes in understanding, and ways of knowing the logarithmic function were investigated. Phenomenological interviews are designed to gather data about specific to help the researcher build a description of the participant's view of his or her world. The fundamental assumption made in a phenomenological study, according to Kvale (1993), is that "the important reality is what people perceive it to be" (p. 52). This perception was needed to build a description of a student's conception of the logarithmic function.

Thus, the phenomenological interview is a technique ideally suited for data collection in this study.

Clinical interview. According to Brownell and Sims (1972), “Understanding is inferred from what the pupil says and does, and from what he does not say and do in situations confronting him” (p. 41). In this study, I inferred understanding from students’ actions and utterances. The clinical interview provides a forum for the study of actions and utterances in a problem-solving environment. During a clinical interview, the researcher attempts to gather information about a student by watching him or her perform tasks and asking questions about the decision making process. Much of the interview is planned prior to meeting with a participant: however, questions and tasks that are not preplanned may be posed. Zazkis and Hazzan (1999) identify Piaget as the originator of the clinical interview. If the goal of the research is “the explication of thought” (p. 430), then the clinical interview is an effective tool for achieving it. The flexibility to follow a line of questioning that is not preplanned and the opportunity to observe a student in action that this technique allows made it an appropriate data collection method for this study.

Mapping. “To map is to construct a bounded graphic representation that corresponds to a perceived reality” (Wandersee, 1990, p. 923). This is the perceived reality I hoped to see when I asked the participants to draw maps of the logarithmic function. The mapping technique I taught the participants to use was proposed by Novak (1972, 1990) for studying students’ understanding of science concepts. Gathering a student’s graphic depiction of his or her perceived reality provided data from which I developed a description of his or her understanding of the logarithmic function

Drawings. When a student visualizes a picture of his or her process of understanding, he or she often includes unarticulated representations of his or her beliefs. For example, if a preservice teacher is asked to draw a picture of what he or she thinks mathematics is and the drawing is a tool kit, we can infer something about his or her beliefs about mathematics. In this study, drawings were used to gather a student’s impressions of his or her understanding and changes in understanding. I used these impressions to develop the meanings for the students' use of the term understanding. The drawings provided a window into the students' beliefs about *understanding*, mathematics, teaching, and the logarithmic function that might have been impervious to simple questioning.

Participant observation. According to Wolcott (1999), participant observation is a way of experiencing the world in which the participants live. The researcher observes or otherwise engages in the world of the participants. In this study, I used the technique as I observed three college algebra classes. The goal of these observations was threefold. First, I used the technique to collect data about the behavior of the participants in their college algebra classes. The data were to be used to construct a general description of each participant. Second, I used the questions students asked and the responses they provided as evidence of their understanding. Third, I used participant observation to catalog the curriculum used to teach the logarithmic function. The use of this technique provided me with information about the student's understanding of the logarithmic function during instruction.

Instruments and Interviews

Interview protocols and tasks were used in combination to gather evidence from the participants. The instruments in this study were the interview protocols (see Appendix A for the complete interview protocols). A description of each of the protocols and of the question it was designed to provide evidence for is given in this section.

Interview 1

The first interview consisted of five activities: mapping instruction, phenomenological questions on the student's prior experience with the logarithmic function, skills assessment, phenomenological questions on understanding the logarithmic function, and mapping of the function.

The mapping instruction used in this study was adapted from Novak and Gowen (1984). Two adaptations were made. First, the student was not instructed to rank concepts he or she generated related to the concept being mapped. This adaptation was necessary because I disagree with Novak’s view of concepts as being related in a hierarchy. Second, the student and I constructed two maps together, before he or she was asked to construct a map independently. The two maps we constructed together were for the concepts *car* and *pet*. The student and I each mapped the third concept, *high school*, and then we described and compared our maps.

The primary purpose of the mapping instruction was to train the student to make maps. The secondary purpose was to become more familiar with the student. Discussions regarding cars, pets, and high school were used as data in the development of a description of the student.

Following mapping instruction, the student was asked about his or her prior experiences with the logarithmic function. Specifically, the student was asked to recall and recount any experiences he or she had had with the logarithmic function. The purpose of this phenomenological section of the interview was to gather data regarding the student’s understanding of the logarithmic function prior to instruction.

The student was then given the skills assessment activity. The skills assessment consisted of definitions of terms, recall of properties, and questions associated with representations and applications of the logarithmic function. In particular, the student was asked to define the terms *function*, *logarithm*, and *logarithmic function* and to list all properties of the logarithmic function he or she could remember. The remainder of the assessment is best described as a traditional mathematics examination based on the logarithmic function. Three sample problems for the skills assessment are given in Figure 5.

This activity had two purposes. First, it was used to gather information about the students’ understanding of the logarithmic function prior to instruction. This information was used to develop descriptions of the student’s understanding of the logarithmic function. Second, it was meant to stimulate recall of any concepts associated with the logarithmic function in preparation for mapping the concept.

1. Simplify the following expressions:

   a. $\log_3 4 + \log_3 5$

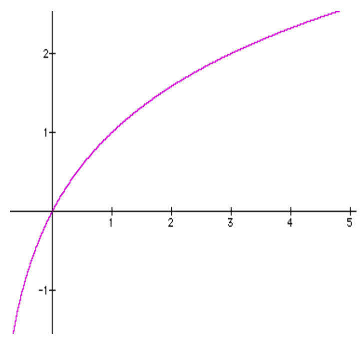
   b. $\log_3 4 - \log_3 5$

   c. $\log_3 9$

   d. $\frac{1}{2}\log_3 25$

   e. $\log_3 1$

2. What function is graphed on the axes below? ____________________



3. The 1980 population of the United States was approximately 227 million, and the population has been growing continuously at a rate of 0.7% per year. Predict the population in the year 2010 if this growth trend continues.

Figure 5. Sample problems form the skills assessment.

Following the skills assessment, the student was asked to recall and recount what he or she understood most while completing the skills assessment and then what he or she understood least. The purpose of these phenomenological questions was to gather data on the student's perspective of his or her understanding of the logarithmic function.

Mapping of the concept of the logarithmic function followed the phenomenological questions. I asked the students to include anything that came to mind when they thought of the logarithmic function and to map those concepts. The purpose of this activity was to gather evidence of the student's understanding of the logarithmic function.

Interview 2

The second interview began with two questions adapted from Brookfield (1990). I first asked the student to recall and recount learning experiences that he or she felt were significant to him or her as a learner. I then asked the student to recall and recount experiences he or she felt were frustrating. These two questions encouraged communication and trust between myself and the student. In addition, in several cases these questions elicited responses about learning that occurred outside of school and thus provided more background information about the student.

The focus of the interview was then shifted to mathematics. The student was asked to describe him- or herself as a mathematics student and to identify any goals he or she had when taking a mathematics class. These questions were meant to elicit background information about the student and his or her view of mathematics. In particular, the students' view of mathematics and goals associated with mathematics classes can influence the student's perspective regarding understanding. The student was then asked about his or her educational goals. This question was designed to gather information that I could use in my description of the student.

The third series of questions focused on the phenomenon of understanding a mathematical concept. The student was asked to identify a mathematical concept that at some time he or she did not understand. After sharing the concept, the student was asked recount his or her experience of not understanding. The student was then asked to recall and recount a time when he or she felt they had understood a mathematical concept. In addition, the student was asked to define understanding. The purpose of these questions was to collect data about the student's definition of understanding and to develop my view of the student meaning for the term.

Further evidence of the student's definition of understanding was gathered using a drawing task. The student was given a pen and a sheet of paper and asked to visualize and draw a diagram or picture of his or her process of understanding. After completing the drawing the student was asked to explain how the drawing related to his or her process of understanding. Specifically, the student was asked to refer to his or her drawing while recounting a time when he or she first did not understand a mathematical concept but then later did. This activity provided evidence that I used to build a description of the student's view of understanding mathematical concepts.

The questions and activities regarding understanding used in Interview 2 where adapted from those I used in a previous study of understanding. The study, Acts of Understanding: A Phenomenology (Kastberg, 2000), was designed to help me investigate and build a theory of students' perceptions of their experiences of understanding a mathematical concept (see Appendix B for a complete protocol). The only change in the protocol was a question about the student's definition of understanding. The descriptions and definitions were necessary to discern the student's meaning for the term *understanding.*

Interview 3

In the third interview, the student was asked to recall and recount a time during class when he or she did not understand the mathematics being presented and to repeat the activity for a time when they did understand. The student was then asked to construct a map of the logarithmic function.

The purpose of this phenomenological interview was to collect the student's perceptions of his or her experience of understanding the logarithmic function and to gather data about the student's understanding of the logarithmic function. The student's description and map were used as evidence of the student's understanding of the logarithmic function during instruction.

Interview 4

During Interview 4, the student was given the skills assessment used in Interview 1. This assessment was then followed by the phenomenological questions about understanding. Specifically, students was asked to recall and recount times during the interview they understood mathematics. A similar question was asked focusing on what was not understood. The student was then asked to construct a map of the logarithmic function. The purpose of this interview was to gather evidence about the student's understanding of the logarithmic function following instruction and, in particular, about the student's conception of the logarithmic function.

Interview 5

Interview 5 originally consisted of three different activities presented in pictorial representation. The first is described in the next paragraph. The second was the interpretation and use of the graph of $y = \log_3 x$. The third was an approximation task. Given the logarithms of several integers, the student was asked to approximate the logarithm of others. During the pilot, I realized these three tasks were time intensive and eliminated the third task. During the study, each interview was 90 minutes long. The time was insufficient for the completion of both tasks. This realization occurred to me as I interviewed the first participant. I elected to eliminate the second task and focus on probing the students' responses on the first task.

The task used in this interview was adapted from Jacobs (1970). The student was given a number line with the integers from 0 to 10 on it. Above the integers 1, 2, 4, and 8 on the number line were the numbers 0, 1, 2, and 3, respectively, enclosed in boxes. I told each student a short story about the picture before he or she began the questions on the task sheet. The following is an example of the story I used during the interviews:

> I'm going to start out by telling you a little story about this [the pictorial representation of the logarithmic function]. The story is there is this guy and he gets in his car in the morning. He is like 'I'm going to take a trip.' And so he starts from his house and he drives one mile and he sees a giant sign with a zero on it. He is like whew, what's this. This is very strange. So he drives two miles from his house and he sees another sign that has a giant one on it. He is like uh, what is going on here. So he drives from his house, he drives four miles and he sees a two, he drives eight miles, he

sees a three. He's like what is going on. But, then a light bulb comes on and he says ok, I know what's going on. I know where I'm going to see the next sign.

The student was then asked to predict what number he or she expected to be above each of 64, 256, $\frac{1}{2}$, $\sqrt{2}$, and 3. (For the sake of brevity the numbers above the number line will be called *sign numbers* and those below the number line will be called *number line numbers.*) This question was designed to invoke generalizations and alternative representations to help the student predict the sign numbers that corresponded to the number line numbers. To evoke generalizations about the number line numbers, the student was asked if there were any number line numbers that could not have signs above them. In an attempt to see if the students could reverse his or her prediction procedure, the student was asked to predict number line numbers corresponding to the sign numbers 7, -7, $\frac{1}{2}$, $\frac{3}{4}$, and $\sqrt{2}$. This question was followed with another generalizing question: Are there any numbers that cannot be on signs? Finally, the student was asked to explore the correspondence between the number line numbers and the sign numbers by generating properties based on two arbitrary number line numbers *A* and *B* corresponding to sign numbers *m* and *n*, respectively. Specifically, the student was asked what sign number would be above *AB*, $\frac{A}{B}$, and $\frac{B}{A}$. To invoke translations and transformations to other representations, the student was asked how he or she would organize and display all the data generated in this activity and to write down everything he or she knew about the relationship between the number line numbers and the sign numbers. He or she was then asked to recall and recount experiences of understanding and not understanding during the activity. The purpose here was to generate evidence regarding the student's perception of his or her understanding of the logarithmic function.

In this interview and in Interview 6, the logarithmic function was not mentioned. One reason for this was to see if the student was able to apply his or her understanding of the logarithmic function to solve the problem. Another reason was to gather evidence of the ways of knowing used by the student.

The purpose of this interview was to gather evidence of representations, connections, and applications of the logarithmic function that the student used to complete the task. This evidence was used to conjecture the student's beliefs about the logarithmic function following instruction.

<u>Interview 6</u>

Roy Smith, a university mathematician, and I developed the activity used in Interview 6. The student was given a function, *f*, and was told the function obeyed two rules:

$f(AB) = f(A) + f(B)$ and $f(2) = 1$. Based on this information, the student was asked to find the value of the function for 4, 8, 16, 256, $\frac{1}{2}$, $\frac{1}{8}$, $\frac{1}{256}$, $\sqrt{2}$, $\sqrt[4]{2}$, 0, -4, 3, and $\frac{3}{2}$. In practice this task was very difficult for the students because they could not decipher the notation. Using the data gathered from a pilot study using these tasks, I modified the task and included an example. Before the student began, I wrote the following example using the notation on the task sheet:

$$f(4) = f(2\cdot 2) = f(2) + f(2) = 1 + 1 = 2.$$

Students had less difficulty with the task after I presented the example. After the student generated function values for each of the real numbers, he or she was asked how the data could be organized and displayed. The student was also asked to record everything he or she knew about the function *f*. These questions were designed to invoke representational translation or transformation. The task was followed by the phenomenological questions regarding the student's experiences of understanding and not understanding while completing the task.

The purpose of this interview was to collect evidence of the student's representations, connections, and applications of the logarithmic function when a problem is presented in a written representation. This interview was also designed to collect evidence of the student's ways of knowing. Evidence of the student's perspective of his or her own understanding was also gathered.

Interview 7

Interview 7 involved two table-completion tasks. In the first task, the student was given a table with the numerals 1–9 and the first column and the numbers 10, 20, and 30 in the first three cells of the third column. The last six cells were left empty. Approximations for the logarithms of 1, 3, 5, 7, 8, 9, and 10 were provided in Columns 2 and 4. The student was then told the second and fourth columns contained an approximate value of the logarithm of the numbers in the first and third columns and was asked to complete the table. The student was also asked to find log 9000, log 0.09, and $\log\frac{5}{8}$ using the table. The student was given a TI-15 calculator for the completion of this task. This calculator is designed for elementary school students and does not have a logarithm or a natural logarithm key.

The second table-completion task was based on the function $y = \log_3 x$. The numerals 1 - 18 are entered in Columns 1 and 3; however, the only approximation in the second column was for $\log_3 2$. The student was then asked what other information would be need to complete the table, what other ways the data might be represented, how these other representations might help to fill in the table, and what the best way to represent the data would be. Following the table-completion activities, the student was again asked to recall and recount his or her experiences of understanding and not understanding during the activities.

The purpose of this interview was to gather evidence of the student's understanding of the logarithmic function, to identify ways of knowing, and to gather evidence of the student's perception of his or her understanding.

Interview 8

In order to determine how students used representations and in their oral representations of a mathematical concept, the student was asked in Interview 8 to explain what he or she knew about the logarithmic function. Specifically, the student was asked to pretend that I was a new college algebra student who knew the concept of function and to explain the logarithmic function to me.

This activity was followed by the mapping activity. The student was asked to draw a map of the concept of the logarithmic function. As in the other interviews, the maps were used to gather evidence about the student's understanding of the logarithmic function.

Interview 9

The purpose of the final interview was to gather the student's perspective on changes in his or her understanding of the logarithmic function. First, the student was asked to recall and recount a time in his or her study of the logarithmic function when he or she did not understanding the logarithmic function. Similarly, the student was asked to describe a time when he or she did understand the logarithmic function. Following these phenomenological questions, the student was asked to visualize and then draw his or her process of understanding the logarithmic function. After completing the drawing, the student was asked to explain the drawing.

In addition to the activities described here, the student was also provided with summaries of all of his or her comments regarding understanding in general and understanding the logarithmic function in particular. I had planned to ask the student to compare the summaries to his or her drawing, but no student actually read the summaries. Therefore, I had to eliminate this question from the interview.

Attention was then drawn to the maps that the student had produced during the course of the study. The student was asked to compare and contrast the maps, to give an example of how his or her understanding of the logarithmic function had changed since the beginning of the study, and to illustrate how that change was manifested in the maps.

Developing the Interview Protocols

The interview protocols and tasks described above were the result of a refinement process that included three developmental stages. First, as the protocols and tasks were developed, they were reviewed and discussed by five doctoral students (writing group) in the mathematics education program at the University of Georgia. Based on the feedback I received from this group, the interviews and tasks were revised. Second, the protocols for Interviews 5, 6, and 7 were trialed with two graduate students in the master's program at the University of Georgia. These trials allowed me to see how students might attempt to respond to the questions and solve the problems I posed. The third developmental phase was a pilot study.

The pilot study was conducted over a 2-week period during October 2000. Two former students of a doctoral student in the mathematics education program at the University of Georgia were asked to participate in the pilot study. Both students were taking precalculus at the time of the study. In fact, the interviews happened to coincide with the presentation of concept of the logarithmic and exponential function the students were seeing in their precalculus class. The two students were paid $100 each for their participation.

Despite the condensed timeline, the data I gathered during the pilot study provided me with an additional opportunity to critique and modify both the interview protocols and the tasks. It also allowed a chance to check the coding scheme that I intended to use to analyze the data. A fellow graduate student and I used the coding scheme to analyze a transcript from Interview 4. The students' actions during the interviews allowed to me prepare for what I was going to see during the main study.

Procedure

Research Site

The study was conduced at a rural community college, RC, in the Southeast serving an agrarian community. RC, a two-year college, is a community-based residential institution offering programs in the natural and physical sciences, the liberal arts, the social sciences, business, physical education and recreation, and the health occupations. It is also a specialized institution serving a unique role through programs in agriculture and related disciplines. According to the college catalog, when it was founded in the early 1900s the institution's mission was the development of technological expertise in young men and women wishing to embark on careers in agriculture, home economics, and related fields. Later, the role of the institution was expanded to include college transfer programs designed to prepare students to enter senior institutions in the state university system. All majors at RC are required to take at least one mathematics course. Students in the technological programs take a technical mathematics course or a mathematical modeling course; those in most transfer programs take calculus preparatory courses such as college algebra, trigonometry, and precalculus. Students who intend to transfer into science or business programs at senior institutions in the state are required to take calculus. Most RC students enroll in or are placed in calculus preparatory courses even though calculus can be taken in the first year. For students in the humanities and in the elementary education program, college algebra is a terminal mathematics course.

College algebra, a 3-semester-hour course at RC, was designed around the concept of algebraic and transcendental functions. The functions studied in the course included polynomial, rational, exponential, and logarithmic functions. The final topic of the semester in college algebra was logarithmic functions. Approximately 5 hours of instruction were spent on this concept.

Participant Selection

The participants for the study were students enrolled in college algebra at RC. All fulltime tenure track instructors teaching college algebra at RC during the fall, 2000 term were contacted and asked to volunteer to be observed while teaching the logarithmic function. To reduce the number of classroom visitations that I made, only three volunteers were selected. Selection was based on time and day that each instructor was teaching and the date that the presentation of the logarithmic function was to begin. Those who planned to start teaching the logarithmic function before my arrival at the research site were not considered for the study. No afternoon or evening classes were selected in an attempt to observe classes with primarily traditional aged students.

To solicit participants, I visited each of the three instructor's classes and gave a short 5-minute presentation about the study. I explained what participants would be asked to do and how they might benefit from participation. I then distributed forms (see Appendix C) with the same information. At the end of class, the instructors asked the students to submit the forms as they left class. A total of 29 students volunteered to participate in the study. Every attempt was made contact all volunteers by phone and by e-mail for an initial meeting.

During the 5- to 10-minute initial meeting, the volunteers were asked questions to gather background information. I asked how and why they chose RC and how they might rate themselves as a mathematics student. In addition, volunteers were asked whether they had ever sought help in mathematics and whether they would feel comfortable being observed as they worked on mathematics problems. At the conclusion of the initial meeting, each volunteer was asked whether he or she was still interested in participating in the study.

Six students, two from each instructor's class, were selected to participate in the study. The selection of was based on three criteria: my impression of the student's comfort level with me and willingness to respond to questions, my impression of student's willingness to seek help with mathematics if and when difficulty was encountered, and the student's reported mathematical ability. I chose the first two criteria in an attempt to maximize responsiveness during the interviews. The third criterion was used in an attempt to select students whose goal it was to pass the course.

Each of the participants was paid $150 for his or her participation in the study. The monetary award was a strong motivator for at least three of the participants, each of whom noted that he or she might not have volunteered if compensation had not been offered. I chose to compensate the students for two reasons. First, the participants were committing a substantial amount of time to the study. Each interview was scheduled to last between 60 and 90 minutes, and nine to twelve interviews were conducted. Hence the money was offered as partial compensation for the participant's time. Second, because of the small size of the sample, I needed to retain all of the selected participants for the duration of the study. The cash award provided incentive for the students to complete all of the interviews.

One of the instructors (Teacher 2) began presenting the logarithmic function the day I attended his class to solicit participants for the study. It was his practice to introduce concepts slowly over a number of days. Although his focus during the first few days was not the logarithmic function, several of the activities he used were meant to stimulate students' thinking about the function. I was only able to select the participants from Teacher 2's class after he had presented some concepts associated with the logarithmic function. Hence, although I had selected two students from each of the three classes to participate in the study, the data collected from the students in Teacher 2's class were not analyzed.

Data Collection Phases

The procedure for the study was broken into three phases: preinstructional, instructional, and postinstructional. During each phase, evidence of the student's understanding of the logarithmic function was collected. This evidence was used to develop conjectures about the student's *beliefs* about the logarithmic function during each of the phases. My decision to gather data in these three phases was based in part on Brownell's (1972) comments regarding researcher's claims of learning. In particular, Brownell noted that learning claims should only be made if some time between instruction and testing was built into the study. Naturally, when students are tested immediately following instruction, they often demonstrate increased computational proficiency on the concepts presented. A better measure of what a student understands is what he or she knows after some time has elapsed. Hence in this study, data on the students' understanding were gathered 6 weeks after instruction was completed. Another reason I collected data during the three phases of the study was to gather evidence of changes in the student's understanding.

Preinstructional Phase

The preinstructional phase was conducted during the second week of November, prior to classroom instruction on the logarithmic function. In these classes I took notes from the board and observed the general structure of the class. In particular, I looked for general procedures used by the instructors and how each of the participants responded to those procedures.

During this phase Interviews 1 and 2 were scheduled and completed. Both interviews for each participant were 60- to 90-minutes long and the two were completed within a five-day period. The interviews were conducted in a large office in an academic building on the RC campus. The office was comfortable and contained a large office desk with a pull-out surface on which the students could work. For each interview, the student was positioned to my left with the pull-out work surface between us. I used a boundary microphone and a standard cassette recorder to tape the interviews. I chose a boundary microphone because it was less conspicuous and it picked up even very soft speech. During the pilot study only the cassette recorder was used. When students became confused, they spoke softly and the recorder did not pick up their comments.

Instructional Phase

The instructional phase of the study was concurrent with instruction on the logarithmic function. This phase consisted of classroom observations and student interviews. During the observations, I developed a set of class notes and recorded various actions of the participants. My field notes were recorded on 8.5-by-11 paper, in two columns. The left column contained the board notes and relevant comments made by the instructor; the right column contained observations regarding student behavior during class (e.g. student responds to instructor's question).

Within 24 hours of any class meeting during which the logarithmic function was discussed each participant was required to complete Interview 3. This interview lasted between 15 and 30 minutes. In several cases students could not recall exactly what they understood or not understood during class. In these cases I provided my class notes to help stimulate recall. For two of the participants this interview was repeated three times. If the student allowed me to interview him or she more than once during the instructional phase, the mapping activity was only conducted during the first iteration of the interview.

Postinstructional Phase

The postinstructional phase of the study took place during the last 3 weeks of January 2001. By this time each of the participants had completed college algebra. During this phase, Interviews 4 - 9 were conducted. Although Interview 5 was to last no more than 90 minutes, two of the participants worked on the activity for approximately 100 minutes. This amount of time proved to be too much for both the participants and me, and hence I enforced a strict 90-minute time limit on the remainder of the interviews.

General Procedures

During the study I kept an electronic journal on each of the participants in the study. Following each interaction with or observation of a participant, I made notes in the journal. After class I recorded my general impressions of the class, the student's behavior in class, and any interpretations I had regarding that behavior. Following each interview, I recorded in the journal comments the student had made either before or after the interview regarding personal activities, goals, and attitudes. In addition, I made notes regarding my impressions of the student's activity and major questions I had about the student's behavior during the activities. I used these notes as data in the development of student profiles and in the preliminary stage of data analysis.

Each day I had lunch in the instructor's lounge. During this time I was able to hear conversations between two of the instructors (Teachers 1 and 3) whose classes I observed. These comments provided me with additional information about the curriculum and philosophy the two used to teach the logarithmic function.

Data Analysis

Transcription. I personally transcribed each interview and added comments regarding the student's actions based on my interview notes. This process was extremely valuable, since it allowed me to get to know the student's intonations and ways of speaking. I finished the transcription before proceeding with the development of case summaries. A three-ring binder for each student containing the transcripts, task sheets, maps, drawings, any scratch work produced by the student, along with a hard copy of my journals for the student became the case books for the study.

Case notes and summaries. I read and made notes on each interview, marking what I felt were especially significant passages in the transcripts or images in the drawings, maps, and task sheets. I identified and described important categories from each interview, which I called *case notes*. I then summarized the events of each interview and elaborated on the categories I identified in the case notes. I called these summaries *case summaries.*

Coding and identification of evidence. After an interview was summarized, I then coded it using the following coding scheme developed from the framework described in chapter 1 and from the case summaries. I also asked the writing group of mathematics education students at UGA to code one interview. Based on our discussion, the codes were revised to include categories of representations used by the students.

1. *Conception*: Student explicitly communicates feelings or ideas about the logarithmic function.
2. *Representation*: Student uses written or oral symbols to think about or communicate about the logarithmic function.
    a. *Written*: Student uses written notations and words to communicate or investigate a problem.
        i) *Name*: Student uses a written name for an action or procedure.
        ii) *Notation*: Student uses written notation for an action or procedure.
        iii) *Maxim*: Student uses a written phrase for an action or procedure
        iv) *Description*: Student writes a description for an action or procedure.
    b. *Oral*: Student uses words to communicate about the logarithmic function.
        i) *Name*: Student uses a name for an action or procedure.
        ii) *Notation*: Student uses notation for an action or procedure.

iii) *Maxim*: Student uses a phrase for an action or procedure.

iv) *Description*: Student gives description for an action or procedure.

c. *Pictorial*: Student uses a picture or a graph to communicate or investigate a problem.

d. *Tabular*: Student uses a table to communicate or investigate a problem.

3. *Connection*: Student translates or transforms a representation.
4. *Applications*: Student uses the logarithmic function to solve a problem.
5. *Ways of Knowing*: Student uses a procedure to solve a problem he or she does not recognize as a representation of the logarithmic function.

Summarizing evidence. After the data were coded, the evidence of understanding for each of the interviews and then for each phase of the student was summarized on a single 8.5-by-11 sheet of paper. The summary sheet was divided into four quadrants labeled "conceptions," "representations," "connections," and "applications." At the bottom of the summary sheet for Interviews 5, 6, and 7, evidence of ways of knowing was recorded. Sources (transcripts, maps, drawings, participant observation notes, or impressions), locations (line numbers in the transcript), and evidence were recorded on the summary sheets. These summaries were then used to characterize the student's beliefs about the logarithmic function for each phase of the study.

Student's beliefs. Using the summaries of evidence from each phase of the study, I noted ideas that were used in more than one way by the student. For example a student who used an oral and written representation for the sum of logarithms might have a belief about this property of the function. I conjectured possible beliefs and then searched the case summaries for evidence supporting and contradicting my conjectures. Modifications were made to the conjectured beliefs when they did not fit the evidence. In the proposed beliefs I attempted to preserve some of the student's way of communicating.

Themes. I used the student's beliefs to make generalizations about their understanding. In particular, I put each belief from each phase and participant on a piece of paper and grouped the papers into categories. Some beliefs fit into two categories. I then generated written descriptions of the beliefs students used during each phase. These descriptions helped me identify themes in the *beliefs.* For example, during the preinstructional phase, students speculated about the logarithmic function. When I described what the students speculated about, I was able to see themes in the beliefs. I then coded the beliefs with the four themes.

CHAPTER 4: CASE STUDIES

This chapter consists of four case studies of students in two of the classes I observed. Each case begins with a description of the student, his or her view of mathematics, and his or her view of understanding. A summary of the evidence of understanding in terms of the student's conceptions, representations, connections, and applications of the logarithmic function and its properties for each phase of the study is presented. These summaries serve as the background for the beliefs that constitute the student's understanding of the logarithmic function for each phase. A discussion of the student's changes in understanding is presented. To close each case, I argue that one or several of the ways of knowing used by the student during the postinstructional phase could be used as the basis for further growth of understanding of the logarithmic function.

Jamie

Getting to Know Jamie

Jamie was 18 and a first-year student at RC. She commuted to school from a small town about 30 miles west of the college. I was quickly impressed with Jamie because of her interest in mathematics. She was very enthusiastic about the subject and planned to become a middle school mathematics teacher. She was so interested that since her high school graduation, she had observed one of her high school mathematics teachers to investigate the possibility of becoming a secondary school mathematics teacher. One visit was enough to convince her that she did not want to teach secondary school mathematics. Jamie’s positive experiences tutoring her sister, a seventh grader, convinced her that she would prefer teaching middle school students.

Jamie was a very busy young woman, working 37 - 40 hours a week as a server at a regional restaurant chain while also taking 12 semester hours at RC. Occasionally her home and school environments as well as her workload were sources of stress for her. I asked Jamie to tell me more about being stressed out. She said,“I had so much to do and so little time to do it in. And I was trying to work and trying to do my algebra and I had a test and…it was stressful. Very, very stressful.” School was not Jamie's only source of stress. She also spent time defending her decisions to her colleagues at work and debating them with her mother. Some of the women that she worked with took pleasure in reminding Jamie that she was a server and would likely be one for the rest of her life. Her mother, on the other hand, had high expectations for her daughter. She wanted Jamie to do well in school and to graduate. The hopes that Jamie’s mother had for her, while helpful at times, caused conflict when they were manifested in advice. Jamie resented her mother’s advice and felt that since her mother had never attended college, she did not understand the stress involved. Hence, any advice that Jamie’s mother gave was unwelcome. This tension caused conflict between the two during the spring 2001 semester. Jamie realized that she would not be able to take 12 hours of course work and work full time. Since the money she earned was used to pay for her car and gas to commute to school, the job was necessary. In order to maintain a B average and to achieve her long-range goal of becoming a mathematics teacher, Jamie felt that she had to drop a class. Her mother did not agree. The two argued about Jamie’s decision, but ultimately Jamie did drop a class.

Jamie loved the freedom that professors at RC allowed. Speaking of her first month in college, Jamie said, “It made me feel like more of an adult.” She noted that teachers at RC differed from those she had in high school because “they don’t go behind you every step of the way to make sure you are doing everything that you’re suppose to do as in high school they did.” This new autonomy suited Jamie, since she did not need anyone to check up on her.

Jamie As a Mathematics Student

During Interview 2, Jamie described herself as a mathematics student: “Focused. I like to explore new things. Learn new things. … And I like to be organized. I have to have everything really, really organized." Being focused and organized were necessary characteristics for Jamie. Her hectic schedule did not allow time to think too long about the concepts that were presented to her. Instead, if she simply copied the examples from the board and reviewed them later, she was usually able to figure out how to do the homework problems:

> The thing that helps me learn most, like whenever she (Teacher 1) puts the examples on the board, if I copy those down I pretty much, I can look back. She gives really, really good examples, and she writes everything, all the steps and stuff on the board. So that helps me out a lot whenever I write them down, because I can go back to them whenever I'm doing my homework.

