TEACHING THROUGH PROBLEM SOLVING: PRACTICES OF FOUR HIGH SCHOOL MATHEMATICS TEACHERS

by

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(Under the Direction of Andrew Izsak and Jeremy Kilpatrick)

ABSTRACT

If problem solving is what mathematics is all about, then mathematics teachers should be in the business of helping students develop their problem-solving abilities. One way to help is to teach mathematics through problem solving. In this approach, problems are a means by which students learn new mathematical concepts and synthesize mathematical knowledge. During the last few decades, mathematics education researchers have called for studies that focus on the role of the teacher in problem-solving instruction. The purpose of the present study was to investigate the teaching practices used by those who teach through problem solving. Four high school mathematics teachers participated in the study. Although each teacher was unique, five common practices emerged: (a) teaching problem-solving strategies, (b) modeling problem solving, (c) limiting teacher input, (d) promoting metacognition, and (e) highlighting multiple solutions. These practices were consistent with the advice given by mathematics education experts.

INDEX WORDS: problem solving, teaching through problem solving, teaching practices, metacognition

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A Dissertation Submitted to the Graduate Faculty of The University of Georgia in Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

2011

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May 2011

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CHAPTER 1

BACKGROUND AND RATIONALE

Mathematicians and mathematics education researchers have long claimed that problem solving is the essence of mathematics. Wilson, Fernandez, and Hadaway (1993) expressed a widespread belief when they said, “The art of problem solving is the heart of mathematics” (p. 66). There is consensus among mathematics education researchers that problem solving is fundamental not only to doing mathematics but also to teaching and learning mathematics (e.g., Lester & Charles, 2003; National Council of Teachers of Mathematics [NCTM], 1980, 1989, 2000; Schoen, 2003). In its *Principles and Standards for School Mathematics*, the NCTM (2000) stated, “Problem solving is the cornerstone of school mathematics. Without the ability to solve problems, the usefulness and power of mathematical ideas, knowledge, and skills are severely limited” (p. 182). In Polya’s (1965) view, “one of the principal aims of the high school mathematics curriculum is to develop the students’ ability to solve problems” (p. 100). According to the NCTM, “The goal of school mathematics should be for all students to become increasingly able and willing to engage with and solve problems” (p. 182).

If one agrees that problem solving is central to teaching mathematics, then it is natural to ask how teachers can help students develop as mathematical problem solvers. Can problem solving be taught directly, for example, by teaching particular problem-solving strategies? Should teachers teach mathematics in order to help students become better problem solvers, or should they teach problem solving in order to help students become better mathematicians? Are these goals mutually exclusive? Is there a difference between teaching problem solving and teaching *through* problem solving?

Definitions of Problems and Problem Solving

Through the years, mathematicians and mathematics education researchers have offered many definitions of *problem* and *problem solving*. Some differences among definitions reflect different opinions about what constitutes a problem. Others simply reflect different ways of expressing compatible ideas about what is important in problem solving. For example, Polya (1962) described problem solving as “finding a way out of a difficulty, a way around an obstacle, attaining an aim which was not immediately attainable” (p. v). Polya expressed his perspective on what constitutes a problem and problem solving as follows:

> Getting food is usually no problem in modern life. If I get hungry at home, I grab something in the refrigerator, and I go to a coffeeshop or some other shop if I am in town. It is a different matter, however, when the refrigerator is empty or I happen to be in town without money; in such a case, getting food becomes a problem. In general, a desire may or may not lead to a problem. If the desire brings to mind immediately, without any difficulty, some obvious action that is likely to attain the desired object, there is no problem. If, however, no such action occurs to me, there is a problem. Thus, to have a problem means: *to search consciously for some action appropriate to attain a clearly conceived, but not immediately attainable, aim*. To solve a problem means to find such action. … Some degree of difficulty belongs to the very notion of a problem: where there is no difficulty, there is no problem. (p. 117, emphasis in original)

Elsewhere, Polya (1962) specified this broad conception of problems and problem solving in terms of mathematics: “Our knowledge about any subject consists of information and know-how. … What is know-how in mathematics? The ability to solve problems—not merely routine problems but problems requiring some degree of independence, judgment, originality, creativity” (p. vii–viii).

Schoenfeld (1988) distinguished between mathematical tasks that are *problems* and those that are *exercises*. He claimed that both are important but that students in many high school mathematics classrooms engage primarily in completing exercises and rarely, if ever, are challenged to solve problems. A problem, in this sense, is a task for which the method of solution is not immediately obvious, and which is likely to take more than just a minute or two.

Mayer (1985) described problems and problem solving as follows:

> A problem occurs when you are confronted with a given situation—let's call that the *given state*—and you want another situation—let's call that the *goal state*—but there is no obvious way of accomplishing your goal. … Problem solving refers to the process of moving from the given state to the goal state of a problem. (pp. 123–124, italics in original)

Like Polya's description above, Mayer's definition applies to problem solving in general and is not unique to mathematics.

Schoenfeld (1992), in his review of the literature on problem solving, noted the broad range of definitions used in discussions of mathematical problem solving. The term *problem solving* "has been used with multiple meanings that range from 'working rote exercises' to 'doing mathematics as a professional'" (p. 334).

The NCTM (2000) offered a definition of problem solving similar to those above but applied it specifically to mathematics. In the *Standards*, the NCTM defined problem solving using different phrasing at different points, but the following is representative:

> Problem solving means engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings. (p. 52)

The following summary of the descriptions of problem solving proposed by Polya, Schoenfeld, the NCTM, and others highlights the ideas most relevant to the present study. I use the following definition of *mathematical problem solving* in this dissertation: Mathematical problem solving is a nonsequential process that involves creativity and the application of mathematical knowledge—resources, strategies, and so on—to solve a nonroutine task for which a solution method is not immediately known. It is important to note that something could be a problem for one person and not for another, and that once a problem is solved it is no longer a problem for the person who solved it. Kilpatrick (1985) noted, “To be a problem, it has to be a problem for *someone*” (p. 2, emphasis added) and also, “Researchers in mathematics education have long accepted the truth that a problem for you today may not be one for me today or for you tomorrow” (p. 3).

A problem must present a challenge to the solver. Recall Polya’s (1962) comment: “Where there is no difficulty, there is no problem” (p. 117). According to Hiebert and Wearne (2003), a problem should be difficult, but not *too* difficult: “Allowing mathematics to be problematic does not mean making mathematics unnecessarily difficult, but it does mean allowing students to wrestle with what is mathematically challenging” (p. 6).

Problems can be open-ended or have a single answer, but every problem has multiple solution paths. Problems can either be set in a “real world” context or have virtually no direct application to the world outside pure mathematics. A problem may be stated in a single sentence or involve an elaborate description. Some problems may be solved quickly, whereas others may take hours, days, or even longer. In fact, some problems cannot be solved at all. The following is one of hundreds of examples from Polya’s (1962) *Mathematical Discovery*: “Construct a parallelogram, being given one side and two diagonals” (p. 17). Such a problem has a single answer in that there is only one such parallelogram, but there are many ways to complete the geometric construction. Although there may be applications of this problem to “real world” situations, the problem as written is set in a strictly mathematical context. An example to contrast with Polya’s parallelogram problem is the following open-ended problem for which there are many correct answers:

> Try to make every number between 0 and 20 using only four 4's and any mathematical operation (such as multiplication, division, addition, subtraction, raising to a power, or finding a square root), with all four 4's being used each time. For example
>
> $$5 = \sqrt{4} + \sqrt{4} + \frac{4}{4}$$
>
> How many of the numbers between 0 and 20 can be found? (Boaler, 2008, p. 235)

Teaching Through Problem Solving

Teaching through problem solving is an instructional approach in which teachers use problem solving as a primary means to teach mathematical concepts and help students synthesize their mathematical knowledge. In this dissertation, I use the term *instructional approach*—or *teaching approach*—to denote a set of teaching practices with which a teacher implements his or her philosophy of teaching. I use the term *teaching practice*—or simply *practice*—to denote a specific technique a teacher uses. Teaching through problem solving is based on the premise that "students develop, extend, and enrich their understandings by solving problems" (Hiebert & Wearne, 2003, p. 5). There are two main goals of teaching through problem solving: (a) for students to grow in their mathematical understanding and (b) for students to become better problem solvers. Some proponents of teaching through problem solving favor one of these goals over the other, but the goals can be achieved simultaneously.

In the introduction to *Teaching Mathematics Through Problem Solving: Grades 6–12*, Schoen (2003) described this instructional approach:

> As students attempt to solve rich problem tasks, they come to understand the mathematical concepts and methods involved, become more adept at mathematical problem solving, and develop mathematical habits of mind that are useful ways to think about any mathematical situation. (p. xi)

In this description, both mathematical understanding and increased problem-solving ability are desired outcomes. The ability to solve problems is not simply a skill, like factoring polynomials or taking a derivative, but incorporates creativity, intuition, and other habits of mind.

Whether teaching problem solving directly or teaching mathematics through problem solving, the underlying philosophy—that students should be doing the mathematical work themselves and gaining expertise in solving problems—is basically the same. The goals, too, are similar. In both approaches teachers strive to help strengthen students' proficiency in solving mathematical problems. In contrast to teaching problem solving explicitly, however, the ultimate aim of teaching *through* problem solving is that students will be able not only to solve more and harder problems, but also to deepen their mathematical understanding. That is not to say that mathematical understanding is not a goal of those who advocate explicit problem-solving instruction, but the focus of these two approaches is slightly different.

The difference between teaching problem solving and teaching *through* problem solving is subtle because these instructional approaches arise out of similar goals. Nevertheless, Stein and colleagues (2003) made a distinction between the approaches in their review of research on teaching through problem solving. Stein and colleagues noted that although there is copious research on problem solving and problem-solving instruction, particularly from the 1980s, "very little of this vast research base has explicitly investigated [teaching through problem solving]" (p. 246). They went on to state specific differences between research from the 1980s and research on teaching through problem solving:

> Research on mathematics problem solving has generally been prompted by a desire to understand the nature of problem solving as a process and to specify some ways to help students acquire proficiency as problem solvers. The instructional implications of much of this research directly pertain to the teaching *of* problem solving and teaching *about* problem solving. Yet the chapters in this book propose ideas related to teaching *through* problem solving. Although this perspective is not new in mathematics education literature (see, e.g., Branca, 1980; Silver, Kilpatrick, & Schlesinger, 1989; Wilson et al., 1993; Wirtz, 1976), far less research has been conducted from this perspective. (p. 246, italics in original)

Roles of Problem Solving in Mathematics Instruction

To highlight what is unique about teaching through problem solving, I describe various roles problem solving can play in the mathematics classroom. Stanic and Kilpatrick (1988) summarized historically held views of problem solving in mathematics education by describing three perspectives: (a) problem solving as context, (b) problem solving as skill, and (c) problem solving as art.

Problem solving as context refers to a perspective in which problem solving is a means to an end. The end—or goal—of problem solving varies depending on a teacher's objective. First, a teacher may use problem solving as a means to persuade students of the usefulness of mathematics, for example by highlighting real world problems. Stanic and Kilpatrick (1988) called this emphasis "problem solving as justification." Second, a teacher may want to gain student interest; in other words to use "problem solving as motivation." Third, a teacher may use problem solving to provide students with a fun experience; this emphasis is "problem solving as recreation." Fourth, teachers may use problem solving to teach new mathematical content; in other words "problem solving as vehicle." Finally, a teacher's objective may be for students to apply mathematical content they already know; in other words, to use "problem solving as practice" (pp. 13–14).

Problem solving as skill is a theme in which problem solving is seen "as a valuable curriculum end deserving special attention, rather than as simply a means to achieve other ends or an inevitable outcome of the study of mathematics" (Stanic & Kilpatrick, 1988, p. 15). Stanic and Kilpatrick noted that in this view problem solving might be viewed as a high-level skill that students practice only after they have mastered lower-level skills such as solving routine exercises. A potential negative result of this view is that solving nonroutine problems is available only to certain students.

Problem solving as art is the perspective Stanic and Kilpatrick (1988) claimed is most in line with Polya's notion of problem solving. In this view, problem solving involves creativity, reasoning, and discovery of mathematical truth. Stanic and Kilpatrick warned that "problem solving as art gets reduced to problem solving as skill when attempts are made to implement Polya's ideas by focusing on his steps and putting them in textbooks" (p. 17). Furthermore, Stanic and Kilpatrick acknowledged that problem solving as art is a difficult theme for teachers to put into practice.

Schroeder and Lester (1989) made distinctions similar to those Stanic and Kilpatrick (1988) made but described the possible roles of problem solving slightly differently. In their chapter, "Developing Understanding in Mathematics via Problem Solving," Schroeder and Lester identified three ways to include problem solving in school mathematics instruction: (a) teaching *about* problem solving, (b) teaching *for* problem solving, and (c) teaching *via* problem solving. They argued that although the three approaches are not mutually exclusive, a focus on teaching *via* problem solving is most in line with the goal of promoting conceptual understanding in mathematics. When teaching *about* problem solving or teaching *for* problem solving, a teacher runs the risk of making problem solving, rather than mathematical understanding, the primary goal of instruction. An additional danger of teaching only *about* problem solving is that students may see problem solving as an isolated skill.

The similarities between Stanic and Kilpatrick's (1988) list of perspectives on problem solving and Schroeder and Lester's (1989) list of roles of problem solving are clear. Teaching through problem solving aligns with teaching *via* problem solving in Schroeder and Lester's language and with teaching new mathematical content within a *problem solving as vehicle* perspective as Stanic and Kilpatrick described. The NCTM (2000) claimed that problem solving can and should be: (a) a vehicle—or means—for gaining new mathematical knowledge, (b) an opportunity to use mathematical knowledge, and (c) a goal of mathematics instruction. The NCTM stated, "Solving problems is not only a *goal of learning mathematics* but also a *major means of doing so*," and "good problems give students the chance to *solidify and extend what they know*" (p. 52, emphasis added).

Is Teaching Through Problem Solving the Same As a Standards-Based Instructional Approach?

Teaching through problem solving is not synonymous with *standards-based* instruction. Mathematics teachers with a standards-based instructional approach seek to teach in a way that is consistent with the recommendations of the NCTM (1989, 2000) *Standards*. Such teachers focus on conceptual understanding and seek to help students build knowledge on what they have previously learned in a way that is meaningful to the students. Certainly a philosophy of teaching that emphasizes conceptual understanding and meaningful learning is the basis for teaching through problem solving, but teaching through problem solving is only one of several approaches that teachers who follow the recommendations of the NCTM *Standards* may employ. The *Standards* cover a wide range of issues including reasoning, communication, connections, and so on, only one of which is problem solving.

The following is an example to help clarify the distinction. Suppose a class is studying the formula for the area of a circle. A teacher with a standards-based approach might lead the class through an activity in which students cut a paper circle into narrow sectors and arrange the sectors in such a way that the shape resembles a rectangle. Students may then use their knowledge of the area of a rectangle to derive the formula for the area of a circle. Using this activity is a way to guide students into meaningful learning rather than simply to have them memorize the formula for the area of a circle. This activity does not present students with a problem to solve, however, particularly if the teacher leads them through the steps of the activity. Therefore, the circle activity is *not* an example of teaching through problem solving although standards-based principles are at work.

Another term that is commonly used to describe a teaching approach that emphasizes problem solving is *problem-centered* (e.g., Lester, 1994). Also, there are instructional approaches that are closely aligned with teaching through problem solving, including an *activity-based* approach, but a teacher may effectively use activities without necessarily creating a problem-solving environment. For example, if a teacher or a set of instructions on a worksheet guides students step-by-step through an activity, the students have not necessarily encountered a problem-solving situation.

Why Teach Mathematics Through Problem Solving?

Mathematics educators have long assumed that problem solving and mathematical understanding are linked. One of my assumptions in this study was that teaching through problem solving is a particularly effective way to help students gain understanding of mathematical concepts. Teaching through problem solving has other advantages as well. For example, it can: (a) increase students' interest in and enjoyment of mathematics (Kahan & Wyberg, 2003; Lambdin, 2003), (b) help students develop mathematical habits of mind (Levasseur & Cuoco, 2003), and (c) demonstrate the usefulness of mathematics for solving a wide range of problems (NCTM, 2000). Mathematics educators offering their expertise and advice, as well as those who have researched problem-solving instruction, have highlighted these advantages.

The most prevalent argument for teaching through problem solving is that it aids mathematical understanding. Hiebert and Wearne (2003) claimed, "Students develop, extend, and enrich their understandings by solving problems" (p. 5). Elsewhere Hiebert et al. (1996) justified "the practice of problematizing the subject by claiming that it is this activity that most likely leads to the construction of understanding" (p. 15). Lambdin (2003) noted, "A primary tenet of teaching through problem solving is that individuals confronted with honest-to-goodness problems are forced into a state of needing to connect what they know with the problem at hand" (p. 7). Throughout the *Standards*, the NCTM (2000) asserted a link between problem solving and mathematical understanding:

> Problems and problem solving play an essential role in students' learning of mathematical content and in helping students make connections across mathematical content areas. … Accordingly, much of the mathematics that students encounter can be introduced by posing interesting problems on which students can make legitimate progress. (p. 334)

Not only is problem solving a means by which students can deepen their mathematical understanding, but also many students enjoy solving problems. Lambdin (2003) claimed, "Learning mathematics through problem solving is engaging and rewarding" (p. 11). When students are the ones doing the mathematics, they can see mathematics as meaningful. Many students find problem solving to be more enjoyable than rote memorizing or learning only by watching and listening to the teacher. That enjoyment, combined with the likelihood that students will understand and retain mathematical concepts, points to the value of teaching and learning through problem solving.

Another advantage of teaching through problem solving is that students develop mathematical habits of mind. Levasseur and Cuoco (2003) noted, "Students develop these habits of mind as a by-product of learning mathematics through problem solving" (p. 27). Examples of such habits are guessing, challenging a solution by looking back, looking for patterns, analyzing a special case, and representing a problem in various ways.

Finally, teaching through problem solving allows students to appreciate the usefulness of mathematics, whether a problem is set in a "real world" situation or taken from a purely mathematical context. According to the NCTM (2000), "Through problem solving, students can experience the power and utility of mathematics. Problem solving is central to inquiry and application and should be interwoven throughout the mathematics curriculum to provide a context for learning and applying mathematical ideas" (p. 256).

Calls for Research

There have been calls for research on problem-solving instruction for 25 years or more. It seems every 10 years or so, researchers turn their attention back to problem-solving instruction. As an example of the ongoing nature of such calls for research, consider the following statements from mathematics education researchers.

In the mid 1980s, Grouws (1985) and Lester (1985) separately stated the need for research in problem-solving instruction. According to Grouws, "Explicit attention must be given to instruction in order to make progress toward the long-range goal of improving student problem-solving performance" (p. 297). Lester said, "An ultimate goal of research on teaching mathematical problem solving is to develop instructional theories that contain the essence of what it is that a good teacher does" (p. 55).

A decade later, Lester (1994) stated that there was still much more work to be done: "Very little of the literature on mathematical problem-solving instruction discusses the specifics of the teacher's role. … In my view, attention to the teacher's role should be the single most important item on any problem-solving research agenda" (p. 672). He went on to say,

> What actually takes place in problem-centered classrooms? … [Researchers] have noticed an absence of adequate descriptions of what actually happens in the classroom. In particular, there has been a lack of descriptions of teachers' behaviors, teacher-student and student-student interactions, and the type of classroom atmosphere that exists. It is vital that such descriptions be compiled if there is to be any hope of developing sound programs for teaching problem solving. (p. 672)

In the early 2000s, there were still calls for research in problem-solving instruction. Stein and colleagues (2003) argued, "One of the crucial research questions for the next decade [is], 'What happens inside classrooms in which problem-solving approaches are used effectively?'" (p. 250). Stein and colleagues described research done in the 1990s, but most of the studies they cited considered curriculums that supported problem-centered teaching and did not address practices used by teachers who implemented such curriculums. Only a few studies they found addressed what happened in classrooms where teachers were teaching through problem solving.

These calls for research in problem-solving instruction motivated the present study. I became interested in what happens on a day-to-day basis in the classrooms of teachers who believe that problem solving is central to mathematics teaching and learning. Specifically, I found the shift from teaching problem solving to teaching *through* problem solving to be intriguing, and I wanted to know what practices teachers use to implement this instructional approach.

Purpose of the Study

The primary purpose of the present study was to identify the practices used by four high school mathematics teachers who taught through problem solving. The following research questions refer to a select group of high school mathematics teachers who have a strong reputation for effective teaching and have been identified as those who teach through problem solving:

1. What do these teachers believe about mathematical problem solving?
2. What do these teachers believe about what makes a good problem, and what kinds of mathematical problems do they pose?
3. What practices do these teachers use when teaching through problem solving, and how do they implement those practices?
    a. What problem-solving strategies do teachers introduce?
    b. How do teachers model mathematical problem solving?
    c. What do teachers do to limit the amount of input they give students? For example, how do teachers incorporate group work?
    d. How do teachers encourage metacognition?
    e. How do teachers highlight multiple solutions?

I interviewed four high school mathematics teachers to investigate what they believed about mathematical problem solving, problem-solving instruction, and what makes a good problem. In addition, I conducted classroom observations to investigate practices used by the teachers. On two—or, in the case of one teacher, three—separate occasions, I observed instruction every day for a week to get a sense of their typical behavior and to see the flow of mathematics lessons from one day to the next.

Various publications of the NCTM in the last 10 years (e.g., Lester, 2003; NCTM, 2000; Schoen, 2003) have highlighted the importance of problem solving and have advocated teaching through problem solving, but these publications are not research studies. They are compilations of advice from experts about how and why to teach through problem solving and anecdotes from classroom teachers about successful lessons centered on good problems (e.g., Barrett & Compton, 2003; Bellman, 2003). Advice and examples provide general—and helpful—descriptions of teachers' practices, but they are not in-depth investigations of teachers who successfully teach through problem solving. The present study can contribute to the mathematics education literature by providing detailed descriptions of practices used by four teachers. In contrast to stories of isolated lessons, the present study includes observations of teachers over several days to provide a more informed picture of their practices for teaching through problem solving.

CHAPTER 2

LITERATURE REVIEW

This chapter begins with an overview of two frameworks for mathematical problem solving. The first is a problem-solving model developed by Polya (1957) that includes four phases of problem solving: understanding the problem, devising a plan, carrying out the plan, and looking back. The second framework I describe is Schoenfeld's (1985, 1992) framework for mathematical problem solving. He suggested that successful problem solving depends on four main elements: resources, strategies, control, and beliefs. This description of frameworks is followed by a brief discussion of the importance of metacognition as it relates to problem solving. I end the section by explaining that I used Schoenfeld's framework as the theoretical framework for the study. Specifically, I used the framework to connect elements of successful problem solving to practices for teaching through problem solving.

In the second section of the chapter, I describe the history of research in problem solving and problem-solving instruction. I have divided the history into two parts: research from 1960 to 1985 and research from 1985 to 2010. This description is followed by a summary of seven practices for problem-solving instruction that occur as common themes in the mathematics education literature.

The third section of the chapter contains a description of research on the benefits of a problem-based approach to instruction. This description includes studies of problem-based curriculum and studies of problem-based instruction. Following this description I note that mathematics education experts have offered lots of advice on teaching through problem solving, but research is scarce. I briefly describe the Japanese model for teaching through problem solving and conclude the chapter by reiterating calls for research on teaching through problem solving.