When Jamie could not figure out how to do a particular type of problem, she sought help from multiple sources: Teacher 1, a friend that sat next to her in class, the textbook, and the academic assistance center (AAC), a free on-campus tutoring center. Having a friend who was willing to help was nice, but not essential for Jamie:

> S: So you meet to study for the test. Okay ...would you say that she is very helpful to you?
>
> J: She helps me out a lot because there's some stuff that I understand and there's some stuff that she understands that I don't understand, so we kind of counteract each other.
>
> S: Do you think it's really important for you to discuss it with her to get a good understanding.
>
> J: It is not so much that it's important. It helps a lot. I mean, if I didn't have her to study with, I could go to the AAC. But it's a lot easier whenever you know somebody that's in your class and you know where they live and if you need them. You know they will be there.

Jamie's friend was accessible and available 24 hours a day. In addition, Jamie's friend knew what had been presented in class. That knowledge made her the best out-of-class source of help. Jamie did not hesitate to ask for help when she could not do a problem. Once, during the preinstructional phase of the study, she even came by and asked me for help on a graphing problem from the class handout.

Since she wanted to become a mathematics teacher Jamie was very interested in doing well in her mathematics classes. During Interview 2, she explained her goal in her mathematics classes:

> To do my best to make the best grades that I can which is not very good sometimes. To try to understand exactly what she is talking about and not just kind of have the fuzzy idea. To understand exactly what she's talking about, and if I don't understand, and then I will just go to her later and talk about it.

If she could not do the problems that were presented in class after trying them on her own Jamie sought help. She described going in for help twice when she was trying to learn how to find "holes" in rational functions. Jamie did not like feeling that she could not do a problem.

> I was scared that it was going to be like a major part of the test. And I didn't know how to do it. And I was like, oh gosh, this isn't going to help me at all. But since it was just a bonus, and I saw that it kind of helped me a little. I mean I understood the concept, like on the test I got it right, but I missed one part of it. Because you had to explain some stuff, and I missed one part of it, but other than that I got it all.

Jamie wanted to do well on the test and was happy that "holes" were not a major part of it. She only missed a problem that asked for an explanation. She knew how to do the "hole" problems.

Jamie sat on the right side of the classroom, three chairs from the front. She stayed busy during class taking notes on the handout, doing problems, or using her calculator. Jamie did not like to miss class. She felt that the material was cumulative and that she would have a difficult time catching up if she missed class. While I observed the class, she missed one day because her mother was having outpatient surgery and she wanted to be with her. According to Jamie, missing class was “not fun.” During interview 2 she elaborated:

> It’s not fun. Especially if you have ... missed a couple days or if you were sick. You just don’t know what is going on. That’s awful. In algebra I’ve only missed like 3 days, because I try to be there as much as I can, because if you miss a day then you are totally thrown off. Because in her… and if I do miss class… stuff that she does the next day is based on the day before.

If Jamie did miss a class, it simply meant she would have more work to do, thereby complicating her already hectic schedule.

In class Jamie followed along on the handout and even got ahead of the teacher if she knew how to do the problems. She asked questions if she needed clarification on a problem. Although she felt that doing homework was important and helped her “understand” more about the concepts, Jamie did not always do all the homework. She worked on assignments regularly, but in her notebook her assignments were not complete.

Jamie earned a B in college algebra. She explained that although the course had not been particularly difficult, she had earned a B in part because of her hectic schedule. Jamie was satisfied with her grade but had wanted to do better.

Understanding Mathematical Concepts

When I asked Jamie to define the term *understanding* she described it as a feeling.

> Feeling...knowing what is going on. Feeling sure. Feeling sure that you are familiar with the concepts. And it’s not being fuzzy or that you don’t know what is going, but that you are...that you’re...I don’t know, I can’t think of a word to use. I guess comfortable with it.

Feeling comfortable with a concept was not the only way that Jamie evaluated her own understanding. In practice, Jamie said she understood a mathematical concept if she could do the problems associated with the concept. Jamie's view of understanding was depicted in her comments about her actions when she did not understand a mathematical concept.

> J: If I don't understand it, then...I know that either I'm going to have to go to the AAC or I'm going to have to try to work.... Working more problems helps, like more of the same kind of problem that I don't understand. And those answers in the back of the book help too because... if you get the right answer, then you will know it immediately, if you work the odds. It is just the more that you do it, the more that you understand. Like if you start out with not understanding it, and you work more and more problems, you will get it eventually. It might take a while, but you'll get it eventually.
>
> S: Right. Does it create some kind of meaning, or do you just learn how to do the process?
>
> J: Learn how to do the process. Once you get the hang of how to do it then any problem, pretty much that she puts down there, you can do it if you understand like the process of how to do it.

Jamie wanted to feel comfortable, but the feeling was associated with being able to do the problems. Understanding a mathematical concept for Jamie meant she felt comfortable because she could do the procedures associated with the concept.

Jamie's Understanding of the Logarithmic Function: Preinstructional Phase

Evidence of Understanding

Three categories of evidence were collected during the preinstructional phase of the study: conception, representation, and connections. There was no evidence of application of the concept because Jamie did not recall ever having seen the function before.

Conception. Jamie’s first remark about the logarithmic function was “I don’t know that we did it.” She did not remember “doing” the logarithmic function in high school, and explained that her class had not gotten very far in the textbook.

> Our teacher was really, really detailed so we didn’t get very far. We did like a lot a lot of stuff. We didn’t get very far in the book. I think that might have been in the back, because I don’t think we did it.

Not recalling any instruction on the logarithmic function made the skills assessment difficult for Jamie. When we discussed Jamie’s feelings about not understanding she focused on the notation $\ln_3$: “If I knew what L-O-G (she spelled out log) three was, I might could do them.” Jamie felt if she knew what the notation meant, then she would have been able to do the problems. This thinking was consistent with her drawing and explanation of her process of understanding (see Figure 6). Jamie explained her drawing:

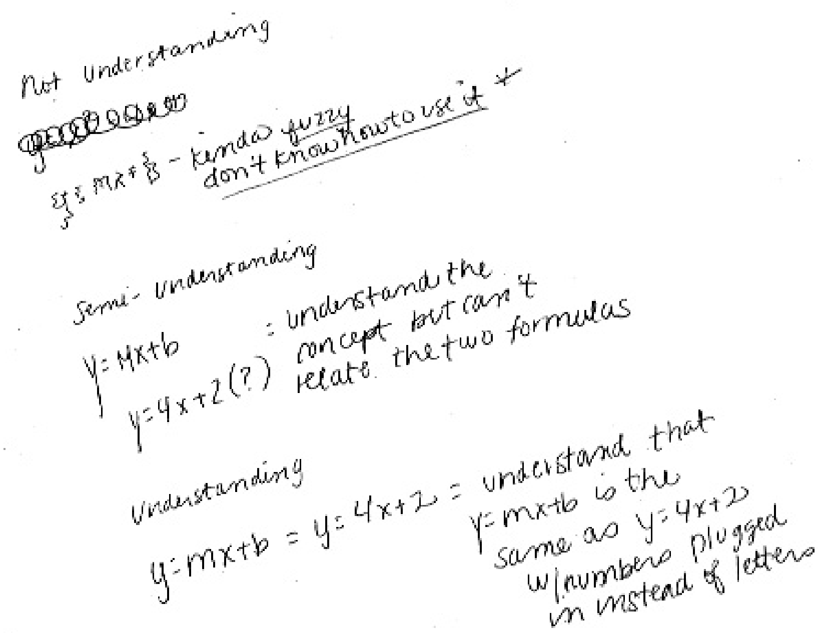


Figure 6. Jamie's drawing of her process of understanding.

> At first, she gives you the formula $y = mx + b$, and you’re like, that doesn’t really relate to numbers. And then she gives you numbers, and you have to plug them in and work the problem out. Like at first whenever she gives you the formula with no numbers, you don’t know how to work it.

But then whenever she gives you some numbers to fill in, it's a lot easier to understand because you can relate the numbers to the letters.

The method that she explained is the same one she used on the skills assessment. She saw the notation for the logarithm as a formula and tried to figure out what numbers should be plugged into it. Jamie's conception of the logarithmic function was that it was an interesting formula. Interesting because "it looks cool," and " I'm ready to learn how to do those things." Jamie wanted to learn how to "do" the logarithmic function by learning how to use the formula.

Representation. In the preinstructional phase of the study, Jamie used notations and names that she read on the skills assessment to represent the logarithmic function. Her first map of the logarithmic function gives an indication of the influence that the skills assessment had on her understanding during this phase (see Figure 7). Jamie's notation

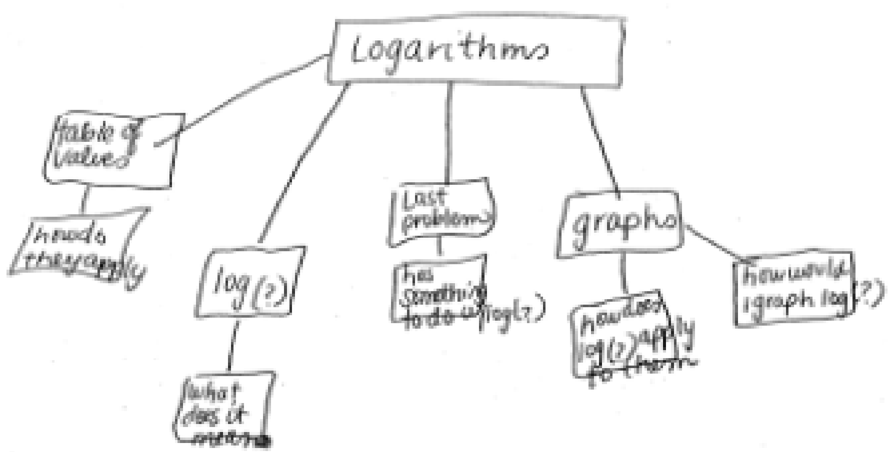


Figure 7. Jamie's map of the logarithmic function from Interview 1.

$\log_{(?)}$ was a generalization of notation used in the skills assessment. Her use of the question mark as a variable was consistent with other notations she used during the study. In solving problems or doing tasks, Jamie used "?" as an unknown or variable. So, although she included the question "What does it mean?" on her map, Jamie was not literally searching for meaning. She simply wanted to know how to use formula to do the problems.

As with the notations, the names Jamie used were all taken from the skills assessment. Logarithms, graphs, and table of values were depicted on her map as associated with problems she could not do.

Connection. Jamie was sure she had never seen the logarithmic function before Interview 1, so the connections she made during the preinstructional phase of the study were the result of her experience with the skills assessment. These connections are depicted on Jamie's map of the logarithmic function (Figure 7). Jamie linked all three: the notation $\ln_3$, graphs, and tables. Based on her experience with the skills assessment, Jamie hypothesized that there were links between the notation she saw, tables, and graphs. During the instructional phase, Jamie quickly abandoned this hypothesis.

Beliefs

Jamie saw the logarithmic function as a formula that she needed to learn how to use. She was interested in learning how to do the problems presented on the skills assessment. In particular she wondered what the notation stood for in terms of numbers and how it might help her graph the function and create a table of values. Thus the following beliefs represent Jamie's preinstructional understanding of the logarithmic function:

1. The logarithmic function is associated with the notation $\log_{(?)}$, which might represent a number.
2. The logarithmic function is interesting, and I would like to know how to do the problems.
3. If I could figure out how to use the notation $\log_{(?)}$, I could do the problems.
4. The logarithmic function is related to at least two types of problems: creating a table of values and drawing graphs.

Jamie's Understanding of the Logarithmic Function: Instructional Phase

The presentation of the concept of the logarithmic function in Teacher 1's class consisted of a series of handouts that the students filled in during the lecture. The title of the handouts always included a section number from the book. For example, the handout distributed on November 29th was titled 4.3 Logarithmic Functions. 4.3 was the number of the section from the text that introduced the logarithmic function. Problem 1 on the handout was stated as follows:

Graph $y = 2^x$ and its inverse. Graph $y = (0.5)^x$ and its inverse.

Space was left beneath the problems to allow the students to copy Teacher 1's procedure as she worked the problems on the board. During each class, time was set aside for student questions and practice. Teacher 1 circulated during student practice, helping individual students with the problems on the handout.

Jamie particularly liked this method of instruction. She could go at her own pace, which was a bit faster than the pace of the class. She explained how she worked in class.

S: So when she is saying stuff and you see where it's going, you sometimes do things on your calculator...

J: Yes ma'am but one thing I'm bad about is like...whenever we are doing the handout that she gave us today on the back. I went ahead and worked all the problems before she did.

S: Well, I don't think that's a bad thing.

J: Like whenever she explains the first one...I'm bad about that. I've always done that. If I understand something I'm going to go ahead and work them all. I know she was thinking, "Jamie is not doing anything," but I had already done them.

When Jamie knew how to do a problem on the handout ("If I understand something") she "worked" all the problems similar to it.

Evidence of Understanding

Conception. Jamie's conception of the logarithmic function during the instructional phase consisted of two elements: interest and connection with exponents. Before I started taping Interview 3, Jamie came in very excited, saying that she had enjoyed class. "I don't understand where I'm going to apply it (the logarithmic function) but it was interesting." Her excitement and interest in the logarithmic function was based on her feeling that she knew how to "do" the logarithm notation. In particular, she knew how to calculate logarithmic expressions using her calculator and how to convert expressions from exponential form to logarithmic form.

Jamie associated her knowledge of exponents with the logarithmic function. "Yesterday we learned about how to convert ... from exponents to logs. And exponents are something that I kind of knew about, so that kind of helped me convert them to logs." The origin of this association was instruction. Teacher 1 motivated the connection by asking the students to explore $2^x = 10$ and then introducing the logarithm as an exponent. Jamie used her calculator and a trial-and-error method to quickly approximate the solution. Hence Jamie associated her knowledge of exponents with the logarithmic function.

Representation. Jamie worked in two representational modes during this phase of the study: oral and written. Her use of oral representations of the logarithmic function and its properties consisted of naming procedures and notations. The four primary names she used were *converting, exponents, logs,* and *properties.* Teacher 1 had used each of these terms in class during her presentation of the logarithmic function. Terms such as *base* were not defined. Sample problems were presented, and the terminology was used in the demonstration of solutions to the problems (see Figure 8). It was up to the students to

Log form	*Exponential* form
$\log_2 32 = 5$	______________

Figure 8. Example of converting problem from a class handout.

define the terms. For Jamie the term *converting* became a procedure one used to change an exponential expression to a logarithmic one.

Jamie used the term *properties* to refer to the following mathematical notations: $\log_a MN = \log_a M + \log_a N$ and $\log_a M^p = p \log_a M$. She noted during Interview 3 that she had not understood the properties. During class, Jamie was confused about a question that involved expanding an expression (see Figure 9 for the problem and the directions given in the text). Jamie asked: “Are you suppose to work it out?” Teacher 1 responded to Jamie’s

Express in terms of sums and differences of logarithms

$$\log_a 6xy^5z^4$$

Figure 9. Jamie’s homework problem.

question by doing the problem on the board. Her answer was $\log_a 6 + \log_a x + 5 \log_a y + 4 \log_a z$. After the teacher finished, Jamie asked: “So if it says write as a sum, you don’t work it out? Just go that one step? I worked it out and found a number.” Jamie thought the properties of the logarithmic function were used to find numeric answers. This thinking is consistent with her view that to understand mathematics, one learns how to plug numbers into formulas. Hence the properties in notational form, $\log_a MN = \log_a M + \log_a N$ and $\log_a M^p = p \log_a M$, were formulas she had difficulty using.

The written representations that Jamie used corresponded to the names and notations that she had seen in class and used during Interview 3. Her map (see Figure 10) of the logarithmic function during this best illustrates Jamie’s use of written

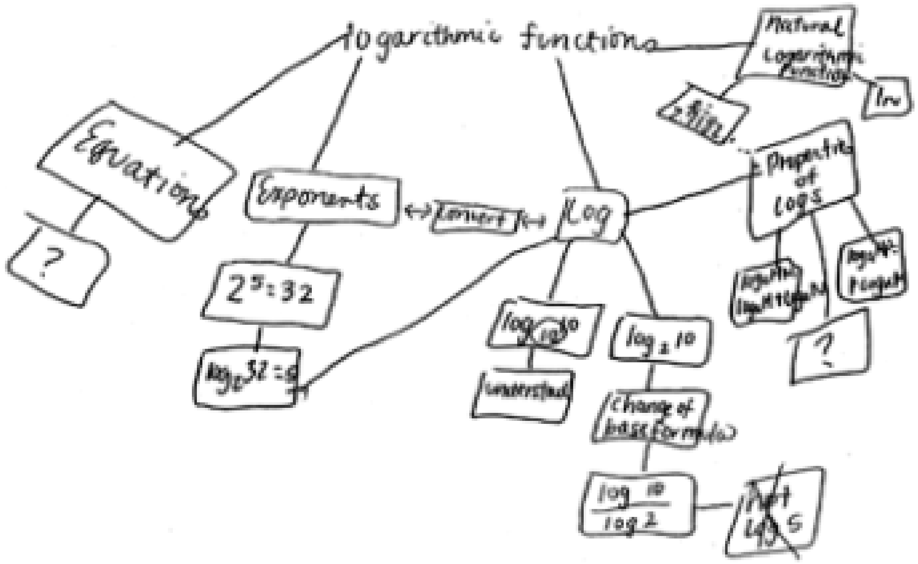


Figure 10. Jamie's map of the logarithmic function in the instructional phase.

representations. "Logarithmic function" on Jamie's map is linked to the categories of *exponents, convert*, *log, properties of logs*, and *the natural logarithmic function.* (The category labeled *equations* has a question mark below it because Jamie had not studied equations in class yet.) Below the word *exponents*, an example of an exponential expression and its conversion to logarithmic form are given. The notation and examples Jamie used on her map were presented in class that day. Jamie's use of names and notations in our interview and on her map illustrates how she understood the logarithmic function. The names and notations were related to problems that she needed to know how to do.

Connection. During the instructional phase of the study Jamie made connections between written representations of the logarithmic function. She connected names, notations, and procedures. The term *convert* is a good example of this practice. On Jamie's map of the logarithmic function, she linked *exponents* and *log* with the name *convert*. This name represented a procedure that she illustrated notationaly with an example.

Jamie also connected names and notations for which she did not yet have procedures. On her map Jamie included a category: *properties of logs*. This name was connected to two notations: $\log_a MN = \log_a M + \log_a N$ and $\log_a M^p = p \log_a M$ linked to a question mark. Jamie realized that these notations were important, but she did not have a procedure for them yet. Teacher 1 had talked about them in class, but Jamie could not use the properties.

Jamie's connected names and notations with problems and the procedures used to solve them. When she could not associate the names and notations with problems, she "did not understand" them.

Application. I defined *application* as the use of the logarithmic function or its properties during problem solving. During the study the logarithm key on the calculator was referred to and used by the students as a resource to generate information about the logarithmic function and to compute logarithms. For example, Jamie used the calculator to determine that zero was not an element in the domain of the logarithmic function. She entered log 0, and when an error message was returned, she assumed that zero was not an element of the domain. Whether or not students used the calculator to generate information or calculate logarithms, how they used calculator to help them solve problems was part of their knowledge of the logarithmic function.

The TI-83 graphing calculator is standard equipment for all students taking mathematics classes at RC. The demonstration and use of the calculator during instruction via an overhead projection panel was a daily occurrence in all of the classes I observed. The logarithmic function unit was no exception. Teacher 1 encouraged the students to use their calculators to evaluate logarithmic expressions. In fact, the handouts used in class had a section entitled "Evaluating Logarithmic Function on a Calculator." Throughout her presentations, Teacher 1 referred to the calculator. For example, while she demonstrated how to "evaluate logarithms on a calculators" she remarked, "You can do some of these without a calculator, but in the interest of time we are going to do them all with our calculator." She showed the students how to rewrite $\log_2 10$ using the change of base formula so that they could "find" the logarithm with their calculators. At the end of class, Teacher 1 said, "Make sure you know how to use your calculator just like we were doing today."

In part as a result of this type of instruction, Jamie came to see the natural logarithm and logarithm keys on her calculator as useful tools for finding the logarithm of a number. She used her calculator in each class. When she entered class she took out her notebook and calculator and laid them on her desk. She used the calculator to explore and check answers she or the teacher got during class. Jamie applied the logarithmic function when she used the logarithm key on her calculator.

Beliefs

During instruction, Jamie's *beliefs* about the logarithmic function shifted. She still found the function interesting but was now focused on becoming proficient with terms, notations, and procedures associated with the logarithmic function. This practice was consistent with Jamie's desire to do well on the tests. She believed that the logarithmic function was like any other mathematical concept. If she could learn how to associate problem types with names, the notation used to solve the problem, and the procedure for doing the problem, Jamie felt she would do well on the test. Jamie's *beliefs* about the logarithmic function are related to this general belief:

1. Exponents are related to logarithmss.
2. The terms *log*, *exponents*, *convert*, and *properties* are important to know if you want to be able to solve problems.
3. The properties $\log_a MN = \log_a M + \log_a N$ and $\log_a M^p = p \log_a M$ are important to memorize for the test.
4. Converting is used to change an exponential expression to a logarithmic one.
5. To evaluate $\log_2 10$, use the change of base formula $\frac{\log 10}{\log 2}$ and your calculator.
6. Use your calculator to evaluate logarithms.

Jamie's Understanding of the Logarithmic Function: Postinstructional Phase

Evidence of Understanding

Conception. Jamie's conception of the logarithmic function during the postinstructional phase was that the function was hard. Her rationale for this assertion was that the logarithm is a word not a number. According to Jamie, that made it hard to remember. She also explained that after her second attempt at the skills assessment, she knew much less about the logarithmic function than she had known during the instructional phase. When I asked her how she felt about the skills assessment activity, she replied, " I felt bad since I thought I knew a lot about logarithms." Because she was not able to answer most of the problems, she did not feel she understood logarithms. The best illustration of Jamie's evaluation of her understanding during the postinstructional phase of the study is her drawing of her process of understanding. During Interview 9, I asked her to visualize her process of understanding the logarithmic function and to draw a diagram or picture of that process (see Figure 11). Jamie depicted her understanding as a series of hills, each one

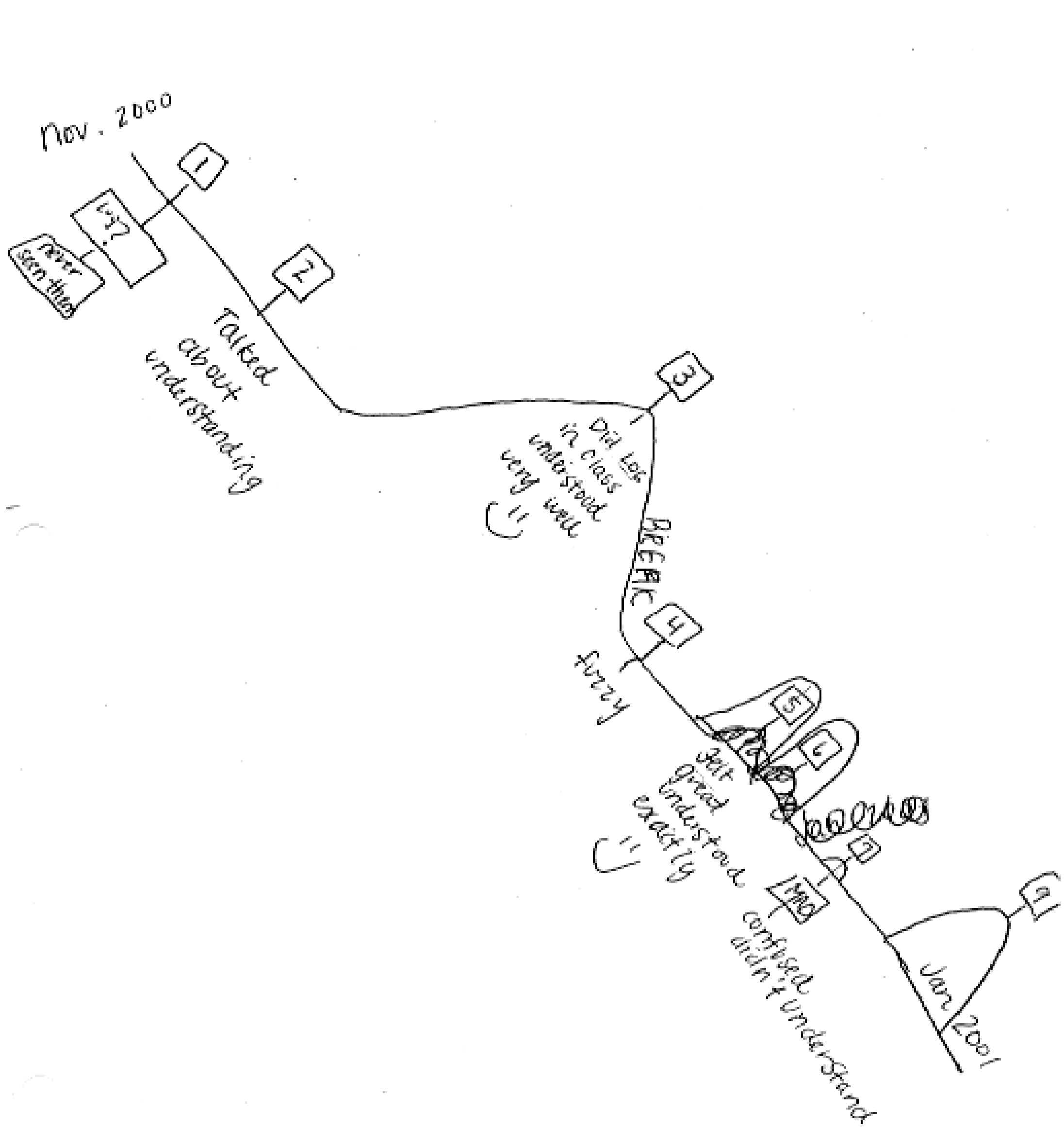


Figure 11. Jamie's drawing of her process of understanding.

corresponding to an interview. The height of each hill represented how much Jamie felt she understood about the logarithmic function at that time. The hill drawn above the heading "Interview 3" is the tallest, whereas there is no hill drawn for Interview 4. Jamie labeled this interview "fuzzy." Jamie defined the term *fuzzy* as "not understanding what is going on. You know kind of what to do, but not really how to do it. And you know kinda how to do the problem, but you are not getting the right answer. Just kind of uneasy I guess." She used the term when she did not know exactly how to do the task or problems she was asked to do. Her use of this label on her drawing during Interview 4 indicates that she felt she did not understand the logarithmic function. During Interview 7, Jamie once again called the logarithmic function hard. She noted that doing the task made her head hurt. Because Jamie was not able to complete skills assessment or the tables in Interview 7, she referred to the logarithmic function as hard.

During Interview 5, Jamie introduced a rationale for her difficulty with the logarithmic function during the postinstructional phase that explained her conception of the logarithmic function as hard:

> J: I thought about this interview during English. Why, I don't know. But I think the reason that we don't understand logarithms after such a short time is because there are words associated with it, and with math you think totally about numbers. And the word and number association just doesn't stay clear. I don't know why I thought of that, but I did.
>
> S: That would make sense with some of the other comments that the participants are making actually.
>
> J: I think that word log is just,...I think if it was like a certain number it would help.

Jamie's theory that the logarithmic function was a word and should be related to numbers is reminiscent of her preinstructional understanding of notation $\log_3$ as a number. Jamie commented in Interviews 8 and 9 about the difficulty she was having remembering that the logarithmic function was based on its representation as a word. During Interview 9, when I asked Jamie to define understanding logarithms, she noted:

> I understand like more, uhm, by the activities (in Interviews 5 and 6) we did, because they were dealing with it (the logarithmic function), but they didn't have that word (*logarithms*) in there. That word just,...I think that's what throws everybody off. Because it is not numbers. It is just words.

Jamie's conception of the logarithmic function was as a confusing word that made problems harder to solve. So the first component of Jamie's conception of the logarithmic function during the postinstructional phase was that the function was hard because it is a word and not a number.

Jamie felt that she was more successful with tasks during the postinstructional phase that did not involve the word logarithm. During Interviews 5 and 6, Jamie represented the problems using exponents, but she was not aware that either of the tasks was related to the logarithmic function. When I asked her to make a map of the logarithmic function during Interview 8, she asked about including the interview tasks on her map: "What we did in this study was what I was thinking about. It didn't have to do with it (the logarithmic function) I don't think. Those exercise (tasks) we did didn't. Did they?" When I told Jamie that all of the postinstructional interview tasks were related to the logarithmic function, she included the tasks in her map of the logarithmic function and reasoned that she understood the logarithmic function better when the word was not used in the problem.

Jamie's conception of the logarithmic function as hard was a change from her conception during the instructional phase that it was easy. As we have seen, this change in conception was due to Jamie's inability to perform the interview tasks. Her conception of the logarithm as a word appears to be connected to her view of the logarithmic function during the preinstructional phase as a notation that "might be a number." During instruction, Jamie never indicated that logarithm being a word made it a harder concept. This postinstructional conception is a reappearance of her preinstructional conception of the logarithmic function as letters that stand for a number.

Representation. During the postinstructional phase, Jamie focused on three modes of representation: oral, written, and pictorial. The oral representations that she used were names. The written representations were names, notations, and descriptions. She used the pictorial mode to explain and illustrate procedures. The representations Jamie used were often to answer questions that I asked. She rarely used anything but notation to do problems on her homework.

Jamie's oral representations in the form of names were *trial and error* and *convert*. Trial and error was Jamie's name for any procedure that involved using the calculator to investigate a mathematical question. Jamie used the term to explain how she found exponents during Interviews 5, 6, and 7. During Interview 5 Jamie used the procedure to find the sign above the product 64· 256.

S: Okay, now how can you figure out the sign that is above that [64 · 256]?

J: Trial and error is how I have been for 15 minutes. Oh, I just totally lost my number [64 · 256]. This takes forever. This is frustrating. I want to figure it out.

S: You are doing great.

J: Two raised to the...(uses calculator to find the exponents). Ha! That number was? Where did it go? Is that the same number?

Jamie evaluated 2 raised to various exponents on her calculator until she matched the calculator display with the answer she was holding in her mind. The match told her that the exponent she used was the answer. She also used the term *trial and error* to refer to finding values that were not in the domain of the logarithmic function and for finding the correct exponential expression for a given logarithmic one. An example of the latter occurred during Interview 8. Jamie demonstrated the procedure for *converting*, the second name she used. She showed me how to transform $\log_{10} 2 = .30$ to $10^{.30} = 2$.

S: Oh, okay, there is an exponential and a log, and you can go back and forth.

J: Right. You can convert it or whatever.

S: So how did you know to put the 10 there [as the base in the exponential expression]?

J: You just have to remember that. That is the formula, but you can use *a*'s and *b*'s or something. I don't remember how the formula goes, but this base number would be your number that you are raising to the power, and this would be the exponent [points to .30 in the logarithmic expression] and that would be the answer [points to 2 in the logarithmic expression].

S: Oh, okay, so you just have to remember where everything goes.

J: Right. But, if you don't remember where everything goes, you can kind of work with it in your calculator to figure it out.

S: So you just what? Try some stuff.

J: Right, you can try trial and error. That always works really well.

Here Jamie used the term *trial and error* to refer to how one can figure out the positions of the numbers in the exponential notation. She did not remember the positions and suggested that the calculator could be used. For Jamie the two uses of the calculator were similar, so she named them both *trial and error*.