Frameworks for Problem Solving

Polya's Problem-Solving Model

Polya (1957) described four phases of problem solving: understanding the problem, devising a plan, carrying out the plan, and looking back. Many people have interpreted the use of these elements as a sequential process. There are numerous mathematics textbooks that reduce problem solving to a four-step procedure (e.g., Bennett, 2007; Larson, Kanold, & Stiff, 1993; Smith, Charles, Keedy, Bittinger, & Orfan, 1988), but as Wilson and colleagues (1993) observed, one can better interpret Polya's work by considering problem solving as a dynamic and cyclic process in which problem solvers move among the phases as they work through a problem. For example, one may find that devising a plan helps in understanding the problem or that looking back leads to better ways to solve the problem. Polya (1957) explained this nonsequential process:

> Trying to find the solution, we may repeatedly change our point of view, our way of looking at the problem. … Our conception of the problem is likely to be rather incomplete when we start the work; our outlook is different when we have made some progress; it is again different when we have almost obtained the solution. (p. 5)

Even though the phases are nonsequential and a problem solver does not necessarily leave one phase before entering another, it is helpful to describe each phase individually.

Understanding the problem requires identifying the unknown, the data, and the conditions of the problem. As Polya (1957) stated, "It is foolish to answer a question that you do not understand" (p. 6). He also noted that a problem solver must be motivated to solve the problem, and that teachers can motivate students by choosing good and interesting problems that are at the right difficulty level.

One must have at least a cursory understanding of the problem in order to come up with a plan of attack. *Devising a plan* is often the crux of the solution process. In fact, according to Polya (1957), "the main achievement in the solution of a problem is to conceive the idea of a plan" (p. 8). The plan may develop slowly or dawn on the problem solver rather suddenly. One may devise a plan by comparing the problem to a previously solved problem or by solving a simpler—or similar—problem.

Carrying out the plan is the next phase Polya (1957) described. He compared devising a plan to carrying it out:

> To devise a plan, to conceive the idea of the solution is not easy. It takes so much to succeed; formerly acquired knowledge, good mental habits, concentration upon the purpose, and one more thing: good luck. To carry out the plan is much easier; what we need is mainly patience. (p. 12)

Patience is required not only for carrying out the plan, but also for making necessary adjustments to the plan or even abandoning the plan altogether and devising a new plan.

As good problem solvers work on a problem, they pay attention to their solution process both during and after they have solved the problem. *Looking back* includes checking the answer, but it is much more than that. It involves reviewing both the problem and the solution; looking for other solution methods; considering extensions, connections, and related problems; and reflecting on one's solution process. Polya (1962) claimed, "The best time to think about methods may be when the reader has finished solving a problem" (p. xii). It is particularly difficult to motivate students to look back after solving a problem, but according to Polya (1957), "a good teacher should understand and impress on his students the view that no problem whatever is completely exhausted" (p. 15).

Polya's phases provide a helpful framework for looking at problem solving, but they constitute only part of his contribution. A distinctive feature of Polya's conception of problem solving is the notion of *heuristic*. The bulk of *How to Solve It* (1957) is a section titled "Short Dictionary of Heuristic" in which Polya identified strategies that apply to a wide range of problems. He described heuristic as the "study of methods of solution" (p. vii). More recently, some mathematics educators have considered heuristic strategies, or simply *heuristics*, to be synonymous with problem-solving strategies, whereas others describe heuristics as being contained in a larger set of problem-solving strategies. Schoenfeld (1987b) described heuristic strategies this way:

> Heuristic strategies are rules of thumb for making progress on difficult problems. There are, for example, heuristic strategies for understanding a problem (focusing on the unknown, on the data, drawing a diagram, etc.), for devising a plan (exploiting related problems, analogous problems, working backward, etc.), and for carrying out and checking a solution. (p. 284)

In this dissertation, I use the terms *heuristic* and *heuristic strategy* to refer to items such as those Polya suggested in *How to Solve It* (e.g., add auxiliary lines to a geometric figure, solve a related problem, examine a special case, or work backwards), and I use *problem-solving strategy* to denote elements of a larger set of both general and specific strategies for problem solving. For example, there are specific strategies that a problem solver can use to perform tasks such as simplifying an algebraic expression, and there are general strategies that a problem solver can use with any problem. An example of a general strategy is using intuition to make a conjecture.

Schoenfeld's Framework for Problem Solving

Several decades after Polya's work, Schoenfeld (1985, 1992) developed a framework for mathematical problem solving that built on Polya's framework and added to our understanding of what it means to be a good problem solver. He suggested phases of problem solving akin to Polya's: read, analyze, explore, plan, implement, and verify. In Schoenfeld's description, as in Polya's, a good problem solver moves among the phases in a nonsequential fashion. Schoenfeld

(1985) employed a think-aloud protocol to gain insight into problem solvers' decision making. He devised a method for charting the progress of problem solvers as they work on a problem and noted that novice problem solvers moved in one direction, from reading the problem statement to implementing a strategy, without considering whether the strategy was leading to a solution. On the other hand, expert problem solvers analyzed the problem before implementing a strategy and then moved back and forth among the different phases of problem solving (see also Schoenfeld, 1992).

For someone to be a successful problem solver, a number of elements must be in place. In Schoenfeld's (1985, 1992) framework, these elements are resources, problem-solving strategies (including heuristics), control, and beliefs and affects. A set of *resources*—or the *knowledge base*—refers to mathematical knowledge at a problem solver's disposal. Resources include facts, concepts, algorithms, and routine procedures. Schoenfeld (1992) made a distinction between algorithms and routine procedures, noting that algorithms are guaranteed to work, whereas "routine procedures are likely to work, but with no guarantees" (p. 350). For example, the long division algorithm for dividing polynomials is guaranteed to work if one follows the steps correctly. As an example of a "routine [procedure that is] likely to work, but with no guarantees," Schoenfeld described a common strategy for proving elements of a geometric figure are congruent: First show that the elements are corresponding parts of congruent triangles. He noted that this strategy is one of several "proof techniques [that are] not algorithmic, but they are somewhat routine" (p. 350).

Mathematical knowledge alone is not enough to make someone a good problem solver; problem-solving *strategies* are necessary in order to help problem solvers use their resources effectively and efficiently. For example, suppose a problem requires one to calculate the area of an irregular shape. A problem solver needs resources such as the area formulas for rectangles, triangles, or other figures, but the solver also needs strategies in order to make use of these resources. One potential strategy is to first divide the irregular shape into familiar shapes whose area formulas are known, then add the areas of the individual shapes in order to determine the area of the irregular shape.

The third component of problem solving Schoenfeld mentioned is *control*. Control falls under the category of *metacognition*, a broad term that includes knowledge of one's own cognition, monitoring—or control—of cognitive processes, and reflection. Schoenfeld (1985) described control as "resource allocation during problem-solving performance" (p. 143). More specifically, control involves deciding what resources may be useful, identifying what strategies will provide an efficient way to solve a problem, "recovering from inappropriate choices," and monitoring one's progress while solving a problem (p. 99).

Kilpatrick (1985) also noted the importance of resources, strategies, and control: "Successful problem solving in a given domain depends upon the possession of a large store of organized *knowledge* about that domain, *techniques* for representing and transforming the problem, and *metacognitive processes* to monitor and guide performance" (p. 11, italics added).

Finally, *beliefs and affects* refer to "an individual's understandings and feelings that shape the ways that the individual conceptualizes and engages in mathematical behavior" (Schoenfeld, 1992, p. 358). Confidence in one's ability to solve a problem, belief that the problem is worth solving, and conviction that mathematics itself is a sensible and worthwhile endeavor all play a part in successful problem solving. According to Silver (1982), one of the "components of a mathematical belief system which may have important implications for how one approaches mathematical problems [is] the belief that there is usually more than one way to solve a problem" (p. 21). Schoenfeld discussed typical student beliefs that can be a hindrance to the students' ability to be good problem solvers. For example, "Mathematics problems have one and only one right answer," "Students … will be able to solve any assigned problem in 5 minutes or less," and "Mathematics is a solitary activity" are beliefs that have a negative impact on students' ability to solve problems (Schoenfeld, 1992, p. 359; see also Schoenfeld, 1988).

The Importance of Metacognition

Metacognition warrants special attention because of the significant role it plays in problem solving. As Polya implied (1957, 1962, 1965), and as Schoenfeld (1985) stated explicitly, metacognitive behavior can be the difference between success and failure for the problem solver. Simply stated, metacognition is *thinking about thinking*. Evidence of problem solvers' metacognitive behavior includes awareness—not simply use—of some or all of the following: understanding what the problem is asking, choosing a particular strategy to solve the problem, evaluating whether the strategy is leading closer to a solution, and examining whether the answer makes sense.

One could argue that Polya valued metacognition in mathematical problem solving even though he never used the term. According to Silver (1982), "If we adopt a metacognitive perspective, we can view many of Polya's (1957) heuristic suggestions as metacognitive prompts" (p. 21). In Polya's (1962) *looking back* phase, the problem solver

> may ask himself many useful questions: "What was the decisive point? What was the main difficulty? What could I have done better? I failed to see this point: which item of knowledge, which attitude of mind should I have had to see it?" (p. xii)

Metacognition includes not only knowledge of one's own cognition, but also regulation of one's behavior in response to that knowledge (Lester, 1985). This concept is known as self-regulation, which is closely related to control (to use Schoenfeld's term). Schoenfeld (1987a) summed up the notion of self-regulation: “It’s not only what you know, but how you use it (if at all) that matters” (p. 192).

Metacognitive behavior during problem solving was a hot topic during the 1980s (e.g., Campione, Brown, & Connell, 1988; Garofalo & Lester, 1985; Schoenfeld, 1983, 1987a; Silver, 1982), but research on metacognition is no longer prominent in mathematics education. Metacognition does, however, remain a tacit part of the discussion of problem solving. For example, the mathematics education literature is replete with terms such as *monitoring* and *reflection*, and ideas such as self-assessment and knowledge of one’s own cognition. The NCTM (2000) claimed that development of students’ metacognitive abilities is an important part of classroom instruction: “Students … should be encouraged to monitor and assess themselves. Good problem solvers realize what they know and don’t know [and] what they are good at and not so good at” (p. 260).

The following quotation shows the multiple dimensions of metacognition that the NCTM (2000) values in mathematics instruction. These dimensions include reflection, metacognitive questions, and monitoring.

> Reflective skills (called *metacognition*) are much more likely to develop in a classroom environment that supports them. Teachers play an important role in helping to enable the development of these reflective habits of mind by asking questions such as “Before we go on, are we sure we understand this?” “What are our options?” “Do we have a plan?” “Are we making progress or should we reconsider what we are doing?” “Why do we think this is true?” Such questions help students get in the habit of checking their understanding as they go along. … As teachers maintain an environment in which the development of understanding is consistently monitored through reflection, students are more likely to learn to take responsibility for reflecting on their work and make the adjustments necessary when solving problems. (pp. 54–55, italics in original)

Note the emphasis on the teacher’s role in fostering metacognition. Teachers are responsible for creating classroom environments in which they encourage metacognitive behavior and give students opportunities to reflect on their work. Teachers encourage metacognition by modeling metacognitive behavior—for example, by thinking aloud—and by asking metacognitive questions.

Theoretical Framework

Schoenfeld's (1985, 1992) framework for mathematical problem solving summarizes my assumptions about problem solving and served as the theoretical framework for the present study. As Schoenfeld's framework suggests, problem solving is multifaceted. One consequence of this multi-faceted nature is that the teacher's role in helping students develop their problem-solving ability is complex. Because successful problem solving requires mathematical knowledge, problem-solving strategies, metacognitive control, and positive beliefs, it is fitting to investigate actions teachers can take to facilitate their students' development of these aspects of problem solving. These actions include practices for teaching through problem solving that I describe briefly in this chapter, and more thoroughly in chapter 4 where I discuss how the teachers in this study used the practices. In chapter 5, I discuss connections between the elements of Schoenfeld's framework and practices for teaching through problem solving.

History of Research in Problem Solving and Problem-Solving Instruction

An emphasis in mathematics education on problem solving has a long history. Stanic and Kilpatrick (1988) traced the history of problem solving in mathematics education, giving examples of problems from as far back as ancient Egypt and China. They noted that whereas "problems have occupied a central place in the school mathematics curriculum since antiquity, … problem solving has not" (p. 1). It was not until the second half of the 20th century that problem solving came to the forefront of research in mathematics education. In the 1980s in particular, problem solving had the attention of many of those seeking reform in mathematics education. The recommendation of the NCTM (1980) that "problem solving must be the focus of school mathematics in the 1980s" (p. 1) began a decade of research into various aspects of problem solving, from how experts solve problems to effective ways to teach problem solving.

The importance of problem solving in doing, learning, and teaching mathematics was recognized well before the NCTM and others made it a primary focus in the 1980s. Certainly humankind has always faced problems, mathematical and otherwise, and has devised ways to solve them. To understand a modern view of problem solving in the context of teaching and learning mathematics, one must look to the mid 20th century. Most notably, Polya (1957, 1962, 1965) emphasized problem solving in school mathematics and published several books on the topic including the seminal work *How to Solve It* and two volumes of *Mathematical Discovery*. Many years before research on problem solving became popular, Polya described the nature of problems, problem solving, and the teaching of problem solving.

In the sections that follow, I elaborate on some of the specific areas of research on problem-solving instruction. I have divided this review into two main time periods: 1960–1985 and 1985–2010. Enthusiasm in the mathematics education community for problem solving and problem-solving instruction was particularly high in the 1980s. According to Lester (1994), this enthusiasm began to decline in the late 1980s. At the very least, the late 1980s saw a change in the kinds of questions mathematics education researchers were asking about problem solving. For example, in the 1990s many researchers shifted their focus from the role of the teacher to the role of curriculum in helping students become better problem solvers.

Research in Problem-Solving Instruction: 1960–1985

Reflecting on research on problem solving instruction from 1960 to 1985, Kilpatrick (1985) stated, “Research over the past two and a half decades suggests that ‘slowly and with difficulty’ is probably the best answer to the question of how problem solving is learned” (p. 8). Presumably mathematics education research in problem-solving instruction has made progress since the mid 1980s, but reflecting on the research as of 1985, Kilpatrick noted, “We do not have a final vision of what problem solving is and how to teach it, but we are much more keenly aware of the complexity of both” (p. 13).

In his summary of research on teaching mathematical problem solving, Kilpatrick (1985) highlighted various definitions of *problem* and gave examples of how mathematics educators have used problems. He described a spectrum of problem types as Polya (1965) had classified them, from those requiring mere “mechanical application of a rule that has just been presented or discussed” (Kilpatrick, 1985, p. 4) to those requiring “a novel combination of rules or examples … and [requiring] a high degree of independence and the use of plausible reasoning” (p. 4). Some researchers, primarily psychologists, characterized problems as items Kilpatrick described as “straightforward ‘word problems’ … meant to give students an opportunity to apply what they might have learned” (p. 4). These researchers searched for ways to increase students’ abilities to solve those kinds of problems. Other researchers, primarily mathematics educators, did not focus primarily on routine word problems but gave attention to ways of helping students to solve “nonroutine mathematical problems of greater complexity and greater mathematical interest than the ordinary word problem” (p. 5).

Stein, Boaler, and Silver (2003) stated, “The 1970s and 1980s were particularly productive times, with a consolidation of research on the ways problem-solving activities can support students' learning" (p. 245). During those decades, problem solving was a prominent topic of discussion and research, with much of the focus being on expert problem solvers, heuristics training, and metacognition in problem solving (e.g., Charles & Lester, 1984; Garofalo & Lester, 1985; Lester, 1985, Schoenfeld, 1980, 1985, 1987a, 1992; Silver, 1982). Mathematics education experts remain interested in characteristics of successful problem solving, including metacognitive behavior and proficiency with a wide range of heuristic strategies. For example, the NCTM (2000) stated, "A significant part of a teacher's responsibility consists of planning problems that will give students the opportunity to learn important content through their explorations of the problems and to learn and practice a wide range of heuristic strategies" (p. 341).

Heuristics training. Kilpatrick (1985) described researchers' efforts to use what was known about expert problem solving behavior to formulate a plan for instruction. For example, expert problem solvers make use of heuristic strategies (Schoenfeld, 1985). Arcavi, Kessel, Meira, and Smith, (1998) researched Schoenfeld's problem-solving course for undergraduates. They focused on the first two days of class in which Schoenfeld set the tone for the course. A key feature of doing mathematics in the course was the use of heuristics for solving mathematical problems. At the beginning of the course, Schoenfeld provided problems and, after giving the students ample time to work on solutions, introduced particular heuristics that were helpful for solving those problems and that would apply to other problems students would see later in the course. Examples of heuristics Schoenfeld highlighted were *try specific values* (in order to find a pattern, for example), *solve an easier related problem*, and *exploit extreme cases*. Schoenfeld relied heavily on *How to Solve It* (Polya, 1957) as a source of heuristic strategies and good mathematics problems.

Several researchers conducted studies in which they investigated the effects of heuristics training. Results were mixed. Some researchers found that students who were taught heuristics performed better on problem-solving tasks in which those heuristics were useful (e.g., Guernon, 1989). Charles and Lester (1984) found that children's problem solving improved after receiving instruction that emphasized use of heuristic strategies. In other words, some evidence suggests students can learn problem-solving strategies. Other researchers pointed out the limitations of heuristics training. For example, Kilpatrick (1985) noted that heuristics training often reduces a problem-solving strategy to an algorithm or other procedure that "cannot be used with problems to which the algorithm does not apply" (p. 9). That is, knowledge gained from heuristics training may not transfer to novel problem situations. Schoenfeld (1987b) summed up the disappointment he and others felt when they saw the limitations of heuristics training, particularly the limitations of training students to use Polya's dictionary of heuristic strategies:

> Largely because Polya's ideas seemed so right, the math-ed literature was chock full of studies designed to teach problem solving *via* heuristics. Unfortunately, the results—whether in first grade, algebra, calculus, or number theory, to name a few—were all depressingly the same. … Despite all the enthusiasm for the approach, there was no clear evidence that the students had actually learned more as a result of their heuristic instruction, or that they had learned any general problem-solving skills that transferred to novel situations. (pp. 287–288, italics in original)

It is important to note that Schoenfeld's comments were about the limitations of heuristics training and not a criticism of Polya's model. It is also important to note that Schoenfeld had some success teaching heuristic strategies in his problem-solving course (e.g., Arcavi et al., 1998, Schoenfeld, 1985). His comments simply highlight that heuristics training was not as widely successful as many mathematics educators had hoped.

Metacognition training. One reason heuristics training may not have been as effective as researchers hoped was that metacognitive considerations were not taken into account (Lester, 1985). Some research indicates that students can learn metacognitive skills through instruction (Posamentier & Jaye, 2006). For example, Montague (2007) found that students could be taught to effectively use metacognitive techniques such as self-questioning and self-monitoring while solving problems. Schoenfeld (1987b) explained the value of emphasizing metacognition when teaching problem solving:

> There is evidence that when students get coaching in problem solving that includes attention to such things—when they are encouraged to think about issues like "What are you doing? Why are you doing it? How will it help you solve the problem?"—their problem-solving performance can improve dramatically. (p. 290)

Posamentier and Jaye (2006) noted that as students develop metacognitive skills, particularly the use of metacognitive questions, they become more successful problem solvers. Examples of questions teachers can encourage students to ask themselves include the following:

> "What technique did I use to solve a similar problem in the past?" "How do I find the derivative?" "What is the problem asking for?" "What information am I given?" Students should also ask themselves general questions designed to self-regulate their performance, such as, "Is there anything I don't understand?" "Am I headed in the right direction?" … "Have I made any careless mistakes?" (p. 80)

Research in Problem-Solving Instruction: 1985–2010

Following the mid 1980s, there was a lull in research in problem solving and problem-solving instruction. Lester (1994, pp. 667–668) offered a few explanations:

1. "Other issues have drawn attention away from problem solving"—issues such as "beliefs about the nature of mathematics, sociocultural influences on mathematics learning, applications of mathematics, and assessment."
2. "We think we already know all about problem solving."
3. "Constructivism has replaced problem solving as the dominant 'ideology' driving mathematics education research."
4. "Problem solving is even more complex than we once thought."

Lester went on to say that more problem-solving research is necessary. He pointed in particular to the need to investigate the role of the teacher in problem-solving instruction.

Beginning in the mid 1990s many mathematics education researchers (e.g., Harwell et al., 2007; Huntley, Rasmussen, Villarubi, Sangtong, & Fey, 2000; Riordan & Noyce, 2001; Schoenfeld, 2002; Thompson & Senk, 2001) have focused their efforts on investigating the effectiveness of standards-based mathematics curriculums—that is, curriculums that were written in response to the NCTM (1989) *Curriculum and Evaluation Standards for School Mathematics* and, more recently, *Principles and Standards for School Mathematics* (NCTM, 2000). The focus of this research has been on curriculum, much of which is problem-centered, rather than specifically on teaching practices. A few investigations of teacher behavior have occurred in the context of implementation of problem-based curriculums.

Now, several years into the 21st century, some mathematics education researchers have turned their attention to *teaching through problem solving*. Teaching through problem solving is a relatively new area of interest, so research is scarce; advice, however, is plentiful. The NCTM published a two-volume series titled *Teaching Mathematics Through Problem Solving* (Lester & Charles, 2003; Schoen, 2003) that gives descriptions of, and arguments for, teaching through problem solving.

Practices for Problem-Solving Instruction

There are at least seven practices for problem-solving instruction that mathematics education researchers have claimed are important for helping students grow in their problem-solving ability (see Figure 1). I compiled this summary based on the research of Kilpatrick (1985), Lester (1985), Grouws (1985); Polya's (1957, 1962, 1965) advice; and the more recent work of mathematics educators and mathematics education researchers (e.g., Arcavi et al., 1998; Hiebert & Wearne, 2003; Kahan & Wyberg, 2003; Marcus & Fey, 2003; NCTM, 2000; Posamentier & Jaye, 2006; Schoen, 2003; Stein et al., 2003). The following paragraphs contain a review of some of the literature I consulted to compile the list in Figure 1. Although much of the literature I cite was written over 25 years ago, more recent literature confirms the advice researchers have given over several decades.

1. Give lots of problems.
2. Give "good" problems.
3. Teach specific or general problem-solving strategies (including heuristics).
4. Model problem solving.
5. Limit teacher input—for instance, by having students work in small groups.
6. Promote metacognition—for instance, by asking metacognitive questions or encouraging students to be reflective.
7. Highlight multiple solutions.

Figure 1: Seven practices for problem-solving instruction.

Kilpatrick (1985) identified five categories among the many perspectives on how to teach problem solving and noted "most programs of problem-solving instruction combine features of several categories" (p. 9). The five categories are as follows:

1. Osmosis: Give students lots of problems to solve. Kilpatrick commented that solving lots of problems is necessary, but probably not sufficient, for becoming a better problem solver.
2. Memorization: Teach heuristic strategies as procedures or algorithms to be memorized and applied. In Kilpatrick's view, "such approaches can be effective within narrow limits, but they cannot be used with problems to which the algorithm does not apply" (p. 9).
3. Imitation: Model problem solving for students.
4. Cooperation: Have students work in groups to solve problems.
5. Reflection: "Get the problem solver to reflect on his or her progress in problem solving and to assess the effectiveness of the procedures being used" (p. 10). Reflection is one element of metacognitive behavior.