The second name Jamie used was *convert*. What Jamie meant by this term was illustrated during the preceding interview excerpt. The primary difference between Jamie's use of the term during the instructional phase of the study and the postinstructional was that during the latter she used the term to refer to conversions from an expression in logarithmic form to one in exponential form. She did not convert expressions in exponential form (such as those in Interviews 5 and 6) to logarithmic form. She did not think the tasks had anything to do with the logarithmic function. She asked me whether they were related during Interview 9. Seeing the signs above the number line as exponents was not associated with the logarithmic function for Jamie. She used the term *convert* to refer to transforming expressions in logarithmic form to exponential form.

Jamie's written representations included names, notations, and descriptions. *Convert* was the name Jamie used most often. The name occupied a central position in both

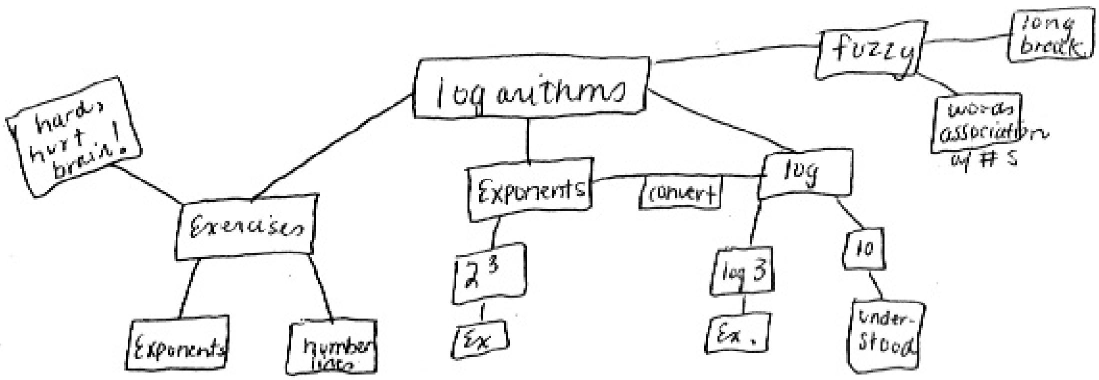


Figure 12. Jamie's map of the logarithmic function from interview 8.

maps that Jamie constructed during this phase (see Figure 12). Jamie saw her convert procedure as her most effective tool to do problems with logarithms in them.

Trial and error was not depicted on any of Jamie's maps, but she did write the term during Interview 5 when she summarized her actions (see Figure 13). For Jamie, the

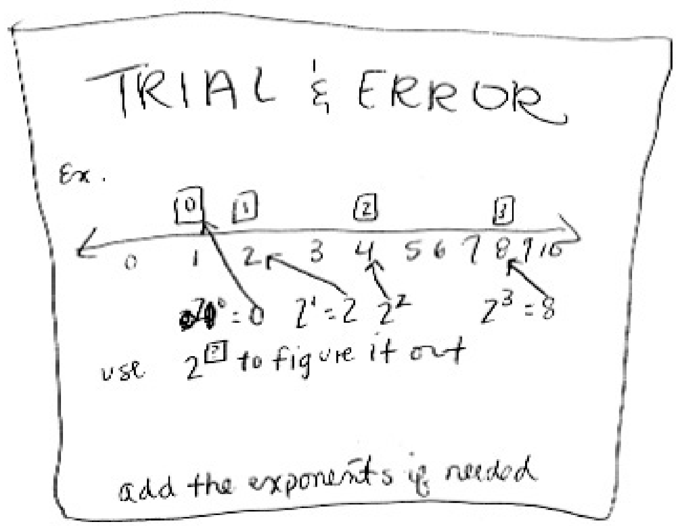


Figure 13. Jamie's written representations of the data generated in Interview 5.

overarching theme for the task in Interview 5 was trial and error. She used the procedure to find the sign above the number $\frac{1}{2}$ as well as the number 3. In her diagram Jamie also used another type of written representation: notation. The notation she developed contained her representative for an unknown, a question mark. This self-generated notation stands in contrast to notations that she tried to remember. During Interviews 4 and 8 Jamie tried to remember the rule for adding logarithms (see Figure 14). Jamie referred to problems like

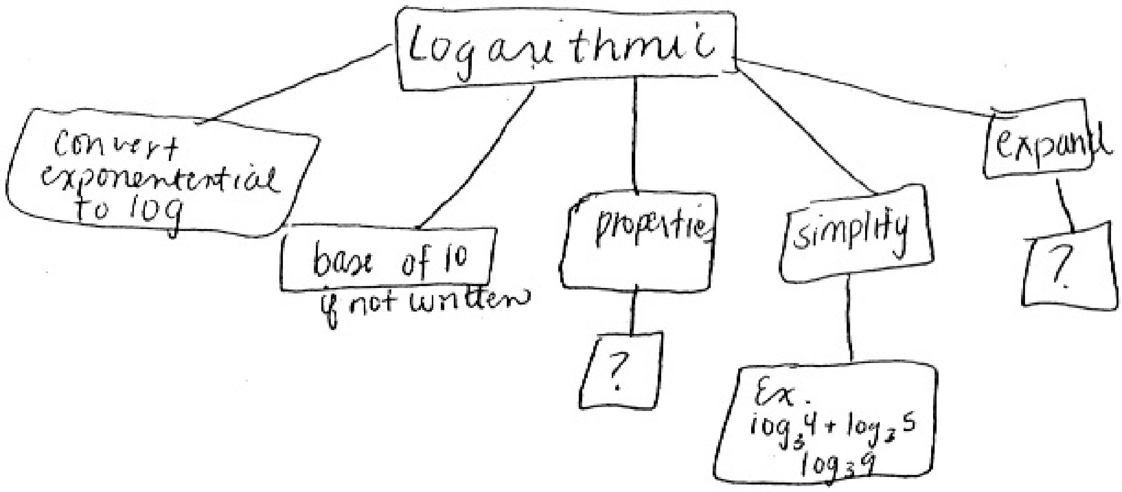


Figure 14. Jamie's map of the logarithmic function from Interview 4.

the one illustrated on this map as "easy," because she had done so many of them during the instructional phase that she thought she knew how to do them. Jamie had no way of checking her calculations, so she trusted her memory. Both the notation illustrated and the faulty rule were used often during this phase of the study. During Interview 8, when I asked Jamie what problems one might use the logarithmic function to solve, she again presented $\log_3 4 + \log_3 5 = \log_3 9$. Jamie also used the notation while demonstrating or implementing her convert procedure. During Interview 7, I wrote a notational representation for one of the table entries. She used my written representation, $\log_3 2 = .631$, as the basis for her solution path. Using this representation, she applied the convert and a trial-and-error procedure to find the combination of 3, 2, and .631 that resulted in a true statement in exponential form. She then used the notation in Figure 15 to help her fill in some of the cells in the second table completion task. Jamie was mimicking, with modification, the

$$\log_3 4 = ?$$

$$3^{?} = 4.$$

Figure 15. Representation Jamie used to fill in *y* values.

written notation that I gave her and combining it with the exponential form that she generated.

The final type of written representation Jamie used was description. As a rule, descriptions were not a part of Jamie's problem-solving process. She used them during Interviews 5 and 6 to describe the relationship between variables. For example, during Interview 5 Jamie described the relationship between the line numbers and the sign numbers: "The relation between the number and the signs is 2 raised to the sign or # in the box is the answer to the # on the number line." Jamie's description is a summary of her procedure more than an illustration of a relationship.

Jamie used very few pictorial representations during her interviews. As we have seen, she summarized her trial-and-error procedure from Interview 5 with a picture of the number line and arrows indicating the positions of numerical expressions she wrote. In addition, she provided the diagram in Figure 16 of her convert procedure when I asked her to define logarithm. The arrow represents the procedure. Her notation indicates that she

$$\text{exponential} \rightarrow \text{logarithmic}$$

$$10^{10} = 1000 \qquad \log_{10}{}^{10} = 1000$$

Figure 16. Jamie's definition of logarithm during Interview 5.

knew there was a procedure involved but could not perform it. Jamie also used pictures when drawing bulletin boards during Interviews 5 and 6. In each of her drawings, Jamie represented the relationship between exponential expressions and the pictorial representation by drawing arrows

During this phase of the study, I was able to see Jamie use all three types of representations. Her primary representational tool was written representation. She adopted notations used in class and developed her own in several of the interviews. Those Jamie developed out of procedures were representational for her thinking, whereas those she adopted were difficult for her to remember. When she felt compelled to use standard notation, she often became confused just as she did when converting an expression in exponential form to logarithmic form. Her attempts to remember also resulted in errors such as those that arose during her newly invented method of adding logarithms. Jamie's representations illustrate that she understood the logarithmic function as a collection of problems to be solved using procedures.

Connection. The diagram in Figure 17 illustrates Jamie's connections. She viewed the logarithmic function as a collection of procedures described with names and notations. The names signified procedures or objects. The notations were used to demonstrate how to do the procedures. Of particular importance were the procedures she called *convert and trial and error*. Jamie used connections between these procedures to solve problems that involved the logarithmic function.

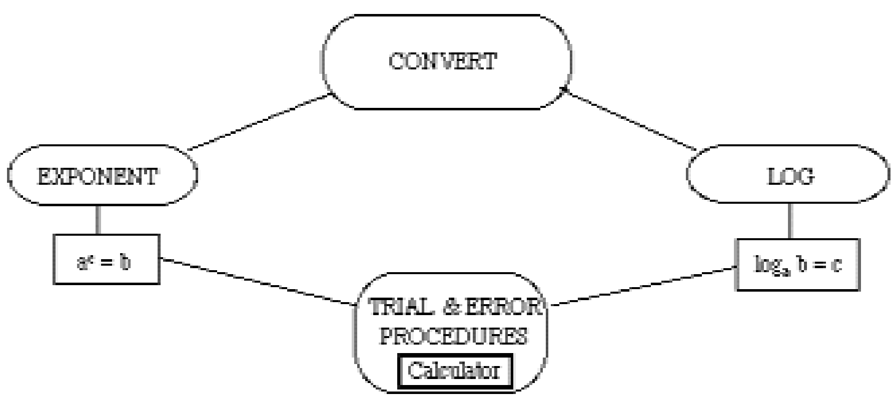


Figure 17. Connections between Jamie's representations and procedures.

In Figure 17, the ovals represent names Jamie used to communicate about or solve problems involving the logarithmic function. The rectangles represent generalizations of the notation that Jamie used. The bold rectangle illustrates Jamie's use of the calculator as an integral component of the trial-and-error procedures she used to solve problems. The connections Jamie drew were between her names, notations, and procedures.

Application. Jamie's knowledge of how to use the calculator was an integral component in her attempts to solve problems. She saw the logarithm key as an answer key. If she was able to put an expression in the correct form, she could use the calculator to find the logarithm of a number. The centrality of the calculator to Jamie's understanding of the logarithmic function was illustrated during the table completion task in Interview 7. Jamie was given a TI-15 calculator instead of a TI-83. Since the TI-15 does not have a logarithm key Jamie said that she did not know how to fill in the first table:

J: I have no clue.

S: Well, can you use anything about logarithms to help you solve it?

J: I don't have a key on here.

Jamie was aware that she could fill in the table easily if she had a TI-83 calculator. This was one of several incidents that illustrate Jamie's understanding of the logarithmic function as a collection of procedures. During Interview 8, when I asked Jamie how *x* and *y* were involved in the logarithmic function she said they were not:

S: So is there any other way to represent this function. I mean, so far everything you have written isn't really a function, because my teacher told me in a function you have *x* and *y*.

J: Uhm...I don't think so. Because you can do the natural log that's just ln [writes ln], but...and you get a number. Like whenever you do like log of five (on the calculator) is point six nine.

Jamie thought of the logarithmic function to something "you do," with your calculator to get a number. When Jamie saw or heard the term *logarithmic function,* she associated it with a key on her TI-83 calculator.

<u>Beliefs</u>

Jamie used several beliefs during the postinstructional phase. First, she believed she knew less about the logarithmic function than she had during instruction. She believed the notation used to represent the function, —a word and letters,— made it harder to remember. Second, Jamie's attempt to remember properties of the logarithmic function from the instructional phase resulted in an application of the distributive law to the sum of logarithms. Third, Jamie was able to use her calculator and a procedure she called *convert* to solve problems that used a representation of the logarithmic function.

1. The logarithmic function is hard, and I do not understand it now, because I cannot always do the problems that use it.
2. Since the log is a word or letters, not a number, it is difficult to remember.
3. $\log_a b + \log_a c = \log_a (b + c)$.
4. To solve $a^x = b$ use a calculator and trial and error.
5. To evaluate a logarithmic expressions in the form $\log_a b = c$, convert it to exponential form.
6. The calculator is necessary tool for evaluating logarithms.

These beliefs represent a change from both the preinstructional and instructional phases of the study.

<u>Changes in Understanding</u>

Jamie's beliefs about the logarithmic function changed over the course of the study. During the preinstructional phase, Jamie was interested in learning how to do the logarithmic function. She focused on what the notation might mean.

During the instructional phase the function was still interesting to Jamie, but now her goal was to pass the test. To accomplish this goal, she attempted to adopt terminology and notation used in her textbook and by Teacher 1 in class. She also tried to learn procedures that would be of use on the test. In particular, Jamie noted the importance of a procedure called *convert* that became the cornerstone of her problem-solving activity.

The postinstructional phase was a bit of a shock for Jamie. She was unable to remember what she had learned during the instructional phase to help her solve problems. As a result she reconstructed rules that made sense to her, but were not correct (log 4 + log 5 = log 9). Not being able to do problems that mentioned the logarithmic function changed Jamie's conception of the function from interesting to hard. She was still able to do problems in which the logarithmic function was not mentioned, using her collection of trial- and-error procedures. Various trial-and-error procedures were used, both in service of the convert procedure she had learned during the instructional phase and to find exponents in problems that she did not know involved the logarithmic function. The trial-and-error procedures all involved the intelligent use of a calculator as a tool.

The first important observation about Jamie's changes in understanding is the role that her memory played during the postinstructional phase. Jamie's attempts to remember how to use the logarithmic function almost always resulted in either incorrect answers or faulty reasoning. The primary example of this was her memory of how to add logarithms. She used the property correctly during the instructional phase, but had not understood why it worked. She simply mimicked the notation. During the postinstrucitonal phase, Jamie reconstructed (Bartlett, 1932) what she believed was a correct rule for adding logarithms. It made sense to her, but then she had no way of checking her results because she did not know what a logarithm was.

A second important observation regarding changes in Jamie's understanding is her ability to use trial-and-error procedures to help her solve problems. Jamie's performance on problems involving logarithms improved when the function itself was not mentioned. Being aware that the logarithmic function was involved in a problem limited her search for a solution path in both time and scope. If a logarithmic function was mentioned, Jamie associated it with keys on the calculator or the convert procedure. She did not attempt to look beyond these two techniques for solution paths. If she was not able to solve the problem using either of these methods, she gave up. Her attempts to solve problems in which the logarithmic function was not mentioned began with the four operations. When these did not work, she tried an exponential expression. Since this attempt worked, Jamie then developed a trial-and-error procedure with her calculator to solve the problems. This combination of mathematical knowledge and the use of a tool made it possible for Jamie to solve problems.

Despite Jamie’s initial enthusiasm about the logarithmic function, her realization that she knew little about the function during the postinstructional phase led to frustration. The only success that Jamie had during the postinstructional phase occurred when she was unaware the problem involved the logarithmic function.

Ways of Knowing

Jamie’s work during Interviews 5 and 6 illustrates a procedure that could be used to develop her understanding of the logarithmic function. In particular, although Jamie did not know how to find signs numbers during Interview 5, she quickly realized that each of the number line numbers with a sign above it was a power of 2. When she rewrote the number line number as a power of 2, Jamie realized that the sign number was just the exponent in her expression. Despite the inefficiency of the trial-and-error procedure she had used to find the number line numbers as powers of 2, the procedure made sense to Jamie. In addition, it resulted in correct solutions and was consistent with standard mathematical thinking associated with the logarithmic function.

Jamie’s way of knowing in Interview 5 could be used to develop her understanding of the logarithmic function. Although she knew what she was doing, Jamie had great difficulty articulating her procedure. For understanding to grow, connections between Jamie’s trial and error method and other representations of the method needed to be developed. In a teaching situation, I might have asked Jamie to develop a table of values for her data and then graph her data. Answering these questions would help Jamie associate her actions with other representations.

Of further interest is Jamie’s avoidance of standard mathematical notation such as the use of *x* as an unknown or as a variable. For growth of understanding of mathematical concepts in general and the logarithmic function in particular, a connection between Jamie’s representations of functional relationships and standard mathematical notation should be developed. In Interview 5, when Jamie tried to communicate her data, she drew a picture and included the notation $2^?$. Jamie’s use of notation indicates she was aware notation conveys meaning to the reader. She seemed unaware that standard mathematical notation could convey the same meaning with more precision. If Jamie was able to see her notation as imprecise, she would seek alternatives. This search would be the perfect opportunity for instruction.

Jamie's use of representations and the calculator during Interviews 5, 6, and 7 provide the basis for growth of understanding. In particular, she noted the inefficiency of her trial-and-error procedure for finding exponents during Interview 7. This desire for a more efficient tool could be the beginning of a search for one.

Rachel

Getting to Know Rachel

Rachel chose to attend RC because it was a two-year residential college with a pre-veterinary medicine program. After her arrival at RC, Rachel quickly decided to change her major. She explained that the course work in the pre-veterinary program had become too difficult for her and that she changed her major to family and consumer sciences. While she was growing up, Rachel's mother had owned and operated a family day care center in her home. Rachel had worked with children, —babysitting and in summer camps—, during high school but had never thought of pursuing the work as a career. So when she decided to change her major, family and consumer science was a natural choice. Although she knew she wanted work with children, Rachel did not know exactly what she wanted to do. She described her goals during Interview 2:

> I really don't know where I'm going with it. I know that I'm working with kids. I know that I'm not going to be a teacher. I'm iffy about maybe running my own [child care] center. I'm probably...you know...maybe working with DFACS (Department of Family and Consumer Services), something like that. I want to help the kids that can't help themselves. I don't want to...not to say that teaching is bad, but your hands are really tied as far as helping the child grow in ways other than educationally. And I am more interested in the personal child.

This focus on the emotional needs of children was the result of both her home environment and her school experiences.

School had always been difficult for Rachel. She described it as “awful.” In elementary school in particular, Rachel explained, “I just wanted to crawl in my little cubby hole and never come out.” She disliked school because of her relationships with teachers and peers. Rachel suffered from a neurological disorder that resulted in extreme hand tremors. By the time I met her, Rachel was taking medication that enabled her to write clearly, something she had done with difficulty in elementary school. In addition to the tremors, Rachel had attention deficit hyperactive disorder (ADHD). She was very conscious of how her disability set her apart from her peers. In elementary school, she had been sent to a resource room teacher for several of her subjects. She described her experience during Interview 2:

> I have a learning disability, and I remember the humiliation of having to leave the classroom and go to a resource classroom for math and English and reading. I had to get up and leave, and it was just like public humiliation to have to go and leave and go to another classroom to a different teacher to get special attention.

Her school experiences had made Rachel very conscious of her classroom behavior. She rarely spoke in class and stayed very busy looking in the book, using her calculator, and taking notes. Rachel never spoke to anyone in class, other than the teacher, and admitted to me that she did not try to get to know anyone in her classes.

Rachel was a 2-year-old sophomore at RC in the middle of her third year. Two events contributed to the length of Rachel’s stay at RC: changing her major and her placement in developmental studies for mathematics. It had taken Rachel three semesters to exit developmental studies, the maximum time allowed by the state system. Following her successful completion of the developmental studies program in the spring 2000, Rachel had taken college algebra with Teacher 1. She earned a D. Since she needed a C in the course to take any subsequent mathematics courses, she re-enrolled in the course during the fall 2000 semester.

Rachel was from a large metropolitan area approximately 300 miles north of RC and only went home on the weekends. Despite coming to school from such a distance, Rachel felt she was part of the RC community. During a regional disaster that had forced several hundred families from their homes into a temporary shelter at RC, Rachel had volunteered to baby-sit and entertain children while their parents rested. She enjoyed being with the children and helping the families.

Talking to Rachel was a real treat. She was especially knowledgeable about child development. During one of our early meetings, she recommended a helpful book on the subject designed for parents of children age 0 to 18. Rachel and I talked a great deal while I was at RC. She dropped by my office to discuss school, her roommate, her boyfriend, and photography, or to have me proofread a paper.

The single character trait that affected Rachel most deeply was her quest for perfection, the most obvious manifestation of which was her obsession to organize. She talked about this obsession during Interview 9. I noticed that Rachel seemed particularly concerned about a wrong answer she had given during Interview 6. I mistook this concern for mathematical interest. Curiosity about mathematics would have been very unusual for Rachel. She quickly set me straight:

> It is the perfection thing, really I think it is. I'm really bad at that with all things. It drives me nuts. It has to be, it has to be just so. Like I have everything just so in my room. And when it gets unorganized, it drives me nuts. I mean when I have nothing better to do or when I am bored I will go organize my room....Whenever I organize my room, I will have a box of junk that I really couldn't do anything with, and I will just put everything in there and shove it under my bed. And that's what I start with when I go back to organize the next time. So it is like an ongoing thing. It is like, okay, I couldn't figure out what to do with this, so I'm

> just going to sit it here. And then I will come back … a couple days later, and I will pull it back out and say okay, I can put this here, this here, this here, this can get thrown away, you know. And then, then I start back on everything else. I guess it is the same system. Pretty much.

As she noted at the end of this quote, Rachel thought how she did mathematics was similar to the system she used for organizing her room. She was not interested in the problem from Interview 6, but was just tying up loose ends. Organization for Rachel was a compulsion.

Rachel As a Mathematics Student

Rachel attributed her quest for perfection to her parents, particularly her father, and their expectations of her. In each Interview, Rachel mentioned her father and his mathematical abilities or expectations of her. He had spent many hours trying to help her with her mathematics homework during middle and high school. According to Rachel, her father considered the subject rather easy, whereas she found it extremely difficult. She spoke about her father and mathematics during Interview 3.

> Because my dad was so hard on me about math...I would sometimes not do my homework or not say that I had homework, because I knew that it was like a 1-hour lecture per problem. And when I come home with my homework now that I'm in college, he will try and help me. And I'm like "Don't help me. Don't bug me. Don't help me. Don't try." It's like he is such a...he's so good at math. And he expects me to be this genius, and I'm not. I get it my own way, my own pace, my own time. You know. But, I will eventually get it, but it's just the mental process of telling myself...you know you've got to tell yourself that you will get it and you will figure it out.

When I asked Rachel how she would describe herself as a mathematics student, she replied "extremely weak." She was still optimistic that she would eventually understand mathematics: "I hoped [I would eventually understand it]. I do hope still. I gain a little bit of knowledge every time I take a math class." Rachel's hope was preceded by numerous failures in mathematics classes. She became extremely frustrated when she could not do a problem and explained that the frustration was with her teacher for assigning the problem, with herself for not "being able to comprehend like other people," and with "other people that are like 'oh, that is sooo easy, and dudada.'"

Rachel identified the root of all her trouble in mathematics as the introduction of "letters." She explained her confusion during Interview 2:

> I think probably in high school They started putting… letters in to equations, and all letters have different values, and which value in which problem. Because it varied from problem to problem, the value of the letter. And in formulas there's letters. *K* stands for something, and *h* stands for some thing. *P*'s and *Q*'s, and like why, why? I don't get it.

Rachel's frustration extended to having to learn anything about mathematics, which she saw as useless. To Rachel, some mathematics just "didn't make sense." She singled out the square root and asked, "Who invented it anyway?" Being a perfectionist, however, Rachel still wanted to pass her mathematics class. Her goal in a mathematics class was to "get through it and pass it. Get it over with."

In Teacher 1's class, Rachel sat in the center of the front row. She took very careful notes on both the handout and on separate sheets of notebook paper. She tried to get down "exactly" what the teacher had on the board. She thought that if she copied down everything exactly as Teacher 1 had written it, later when she tried to figure it out, she would know exactly what the teacher meant. Rachel asked questions in class, but they were usually regarding alternative answers or procedures.

Rachel was very organized and did all of her homework. During past semesters she had relied on a tutor at the AAC, but the tutor had graduated. So Rachel used her book and notes from her last attempt at the course. She found the book particularly helpful in that it gave formulas and examples to follow. Rachel's second attempt at college algebra resulted in a grade of B for the course.

Understanding Mathematical Concepts

During Interview 2 Rachel defined understanding as being “able to comprehend what you are trying to do.” She clarified this statement a little later in the Interview. After she had finished her drawing, she noted that understanding was “to be able to do the process.” Understanding for Rachel, like Jamie, was being able to do the problems.

She used several expressions for understanding. She would say or write, “I get it.” She also called understanding “making sense.” Although these expressions sound more like conceptual understanding than being able to “do the process,” Rachel only used these phrases when she was able to do a problem or saw how to find an answer. During Interview 2, I asked Rachel to give me an example of an experience of understanding a mathematical concept, and she explained her experience of understanding the quadratic equations:

S: Think of a time in your study of mathematics that you felt you understood an idea or concept and tell me about that.

R: Uh...I can’t think of the name of what the formula is called but, I can tell it to you. The one that we are working on now. That we have been working on. It is like $ax^2 + bx + c$.

S: The quadratic formula.

R: Yeah, the quadratic formula. I really feel like I get that.

S: Okay, when you first saw it, did you understand it?

R: No the first time I saw it was in the…. I probably didn’t get it when I first saw it, but I get it now.

S: Okay, do you recall why it is you get it now, but you didn’t get it then.

R: Probably repetition….That was just a lot of practice at it. Imprinting the formula in my head really….

S: Do you think anyone helped you to get there?

R: I did have help understanding it. But, it is really just memorizing the formula and being able to plug stuff in and so.…

S: How do you feel about it now that you understand it?

R: I feel good about it. I feel that, okay there is something I can do.

Being able to memorize procedures was difficult for Rachel, but she hoped it would help her be able to retrieve the information from her brain. She thought of her brain as a huge "card catalog" with information written on cards. Although Rachel felt she understood mathematical concepts, she explained that sometimes she could not find the card in her mind the information was written on.

Rachel's Understanding of the Logarithmic Function: Preinstructional Phase

Evidence of Understanding

Because Rachel had taken college algebra just two semesters prior to the beginning of the study, her experience with the logarithmic function was in Teacher 1's class. She recalled the logarithmic function was "to a power" and the power was "on the bottom." These comments did not give me any insight into her experience with the logarithmic function, but they made more sense after I observed Rachel doing the skills assessment during Interview 1.

Conception. Rachel's conception of the logarithmic function had two elements. First, she felt the logarithmic function was "hard." She could not do any of the problems on the skills assessment and repeatedly remarked, "I don't remember how to do it," as she attempted to do various problems. Not remembering frustrated Rachel, and when I asked her to draw a map of her concept of the logarithmic function, she connected it the word *hard* and the phrase "don't get it." She described the function as *hard* because she could not do the problems associated with it.

The second element of Rachel's conception of the logarithmic function was that it was "to a power." She used the term *power* to signify the base of the logarithmic function. Her list of the properties of the logarithmic function included "can't be a neg. power." Rachel saw the base as significant. After we discussed some of the problems on the skills assessment, Rachel wrote "$\log_{base}$ #" as another property of the logarithmic function. She had a template for the notation of the logarithm, but did not know what the terms *base* and # in her template meant. She knew where numbers went, not what they did.

Rachel's conception of the logarithmic function during the preinstructional phase was based on what she remembered about the function from her previous attempt at college algebra. Since she could not do the problems the concept was hard and she remembered that the base, which she called the power, was important and could not be negative.

Representation. Rachel's representations of the logarithmic function were primarily oral and were limited to trying to use the calculator to evaluate $\log_3 4$. She knew there was a way to calculate logarithms using the button on her calculator, but she could not figure out what to put in the parenthesis. She could not decide whether "log of three to the fourth" was correct or whether it was log of three times four. Rachel eventually decided to use log of three times four, but was very unsure of the answer and only wrote an answer for the first simplification problem on the skills assessment. She was attempting to read the written notation in a way that allowed her to use it as a basis for calculating the logarithm on her calculator.

The only written representations that Rachel used during the preinstructional phase of the study were notations, $\log_{base}$ #; a maxim, "base can't be negative"; and an example of her idea of a logarithm, $\log_{6.}$ Each of these representations provides information about how to write a logarithm.

Connection. Rachel used two different oral representations for $\log_3 4$. The only connections that she made during this phase were between given written representations and oral ones.

Application. Rachel applied her knowledge of the logarithm as a computation done with the calculator to evaluate $\log_3 4$. She recalled that calculating logarithms involved the calculator, but she did not know how to use it.

Beliefs.

During the preinstructional phase of the study Rachel used three beliefs about the logarithmic function. First: logarithms are hard to remember. Although Rachel said logarithms were hard, her references to not being able to remember how to do the problems in conjunction with her definition that knowing how to do the process was "understanding," led me to believe that when Rachel wrote "hard," she meant "hard to remember what to do."

Second, logarithms have the form $\log_{base}$ #, where the base cannot be a negative number. The template $\log_{base}$ # allowed Rachel to appear competent. Using the template and her maxim about the base, Rachel could write correct logarithmic expressions. Although she associated the notation with the topic, I was unable to associate her use of the notation with any valid concept about the logarithmic function.

Third, the logarithmic expressions could be calculated using the logarithm key on the TI-83 graphing calculator. Rachel was confident that the calculator would help her solve the simplification problems if she knew how to use it. In summary, Rachel's beliefs about the logarithmic function prior to instruction were

1. Logarithms are hard to remember because I cannot do these problems.
2. Logarithms have the form $\log_{base}$ #, where the base cannot be a negative number.
3. Logarithmic expressions can be calculated using the logarithm key on the TI-83 graphing calculator.

Rachel's Understanding of the Logarithmic Function: Instructional Phase

Evidence of Understanding

Conception. During the instructional phase of the study, Rachel learned how to do problems, and hence her conception of the logarithmic function changed. Now logarithms were "really easy."

> I knew that I knew logs, but it was just a...It was all about pulling it out of my memory bank. Because I was, like, okay I know this...I know that I know this, and I know that I thought it was kind of easy when I did it the

> first go-round. So I was like, okay, why am I having trouble with this? Because I know I know it. Like I said, memory recall.

For Rachel, logarithms were easy now because she remembered how to do them. When I asked her how she understood logarithms, she told me how she did the problems:

> Just plug it in. That's really all. That's all I can think of. That's all I do. With my roommate… she would be, like, "How do you do this problem?" I said, "go back to ...go back to [page 331] 321, look at the little box with the formulas in it and match it up (laughs)." I was just like just match it. That's all you have to do is match it.

The formulas that Rachel was referring to were on page 331 of her textbook in a box titled, "Summary of the Properties of Logarithms." Inside the box are what the authors referred to as rules, formulas, and properties: the product rule, the power rule, the quotient rule, the change of base formula, and other properties. The other properties included were

$\log_a a = 1$, $\log_a 1 = 0$, $\log_a a^x = x$, and $a^{\log_a x} = x$. Rachel used these "formulas" to transform logarithmic expressions from one form to another. She saw "matching formulas" as the essence of "understanding" the logarithmic function. Rachel's conception of the logarithmic function was that it was easy since all she had to do was match the "formulas" to the problems.