Lester (1985) summarized the research on problem-solving instruction and noted that effective instruction includes an emphasis on the development of general and specific problem-solving strategies, including heuristics. He also claimed that problem-solving instruction should include having students solve lots of problems and giving attention to "the 'guiding forces' of problem solving (i.e., the metacognitive aspects)" (p. 45).

Polya (1962) noted how important it is that students have the opportunity to imitate a teacher who models problem solving. In addition, Polya advocated giving students lots of problems to solve because "imitation and practice" are vital for improving problem-solving ability:

> Solving problems is a practical art, like swimming, or skiing, or playing the piano: You can learn it only by imitation and practice. … If you wish to learn swimming you have to go into the water, and if you wish to become a better problem solver you have to solve problems. (p. v)

Teachers can model problem-solving behavior, including elements of metacognition, by thinking aloud when demonstrating problem solving. Polya (1957) noted that sometimes teachers should act as if they do not know how to solve the problem in order to model authentic problem-solving behavior. More recently, Grouws (2003) claimed that it is important that the teacher model problem solving:

> A teacher must do some acting as he or she solves a task for the class, as Polya (1957) pointed out long ago. … As we all know, solving a mathematics problem involves advances and retreats, and moments of frustration and excitement, to name but a few of the cognitive and emotional components. A teacher should communicate these components to students while demonstrating a solution to a task, teaching students not always to expect a smooth march to a mathematical problem's solution. Accepting that difficulties are normal and that they should not be a cause for distress or quitting is a lesson that can have positive long-term effects. For this reason alone, teachers need to regularly solve problems in front of the class. (pp. 138–139)

In other words, modeling problem solving not only has the positive effect of helping students see how mathematics can be used to solve problems but also shows that perseverance and patience are part of the process.

The descriptions Lester, Kilpatrick, and Polya offered summarize the research and perspectives on problem-solving instruction as of 1985. These themes continue to be evident in more recent literature (e.g., Arcavi et al., 1998; NCTM, 2000; Posamentier & Jaye, 2006; Schoen, 2003; Stein et al., 2003) and are frequently cited as being important in problem-centered instruction. Another theme that is implicit in the lists above and that has been made more explicit in the last 10 years is the importance of giving students engaging and mathematically rich problems (e.g., Hiebert & Wearne, 2003; Kahan & Wyberg, 2003; Marcus & Fey, 2003; NCTM, 2000; Posamentier & Jaye, 2006).

Experts, particularly in the last 15 years or so, have recommended that teachers highlight multiple solutions (e.g., Grouws, 2003; Hiebert et al., 1996; Hiebert & Wearne, 2003; NCTM, 2000). Highlighting multiple solutions includes comparing the merits of different solutions. Hiebert and colleagues (1996) explained:

> The teacher bears the responsibility for developing a social community of students that … shares in searching for solutions. A critical feature of such communities is that the focus of examination and discussion be on the methods used to achieve solutions. Analyzing the adequacy of methods and searching for better ones are the activities around which teachers build the social and intellectual community of the classroom. (p. 16)

Research on Benefits of a Problem-Based Approach to Instruction

As I mentioned above, beginning in the mid 1990s, much of the research related to problem solving centered on the standards-based curriculums (e.g., Harwell et al., 2007; Huntley, et al., 2000; Riordan & Noyce, 2001; Schoenfeld, 2002; Thompson & Senk, 2001). A few researchers have investigated teaching practices—that is, the role of the teacher in problem-solving instruction—but most studies focused on the curriculums themselves. In the following sections I discuss studies of both types: first those that focus on curriculums and then those that focus on instruction.

Problem-Based Curriculum

Research on the value of standards-based curriculums suggests that a focus on problems and problem solving in the mathematics classroom enhances student learning. Standards-based curriculums typically emphasize group work, multiple solutions, student reasoning, multiple mathematical representations, complex problems, and real-world applications. In many cases, these curriculums are specifically problem-based in that mathematical concepts are presented using problems. One such curriculum, the Connected Mathematics Project (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1991–1997, 2002, 2006) is explicitly “problem-centered” in that each lesson has three parts: “launching the problem,” “exploring the problem,” and “summarizing the problem” (Riordan & Noyce, 2001, p. 374).

Some studies have shown the benefits of using standards-based curriculums (e.g., Harwell et al., 2007; Huntley et al., 2000; Riordan & Noyce, 2001; Schoenfeld, 2002; Thompson & Senk, 2001). For example, Hartwell et al. (2007) found that after spending 3 years using a standards-based curriculum—either Core-Plus Mathematics, the Interactive Mathematics Program, or Mathematics: Modeling Our World—high school students met or exceeded the national average on standardized achievement tests. These students also scored higher on the New Standards Reference Examination in Mathematics, a test focused on mathematical skills, concepts, and problem solving. Moreover, students using standards-based curriculums tended to have a more positive view of mathematics and its usefulness than did students using traditional curriculums.

Riordan and Noyce (2001) compared the mathematics achievement of students using a standards-based curriculum—Everyday Mathematics in elementary school or the Connected Mathematics Project in middle school—to those using traditional curriculums. Results indicated “that students using either of the [standards-based] programs are still capable of performing procedural arithmetic items … and doing so in a traditional, multiple choice format while also demonstrating an ability to solve higher order mathematics problems” (p. 390). Riordan and Noyce concluded, “The results of the study reported in this article add to the accumulating body of evidence that standards-based mathematics programs have a positive impact on student achievement" (p. 392).

As notable as these studies are, they focus solely on the use of standards-based curriculums and, for the most part, ignore particular teaching practices that accompany use of the curriculums. Research that focuses solely on the apparent effects of a particular curriculum neglects the most important factor in classroom instruction: the teacher. Boaler (2008) stated,

> The most important factor in school effectiveness, proved by study after study, is not the curriculum but the teacher. Good teachers can make mathematics exciting even with a dreary textbook. Conversely, bad teachers do not become good just because a book is written well. (p. 32)

Problem-Based Instruction

There is very little research on the details of teaching through problem solving, but there are studies that indicate that the broader set of reform or standards-based teaching practices can enhance student learning, particularly when implemented in combination with a standards-based curriculum. Wood and Sellers (1997) conducted a longitudinal study in which they examined the impact of a problem-centered approach to instruction. The researchers investigated the mathematical achievement and beliefs of three groups of elementary school students. One group received 2 years of problem-centered instruction, a second group received 1 year of problem-centered instruction and 1 year of traditional instruction, and the third group received 2 years of traditional instruction. Problem-centered classes were described as those in which "a typical mathematics lesson frequently began with children working in pairs on the problem-centered instructional activities. This was followed by class discussion in which the children generally gave explanations for their solutions to the activities" (p. 166). Wood and Sellers found that students with 2 years of problem-centered instruction scored higher on standardized achievement tests and demonstrated better mathematical understanding than students in the other two groups.

Furthermore, results of the study indicated that these changes remained even after students returned to traditional instruction. In addition, students in the problem-centered classes experienced changes in their beliefs about mathematics and about themselves as problem solvers. In contrast to the students receiving traditional instruction, which is frequently accompanied by a spirit of competition, students in the problem-centered classes "were not motivated by a desire to be better than others (ego orientation) but rather by a belief in the importance of finding their own ways to solve problems (task orientation)" (p. 171).

McCaffrey, Hamilton, Stecher, Klein, Bugliari, and Robyn (2001) investigated the relationship between reform teaching and student achievement in high school mathematics courses. They examined this relationship in two contexts: classrooms in which teachers used a traditional mathematics curriculum and classrooms in which teachers used a problem-based curriculum. According to McCaffrey et al., reform teaching practices include group work, investigations, open-ended questions, emphasis on multiple solution methods, use of real-world problems, and long-term projects. They noted that reform teachers focus on problem solving and allow students to rely on mathematical reasoning to evaluate solutions rather than looking to the textbook or the teacher for answers. Results of this study indicated that reform teaching practices in combination with the use of a problem-based curriculum were associated with growth in student achievement.

Schoen, Cebulla, Finn, and Fi (2003) investigated the effects of reform teaching on student achievement in high school mathematics. Teachers in the study used Core-Plus Mathematics, a standards-based curriculum that emphasizes problem solving. Researchers found that reform teaching practices were associated with growth in student achievement. Student achievement, in this case, was measured using the Iowa Test of Educational Development

(ITED-Q), a standardized test designed to assess students' mathematical reasoning, conceptual understanding, and problem solving. Classroom observations were a crucial component of the study. By observing teachers, researchers ascertained the qualities of reform teaching practices that each teacher exhibited. Schoen and colleagues used a list of criteria developed from the NCTM (2000) standards and the Core-Plus curriculum to determine teachers' use of reform teaching. These criteria included "open-ended questioning to facilitate student thinking and exploration," evidence that students were encouraged to use mathematical reasoning to assess solutions rather than depend solely on the teacher or textbook, ample time for students to engage in investigations, group work, use of technology and manipulatives, and questioning of students' "understanding and problem-solving strategies" (Schoen et al., 2003, p. 236). Results of this study indicated that "teacher [practices] … that [were] significantly and positively associated with growth in student achievement" included collaborating with other teachers, having students work in small groups, and attaining "a high observer rating of teaching based on the criteria for effective reform teaching" (p. 255). Schoen and colleagues noted that the increase in student achievement "was consistent across a wide range of students in schools with varying SES [socioeconomic status] levels, sizes and ethnic mixes of school populations, beginning achievement levels of the students, lengths of classes, and numbers enrolled in classes" (p. 255).

As these studies indicate, a standards-based approach to teaching can have a positive impact on students' learning of mathematics. The following section looks more specifically at the standards-based approach to mathematics instruction that was the focus of the present study: teaching through problem solving.

Teaching Through Problem Solving

There is not much research on the practices of teaching through problem solving specifically—that is, what this instructional approach actually looks like in practice. But in their review of research on problem-solving instruction, Stein and colleagues (2003) noted a few studies that indicated that teaching through problem solving led to deeper mathematical understanding among students than traditional teaching approaches did. For example, "Boaler (1997) found that students who had learned through open-ended projects developed more flexible and adaptable forms of knowledge and understanding than did those students who had learned through more traditional methods of teaching" (p. 251). Stein and colleagues cited a 1993 study by Hiebert and Wearne in which students in a traditional classroom were compared with those in a setting in which the teacher emphasized problem solving. Hiebert and Wearne "found that students who learned the most were those who spent more time on each problem, with the additional time being used by students to describe the strategies they used and explain why those strategies worked" (Stein et al., 2003, pp. 252–253).

There are many advocates for teaching mathematics through problem solving. Schoen (2003) claimed that problem solving should be a "means for acquiring new mathematical knowledge," rather than something students wait to do until after learning a concept or skill (p. x). According to the NCTM (2000),

> Problem solving … can serve as a vehicle for learning new mathematical ideas and skills (Schroeder and Lester, 1989). … Good problems can inspire the exploration of important mathematical ideas, nurture persistence, and reinforce the need to understand and use various strategies, mathematical properties, and relationships. (p. 182)

Hiebert et al. (1996) argued that "reform in curriculum and instruction should be based on allowing students to problematize the subject. Rather than mastering skills and applying them, students should be engaged in resolving problems" (p. 12). Kahan and Wyberg (2003) noted three benefits of teaching through problem solving: “(a) it helps students understand that mathematics develops through a sense-making process, (b) it deepens students’ understanding of the underlying mathematical ideas and methods, and (c) it engages students’ interest” (p. 20).

Despite the enthusiasm for this approach to instruction, as I mentioned previously, research is scarce on what it looks like to teach through problem solving on a day-to-day basis. Most of what mathematics education researchers have to say about specific practices associated with teaching through problem solving is in the way of advice. For example, Grouws (2003) explained his vision of the teacher’s role in teaching through problem solving:

> Successfully implementing such an approach [i.e., teaching through problem solving] involves many teacher decisions and actions, which include, to name a few, choosing appropriate tasks, conveying tasks to students in ways that stimulate interest, maintaining students’ engagement in tasks, and leading discussions in which the important mathematical ideas embedded in the tasks are brought to the surface. (p. 130)

Such advice is well founded because suggestions researchers have given logically follow from what the research says about the benefits of problem-based curriculums and instruction. But as Boaler (2008) suggested, simply putting a problem-based curriculum in the hands of a teacher does not guarantee that teaching through problem solving will occur.

There is no single model for teaching problem solving most effectively; rather, there are many ways to have success in problem-solving instruction. Alan Schoenfeld, who is widely considered to be a successful teacher of problem solving, provides one example of how to teach problem solving. Arcavi and colleagues (1998) and Santos-Trigo (1998) studied Schoenfeld’s problem-solving course for undergraduate students and observed that he employed many of the practices listed above, particularly teaching specific problem-solving strategies. These researchers were clear—as was Schoenfeld—that his is just one way to teach problem solving.

Posamentier and Jaye (2006) compiled expert advice drawn from mathematics education researchers. They examined and distilled research in order to give teachers practical ideas for the classroom. In particular, they summarized research on problem solving and problem-solving instruction and recommended several specific teaching practices. They reiterated the advice I have already discussed about problem-solving instruction—for example, the practices in Figure 1—but also advocated teaching through problem solving specifically. They recommended that teachers “structure teaching of mathematical concepts and skills around problems to be solved, using a problem-centered or problem-based approach to learning” (p. 141). That is, “instead of starting a unit by using the textbook and telling students about a mathematical topic and explaining and demonstrating various concepts, problems, and solution methods, start by giving students a meaningful problem to solve” (p. 142). This instructional approach is commonly used in Japanese mathematics classrooms, as I describe in the next section.

The Japanese Model

Takahashi (2008) described how many Japanese teachers taught mathematics through problem solving. He made a distinction between the tendency of many American teachers to first show students how to solve a particular kind of problem and then give them the opportunity to solve similar problems, and the practice of Japanese teachers to begin the lesson by giving the students a problem without first showing them how to do it. The Japanese perspective, as Takahashi described it, is that mathematical problems are a means of learning, not just of demonstrating what has already been learned.

Allevato and Onuchic (2008) proposed a teaching approach similar to the Japanese model called “Mathematics Teaching-Learning-Evaluation through Problem Solving.” A key element of this approach is that when students initially see the problem, “the mathematical content necessary, or most appropriate, to solve the problem has not yet been presented in class" (p. 6). Rather, "a problem is the point of departure for learning, and the construction of knowledge occurs in the process of solving it" (p. 5). It is only at the end of the lesson, after several students have presented solutions, that the teacher summarizes new mathematical concepts and presents them explicitly.

Smith (2004) analyzed video data from the Third International Mathematics and Science Study and examined practices teachers from the United States and Japan used. Specifically, she compared the behavior of teachers from both countries who used a problem-based approach to teach mathematics. She found that even when the problems the teachers assigned were basically the same in both countries, more teachers from Japan than from the U.S. helped students see mathematical connections and develop conceptual understanding.

In the following description, "Mrs. Jones" represents a sampling of typical behavior U.S. teachers exhibited, and "Mrs. Hamada" represents a sampling of typical behavior Japanese teachers exhibited. Mrs. Jones and Mrs. Hamada began their lessons the same way, giving their students a problem and allowing ample time to solve it. The problem Mrs. Hamada gave is in Figure 2—Mrs. Smith's problem was comparable but was set in a different context. For each of these teachers, one of their goals was to have the class come up with several solution methods so that they could discuss the various solutions, compare them, and make mathematical connections. When students in Mrs. Jones's class got stuck, she gave a suggestion for a solution method. When she then asked students to devise other methods, they were unwilling. On the other hand, when Mrs. Hamada's student needed help, she directed their attention to the problem statement and encouraged them to think about how they could model the situation. Her actions prompted the students to come up with various ways to solve the problem. As a result,

> in the Japanese lesson, the solution methods presented were analyzed and compared. Mrs. Hamada and Mrs. Jones valued students' solution methods. Unfortunately, Mrs. Jones's students presented only one solution method, allowing little room for developing mathematical connections across solution methods. Because Mrs. Hamada's students presented more than one solution method, she was able to have them highlight mathematical relationships. (Smith, 2004, p. 105)

The Japanese model highlights a defining characteristic of teaching through problem solving: the use of problems as a means of learning mathematical content.

Ken and his brother enjoy chewing gum. One day, the boys go to the candy store and buy several packages of gum. Ken bought 18 ten-piece packages of gum, and his brother bought 24 five-piece packages of gum. Every day, each of the boys finishes one whole pack of gum. One day, they looked at how much gum each boy had. Ken noticed that his brother had more pieces of gum than he had. How many days has it been since the boys bought gum?

Figure 2: Gumball task (Smith, 2004, p. 100).

Calls for Research

Recall the argument of Stein and colleagues (2003) who said, "One of the crucial research questions for the next decade [is], 'What happens inside classrooms in which problem-solving approaches are used effectively?'" (p. 250). In addition, according to Grouws (2003),

> The teacher's role in fostering students' mathematical learning is central and deserves greater attention than it has received in recent years. Does effective mathematics teaching involve more than forming small groups and assigning good problems for students to solve? Of course it does. (p. 129)

I sought to address these questions in this dissertation study. I was interested in what actually happens on a day-to-day basis in the classrooms of teachers who believe that problem solving is central to mathematics teaching and learning. Specifically, I wanted to know what practices teachers used to teach through problem solving.

CHAPTER 3

METHOD

Kilpatrick (1978) noted that researchers should formulate research questions before considering specific research methods. When it comes to investigating teaching practices, I believe that because teaching is complex and nuanced, more meaning can be found in a close examination of a few teachers than in a broad look at a large sample of teachers. For that reason, I chose a *qualitative* method to address the research questions in the present study. Berg (2007) described this kind of research: “Qualitative research properly seeks answers to questions by examining various social settings and the individuals who inhabit these settings. … Qualitative procedures provide a means of accessing unquantifiable facts about the actual people researchers observe and talk to” (p. 8). Qualitative research methods are useful for addressing different kinds of questions than those that quantitative methods can address. Certainly any data that can be counted or measured are best gathered and analyzed using quantitative methods, but most human behavior is complex and cannot be reduced to quantifiable data.

Lester (1985) advocated qualitative methods for conducting research in problem-solving instruction. He stated, “Adopting a holistic view of problem solving and problem-solving instruction necessitates the use of naturalistic [inquiry] rather than traditional scientific research paradigms” (p. 52). By *naturalistic inquiry* he was referring to qualitative research done in a natural setting such as a classroom.

Of the many qualitative research methods, I determined that a *descriptive case study* (Berg, 2007) was most appropriate for addressing the research questions in the present study. Berg defined *case study* as “a method involving systematically gathering enough information about a particular person, social setting, event, or group to permit the researcher to effectively understand how the subject operates or functions" (p. 283). The present descriptive case study involved spending time in teachers' classrooms and talking with each of them to get a sense of their beliefs about mathematical problem solving and the specific teaching practices they employed. Of the subjects Berg listed, a *group* best describes the focus of the present study. That is, I was interested in describing teaching practices of a group of teachers who taught through problem solving rather than conducting four individual case studies or creating a profile of each teacher.

Some have criticized case study research for not producing findings that are generalizable, but Berg (2007) argued,

> When case studies are properly undertaken, they should not only fit the specific individual, group, or event studied but also generally provide understanding about similar individuals, groups, and events. … The logic behind this has to do with the fact that few human behaviors are unique, idiosyncratic, and spontaneous. (pp. 295–296)

Specifically, the present study was an *intrinsic case study* in which "the role of the researcher is not to … test abstract theory or to develop new or grounded theoretical explanations; instead, the intention is to better understand intrinsic aspects of the particular [participant or group]" (p. 291).

In the present study, I was not trying to develop or test a teaching theory or theory of problem solving. Rather, I began with the assumption that teaching through problem solving is a valid, and in many cases superior, approach to teaching mathematics. I based this assumption on advice given frequently over many decades by experts in mathematics education (e.g., Grouws, 1985; Hiebert & Wearne, 2003; Lester, 1994; Lester & Charles, 2003; Polya, 1957; Schoenfeld, 1985, 1988; Schoen, 2003) and authorities such as the NCTM (1980, 1989, 2000), as chapter 2 indicated. I also worked under the assumption, based on the reputation and past performance of the teachers in the present study, that these teachers had had success in teaching through problem solving. I sought to describe teaching through problem solving by giving examples of how four teachers put this instructional approach into practice. My goal was to investigate practices for teaching through problem solving in order to better understand the approach and to promote it as well.

In any research study, specific methods are chosen with which to gather the data. In order to address the research questions in the present study, I determined that interviews with teachers and classroom observations were most appropriate. Researchers have called specifically for classroom observations as a method for addressing such questions. Grouws (1985) claimed, "What is needed are observational studies of problem-solving instruction" (p. 298). Lester (1985) suggested that in order to develop a theory of problem-solving instruction, data should be "gathered from extensive observations of 'real' teachers teaching 'real' students 'real' mathematics in 'real' classrooms" (p. 56).

Participants

In the spring of 2010, four high school mathematics teachers who considered problem solving to be a priority in their mathematics teaching participated in the study. In considering the selection of participants, I took into account several qualities: (a) experience in teaching through problem solving, (b) a commitment to that kind of teaching, and (c) a reputation for being effective in teaching through problem solving. I relied on word of mouth to locate possible participants and made the decision to include these particular teachers using *purposeful selection* (Maxwell, 2005). That is, the teachers were "selected deliberately in order to provide information that [could not] be gotten as well from other choices" (p. 88). Three of the teachers—Miss Atkinson,[1] Mr. Dalton, and Mr. Fulbright—taught at Northridge High School, and the other teacher, Mr. Bailey, taught at Tanner Academy.

[1] All names of teachers and schools are pseudonyms.

Northridge High School was a public boarding school for Grades 11 and 12 in a midsize city in the southeastern United States. The school curriculum focused on science and mathematics, though students took the typical range of high school courses. Northridge had approximately 650 students, all of whom were boarders. The school was open only to in-state residents, and students came from across the state in proportion to population distribution (i.e., Congressional districts). Students had to complete an application process in order to attend, so Northridge could be somewhat selective in admitting students. Applicants were evaluated based on academic performance, extracurricular or leadership activities, and recommendations from a teacher and guidance counselor. Furthermore, applicants were required to show an interest in mathematics and science. Students were not required to be mathematically gifted to attend the school, and there was a range of mathematical ability in the student population.

Northridge operated on a schedule that divided the school year into three 10-week trimesters rather than the typical two 15-week semesters. Most mathematics courses met for the entire year, but there were several that met for only 1 trimester. Classes met 4 days a week, and one of these classes was a *lab*. Labs were 90 minutes, and the remaining 3 class periods each week were 50 minutes. The purpose of the lab was to allow time for investigations and activities.

Miss Atkinson, Mr. Fulbright, and Mr. Dalton taught at Northridge High School. These teachers had each participated in an annual conference held at Northridge that drew mathematics teachers from across the country. The purpose of the conference was to give teachers ideas about how to implement an instructional approach that emphasized active student learning and problem solving.