Representation. During instruction Rachel used written and oral representations. As she tried to learn how to do problems, she developed a vocabulary associated with her actions or procedures. She used the terms *formula* and *match it* to describe how she did problems. Rachel's map from Interview 3 illustrates what she meant when she used the term formula (see Figure 18). As this map illustrates, Rachel thought of mathematics as a

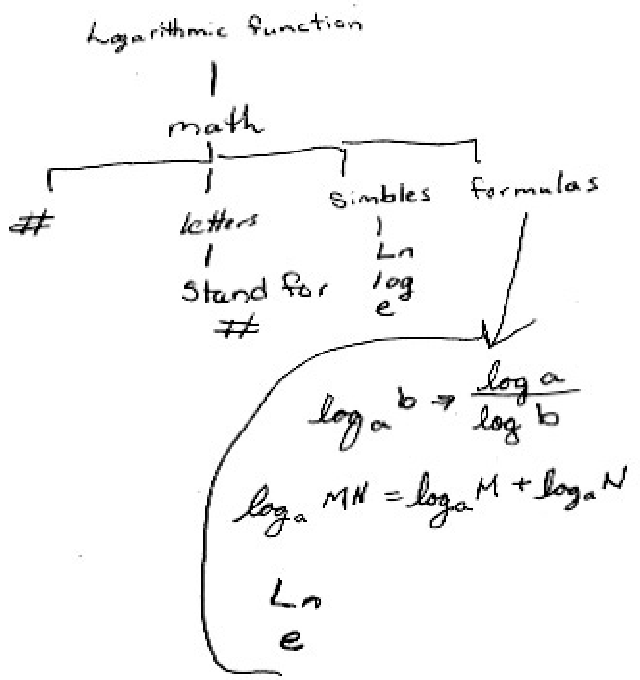


Figure 18. Rachel's map of the logarithmic function from Interview 3.

collection of numbers, letters, symbols, and formulas. Because the logarithmic function was a part of mathematics, it was composed of these essential parts. This association was the basis for Rachel's thinking about the logarithmic function.

Connections. Rachel connected both names of objects and procedures, descriptions of the procedures, and notations associated with the objects. Her oral representation "match it" was related to her oral representation "formulas" because it was part of her description of the procedure itself. Finding the appropriate formula was part of the matching procedure.

The term *formulas* was also written on Rachel's map of the logarithmic function. Rachel connected this category to notation and gave written examples of the formulas on her map. Hence there is a connection between the written term *formulas* and the notation.

<u>Beliefs</u>

Rachel's understanding of the logarithmic function during this phase of the study was as a collection of formulas that were easily applied to do problems. If the list of formulas was given, Rachel "matched" the formulas from the text with the problem types to solve the problems:

1. Logarithms are easy because I can do the problems.
2. To do problems associated with the logarithmic function, match the problems to an appropriate formula in the textbook.
3. Important formulas associated to the logarithmic function are: $\log_a b = \frac{\log a}{\log b}$,

   $\log_a MN = \log_a M + \log_a N$, Ln, and e.

Rachel's Understanding of the Logarithmic Function: Postinstructional Phase

Evidence of Understanding

Conception. Rachel's conception of the logarithmic function was as a collection of problems that were easy to solve with the table in the book. She saw the function as a collection of problems for two reasons. First, since the logarithmic function was a function:

> It is always something being plugged into something. You are always given some sort of function that you've got to solve. It bothers me that they are called functions and not problems. I mean, it's the same thing.

Rachel saw no distinction between functions and problems. The goal was to find the answer. Second, the logarithmic function was a type of mathematics. During Interview 9, she drew a diagram of and explained her process of understanding the logarithmic function:

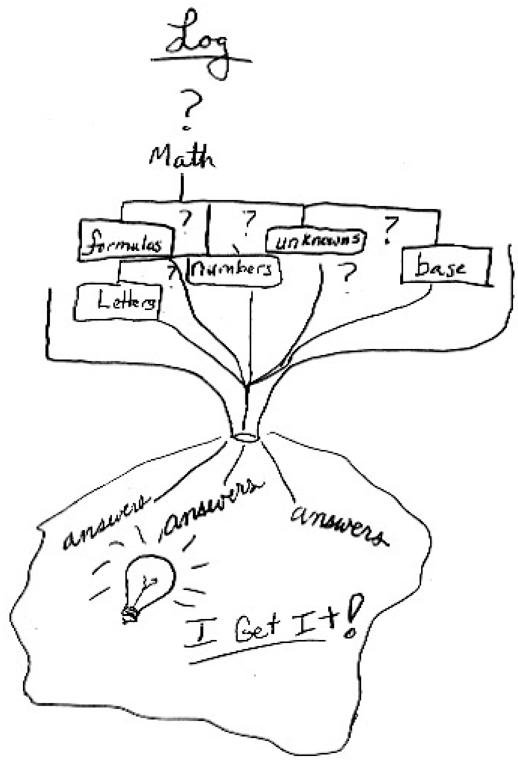


Figure 19. Rachel's process of "understanding" the logarithmic function.

> Log is...well, we are really not sure what log is at first. And then when we figure out that it is a kind of math. And through that we figure out that there are formulas, numbers, letters, unknowns, and of course you always have your base. And there are a lot of questions as to how to get from place to place, the question marks in between. You know you know this, but how do you relate it to this and how do you relate it to that? And then it kind of slowly all comes together, and then we get our answers, and then a light bulb goes on. I get it. It makes sense.

For Rachel the collection of problems associated with the logarithmic function was typical of all mathematics. The goal was to get answers.

Rachel felt confident that she could get answers using the properties of the logarithmic function in her textbook and using her calculator. She explained her process during Interview 8:

S: Okay, what else do I need to know? I want to be ready. I want to get an A in this class.

R: Basically, all you need to know is the.… You will have a chart. It should be in your book. You have a chart that is going to give you like log base times the.… Oh gosh, I don't know what the letter is. We will just call it *a* for lack of a better thing [Rachel writes $\log_b a$]. So this is your base [Rachel points to *b*] and this is your, your number [Rachel points to *a*] and then you would convert that into log of *a* over log of *b* [$\log_b a = \frac{\log a}{\log b}$]. Right, you know how to do that.

S: Yeah, you showed me that. That was cool.

R: Then there will be, like this one [points to $\log_b a = \frac{\log a}{\log b}$] will be there. And a couple of others will be there, but there will be a chart in your book that gives you all the different conversions. So if you find that in your book, then it makes it very simple to follow.

S: So I need to get that book, no matter what the teacher said.

R: Yeah. The book is not an option.

Understanding the logarithmic function was simple for Rachel, it was a collection of problems easily solved using the properties in the text and the calculator.

Representation. During the postinstructional phase of the study, Rachel used three modes of representation: oral, written, and pictorial. Like Jamie, Rachel explained her work with several names: *convert*, *formula*, and *guess and check.* Although both Jamie and Rachel used the term *convert*, Rachel's definition was different. She used *convert* to refer to the action she took to transform a logarithmic function or expression into a form she could use her calculator to evaluate. For example, she called transforming a logarithmic expression with the change of base formula *converting.* She explained during Interview 8:

R: If it is blank [pointing to the space between log and 5 in her written expression log 5], it's ten. Now see here we go. We have to have a calculator. And then ... If you wanted to solve this equation [referring to $\log_4 5$] as an equation,... you would have convert it because you can't solve it this way.

S: Why?

R: Because there is no way to put. This is your base, and you have to put it into a form that your calculator can read. Because you like to use a calculator.

S: Oh, the calculator can't do base four?

R: No.

S: That's not very good.

R: But it can, if you rewrite it. So you would write log of five and divide log of five by log of four [writes $\frac{\log 5}{\log 4}$]. So then you can get it [the numeric value for the expression], and this is your log key right here. So you would say log, and it automatically gives you the parenthesis. So five, and you have to make sure you close them. Then you divide by log of four and then close the parenthesis, and you enter. And that gives you a number. Then you just round to like the nearest decimal, so you would say that [writes 1.16].

Converting did not refer to simplifying an expression, such as $\log_3 4 + \log_3 5$ into $\log_3 9$. *Convert* was a name used to refer strictly to actions performed with the change of base formula and the transformation of an expression in logarithmic form to one in exponential form. The second term Rachel used during the postinstructional phase was *formula*. While *convert* referred to her actions, *formula* referred to the notations she used to direct her actions. The third term Rachel used during the postinstructional phase was *guess and check*. The procedure Rachel called "guess and check" was similar Jamie's use of “trial and error.” Rachel used the procedure during Interview 7 to fill in the table. Since she did not have a logarithm key on her calculator, with my help she developed an exponential equation and used “guess and check” to solve it. The primary difference between Jamie’s procedure and Rachel’s was Rachel’s use of a relevant domain for her selections of the *x* values she should try for the exponents. For example, since log 4 was between log 3 and log 5, Rachel tried numbers between .699 and .845 in her “guess and check” procedure.

In her explanations of her work, Rachel used incorrect representations for the exponential and logarithmic function. For example, during Interview 4 Rachel saw the expression $\log_3 9$ as said: “log of three to the six and log of three to the three” and wrote $\log_3 6 + \log_3 3$. She thought of the log and 3^6 as separate expressions and read them that way. She only used the correct oral representation for the logarithm notation when the base was not written. She was aware the base of the function played a role in the oral representation, but not what role it played.

Rachel also used maxims about the base to explain to do problems. For example, she noted during Interview 8 she said the base can “never, never be negative. No matter what.” and if “it is not there, it is a ten.” When I asked Rachel why this was so, she replied, “It just can’t. It is one of those weird rules.”

Rachel used two types of written representations frequently during the postinstructional phase. She wrote names and notation. Rachel wrote the names: *formulas*, *math*, *base*, and *numbers* on each map and drawing she created during this phase of the study. Rachel’s used the term *base* in conjunction with her maxims. In particular she associated “never negative and “if unknown 10” with *base* on her map of the logarithmic function. I am uncertain of the meaning of Rachel’s written name: *number*. The most plausible interpretation is that she associated finding and using numbers with plugging into formulas.

The second type of written representation that Rachel used was notation. She transformed expression from one written notation to another so she could use her calculator to find a numeric expression. For example, during Interview 4, Rachel transformed $\log_3 9$ into $\frac{\log 9}{\log 3}$ saying, "Yeah, you can change it from that [$\log_3 9$], and then you can do this [picks up calculator]." The notation was used to enable Rachel to write an expression she could evaluate with her calculator.

Rachel used one pictorial representation of the logarithmic function during the postinstructional phase. During Interview 6, she used a picture to help illustrate $f(x)$ when x is a negative integer. Prior to this extension, the only domain that was meaningful to Rachel were positive integer powers of 2. Rachel, however, did not see x as a power of 2, but rather as repeatedly divisible by 2, hence she was able to find $f(4)$, $f(8)$, $f(16)$, and $f(256)$ easily. For example, she found $f(256)$ by dividing 256 by 2 and writing $f(128) + f(2)$. She then repeated the process. She divided 128 by 2 and wrote $f(64) + f(2) + f(2)$. Using this method, Rachel was able to substitute 1 for $f(2)$ and find $f(128)$. Unfortunately, this method did not work for $x = \frac{1}{2}, \frac{1}{8}$, and $\frac{1}{256}$. In part because of the failure of this division algorithm to produce answers, Rachel tried a new predictive method to evaluate $f(-4)$. She reasoned that the $f(-4) = -f(4)$ because of the symmetry of the real line and the order of the "answers." To illustrate this perspective, Rachel drew the picture in Figure 20. This

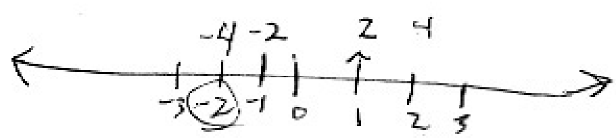


Figure 20. Rachel's pictorial representation for *f*.

diagram illustrates Rachel's use of the symmetry of the real line and the relationship between the x and y values to predict y values when x was less than zero. Hence, she predicted that $f(-2) = -1$ and $f(-4) = -2$. "Well, ...it's on the number line.... So it would only be logical that if this [pointing to 1] equals two, then this [pointing to –1] should equal negative two." Not only was Rachel using symmetry to predict *f*(-4), she has also found a way to eliminate the notation for the function *f* from the process.

Connection. The two primary connections that Rachel used during the postinstructional phase of the study were between oral and written representations and among written representations. The connection between oral and written representation was artificial because she only attempted oral representations to explain her work to me. Thus the primary connection used by Rachel was from one notational form to another. These connections enable her to evaluate expression with her calculator. For example during Interview 8, she transformed $5^2 = 3$ to 3 log 5 = 2, saying, "This [$5^2 = 3$] is without the log, but it means the same as that [3 log 5 = 2]." Again, although Rachel connected notations, she rarely did so correctly.

Application. Rachel believed a calculator was important in evaluating logarithmic expressions. She converted or matched a problem with the right formula and then used the calculator to compute answers. For example, during Interview 8 Rachel explained "We have to have a calculator...to solve this [$\log_{10} 5$]."

<u>Beliefs</u>

Rachel used four beliefs during the postinstructional phase. First, she saw the logarithmic function as a type of mathematics that involved converting from one form to another using a collection of formulas. Second, she saw the calculator as integral to the process of finding the answer to problems associated with the logarithmic function. Third, she knew the formulas were important and could use one or two of them from her book. Fourth, she often used the formulas incorrectly but knew that if she had her book she would be able to match the problem to the correct formula.

1. A logarithmic function is a type of mathematics that involves converting from one form to another by matching the problems with the formulas found in my textbook.

2. Some formulas are very important (change of base and changing exponential to logarithm), and I need to know how to use them to convert logarithms to a form my calculator can evaluate.
3. The book contains all the conversion formulas for the logarithmic function.
4. The calculator is a necessary tool for evaluating logarithms.

<u>Changes in Understanding</u>

Rachel's understanding of the logarithmic function was influenced by her attempts to memorize enough formulas and their applications to pass the test. Prior to instruction, Rachel was frustrated by her inability to remember how to do the problems on the skills assessment. Following the assessment, I asked Rachel what she understood least during the activity.

> S: The next thing I want to ask you what you felt you understood least. Tell me about that.
>
> R: I didn't remember very much about the logarithmic function at all.
>
> S: And how did you feel about that?
>
> R: Frustrated...I wanted to be able to do it, but I can't remember what I was ... how to do it.

Rachel's comments indicate her desire to do the problems by remembering the procedures associated with the logarithmic function. On her map of the logarithmic function, she had only two categories "hard" and "I don't get it."

On numerous occasions during the course of the study, Rachel and I discussed mathematical notation. Rachel felt she had done well in school mathematics "until they started throwing letters in. Letters and shapes that are suppose to mean a number." She felt the notation was useless but realized her success was predicated on learning the symbols and formulas. During the instructional phase, she attempted to coordinate the formulas and the problems she was asked to solve. Because she had taken the course in spring 2000, she was able to coordinate the formulas and problems quickly. She was excited when she came into the office for the instructional interview because she felt she had understood the whole lesson. “As long as I have a formula and I know what goes where, I can plug it in and make it work.” Rachel had memorized the change of base formula and the sum of logarithms formula and included these on her map of the function. By the end of the instructional phase, she matched simplification problems with the formulas in her textbook. She was able to get the answers she needed to compute for the test.

During the postinstructional phase of the study Rachel’s *beliefs* about the logarithmic function, like Jamie’s, were associated with the notation “log.” If she saw or heard the word, she was aware that the formulas from her textbook should apply. If not, she used other strategies. Rachel was unable to do any of the problems on the skills assessment. She was able to remember her maxims about the base but could not recall a single formula from the textbook. Like Jamie, she recalled the log of a sum was the sum of the logarithms. She also incorrectly recalled the change of base formula. Rachel explained that the information was still in her “card catalog,” but she could not find the card.

During Interviews 5 and 6, Rachel was able to perform but was not aware that either of the tasks were associated with the logarithmic function. During Interview 5, she used a successive differences scheme to predict the signs above the numbers greater than 8 and linear interpolation to predict the signs above numbers such as $\sqrt{2}$ and 6. Her performance during Interview 6 illustrated the difficulty Rachel had with notation. She saw f as a formula and conjectured that the operations used were dependent on the numbers that she was given to “plug in.” For example, to evaluate f at 2 would require one formula and to evaluate f at –2 would require another formula. In general, for these two interviews Rachel relied on strategies associated with the four arithmetic operations.

During Interview 7, Rachel was able to convert the table entries into exponential form (with prompting from me) and used the form and a guess and check strategy to complete the table. Her explanation of the logarithmic function during Interview 8 consisted of a description of her converting procedure and the advice “with table on page

331 of the text I should be able to solve any problem with logarithms." Despite this impoverished understanding Rachel felt that she understood the logarithmic function. During Interview 9 she explained, "I went from knowing absolutely nothing about the logarithmic function to knowing and understanding it." Despite Rachel's subjective view that she understood the logarithmic function, from my perspective she understood very little. Rachel's beliefs about the logarithmic function during this phase were not useful in doing standard problems or in trying to make sense of nonstandard ones.

Ways of Knowing

Rachel used four ways of knowing during the postinstructional phase of the study that have potential as tools for the growth of understanding: linear interpolation, guess and check, validation of answers, and awareness of inconsistencies in answers. First, I discuss Rachel's use of validation and awareness of inconsistencies in answers. I then illustrate how Rachel's use of linear interpolation might be used to provoke growth of understanding of the logarithmic function.

Any time Rachel did a problem, she tried to find a way to validate her answer either by "working off" other information in the task worksheet, by checking her answer on the calculator, or by checking her answers with an authority such as the teacher. One example of this practice was Rachel's attempt to validate her answer to Problem 3c on the skills assessment during Interview 4. Rachel used a change of base formula incorrectly to find $\log_3 9 = \frac{\log 3}{\log 9} = \frac{1}{2}$. Rachel explained what she was doing:

> R: Umhum....[thinking]. Um...oh, unless we do this. Log of three over log of nine [writes $\frac{\log 3}{\log 9}$]. Like that?
>
> S: So you are remembering something about changing the form from this [$\log_3 9$] to this [$\frac{\log 3}{\log 9}$]?
>
> R: Yeah, you can change it from that, and then you can do this [picks up calculator].

S: Go to the calculator.

R: Log of three divided by log 9 is .5 or $\frac{1}{2}$ which would be the same answer for this [3a] as well.

S: Okay.

R: [inaudible] Try this. Log of three over...I am just going to verify this. (Because Rachel got $\log_3 9$ for both 3a and 3c, she was using the calculator to check that she got $\frac{1}{2}$ by adding $\frac{\log 3}{\log 4}+\frac{\log 3}{\log 5}$). Because I think this [referring to the change of base formula] is the one that works all the time [tries it in the calculator]. Point seven nine.

S: What was that: log of 3 divided by log of 4?

R: Umhum. It shouldn't be that number. Log 3 divided by...so....

S: So what did that come out to be?

R: 147 over 100.

S: Okay.

R: [tries to reduce] You can't reduce that, can you?

S: What?

R: 147 divided by...

S: No, it doesn't reduce.

R: No.

S: So you got .79 or is that the final answer 1.47? Is it like when you add these two [.79 and .64], you get 1.47, or was that just for this [$\frac{\log 3}{\log 4}$] part?

R: No, 1.47 is this [$\frac{\log 3}{\log 4}$] plus this [$\frac{\log 3}{\log 5}$].

S: Okay, so you didn't get one half [for 3a], but you got one half down here [for 3c].

R: This is almost one half.

S: It is almost one and a half. So it is pretty close.

R: Yeah, one and a half.

S: But, something must be wrong....

R: Because all three of these [3a, b, c] should have the same answer. Well, a and c should have the same answer. I look for similarities. So they should have the same answer, but I thought that the base always went up top, but does it not?

In this example, Rachel attempted to validate her answer which resulted in an inconsistency between two problems that she believed should have the same answer. Rachel did finally resolve the inconsistency in her two answers. She decided that $\frac{\log 3}{\log 9}$ was $\frac{1}{2}$. The calculator had verified that, but that she was incorrectly adding $\log_3 4 + \log_3 5$. She decided that this expression could be rewritten as $\log_3 (4 + 5)$ and hence was $\log_3 9$, which her calculator had verified was $\frac{1}{2}$. Although she drew incorrect conclusions and used faulty reasoning (that her calculation for $\log_3 9$ was correct, but for $\log_3 4 + \log_3 5$ was incorrect), Rachel was aware that the two answers should be the same and attempted to resolve the conflict. This awareness could be used to help search for meaningful reasons for the inconsistencies. Why are the answers inconsistent? Is one of the solution paths flawed? This type of thinking can be capitalized on in the classroom.

Rachel's use of linear interpolation in combination with her awareness of inconsistencies could be used in promoting the growth of understanding of the logarithmic function. During Interview 5, Rachel used linear interpolation to predict the sign number above the number line number that she was not able to find using her successive differences algorithm. When Rachel was asked to predict the sign over the 6, she predicted it would be $2\frac{1}{2}$, halfway between the sign above 4 and the sign above 8. This use of linear interpolation is an example of what Stavey and Tirosh (2000) called an *intuitive rule*: “More A-More B.” Rachel assumed that if 6 is half way between 4 and 8, then the sign should be halfway between 2 and 3. The results of Rachel’s predictions could be graphed (see Figure 21).

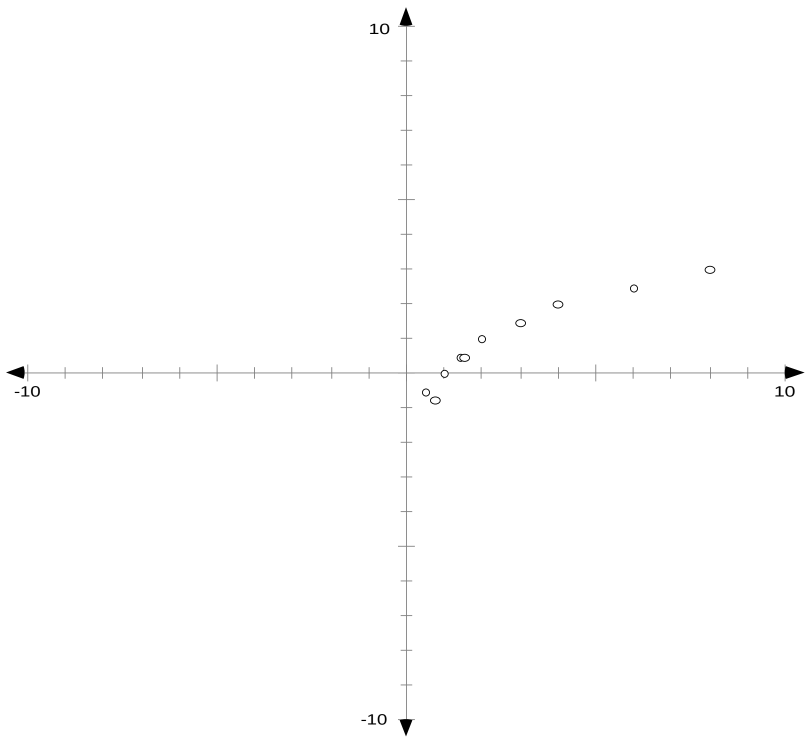


Figure 21. Plot of Rachel’s predictions for Interview 5 task.

Given Rachel’s awareness of inconsistencies, she would have seen the graph as inconsistent or at least problematic because of, the dips and the flat section. This awareness is likely prompt a further examination of her prediction algorithm.

Nora

<u>Getting to Know Nora</u>

Nora did not enroll in college immediately following her high school graduation. She could not afford RC. After a year off, she applied for and was awarded financial aid to attend RC. Nora was 19 during fall 2000. She commuted 30 miles to school each day. She also had a part time job at a nationwide sandwich chain.   Her work schedule was hectic, and Nora fell behind in her studies. During Interview 2, Nora talked about how difficult working and going to school was for her:

> It wasn't the last chapter, but I had one chapter where I had worked a midnight shift all...like two weeks straight. And I wasn't having time to do homework, so I went to a friend's house, and we did the homework. I understood it very well. It is just, you know, pulling those midnight shifts You know, it was kind of hard…

Nora quit her evening job because of the long hours she was asked to work. To pay for gas and expenses she took a campus job. It was fewer hours, but the workday ended at 5:00 p.m. Unfortunately, a family financial crisis came up, and Nora returned to her evening job, while she continued to work on campus.

Nora wanted to be successful in her work and in school. Her motivation came from her pride in past accomplishments and her competitive nature. She described herself as competitive during Interview 2:

> S: What are your educational goals?
>
> N: Uh, ...not really many. I just want to make it. Get a degree. It is kind of a competition between me and my brothers. I really...I'm a competitive person when it comes to stuff like that. None of my brothers are in college and I've got a twin brother that is still in high school so. But there is still that "I got to do better, I got to do better."

Nora's primary life goal was to "make it," which she defined as getting an associate's degree in business. After getting her degree, she planned to work for a while and then return to school for a bachelor's degree.

Nora As a Mathematics Student

Nora felt that she was a very strong mathematics student, especially in high school. She was always the best in her mathematics classes and had earned an A in every high school mathematics course she had taken. Although she waited a year after high school before enrolling in college, she still expected to be the best student in her college mathematics classes. When I asked Nora what her goal in a mathematics class was, she replied, "Mainly to have the highest grade. That is actually...my goal is to have the highest grade in the class. Not actually the school, but in that particular class." Getting the highest grade in the class got harder for Nora when she decided to attend college. When I asked Nora during Interview 2 to describe herself as a mathematics student, she replied:

> In high school exceptionally well. I really thought that I would major in something math-wise or even…. I considered being a math teacher for high school because I actually loved it and I associate everything with math. In fact I tried to...before you even showed me… concept mapping, that is how I associate math and English. That is how I do an outline with a concept map. But in college I am trying to adapt. I feel it is getting easier, but it is not as easy as it was in high school.

The demands of college, work, and home, where she often cared for her four-year-old brother, had made it more difficult to achieve her goal of getting the highest grade in the class. On a test that was returned the first day that I observed her class, Nora earned a 79, which she explained was far below her standards. "I felt disappointed and it kind of hurt my feelings because, I mean, a 79. I mean it is still good, but it is not up to my standards. And I do kind of set my standards high." She valued her grades and the status she felt they gave her.

In class Nora, sat in the center of the front row, a practice she had begun in high school. She had observed that students who sit in the front get more help from the teacher. She elaborated on her choice of seats during Interview 2:

> Well, I am as blind as a bat, so it helps one for seeing, and sometimes I can't hear too well because I get a lot of ear infections. So it helps my hearing. And plus if I have him (Teacher 3) standing up there watching what I'm doing, he can point out something that I'm doing wrong.

After Nora remarked about this phenomenon, I observed Teacher 3 in class and noticed that he did provide more help to those sitting on the front row. There were seven columns of desks with six desks in each column, and the desks were extremely close together. I had difficulty getting to my seat near the back of the class. During my two weeks of observation, Teacher 3 only helped students beyond the first row once. Nora got help more often than the other students.

Nora never interacted with her peers in the class. Her only communication was with Teacher 3. She explained that even outside of class, she felt most comfortable in one-on-one situation as opposed to crowds. Despite this preference, Nora participated much more than any other student in the class. She often waited for her peers to answer questions posed in class but became frustrated when they would not respond to even the simplest questions. "That gets on my nerves sometimes. I feel like if, well, if nobody else is going to say it I'll say it." As is indicated in this quote, Teacher 3's questions were primarily simple recall, requiring only a single word answer. When students did not respond to his questions, Teacher 3 encouraged and cajoled them saying "call it out." Despite these attempts, students rarely participated. Many came to class, unprepared. One student never brought a pencil and paper to class and several students regularly slept in class. Nora was not one of those students. She brought her learning materials, took careful notes, and after listening to the silences that often followed Teacher 3's questions, she would call out an answer.

Although Nora almost never asked a question in class, if she was having difficulty with a mathematical concept she felt free to ask Teacher 3 for help either before or after class. She also went to go to the AAC for help. She felt certain that someone in the center could show her how to do the problem she was having trouble with.

Nora did her homework and circled any problems she had difficulty with to review for the test. Her class notes were very neatly written, as were her homework papers. Nora was what most teachers would call a model student. She showed respect for the teacher, participated in class without dominating it, did her homework, was concerned about her grades, and loved the subject. Both her attitude and her study habits contributed to the A that Nora eventually earned in college algebra.

Understanding Mathematical Concepts

When I asked Nora how she defined understanding during Interview 2, she described it in terms of doing.

> S: Okay we have talked a lot about understanding and not understanding, but I want to know what your definition of understanding is as it relates to math.
>
> N: My understanding. If I understand something, that means I can do it. If I understand it, then I can walk in on a test and be done with it. You know ten minutes, all right, I'm done. And if I don't understand it then I tend to take longer than a minute trying to work on it. And then that means that I really don't understand it or I am having a hard time at it and I probably need help.

Understanding a concept meant being able to do problems quickly. Nora commented that memorizing was different than understanding. She believed that practicing problems was a great way to understand them. "Like the vertex [of a parabola], negative *b* over two *a*. I didn't memorize that I just understood it and practiced it." Nora knew how to find the vertex of a parabola, so she felt she understood parabolas. When she practiced a problem Nora felt she was learning how to do the problem not memorizing it, hence memorizing was not understanding.

Nora's view of understanding as doing was especially evident during Interview 9. When I asked her to draw a picture of her process of understanding the logarithmic function Nora drew a map (see Figure 22). Each of the categories illustrated how one might do a type of problem. This drawing was markedly different from her depiction of her process of understanding a mathematical concept from Interview 2 (see Figure 23). Nora described not understanding and understanding in terms of feelings she had when she experienced them. When she did not understand, she said, "I get mad." When she understood, she described her mood as bubbly, she explained she just bobbed her head and said okay.

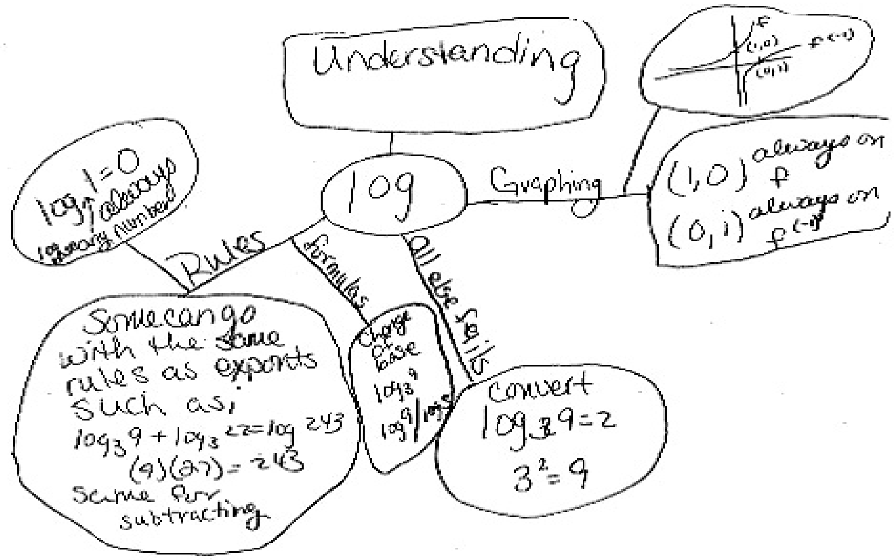


Figure 22. Nora's drawing of her process of understanding the logarithmic function.

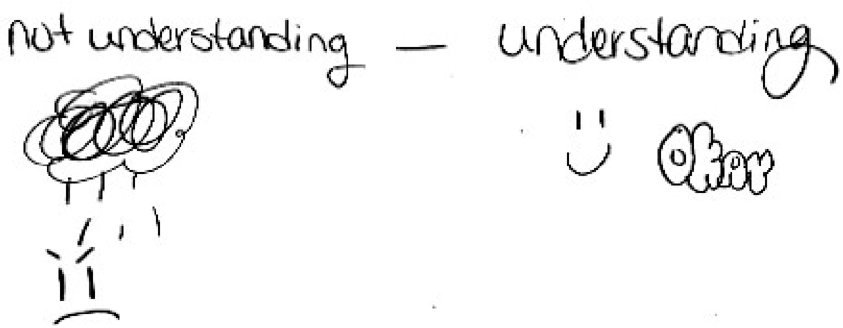


Figure 23. Nora's drawing of her process of understanding.

Nora drew her diagram after I explained to her four times what I wanted her to do: visualize her process of understanding a mathematical concept and draw what she saw. Nora wanted to know what I was looking for and said, "I'm a little confused at what you want." These types of remarks were characteristic of Nora's approach to learning. She tried to find out what was required, of her and then she did exactly that. When it was not clear what was required Nora became frustrated. During Interview 2, she expressed her frustration with Teacher 2:

> It is like the homework … He doesn't quite teach to the homework. And I'm not sure if he is teaching to the test or if he is just teaching to teach the subject because there was some stuff from the homework that he didn't

> show us that was like first off that I felt he should have showed us.… I figured it out by going back into the book and reading, but… I felt like he should have taught that in class.

Nora wanted to know what to focus on, and when Teacher 3 did not provide that focus, she became frustrated. She needed the information to be efficient in her exam preparation.

Nora’s Understanding of the Logarithmic Function: Preinstructional Phase

Nora recalled very little about the logarithmic function from high school. When I asked her if she had seen the function before, she replied, “I don’t know. I would have to see it to tell you if I have.” Later in the same Interview Nora recalled that with the logarithmic function you “start with one, make it into another.” Her remark seemed to apply to any function until she described transforming an expression in logarithmic form into one in exponential form as “easy” for her since she had seen it before in high school. Hence “start with one, make it into another” is a remark about a procedure Nora called converting.

Nora took great pride in her grades. She recalled during Interview 1 that in high school she had "made an A on” the logarithmic function, but she did not remember it. Because she was an A student in mathematics in high school and the logarithmic function was a high school mathematics topic, Nora reasoned she must have earned an A on it. This view was characteristic of Nora’s general view of grades. She felt that if she made an A on a chapter test, she knew everything in the chapter. Similarly if she made an A in a course, she knew all the material in the course.

Evidence of Understanding

Nora did not recall any specific information about the logarithmic function during the preinstructional phase of the study. There was no evidence that Nora applied the logarithmic function or its properties in solving any of the problems on the skills assessment. Her comments and actions during the skills assessment and her map provided evidence of Nora’s conceptions, representations, and connections.

Conception. Nora's conception of the logarithmic was that she could do the problems if she could recall the necessary procedures and knew how to use her calculator to solve them. Since Nora could not remember "how to do" logarithms she saw them as generic problems. Nora's map of the logarithmic function drawn during this phase included two categories: *problems* and *function.* Nora reasoned the function was a topic in mathematics so it was related to problems. And since it was a function it must be related to functions in general. She thought of functions as formulas and depicted a function as $y = ax^2 + bx + c$ on the skills assessment. "That's what I think a function is." Later in the interview I asked Nora how she had developed a table of values from a graph on the skills assessment: "Well, I didn't have a formula, you know a function,…So I didn't know what to do without that function." Based on Nora's view of function, I anticipated that she would be looking for a written representation when she was introduced to the logarithmic function in class.

Following the skills assessment, Nora remarked her calculator might have been of use in helping her figure out the problems. "If I knew how to do it [the logarithm] on the calculator, I could do it." Nora's comment illustrates her view of functions as something that a calculator can help you solve.

In addition to seeing the logarithmic function as a set of problems associated with the general definition of function and with procedures involving the calculator, Nora also noted she recalled doing problems involving the logarithmic function but "not how to do them." Hence, Nora's conception of the logarithmic function was as collection of problems she might be able to use her calculator to help her solve.

Representation. Although Nora did not remember doing problems with logarithms, she still attempted the problems on the skills assessment. During these attempts, she used oral and written representations. Her oral representations were simple attempts to read the notation. For example, Nora read $\log_3 1$ as "log three one." This oral representation illustrates that Nora was unaware of the importance of positionality in the notation. Nora's written representations of the logarithmic function and its properties were in the form of notation. This is not surprising, because her only access to information about the logarithmic function was the skills assessment. She used two rules to simplify the logarithmic expressions given in Problem 3: a generalization of the distributive property and a generalization of the associative property. Nora found the sum of logs to be the log of the sum, a generalization of the distributive law. For example, she calculated:

$\log_3 4 + \log_3 5 = \log_3 9$. She also generalized the associative property, simplifying $\frac{1}{2}\log_3 25$ to $\log_3 12.5$. Both of these generalizations are based on a view of $\log_3$ as a variable like *x*. The representations used by Nora during this phase of the study were attempts to mimic or make sense the notation presented in the skills assessment.

Connection. During the preinstructional phase of the study, Nora made connection between written representations and between written and oral representations. She attempted to mimic the notation she saw used in the problems on the skills assessment. Her answers to Problems 3 and 4, simplification and expansion problems, used the notation $\log_3$ that was given. The answers were incorrect, but the notation was used correctly. Nora was less successful in her attempts to transform the given notation into oral form. She read

$\log_3 1$ as a sequence: "log three one."

<u>Beliefs</u>

Two groups of beliefs were important in Nora's thinking about the logarithmic function. First , Nora felt she had seen the logarithmic function in high school, and she conjectured she had done well on it because she had been successful in all of her mathematics classes. This reconstruction of her experience was a source of confidence for Nora. Second, based on information she drew from the problems on the skills assessment Nora attempted to piece together a reasonable written representation of the logarithmic function. She felt certain that the log key on her calculator was involved and that if she knew how to use it she would be able to solve problems.

1. To do problems involving the logarithmic function, I need to learn how to use my calculator.
2. I have seen logarithmic functions before and since I am a good mathematics student, I must have done well on them.
3. The logarithmic function is a type of function, and so it has problems associated with it.
4. The logarithmic function has a special written representation I have to learn how to use.

Nora's Understanding of the Logarithmic Function: Instructional Phase

The primary mode of instruction used by Teacher 3 was lecture. Each day he made announcements about campus events and then began lecturing. No demands were made on the students. Some slept, others did homework that they had from other class, and several passed notes during class. Teacher 3 carefully wrote examples and explanations on the board and a few students like Nora took notes. The board notes were a summary of what was in the textbook. Teacher 3 followed the book very closely. He referred to it in class and assigned homework problems from it.

Teacher 3 lectured about the logarithmic function for two 75-minute periods. During that time, he demonstrated how to do a number of procedures, all of which were included in a handout that he gave the students one day prior to introducing the function. The handout included examples of what were called "'powerful' log rules that we will find helpful" (see Figure 24). It also contained explanations of how to use the calculator to

Log Rule	Examples
$\log_a M \cdot N = \log_a M + \log_a N$ because $a^M \cdot a^N = a^{M+N}$	$\log_2 8 + \log_2 4 = \log_2 32$ [3 + 2 = 5]

Figure 24. Example from Teacher 3's handout.

evaluate logarithmic expressions and graph logarithmic functions. In addition, the handout contained problems for the student to try. For example, evaluate log 10,000,000. As in class, the students were to find the exact answer if possible or the approximate answer to four decimal places. Also included on the handout were logarithmic equations and applications problems such as finding the pH of a substance.

Nora criticized Teacher 3's ability to convey the information that she needed to know on the test. She questioned the relationship between of the classroom presentations the tests. She also became frustrated when Teacher 3 made mistakes with his calculator. In particular, when he was solving a system of nonlinear equations using his calculator, Teacher 3 could only find one point of intersection when the system had two. He tried several times but failed each time. Nora knew how to find both solutions, but she did not offer any assistance. Later she expressed frustration at Teacher 3's failure:

> I don't think he knows…everything, because there was some stuff … that he couldn't quite get, and I already had the answer for it. And so I think it is kind of an understanding problem.... He knows what he is doing, but he can't quite get it [the calculator] to do what he wants it to do, and then I don't understand what he is wanting.

Because Nora did not know what Teacher 3 wanted or what he would be looking for on the test, she felt frustrated. Her frustration intensified during the postinstructional phase. Nora felt that Teacher 3 had not taught her enough and had covered the material on the logarithmic function too fast. She made numerous comments about the content and how Teacher 3 had presented it. Although none of her remarks was unkind, she faulted Teacher 3 whenever she could not do a problem or remember a procedure.

<u>Evidence of Understanding</u>

Conception. During Interview 3 Nora explained how she viewed the function:

> The main thing I try to do in mind is remember exponential things, the way we combined them, and all that kind of stuff…to remember the principles of them [the logarithmic function]. And if you know the

principles then you can pretty much figure out the log itself. The problem you are trying to do.

Nora's conception of the logarithmic function was as a collection of problems she had to learn how to solve. She saw the *principles*, the properties of the logarithmic function, as tools for solving the problems. She used the properties of exponents to help her remember the properties of the logarithmic function. When I asked her to tell me about the function, Nora explained that her main concern was being able to "do" the function: "Can I do it?…Can I figure out a way to do it?"

Nora was most interested in "understanding" test problems. When I asked her to think of a time when she did not understand the mathematics that was being presented to her, she identified the written notation: $b^{\log_b x} = x$. She was not particularly concerned with learning about the notation, but rather about whether or not she would need to use the property on the test:

S: So are you going to go ask him about it [the property $b^{\log_b x} = x$] or do you think that's just going to be a done deal?

N: I might ask him the day of the test if it's going to be on the test, because he really didn't go over it any of the days.

Indeed, on the day of the test, Nora did ask Teacher 3 if anything using the property was going to be on the exam. Teacher 3 replied it was not, and Nora did not worry about the property again. She focused her attention on those problems and properties she needed to know for the test.

During the postinstructional phase, Nora saw the logarithmic function as a collection of problems she was "confident" she could solve. She used the properties of exponents to help her remember the properties of the logarithmic function.

Representation. Nora used two different modes of representation during the instructional phase of the study: written and oral. Nora wrote names and notations associated with the logarithmic function. The names that she chose for her map of the logarithmic function included *principles*, *do*, *exponential*, and *calculator*. (see Figure 25).

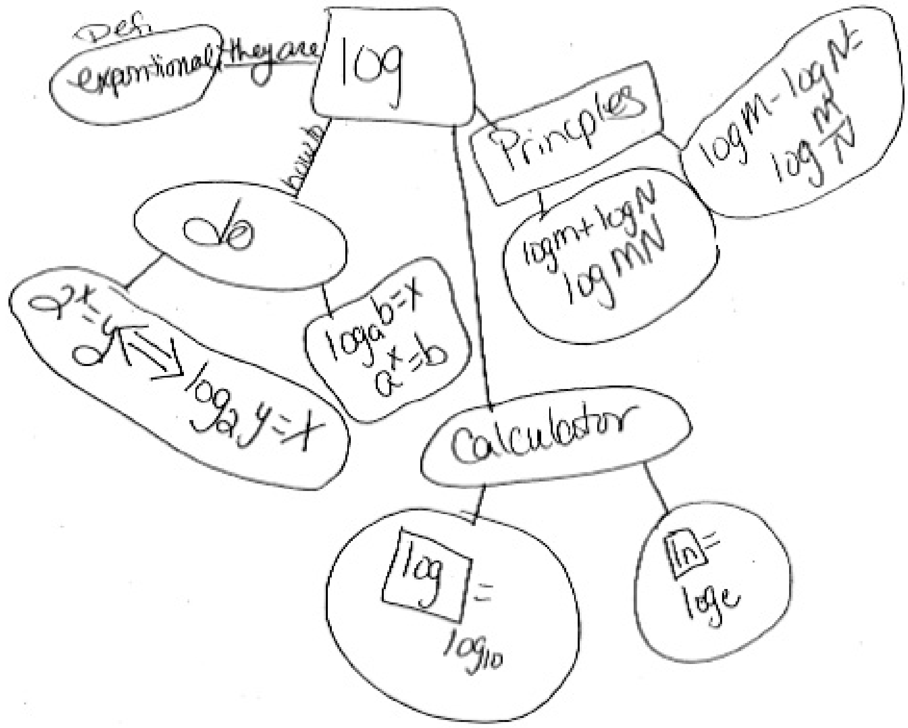


Figure 25. Nora's map of the logarithmic function constructed during Interview 3. With the exception of *exponential*, the names on Nora's map were associated with doing problems. The principles from the textbook, Teacher 3's handout, and class could be used to do simplification problems. The calculator could be used to evaluate logarithms. Transforming an exponential function into logarithmic form was characterized as "how to do" problems. Although Nora's map includes "exponentials are logs," her explanation of how she remembered the "principles" of the logarithmic function contradicts a literal interpretation. She noted that she could remember the principles of the logarithmic function if she just remembered properties of exponents. Nora believed the two were related and not the same.

The second type of written representation Nora used was notation. She always made an effort to get the notation correct, because using correct notation would help her perform on tests. Nora used two notations for the same transformation of a function in exponential form. This use of notation illustrates that Nora saw converting a function as different from converting an expression such as $\log_3 2 = x$. In Nora's notation, *a* and *b* were considered constants, where *x* and *y* were variables.

Nora did not put the same value on oral representations that she did on written ones. For example, she described a problem from class as "logarithmics." This excerpt from Interview 3 was a description of what Nora did not understand during class.

> The logarithmics, where you multiply the two functions. Where he [Teacher 3] did log base of something of something else plus log base of something else and you multiplied it together. I just thought skip that go to that.

When Teacher 3 solved the equation $\log x + \log (x + 3) = 1$, Nora explained that she would have skipped some steps. Nora consistently used incorrect terminology and read notation incorrectly. For example, she read $b^{\log_b x} = x$ as "*b* raised to the log power of base *b* to the *x*." It was not vital for her to use correct oral representations. In class, Nora answered Teacher 3's questions using nonstandard language, but precise language was neither used nor expected. For example, when Teacher 3 asked why ln e = 1. Nora replied, "Because it is itself." This comment was not discussed or questioned; rather, the lesson continued with no further explanation. Knowing how to represent the problem orally was not necessary. Teacher 3 understood Nora's response and her response was sufficient.

Nora's written representations differed from her oral ones. Her written notations were used to perform on tests, whereas her oral ones were used to participate in class. Because getting the highest grade in the class was Nora's goal, knowing how to use correct notation was very important to her. Oral representations were simply for finding out how to do the problems and for being recognized in class. The onus was on Teacher 3 to find a way to make sense of Nora's utterances. Both Teacher 3 and Nora accepted this unspoken contract regarding classroom communication.

Connection. During the instructional phase of the study, Nora used connection between written representations most often. She focused on adopting and creating written representations. Nora's homework illustrated she moved easily between written representations of the logarithmic function. She used all the notations from her map in her homework. Knowing how to use these representations to do problems made Nora feel "extremely confident." She completed the test on inverse, exponential, and logarithmic functions in 25 minutes. Following the examination, she came by my office bubbly and excited and explained that the test was "easy." The exam consisted primarily of simplification, evaluation, and equation problems. Nora had no difficulty with these problems because they were simple applications of the written notation she was so good at using. Her focus on written representations and connections between them produced the desired result.

The categories Nora chose for her map were connected to notation as well. Each referred to or was related to how to do problems using written notation. Hence Nora's written names and notations were connected.

All the connections that Nora drew were focused on being able to do problems and represent what she was doing correctly. These connections were developed to help Nora perform on the test.

Application. I saw very little evidence of application during the instructional phase. Nora used properties of the logarithmic function, such as $\log_a b^x = x \log_a b$, during class discussions and referred to this property as "pull the *x* out front." She also used her calculator in class to evaluate logarithms such as ln 5 and $\log_5 23$. In general, Nora attempted to learn how to apply the properties of the logarithmic function to solve exponential and logarithmic equations.

<u>Beliefs</u>

During the instructional phase, Nora's efforts to understand the logarithmic function focused on preparing for the test. Initially, as she confessed during Interview 9, she found the function both exciting and scary, but she quickly acquired written notations and learned when and how to apply them. Her capability with these representations made her feel confident. She believed the logarithmic function was easy.

1. I know how to do the problems associated with the logarithmic function, so I understand it.
2. The logarithmic function is related to the exponential function.
3. The properties of the logarithmic function are adapted from the exponential.
4. Use your calculator to compute logarithms.
5. It is important to learn the written representations involved in translating between exponential and logarithmic forms and in applying the properties of logarithms.

Nora's Understanding of the Logarithmic Function: Postinstructional Phase

Evidence of Understanding

Conception. During the preinstructional phase of the study, Nora viewed her mathematics grades as a source of confidence and pride, and as a measure of her knowledge of mathematics. During the postinstructional phase, Nora's view of her grades shifted. During Interview 4, while Nora was doing skills assessment, she commented, "If I can make an A on the test, I should be able to do well on this." She made similar claims later in the same interview. Two weeks later, during Interview 9, Nora's view of what her grade in college algebra meant had changed. She felt she had not learned enough in college algebra and that "the A wasn't completely earned." She even remarked, "Since coming back from the semester and me not understanding it [the logarithmic function], I feel like I should go back and do college algebra again." Nora was not able to finish any of the tasks I gave her during the postinstructional phase and was dismayed that she "didn't know how to use" the logarithmic function to solve them. Since Nora could not do the problems, she felt she did not understand.

Nora generally viewed the logarithmic function as related to the exponential function. She was able to remember the properties of the logarithmic function by recalling the properties of the exponential function. According to Nora, she could then use the properties to "do" problems involving the logarithmic function.

Nora’s conception of the logarithmic function during this phase was that she no longer understood it because she could not do the problems in the interview tasks. She also related the function to the exponential because of the similarities in their properties.

Representation. During the postinstructional phase of the study, Nora used all four representational modes to illustrate or investigate various characteristics of the logarithmic function. Although, Nora's primary mode of representation was written, how she used oral and pictorial representations is also of interest here.

The oral representations that Nora used were names, notations, and maxims. Like the other participants, Nora had a procedure she called *convert*. She described the procedure during Interview 8 as a two-step process:

> N: To me all it was was changing it [logarithms] to exponential form. So that is all I did was change it to exponential form. And if I didn’t know the power, then I put it back in log form and did a change of base formula and got the exponential that it needed to be powered by.

She used the procedure to solve most of the problems involving the logarithmic function. First, she changed the logarithmic expression (e.g., $\log_b a = c$) to an exponential one ($b^c =$ a) to see if *a* was a power of *b*. If it was not, second, she converted the expression back to logarithmic form and used the change of base formula and her calculator to compute the answer. The terminology Nora associated with this procedure was correct, but her use of terminology in general was flawed. For example she used the term *similarity* instead of *symmetry* to refer to the relationships between the graphs of the exponential and logarithmic functions. Nora’s confidence in and familiarity with her convert procedure made it easy for her to talk about it.

Although Nora made errors in her reading of notation during the skills assessment, — she wrote $\log_3 .8$, but said “log base three to the point eight”— during subsequent interviews she read notation correctly. She also used maxims to remember the “rules” of the logarithmic function. For example, she recalled that “when you multiply, you add.” Her first translation of this maxim to written notation was log *a* + log *b, ab*, but she noted that her representation did not "look right" and changed it to log *a* + log *b* = log *ba*.

As in the instructional phase of the study, Nora considered written representations to be most important. When I pointed this out to her during Interview 9, she was not surprised.

> S: Can you shed any light on my whole idea that you think of the written representation first and then you try to work off of that?
>
> N: I think you've got it right there, because I did that in high school and I do it now. I think for me it benefited me, because I've always done pretty well.

Nora had learned how to write notations and do problems, and she had been very successful. She attempted to use the same strategy during this phase of the study. Regardless of the mode of representation in which the logarithmic function was presented, Nora attempted to use written representations to make sense of them.

The map that Nora drew during Interview 4 (Figure 26) is indicative of Nora's use of written representations.

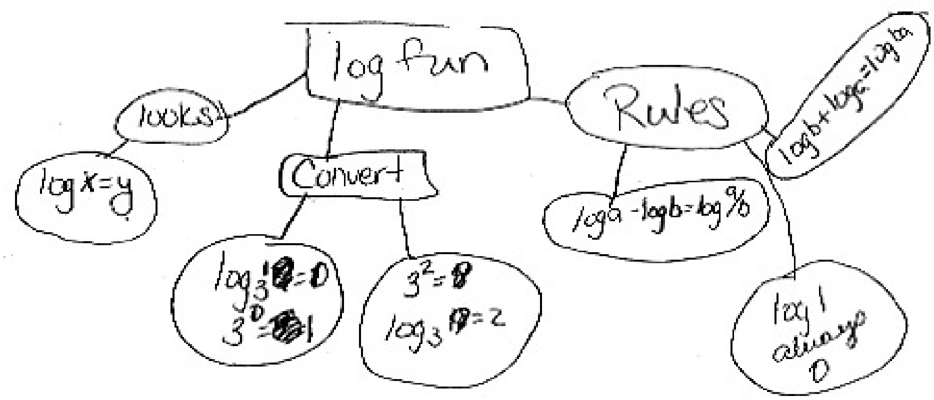


Figure 26. Nora's map of the logarithmic function constructed during Interview 4.

Included on the map are names and notations. In particular, the category *looks* refers back to Nora's impression of how the function should look. Also included are the terms *convert* and *rules*. Nora saw these two names as referring to different objects.

Convert was the name of a computational procedure. *Rules* was the name of a group of properties of the logarithmic function, that were useful in solving problems that convert could not.

Although the notation Nora used on her map was standard, she also generated her own notations and modified given notation. During Interview 5, Nora represented the relationship between the signs and the numbers on the number line as *number on the line* = $2^{\textit{sign number}}$. She did not see the data in the task as indicative of logarithmic relationship. Instead she wrote the relationship in exponential form and transformed it into logarithmic form to find the sign number corresponding to the number 3. During Interview 6, she used function notation to represent her procedure. Nora wrote $f(\ ) \times 2$ and then *f*(2û2) = one greater. She explained, " multiply it [domain value] by two. It [the range value] is always going to be one greater." This use of notation illustrates Nora's confusion with the function notation. She often used the notation incorrectly. To explain the graph of the logarithmic function passed through (0,1), she wrote *f*(log) (0, 1). When she was asked to construct a table of values for the function $f(x) = \log_3 x$, Nora asked me, "What is *y* going to be?" I could not associate Nora's use of notation to any standard meaning.

The pictorial representations that Nora used were graphs I asked her to generate. On the skills assessment Problem 5 asked for the graph $f(x) = \log_2 x$. Nora used her calculator and graphed $f(x) = \dfrac{\log 2}{\log x}$. She hesitated for a moment saying "one comma zero should have been [on it]," but quickly moved on to the next problem. When I asked Nora to compare Problem 5 and 6 (develop a table of values for $f(x) = \log_3 x$) she noted that both should pass through the point (1,0). It also became clear the graph of the logarithmic function in Nora's mind was both the logarithmic function and its inverse. She drew a graph of the function during Interview 8 (see Figure 27). When she drew this graph Nora

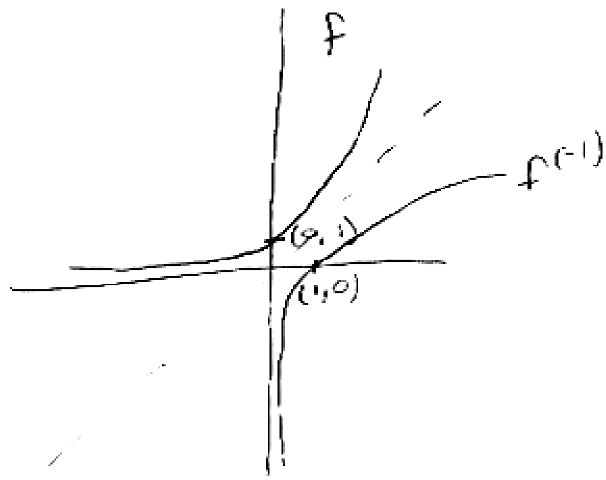


Figure 27. Nora's graph of the logarithmic function from Interview 8

When she drew this graph Nora also explained a bit about it:

> N: I don't remember too much about the graphs. I know it is going to look something [gets paper] the log graph. Just the basic log graph is going to look something like… It is going to have an asymptote at zero [$y = 0$, draws while talking]. And that is going through zero comma one and it is going to go through zero comma one [draws the exponential function]. And *f* inverse is going to go through one comma zero and it is going to have an asymptote like that [referring to the *y*-axis] and then it will have symmetry [draws $y = x$ dotted line].
>
> S: So which one of those is the log graph? Or is it all the log graph?
>
> N: It is all log, but this is just the *f* and this is the *f* inverse [labels the exponential function *f* and the logarithmic function $f^{(-1)}$].

Nora's drawing and her comments resolved much of my confusion about her view of the graph of the logarithmic function. Teacher 3 drew this picture on both days the logarithmic function was discussed in class. In addition, on the test, both functions were drawn on the same axes. Nora failed to distinguish between the graphical representations of these two functions. She saw both curves as part of the logarithmic function.

The representations Nora used during this phase of the study were oral, written, and pictorial. She used names and notations in oral and written form. She used maxims to help her remember the properties of the logarithmic functions. Her graph of the logarithmic function included the exponential function as a component.

Connection. The connections Nora used during the postinstructional phase of the study were related to her convert procedure and properties of the logarithmic function. Connections between oral and written representation of the convert procedure occurred in that order. For example, during Interview 5 after many trials, Nora mapped $\sqrt{2}$ to $\frac{1}{2}$. Following this breakthrough, I asked her to find the sign that corresponded to 3.

> S: Okay, now what is going to go with three?
>
> N: Three would be where the square root of two would be [in the written representation]. Where is my scratch [paper]?
>
> S: There it is.
>
> N: Three would be where the square root of two is going to be, so it would be two to some power, right? Two to some power would equal three (Writes $3 = 2^x$ on scratch paper). Now I could convert that to log form.
>
> S: Maybe so.
>
> N: Yeah. Because you could do the log base two to the third equals *x* (writes $\log_2 3 = x$). And then I could do log two divided by log three. (Nora is still using the wrong change of base formula) I get point six three oh nine (.6309).

Nora used her map for the square root of two as a template for this problem. Her oral representations, for example "Two to some power would equal three," preceded the written ones, $3 = 2^x$. Nora’s convert procedure in oral form was connected to written notations.

Nora also connected her oral maxim when you add you multiply to the written notation log *a* + log *b* = log *ab*.

The primary connections Nora formed among representations during the postinstructional phase were between oral and written. She expressed a relationship orally and then wrote it on paper. This is distinctly different from how Nora operated during the instructional phase of the study, where the onus was on the listener to establish mathematical meanings for her utterances. During the postinstructional phase, she attempted to coordinate her oral and written representations. Since I was not willing to tell her if she was correct, she attempted to draw conclusions from my reactions to her remarks.

Application. Nora applied her *convert* procedure to logarithmic and exponential equations. Her application of the procedure included the change of base formula and the calculator. Nora explained how she used the change of base formula during Interview 4 when asked her to explain why she felt her simplification problems on the skills assessment were incorrect:

> I distinctly remember, if you can't find the answer just do log base. Do it in the calculator. Hit log, then you do whatever the base is, hit your parentheses, divide log again. And then do the number that is beside [referring to the argument] whatever the number is called, and then hit the parentheses again and then you can solve it that way. That's where I remember decimals. That's why I didn't feel like that one was right [referring to Problem 3b. $\log_3 4 - \log_3 5$]. I didn't remember too many decimals on those [simplifying problems].

Nora explained that when she could not evaluate $\log_3 4$, she used the change of base formula to rewrite the expression and her calculator to evaluate it. For Nora the convert procedure, change of base formula, and the calculator were all associated. During Interview 7, when she did not have her calculator she had a difficult time filling in the table. Half-way through the interview, after several solution attempts, she said: "I need that log button. I keep thinking I can do the change of base formula." One of the key components in her *convert* procedure was missing, so Nora could not fill in the table.

Nora was not aware she linked her convert procedure, the change of base formula, and the calculator. In general, she viewed using the calculator as cheating. When I asked her to graph a logarithmic function during the skills assessment, she said, "Let's cheat." When I asked her if she could match a function with its graph, she said "I would cheat," and graph each function using her calculator. When Nora found a solution path using her calculator and successive approximation to fill in the table during Interview 7, she called her method cheating. Nora claimed that she was "addicted" to the graphing calculator, but in my view she had become adept at using it. She knew how to convert logarithmic functions to exponential form, and how to use the calculator to find values. She applied her knowledge of both the logarithmic function and the calculator to calculate answers.

<u>Beliefs</u>

The central elements of Nora's understanding of the logarithmic function during this phase of the study were: her convert procedure, properties of the logarithmic function, and the graph of the function. Her convert procedure involved transformations between logarithmic and exponential form, the change of base formula, and her calculator. The term convert was connected to both oral and written representations. She remembered several properties of the logarithmic function because she associated them with laws of exponents, but in practice Nora only applied them when they were presented in forms she had seen before. For example she knew "if you subtract you divide," but she could only apply the property when she saw $\log_3 4 - \log_3 5$. She was unable to apply the properties to the table completion problems in Interview 7. Finally, Nora's idea of the graph of the logarithmic function was what I would call *iconic.* She saw it as a picture with components that she remembered, but combined the exponential and the logarithmic functions saying they were both pieces of the graph of the logarithmic function.

1. Getting an A in college algebra does not mean that I understand the logarithmic function, since I cannot do all the problems I am being asked to do.

2. To solve logarithmic equations convert to exponential form or use the change of base formula and the calculator.
3. The graph of the logarithmic function passes through the points (0, 1) and (1, 0), has asymptotes on the axes, and has a line of symmetry at $y = x$.
4. To solve problems with logarithms you need to know three rules that are some how generated from the laws of exponents: “if you subtract you divide,” “if you multiply you add,” and “log of one is always zero.”
5. The calculator is a necessary tool for evaluating logarithms.

Changes in Understanding

Nora’s understanding was influenced by her attempts to make the highest grade in her mathematics class. During the preinstructional phase, Nora remembered very little about the logarithmic function from high school. The strategies she used to do the problems on the skills assessment were based on notations she had seen before. She treated the $\log_3$ as if it was a variable, applying the distributive property to the sum of logarithms and the associative property to the product of a constant and a logarithm. In addition, Nora pointed out the log key on her calculator and commented that if she knew how to use it she could solve the problems. Nora mimicked the written notation from the skills assessment and used it to write answers to the simplification problems. Nora wanted to make sense of the marks on the page and made logical attempts at doing so. Noticing that the calculator could help and trying to acquire notation were Nora’s attempts.

During the instructional phase, Nora was told how to do particular types of problems and what properties were important and useful in solving logarithmic and exponential equations. In her quest to do well on the exam, Nora paid close attention to what was said in class and faithfully did her homework. She quickly learned how to do the problems that Teacher 3 showed in class using a transformation of the logarithmic function to exponential form, the change of base formula, and her calculator. Nora gained what Skemp (1976) called instrumental understanding of the logarithmic function.

She paid close attention to the written representations that were presented in class, but virtually ignored any other form of representation. On the test, she was successful. She could do the problems with ease. Nora felt she understood the logarithmic function.

During the postinstructional phase, Nora was surprised that she had difficulty applying logarithmic function and its properties to the tasks I gave her. She believed making an A in the class meant she knew virtually everything about the logarithmic function. By Interview 9, Nora's view of her own understanding of the logarithmic function had changed. She felt that she did not understand the logarithmic function. She pointed out that she knew each of the tasks was somehow related to the function, but "I just didn't know how to use them." Nora was able to apply the convert procedure and could state and give examples of some of the properties of the logarithmic function. She was unable to explain how the properties were derived and could not apply them to situations other than those she had seen in class. Nora also had trouble with graphing.

Nora's understanding of the logarithmic function changed in each of the three phases of the study. She acquired skill in solving standard problems using written notation during the instructional phase. During the postinstructional phase, Nora found the problems difficult, but did not give up easily. She eventually constructed solution procedures for the tasks, in part due to her persistence. Her reliance on written representation generally hampered her problem solving. She never used other representations to gain insight into the problems. Nora's awareness that making an A in a course is not a measure of her mathematical knowledge was a marked change from her preinstructional and instructional views. Still Nora's belief that understanding a mathematical concept means one can do problems associated with it did not change during the course of the study.