Miss Atkinson was in her second year of teaching at Northridge in 2009–2010 and had 9 years of experience teaching high school mathematics. She had a Ph.D. in Mathematics

Education and was a member of NCTM. She had been a frequent presenter at conferences for the state chapters of both NCTM and the Association of Mathematics Teacher Educators. During the 2009–2010 school year, she taught Precalculus and Algebra 3. In addition to being a presenter at Northridge's annual conference, Miss Atkinson served as the conference organizer in 2009 and 2010.

Mr. Fulbright had 12 years of teaching experience and had taught at Northridge for 7 of those years. During the 2009–2010 school year, he taught Advanced Placement (AP) Calculus AB and AP Statistics. He had been a leader in the College Board's AP Statistics program and had written resource materials for AP Statistics teachers. Mr. Fulbright had a Ph.D. in Statistics and was a member of NCTM and the American Statistical Association (ASA). He had been a frequent presenter at various conferences for high school mathematics teachers, including Northridge's annual conference.

Mr. Dalton had taught at Northridge for 29 years and had a total of 35 years of teaching experience. In 2009–2010, he taught AP Statistics, Mathematical Modeling, Graph Theory, Modeling with Matrices, Combinatorics, and Modeling with Differential Equations. Northridge's trimester system allowed him to offer so many different classes during a single school year. Mr. Dalton had a Ph.D. in Mathematics Education and was a member of NCTM, ASA, and the Mathematical Association of America (MAA). He was a contributing author of several high school mathematics textbooks and had participated in the development of various NCTM standards projects. Mr. Dalton also had been a frequent presenter at regional and national meetings of the MAA, various AP meetings and workshops, and other conferences focused on mathematics education, including the annual conference at Northridge.

Tanner Academy, where Mr. Bailey taught, was a private college preparatory school in a midsize city in the southeastern United States. It was a boarding school, and approximately 30% of the high school students (Grades 9–12) were boarders. There were approximately 1100 students in Grades 6–12, with about 700 students in Grades 9–12. The daily schedule at Tanner Academy was such that classes met 4 days a week for two 55-minute periods and two 60-minute periods. This schedule allowed time during the week for students to be involved in activities and to attend assemblies and colloquia.

Mr. Bailey had taught at Tanner Academy for 37 years. He was an active member of NCTM and had been a frequent presenter at its annual national conference. He had a Ph.D. in Mathematics, was a long time member of the MAA, and had co-authored a number of high school mathematics textbooks. For many years, Mr. Bailey had a national leadership role in the College Board's AP Calculus program. In 2009–2010, he taught Finite Math and AP Calculus BC.

Each of these teachers had come in different ways to be convinced of the importance of teaching through problem solving. A change in perspective on mathematics teaching and learning is usually necessary for teachers who come to value problem-solving instruction because many of them, including some in this study, begin their careers by teaching in a way that they were taught (see Cohen, 1990). In the case of mathematics, teaching the way one was taught often includes a focus primarily on skill development and procedural knowledge.

A number of events led Miss Atkinson to a philosophy of teaching that emphasized problem solving. First, after gaining some experience teaching high school mathematics, she began a graduate program in mathematics education. Her experience in graduate school played a significant role in opening her eyes to the difference between how she had been teaching, which was very procedurally, and better ways to help students understand mathematical concepts. Second, she had the opportunity to observe various teachers and compare their teaching styles. In doing so, she was impressed by what certain teachers were doing to challenge their students with good problems, and she saw the students responding positively. In our first interview, she said her response to observing these teachers was, “This is it. This is what we need to be doing in our classrooms. This is where we need to be.” A third factor in Miss Atkinson’s development as a teacher who taught through problem solving was her positive interaction with colleagues at Northridge who encouraged one another to use problem solving in their classrooms and shared ideas about how to do so.

Mr. Fulbright began his teaching career using a lecture-based approach in the classroom, but he said his use of lecturing had more to do with what was expected of him at the school where he taught than with his own thoughts about ideal teaching practices. He indicated that when he was a mathematics student, he had been exposed to a problem-based approach, so when he became a teacher, he “unintentionally infused a little more problem solving than they intended me to” into his mathematics lessons. As Mr. Fulbright continued teaching in other schools, including Northridge, he found colleagues who were interested in a problem-based approach. This common interest led him to talk explicitly with fellow teachers about incorporating problem solving into his instruction.

Mr. Dalton noted that most teachers start out as lecturers, he being among them. But he recalled that his journey away from the lecture mode and toward a problem-solving approach began as he came to the point of “recognizing how much of what you were telling [students] they could actually figure out on their own.”

Mr. Bailey, in presentations, articles, essays, and his own teaching, had advocated a problem-solving approach to teaching mathematics since the mid 1980s. He described how he came to be convinced of the importance of problem solving in his classroom:

> I was persuaded by a variety of "reform" documents around 1990, but the two I found most compelling were *Everybody Counts* from the National Academy of the Sciences [1989] and [*Every Minute Counts*] by David Johnson [1990] (NCTM). I had also come to the conclusion that I was doing all the important mathematics in my classroom, while my students were doing all the routine mathematics at home, where I could not even see them doing it. This did not seem like the ideal model.

As their credentials attest, these teachers were not typical. Furthermore, they taught in very specialized settings and had advantages over many public school teachers in terms of student behavior, resources, and so forth. Negative student behavior was generally not a problem at Northridge and Tanner, so the teachers could focus on instruction without spending time or energy on classroom management. These characteristics provided good access to the beliefs and practices of teachers who taught through problem solving.

Given the advantages of these teaching environments, one could argue that the teachers' circumstances were too exceptional to be of use when discussing teaching practices. After all, it is no surprise that these teachers had the freedom to teach in a nontraditional way. And with such a privileged student population, it is no wonder that they had success in teaching through problem solving. I would argue, however, that there *is* value in examining these teachers' practices even though their situations were atypical. Although it is true that having cooperative and capable students allows teachers to more easily implement a particular teaching approach, that advantage does not diminish the value of the teaching approach itself or the teacher's ability. Consider the analogy of a music performance. It is quite possible that a Stradivarius violin enhances the performance of a concert violinist, but it is primarily the violinist—not the violin—that creates the beautiful music. We give credit to the performer rather than to the instrument with which the performer displays his or her talent. Similarly, no matter who the students are, if their teacher is exhibiting good teaching, it is worth taking notice and studying that teacher's practices.

One might also argue that if these teachers are exceptional, then their practices are unattainable for the average teacher, and therefore it is unproductive to study them. I would argue, however, that just as researchers have found value in studying expert problem solvers to gain insight into the nature of problem solving (e.g., Schoenfeld, 1985), so it is important to study experts in the area of teaching through problem solving to gain insight into what this approach actually looks like in practice.

Procedure

Classroom Observations

From January to April, 2010, I conducted a series of classroom observations at Northridge High School and Tanner Academy. Each class was audiorecorded, and I also took fieldnotes to record my thoughts and ideas during each observation and to note important inaudible data such as items written on the board and teacher movement around the classroom. My goal was to identify practices that the four teachers used to teach mathematics through problem solving. I visited the three teachers at Northridge on two separate occasions, spending 1 week in January and 1 week in March. I visited Mr. Bailey at Tanner Academy for 1 week in January and 2 weeks—separated by a week—in April.

It was advantageous to visit each teacher more than once—that is, for more than just 1 week—and to have several weeks between visits. This advantage was particularly evident in the case of Mr. Dalton. I was able to observe the introduction and first week of his Modeling with Differential Equations course because the week in March that I visited Northridge was the beginning of the third trimester and therefore the beginning of the course. If I had visited only in January, I would have missed the opportunity to observe how he began his course.

It was also important that I observed the teachers for a number of consecutive days. This schedule allowed me to get a sense of the flow of things from one lesson to the next, to have a better vision of the patterns that emerged, and to have an opportunity to observe typical behavior. An alternate strategy of conducting observations during nonconsecutive lessons scattered over the semester, even if the number of lessons observed had been the same or greater, may have resulted in a misrepresentation of what normally occurred in the classroom. An individual lesson may have been atypical for a number of reasons: for example, if it included the administering of a quiz, reviewing for a test, or spending an uncharacteristic amount of class time going over homework. Each teacher indicated that the classes I observed were fairly typical. Scheduling restrictions also factored into my decision about when to visit the teachers. Northridge High School was several hours from my home by car, so when I traveled there, it made sense to stay for a week. There were only 2 weeks—one in January and one in March—when I could make the trip. In order to be consistent, I adopted a similar schedule for visiting Mr. Bailey at Tanner Academy, but the close proximity of Tanner to my home allowed me to visit Mr. Bailey during 3 different weeks rather than 2.

The choice of what specific classes to observe was based on two factors: (a) convenience sampling (Berg, 2007) because of scheduling restrictions and (b) the teachers' indications about which classes they thought I would find helpful for my research. During my January visit, Northridge High School was in its 2nd trimester. I observed Miss Atkinson's Algebra 3 and Precalculus classes, Mr. Fulbright's AP Calculus class, and Mr. Dalton's Combinatorics and Modeling with Matrices classes. In March, I was at Northridge during the first week of the 3rd trimester. I observed the same courses as I had in January with the exception of Mr. Dalton's: Instead of the matrices course—which was offered in the 2nd trimester—I observed his Modeling with Differential Equations course—which was a 3rd trimester course—and his AP Statistics course, which was a year-long course. At Tanner Academy, I observed Mr. Bailey's Finite Math and AP Calculus courses in both January and April. One advantage of observing the same teacher in two different courses was that I could see if there were teaching practices that were specific to a particular course, or if a teacher was consistent across different courses in the practices he or she implemented.

I looked for several things as I conducted the classroom observations. In view of the literature on problem solving and recommended practices for teaching through problem solving, I hoped to find some or all of the following. With each main category I had in mind the accompanying questions to guide my data collection:

1. Teaching general or specific problem-solving strategies
   a. Does the teacher talk explicitly about problem solving?
   b. Does the teacher mention or demonstrate *general* problem-solving strategies?
   c. Does the teacher mention or demonstrate *specific* problem-solving strategies?
2. Modeling problem solving
   a. Does the teacher demonstrate problem solving or particular problem-solving skills?
   b. Does the teacher ever highlight, either implicitly or explicitly, Polya's four phases of problem solving?
3. Limiting teacher input
   a. Do students work together to solve problems?
   b. Do students explain or demonstrate their solutions to classmates?
   c. What assistance or guidance does the teacher provide to students as they work on problems?
   d. How does the teacher respond when students pursue unproductive solution paths or dead ends? How far does the teacher let students go before intervening?
4. Promoting metacognition
   a. How does the teacher promote metacognition in the classroom?
   b. Does the teacher ask questions or make comments that encourage students to be reflective about problem solving? If so, how?
   c. Does the teacher model metacognitive behavior in regard to problem solving?

5. Highlighting multiple solutions
    a. Does the teacher encourage students to find various ways to solve a particular problem?
    b. Do students share their solutions with one another?
    c. Does the teacher discuss connections between different solutions?
    d. Is there discussion of advantages and disadvantages of particular problem-solving strategies?
    e. Does the teacher encourage students to develop more efficient problem-solving strategies?

I recorded in my fieldnotes those instances in which a teacher used one of the above practices. I was also attentive to behavior that gave a more complete picture of how the class period was spent—for example, questions the teachers asked, tasks the teachers assigned, the teacher's movement around the classroom, or instances of students going to the board—or that indicated teaching practices that I had not anticipated. Essentially I was watching and listening for anything that seemed to be related to problem solving.

During each observation, I also wrote down any questions I wanted to ask the teacher at the conclusion of the class period. These questions had to do with the teacher's reasoning behind a particular decision, whether or not a particular comment or event was typical, or other issues that came to mind as a result of something that happened during class. When I had questions about a lesson, the teacher and I had a brief conversation after class.

The specific mathematical content of the lesson was not the focus of my attention during the observations. I certainly took note of the mathematics and followed along with the examples and problems posed during class. These notes gave me a context in which to describe a particular practice the teacher used. For example, one practice for teaching through problem solving is to emphasize that a problem can be solved in more than one way. To give a full description of such an instance, it was important that I write down the particular problem and the solutions that teachers and students discussed.

The kinds of problems posed in class as well as in homework assignments were essential to learning about these teachers' approaches. Rather than describe the problems in the context of this discussion of classroom observations, I describe the types of problems later in this chapter.

Having audiorecorded each lesson in its entirety, I transcribed relevant portions of each recording and noted the length of time spent on each part of the lesson. The format for this documentation, created in the process of transcribing the data, was a three-column table. The first column was the time stamp; the second column the description of each activity or a quotation, including the time it occurred; and the third column, my notes and coding. See Figure 3 for an example of a portion of one of these tables. At the top is the name of the teacher, the date of the observation (YYMMDD), and the class observed.

	Miss Atkinson 100104 Algebra 3	
0:00-1:51	Intro/review (first day back after Christmas break).	
1:51-17:33	Example (review): Solve the system: $2x - y - z = 7$ $3x + 5y + z = -10$ $4x - 3y + 2z = 4$	
	Students working in groups, teacher circulating.	Group work.
	4:49 “Are you guys checking behind each other and trying to avoid those dreaded careless mistakes?”	Metacognition: monitoring.
	8:52 “Is it right? How can you find out if it’s right?”* 10:57 “You’re eliminating your *x*’s. That’s interesting. That’s ok. It doesn’t matter. …Well, the *z*’s are probably the easiest to eliminate, but that doesn’t mean that that’s the only way to do it.”** 15:15 “You’re gonna have to convince *yourself* whether it’s right or wrong.”* 15:32 “Find the whole solution. Check your answers.”*	**Looking back*. Also, sharing authority for correct answers. **Multiple solution paths.
	16:56 “Well, right now you only know one variable. You only know about one variable. So how, what can you use to help you find out [the rest]?”	Metacognition: monitoring. How can you get from where you are to where you are going?
17:33-24:50	Teacher leading class discussion about solving systems of equations (3 variables).	

Figure 3: Partial transcript of a class observation, with notes.

Rather than transcribing entire lessons, I transcribed parts of each lesson that I considered relevant to the study. These relevant passages were quotations from the teacher or exchanges between the teacher and one or more students. Some of these exchanges involved the whole class, and others involved the teacher and an individual student or a small group of students.

Interviews

The data for the study included three interviews with each teacher. The first two were face-to-face interviews conducted during each of my visits. The third was an email interview I conducted in May. Each face-to-face interview lasted approximately 30–40 minutes and was held in the teacher's office or classroom. Each interview was audiorecorded and later transcribed. In addition, informal conversations followed some of the lessons, as described above. These were also audiorecorded and later transcribed.

The interviews were semi-structured, or *semi-standardized* (Berg, 2007). As such, although I generally kept to my questions as written, there were times when I asked a follow-up question based on a teacher's response. At other times, I omitted a question when I judged that the teacher had addressed the content of that question in a previous response. Interview questions appear in the Appendix.

The interviews were an important part of the study because there are many different views of what problem solving is and what makes a good problem. Some believe, for example, that *problem solving* and *doing word problems* are the same thing. Some see problem solving as a step-by-step process; for example, some teachers use Polya's four phases as a formula for solving problems. Some think that in order for something to be a problem, the way to solve it cannot be immediately evident. It was necessary to know how each teacher viewed problem solving as well as what they considered to be good ways to help students become better problem solvers.

The interviews allowed the teachers to express their views about problem solving and what they believed was important in teaching through problem solving. The interviews also provided access to the teachers' perspectives on what they saw as significant in their own teaching. It is one thing to observe a teacher's actions and bring one's own interpretation to what is occurring, and another to hear a teacher's explanation or interpretation of his or her actions. Beyond providing information, the interviews gave the teachers the opportunity to reflect on their own beliefs and practices. In some cases, they expressed having never thought before about an issue I raised in an interview question.

Documents

During each visit, I saw examples of problems the teachers assigned, both in class and for homework. In-class assignments included everything from problems that could be solved in just a minute or two to activities that took the whole class period to complete. The teachers also assigned out-of-class work, both daily homework assignments and long-term projects. I asked the teachers to provide me with problems, activities, and projects—other than those I had seen during my visits—that they thought were representative or covered a spectrum of mathematical topics. Some of these problems are discussed in chapter 4. As I examined the assignments, I looked for characteristics that would give me insight into the teachers' views about what makes a worthwhile problem. I had asked the teachers to discuss this matter in the first interview, but used the assignments to provide actual examples.

Coding

After transcribing relevant portions of the audiorecordings of the classes I observed, I coded the transcripts. I noted instances in which the teacher engaged in a teaching practice related to problem solving. I wrote these codes in the last column of the transcript (see Figure 3 above for an example).

Codes that appeared most frequently were the following:

1. teach problem-solving strategies
2. model problem solving
3. group work
4. share authority for correct answers
5. metacognition
6. multiple solutions
7. Polya's phases

I color-coded each instance so that I could more easily refer back to particular teaching practices. Sometimes codes were more specific than those listed above—for example, "teach specific problem-solving strategy" or "metacognition: monitoring"—because they were particular examples of each practice.

I constructed an initial list of codes based on what I found in the literature on problem-solving instruction. That initial list consisted of the following:

1. teaching general problem-solving strategies
2. teaching specific problem-solving strategies
3. modeling problem solving
4. group work
5. metacognition
6. multiple solutions

To that list I added the following:

7. Polya's phases
8. students share solutions

Since Polya had been such an influential figure in the history of problem solving in the 20th century, I wondered if I would find instances of teachers referring—either explicitly or implicitly—to his four phases of problem solving. *Students sharing solutions* is not a practice I found to be widely emphasized in the literature on problem-solving instruction, but it is a key component in the Japanese model of teaching through problem solving, so I wondered if teachers in this study would incorporate the practice into their instruction. I later included *students sharing solutions* in the broader practice of *highlighting multiple solutions*.

Finally, I added the following codes based on what I saw during my observations:

9. salvaging good ideas from incorrect solutions
10. non-mathematical problem solving (for example, What to do when technology does not cooperate.)
11. mathematical authority (that is, sharing authority for correct answers)
12. warning of common errors

These were practices that I had not anticipated when I began my observations, but in the process of conducting the observations, I noticed the practices. Some of these practices—for example, *salvaging good ideas* and *non-mathematical problem solving*—did not occur frequently enough to be part of my data analysis. Other practices were incorporated into broader categories. For example, I considered *sharing authority for correct answers* to be a specific instance of *limiting teacher input*, and *warning of common errors* is a practice that I discuss as a contrast to *limiting teacher input*.

Certain teaching practices—for example, *group work*—were easily identifiable. Instances of group work included the following: (a) students working together on a problem during class and (b) students sharing their solutions with one another—for example, sharing solutions to homework problems. The mere fact that students sat in groups, as they did at Northridge, did not constitute group work unless students actually worked together.

Other teaching practices were not as clear as group work. For example, it was sometimes difficult to discern whether a strategy a teacher highlighted was a problem-solving strategy or something else, like a test-taking strategy or a strategy for using technology. I made the decision to code an instance as a problem-solving strategy by considering whether the strategy was likely to help students solve a mathematical problem. An example of a strategy that I decided was *not* a problem-solving strategy was a piece of advice about how to phrase a written response in order to gain maximum points on the AP exam. I considered this to be a test-taking strategy rather than a problem-solving strategy.

Modeling problem solving was also sometimes difficult to code because I made a distinction between a teacher showing students how to solve a problem and actually modeling the problem-solving process. If a teacher simply engaged in “show and tell,” I did not code that as modeling problem solving. If, however, a teacher demonstrated his or her thought process or highlighted particular problem-solving strategies while solving a problem in front of the class, I coded that as an instance of modeling problem solving.

Instances of *promoting metacognition* were particularly difficult to code. I had a broad understanding of what I considered to be metacognitive behavior, because the literature on metacognition indicates a wide range of what researchers consider to be metacognition. It was sometimes difficult to distinguish between a teacher encouraging students to *think* and the teacher encouraging them to *think about their thinking*—that is, to be metacognitive. If a teacher simply asked students to think about a particular problem or mathematics concept, I did not code that as metacognition, but if a teacher called attention to *how* the students were thinking, I considered that to be an instance of promoting metacognition. I paid particular attention to metacognitive questions, so any question teachers asked that drew attention to thought processes, knowledge, or decision making—either theirs or the students—I coded as metacognition. Furthermore, any time a teacher encouraged students to monitor their problem solving—including checking their work—or reflect on the problem or solution, I coded it as metacognition. If a teacher modeled metacognition while modeling problem solving, I coded the instance as both modeling problem solving and metacognition.

Coding multiple solutions was sometimes difficult because I wanted to make a distinction between a solution and an answer, but this distinction was not always clear-cut. I considered a *solution* to be an approach to solving a problem—that is, a solution path—and an *answer* to be simply the final result. It was possible for a problem to have more than one correct answer, but having more than one correct answer did not mean that the teacher necessarily highlighted multiple ways to solve the problem. For example, there may be multiple answers to problem requiring an equation for a trigonometric function. But if those answers do not correspond to distinct methods of solving the problem, then they do not qualify as multiple *solutions*.

I coded each teacher's interviews using a similar method, noting instances in which the teacher referred to any of the items listed above. In addition, I noted instances in which teachers expressed their beliefs about problem solving and what makes a good problem—this process was more like note taking than coding.

CHAPTER 4

RESULTS

The present study addressed the following research questions about a select group of high school mathematics teachers who had a strong reputation for effective teaching and had been identified as teaching through problem solving:

1. What do these teachers believe about mathematical problem solving?
2. What do these teachers believe about what makes a good problem, and what kinds of mathematical problems do they pose?
3. What practices do these teachers use when teaching through problem solving, and how do they implement those practices?
   a. What problem-solving strategies do teachers introduce?
   b. How do teachers model mathematical problem solving?
   c. What do teachers do to limit the amount of input they give students? For example, how do teachers incorporate group work?
   d. How do teachers encourage metacognition?
   e. How do teachers highlight multiple solutions?

The data for the study included classroom observations with four teachers—Miss Atkinson, Mr. Fulbright, Mr. Dalton, and Mr. Bailey. Each teacher had his or her own style of teaching through problem solving. There were also practices common to all the teachers, as I discuss in this chapter.

This chapter is divided into two main sections: teacher beliefs and teacher practices. In the first section, I address the first and second research questions: What do the teachers believe about problem solving, and what do they believe about what makes a good problem? The bases of these descriptions are interviews with the teachers and examples of problems they assigned.

The second section of the chapter addresses the third research question and consists of descriptions of teaching practices the teachers either demonstrated in the classes I observed or described in interviews. There are five practices that correspond to the five parts of the third research question. The practices are as follows: (a) teaching problem-solving strategies, (b) modeling problem solving, (c) limiting teacher input, (d) promoting metacognition, and (e) highlighting multiple solutions. In the final part of the section on teaching practices, I give evidence that the teachers valued Polya's phases of problem solving—particularly *understanding the problem* and *looking back*—even though none of the teachers mentioned Polya by name. The chapter ends with a discussion of teacher collaboration. The three teachers at Northridge were part of a mathematics department for which collaboration was important. Collaborating with colleagues is not a teaching practice but was one way the teachers were supported in their efforts to teach through problem solving.

Teacher Beliefs

This section contains descriptions of the four teachers' beliefs about problem solving and what makes a good problem. In interviews, I asked the teachers what they thought about mathematical problem solving-and whether they viewed problem solving as a way for students to learn new material or as an opportunity to apply mathematical content they had already learned. In order to understand what the teachers believed makes a good problem, I asked them directly in our first interview: "What makes a good problem? What do you look for in a good problem?" I also collected examples of problems they assigned.

Teacher Beliefs about Problem Solving

In the first interview, I asked the teachers to talk about what came to mind when they heard the term *mathematical problem solving*. Each teacher began with a concise response and then elaborated. As they spoke, it became apparent that they had pondered this question before, and that they recognized the multi-faceted nature of problem solving. Each of the four teachers expressed the belief that problem solving is an essential part of learning and doing mathematics. They indicated that simply watching a teacher do the work of problem solving is not likely to help students develop as problem solvers and mathematical thinkers—students need experience in solving mathematical problems for themselves.

In Miss Atkinson's view, problem solving is at the core of mathematics. In her first interview, she said, "I think mathematics should be all about problem solving."[2] She noted that mathematics is a means to an end, the end being to solve meaningful problems: "I think the whole reason we have mathematics as a society is as a tool for solving problems." She said she tried to help her students see that part of the value of mathematics is its usefulness for solving problems:

> [I am] trying to develop along the way this idea that we have developed math out of this need to solve our problems, to keep track of stuff, and to explain phenomena in the real world. … That's what we do as a society. We create things that we need to solve problems.

She pursued this goal by assigning problems involving real-world phenomena such as energy consumption and the motion of a swing.

In her first interview, Miss Atkinson said she was familiar with Schoenfeld's framework for mathematical problem solving—resources, strategies, control, and beliefs—and believed the framework to be a good summary of what is important in problem solving:

> Problem solving is a lot of different things. Everything from the critical thinking to, you know, having your mathematical resources and tools, to just general heuristic strategies like "Is working backward appropriate here?" "[What about] breaking the problem down into a lot of different component parts and then putting it all together at the end?" … And then having some sort of metacognition going on. … You know, having some control over what you're doing. I think all of those aspects are what should go into problem solving—that and just believing in yourself and in your ability to do it and having some persistence and perseverance.

[2] Unless otherwise noted, all quotations are from teacher interviews.

Miss Atkinson mentioned ways to help students believe in themselves and develop perseverance, including encouraging students to share their ideas and having confidence in her students' abilities to solve problems.

Making mistakes is part of problem solving, and Miss Atkinson believed it was important to create a safe, supportive environment when teaching through problem solving:

> Another big part of problem solving is obviously making your environment really safe for your kids to try things, and to take intellectual risks, and to make some mistakes and that be ok. … I'm trying to make it ok for them to mess up. … It's ok to try something; it's ok if it doesn't work out because then you can try something else.

She pointed to group work as a key element in creating a safe atmosphere: "I do feel like my classroom is a pretty comfortable place to be. Maybe not for every single student, because some of 'em are so shy. But the small group atmosphere does alleviate some of that."

In her second interview, I asked Miss Atkinson whether she viewed problem solving as a vehicle for learning new material or as an opportunity for students to use content they already knew. She quickly responded, "Both," and gave an example of each. Some of the investigations she assigned were meant to help students discover mathematical concepts, and others were intended to help students synthesize concepts they already knew.

To give an example of using problem solving to introduce new material, Miss Atkinson described an investigation in which students "didn't know anything going in, and they figured a lot of stuff out just by doing it." It was an investigation in which students graphed a series of composite functions and looked for patterns in order to discover how to use "the idea of compositions of functions as a graphing tool." She gave the example of graphing $f(x) = \sqrt{4 - x^2}$ by starting with the graph of $f(x) = 4 - x^2$. She noted, "The kids have never done this before. They'd never even thought about this before, but it's a really powerful and somewhat sophisticated graphing technique … that was totally new content for my students."

Using problem solving as a means to synthesize existing knowledge is similar to using problem solving as practice (Stanic & Kilpatrick, 1988), but synthesis requires a higher level of thinking than practice does. Miss Atkinson described a precalculus activity in which students synthesized their knowledge about parametric equations and sinusoidal functions. In the activity, students measured the horizontal and vertical movement of a swing over time. They graphed these data points and developed a model for the periodic motion of the swing using parametric equations. Miss Atkinson concluded,

> So for that investigation they're actually applying their sinusoidal concepts along with their parametric concepts. It's a nice synthesis, if you're a fan of Bloom [Bloom's Taxonomy (Bloom, 1956)], which I am. … I would consider that to be an investigation with existing content.

Mr. Fulbright shared some of the beliefs Miss Atkinson expressed, but discussed other views as well. He believed problem solving involved confronting situations in which neither the solution nor the method for finding the solution was obvious. He contrasted problem solving with "going from question to answer with a predetermined, dictated approach," and noted in his first interview, "When I think of problem solving, I think of the students having to figure out what approach, on their own, is appropriate." Problems that were conducive to students figuring out an appropriate approach were "open-ended problems with very few directions." Such problems could be solved in various ways, and some even had more than one correct answer.

Mr. Fulbright also said it was important to "give [students] the material they need to confront apparent contradictions … and then they have to resolve things in their own mind." In other words, it was important to him to help students acquire the proper resources so that they could solve problems on their own. He contrasted his approach with simply telling students what to do or just teaching skills:

> [Students are] not learning anything useful unless it changes the way they think about things somehow. If you can't change the way people think about things, they never have to really think. If all they're doing is just mimicking skills, then 2 years after the course is over, they've really gained next to nothing.

When I asked Mr. Fulbright in his second interview whether he saw problem solving as an opportunity for students to learn new material or as something that came after mathematical content had been learned, he replied, "I think it can serve both purposes." He added, "But I think the way you design it is different depending on what your purpose is." The main difference was the amount of guidance he gave students, particularly in written instructions. He gave students more guidance when his goal was for them to learn new material.

If Mr. Fulbright used a problem to introduce new mathematical content, he worded a problem very carefully so that students would discover what he wanted them to discover. He explained,

> If your purpose is to have them learn something new that you haven't taught them before, it needs to be … quite a bit more guided because you want them to achieve a *particular* goal. You don't want 'em to just go off in some weird direction.

He gave an example of a homework assignment in which students worked through a series of differentiation and integration problems that led them to discover that "integration by substitution is essentially reversing the Chain Rule when you differentiate. … I hadn't taught them the technique. I was hoping they would kind of figure it out on their own." Mr. Fulbright noted that there were times when he adjusted a problem statement in order to give students more guidance, but he was careful not to diminish the problem-solving nature of the experience for students. He gave the example of adding two sentences to a problem in order to guide students where he wanted them to go:

> And just adding in those two sentences, it gave [the students] enough guidance. I think it was still a problem-solving kind of a problem because there was nothing in the problem

> that said what the differential equation should be. … They still had to figure some things out on their own, but those two sentences were just what was needed to push them along.

Mr. Fulbright also assigned problems so that students could apply mathematical concepts or synthesize their knowledge. In this case, he gave very few instructions and very little guidance. In describing what he believed were characteristics of a good problem, he referred to using problems that required students to apply what they knew:

> What I think makes a good problem for problem solving is the students need to be at a point where … they've learned new tools, and now you want to help them figure out on their own how to apply those tools in a new context, new situations, and so on. When we do investigations, that's usually what we're doing: We give them very open-ended word problems, with hardly any guidance and hardly any suggestions as to how they might go about solving it. But they've got enough tools at that point.

Unlike his desire that students *not* "go off in some weird direction" when he used a problem to introduce a new concept, with application problems, Mr. Fulbright was "thrilled if they [went] off in a strange direction." The Subway Problem (see Figure 5) is an example of an application problem that I discuss in the next section.

Mr. Dalton's comments revealed some beliefs that were similar to Miss Atkinson's and Mr. Fulbright's, but he had additional views as well. He described teaching through problem solving as "having students work through problems that they haven't been taught how to solve." He listed some of his goals in teaching through problem solving: (a) to have students think rather than him telling them what to do, (b) to help students see what mathematics is and what mathematicians do, and (c) to get students interested in mathematics.

First, it was important to Mr. Dalton that students engage in problem solving, and he stated that he was "trying to get them to think through things." He talked about the resistance of some students to engage in mathematical thinking, saying many of them had never been challenged to wrestle with difficult mathematical concepts until they went to Northridge High

School. Second, Mr. Dalton used problem solving to give his students a glimpse into what mathematics is all about and what mathematicians do:

> What you're trying to do is to get them engaged in mathematical thinking and to recognize that that's what mathematics is. It's not repeating stuff that you've been taught. It's figuring out stuff that you haven't been taught *using* the stuff you've been taught. That's more of what mathematicians do.

One way he attempted to get students engaged was to have them work on problems that had no known solution. He emphasized, "You don't have to do the whole thing, but the way mathematics proceeds is with partial results." In other words, making progress on a problem or only solving part of it—for example, solving it for a few cases—is crucial in the development of mathematics, and he wanted students to appreciate that. Third, Mr. Dalton believed problem solving was a good way to get students interested in mathematics. He wanted to guide students into further mathematical studies at the collegiate level and "to get as many of them going into mathematically oriented careers as we can."

In his second interview, Mr. Dalton said he believed problem solving could be both a vehicle for learning new content and an opportunity for students to apply and synthesize content they already knew. Sometimes he used a problem to motivate the study of new material:

> There are times when we'll look at a problem and reach a point where we just don't know, we don't have anything to do, and there's a technique that we need to learn to get us past that hump. So there are times when you'll use a problem like that to sort of point out that you don't have anything that'll work here. And so that's the need for the new topic.

He said using problems to introduce new content was not as customary for him as assigning problems after students learned the mathematics necessary to solve them. The latter use of problems helped students synthesize their knowledge by "pulling ideas together." The Tractrix Problem (see Figure 7) fulfilled both purposes: It required students to synthesize content they already knew and also provided an opportunity for Mr. Dalton to introduce new techniques of differentiation.

The fourth teacher in the study, Mr. Bailey, shared some of the beliefs described above, but also had his own way of thinking about problem solving and teaching through problem solving. When I asked him in his first interview what came to mind when he heard the term *mathematical problem solving*, he responded by describing problem solving as both a means and a goal of learning mathematics. He contrasted his current beliefs with the views he held early in his teaching career:

> For many years, when I was a lecturing kind of guy, I thought that my main thing was to show them how mathematics works, and then turn them loose on solving the problems. But then I came to realize that really, solving the problems is what it's all about, and that in the process they can find out how mathematics works, and maybe have a little more of a hunger for what else they can learn about this to make it easier to solve the problems. So I've just come around to the modern realization, I guess, that that's what people need to be able to do: to solve problems.

In other words, on the one hand, mathematics is a means to an end, the end being the ability to solve problems. On the other hand, in the process of solving problems, students learn mathematics. He explained that one of the disadvantages of his previous way of teaching was that although he prepared students for the test, all they could do was perform well on the test. They were not able to engage in creative problem solving, because creativity had never been required of them.

Mr. Bailey's primary means of presenting new material was by having students work on problems, especially the Problem of the Day (POTD). He said, "It's about the same thing every day: We use the problem as an entry," and gave an example of a calculus POTD (see Figure 9) that led into a discussion of partial fraction decomposition:

> I find that there's a particular entry that I wanna make into what it is I wanna teach 'em that day. And I know that I wanna have a Problem of the Day that exposes that so that

> they all see that entry, and they all get their toe in the door. … So like today [in calculus class], I mean, I know that I wanna teach them partial fractions, and I want them to come up with an idea that it comes from somewhere. So let's have 'em make the jump from a fraction with real numbers to a rational function and see that, "Oh yeah, this makes sense."

Mr. Bailey also noted,

> [The students] didn't know partial fractions when they started that Problem of the Day. But then, what I'm trying to do is give them the idea that they can actually discover this without being told it in advance if you show them the door the right way.

Mr. Bailey said he believed it was important for students to do the mathematics rather than just watching him do it. He noted that despite the unpredictability of what students might do, he preferred turning the responsibility over to them rather than doing all the mathematical problem solving himself:

> With [a problem-solving approach] it takes a little faith, because I'm turning a lot of it over to [the students] basically. Sometimes it's a little harder on some days to reel it all back in 'cause it gets a little crazy depending on which roads they go down. But it's still better than the old days, I think. … And it's very refreshing to have them doing the math in my classroom rather than just me. It makes more sense to have them doing it.

Teacher Beliefs About What Makes a Good Problem

It seems obvious that teaching through problem solving requires choosing the right problems. Choosing the right problems includes judging not only which problems are suitable, but also how many problems to assign in order to give students sufficient experience with problem solving without overloading them with an impossible amount of work. The teachers in this study were all very thoughtful in their choice of problems. There was a range of difficulty as well as variation in the time required to solve the problems. There were problems students solved quickly during class, and others that were in-depth projects that required several days to complete.

Giving students lots of problems to solve did not seem to be as high a priority for the teachers in this study as giving students worthwhile tasks to do. That is not to say the teachers did not give students ample opportunities to solve a wide range of problems. Both Mr. Dalton and Mr. Fulbright spoke about the importance of practice in becoming a good problem solver. In his first interview, I asked Mr. Dalton to summarize what he did to emphasize problem solving and he replied, “Well, using lots of problems in class.” Mr. Fulbright said it was important to let students practice problem solving for themselves rather than just watching him solve lots of problems:

> Watching somebody do it is not the same as doing it yourself. [My students] all think that’s all it takes: watching me do problems at the board. … Mr. Dalton likes to say, “If all it took was watching to get good, I’d be a great tennis player.” But you have to practice tennis. You gotta do it yourself, but the students don’t like that. “I understood it in class. I understood when you did it.” “But you didn’t try it yourself, did you? You didn’t sit down and practice, practice, practice.”

Practice is essential to a student becoming proficient with mathematical skills and procedures. Practicing procedures is not the same as problem solving, but as Miss Atkinson noted, “We need all kinds of thinking; we need procedural thinking ‘cause that makes the grunt work later on easier.” If the “grunt work” is easier, students’ cognitive energy is freed up to engage in problem solving. Mr. Bailey had his students complete practice exercises for homework in order to improve their procedural skills. He described one homework assignment as “line calisthenics,” meaning a set of exercises involving linear equations. Mr. Bailey encouraged his students to make the most of opportunities to practice, and said to them, “That’s how the game is played, and the way you learn how to play the game is by practicing, just like any other game, and that’s what the homework was all about.”

Choosing good problems is central to teaching mathematics through problem solving. Good problems are those that are interesting, challenging, and mathematically rich (Grouws, 2003; Marcus & Fey, 2003; NCTM, 2000; Schoen, 2003). They need not be “real-world” applications (Hiebert et al., 1996; Kahan & Wyberg, 2003), but problems that are relevant to students’ lives can be particularly engaging. Assigning good problems was a priority for each of the teachers in this study, and the quality of the problems was clearly more important to the teachers than the quantity of problems.

When Miss Atkinson described problem solving in her first interview, she highlighted the importance of assigning problems that are meaningful and require students to synthesize their mathematical knowledge:

> For me, problem solving is taking, as much as possible, meaningful problems that could have some impact on society … and [having] students find ways to use math to model those problems. … I like this idea of synthesis and putting together a lot of different mathematical skills.

An example of a task Miss Atkinson assigned that was meaningful as well as “interesting” and “eye-opening” was the Energy Lab: an investigation and series of questions involving energy savings. She summarized the main problem:

> “What if, for every baby born, we replace one incandescent light bulb with a CFL [compact fluorescent light bulb]? How much energy can we save in [10 years]?” So it ends up being an exponential problem, which is exactly the kind of stuff that we’ve been working on. … It’s interesting, and it’s eye-opening.

Miss Atkinson noted that good problems are challenging and require students to think differently about mathematical concepts and what it means to do mathematics. Thinking differently is particularly important for students who are new to a problem-based approach to learning mathematics. Miss Atkinson described how she began her algebra course at the beginning of the year:

> On the first day, I give them a problem with data to model. The data [are] far from linear, so their preconceived ideas about linear regression (which none of them truly understand) fail, and I tell them that they can only use calculators to do arithmetic. They have to re-express the data to make [them] linear, find some way to model the data linearly. … The

> activity forces them to think differently about data, think through and synthesize a variety of the concepts they know, defend their decisions, work collaboratively, and realize that there isn't always just one right answer in math.

Miss Atkinson sought to maintain this level of challenge throughout the year: "I just don't let up on assigning challenging problems."

As part of an algebra activity that I observed, Miss Atkinson assigned the problem in Figure 4, and the students worked in groups to solve it. It is an example of a problem that is interesting, challenging, and mathematically rich. It is unlike typical quadratic "application problems" that give a height equation, the initial height, and the initial velocity and ask students to fill in information such as maximum height of an object and the amount of time the object is in the air. That information is required by Miss Atkinson's problem, but students had to first use the given information to write the height equation. The initial height is given—the rock left the slingshot at a height of 4 feet—but the initial velocity is not. In fact, figuring out the initial velocity of the rock is Part (d) of the problem. Students were also required to interpret the data in the problem statement rather than simply substituting the data into a known formula. Additional requirements of this problem are reflection and interpretation. In Part (e), Miss Atkinson asked students to look back on the solution and compare it to their expectations about the coefficient on x^2. And finally, after reflecting on the solution, students were to interpret any discrepancy between their solution and their expectations. In summary, this problem required students to apply their knowledge of quadratic equations to a new situation. The problem was interesting and challenging. It required students to think mathematically and to interpret and reflect on the solution.

More Linear Systems Applications

Some (not-so-bright) physics students shot a rock straight up using a slingshot and planned to gather some data about its flight. The rock left the slingshot at a height of 4 feet. One student with a stopwatch determined that the rock passed the window where he was standing in the physics building 1 second after the shot, and the window was 58 feet high. The student who launched the rock was supposed to measure it when it landed, but it wedged in a tree branch on the way down after 5 seconds. So, the student measured the height of the branch to be 34 feet.

a) Use the data above (you should have three data points) to write a quadratic function that models this scenario.
b) Use your function to determine when the rock reached its maximum height. What was the maximum height?
c) When would the rock have landed if it hadn't gotten stuck?
d) According to your function and similar past examples, what was the initial velocity of the rock when it left the slingshot?
e) Was the coefficient on x^2 what you expected? If not, what do you think could account for the difference?

Figure 4. Example of an algebra problem.

When I asked Mr. Fulbright in his first interview to describe what makes a good problem, he responded that it is an open-ended task without an obvious solution. In addition, there are multiple ways of solving good problems. He described good problems as

> open-ended problems with very few directions so that students not only have to solve the problem but they have to figure out what approach is required to solve the problem. … They might get different solutions, assuming a problem permits different solutions. Even if it permits only a single answer though, there still might be different approaches to get to it.

The wording of a problem was very important, according to Mr. Fulbright, particularly if the purpose of the problem was to help students discover some mathematical concept:

> You do have to give them enough guidance that they'll discover what you want 'em to discover, and not just go off in left field. But then you just have to come up with a very carefully worded … problem so that when they think about it, they will discover on their own what you want 'em to. But the wording is everything—coming up with just the right question.

According to Mr. Fulbright, good problems are interesting, relevant, and authentic. He said he assigned problems arising from current events and used original documents and raw data whenever possible. He noted,

> Students are more engaged by material that they feel is not predigested. … A radio story that I just heard on the news that morning and I share with them is something that they're getting the same version of it that I heard. They're not getting a version that I distilled for them. And if there are questions to be asked, questions to be solved, and so on, I think they have much more of a problem-solving bent to them when the raw data that they see [are] not predigested.

Mr. Fulbright also kept his students engaged and interested by letting them choose problems or investigations that appealed to them.

As an example of a problem that was engaging for students and also had a connection to something outside pure mathematics, Mr. Fulbright described a "variation on the 'Three Body Problem'" from physics. He explained,

> It was hard to come up with something that was simultaneously engaging and doable. … And something clicked in my mind and I thought, "Earth, moon, spaceship—three body system—this is perfect." And so I designed this investigation where the students' goal was to write the differential equations describing the motion, the gravitational influences on the spacecraft, and then when they finished all that, come up with a trajectory that'll do the slingshot [of the spacecraft] around the moon.

Three of Mr. Fulbright's students were "super excited" to work on the problem, not just because they found it interesting, but also because it was an original problem: "It would be the first time any student in the country had worked on this problem." Mr. Fulbright added,

> This goes back to that "original documents," "predigested information" [discussion]: Whenever students work on something that's kind of predigested, [and] they know that somewhere out there there's an answer and people have done this before, it's less interesting. When they know—or think—that they're doing something for the very first time, it's much more exciting.

He went on to say that the students who worked on this problem did some of the best work he had ever seen students do.

Figure 5 contains an example of a problem Mr. Fulbright assigned as a long-term project that students worked on in groups to solve. The problem was open-ended—not only were there various solution methods, but there also were various mathematical models that would adequately answer the question. The problem statement was brief and gave no guidance about how to begin the problem. Students had to use their mathematical resources to solve a problem that was truly problematic.

> The question: A city is planning on building a new subway line. How far apart should subway stations be placed if the goal is to minimize the total time that a typical commuter takes getting from her home to the place where she works?
>
> Note: If you are not familiar with what subway systems look like, you can refer to a subway map of New York City on the Internet or ask me for a paper one. If you do refer to the map, do not suppose that the separations of stations on existing subway lines are necessarily optimal. The map is only to help you get a better understanding of the problem.

Figure 5: Subway Problem.

I observed Mr. Fulbright's calculus class working on an investigation involving differential equations, and Part 1 of the investigation is shown in Figure 6. The problem emphasized interpretation and conjecture: What does the variable *k* represent? What might influence the value of *k*? The problem also included an opportunity for students to investigate the differential equation using a graphing program. Finally, the problem invited students to reflect on what they discovered: "Is this what you expected?"

Calculus Lab
Differential equations involving contact frequency

Part 1: Modeling the spread of a disease using differential equations.
If there are 650 people in all, and if P people have the cold at present, then the number of susceptible people must be $650 - P$. Thus, the differential equation modeling the spread of the disease is:

$$\frac{dP}{dt} = kP(650 - P)$$

Think about the meaning of k. What might make k larger? What might make it smaller?

Let us suppose that $k = 0.0013$, when t is measured in days. Use [a graphing program] to plot the slope field for this differential equation, and also the particular solution corresponding to the situation in which one person comes to school with the cold at $t = 0$.

Replace the number 0.0013 with a parameter k, and then animate on k. What happens to the slope field and the solution for larger or smaller values of k? Is this what you expected?

Why does k need to be so small?

Figure 6: Calculus investigation.

Mr. Dalton described problem solving in his classroom as "having students work through problems that they haven't been taught how to solve." This comment gave evidence of what he thought about good problems: Their solutions are not obvious, but the students have the resources to solve them. He said that students "should expect to see, on a regular basis, problems that do not fit any mold that they have seen, but which they know how to solve if they just think about it."

Good problems, said Mr. Dalton, require students to synthesize mathematical concepts—not just concepts from the previous lesson, but things they learned months or even years previously:

> So a problem that you're working on may use the technique you learned yesterday in combination with something that you learned 4 weeks ago, and finally using a technique that you learned a year ago in calculus [class], so that, even though you learn them over time, you want them to be interchangeable within your mind.