Ways of Knowing

Nora exhibited powerful ways of knowing during the postinstructional phase of the study. First, she knew how to transform expressions in exponential and logarithmic forms. When she saw a logarithmic expression or an exponential equation, she was able to simplify, evaluate, or solve it. No other student in the study displayed this level of procedural proficiency. Connections between her procedures and why they work could easily be developed. When I asked Nora to show me how the properties of exponential and logarithmic functions were related, it bothered her that her example (sum of the logarithms is the logarithm of the product because $3^2 + 3^3 = 3^6$) was incorrect. She expressed a general dissatisfaction with her knowledge. She wanted to know how and why the logarithmic function worked. Nora's curiosity and persistence could be used to encourage the growth of her understanding of the logarithmic function.

Second, Nora realized during Interview 7 that her table completion procedure was inefficient. She looked for a simpler method, but eventually used successive approximation as Jamie and Rachel had. During the interview, I encouraged Nora to look for a relationship between $\log_3 2$, $\log_3 4$, $\log_3 8$. Nora did not appear to connect the three expressions, she did notice that $\log_3 4$ was twice $\log_3 2$, and that $\log_3 8$ was three times $\log_3 2$, she never connected this observation to a property of the logarithmic function or to powers of two. Nora had difficulty seeing number patterns due to a very limited view of number. She preferred to use decimals not fractions. She insisted on converting fractions such as $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$ to decimals, which made the commonalties between the numbers during Interviews 5 and 6 more difficult to see. After much prompting on my part, Nora did identify the number line numbers in interview 5 as powers of 2. If Nora learned more about numbers and their various representations, she would understand both exponential and logarithmic function better.

Third, Nora was aware of inconsistencies in her answers. During Interview 5, Nora initially used linear interpolation to find the sign above the number $\sqrt{2}$. Later in the interview, when she conjectured the product of any two numbers on the number line should correspond to the sum of their signs, I questioned Nora about $\sqrt{2}$. I noted if $\sqrt{2}$ did correspond to 0.4 as she had suggested, then 2 should correspond to 0.8. This counterexample eventually resulted in Nora's awareness of a relationship between the number line numbers and the sign numbers that she used to find the sign number above any number line number.

Nora's awareness of inconsistencies and desire to eliminate them, could be used to develop her understanding of the graph of the logarithmic function. During the skills assessment, Nora had no difficulty constructing a table of values from a graph and a function. If Nora were to construct such a table from her graph of the logarithmic function and develop a table of values for the logarithmic function, she would be interested in resolving the inconsistencies she would find.

Nora's ways of knowing provide opportunities for the growth of understanding. She had developed instrumental understanding of some aspects of the logarithmic function, in particular, how to solve and simplify problems. She was curious and when she was aware of inconsistencies, she attempted to eliminate them. These ways of knowing could be used as the basis for growth of understanding of the logarithmic function.

Demetrius

Getting to Know Demetrius

In the fall of 2000, Demetrius transferred to RC as a 20-year-old sophomore. His first year he attended a two-year college in Alabama to play basketball. Demetrius loved the sport and often used basketball analogies to explain his interpretation of the world. For example, during Interview 3a Demetrius commented that Teacher 3 explained how to do things the "hard way" using notations and formulas like the book, "I know it [math] is suppose to be hard. It is suppose to be art or perfection, you know, but he [Teacher 3] can just state it. Like. it is simple really. But he wants to make it all confusing." This comment mystified me, so during Interview 3b I asked him to explain it.

> I work hard in the game and I try to get as good as I can. I try to perfect that sport. I try to be the best in that sport because I love that. For the love of basketball…. Some people love math like Teacher 3. I feels he loves math or he wouldn't want to go in that field and teach it. He loves

> math and tries to be the best he can in it. He has a big smile, a grin from ear to ear, when he is up there teaching because he loves it. Because it works out and he can see it and he understands it. You want to make people feel about math like you feel. Like, if I like math, I want to make people see what I see and love math like I love math. So for him to be a math teacher, you know, he is going to do the best he can and he is going to perfect it. He just loves it. He doesn't want to see it wrong. Just the right way. The way you should write it. That's how he wants it. He wants it done that way because it's just, that's the way it is.

Demetrius believed Teacher 3's love of mathematics was like his love of basketball. According to Demetrius, Teacher 3 wanted to see mathematics written perfectly like Demetrius wanted to see basketball played. While at RC, Demetrius practiced with the team but was redshirted to preserve his years of eligibility. He planned to transfer to a four-year college in fall 2001 to play basketball and pursue a bachelor's degree.

Demetrius wanted to become a special education teacher. He started college with the goal of becoming a physical therapist. As the courses became more difficult and Demetrius realized how long it was going to take to get his degree, he decided to change his major. Demetrius explained he selected special education because he wanted to help people. He volunteered for the study because he needed the money and saw it as an opportunity to help me. Demetrius planned to earn a master's degree before he began teaching. He explained his educational goals during Interview 2:

> Maybe, if I am lucky enough,... if I'm fortunate enough to get my 6-year specialist or whatever. You know, I feel like my brother has set the path for me. He is a doctor in education. So, maybe, if I still have the will and drive by then, and don't get satisfied with the income, I will go on and become a doctor, get my Ph.D., become a doctor in education.

Each of Demetrius's siblings had attended college, a tradition he hoped to carry on.

Demetrius's family was an extremely important part of his life. He and his older sister traveled 85 miles home each weekend. Demetrius was the youngest of six children. His oldest brother was the principal of a middle school and was like a father to him. Demetrius's other brother was a probation officer, two of his sisters were teachers, and one was a registered nurse. The sibling closest to him in age was 29. The others were "grown folks," whom Demetrius admired and respected because of their ages and accomplishments. Demetrius credited his mother with his family's academic success. She had worked in a factory all her life to support her children. When each of her children was old enough to get a summer job, she found some sort of assembly line work for them. This experience was pivotal for Demetrius. After a long summer of assembly line work, when the foreman asked Demetrius to continue working in the fall, he replied, "I'm finished. I'm getting my education."

Demetrius lived on campus and did not work a traditional job. At our first meeting he gave me a pamphlet for prepaid legal services. For a certain monthly fee the service provided legal counsel if it was ever needed. Demetrius was an associate for the company and sold the policy in his spare time.

<u>Demetrius as a Mathematics Student</u>

Demetrius described himself as a caring and committed mathematics student.

> I don't have really strong math skill, … but I describe myself as a person that is eager and willing to learn it. Even though I might not use it in life, just to accomplish it. Because I know it is a challenge to me. Myself...so I feel like, you know, achieve, master learning it. I feel like, it will be,...that it is a self-achievement for me. You know what I'm saying?…Because that is just like anybody else. If there is something they can't do that well or something they want to know more about, even though they can't do it that well, they are still going to work hard to achieve it, even if they don't use it in life…. And I know it is something I've got to have anyway to go on. So, I would describe myself, as a person that is eager and willing to

> learn and to do the best I can in it. Even if that's a C, you know, that's the best I can do. Or if it's a B, the best I can do.

Demetrius saw mathematics as a challenge, but useless for his career. It was also an obstacle he had to overcome to achieve his goal. He had never done well in mathematics and had been frustrated in his high school mathematics classes:

> S: Think of a time when you felt despair or frustration about your learning activities and tell me about that.
>
> D: Okay, once again going back to high school. Well, I felt like this every day in math class actually. Because, like I said, I wasn't a math student. I wasn't really math oriented, smart. I wasn't that smart in math. Just being in college prep. I'd say algebra in 9th grade. Just being in that class, feeling so confused. And then frustrated and confused, to where I didn't want to raise my hand and ask questions because they might have been stupid questions. Or they might have been a question and she would look at me like, you know, because there were a lot of people that did understand and a lot of people that were smart. So I didn't really want to ask questions, and I felt like in that class, I'm by myself, frustrated, and … that I wasn't that smart, and I shouldn't be in there…. I used to get most of my help, like I said tutoring, or even after class, because I didn't really ask any questions in class. That was probably part of my problem. And I came out with a C in it, but I probably would have come out with a B or maybe even. Well, I won't say an A or maybe a B, if I would have asked questions.

Demetrius felt he did not understand mathematics and was not "smart in math." During class he would listen, but still be confused. His feelings about his own performance and how he behaved in class had not changed since high school. Each 75-minute lecture in college algebra was a test of his attentive powers. Each day Demetrius came to class determined to listen and be persistent, during class as the minutes ticked away and he became perplexed. He listened less and daydreamed more, “after a while you listen and you lose them or they lose you…and you just go to daydreaming after that.” Demetrius daydreamed to escape his frustration with not understanding. He mentioned the importance of listening in 8 of the interviews. In Demetrius's view, a good student is one who listens and tries.

Demetrius participated in class by asking questions and doing the seatwork that Teacher 3 suggested. Although he did not participate nearly as much as Nora, Demetrius asked questions when he was confused or when he wanted to know if his answers were correct. Demetrius sat near the back of the room in a column of desks near the door. All four of the African American students in the class sat in these first two columns. Demetrius often talked to the young woman in front of him, asking her questions about mathematics problems and daily events before and after class. Demetrius’s notes were extremely neat and he was judicious in what he wrote down as long as he was not feeling confused. When he gave up and started daydreaming, he was more indiscriminate about what he recorded and spent time doodling in his notebook.

Demetrius’s goal for the class was to pass. During Interview 2, he explained:

> The most important thing to me is that grade. You know. Okay...I’m not going to say that. The most important thing should be understanding, but me personally, the most important thing is that grade. Because, you know, I’ve got to have that grade to move on. The most important thing should be…understanding it. Because that is what education is all about, you know, learning. But, then again that grade is the most important thing to me because it is how you move on.

To achieve his goals he had to pass college algebra. Demetrius explained during a later interview, only mathematics majors needed to understand the material. He would not need to know mathematics for his major, so passing the course was sufficient.

Using a combination of techniques, including regularly attending the AAC, getting my help on a regression project, taking notes, and studying Demetrius was able to pass college algebra with a C. He had hoped for a B, the grade he went into the final exam with, but the C was acceptable to him. College algebra was Demetrius's terminal mathematics course.

Understanding Mathematical Concepts

Demetrius had idealistic and realistic views of understanding. He explained his idealistic view during Interview 2, when I asked him to define understanding:

> The term understanding. Not memorizing. Not just knowing how to do it when you see it, but understanding what you are doing and why you are doing it. But you can memorize some…. But you've got to understand exactly what you are doing and why you are doing it, because there might come a case when it is a different situation from that. You know what I'm saying. It is a different situation in that case and you've got to know exactly what you are doing to maneuver around the differences in those two cases. So, you've got to fully understand what you are doing to a problem when you are doing it. Instead of just memorizing how the teacher showed you to do it. Because you can't do every problem the same exact way…I think understanding is just...understanding thoroughly and about what exactly are you doing to the problem and just...well, it is just the opposite of memorizing.

Ideally, he felt he should know both how to do problems and why he was doing them a particular way. Realistically, his primary concern was passing the college algebra tests. Knowing why was not something he was going to be tested on as he explained during Interview 9:

> Okay, let me see how to put this. I think I...memorized [the material associated with the logarithmic function] because there were some things that if I understood it completely, I would have known in our interview. There where a couple of things that let you know then that I didn't really fully understand what was going on, but I knew how to do it…I just knew

> what I needed to know to pass the test. You know, the test wasn't about telling why this happens and why this. You know what I'm saying? I just understood what I needed to know and what I needed to do. So I can say, in a sense I understood what I needed to do, but I didn't understand why it was going on.

Demetrius was always very candid and reflective in his responses. He realized understanding meant knowing both how and why, but he also knew sometimes in a mathematics class you have to be practical. Being practical in college algebra meant learning to do enough problems to pass the tests.

Demetrius' Understanding of the Logarithmic Function: Preinstructional Phase

Demetrius remembered seeing the logarithmic function in high school. "I remember it was a certain base or a certain thing you go by every time. Like,...this part right here means this and this part right here means this and this part right here means this." This vague description was similar to Nora's, but includes the term *base*. Demetrius remembered very little about the function prior to attempting the skills assessment.

Evidence of Understanding

During this phase of the study three categories of evidence were significant.

Conception. Demetrius remembered the logarithmic function as "kind of easy." His Algebra II teacher had presented the concept as one that would help students improve their averages. This view did not persist. It took Demetrius approximately 70 minutes to complete the skills assessment. He spent more time on it than the other participants. He worked very carefully, but was disappointed in his performance. While doing the assessment, he became frustrated but wrote an answer for each problem. His answer for the properties of the logarithmic function was: "Whatever you do, it's basically the same concept you do all the time." Following the assessment, he concluded, "I didn't remember anything." Demetrius remembered the logarithmic function being presented as an easy mathematical concept, but he was unable to do the problems I asked him to do.

One element of his conception of the function was that it should be easy, but was not for him.

Demetrius thought of the logarithmic function was as a collection of symbols and problems. On the skills assessment, he defined the logarithm with the notation *3* log, *base,* and the logarithmic function with the notation *2* $\log_2$. His meaning became clear as he completed the mapping task (see Figure 28).

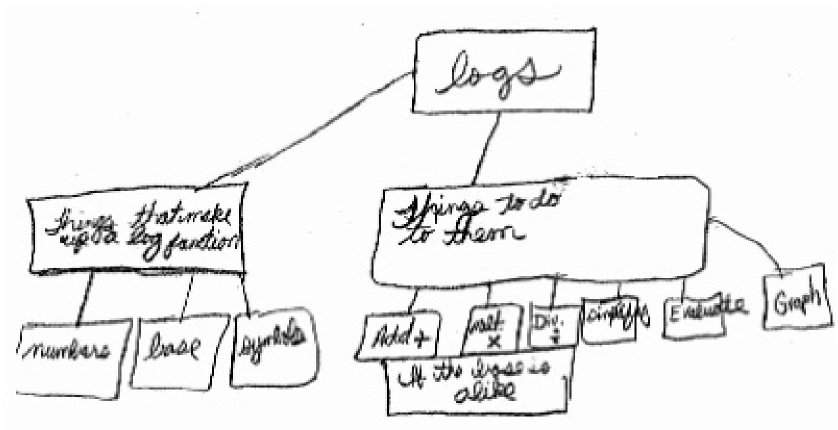


Figure 28. Demetrius's map of the logarithmic function from Interview 1. *Logs* were associated with two categories: one as objects that were used to *make up* a logarithm, and as types of problems to be done. The category, *things to do to them,* is a collection of problem types including some from the skills assessment problems*:* *simplify, evaluate,* and *graph*. The second element of Demetrius's conception of the logarithmic function, was as a collection of symbols and problems.

Representation. During the preinstructional phase, Demetrius used both written and oral representations for the logarithmic function. His written representations were like Nora's. He also used the distributive and associative properties to simplify and evaluate the logarithmic expressions on the assessment. When I asked him what he understood most from the activity, he selected the simplification problems.

> It seemed like it [the simplification problems] was just a regular addition problem. That is something that will stick with you, something that easy. Probably seemed easy, like, because, you know, with regular addition it is

easy anyway. You get problems like that and you are taught how to do them and they are that easy. I guess they stick with you.

Demetrius recalled his teacher had said the concept was easy, so he did what seemed easy. He treated the logarithm as a variable and found $\log_3 4 + \log_3 5 = \log_3 9$.

Demetrius also used the written name *base* on his skills assessment and his map. He included it on each of the maps he constructed. For Demetrius the base was a significant feature of the function, although he admitted during Interview 9 he did not know why it was important.

Demetrius's oral representations included names and maxims. Demetrius recalled the term base and even knew where it belonged in the notation. He stated and applied the maxim, "if the bases are the same you can add logarithms." Unfortunately the maxim provides very little detail and Demetrius's application produced incorrect answers.

Connection. In Demetrius' written work, there were connections between names and notation and among notations. In particular, early in the skills assessment, Demetrius indicated he knew *base* was associated with the subscript in a logarithmic expression. He also connected the notation $\log_3 4 + \log_3 5$ to the notation $\log_3 9$.

Demetrius associated his oral maxim with written notation. He generalized his action of adding two logarithms into an easy to remember phrase. "If they have like bases, the logarithms can be added." His actions and his map suggest that Demetrius had and used other maxims about multiplying and dividing logarithms that he did not state.

Beliefs

Demetrius's beliefs were primarily derived from his algebra teacher's remarks about the function and the skills assessment. Since he felt his teacher was "very good," he trusted her judgement about the logarithmic function. He used his belief that the logarithmic function was easy and maxims about like bases, to simplify logarithmic expressions. He also recalled that the logarithmic function had a base and knew its position in the notation. The only other student to remember this terminology was Rachel, and she had taken the course just two semesters earlier. For Demetrius, the skills assessment provided information about the types of problems that could be solved with the logarithmic function and the notation he should use when writing the function.

1. The logarithmic function should be easy because that is what my teacher told me.
2. The base is part of the logarithmic function and is written as a subscript.
3. If two logarithms have the same base you can add, subtract, multiply, or divide them.
4. Knowing how to combine logarithms means that you know how to evaluate, simplify, and graph.
5. The logarithmic function is a collection of symbols that I need to know how to combine.

Demetius's Understanding of the Logarithmic Function: Instructional Phase

Evidence of Understanding.

Conception. Demetrius' conception of the logarithmic function during this phase of the study, was that it was easy and associated with a collection of procedures. He explain his view during Interview 3:

> S: Is there anything else you can tell me about what went on in class today that will help me understand how you are understanding the logarithmic function?
>
> D: It is basically a routine. You are doing the same pattern. Everything you are doing the same thing depending on the situation. And basically you've got to get familiar with what you've got to do. What they ask you to do. Basically you are doing the same pattern. Going from one form to another to work out the problems, to evaluate the problems. And just go by your rules. That is basically it. *Logs* could be simple, you just learn the rules and learn what you do for a situation. There isn't anything real hard about it.

For Demetrius logarithms were *simple.* He learned how to apply the logarithmic function to evaluate expressions. Since he could solve problems, the function was easy.

Demetrius saw the logarithmic function as a collection of problems that were easy to solve if he knew the *rules*.

Representation. During the instructional phase of the study, Demetrius used oral and written representations. His oral representations were names and maxims. Demetrius used names to identify problem types and to refer to properties of the function. He identified two problem types by name: *evaluating* and *going from logarithmic form to exponential form*. He used these names during Interview 3, as he explained why students might find the logarithmic function difficult to learn:

> Maybe because there are so many ways that you can do a problem. You try to apply this way, … when you were suppose to have done it this way, for this problem. Like, as far as those certain steps. Like when you are evaluating. You can go about doing it different ways. Sometimes you like to get mixed up or do the wrong thing. You have to pay attention to what exactly the question is asking for. Because you can even evaluate something,…you know, you might try to do something to it when you were suppose to have left it in that form. For instance, like they said go from logarithmic form to exponential form. If you don't catch that,…you might try to evaluate the exponential form, if you're not listening carefully.

For Demetrius, knowing what procedure to apply to a problem was important. He felt if he did not apply *the* correct procedure, his answer would be wrong.

The second type of oral representations Demetrius used was maxims. When the properties of the logarithmic function were introduced in class he had difficulty remembering them. The notation $\log A + \log B = \log AB$ was confusing for Demetrius, so he adopted maxims presented in class to help him remember the properties. He used the maxims "if the two logs are subtracted, then you divided them" and "when you add them (logarithms) you multiply." Using these maxims, Demetrius was able to evaluate expressions in logarithmic form, but could not solve logarithmic equations such as

$\log_2 x + \log_2(x - 2) = 3$ from his test.

The primary sources of Demetrius's written representations of the logarithmic function, were his map, class notes, and homework. He used standard notation for the logarithmic function in his notes and on his homework. His map included names, but no notation (see Figure 29).

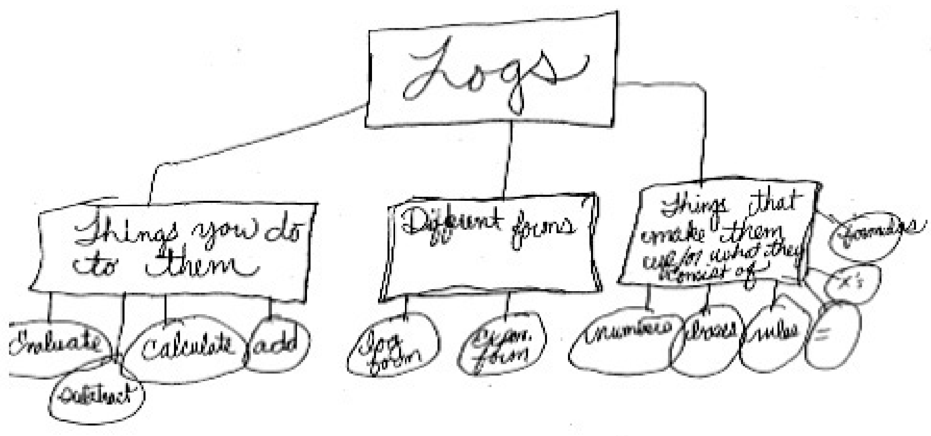


Figure 29. Demetrius's map of the logarithmic function instructional phase.

This map is similar to the one Demetrius constructed during Interview 1. He still depicted the logarithmic function as a collection of problems (*thing you do to them*) and symbols (*things that make them up*). Names of problems were those commonly used in class, *calculate* and *evaluate*. Names of symbols were *base* and *numbers*, as they had been during the preinstructional phase, but he also included the names of collections of symbols: *rules* and *formulas*. Like Nora, Demetrius identified the logarithmic and exponential forms on his map. *Log form* and *expon. form* refer to the translation of an expression from one form to another.

Connection. Demetrius connected his oral representations. In particular, he connected the term *rules* with his maxims. He talked about applying the rules during Interview 3 as he explained how to solve the logarithmic equation ($\log_2 x + \log_2(x - 2) = 3$) he did incorrectly: "Our rules apply. When you add them, then you multiply. They have the same bases." Demetrius associated the properties he called rules with the maxims. These connections helped Demetrius do problems.

Application. Demetrius applied the properties of the logarithmic function to evaluate logarithmic expressions. He used maxims to solve problems with addition and subtraction of logarithms and was able to simplify expressions such as $\log_5 625$ by expressing 625 as an integer power of five and in his words the answer "equals the exponent." Hence, if Demetrius saw the symbol log, he knew there was a collection of problems to solve and procedures to solve them. The challenge was matching the two.

Theories

The primary focus of Demetrius's efforts during the instructional phase, was development of procedures and learning when to apply them to the correct problems. His beliefs reflect this focus.

1. Logarithmic functions are simple, if you just know what rule to apply to a problem.
2. Two types of problems one needs to know how to do are evaluating and how to go from one form to another.
3. Two forms of the logarithmic function are exponential and logarithmic.
4. There are many rules that are used to solve the logarithmic function, but they can be explained most simply with maxims.
5. Two maxims that are especially useful are "when you add you multiply" and "when you subtract you divide."

Demetrius's Understanding of the Logarithmic Function: Postinstructional Phase

Evidence of Understanding

Conception. Demetrius's concept of the logarithmic function during the postinstructional phase of the study, was that he understood the function, despite his inability to do problems with it. Being able to remember doing problems from class, but not being able to do similar problems on the skills assessment, frustrated him. He explained during Interview 4:

S: Can you pick out a problem and tell me about not understanding?

D: I can pick out several.

S: Well, just pick out one.

> D: For the most part, over all, I think it was mind boggling because I didn't remember hardly anything. And I had to stretch my brain and I was, "dog, I should have known some of these." So I felt, kind of like, I didn't learn anything in logs. Some of these graphing problems. Graph the function *f* of *x* equal log base two *x* on the axis provided. I had no idea. I didn't even attempt that one. I didn't even know where to start. I don't even really know if I even knew it some weeks ago, when we were doing the chapter. I had no idea of where to even think of where to start graphing.

Demetrius was disappointed in his performance and viewed his knowledge of the logarithmic function as lacking. During Interview 7, with prompting from me, he eventually transformed a logarithmic expression into exponential form and used the calculator to solve the exponential equation, but commented, "I still really don't understand it. I can memorize it, but that doesn't mean I understand it." He knew how to do the transformation, but could not explain why it worked. Despite making these remarks, when I asked Demetrius during Interview 9 if he understood the logarithmic function he replied, "I would say yeah, I understand them." His rationale was that he knew enough to pass the test:

> For somebody that is probably a math major, they need to know thoroughly why something is done. I just need to know what I needed to do to pass, but they needed to know completely. They need to know why you need to do this to apply to in life, to apply to life, if that is going to be their occupation.

Mathematics majors need to know why, Demetrius explained, he did not need to know mathematics for his career. Although I saw his explanations in the postinstructional phase as contradicting his original definition of understanding, knowing how and why, Demetrius disagreed. Hence, for him, being able to do problems during class meant that he understood the function well enough for his major. His conception of the logarithmic function was as a small collection of problems that he could do without knowing why.

Representation. Demetrius' primary modes of representation during the postinstructional phase were oral and written. In speaking about the logarithmic function, Demetrius used names and maxims. The names he used were *logarithmic* and *exponential* function or form and *base*. These terms were key to Demetrius' view of the logarithmic function because both referred to doing problems. He could transform an expression in logarithmic form to one in exponential form to evaluate it. In addition to names, Demetrius often used oral representations of the properties of the logarithmic function to help him simplify problems. Three of these maxims included the term *base*. “Logarithms can be added and subtracted if they had the same base,” “the base cannot be negative,” and the base of the logarithm on the calculator is ten.” Although the maxims were used in reference to the properties of the logarithmic function, during the postinstructional phase, they were not helpful. Like the other participants, on the skills exam Demetrius used the distributive property to find $\log_3 4 + \log_3 5 = \log_3 9$. He had more success applying “the log is the exponent.” This maxim was introduced during class. When Demetrius heard it, he claimed that it helped him remember. During Interview 8, he illustrated and explained how he used the maxim. After writing the expression $\log_{10} x = y$, Demetrius explained:

> Okay, that is the logarithmic form. In the exponential form, you need to remember that, you need to remember always that, the log equals the exponent. That will help you whenever you are doing the exponential form. You need to know the log equals the exponent.

He was able to use this maxim to successfully transform the logarithmic form to the exponential one. Demetrius used names and maxims as keys to solving problems. He was not always successful with his maxims, but for him, they were more useful than notation.

Demetrius used three types of written representations: names, notations, and descriptions. There was no significant change in the written names he used during the postinstructional phase. The categories on his maps remained the same: *things they* [logs] *are made of, 2 different kinds of logs, things you do to them,* as was the term *base*. During Interview 9, he explained the category *things you do to them.*

> Things you do to them, because everybody knows what you can do to a problem. As far as add them, subtract them, divide them, multiply them, or evaluate them. All of them are basically the same.

According to Demetrius, add, subtract, multiply, divide, and evaluate were how everybody did mathematics problems. The name of the category referred to mathematics problems in general. Like the names he wrote during the instructional phase, the categories and names *exponential* and *logarithmic forms* referred to problems and procedures he used to solve them.

The second written representation Demetrius used was notation. He used it only to illustrate solutions to problems on the skills assessment, during the table completion task in Interview 7, and when he was explaining the logarithmic function to me during Interview 8. Although the form of his notation was correct, his application of his maxims in the form of notation caused errors. For example, during Interview 8 he illustrated his sum of logarithms maxim using the example log 7 + log 3 = log 10.

Demetrius did not feel comfortable using notation. He did not use it on any of his maps. Instead of using notation on the bulletin board tasks during Interviews 5 and 6, he used descriptions. For example, during Interview 5 he illustrated the relationship between the sign numbers and the number line numbers with the following written description:

> Each time you multiply the # under the sign by 2, go up and add one more # than the # on the previous sign to that sign above that #. Moving on the # line back to the left, divide by 2 and subtract one from the previous sign and place the # above.

Demetrius described how he found the sign numbers using coordinated actions. Multiplication by 2 was associated with adding 1, whereas division by 2 was associated with subtracting 1. The descriptions were easier for him to use than notation.

In summary, Demetrius oral and written representations to do problems. He used maxims to remind himself of the properties of the logarithmic function and notation to illustrate his solutions.

Connection. Demetrius made connections between his oral and written representations and among his written representations. He used oral maxims as guides for doing problems in written form. He used oral names in a similar way. For example, as he attempted the first table completion task during Interview 7 he commented, "I was thinking about logarithmic and exponential form." He knew the table had something to do with the logarithmic function and his principle tool was transformation between the logarithmic and exponential forms. Following his comment, he wrote $\log_{10} 1$, transformed it into exponential form, and concluded $\log_{10} 1 = 0$. Demetrius used oral names and maxims prior writing a representation for the logarithmic function.

Demetrius made connections between written representations when he was solving problems. He transformed expressions in logarithmic form to exponential form and simplified expressions such as log 7 + log 3, using the properties of the logarithmic function. In addition, Demetrius associated the term logarithm with the exponential and logarithmic form and the base.

Application. Demetrius applied the logarithmic function and its properties in two ways during the postinstructional phase. Much like the other participants, he saw the calculator as a source of information about the logarithmic function. During Interview 7, he noted if he had a calculator with a log key on it, it would be easy to fill in the table. However, not having the log key did not bother him as much as it did the other participants. He relied on it less in all of the interviews than the other participants. For example, he found the product of 16 and 2 during Interview 5 by using the standard paper and pencil algorithm. During Interview 8, he explained how to use the calculator to compute logarithms:

> D: And uhm...(pause) You have a log key on the TI-83 and uhm...which is right here. You have a natural log key, which is basically the same thing.
>
> S: Oh, yeah.
>
> D: Except the natural log key, uhm...I don't work with it...it is the same thing (laughs).
>
> S: They are the same?
>
> D: Yeah.
>
> S: So I can push either one. It doesn't matter?
>
> D: Doesn't matter.
>
> S: Okay, so how do I use that key?
>
> D: So say you are looking for the log of ten you just press...you just press the log of the number... Whatever number you are looking for the log of. I will just say log of ten, close parenthesis [types in log 10] and enter.

Demetrius could compute base ten logarithms with his calculator and believed the log and ln keys produced the same result. This idea was quickly dispelled when I asked him to compute log10 and ln10. He was not exactly sure why the approximations given by the calculator were not the same. This notion that the two buttons produced the same approximations, was the result of a classroom exchange that took place during the instructional phase. On November 21, Teacher 3 commented, it did not matter if one used the log or the ln in the change of base formula, the same answer would result. Demetrius asked Teacher 3, "What is the difference if you go [use] natural log?" Teacher 3 replied there was no difference. Hence, Demetrius assumed pushing either button on his calculator, would result in the same approximation.

Despite Demetrius's awareness that the calculator could approximate logarithms, he did not use it to do so on the skills assessment as other participants had. In fact, when Demetrius mentioned using a calculator as he was introducing the logarithmic function to me during Interview 8, I asked him if I would need a calculator for this topic. He replied, "not really." He saw the calculator as useful, but not necessary. He used the calculator at my request, to verify the sums and differences of logarithms. In general, he relied on his maxims.

Although Demetrius saw the calculator as an optional tool and was confused about the ln and log keys, he knew how to use it correctly to evaluate base ten logarithms. He was also able to use it in conjunction with his transformation of expression in logarithmic form to exponential form to perform the table completion task.

Beliefs

During the postinstructional phase, Demetrius's understanding of the logarithmic function had changed. He no longer talked about the logarithmic function as being easy. Instead, he characterized his understanding of the function as memorization of rules, he no longer could or needed to remember. He still associated the function with a collection of problems and he realized logarithmic and exponential forms were important to solving these problems. He knew the calculator could be used to approximate logarithms, but it was not a necessary tool. His maxims were vital to generating solution paths, when he knew a problem involved the logarithmic function.

1. Knowing how to do problems involving logarithms is not a complete understanding, but it was all that was required in college algebra and it is enough for me
2. The logarithmic function is a collection of notations and problems that can be solved, if I can remember the maxims.
3. The log is the exponent.
4. When the bases are the same you add (subtract) logarithms.
5. The log and ln keys on the calculator help you compute logarithms.
6. The most important thing to remember about logarithms is that there are two forms: exponential and logarithmic.