He also said good problems have more than one step and may require students to use part of the answer to address the next part of the problem. The Tractrix Problem, shown in Figure 7, is an example of a differential equations problem that required students to bring together several mathematical concepts. Mr. Dalton led the class through a solution of this problem and highlighted various mathematical concepts and procedures they would have to synthesize in order to solve it: the Pythagorean Theorem, integration by substitution, trigonometry, and some complicated algebraic manipulation.

(Mr. Dalton verbally described the problem while drawing an accompanying diagram; the problem was not presented as a written statement.) A motorboat begins at the origin and moves with constant velocity v vertically along the y-axis, pulling a water skier using a rope of length r. When the boat is at the origin, the skier is on the x-axis so that the skier's initial position is $(r, 0)$. At time t the water skier is at (x, y). What is the equation of the path that the water skier follows?

Figure 7: Tractrix Problem.

In Mr. Dalton's view, good problems may or may not be set in "real-world" contexts. The projects he assigned in his matrices course typically related to applications outside pure mathematics, but many of the combinatorics problems, he said, did not: "There aren't that many combinatorially rich application problems. … The combinatorics [problems] are more sort of 'mathy for math's sake' problems."

Many of the problems Mr. Dalton assigned were open-ended. For example, he had his statistics students design an experiment to test the jumping distance of two sizes of origami frogs. He gave minimal guidance, so the students had to come up with ideas on their own, including how large a sample to take and how to measure the jumping distance. Mr. Dalton explained that he started his mathematics courses every year with simple, open-ended problems. Later in each course, the problems got more difficult and become even more open-ended: “As the year progresses, the problems get more complicated and have less scaffolding.”

Mr. Bailey began each class period with the POTD that was usually a preview of new material in the upcoming lesson. That is, the problem contained the content to be learned and Mr. Bailey used the problem as a starting point to launch the lesson. He explained that he used problems as a means to help students make discoveries. I asked him how his teaching had changed since he began focusing on problem solving, and he replied,

> I stopped doing the important math (the instructive problems, the significant discoveries) and became focused on how I could get the students to do it instead. I used to ask myself, “How can I make this clearer to my students?” Now I ask myself, “What sort of question can I pose so that they can discover the path to the answer without thinking it came from me?”

It is important, he said, to “[choose] the right questions to elicit those little mathematical epiphanies.”

Some of the POTDs had the potential to be routine in that students could use procedures or algorithms to solve them. But often a procedure was unknown to the students, so the problem truly was a problem for them. The POTDs varied in terms of open-endedness, level of difficulty, and length of time required to solve them. Some had many correct responses, and some had only one answer. Some were difficult and required most or all of the class period to solve, and others took only a few minutes. Figure 8 gives an example of a very open-ended problem. There are many correct answers for the graph as well as the real-world scenario.

Sketch a scatterplot that shows two clusters of points, each with a clear positive association, such that the graph of the two clusters together shows a clear negative association. Can you think of a real-world scenario that this graph could describe?

Figure 8: Problem of the Day for a finite mathematics course.

One of Mr. Bailey's calculus classes began with the problem in Figure 9, which is an example of a problem that was not routine for the students even though there was a procedure for solving it. The problem provided an entry point for a lesson on antidifferentiation through partial fraction decomposition. Mr. Bailey did not simply give his students the tool and show them how to use it. Rather, he allowed them to discover the process of partial fraction decomposition while solving the Problem of the Day.

> Advice: pay attention to the hint.
>
> Find $\int \frac{dx}{x^2 + x}$ without a calculator.
>
> Hint: $\frac{1}{12} = \frac{1}{3} - \frac{1}{4}$

Figure 9: Problem of the Day for a calculus course.

In his second interview, Mr. Bailey described an evolution in how he gave tests. Early in his career, before he began focusing on problem solving, his tests contained primarily computation questions: "You can make up 20 different kinds of questions about logs [logarithms], and each one with a little new twist to it." He came to realize the limitations of this kind of test: "I found that when all is said and done, you've just done 20 questions on logs." So he broadened his focus to include a wider range of problems than those requiring just computation: "Now I'm much more conscious about having variety [of problems] in there, as much application kind of stuff as I can, so that it looks more interesting and applicable and worthwhile."

Miss Atkinson and Mr. Fulbright both mentioned a feature of a good problem that I had not considered: students' affective responses to the problem. For example, Mr. Fulbright took students' emotional responses into account when choosing problems. He gave an example of a problem in which students were to take data from children's growth charts, investigate the change in average height and weight of boys and girls over time, and create graphs of these changes. The goal of the problem was to see what the graph of a derivative reveals about a set of data. Mr. Fulbright was sensitive to the fact that discussions of height and weight could be uncomfortable for some students: “If I had any students that I thought would be sensitive about their weight, for example, or their height, I probably wouldn’t have done this.”

Miss Atkinson assigned a problem involving energy savings—the Energy Lab Problem—in place of the Garbage Disposal Problem that she used the year before. She explained why she made the change:

> The energy problems are replacing this Garbage Disposal Problem that we did last year where, you know, as the population increases, … trash disposal is getting exponentially out of control. … And so that seems like the disheartening lab, whereas the Energy Lab [Problem] is more positive and, “This is what I can do, I can save this way.” … The math is the same. The Energy [Lab] Problem is very empowering versus the Garbage Disposal Problem is kind of just disheartening when you think about landfills and already the bad situation we’re in is getting exponentially worse.

Both problems addressed environmental concerns and involved the same mathematics: exponential growth. But Miss Atkinson chose the problem that would be empowering and foster a more positive outlook on the part of her students.

Practices for Teaching Through Problem Solving

The last few decades of research on problem-solving instruction have yielded a handful of teaching practices that experts say can help students develop their problem-solving ability. The quantity and especially the *quality* of the problems are important, and in the previous section I indicated the kinds of problems the teachers in this study used. This section contains descriptions of practices the teachers used and how they used them. The practices are: (a) teaching problem-solving strategies, (b) modeling problem solving, (c) limiting teacher input, (d) promoting metacognition, and (e) highlighting multiple solutions.

Teaching Problem-Solving Strategies

There was a time when many mathematics educators relied on teaching problem-solving strategies—heuristics in particular—as a primary means of helping students become better problem solvers (Kilpatrick, 1985). Unfortunately heuristics training generally failed to produce the positive results researchers hoped for. Recall Schoenfeld's (1987b) comments:

> Despite all the enthusiasm for the approach, there was no clear evidence that the students had actually learned more as a result of their heuristic instruction, or that they had learned any general problem-solving skills that transferred to novel situations. (p. 288)

Teaching problem-solving strategies, though not the only way to help students develop as problem solvers, can be one of the practices teachers use to teach through problem solving. These strategies need not be taught in isolation. Rather, teachers can teach problem-solving strategies while modeling problem-solving behavior by highlighting particular strategies as they occur in the course of solving problems. Two of the teachers I observed—Mr. Dalton and Mr. Bailey—used this practice.

Mr. Dalton taught both general and specific problem-solving strategies as he modeled problem solving. He was often explicit in his suggestions about how to think about problems, but he did not teach problem-solving strategies in isolation. He highlighted general problem-solving strategies along the lines of Polya's (1957) heuristics. Some examples of these strategies were (a) assume there is a solution, (b) consider domain restrictions, (c) use unsuccessful attempts to lead to successful ones, and (d) try a specific case. Mr. Dalton talked about the third and fourth strategies in several of the classes I observed as well as in interviews, so they warrant special attention.

Every problem solver runs into roadblocks. According to Mr. Dalton, that is no reason to give up, in part because a roadblock can indicate something useful for solving the problem.

Polya (1962) made a similar observation, noting that even a failed attempt at solving a problem is not a waste: "Even such a misconceived trial need not be quite useless; it may lead us to a more adequate appraisal of the proposed problem" (p. 63). During the first week of the differential equations course, Mr. Dalton gave students the following advice:

> Don't spend overly long thinking through "Is this gonna be exactly the right thing?" Make sort of the most obvious choice. Don't be surprised if it doesn't work, but think about why it didn't work. We'll see that multiple times during the year, where we'll try something, [and] it didn't work. The way in which it doesn't work tells you exactly what you should have done. So making that first mistake early, and looking thoughtfully at it, reflectively on it, is always a really good strategy when you're trying to solve a problem.

Note the value Mr. Dalton placed on perseverance and reflection, two elements of successful problem solving. He also encouraged his students to resist the temptation to completely abandon an unsuccessful solution attempt. Rather, he suggested making revisions, "modifying [the] answer rather than coming up with a complete answer from scratch."

One of the problem-solving strategies Polya (1957) suggested was *specialization*: "passing from the consideration of a given set of objects to that of a smaller set, or of just one object, contained in the given set" (p. 190). For example, if it is too difficult to solve a problem for every case, one might try a specific case. Mr. Dalton recommended this strategy to his students. In his first interview, I asked him how he responded to students who were struggling to solve a problem during class. He replied, "If you see that they're either heading off in the wrong direction or struggling with one of [the problems], not seeming to get anywhere, then you sort of ask them questions about 'What about this simple case?'" He gave one assignment for the purpose of encouraging students to try specific cases:

> One of the first problems in graph theory … was still an open question. But what I wanted to do, again, was sort of think about, "Can you get any piece of it?" Ok, that in this special case, can I show that it's true? … You don't have to do the whole thing, but the way mathematics proceeds is with partial results. I can do these special cases, but I can't do the thing in general.

By encouraging his students to try specific cases, Mr. Dalton helped them solve problems, but more broadly speaking, he helped them to think like mathematicians.

Mr. Dalton used the first several class periods of the differential equations course to introduce specific strategies the students would need to solve problems they would encounter in the course. He made his purpose explicit, saying to the students,

> These initial problems are trying to lay the groundwork for the kinds of things we'll be doing as we go through the course, and also trying to make some suggestions for how to approach some of the problems that we'll be doing.

He used these initial problems to illustrate general problem-solving strategies—such as using roadblocks to actually make progress on solving a problem—but also to introduce specific techniques for solving various kinds of differential equations. Examples include the following: (1) techniques of integration such as u substitution, (2) uses of calculus or algebra to make a problem or equation clearer, and (3) decisions about what "chunks" to keep together when solving an equation. Mr. Dalton highlighted the third item as a particularly important technique.

By *chunks*, Mr. Dalton meant pieces of an equation that should be held together rather than multiplied out or separated. For example, the equation $(x - 2)^2 = 25$ can be solved most efficiently if one keeps $x - 2$ as a chunk, solves for $x - 2$ by taking the square root of both sides of the equation, and then solves for x. It is inefficient to break up the chunk by multiplying out $(x - 2)^2$ and proceeding from there. Mr. Dalton introduced the idea of chunks during the first week of the differential equations course:

> This is another thing that we'll see a lot in the course—and this may be the first time you've seen it, which is good—but we'll see [instances in which it is useful to ask], "What's the chunk that we want to keep together?" Let's think of that as the most important thing, and let's organize our calculus so that we keep these pieces together.

In fact, Mr. Dalton told his students that he chose a problem (see Figure 10) specifically to introduce the idea of chunks:

> The primary reason that I picked this problem as the first problem is thinking about this expression right here [$x – 2l$] as being … the important variable of the problem. I wanna keep that expression together if I can.

Taking a Whipper (Mr. Dalton verbally described the problem while drawing an accompanying diagram; the problem was not presented as a written statement) A rock-climbing rope is attached at the bottom of a wall ("belay") and at a few points on the wall ("protections"). If the climber falls, he/she will fall twice the length of the rope from the last protection. The rope stretches a bit to cushion the fall (i.e., reduce the force on the climber). Suppose climber has used a length of rope L (total height climbed). The distance from the last protection is l. Let x be the distance fallen ($x = 2l$ is when climber has reached the bottom of the rope). Let D_F be the final (total) distance fallen, including after the rope stretches. a) What is the velocity at $x = 2l$? (How fast is climber going when he/she hits the end of the rope?) b) What is the force on the climber (faller) at $x = D_F$?

Figure 10: Problem to illustrate the idea of *chunks*.

Mr. Bailey gave general advice about problem solving and also taught specific problem-solving strategies. Like Mr. Dalton, Mr. Bailey did not teach these strategies in isolation. Rather, as opportunities arose in the course of solving a problem, Mr. Bailey pointed out ways in which students could think about problem solving. An example of general problem-solving advice Mr. Bailey gave was to use instinct when solving a problem. The calculus class worked on problems about convergent and divergent series, and before beginning a problem, Mr. Bailey asked, "First of all, let's see how we're doing on our instincts here. Does this [series] smell convergent or divergent to you?"

A second example of general advice Mr. Bailey gave that did not necessarily constitute a problem-solving strategy per se was to look ahead and try to see if there might be a way to solve a problem that would be easier than another. The ways in which one applies this advice to particular problems can be considered specific problem-solving strategies. For example, the general advice to anticipate where a solution path will lead could be applied to: (a) making a problem easier by keeping one side of an equation as simple as possible, and (b) deciding how to assign parts of a function when integrating by parts.

In one of the calculus classes I observed, Mr. Bailey solved the following problem using integration by parts:

$$\int (e^{-x} \cos x) dx$$

He explained to the students his thought process for designating part of the function as *u* and the other part as *dv*: He first considered whether one part of the function would be easier to differentiate than another, or alternatively, if one part would be easier to antidifferentiate than another. He began,

> It doesn't quite matter which [part of the function] I make *u* and which one I make *dv* because this is a situation where neither one is going to get really good when we differentiate it, and neither one will get really bad when we antidifferentiate it. So I'm going to arbitrarily choose [e^{-x}] as *u* and [cos*x dx*] *dv*.

In this example, there was no advantage as to which part of the function should be *u* and which should be *dv*, but Mr. Bailey made the point that it was important to look ahead to see if there was one way to solve a problem that would be easier than another, and looking ahead required anticipating where a particular solution path would lead.

One of Polya's (1957) heuristic strategies is to exploit *symmetry*: "If a problem is symmetric in some ways we may derive some profit from noticing its interchangeable parts and it often pays to treat those parts which play the same role in the same fashion" (p. 199). There are problems that have symmetric qualities, but there are also mathematical objects whose symmetry can lead to a strategy for solving problems involving them. For example, Mr. Bailey noted that the symmetry of polar curves could be used to integrate them more efficiently, and he said to his calculus students,

> This is often a very good idea with polar curves. Polar curves all have these symmetries, and it's not a bad idea to play the symmetry card. So in other words I really only have to go from 0 to π and then times 2. So if I take twice the integral from 0 to π, I'll get the same area as I would get if I went from 0 to 2π.

Modeling Problem Solving

In the previous section, I noted that Mr. Dalton and Mr. Bailey taught problem-solving strategies by modeling the use of strategies while solving problems. In this section, I discuss aspects of modeling problem solving other than demonstrating problem-solving strategies.

Polya (1957, 1962) and Grouws (2003) have discussed the importance of modeling problem-solving behavior for students. Kilpatrick (1985) listed "imitation" as one aspect of problem-solving instruction. That is, students benefit from imitating the teacher or another experienced problem solver. Polya (1962) listed "imitation and practice" as means of improving problem-solving ability. Modeling problem solving includes the following: (a) demonstrating mathematical concepts and skills, (b) thinking aloud in order to give students insights into the metacognitive aspects of problem solving, and (c) demonstrating perseverance and a positive attitude when faced with difficulties or roadblocks.

Demonstrating mathematical concepts and skills. Mr. Dalton modeled problem solving in many of the classes I observed, particularly in the differential equations course. He also spoke extensively in interviews about the importance of modeling problem solving. Perhaps the most obvious way a teacher models problem solving is to demonstrate how to use relevant mathematical concepts and skills to tackle a problem. Mr. Dalton demonstrated various problem-solving techniques, some of which were specific to problems involving extensive algebraic manipulation. For example, he said, "If I don't want to deal with these squares, then let me solve for x^2 rather than x." As I described in the last section, a powerful technique Mr. Dalton demonstrated in class was to keep a particular piece—or chunk—of an equation together as a variable or unknown. In a differential equations problem about force and distance involving a climber and a climbing rope (see Figure 10), Mr. Dalton highlighted the expression $x - 2l$ as the useful chunk.

> This is called a chunk, a piece of an equation: That's the piece we want. I'm interested in that expression $x - 2l$. I wanna know its value when x is equal to D_F. So I would like to keep this expression together rather than multiply it all out because the force I'm interested in is in terms of that $x - 2l$. So I would prefer this form of the expression rather than that form of the expression [i.e., $\frac{1}{2}v^2 = \frac{k}{2mL}(x-2l)^2 + C$ rather than $\frac{1}{2}v^2 = \frac{1}{2}\frac{k}{mL}x^2 - \frac{2lxk}{mL} + C_1$] because it keeps my chunk together. It's written in terms of $x - 2l$. So I'm thinking of $x - 2l$ as really being my variable—that whole thing is what I'm after: how far [the rope] stretched.

Modeling metacognition. Metacognition includes being aware of one's own cognition and knowledge—including what one knows and does not know, what is easy and what is difficult, and what mistakes one tends to make while solving problems—as well as monitoring the problem-solving process and how one is thinking about the process and the problem itself. Because metacognition is such a crucial part of problem solving, it makes sense that teachers would demonstrate metacognitive behavior when they model problem solving. Miss Atkinson noted, "One thing I try to do is make my thought processes clear so that I'm modeling how I think about the problem."

When I asked Mr. Dalton in his first interview to summarize the main things he did to emphasize problem solving, the first thing he mentioned was modeling problem solving—in particular, giving students insights into his thinking process. He explained,

> I [try] to illustrate my own problem-solving techniques in problems I do in class, to sort of think about, "Ok, so how do we think about this? How do we approach this problem? … What do we see? What do we think about? … What's going to work? What isn't?" [I try] to mimic my own looking at the problem: Why did I decide to do it this way? What was it about the problem I saw that made me take this approach?

One of the features of Mr. Dalton's teaching approach was that he shared his solutions to the homework problems. For every assignment, he wrote up his solutions so the students could see examples of how to solve the problems. He explained to his students that he tried to be explicit about how he solved the problems so they could get some insight into how he thought about the problems. On a day when he distributed his written solutions to his combinatorics class, he told the students,

> I've tried to write up my solutions so that the way I was thinking about the problem is hopefully understandable. … I've tried to sort of lay it out for you so that you can see some sense about how I was thinking about the problem, how I was looking at the problem.

The quotations above show that Mr. Dalton valued metacognition and metacognitive questions, and he saw the importance of not only solutions, but also ways of thinking about a problem in order to arrive at a solution.

Talking through his thought process in class was a regular feature of Mr. Dalton's modeling of problem solving. He frequently made comments to demonstrate his awareness of his own cognition and knowledge. For example, while solving problems in front of the differential equations class, he asked such questions as, "What is it I know about this situation?" and, "Do we know how to do that?"

Part of being aware of one's own knowledge is knowing what concepts or skills are easy or difficult and knowing personal preferences. At one point in the differential equations course, a difficult integration problem arose. Mr. Dalton was leading the class in solving the problem and asked, "Do I know how to integrate $\frac{\sqrt{u}}{r^2 - u}$? (pause) No. Why not? What makes it hard? What's the problem?" A student responded, "Square root," to which Mr. Dalton replied, "Yeah, the square root.” By asking such questions as “What do I know?” Mr. Dalton was modeling metacognition.

Another part of awareness of one’s own knowledge is recognition of one’s common mistakes. Mr. Dalton was very transparent about errors he tended to make, one of which was losing negative signs along the way while solving a problem. While working through a problem on the board, he explained:

> Why do I write things in the way that I do? Because I know where my mistakes come from. They come from forgetting negative signs, so I’m gonna write it in this form because I don’t want to start with a negative, because three lines down that negative will be missing. I know that. So think about where your mistakes come from, and try to arrange things so that you avoid them if possible.

A second aspect of metacognition is monitoring, which involves overseeing the entire solution process. Monitoring includes choosing the most appropriate strategy, determining whether subgoals have been met, assessing whether a chosen strategy is leading toward a final solution, and making sure one answers the question being asked. Mr. Dalton demonstrated each of these aspects of monitoring at one time or another during my observations. For example, he explained his choice of a strategy for solving for *C* in a differential equations problem:

> We need to find the value of *C* ‘cause we have an initial value. But I wouldn’t do it here. In every problem there’s sort of a place in the process where it’s more easily found than others. And I wouldn’t do it here while we’re dealing with all these natural logs. I would simplify first.

In his second interview, Mr. Dalton talked about helping students think through their choice of strategies: “I try to give [students] some ways of thinking about a problem: ‘How would I think through this?’ in terms of ‘Well, I only know a couple of things. Could I use them?’”

Mr. Bailey also demonstrated the monitoring aspect of metacognition while modeling problem solving. During one calculus class, he solved an integration problem in front of the class, thinking aloud:

> First of all, I would look at this and say, "Well, let's see, is this a u substitution that I'm overlooking?" Well, the only u substitution that would really help me is if this denominator is u and the numerator happens to be du or a factor thereof. Well, [since the denominator is] $x^2 - 3x + 2$, what I'd be looking for [in the numerator] is either $2x - 3$ or some multiple of $2x - 3$ that I might be able to fix up by a constant, and that's not what I have there at all. So I'm gonna want to do this by partial fraction decomposition. But I can't do anything with partial fractions yet because the degree of this numerator is bigger than the degree of the denominator. That means partial fractions won't work here. So what I have to do is actually divide $x^2 - 3x + 2$ into the numerator. Fortunately, you can always divide any two polynomials.

He then proceeded to divide the polynomials and finish the problem.

*Demonstrating perseverance***.** Solving problems requires perseverance and patience, so a teacher who models problem solving is likely to demonstrate those qualities. Hitting roadblocks is part of the process, and those roadblocks need not be a reason for giving up. Mr. Fulbright sent this message as he led his calculus class through an integration problem. Students offered several suggestions, many of which led to dead ends, but their perseverance paid off when eventually one suggestion led to a solution. Mr. Fulbright facilitated the problem-solving process by demonstrating patience in the midst of unsuccessful attempts to solve the problem.

Mr. Dalton encouraged his students to persevere despite roadblocks they encountered while solving problems. In fact, he frequently talked to students about how roadblocks can actually lead to a solution if they are seen in the right light.

> Again, one really important idea is to go ahead and make a decision and try a technique. If it doesn't work, then that's not a bad thing because almost always the way in which it doesn't work is really, really useful information and typically tells you what you need to do to make it right. So it's ok to make a substitution that doesn't work, as long as you don't just throw that substitution away, as long as you think about "What was the problem?"

Mr. Dalton demonstrated the need for perseverance when encountering something he or the students did not know. For example, a differential equations problem required the integral of a secant function. Instead of simply telling the students how to integrate the function, he modeled the process of self-questioning and working through the difficulty of what to do when you do not know what to do: "[The integral of] $\sec\theta$ is one of those things that you either know or you don't. If you don't know it, how can we figure it out? … So how do we integrate $\sec\theta$ if we don't already know?"