Changes in Understanding

During the preinstructional phase, Demetrius recalled his high school teacher had called the logarithmic function as *easy*. He used this recollection as the basis for his ways of operating. He over generalized the distributive property and applied it to the logarithmic function. He was able to remember some relevant notation and names. In particular, Demetrius remembered the position of the base of the logarithmic function. In general, he felt if his memory was refreshed, he would recall how to do the logarithmic function. This view was in contrast with his claim that when he understood a concept, he knew both how to do the problems and why he was doing a particular procedure.

During the instructional phase, whenever I asked Demetrius about his understanding he referred to what he knew how to do. This was consistent with his goal of learning enough about the logarithmic function to pass the test. Initially Demetrius found the notation associated with the logarithmic function and its properties difficult to understand, but he quickly realized the adoption of maxims could help him remember the properties of the function. Demetrius noted that *going from* logarithmic to exponential form was of particular importance. He highlighted this transformation on his map, and singled it out as one of the types of problems he needed to know how to solve.

Demetrius still saw the logarithmic function as a collection of symbols that could be combined, an answer to be calculated, and a problem to be solved. In addition, the objects themselves did not have standard meanings for him. When Teacher 3 attempted to illustrate how the properties of the logarithmic function could be applied, the notation looked so complicated to Demetrius and it was so late in the class period, that he just packed up his materials. He gave up trying and “went to daydreaming.”

During the postinstructional phase, Demetrius was extremely surprised that he did not know how to do problems with the logarithmic function. He felt “ashamed.” To add logarithms he reverted to his preinstructional method of taking the logarithm of their sum. He continued to rely on maxims to help him solve problems; this practice was as much a hindrance as it was a help. He resolved the issue of understanding, noting that he knew how, but not why. For Demetrius, his knowledge of the logarithmic function was sufficient for his major and he had passed the course, so he understood enough. During Interview 9, he illustrated his process of understanding the logarithmic function (see Figure 30).

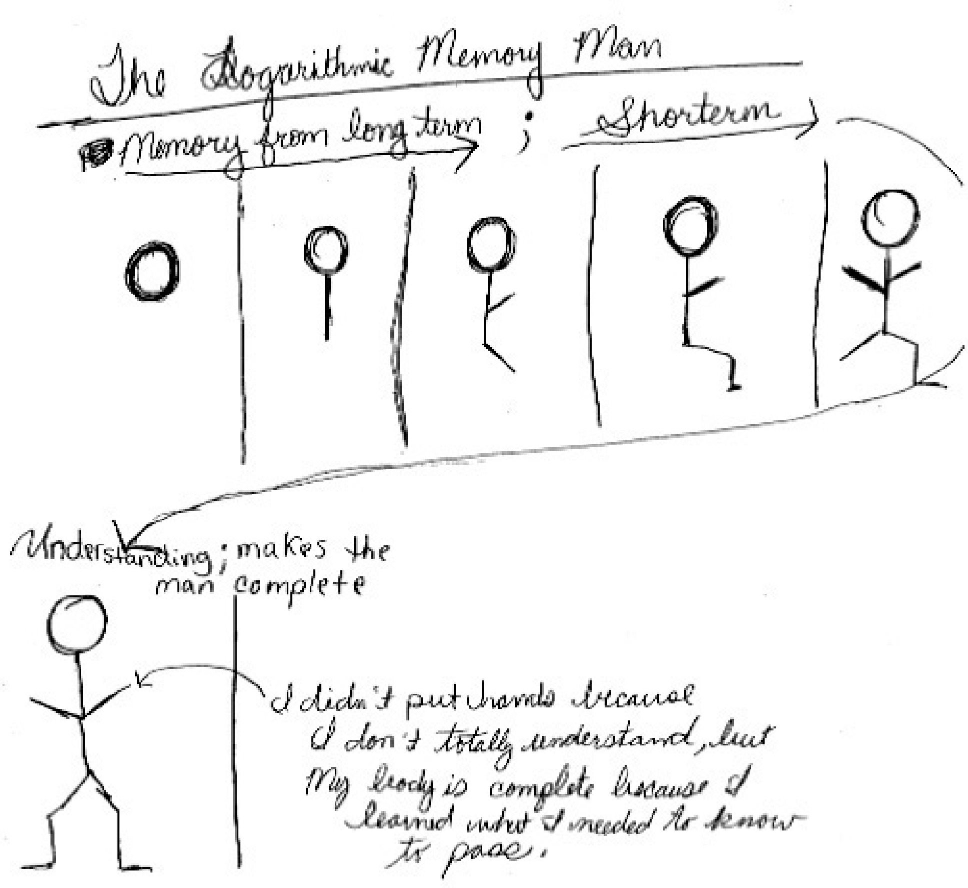


Figure 30. Demetrius's depiction of his process of understanding the logarithmic function.

Demetrius was taking psychology during the semester Interview 9 was conducted. He had recently studied long- and short-term memory. He explained his drawing:

> All right, it is called the logarithmic memory man. Which is a man based on my memory. And this is the memory man from long term, you know, from the long term of talking about logarithms. I didn't know that much, so therefore the man wasn't complete or whatever. And as I'm going from long term, which means as I started to talk about it, because it is short term

> memory I'm here, in the process of doing it. So it was just like the day before or that semester, you know, I just did it. So nothing about logarithms, but I know more about logarithms. I feel like going from there to now, I am complete, because I understand. I know enough to pass and whatever. And I didn't complete the hands because I feel like I understood enough to pass it, but I don't know everything there is to know about them. I don't understand them completely.

Demetrius rationalized his understanding of the logarithmic function, by including on his drawing and in his description reference to knowing enough to pass. Demetrius was very interested in how he remembered concepts. He realized he did not know everything about the logarithmic function, but then he felt no one did. He was satisfied with what he knew. He could transform logarithmic expressions to exponential ones. He also recognized notation in logarithmic form and was able to use some names associated with the logarithmic function.

<u>Ways of Knowing</u>

Demetrius used many of the same ways of knowing as the other participants. He used linear interpolation on the tasks in Interview 5 and 6 to find the logarithms of numbers that were not integer powers of two. In Interview 7, he transformed logarithmic expressions to exponential ones and found approximations for the logarithms using successive approximation and his calculator. He also attempted to eliminate the inconsistencies in his work when he identified them or I pointed them out.

One way of knowing unique to Demetrius, was his confidence in the patterns he discovered. For example, during Interview 5, he identified the pattern among the number line numbers as *doubling*. In this section of the interview he was trying to find the sign above the number $\frac{1}{2}$:

> D: What sign would be above the number one-half? Uhm...one-half?
>
> S: Umhum.

D: Wouldn't it be right here [points to the position for $\frac{1}{2}$ on the number line].

S: Yeah, one-half.

D: (Long pause) Zero...I meant hold up...no, no, no, no (hums, pause, subvocalizes) One-half.

S: One-half? You think one-half goes over one-half?

D: Hold up, let's see. Unum. Two.

S: Two goes over it...what makes you say so.

D: I don't know. That doesn't look right though. Hold up, that can't be right.

S: Why?

D: Because I've got a pattern going. Well, I'd say point five.

S: You would say point five. Okay.

D: No, hold up. (Laughs)

S: That's ok, take your time. I'm not in any hurry.

D: (Subvocalizes) Negative one.

S: Why do you say that?

D: I don't know. I'm just looking at the pattern.

Demetrius tested his conjectures by comparing them with the pattern he found. His first conjectures for the sign over $\frac{1}{2}$ did not fit the pattern, so he abandoned them. Identifying the sign as –1, allowed Demetrius to reverse his procedure. He was able to find the sign numbers corresponding to negative integer powers of 2, by dividing the number line number by 2 and subtracting 1 from the sign numbers. Demetrius was able to account for the sequence of number line numbers and sign numbers and used both to make predictions. He was also aware of another pattern in this problem. When he used linear interpolation to find the sign above the number $\sqrt{2}$, he felt that his answer was incorrect.

His rationale was that distances between signs should be increasing. Eventually, he gave up on this problem because he could not find another prediction method that was consistent with his doubling/halving map. Demetrius was aware of three patterns, the two number sequences and the distances between the signs. The strongest of these were the number sequences, with distance only playing a role, when the number sequences were no longer useful in helping him find the sign numbers.

Demetrius's awareness of the increasing distances between the numbers with signs on them, could be used as a basis for encouraging his growth of understanding of the logarithmic function. In Napier's number lines with simultaneous moving points, it was his awareness of the distance each point traveled on one number line with respect to the other that helped him develop his concept of the logarithmic function. Certainly Demetrius was not Napier, but his awareness of patterns and his ability to coordinate them and consider the actions of three different patterns suggests a promising avenue for development of his understanding.

CHAPTER 5: UNDERSTANDING A MATHEMATICAL CONCEPT AS DOING

In this chapter I describe themes in the students' understanding, changes in their understanding, and their ways of knowing. These descriptions constitute the findings of the study and provide the basis for a discussion of students' understanding of mathematical concepts. The themes are also be used to modify the model of understanding that I used to frame the study. Finally, the analysis of the findings and the limitations of the study suggest avenues for further research into students' understanding.

Understanding

In this study the changes between in the students' beliefs about the logarithmic function were small. The students' understanding of the logarithmic function during the first two phases had an impact on their understanding during the final phase, but their beliefs during the final phase constituted their understanding. Their beliefs both within and across phases had a common theme: *problems*. For the students in this study, the logarithmic function was a collection of problems to be done. In addition, I identified four categories of students' beliefs about the logarithmic function: *level of difficulty*, *problem types*, *tools*, and *character of the function*. In this section I describe the theme and the categories and answer the research questions.

Understanding as Doing

The students viewed and described their understanding a mathematical concept as being able to *do* the function. They used phrases such as the following: "To do the logarithmic function…., I don't know how to do it…., I forgot how to do it…." Their references to doing the function referred to *the* problems associated with the function. *The* problems were those illustrated and class and on their homework. For the students in this study, *the* problems were the concept. Hence, they referred to the logarithmic function as something they had to be able to do.

Level of difficulty. Because all of the students viewed the logarithmic function as doing problems, their beliefs about the function included perceptions of the level of difficulty of the problems. If a student felt he or she did well on a task, the function was easy. If the student performed poorly, his or her assessment varied depending on his or her rationale for the performance. If a rationale for poor performance could be given, the student viewed the function as easy. For example, If the students were asked to compute logarithms without the log key on their calculator, the function was still easy; they just did not have the proper tool. On the other hand, if a rationale for their poor performance could not be found, the function was hard. Students' beliefs about the logarithmic function were based on their perceptions of their performance on problems they associated with the function. When they described the level of difficulty of the function, they were assessing their ability to do the problems.

Jamie and Nora believed that the logarithmic function was hard because they did not perform well on the tasks during the postinstructional phase of the study. Demetrius also felt he had done poorly but rationalized that he had learned to do the problems presented in class and that was all he needed to know. Rachel became frustrated with the tasks during the postinstructional phase but maintained the function was easy to do, if she had her textbook. Each of the participants made some assessment of the level of difficulty of the problems and associated the level of difficulty with the concept.

Problem types. Two problem types, converting and evaluating, were included in the students' beliefs. Converting referred to problems that required a transformation from one form to another. Evaluating referred to problems that required a decimal approximation for a logarithmic expression such as log 2 and $\log_3 2$. These problem types were illustrated in class and were included on the homework.

Tools. The largest category of student beliefs about the logarithmic function involved the tools they used to solve problems. Four types of tools were used and discussed by the students: *facts*, *formulas*, *the calculator*, and *procedures*. *Facts* are maxims or pieces of information students' remember to help them solve problems and identify errors. For example, "the log is the exponent" was a fact that helped Demetrius transform expressions in logarithmic form to exponential form. The students accepted and used the facts even though they did not know why they were true. For example, when Rachel calculated $\log_3 4 - \log_3 5 = \log_3 -1$ she suspected that her answer was incorrect since it violated one of her facts: you cannot take the logarithm of a negative number.

Formulas involve mathematical notation. I make a distinction between facts and formulas because the facts students used were usually correct, the formulas were not. During the postinstructional phase the students' formulas were examples of properties and procedures. For example, Jamie used $\log_3 4 + \log_3 5 = \log_3 9$ as a formula and Nora used $\log_3 4 = \frac{\log 3}{\log 4}$. The students never assumed their formulas were incorrect if an answer they calculated violated a fact. Instead they looked for errors in their calculation.

The *calculator* was an essential tool for the students. They used it to find numerical approximations for logarithms. When they did not have the tool during Interview 7, their initial response to the table completion task was that it could not be done without a calculator with the log key.

The students also used these three tools in sequence to solve problems. I call these sequences *procedures*. One example was Nora's procedure for solving logarithmic equations. She used a fact to transform the logarithmic equation to an exponential one, the change of base formula if the transformation did not produce results, and her calculator.

Character of the function. The students used characterizations of the function to guide their thinking. I call these characterizations *the character of the function*. The students believed the logarithmic function was a collection of problems to be done, but when I asked them what the function ways they characterized it function in a variety of ways. Nora described the logarithmic function as related to the exponential. For Jamie and Demetrius, it was a word or collection of symbols. Rachel described the function as a type of mathematics. These characterizations helped the students do problems. For example Jamie used her characterization of the logarithmic function as a word with simplify sums of logarithms.

What Is a Student's Understanding of the Logarithmic Function?

The student's understanding of the logarithmic function was as collection of problems to do. Components of their understanding included the level of difficulty of the problems, the types of problems that they had to solve, the tools they needed to solve the problems, and the character of the function. The students saw the function as hard or easy. The associated it with evaluating and converting problems. They used a collection of facts, formulas, the calculator, and procedures to solve these problems. And they viewed the function as a type of mathematics, a collection of symbols, and associated to the exponential function.

Changes in Understanding

The students' understanding of the logarithmic function during each phase of the study was different, but comparing the students' understanding for the three phases revealed similarities. In particular, the four categories of beliefs identified in their understanding during the postinstructional phase were present in the other phases. The changes in understanding were in the content of the beliefs. The categories remained consistent. In this section I characterize the students understanding for the preinstructional and instructional phases, describe the students' beliefs in each category, and summarize changes in their understanding across the phases.

Preinstructional Understanding: Speculation

The students' understanding of the logarithmic function during the preinstructional phase was speculative. The basis of their beliefs was their experience with the logarithmic function in courses and with the skills assessment.

Level of difficulty. During the preinstructional phase of the study the students called the function easy or hard to remember. Nora and Demetrius viewed the function as easy, based on their experiences from high school. Rachel performed poorly on the skills assessment, so she called the function hard to remember. Three of the student speculated about the level of difficulty of the function.

Problem types. The students either did not recall details about the function or the function itself from their previous classes; hence their speculation about problem types came from the skills assessment. They speculated that the logarithmic function was associated with evaluating, simplifying, creating tables, and graphing. Each of these terms was used on the skills assessment.

Tools. Beliefs about types of problems and the difficulty of problems were secondary to learning how to do the problems. To do problems, the students had to develop tools. Based on the skills assessment, the students speculated that they needed facts, formulas, and the calculator. The students did not demonstrate knowledge of any formulas during the preinstructional phase, but believed that if they could decipher the notation for the logarithmic function they could do the problems. For example, Jamie explained that if she knew what $\log_{(?)}$ stood for, she could solve the problems. For Nora and Rachel, the key to solving problems was their calculator. Both asserted if they knew how to use the log key on their calculators, they would be able to solve the problems on the skills assessment.

Character. During the preinstructional phase the students speculated that the logarithmic function was a collection of symbols or a notation. For example, Rachel noted the logarithmic function was $\log_{base}\#$.

<u>Understanding during the Instructional Phase</u>

During the instructional phase of the study, the students attempted to stockpile information. The results of these attempts can be seen in their beliefs. Solving problems was the goal, so developing tools was of primary importance during this phase.

Difficulty level. During the instructional phase of the study, the students characterized the logarithmic function as easy and simple because they felt they could do the problems associated with the concept. Each student expressed doubt or confusion about some formula or procedure presented in class, but all were satisfied they knew how to do problems and were ready for the examination. For the students, the problems were easy, hence the logarithmic function was easy.

Problem types. The students identified two problem types during the instructional phase: converting and evaluating. These two problem types were emphasized in class. The majority of instructional time was spent presenting the logarithmic function in algebraic form, converting it to logarithmic form, and evaluating logarithms. Hence converting expressions and evaluating logarithms were two types of problems the students wanted to know how to do.

Tools. During instruction the students' had more beliefs about tools than they did during the preinstructional phase. More facts and formulas were used in class and on the homework. In addition, the students practiced using their calculators to evaluate logarithms and developed procedures to solve problems. The students could state a much larger collection of facts and formulas than during any other phase of the study. Facts they cited were "One cannot take the log of a negative number," "The log of one is zero," and "When you add you multiply." Formulas they used were properties of the logarithmic function and the change of base formula. The students used these facts and formulas to transform expressions into forms they could use their calculators to evaluate. Tools used in sequence on problems became procedures. The students did not know why their tools worked on problems, but only that they did.

Character of the function. The students' beliefs about the character of the function during this phase were adopted from an association presented in their classes: the exponential and the logarithmic function are related. For example, Nora noted "logarithms and exponentials" are related. This statement made the students sound knowledgeable, but for them it was a maxim. truth the student could state but that he or she could not explain.

What are the Changes in the Students' Understanding During the Instructional Process?

Describing changes in the students' understanding of the logarithmic function is much more difficult than describing their understanding because there are exceptions to generalizations made across this group of students. Table 1 summarizes the students' understanding for each phase of instruction. There was no change in their understanding of the logarithmic function as how to do problems or in the categories of beliefs. The changes were in the content of the categories. Beliefs about the level of difficulty of the logarithmic function varied in the pre- and postinstructional phases, but all the students believed the logarithmic function was *easy* during the instructional phase. During the preinstructional phase, the students' used the skills assessment to speculate about the problem types associated with the logarithmic function and tools they would need to solve them. Prior to instruction all the students described the logarithmic function as a collection of symbols. All but Nora returned to this characterization during postinstruction. During instruction all the students characterized the logarithmic function as related to the exponential, but none could explain or illustrate why.

During the instructional phase, the students' beliefs were speculative. During the instructional phase, students' beliefs were consistent with standard mathematical thinking about the function. During the postinstructional phase, their beliefs were fewer in number and were distortions of beliefs they formed during instruction.

Table 1

Characteristics of Students' Understanding During Instructional Phases

	Instructional Phases		
Categories of Beliefs	Preinstruction	Instruction	Postinstruction
Level of Difficulty	Easy, Hard	Easy	Easy, Hard
Problem Types	From Skills Assessment	Convert, Evaluate	Convert, Evaluate
Tools	Notation, Calculator	Facts, Formulas, Calculator, Procedures	Facts, Formulas, Calculator, Procedures
Character	Symbols	Related to the Exponential	Symbols, Related to the Exponential

Ways of Knowing

When one looks at the students' performance on standard problems during the postinstructional phase of the study, the view is rather bleak. The student's tools for solving problems were diminished and distorted. However, as the students' actions during the postinstructional phase suggest, they had ways of knowing that could be the basis for an understanding of the logarithmic function consistent with standard mathematical thinking. In this section, I describe four ways of knowing that were common to all the students in the study: number patterns, successive approximations, more A–more B, and response to inconsistencies.

Number Patterns

During Interviews 5, 6, and 7, each of the students attempted to identify patterns in either the numbers presented or in the numbers that they generated. In Task 5. the students attempted to find a relationships among the number line numbers and the sign numbers, and then to coordinate their actions. Jamie was more successful on this task than any other student because she was able to see the number line numbers as powers of 2 and the sign numbers as exponents. The other students developed various schemes. Nora and Demetrius identified doubling as the pattern in the number line numbers and saw the sign number as a counter for this action. With prompting from me, Nora eventually identified the pattern as a correspondence between powers of 2 and their exponents. Demetrius was able to reverse his procedure to find the signs above numbers less than 1, but he never became aware of the sign numbers as exponents. Rachel used an elaborate addition and subtraction algorithm that used both the sign numbers and the number line numbers. All of the students also used patterns to describe the function behavior in the remaining two interviews.

Although each of the students explained the pattern in Interview 5 differently, each first attempted to determine whether addition and subtraction could be used to describe the pattern. When the explanatory power of their pattern was insufficient, the student sought other mathematical operations, notably multiplication, to describe the pattern he or she saw.

Probably the most surprising use of patterns occurred during Interview 7. Each of the students searched for a pattern between the *y* values in the table. The students tried addition and subtraction first, but then quickly moved to multiplication and division. This strategy helped them eventually identify a pattern in log 2, log 4, and log 8, but they were unable to make a generalization from this finding. Eventually, I asked the students why log 1 was equal to zero, which led to their recall of the conversion procedure they had developed during the instructional phase. They then used exponential equations and their calculators to develop a successive approximation procedure.

In each of the tasks, the students attempted to find and use number patterns to predict answers. This way of knowing played a vital role in the students' attempts at problem solving. In cases where the patterns partially explained the mathematical phenomenon being presented, the students used their patterns. When their patterns could not predict correct answers, the students attempted to modify them.

Successive Approximations

Successive approximation was used by all the students during Interview 7. They converted log 1 = 0 to $10^0 = 1$ and used this example as a template for the other table entries. Using successive approximation and their calculators, the students were able to fill in the tables. Despite its utility, the students recognized the method's inefficiency. They searched diligently for other methods to fill in the table, but always returned to successive approximation.

More A–More B

During the task interviews, at some point the students' prediction strategies broke down or were identified as inefficient. For example, Nora, Rachel, and Demetrius had difficulty predicting the sign over the number line number $\sqrt{2}$. When it became clear their prediction method did not work, they used linear interpolation. Hence for the students, the sign above $\sqrt{2}$ was 0.414…. and the log 4 was $\frac{\log 3 + \log 5}{2}$.

This linear interpolation strategy is one example of what Stavy and Tirosh (2000) have identified as More A – More B. The student incorrectly assumes that two given quantities have a linear relationship. Hence if 0.414… is added to 1 on the number line numbers, then 0.414 should be added to 0 on the sign numbers. Similarly, if 4 is the midpoint between 3 and 5, then log of 4 must be the midpoint between log 3 and log 5.

This type of thinking was the fall-back position for the students when their predictive strategy, based on an identified pattern, failed.

Response to Inconsistencies

During the interviews, when students became aware of inconsistencies in their answers, they attempted to eliminate them. For example, based on a counterexample I gave Nora, she realized that her prediction for $\sqrt{2}$, 0.414… was inconsistent with her generalization that when you multiply two number line numbers the sign numbers are added. Nora, like the other students, attempted to explain the inconsistency.

Responding to inconsistencies in both answers and thinking is a way of knowing that all four students exhibited. Each of the students was bothered whenever he or she noticed an inconsistency in his or her thinking or when I pointed one out using a counterexample. Finding ways to making their thinking consistent was extremely important to these students.

Conclusion

The students did not realize that the tasks in Interviews 5 and 6 were based on the logarithmic function, and, although they did see the logarithmic function as important during Interview 7, initially each of the students claimed that the task could not be done. The solution methods generated by the students were based primarily on patterns. The use of successive approximation was implemented only after an exponential equation was written. These ways of knowing were seen by the students as elementary, akin to finger counting in elementary school. None of the students viewed these techniques as important mathematical tools. For them, tools were facts, formulas, the calculator, and procedures.

The students use of more A–more B reasoning led them to question the answers they generated using number patterns, but ultimately the predictive capability of the number pattern and the students' responses to inconsistencies in their own reasoning overcame the more A–more B reasoning. All the students became aware that the relationship between the sequences in the tasks was not a linear one.

The students' ways of knowing were not connected to their understanding of the logarithmic function. A connection between their methods and their understanding was made only briefly to transform the logarithmic expressions to exponential ones using "converting." The conversion was simply a means to an end. Once the students had exponential equations, they used their ways of knowing to complete the task. No properties of the logarithmic function were used. These four ways of knowing may be useful in the development of students' understanding about the logarithmic function.

CHAPTER 6: DISCUSSION AND IMPLICATIONS

Discussion of the Findings

In this chapter I discuss the findings of this study in terms of the relevant literature discussed in chapter 2.

Understanding and Changes in Understanding

The impetus for this study was students' inability to remember the logarithmic function from one school term to the next. I assumed a description of students would help me explain this phenomenon. One of Brownell's (1972) criteria for learning was that it had to be sustained. Evidence of learning collected immediately following an instructional treatment was insufficient for a researcher to claim the students had learned. Although it is not clear how, researchers studying learning and understanding claim the two are related (Kieran, 1994). Evidence of sustained change in understanding is necessary for understanding to be claimed. In addition, Pirie (1988) asserted students' understanding cannot be described without knowledge of the student's process of understanding. My decision to gather evidence of changes in understanding was based on the work of these researchers. This decision turned out to be crucial. It was only through the examination of the students' beliefs over the three instructional phases that I was able to see the consistency in the beliefs. The descriptions of students' understanding are necessary but not sufficient to explain their understanding. As Pirie (1988) conjectured, the development of understanding is essential for construction a model of students' understanding.

Categories

The existence of the four categories of beliefs across the three phases of instruction indicates a stable structure the student uses to make sense of and take action during his or her learning experiences. The categories act like a filter for the students' attention. Activities or experiences that do not fit the categories are disregarded by the student. For example, the graph of the logarithmic function was illustrated, but not emphasized, in classes I observed. None of the students used or referred to the representation. For the students, the graph was not a problem type since it was not emphasized in class and was not a tool for solving either converting or evaluating problems they identified as important. Instruction on graphing was filtered out.

The appearance of these categories prompts two important questions: What is the origin of the categories? And why do they remain static over the three phases of instruction? Support for my answers to these two questions can be found in the beliefs literature.

Mathematics education researchers have shown that there is a connection between students' beliefs and their behavior (Kloosterman, 1991; Schoenfeld, 1989, 1992; Szydlik, 2000). The students in this study believed that understanding a mathematical concept was being able to do problems. Their behavior was consistent with that belief. They attempted to collect problem types and tools for solving them. Hence the origin of the categories—, problem types and tools— are the students' beliefs about understanding mathematics. The additional categories—, level of difficulty and character— can also be traced to the students' beliefs. Because the students believed performance was understanding, their performance was essential. If a good performance was hard to produce, then the task was difficult; if not, it was easy. The character of the function was also linked to the students' beliefs about mathematics. For these students, mathematics was a collection of formulas, notations, and rules. Hence the character of the function was a notation or rule. Thus the students' general beliefs about mathematics and understanding mathematics were the basis for the four categories of beliefs they used to mediate their activity.

The static nature of these categories is not surprising. As Dossey (1992) noted beliefs are extremely difficult to change even when the curriculum is designed to elicit a change in beliefs. In this study, none of the instruction experienced by the students challenged their beliefs about mathematics. Instead, it supported and thus reinforced their belief structure. Therefore it is not surprising that categories derived from the beliefs did not change. In this study the students' beliefs about mathematics and understanding influenced their categories of beliefs about the logarithmic function.

<u>Content</u>

Although the categories of the *beliefs* remained constant, their content did not. Two factors influenced the content of the students' *beliefs*: instruction and memory.

Instruction. During the preinstructional stage, the students used their categories to determine what they should attend to. In particular, they looked for clues about the logarithmic function in the skills assessment. During the instructional phase, the students processed information presented in class and while doing homework and used it as content for their beliefs. During the posinstructional phase, the content of the beliefs changed dramatically. Part of the reason for this change was the dearth of sources from which information could be derived. When students were given tasks where the logarithmic function was not mentioned, Interviews 5 and 6, they did not relate the tasks and the function. When the logarithmic function was mentioned, the students reconstructed beliefs about the logarithmic function they had formed during the instructional phase. These beliefs were based on their memories and not on logic. Instruction and instructional materials were the primary sources of content for the students' *beliefs.*

Memory. Memory was not a direct source of content for the students' beliefs, but it was an influence. In particular, the students' understanding during the postinstructional phase was in part an attempt to reconstruct (Bartlett, 1932) their *beliefs* from the previous phases. Bartlett noted that reconstruction from memory is rational, based on some reason or association made by the person remembering. This interpretation explains distortion in the students' beliefs about tools. In particular, from an interpretation of $\log A + \log B$ as like terms in algebra, $\log (A + B)$ is a rational reconstruction of the sum. The emphasis is on the term *rational*. The students were certain that this formula was correct. For them the function was reasonable, but in mathematics the formula, log A + log B = log (A + B), is wrong. Why were the students unable to either see the error or respond to counterexamples of the formula?

The answer to this question lies in the content of the beliefs developed during the instructional stage. At no time could the students explain why various formulas worked. The absence of any logical basis for the formulas they adopted and used made the formulas extremely difficult to remember correctly (Sierpinska, 1994). For the students the formulas were a bit more than, but much like, the nonsense syllables of Ebbinghaus (Bruning, Schraw, & Ronning, 1999). The primary difference was that the formulas were remembered for a purpose. Thus problems and the tools used during this phase were associated. This association allowed the students' to reconstruct tools during the postinstructional phase.

The students' reconstructions of formulas during the postinstructional phase were flawed and not derived through logical thinking. Hence the students had no means for checking results obtained using these faulty tools. They simply hoped they had remembered correctly. As Skemp (1987) noted, this instrumental type of understanding is less flexible and more difficult to remember than relational understanding.

Growth Versus Change

Clearly the students' *beliefs* about the logarithmic function changed over the course of the study. But does such change constitute growth of understanding? If growth is defined as a sustained change in beliefs consistent with correct mathematical thought, the only growth was in the problem types and tools category. In the postinstructional phase, the students had beliefs about evaluating and converting logarithms and about tools used to solve these problems. Although the students generally could not use their tools to solve problems correctly, their beliefs indicate an awareness of the existence of logarithms and of solution methods. This awareness is not growth, but could be used in future instruction.

There can be many bases for understanding a concept (Skemp, 1987; Sierpinksa, 1994), but among mathematics educators relational understanding, based in knowing why, is assumed to be superior. The students in this study did not achieve either relational or even instrumental understanding. Instead, they were left with an awareness that the logarithmic function belongs to the mathematical curriculum. It is this awareness that can be used to build subsequent beliefs. Thus my experience teaching students who passed college algebra but appeared to know nothing about the logarithmic function is consistent with the findings in this study. The understanding the students developed during instruction was transitional and soon deteriorated. What remained was awareness, a meager basis for future understanding.

Ways of Knowing

When I planned this study, I conjectured that the students' ways of would be similar to Napier's. To an extent my hypothesis turned out to be true. Students do map multiplication to addition if terms of geometric and arithmetic sequences are given to them. They also notice that there is a common ratio between the given terms of each sequence. These observations are consistent with Confrey and Smith's (1995) findings. Students they studied were able to coordinate actions between arithmetic and geometric sequences. Rizzuti (1991) called these coordinated actions covariation and described them as a precursor to the correspondence definition of function.

In this study, the students used covariation to make predictions from their number patterns. For example, in Interview 6, the students used the examples I provided, $f(2) = 1$ and $f(4) = f(2) + f(2) = 2$, to generate sequences {1, 2, 3, 4, …} and {2, 4, 8, 16, …}. They then abandoned the given function notation in favor of the sequences and used the correspondence between the two to make predictions.

Covariational thinking quickly broke down when the students attempted to insert terms into the sequences. The students began to rely on linear interpolation, a special case of the intuitive rule called More A–More B (Stavey & Tirosh, 2000). Napier was able to insert terms into his sequences so the ratio between the terms would remain constant. The students were not. During Interview 6, when the students attempted to find $f\left(\sqrt{2}\right)$, they could not use their doubling operations and immediately tried linear interpolation.