There were times when Mr. Dalton hit a roadblock while solving a homework problem, and he was honest about the fact that he could not fully solve the problem. In this way he modeled a humble attitude and showed students that errors and roadblocks are just part of the process of problem solving. In his first interview, Mr. Dalton explained,

> When I give a problem set, I'll work the problems myself, and so when they give me their solutions I'll give them my solutions. The first thing we'll do is look at my solutions and see which ones I missed. And I don't ever miss them intentionally, but you know, I can add 2 and 3 and get 6 just as quickly as anyone else. So sometimes I miss them because I've done some silly arithmetic error. Sometimes I miss them because I've just thought about it incorrectly. And sometimes I miss them just 'cause I couldn't figure out an approach. … There are times when my solution is, "Ok, I can't do this. But here's what I thought about. So I thought about doing it this way and I ran into this roadblock. I thought about doing it this way, and I ran into this roadblock."

Mr. Dalton noted that one advantage of being willing to share incorrect solutions with students is that they become more willing to share their solutions even if they are not completely correct.

Limiting Teacher Input

In order for students to develop as problem solvers, they must take on the responsibility of solving problems rather than waiting to be shown what to do every step of the way. Teachers who teach through problem solving allow students to bear this responsibility by limiting the amount of direct instruction they give students. That is not to say students are left entirely on their own, but the teacher encourages them to look to one another, and to their own reasoning abilities, before looking to the teacher for answers. When I began the study, I focused on group work as a practice but later determined that having students work in groups was encompassed in a broader practice of limiting teacher input. Limiting teacher input includes the following: (a) having students work in groups, (b) refraining from telling students too much, (c) allowing students to struggle, and (d) sharing authority for correct answers.

Having students work in groups. Having students work in groups is a common practice among teachers who focus on problem solving. Working in groups allows students to share ideas, explain their thinking, and help each other solve problems. Group work was essential in the classrooms of all four teachers in this study. In fact, it was a department-wide practice at Northridge High School where the desks in every mathematics classroom were arranged in groups of four called *pods*. Students in all the classes I observed at Northridge and at Tanner Academy regularly solved problems together, either in small groups of two to four, or as a class in large group discussion. Students also worked together outside of class on long-term projects and other assignments.

Each day, Miss Atkinson's students began consulting their group mates as soon as class began. They checked their homework and shared solutions with one another, particularly if there was disagreement about the answers. After a few minutes, each group presented one homework problem on the board. Miss Atkinson emphasized the importance of working together on this task, saying to students, "Make sure you have group-wide agreement and then go put all your work on the board." She assigned students to their groups and typically reassigned the groups after each test. Therefore, students had enough time to grow accustomed to working within a particular group, but also had the opportunity to work with several of their classmates over the course of the year.

Students frequently worked together to solve problems in Miss Atkinson's class. She assigned the problems and walked around the room supervising the groups. She gave help as needed but always encouraged students to figure out the solution on their own, check their own answers, or look to one another for help rather than to her. She often made comments such as, “Y’all work together as a group to figure it out,” or “I’m gonna give you these problems, and I’m just gonna let you go.” Miss Atkinson noted that one advantage of group work is that it takes some of the pressure off students—it “eases their stress.” Minimizing stress was particularly important at the beginning of the school year, she said, as students were getting used to a problem-based approach to learning mathematics.

Mr. Fulbright encouraged his students to work together on problems and investigations during class. He said working in groups was “reassuring” for students, particularly at the beginning of the school year when they were unsure what was expected of them. He stated that at the beginning of the year,

> the students work together in groups on numerous problems, which I think they find reassuring—so if they don’t know how to solve a problem at first, but neither does anyone else in their group, they realize that it isn’t “just them,” that I must be really and truly asking them to do something that I haven’t taught them how to do.

When students asked questions during class as they worked on a problem or investigation, Mr. Fulbright frequently responded by having them talk with their group mates and compare answers with one another.

Mr. Dalton said “encouraging [students] to think together and work together” was an important part of his teaching approach. One advantage, he explained, was that “kids like the group work, and that’s one of the things that comes through on the [course] evaluation.” Another advantage was that students “feel like they learn a lot by arguing with each other and trying to work through some common solution.”

Mr. Dalton’s students worked together both inside and outside of class. During class they sat in groups of their choosing, although Mr. Dalton said that if a particular arrangement was not working well, he sometimes encouraged the students to sit with a different group. They frequently solved problems in these groups as Mr. Dalton circulated around the classroom. During one of the classes I observed, he gave explicit instructions on how to best work as a group. In doing so, he expressed a belief about group work: Students learn more by working together than by working alone.

> It is not to your advantage to say, “Ok, I’ll do Problem 1, you do Problem 2, you do Problem 3.” I mean, that certainly cuts down on the amount of work you have to do, but the purpose of doing the work isn’t to do the work. The purpose of doing the work is to learn the things I want you to learn *by* doing the work. So the more you divide up, the less you actually learn in taking the course. So the best way to do this is to go through [the problems] and talk them out together one at a time. You’re not going to have time, probably, to finish them all [during class], so you’ll have to finish it up for homework. So you need to look at each one and have a few minutes of brainstorming in your group to make sure you all know what you’re supposed to do—How do you approach this problem? How do you work it?—before you leave.

One of the features of Mr. Dalton’s classes was that he assigned projects that students worked on in groups outside of class. He gave students a choice among several projects, and usually the choices students made determined the groups. For example, in the matrices course, students chose from projects that involved such topics as population growth, cryptology, and music. Students signed up for the project they wanted to work on, and all the students choosing a particular project worked together on it. Mr. Dalton noted that students often chose a project based on the others in the group rather than their interest in the particular topic. He also said he sometimes encouraged individual students to choose a different topic if he thought they were not giving themselves enough of a challenge with the project they chose initially.

In his second interview, Mr. Dalton expressed the difficulties he had in finding a balance between individual and group work, particularly when it came to assessment. When students worked in groups, he said, “that makes it more speculative as to how to grade those [assignments].” He added, “[With] some groups, it’s easy because everyone really is contributing and doing their share. And [with] other groups, it's not, because there's somebody who does most of the work, and other people get the grade for it." He summarized by saying, "As long as they are in good faith, contributing and listening to try to understand what each is doing, I think that's really all that you can expect them to do."

The students in Mr. Bailey's class worked in pairs every day. He randomly assigned partners at the beginning of each class period so that over the course of the year, students were able to work with most, if not all, of their classmates. As students worked on problems during class, Mr. Bailey often redirected their questions, saying, "Talk to your partner," or, "Compare with your partner." Certain quizzes were "partner quizzes," and even a portion of the final exam was collaborative.

According to Mr. Bailey, group work was important because students learn a lot from each other. In reflecting on the beginning of his teaching career, when he was primarily a "lecturing kind of guy," he said that student learning depended on how well students could listen to him, follow the lecture, and take notes quickly and accurately. In contrast, he described his practices after he began focusing on problem solving: "I run a 'looser' classroom, as [student learning] no longer depends on them listening attentively to me; it depends on them collaborating with each other." During some of the classes I observed, Mr. Bailey explicitly told students that he was going to "try to give [them] very little guidance" so that students would have to work together to solve the problem.

Mr. Bailey encouraged students to work together on homework problems. He noted, "When [collaboration] works well in class, it carries on outside of class. If they come across a homework problem that they can't do, I encourage them to collaborate to find out how to do the problem." He observed that one result of group work was that students improved their ability to clearly communicate solutions:

> I have been delighted at the way the students work together to refine the clarity of their solutions. … Getting students to understand the importance of communicating their methods clearly has always been a challenge, and now it is happening naturally—a happy corollary of our collaborative problem-solving approach throughout the year!

Another advantage of group work, Mr. Bailey said, was to prepare students for life after high school: "[Successful collaboration is] ultimately what I'm trying to get to happen, so that they're comfortable collaborating [while] doing problem solving. Because I know that is going to be a good transferable skill for later: college, then jobs and whatnot."

Refraining from telling students too much. All four teachers in the study indicated that it was important to refrain from telling students too much. Rarely did teachers directly answer a mathematics question from a student. Typically, the teachers either redirected the question to one of the student's group members or responded to the student by asking a question in return.

Miss Atkinson said, "I try to limit how many hints I give and really push them to be more independent. … I'm trying to tell them less." She tried to limit not only her verbal instructions, but written ones as well. For example, in speaking about something that had changed about her teaching in the last few years, she said, "One thing that I'm trying to do less of is give copious instructions for how to go through an investigation. [I'm trying to] make it a little more vague and a little more student decision guided."

In Mr. Fulbright's view, it was important that students gain independence in their development as problem solvers. This process is gradual, and Mr. Fulbright said he gave students fewer explicit instructions as the year went on in an attempt to "wean" them from his help. He explained why it is essential that students do the work of problem solving rather than being told what to do: “In developing a strategy for solving the problem themselves, they’re learning way more than they would if I told them such a strategy.”

In our interviews, Mr. Fulbright talked a lot about not giving students too much guidance when they were solving problems that required them to apply mathematical knowledge in new situations: “The real thinking is going on when they have to figure out things on their own. And that requires not telling them too much.” In his first interview, he described the evolution of the investigations he and his colleagues had developed, noting that over the years the instructions had gotten longer and longer and eventually turned into steps to follow, thereby ridding the activities of their investigative nature:

> As [students] struggled year after year, we kept taking hints that we used to tell them, and we wrote them into the investigations. So what had started out as just a spare, open-ended question gradually evolved into a recipe—you know, first do this, now try this, now do this. No wonder they didn’t do anything investigative! We told them exactly what to do, and they did it.

Noticing this trend, Mr. Fulbright and his colleagues rewrote the investigations to restore the original intent: “I stripped everything away; I stripped it down. Sometimes the questions had been two pages long. I cut [them] down to two sentences. … And that was a vast improvement, it really was.”

Mr. Dalton also talked about the importance of students becoming independent problem solvers. He noted that when students were working on problems during class, “what you try to do is walk around and see whether they’re making progress. If they’re making progress you sort of leave them alone, or ask a question or two, so they don’t feel like you’re ignoring them.” Something that was important to Mr. Dalton was “keeping things [open-ended] and giving them less help rather than more” so that they would solve problems on their own. Not all of his students responded positively to this approach. Rather, he said, “there are some kids who just resent mightily the fact that I won’t tell them how to do this.”

On several occasions, Mr. Bailey’s class worked together to solve problems with limited input from him. He encouraged them to listen carefully to one another and allow everyone a chance to offer ideas. In one of the calculus classes I observed, students filled in missing steps of a general logistic differential equation. One student was at the board as the “scribe,” and others offered suggestions. Mr. Bailey allowed students to share ideas, both good and bad, without interrupting. At times, he gave warnings and asked questions, but he did not take over. At one point, the students made an error, but he did not intervene, and eventually the students found the error. This experience allowed them to really engage in problem solving, a process that includes getting stuck and overcoming obstacles.

Allowing students to struggle. All four teachers expressed the importance of allowing students to struggle when they solved problems. The teachers wanted their students to engage in mathematical thinking rather than following a set of instructions. Each teacher sought a balance between letting students experience the struggle of problem solving—for example hitting roadblocks or making errors—and helping them avoid excessive frustration that may have resulted in giving up on a problem.

Miss Atkinson said that allowing students to struggle was one of the main things she did to teach in a way that emphasized problem solving. She said, “I’m trying to really get better at letting them mess up and see that their method isn’t getting them anywhere. Yet it is getting ‘em somewhere at the same time.” In an algebra class I observed, the students were working together to solve problems involving quadratics. They could not solve one of the problems because they were using a form of the quadratic function that was not helping them reach a solution. I include a transcript of the incident at the end of this section. Recalling the situation, Miss Atkinson explained her decision to let them struggle: “So they had the right idea, and I wanted them to go through that and hit a roadblock and say, ‘But wait just a minute. This isn’t the only form of a quadratic equation.’” In her first interview, I asked her how she decided how much help to give students, or when to intervene if it seemed they were heading down an unproductive path in solving a problem. She replied that she tried to gauge students’ level of frustration:

> If I sense intense frustration and … they’re just stuck, I’ll try to start giving them clues. And how many clues? For some groups a tiny little clue and they’re off. For some, I just kinda have to stand with them for a little while.

According to Mr. Fulbright, letting students struggle was essential to their discovery of mathematical concepts and their development as problem solvers. He noted that although “it takes a lot longer for students to discover something on their own than it does if you just tell ‘em the answer,” the extra time is worth it. He said, “If a person agrees that there’s still value to be had in having [students] struggle and figure things out [by] problem solving … you’ve gotta allow class time for that. … But it’s time well spent.” Recall that Mr. Fulbright gave students lots of guidance when he had a particular goal in mind for them. It may seem contradictory that a teacher could give guidance and also allow students to struggle. One might reconcile this seeming contradiction by noting that the guidance Mr. Fulbright talked about did not necessarily remove the struggle for students of having to think through hard problems. Rather, the guidance helped students think about the hard questions he wanted them to think about. There were times when practical considerations such as time constraints prevented Mr. Fulbright from letting students struggle as much as he might have wanted them to. I discuss this tension below.

Mr. Fulbright distinguished between two reasons the students might struggle in solving a problem or completing an investigative task. On the one hand, they might struggle to understand what he expected of them, particularly regarding how he would grade the assignment. On the other hand, the students might struggle with the mathematics of the task itself. Mr. Fulbright said he tried to minimize the first type of difficulty by completing the first investigation of the year together as a whole class. The second type of difficulty, however, was exactly what he wanted students to confront. Speaking of an instance of this good kind of struggling, Mr. Fulbright said, "They struggled a little bit with the question, but only in exactly the ways that they were supposed to be struggling. They struggled because [the questions] were hard questions."

Mr. Fulbright reiterated the importance of allowing enough time for students to engage in problem solving because struggling takes time. He described an open-ended investigation that students worked on for 5 class days:

> If they've got that much time, and if they struggle on Day 1, let 'em struggle. I mean, that's good. It's good for them to be puzzling through, and they don't feel a lot of time pressure on that first day, so it's not stressful. … I've told them before on the first day, I said, "Look, you're struggling. It's a hard problem—of course you're gonna struggle. Don't worry about it."

I asked Mr. Fulbright in our first interview how he made decisions about intervening when students were heading down unproductive paths while they were working on problems. He said it depended on time constraints. If they were engaged in a long investigation, he allowed them to struggle and did not redirect them. But on other occasions, he was under time pressure to finish a particular problem so the class could move on to other topics. He gave the example of when there were only 15 minutes left in class, and he deemed it "important to reach a resolution" to the problem:

> I hate to leave questions hanging. … So you want to resolve it, and so if they're struggling and I realize, ok they're not gonna get there, then I start giving more and more obvious hints until eventually we get there. The worst possible ending is you stand in front of them at the board and solve the problem for them, but sometimes that happens too.

This example shows the tension Mr. Fulbright sometimes felt as he sought to reconcile the value of letting students struggle and the practical constraints—such as time limitations—that are part of the reality of teaching mathematics.

In several of the classes I observed, when Mr. Dalton assigned a problem in class, he circulated among the groups of students to check on their progress, but he did not intervene too quickly. Rather, he spent the first several minutes simply observing the groups without saying anything. After allowing them to struggle on their own for a while, he offered help as needed. He noted that if a group was making good progress toward a solution he would leave them alone, but in other cases he would handle things differently:

> If you see that they're either heading off in the wrong direction or struggling with one of [the problems], not seeming to get anywhere, then you sort of ask them questions about "What about this simple case?" or "What about that simple case? Could you do that?" So a couple times you'll stop and work through a problem with them to try to get them restarted again.

Mr. Bailey also gave students time to struggle with problems, and he avoided intervening when they took wrong solution paths or made errors. One reason for this hands-off style was his commitment to having students work together: He was more interested in them helping each other solve problems than he was in telling them correct answers. In his second interview, he told the story of a student who was working on the POTD and pursued an incorrect solution path that eventually led her to the correct answer. In fact, even though her route was circuitous and included errors along the way, in the end she made an important discovery. Mr. Bailey recalled,

> And so here's this girl [who] has the right answer, and yet she had continued down this wrong path. So I stared at it, and I realized what she had done was [that] she had invented integration by parts. … This was not one of my top students or anything like that. She had just flat stumbled into it. … I would not have ever dreamed, before that accident occurred, that a student could stumble upon integration by parts.

This story provides a clear example of the value of letting students struggle and not intervening too soon. Not only can they have a rich experience of solving a problem on their own, but they may also discover mathematical concepts in a manner much more meaningful than if the teacher simply told them the answer.

There were times when Mr. Bailey intervened to prevent students from going too far down the road toward an incorrect solution. When I asked him in his first interview how he decided whether to intervene, he indicated that time constraints often influenced his decision. He acknowledged that a downside of intervening was that students were kept from a full experience of problem solving, which often included having to overcome hurdles:

> I'll see them go down a wrong road, and I'll say, "Oh yeah. Now if I could just seal that road off." So that will either require maybe a little hint or maybe change the nature of the problem so that wrong road isn't as tempting. So to a certain extent I suppose I'm saving them from part of the reality of problem solving because you're gonna go down wrong paths. But on the other hand, we do want to streamline it a little bit so that we can get to covering some stuff.

Another practice Mr. Bailey sometimes used to keep students on an efficient solution path was to warn them of common errors such as forgetting to apply the Chain Rule when differentiating a function.

Sharing authority for right answers. The teachers in this study not only refrained from giving students too many instructions, but also encouraged students to rely on their own mathematical reasoning rather than looking solely to the teacher for the right answers. Miss Atkinson explained that she wanted her students to be independent thinkers and to realize that they were capable of solving problems on their own:

> A lot of times if a student says, "Is this right?" I'll be like, "I don't know. How can you figure it out?" … I want 'em to think that, "Well, maybe I could figure out if it's right or not. Maybe I don't need to ask her."

I observed that Miss Atkinson frequently responded to students' questions by asking them questions in return, thereby putting the responsibility back on them to solve the problem and to verify their answers. At one point, a student asked her about using a particular strategy, and she replied, "Well, why don't you try it and see instead of asking me the question?" Later she explained to the student, "What I'm trying to get you to do is try something and not ask me if you can do it."

Mr. Dalton let his students know that he did not have all the right answers. He saw himself as a fellow problem solver with his students rather than the source of all correct solutions: "Oftentimes I don't know how to do [a problem] myself, and I'm interested in figuring it out, so maybe one of [the students] will figure it out and teach me." I observed Mr. Dalton's combinatorics class on a day when he distributed his written solutions to some homework problems. He said to the class,

> These are just my solutions. I don't have an answer book that I'm just copying out of. This is just the way I was thinking about doing the problem. It doesn't mean it's correct. I mean, I may have counted incorrectly.

These statements reveal Mr. Dalton's willingness to learn from his students and his recognition that students were able to solve problems on their own.

Mr. Bailey consistently responded to students' desire for affirmation by reminding them that they were fully capable of making progress on a problem on their own. Several times every class period, while students worked on problems during class, I heard Mr. Bailey say, "Talk to your partner," indicating that he wanted students to look to each other before looking to him for answers. He also relied on students to go to the board to show how they solved the problems. He explained, "I like having them show ultimately how the problem was done rather than me.

Again, there the object is to get them to understand that I'm not the only one around here who can do that problem."

The following is an example to illustrate all four aspects of limiting teacher input: (a) having students work in groups, (b) refraining from telling students too much, (c) allowing students to struggle, and (d) sharing authority for correct answers. Miss Atkinson's algebra students worked in groups on the following problem: "Suppose there is a parabola that contains (1, 2), (-2, 23), (3, 8). Find its equation."

> Student 1 (S1): Ok, I have a question.
> Miss Atkinson (A): I may or may not have an answer.
> S1: Is [this problem] pertaining to what we've been doing most recently at all? Like is there a connection between that and this?
> A: I don't know.
> S1: Well, that's not helpful.
> A: I know.
> S1: Would it be useful to convert these lines to standard form or vertex form?
> Student 2 (S2): Well, they're not lines; they're points.
> S1: Not lines, but the points—like put them in equations, separate equations. Would that help?
> A: Well, why don't you try it and see, instead of asking me the question?
> Student 3 (S3): Miss Atkinson, you're being vague on purpose.
> A: Yes, I am. I surely am. Which form [of quadratic equation] do you want to pick, [S1]?
> S1: Probably vertex [form].
> A: Why?
> S1: Because we're doing a parabola, and the vertex is always pretty important. Well, so is standard [form], but you can't really get much out of standard [form].
> A: What can you always do with standard [form]?
> S1: Convert to slope-intercept?
> S3: What?
> S1: Wait, no.
> A: Slope-intercept? All right, let's stay on the train tracks here. What are you thinking there, [S3]?
> S3: Can't you put each of these [points] in this formula [standard form] and then solve it as a system of equations?
> A: I don't know.
> S3: That means yes.
> S1: Try it.
> A: What I'm trying to get you to do is try something and not ask me if you can do it.

Miss Atkinson purposefully refrained from telling the students too much, both in the problem statement itself and in her verbal comments. When a student asked a question, Miss Atkinson responded either by “being vague on purpose” or by asking the student a question in return. In addition, she encouraged the students to rely on one another and on their own mathematical reasoning to solve the problem, rather than looking to her for the correct answer. Note that even though she allowed the students to struggle and did not tell them how to solve the problem, she also helped them “stay on the train tracks.”

Promoting Metacognition

As I discussed previously, when teachers in this study modeled problem solving, they often modeled metacognitive behavior. In this way, the teachers promoted metacognition. There were other ways the teachers encouraged students to be metacognitive, for example by asking metacognitive questions or drawing attention to the students’ own thinking.

In her first interview, Miss Atkinson gave an articulate description of problem solving that included aspects of metacognition:

> Problem solving is a lot of different things. … [It includes] having some sort of metacognition going on. Are you thinking about what you’re doing? Is what you’re doing getting you anywhere? Are the answers that you’re coming up with … making any sense? If you expect the parabola to be doing this (motions with hands), and it’s doing this (reverses motion), what went wrong? Don’t keep going. Stop! Have some control over what you’re doing.

Miss Atkinson’s questions and comments addressed two aspects of metacognition: monitoring (“Is what you’re doing getting you anywhere?” “What went wrong? Don’t keep going. Stop!”) and reflection (“Are the answers that you’re coming up with … making any sense?”) At various times, the teachers in this study asked metacognitive questions to encourage students to do the following: (a) be aware of their own cognition, (b) monitor the problem-solving process, and (c) reflect on the problem and solution.

Being aware of one's own cognition. Metacognitive problem solvers are aware of how they are thinking. On one occasion, Miss Atkinson asked her students to think about how they were visualizing the graph of the function $f(x) = 10^x$ and its inverse: "How do you picture in your head what the inverse is gonna look like?" … "How did you know it was gonna look something like that?" Miss Atkinson told her students she wanted them not only to know how they were thinking, but also to express how they were thinking about problems as they solved them:

> I want everybody to have all of this work done completely because when we get to class on Monday I want us to be able to finish up these problems and have really good discussion about why these curves look the way they do, ok? And talk about how we thought about these problems when we were solving them.

Mr. Dalton emphasized that solutions to problems included not only a final answer, but also an explanation of the thought process that led to the answer. He told his students, "What I want to see in your solutions is the way you're thinking about the solutions." Miss Atkinson expressed a similar opinion: "I wanted them to share their work, share how they were thinking about all of their steps." In one of Mr. Fulbright's calculus classes, a student presented her solution at the board. Her answer was incorrect, but Mr. Fulbright asked the rest of the class to consider how she might have been thinking about the problem: "What was [she] thinking? … There's something sensible that she's doing."