The More A–More B rule was in direct conflict with the students' use of covariation. When I presented counterexamples to their hypothesized solutions, they attempted to resolve the inconsistencies with little success. The intuitive rule was very strong and seemed logical to them. They used it when they were asked to insert terms that did not fit the sequence into a given sequence and when their exploration of number patterns did not yield a covariational relationship (for example Interview 7).

The use of covariational reasoning was restricted to cases where a geometric sequence and an arithmetic one could be discovered. When no such pattern was identified or when the student could not use this reasoning to produce answers, he or she quickly reverted to predicting terms using a More A – More B strategy. The students responded, usually unsuccessfully, to counterexamples, but never completely abandoned it.

The final way of knowing, successive approximation, used by the students was in all but one case motivated by questions I posed during the interviews. For example, Jamie spontaneously used this way of knowing in Interviews 5 and 6, but only in response to my questions during Interview 7. The students in this study, who were able to generate an exponential equation, were able to approximate their solutions (logarithms) with a TI-15 calculator (no log key). More research on students' behavior and thinking associated with exponential expressions of the form a^n, where a is a positive integer and n is any rational number would help explain the use of this way of knowing.

Discussion of the Framework

In this section I critique the categories of evidence and present a modified theory of understanding. In addition, I discuss the design of the study and suggest improvements.

Modification of the Theory

Categories of Evidence

The beliefs I attributed to the students were developed using four categories of evidence: conception, representation, connection, and application. Each of the categories contributed to the development of my conjectures about the students' beliefs.

Conception. The students' remarks about understanding and their maps of the function allowed me to view the function from their perspective. It is this perspective that has been missing from studies of students' understanding of mathematical concepts. This category of evidence is essential in the development of beliefs attributed to the student. In particular, collection of this data assumes a student's communicated ideas about a mathematical concept provide insight and may help explain the student's performance on tasks that involve the concept.

Representation. When I began collecting evidence of the students' representations of the logarithmic function, I assumed that when they spoke or wrote about the function they were providing representations of their own thinking. Representation of concepts referred to by Hiebert and Carpenter (1992) were not the type produced by the students in this study. Generally, when students were asked to solve standard problems, they generated marks or utterances they hoped would demonstrate their ability to do problems. For the students, their demonstrations meant they understood the function. For me, the demonstrations meant that the students could write correct notation, not that they understood the function. Hence, the representations I collected, especially during the instructional phase, were often those presented in class and mimicked by the students. They were not the students' representations and did not appear to have standard mathematical meanings for them. For example, during the instructional phase, the students could translate an expression in exponential form to logarithmic form and explained that the logarithmic function was related to the exponential function. None of them could explain why. During Interviews 5 and 6 however, when I asked the students to explain their notations or procedures, they hesitated, but could explain why they followed their developed procedures. The representations they generated appeared to be meaningful to the students. The notations were representative of the students' thinking. This observation about the findings suggests this category of evidence be divided into two subcategories: representations adopted by the students and representations generated by the students.

Connection. The importance of the connections between representations has been emphasized in the literature on understanding (Hiebert & Carpenter, 1992, Skemp, 1976) and in the literature on representation (Even, 1998; Janvier, 1987; Kaput, 1998). My definition of a connection between representations was extremely difficult to use. To identify a connection, I had to see the student use one representation and then another. This usage was extremely difficult to see because the two types of representation the students used most often were written and oral. It was difficult to tell whether two written representations were connected or simply part of the same formula. For example, when students simplified $\log_3 4 + \log_3 5$, were they connecting two written representations or remembering one formula? In addition, when students read their notation, were they connecting their written and oral representations or simply reciting names? I resolved this dilemma during data analysis by assuming both of these examples were connections. Hence I defined a connection as the use of one representation followed by another. However, a more precise definition must be generated for this category before evidence of connections can be useful in hypothesizing students' *beliefs.*

Application. The students' application of the definition and properties of the logarithmic function to nonstandard problems was evidence of understanding. In general, the students in this study were unable to apply the logarithmic function it this way. They were able to use apply their knowledge of the calculator to generate logarithms, approximations of logarithms, and facts about the logarithmic function. During the data analysis, I redefined *application* as the use of tools to solve problems associated with the logarithmic function. This new definition allowed me to include the students' use of the calculator as an application of the function.

Beliefs

Identifying beliefs about mathematics and understanding as the framework for a student's understanding of a mathematical concept, suggests that a study of these beliefs is necessary to explain the student's understanding. Schoenfeld's (1989, 1992) work provides a basis for the investigation of students' beliefs about mathematics and their impact on students' understanding. Research that answers the question: What are students' beliefs about understanding of mathematical concepts is needed. This work could clarify the role of students' beliefs about understanding in their understanding of mathematical concepts.

Revised Theory

Based on the findings and observations from the data analysis phase I have revised my theory of understanding:

> *Students' understanding of a mathematical concept is a collection of beliefs derived from four categories of evidence: conceptions, representations, connections, and applications and influenced by their beliefs about mathematics and understanding. Conceptions are the students explicitly expressed ideas and feelings about the concept. Representations are symbols the student uses to think about or communicate a mathematical concept to others. Two categories of representations must be used to generate models of students' understanding: representations they adopt from instruction and representations they develop. Connections among representations are links the students form between types of representations (oral, written, tabular, and pictorial). An application of a mathematical concept is the use of tools to solve problems.*

Modification of Design

One design problem caused difficulty during the data analysis phase. Limited data were collected during the instructional phase of the study. I observed class, conducted at least one interview with each participant, and asked the students to map the function. A richer database was needed to develop hypotheses about the students' *beliefs* during this phase. In particular, I would add another iteration of the skills assessment. The skills assessment would provide needed data on the students' definitions of terms and their ability to apply concepts to problems in various representational modes.

Limitations of the Study

The findings of this study have two limitations. The first limitation stems from my assumption that the goal of teaching is understanding. The second is based on students' familiarity with the logarithmic function.

In the classes I observed, the focus was on doing problems. Teachers demonstrated how problems were done, and students practiced doing them. Some observations about the concept were included in the teacher's presentations, but the majority of each class session was spent demonstrating procedures for particular problem types. The presentation of the logarithmic function was no exception. Very few class sessions were allotted to the function: the concept was the last of the semester. Both teachers admitted that they rushed through their presentations of the logarithmic function because of limited time. Teacher 3 spent only two 75-minute class sessions on the topic.

Skemp (1976) noted that some teachers attempt to teach for instrumental understanding because it is efficient and produces excellent results. Teaching for instrumental understanding worked exceptionally well with the four participants in this study: they all performed well on their examinations. The students could do problems: however, one month later their understanding changed into a collection of beliefs that were not useful in solving either standard or nonstandard problems. At best, the students had an awareness of the existence of the logarithmic function. The limited amount of time available certainly constrained the instructional choices made by the teachers and limited the students' opportunity to build understanding. Hence one limitation of the findings is that they are based on the assumption that students had little opportunity to understand.

Another limitation of the findings is their relevance as a basis for inference about students' understanding of other mathematical concepts. It is not clear that results from a study of students' understanding of the quadratic function, for example, would produce similar results. Students' understanding of the quadratic function could have a different structure for two reasons. First, the students have more experience with the concept in high school. Second, more time is spent on the topic in college algebra. Further investigation of students' understanding of mathematical concepts they are more familiar with could help identify key structures in students' understanding.

Implications

This study has a number of implications for teaching and research. The implications for teaching involve confronting students' beliefs about mathematics and using their ways of knowing to help them develop *beliefs* about mathematical concepts. The implications for research are based on questions the study raised.

For Teaching

My teaching was the origin of the research questions for this study. The findings that resulted suggest some possibilities for the teaching of mathematical concepts.

The curriculum is overloaded with mathematical concepts students are asked to understand. These concepts are taught as if there are connections among them, but for the student these links may not be obvious. A students' experience with school mathematics may suggest that mathematics is a collection of rules to be memorized and applied correctly during examinations. Skemp (1987) noted that teaching for instrumental understanding can achieve a number of short-term educational goals. "If what is wanted is a page of right answers, instrumental mathematics can provide this" (p. 158). Students and teachers are aware of the efficiency of instrumental understanding. Teachers who attempt to present mathematical topics using curricular materials and methods designed to promote relational understanding are quickly made aware of students' preference for instruction designed to result in instrumental understanding. In the face of student resistance to these methods (Cooney, 1985), time constraints, and an overfull curriculum, teachers often turn to instrumental understanding to provide students with small measures of success and to view themselves as successful. This is the system that produced the kind of understanding described in this report.

The findings in this study suggest two avenues for growth of understanding. First, students must confront and question their beliefs about mathematics and understanding. These beliefs are the framework for students' understanding of mathematical concepts, hence: if they change so will students' understanding. Second, students should experience mathematics as sense making. Students develop beliefs about mathematics as they do mathematics (Schoenfeld, 1992). If reflection on their experience develops into a belief that mathematics is a collection of problems that need not make sense, they do not struggle with meaning, but are satisfied with mimicry. In this study when students were presented with standard mathematical problems, they attempted to remember how to do them, with disastrous results. Their answers were illogical and incorrect. If mathematics educators hope to change students' beliefs about mathematics, then students should experience mathematics as sense making.

For Research

The four theories of understanding described in this report (Hiebert & Carpenter, 1992; Pirie & Kieren, 1994a; Sierpinska, 1992; Skemp, 1987) do not suggest a connection between students' beliefs about mathematics and their understanding. The findings of this study and the literature on beliefs suggest that students' beliefs influence their understanding of mathematical concepts. What is needed is a close examination of students' understanding and their beliefs in an attempt to identify and explain connections between the two.

Much current research on students' understanding of mathematics is based on investigation using nonstandard problems. However, the students in this study used quite different approaches to standard and nonstandard problems. The students approached standard problems as tests of memory. Comments such as "I know we have done this; I just can't remember it" were common on these sort of problems. The students also generated answers using faulty formulas reconstructed from their memory of instruction.

In addition, when a student could not remember how to do a problem quickly, he or she gave up. In contrast, students investigating nonstandard problems appear to act in meaningful ways. Further research of students' understanding in these two settings is needed to explore the limitations of the theory to predict and explain students' understanding of both standard and nonstandard problems.

This report suggests that students' understanding of mathematical concepts changes over instructional phases. Students speculate, stockpile, and in the end are left with a collection of beliefs about the concept that are likely to be of little use as the basis for further understanding. This finding suggest students' understanding does not grow over the course of instruction. Instead students' understanding in pre- and postinstruction are similar, whereas their understanding during the instructional phase is transitory and thus not useful in subsequent mathematical activity. Further research on the understanding of mathematical concepts should investigate this phenomenon. Is the transitory understanding of the instructional phase the result of instruction that does not connect to students' ways of knowing? Do students for whom the mathematical concepts have standard meanings retain their understanding into the postinstructional phase?

To investigate these questions a study modified to include teaching by the researcher would be appropriate. The design would be similar to this study. The tasks in the instructional phase would be modified to include an assessment of the student's ways of knowing associated with the logarithmic function. Tasks such as those used during Interviews 5 and 6 could be used. With this knowledge, the researcher could construct instruction based on these ways of knowing. The second phase would be referred to as the instructional phase; however, instead of observing instruction presented to the student in a classroom, the researcher would be the instructor. Following instruction, a model of the student's understanding would be generated. This phase would be similar to the postinstructional phase. This type of investigation could illustrate understanding of mathematical concepts that can be achieved and what factors are necessary for students to develop an understanding of the logarithmic function consistent with current mathematical views. A student with this understanding believes the function is a map with a collection of special properties that can be represented in written, oral, tabular, and pictorial form. He or she can represent and identify the function in these forms, can connect representation, and can apply properties of the function to solve standard and nonstandard problems.

Conclusions and Questions

Students' understanding of mathematical concepts is intimately tied to instruction: however, the influence of instruction is not long lasting. This was the phenomenon I hoped to explain when I began this study. As a teacher I was unable to see the variables at play in the students' understanding. It was mystifying. The findings in this study have provided me with an explanation for what I experienced. They have also suggested to help my students understand mathematical concept, I should find ways to alter their beliefs about understanding and mathematics while providing opportunities for them to use their ways of knowing explore mathematical concepts.

As a researcher, examining these questions has suggested some answers but has raised many more questions. Do students' beliefs about mathematics and understanding provide a framework for their *beliefs* about a mathematical concept? Is students' understanding of different mathematical concepts composed of the same categories of beliefs? How is the understanding of students' who built their beliefs using their ways of knowing different from the understanding of the students in this study? Answers to such questions might benefit researchers, teachers, and students.

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APPENDIX A: INTERVIEW PROTOCOLS

Protocol for Interview 1: Preinstructional Phase

Instruction in Mapping

Adapted from Novak & Gowin (1984, p. 32-34).

1. What do you think of when you hear the word *car*?
    a. Does everyone think the same thing?
    b. These mental images that we have for object words are our *concepts* of objects.
2. What do you think of when you hear the word *is*?
    a. These are not concept words; we call them *linking* words.
    b. Linking words are used together with concept words to make sentences that have meaning.
3. We can make a map of the concept car using other concepts and linking words. The map is a picture of what you think of when you hear the word car. Let's try to make a map of that concept. We will make the map together. I will write down what you think of and organize it in a map.
4. Okay, now let's try to make one for pets. I'll make one and you make one. Then we will talk about them.
5. Now let's make one for high school. I'll make one and you make one.

Experience with the Logarithmic Function

6. When was the last time you saw or used logarithms or the logarithmic function? What was your experience like?
    a. How were you feeling at that time?
    b. What did you do to try and understand?
    c. Who or what was the most helpful to you during that time?

7. (Student is given the skills assessment task sheet) I will give you as long as you need to complete the following activity. I realize that you may not know all that is necessary to do the activity, but do as much as you can. (Student does skills assessment)

Student's Perception of His or Her Understanding

8. Think back on the activity and tell me about what you understood most.
9. Think back on the activity and tell me about what you understood least.

Mapping Activity

10. Make a map of the concept *logarithmic function.*

Skills Assessment Activity

Participant: ___________________________ Date: ______________________________

1. Using your own words and any pictures or diagrams you need to express your ideas, define the terms
   a. Function
   b. Logarithm
   c. Logarithmic function
2. List all the properties of logarithms (or rules about logarithms) that you can recall
3. Simplify the following expressions:
   a. $\log_3 4 + \log_3 5$
   b. $\log_3 4 - \log_3 5$
   c. $\log_3 9$
   d. $\frac{1}{2}\log_3 25$
   e. $\log_3 1$
4. Expand the following expressions if possible. If you can think of more than one expansion please include it in your answer.
   a. $\log_3 \frac{2}{5a}$, where $a > 0$

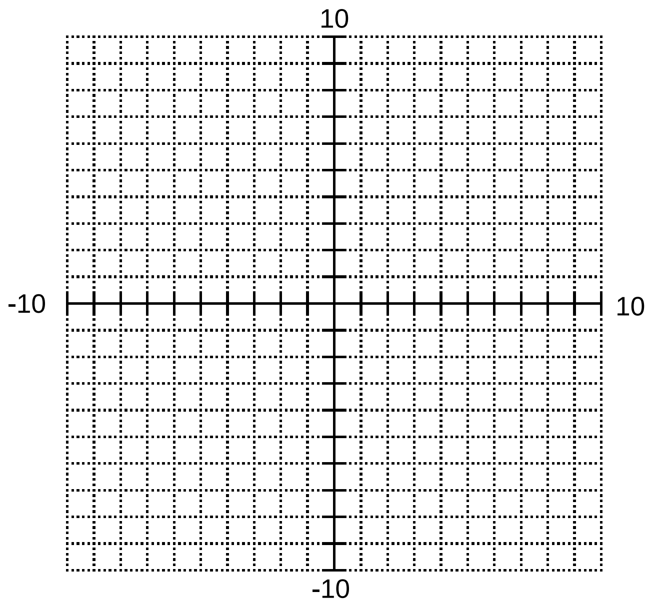
b. $\log_3(3+1)$

c. $\log_3 \sqrt{6}$

d. $\log_3 189$

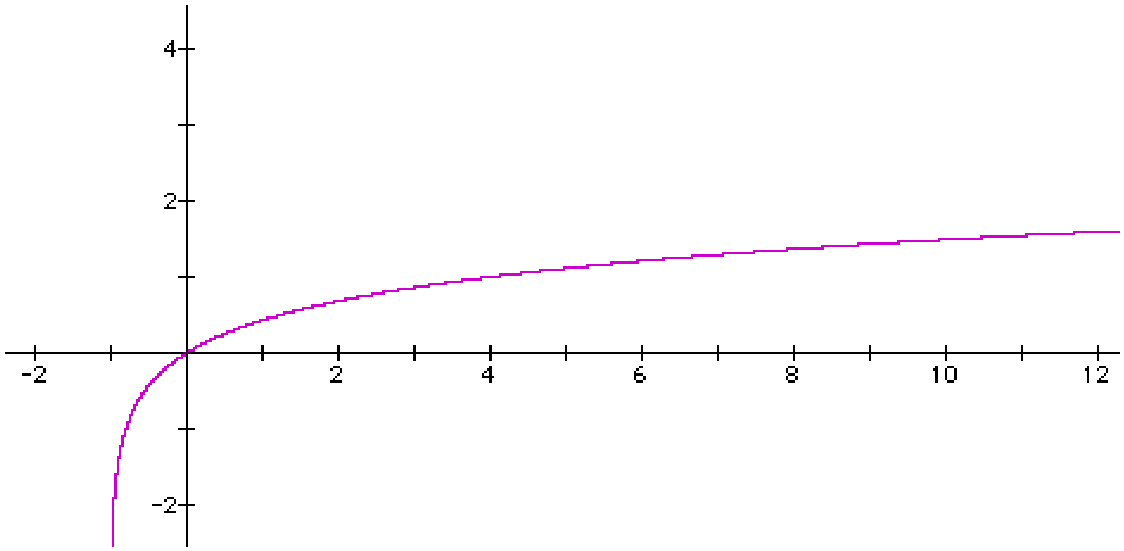
e. $\left(\log_3 \frac{4}{7}\right)^2$

5. Graph the function $f(x) = \log_2(x)$ on the axes provided.

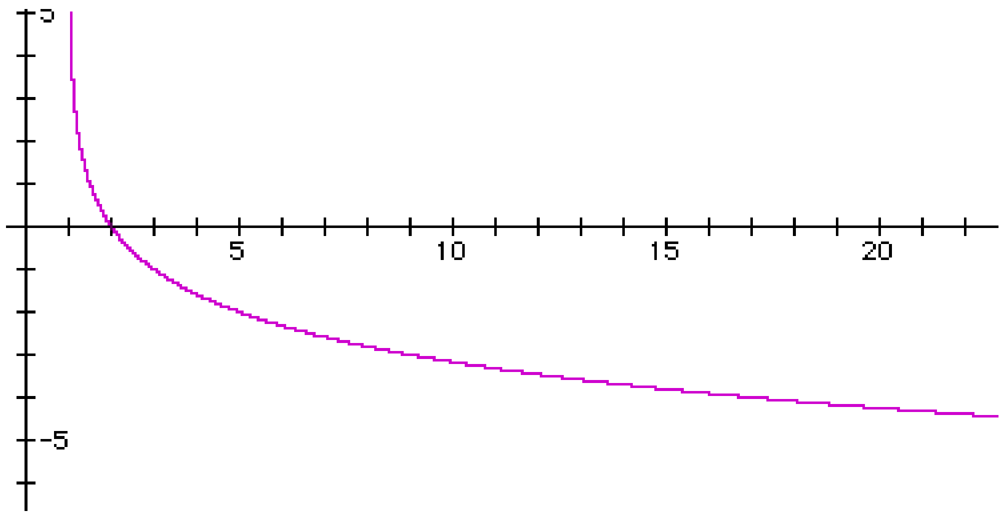


6. Construct a table of values for the function $f(x) = \log_3(x)$.

7. What function is graphed on the axes below? ___________________________



8. Use the graph below to construct a table of values for the function the graph represents.



9. What function could have been used to generate the table of values given below?

x	$\frac{1}{9}$	$\frac{1}{3}$	1	27	81
$f(x)$	-2	-1	0	3	4

10. The 1980 population of the United States was approximately 227 million, and the population has been growing continuously at a rate of 0.7% per year. Predict the population in the year 2010 if this growth trend continues.

Protocol for Interview 2: Preinstructional Phase

Understanding

(Questions 1 and 2 are adapted from Brookfield (1990, p. 32-33))

1. Think of a time when you felt something important or significant was happening to you as a learner and tell me about that time.
2. Think of a time when you felt despair or frustration about your learning activities and tell me about that time.
3. Describe yourself as a mathematics student.
4. When you are taking a mathematics class what are your goals?
5. What are your educational goals?
6. Think of a time in your study of mathematics when you felt that you did not understand an idea or concept. Tell me about that time.
    a. How did you feel about that?
    b. What did you do to try and understand?
    c. Who or what was the most helpful to you during that time?
    d. Did you feel as though you would eventually understand it?
7. Think of a time in your study of mathematics that you felt that you understood an idea or concept. Tell me about that time.
    a. How did you feel about that?
    b. What did you do that helped you understand?
    c. Who or what was the most helpful to you during that time?
8. We have talked a lot about not understanding and understanding. How would you define the word understanding?

9. Task: Now I want you to visualize your process of understanding going from not understanding to understanding and draw what you see on paper.
10. So that I can understand your drawing, I would like you to think of a mathematical concept that you did not understand at first but later did understand.
    a. What was that concept?
    b. Tell me about how you came to understand that concept and explain how your picture illustrates that process.

Protocol for Interview 3: Instructional Phase

1. Think of a time during today's class that you felt that you did not understand the mathematics being presented and tell me about that time.
    a. How did you feel about that?
    b. What did you do to try and understand?
    c. Who or what was the most helpful to you during that time?
2. Think of a time during today's class that you felt that you understood the mathematics being presented and tell me about that time.
    a. How did you feel about that?
    b. What did you do that helped you understand?
    c. Who or what was the most helpful to you during that time?
3. Do you have any questions about either what was presented in class or any homework problems? I would be happy to help.

<u>Mapping Activity</u> (Used during at least one iteration of this interview)

4. Make a map of the concept of the logarithmic function.

Protocol for Interview 4: Postinstructional Phase

1. (Student is given the skills assessment task sheet) I will give you as long as you need to complete the following activity. I realize that you may not know all that is

necessary to do the activity, but do as much as you can. (Student completes skills assessment)

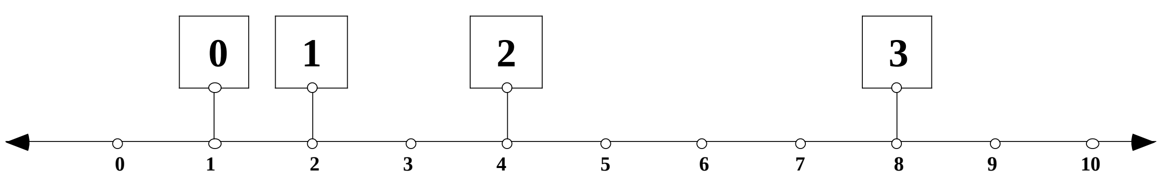
<u>Student's Perception of His or Her Understanding of the Logarithmic Function</u>

2. Think of a time during this activity that you felt that you did not understand the mathematics you were doing and tell me about that time.
   a. How did you feel about that time?
   b. What did you do to try and understand?
3. Think of a time during this activity that you felt that you did understand the mathematics you were doing and tell me about that time.
   a. How did you feel about that time?
   b. What did you do that helped you understand?

<u>Mapping Activity</u>

4. Make a map of the concept *logarithmic function.*

Protocol for Interview 5: Postinstructional Phase



1. Suppose that the numbers in the squares above the number line are signs.

   a. What sign do you think will be above the number 64?
   b. What sign will be above the number 256?
   c. What sign will be above the number $\frac{1}{2}$?
   d. What sign will be above the number $\sqrt{2}$?
   e. What sign will be above the number 3?

f. Are there any numbers that cannot have signs above them?

2.

a. If a sign had the number 7 on it, what number would be below the sign?

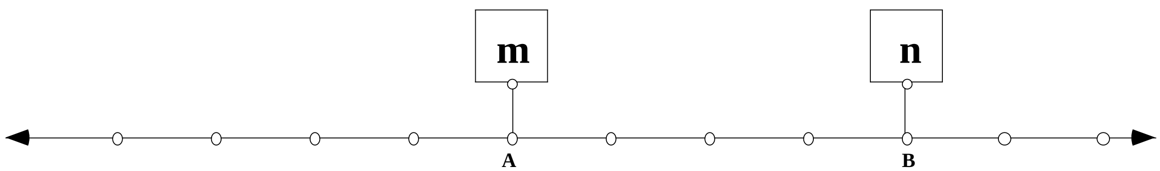
b. If a sign had the number -7 on it, what number would be below the sign?

c. If a sign had the number $\frac{1}{2}$ on it, what number would be below the sign?

d. If a sign had the number on $\frac{3}{4}$ it, what number would be below the sign?

e. If a sign had the number $\sqrt{2}$ on it, what number would be below the sign?

f. Are there any numbers that cannot be on signs? Why or why not.



1. Suppose *A* and *B* are two numbers on the number line. If the sign above *A* has an *m* on it and the sign above *B* has an *n* on it, what would the sign above the number *AB* have on it?

   a. What would the sign above the number $\frac{B}{A}$ have on it?

   b. What would the sign above the number $\frac{A}{B}$ have on it?

2. What is the best way to organize and display all the data that you generated in problems 1 - 3? If you were making a bulletin board for this data, what would you put on it?

3. Write down everything that you know about the relationship between the signs and the numbers on the number line.

<u>Student's Perception of His or Her Understanding of the Logarithmic Function</u>

4. Think of a time during this activity that you felt that you did not understand the mathematics you were doing and tell me about that time.
   c. How did you feel about that time?
   d. What did you do to try and understand?
5. Think of a time during this activity that you felt that you did understand the mathematics you were doing and tell me about that time.
   a. How did you feel about that time?
   b. What did you do that helped you understand?

Protocol for Interview 6: Postinstructional Phase

1. Suppose there is a function f such that $f(AB) = f(A) + f(B)$ and $f(2) = 1$,

   a. What is $f(4)$?

   b. What is $f(8)$?

   c. What is $f(16)$?

   d. What is $f(256)$?

   e. What is $f\left(\frac{1}{2}\right)$?

   f. What is $f\left(\frac{1}{8}\right)$?

   g. What is $f\left(\frac{1}{256}\right)$?

h. What is $f\left(\sqrt{2}\right)$?

i. What is $f\left(\sqrt[4]{2}\right)$?

j. What is $f(0)$?

k. What is $f(-4)$?

l. What is $f(3)$?

m. What is $f\left(\frac{3}{2}\right)$?

2. What is the best way to organize and display all the data that you generated in Problem 1? If you were trying to display the information on a bulletin board what would you include?
3. Write down everything that you know about the function *f*.

Student's Perception of His or Her Understanding of the Logarithmic Function

4. Think of a time during this activity that you felt that you did not understand the mathematics you were doing and tell me about that time.
   a. How did you feel about that time?
   b. What did you do to try and understand?
5. Think of a time during this activity that you felt that you did understand the mathematics you were doing and tell me about that time.
   a. How did you feel about that time?
   b. What did you do that helped you understand?

Protocol for Interview 7: Postinstructional Phase

1. Consider the following table of values where the values in the second column are approximations of the logarithms of the values in the first column and the values in the fourth column are approximations of the logarithms of values in the third column.

x	*y*	*x*	*y*
1	0	10	1
2		20	
3	.477	30	
4			
5	.699		
6			
7	.845		
8	.903		
9	.954		

   a. Complete the table.

   b. Find log 9000 using the table.

   c. Find log 0.09 using the table.

   d. Find $\log\left(\frac{5}{8}\right)$ using the table.

2. The following log table is a base 3 table:

x	*y*	*x*	*y*
1		10	
2	.631	11	
3		12	
4		13	
5		14	
6		15	
7		16	
8		17	
9		18	

   a. Complete the table.

   b. What other information is needed to complete the table?

   c. What other ways are there to represent the data in this table?

   d. Can we use any other representations to help us fill in the table?

e. What is the best way to represent the data in the table?

Student's Perception of His or Her Understanding of the Logarithmic Function

3. Think of a time during this activity that you felt that you did not understand the mathematics you were doing and tell me about that time.
    a. How did you feel about that time?
    b. What did you do to try and understand?
4. Think of a time during this activity that you felt that you did understand the mathematics you were doing and tell me about that time.
    a. How did you feel about that time?
    b. What did you do that helped you understand?

Protocol for Interview 8: Postinstructional Phase

1. Today I want to talk to you about logarithmic functions. Pretend that I am a new student studying college algebra. I already know about functions but have not yet encountered the logarithmic function. Assuming I want to understand logarithmic functions, what would you tell me about the function?
    a. Explain any special properties of the function.
    b. Explain how would you illustrate the properties that you mentioned.
    c. Explain how you might represent the function.
    d. Explain how the function is applied.

Mapping Activity

2. Make a map of the concept *logarithmic function.*

Protocol for Interview 9: Postinstructional Phase

Student's Perspective of His or Her Understanding

During interview 8 the student is presented with and is asked to read excerpts taken from interviews 2, 3, 4, 5, and 6.

1. Think back on your experiences with the logarithmic function both in class and in our interviews. Think of a time when you did not understand something about the logarithmic function and tell me about that time.
2. Think of a time when you understood something about the logarithmic function and tell me about that time.
3. Visualize what you see as the process that you went through to try and understand logarithms. Now draw a picture of that process.
4. Explain your drawing.
5. How does your drawing relate to the summaries of from the interviews that I gave you to read?

<u>Analysis of Maps</u>

6. Compare and contrast the maps that you drew to represent your concept of a logarithmic function.
7. Give me an example of how your understanding of the logarithmic function has changed since we began these interviews in November.
8. Can we see that change by looking at your maps?

APPENDIX B: INTERVIEW PROTOCOL ACTS OF UNDERSTANDING

1. Think of a time in your study of mathematics when you did not understand a mathematical concept or idea and tell me about that time.
2. Think of a time in your study of mathematics when you did understand a mathematical concept or idea and tell me about that time.
3. We've talked about understanding and not understanding. Now think of your process of understanding and draw a picture of that process.
4. Using an example of a concept you did not initially understand, but later did explain your drawing.
5. Is there anything else you can add that will help me get a better idea of your experience of understanding mathematical concepts?

APPENDIX C: CALL FOR ASSISTANCE

CALL FOR ASSISTANCE

During the next three months an in-depth study of college algebra students' understanding will be conducted. This paper is a call for College Algebra students to participate in this investigation.

Each participant will be required to meet with an interviewer for 9-12 interviews. The first two interviews will be conducted during the second and third week in November, at least one and up to four interviews will be conducted following my observation of your College Algebra class, and the remaining six interviews will be conducted during the third and fourth weeks of January. The interviews will each last no more than 90 minutes. Participation (or non-participation) will not directly affect your grade in the course and you may of course terminate your participation in the study at any time during the investigation. All responses made by you, written or oral, will remain completely anonymous unless you request otherwise in a written statement. The tasks done in the interviews are related to the material of College Algebra. I have had many years of experience as a teacher of mathematics. At the end of each session you may ask me specific questions about the material in your College Algebra course. Your participation in the study will provide you with an opportunity to reflect on your own process of understanding and to develop an awareness of how your own understanding of mathematics develops. In addition, if you agree to participate and complete all the interviews you will receive $150 for your participation.

Unfortunately, due to time constraints, only six students can be used in the study. If you would like to be included as one of the participants in the study, please sign your name in the appropriate space below. Participants will be contacted within the next few days to set up an initial meeting time.

Thank you for your cooperation,

Signe E. Kastberg
Principle Investigator

I do not wish to participate in the study.

____

I would like to set up an initial interview.

____

Local Phone # ______________________

e-mail _____________________________