The primary means by which the teachers encouraged their students to be aware of their own cognition was helping students think about what they knew and what they did not know. Miss Atkinson expressed the challenge of getting students to think about their knowledge, describing it as a battle: "For me I guess it's this constant battle of, 'Ok, what else do you know? What other resources do you have?'—just asking 'em to articulate the things that they know." I observed her asking questions during class to get students to think about the knowledge they had that would help them solve a problem involving a quadratic equation: "What else do you know about quadratic equations? … Reach back in your toolbox to stuff you know about quadratic equations, functions. What else do you know? Write down a list of everything you know." Miss Atkinson helped students tune in to their own knowledge by encouraging them to be aware of their common mistakes: "What errors are you prone to making? … When you're going back and checking your work, be aware of the kinds of errors that you are making, and think about those."

Mr. Dalton wanted to help students think about what they knew and what they did not know. He made the interesting comment in a differential equations class that in some ways solving a problem is easier the less you know, because then you have fewer problem-solving strategies to try:

> So how do we integrate $\sec\theta$ if we don't already know? … The good news is I don't know very much about secant. Now why is that good news? I don't have very many choices, right? The more I know about something, the more things I have to sift through to make the right choice.

On another occasion, he told students, "In a lot of these [differential equations problems] just stop and think about them, 'What do I know?' and you're almost always better off not to know a whole lot, 'cause it doesn't give you that many choices." When Mr. Dalton led a discussion of a problem, he regularly asked questions to get students thinking about what they knew:

> So what do we know? … How do we know that? … What else do we know? … So how do we do this? What techniques did we learn when we learned how to integrate? … Is that one we recognize? Do we know how to do that?

Monitoring the problem-solving process. The aspect of metacognition that Schoenfeld (1985) highlighted as critical to successful problem solving is control or monitoring. All four teachers in this study encouraged their students to develop the ability to monitor while solving problems. This encouragement sometimes came in the form of direct comments such as, "Be aware of the kinds of errors that you are making" (Miss Atkinson), but most frequently occurred through asking questions and encouraging students to ask questions of themselves while solving problems.

Miss Atkinson deliberately asked her students metacognitive questions. She explained her intentions in her first interview:

> I try to say throughout the whole process, "You need to be asking yourself questions like, 'Well, let's look at the original scatterplot. What would be potential functions?' You know what the toolkit functions look like. … And then you've linearized the data, you've got this linear function. How do you know if it fits? You should be asking yourself this question like 'Now how can I be sure that this line fits this data well?'" … So I really try to enforce this idea that "What questions are you asking yourself?" Sometimes I'm better about doing it than others. But that's one thing that I'm trying, to build metacognitive skills that way.

I observed Miss Atkinson asking metacognitive questions on a regular basis to encourage students to monitor their problem-solving process:

> Did I do that correctly? … What are you going to need to find? … What can you use to do that? … Is this going to be helpful to you? … Do you have any other way to think about quadratics? … What else could you do? … How do you check and see if it's right?

As she posed such questions, she told students, "These are questions I want you to start asking yourself."

Miss Atkinson gave students advice about monitoring their studying and problem solving even outside the mathematics classroom:

> This is important for all of you for all time: I want you to think about the things that you're doing—in mathematics and in all of your classes. Be thoughtful about how you're studying. Be asking yourself good questions while you're solving your problems. Make sure that your answers that you're coming up with make sense.

Monitoring includes anticipating what a problem will require, what a solution might look like, or where a solution path might lead. What strategy might be useful? Will that solution path be easy or difficult? How might the information in the problem help determine the best plan of action? In an algebra class, Miss Atkinson encouraged her students to anticipate such things:

> Some of you are just going about your usual process and you're coming up with the correct answer. But I want you to start thinking about these on the way in. … Before you start furiously scribbling stuff down, think about the gifts that you are given [i.e., information in the problem that is helpful].

In a precalculus class, she encouraged students by saying, "Think about some strategies you might employ to get a graph of this. Think about how this is different from the functions we've done before. Think about characteristics this type of graph might have. Think, think, think." Mr. Dalton and Mr. Bailey both encouraged students to monitor their expectations by anticipating possible solutions or making conjectures upon encountering a problem. Mr. Bailey encouraged his students to use their intuition to predict an answer before beginning a problem. Before beginning a matrices problem, Mr. Dalton asked, "What do you think's gonna happen? Any predictions?"

Another aspect of monitoring is checking for mistakes along the way when solving a problem. Miss Atkinson encouraged students to check both individually and in groups: "Are you guys checking behind each other and trying to avoid those dreaded careless mistakes?" She also helped students see that knowing what kind of mistakes they tended to make was a step toward avoiding those mistakes: "Think about [the errors you are prone to making] so you can train yourself not to make those errors." Mr. Dalton explained to his students that many problems have natural points at which to stop and check how things are going:

> We don't want to get too far down the road having made a mistake up here. The further we are away from the mistake when we find it, the harder it is to … actually find it. … And this is something we'll do a lot: As we go through a problem, there are natural places to stop and ask, "Does what we have make sense?" Usually when the answer is no, it's because there's a sign error. We've left some negative sign out at one step in the process. That's something we should all do as we go along.

At one point, as Mr. Bailey led his finite math class through a problem involving probabilities, he explained that it was a good idea to check along the way that the sum of the probabilities was 1:

> If I've done anything wrong, chances are good I'll have a last chance to catch myself once I look at these two numbers. And if those two numbers don't add up to 1, I need to back off and do something else because I've done something wrong.

Checking for mistakes in the process of solving a problem is part of monitoring whether one is getting closer to a solution. Another part of monitoring is asking oneself, "Are the decisions I am making actually helpful?" Mr. Dalton noted that when integrating by substitution, the goal of the *u* substitution is to create an integral that is easier to solve than the original one, so students should make sure that the substitution benefits them before proceeding. Mr. Bailey's students frequently solved problems as a class. Although Mr. Bailey generally remained hands off, he sometimes interjected bits of advice about monitoring, such as "Remember where we're headed," "Remember the object is to find the antiderivative," or "Make sure that you all agree with everything up until now."

Reflecting on the problem-solving process. Reflecting on the problem-solving process, both in the midst of solving a problem and after one has found a solution, is an example of metacognitive behavior. It is closely related to Polya's (1957) notion of *looking back* and includes reflecting on decisions and examining the solution to a problem. Miss Atkinson wanted her students to be reflective problem solvers and noted that becoming reflective does not happen automatically:

> I think the one thing that I ask them to work on more and more as the year progresses … is becoming more reflective in their problem solving. They may not like it at first, but I think it does make a difference over time. They write reflections, and I ask them a lot of "reflective" questions as I walk around. For example, "Why did you decide to use that approach?" "Do you think there are things you [could] have done to have solved the problem more quickly or efficiently?"

I observed her asking such questions in an algebra class: “Why did you decide to use that method?” “So what’s your independent variable? What’s your dependent variable? Why did you choose it to be that way?” Mr. Fulbright also encouraged his students to reflect on their decisions. For example, he asked them to explain their choice of u for a u substitution in an antidifferentiation problem. Mr. Dalton encouraged his students to not only check for mistakes in the midst of solving a problem, but to reflect on those mistakes as a means of moving closer to a solution: “Making that first mistake early, and looking thoughtfully at it, reflectively on it, is always a really good strategy when you’re trying to solve a problem.”

Polya (1957) recognized the difficulty of having students reflect on their work once they reached a solution. When students perceive they have finished a problem, he said, they are ready to move on. Despite this difficulty, all four teachers in this study strove to help their students look back on a problem and their solution, at least to check their work. Miss Atkinson spoke in her first interview about the importance of looking back:

> Interpretation is a big part of problem solving to me. Interpreting the results and evaluating the results: Do they make sense in your understanding of the problem? If not, why not? Did you make a mistake? Or is this just somehow unexpected, and I need to figure out a different way to think about the problem to account for what I wasn’t expecting?

During class, Miss Atkinson asked her students to reflect on solutions and confirm that they made sense. She asked her precalculus students to examine the subjective aspects of a solution: “For Number 3, wasn’t that [transformation] interesting? … Which way did you like the graph better? Did you like the first way better?” Mr. Bailey asked a similar question after his students solved a finite math question: “Are you surprised [by this solution]?”

Highlighting Multiple Solutions

The teachers in the study demonstrated and described another aspect of teaching through problem solving in addition to the teaching practices I have described thus far: highlighting multiple solutions. In their descriptions of problems and problem solving, the teachers indicated that there is more than one way to solve any given problem. Highlighting multiple solutions occurs when teachers do any of the following: (a) emphasize that there is more than one solution path for solving a particular problem, (b) ask students to solve a problem in more than one way, (c) have students share their solutions, and (d) compare the merits of different solutions.

Miss Atkinson frequently noted that there are multiple ways of solving a particular problem. Each day, as students worked on a problem during class, she circulated around the room to observe their work and check their progress. There were times when a student solved a problem differently from how she had, but her first response was not to correct or redirect the student. Rather, she wanted to know how the student was thinking about the problem. For example, on one occasion she simply said to a student, "I did mine differently. Tell me [what you did]." There were times when Miss Atkinson saw students using an inefficient strategy, but rather than telling them to change course, she encouraged them to continue down the solution path that made sense to them. For example, a student was working on a problem involving a system of linear equations, and Miss Atkinson said, "You're eliminating your *x*'s. That's interesting. That's ok. It doesn't matter. … Well, the *z*'s are probably the easiest to eliminate, but that doesn't mean that that's the only way to do it."

On one occasion, Miss Atkinson's led a discussion in her precalculus class about the following problem:

$$\text{Solve for } x\text{: } x = 5^{2\log_5 6}.$$

A student in the class suggested using properties of logarithms:

$$\log_a x^n = n\log_a x \text{ and } a^{\log_a x} = x,$$
$$\text{so } x = 5^{2\log_5 6} = 5^{\log_5 6^2} = 6^2.$$

Miss Atkinson added to the discussion a solution that a student in a previous class had used, in which the student changed the original equation into logarithmic form:

$$\begin{aligned} &2\log_5 6 = \log_5 x \\ &\log_5 6^2 = \log_5 x \\ &x = 6^2 \end{aligned}$$

This incident indicated that Miss Atkinson wanted her students to think about multiple ways of solving a problem.

According to Mr. Fulbright, "Solutions can look different and both be correct." Recall that his explanation of what makes a good problem included comments about multiple solution paths. He said that when students solve problems, "they might get different approaches. They might get different solutions, assuming a problem permits different solutions. Even if it permits only a single answer, though, there still might be different approaches to get to it." After students solved a problem in his calculus class, Mr. Fulbright responded, "That's only one way to do it. You could have done this a number of ways."

In his second interview, Mr. Fulbright recalled an incident in which the textbook suggested a particular solution path, but a student came up with a different—and, according to Mr. Fulbright, a better—way of solving the problem. In his description, *solution* refers to the answer or result. It was the way the student approached the problem—that is, the solution *path*—that was unique.

> We have a problem in our textbook. … "When should you sell a baseball card in order to maximize the amount of money that you get for it?" Well the two competing forces are, on the one hand, the value's always increasing. They [the textbook authors] actually give you a model for how the value of the baseball card increases over time, so just assume

> that model is true. But on the other hand, inflation makes every dollar worth less, inherently, every single year. So there's some point where the buying power is actually maximized. … The way it's worded in the textbook is they lead the students through it by suggesting that they always consider the present value of the future sale. But that's a difficult concept to them. I finally had a student who solved it a different way, … which was … just pretend that by the time you retire, … you will have sold the baseball card and you will have taken that money that you got for the sale, popped it in the bank, and it accrued interest from then on until you retire, and that's the money you're gonna use to go on your big retirement vacation or whatever. Then you don't have to worry about looking back in time, but you've still got the exact same problem, exact same solution. The only difference is that instead of looking at the present value of the future sale, you're looking at the future value of the future sale.

Mr. Dalton repeatedly emphasized that there was always more than one way to solve a problem. This emphasis was particularly evident when he talked about his solutions to homework problems. Upon distributing his homework solutions to his combinatorics class, he said, "You certainly do not need to do it the same way I did it, particularly if I did it incorrectly. … Just because you did it a different way doesn't mean anything at all [in terms of its merit]."

On occasion, Miss Atkinson asked students to solve a problem in more than one way, or at least to consider a problem from multiple perspectives. In an algebra class, the students worked on a problem that asked if they could write a quadratic equation that would include three given sets of ordered pairs. One group of students initially solved the problem algebraically, and Miss Atkinson challenged them to think about it differently: "Tell me another reason, other than the algebraic one: Why do those three [points] not [determine] a quadratic function? I want you to look at it a second way."

The daily routine in Miss Atkinson's classes was for the students to share their homework solutions with one another in their small groups. Sometimes groups of students worked together on a problem and then shared their solution with the rest of the class. Miss Atkinson noted that at the beginning of the year, the students had not been accustomed to seeing more than one way to do a problem. After the first mathematics activity of the year, groups shared their solutions, and Miss Atkinson recalled, “The biggest eye-opener at the end of the activity [was] when the groups [shared] their models. They [were] all different, and there [was] no *one* correct answer.”

Miss Atkinson also had individual students share their solutions with the whole class, and she wanted them to include explanations of how they were thinking about the problems: “I wanted them to share their work, share how they were thinking about all of their steps.” On one occasion, two algebra students had different solutions to a problem that asked whether three given points were collinear. The first student demonstrated his method, which was to use two points to write an equation of the line containing them and then determine whether or not the third point satisfied the equation. After this student presented his solution, Miss Atkinson called on another student who had done the problem differently: “You had a delightfully different approach to figuring out how those three points were collinear. Tell us how you did it.” This student’s solution was to calculate the slope between one pair of points and see if it was the same as the slope between a different pair of points.

On occasion, Mr. Fulbright’s students shared their solutions with the whole class by going to the board and presenting their ideas. He saw several benefits of this practice: Students enjoy presenting solutions, the class is more engaged when listening to a fellow student as opposed to listening to the teacher all the time, students gain confidence in their mathematical ability, and they are able to help their classmates learn. In a calculus lesson, the class was discussing solutions to the following integral:

$$\int (\sin x)(\cos x)dx.$$

One solution was $\frac{1}{2}\sin^2 x + C$, but some students had different answers. One student shared her solution at the board: $-\frac{1}{2}\cos^2 x + C$. This presentation was followed by a second volunteer who said, "I got the same answer. … Can I tell you how I did it?" Mr. Fulbright replied, "Yeah, tell us how you got it. Please do. Show us on the board. This sounds very interesting to me. … This is much cooler than I was expecting." The class went on to discover that the integration problem had several solutions that were all correct because they differed only by a constant.

After Mr. Dalton's students worked on a problem in their small groups during class, if a group had a unique way of solving the problem, he said he sometimes asked the group to share their solution: "And particularly interesting things I'll have students come up and present or if they have a nice idea, something that I think could lead somewhere, to say, 'Tell us what you've done.'"

Mr. Dalton wanted his students to see that some solution methods were better than others. He admitted that he did not always solve a problem in the most elegant or most efficient way. In fact, he was glad when students came up with a better solution than he did:

> It's … important for [students] to think that their answer can be better than mine, that their approach was, you know, nicer. I did it by brute force, and they thought of a nice way to do it, and that's always kinda nice.

In Mr. Dalton's comments to his combinatorics class about his written solutions to the homework problems, he emphasized not only that there are multiple ways to solve a problem, but also that some solutions are better than others:

> This is just the way I was thinking about doing the problem. … If you thought of it a different way, that's fine, particularly if you thought of it a better way. In some of these I just looked at all the cases, which isn't a particularly elegant way of doing it, but it's the only way I could think to do it. And so again, [if there's] a nicer way, I'm hoping it's in your paper so I'll see a better way of doing it than the way I did it. So you don't need to recapitulate my answer in order for it to be correct.

In his first interview, Mr. Dalton explained that he graded students not only on the correctness of a solution, but on its elegance and sophistication as well. He was not specific about how he weighted correctness and elegance, but he noted that he gave only partial credit if the solution was especially messy or inefficient. Ultimately, he was more interested in the thinking that went into a solution than the final result. He talked about comparing two correct solutions:

> Here are these two correct solutions, and one of them is better than another. [Students are] fine with partial credit for wrong stuff. But there's also partial credit for right stuff. It's correct, but it's not a particularly elegant solution. It's not a nice solution. … In fact, there may be some [students who used an] approach that didn't quite work out that shows a lot more understanding than the one you'd use that got the correct answer. And again, it's this idea of trying to build up their mathematical sophistication, sort of thinking about solutions and how things work.

Mr. Bailey highlighted multiple solutions as well, either by presenting them himself or by asking students to share their solutions. He encouraged students to explain different ways of solving a problem, for example, by saying, "I'm interested in seeing how different people solve this" or asking, "Did anybody do this a different way? There are other ways to do this." Each class period, after students worked on the POTD, Mr. Bailey randomly selected a student to go to the board and either present a solution or, if the student had not solved the problem, to serve as a "scribe" as others in the class explained their solutions. At times, he had more than one student present: "If I know that somebody has done something a different way, then yes, we always try to have the alternate way up there [on the board]."

Polya's Phases

None of the teachers in this study explicitly referred to Polya or the phases of problem solving. But implicit references to the phases—particularly *understanding the problem* and *looking back*—were evident in the teachers' comments both in interviews and in their classrooms.

Understanding the problem. Mr. Bailey urged his students to understand the problem before diving in: "You have to be able to interpret the question." Miss Atkinson also encouraged students to understand or interpret the problem statement, and asked questions such as "What is this asking?" and "How can we think about this to answer the question?" On one occasion, Miss Atkinson's precalculus students had difficulty interpreting the wording of a problem about a diving board. In a discussion about different ways to interpret the problem, Miss Atkinson encouraged them to "make sense of the problem" by thinking about the actual situation with the diving board: What seemed most plausible or realistic?

Looking back. Reflecting on a problem, either in the midst of solving it or after one has found a solution, is a critical element in Polya's phases of problem solving. Earlier in this chapter, I discussed *looking back* as it related to metacognition, but I add a bit more here. A common part of looking back is checking one's solution, and the teachers in this study encouraged students to think about how they could check their work, as well as to actually check it. For example, Miss Atkinson asked her algebra students, "How can you find out if it's right?" and Mr. Fulbright asked his calculus class, "How could you check that without your calculator and without checking the back of the book?"

Looking back also includes thinking about a problem after one has solved it. Miss Atkinson wanted her students to not shut down as soon as they solved a problem: "I don't wanna necessarily always have 'em thinking that they're shooting for this right answer and once they get it they're done." On one occasion, she asked her students to examine a problem they had just solved and compare it with a problem they had solved previously: "What's happening this time around? … Did anybody notice that this is very similar to the other system [of equations] we just did? … This time what happens, though?" Mr. Dalton also asked his students to think about a problem after they had solved it: "Let's go back to this [problem] and think about it a little bit differently." Mr. Bailey encouraged his students to look back on a problem and think about their response: "Are you surprised [by this solution]?"

Summary

Figure 11 contains a chart showing the frequency with which the four teachers engaged in the teaching practices I have described. For each teacher, the course name and date (MMDD) of the occurrence are listed, followed by the frequency if I observed instances of the practice more than once in a particular class period. The information in the chart summarizes the results of my coding of the transcripts of relevant portions of the audiorecordings from classroom observations. The items listed in the chart reflect the main practices I was looking for as I conducted my observations. Since I originally considered group work to be a practice in itself—rather than part of the broader practice of limiting teacher input—*group work* appears in the chart and *limiting teacher input* does not. Multiple instances of *modeling problem solving* indicate that the teacher modeled problem solving for more than one problem during the class period listed. *Group work* is listed only once for each class period in which it occurred.

	Miss Atkinson	Mr. Fulbright	Mr. Dalton	Mr. Bailey
teach problem-solving strategies		Calculus: 0106, 0107, 0309	Matrices: 0105 Differential Equations: 0308 (×3), 0309 (×5), 0310 (×3), 0312	Calculus: 0125, 0127 (×2), 0412, 0414, 0415, 0426 Finite Math: 0428
model problem solving		Calculus: 0311 (×2)	Matrices: 0105 Combinatorics: 0105 Differential Equations: 0308, 0309 (×2), 0310, 0312	Calculus: 0125 (×2), 0414, 0415
group work	Algebra 3: every class period Precalculus: every class period	Calculus: 0104, 0105, 0106, 0107, 0310	Matrices: 0104, 0105 Combinatorics: 0104, 0105 Differential Equations: 0308, 0309, 0312 Statistics: 0308, 0309	Calculus: every class period Finite Math: every class period
promote metacognition	Algebra 3: 0104 (×6), 0106 (×5), 0107 (×2), 0108 (×5), 0312 (×2) Precalculus: 0104, 0107, 0108 (×2), 0311 (×3)	Calculus: 0311 (×4)	Matrices: 0105, 0106 Combinatorics: 0104, 0105 Differential Equations: 0308 (×5), 0309 (×13), 0310 (×5), 0312 Statistics: 0311	Calculus: 0125 (×2), 0127 (×2), 0128 (×2), 0412, 0414 Finite Math: 0412, 0414 (×3), 0416, 0426
highlight multiple solutions	Algebra 3: 0104, 0107, 0108 Precalculus: 0107 (×2), 0309, 0311 (×2)	Calculus: 0107, 0309, 0311 (×2)	Combinatorics: 0105, 0106 (×3) Differential Equations: 0310 (×2) Statistics: 0309 (×2)	Calculus: 0125, 0412, 0414, 0426 Finite Math: 0125, 0412, 0428 (×2)
implicit reference to one or more of Polya's phases	Algebra 3: 0104 (×5), 0106, 0107, 0108, 0312 Precalculus: 0104, 0309 (×2), 0311 (×2)	Calculus: 0311 (×2)	Matrices: 0104, 0106 Combinatorics: 0105 Differential Equations: 0308, 0309 (×2)	Calculus: 0127 (×2), 0128 Finite Math: 0125, 0127, 0412, 0414 (×3), 0426 (×3), 0428

Figure 11: Instances of teaching practices for each teacher.

The chart in Figure 11 shows similarities and differences among the teachers in the study. Although each teacher had his or her own style of teaching through problem solving, the teachers had many practices in common. The practice that was most consistently displayed by all four teachers was having students work in groups. As I have mentioned, this practice is contained in the broader practice of limiting teacher input. Other elements of limiting teacher input—refraining from telling students too much, allowing students to struggle, and sharing authority for correct answers—were also common to all four teachers even though this fact is not apparent in Figure 11. Note that except for teaching problem-solving strategies and modeling problem solving, every practice was implemented by every teacher on at least one occasion.

Teachers differed in which practices were most prevalent in their teaching, as Figure 11 indicates. For example, Miss Atkinson frequently promoted metacognition. This is consistent with her comments during interviews in which she said she consciously tried to foster metacognitive behavior in her students. Note also the frequency with which Mr. Dalton taught problem-solving strategies and modeled problem solving, particularly in his differential equations course. Recall that the only week I observed the differential equations course was in March and that classes at Northridge met only 4 days per week. So as the chart indicates, Mr. Dalton engaged in teaching problem-solving strategies and modeling problem solving every day that I observed him teaching differential equations.

The chart in Figure 11 does not indicate qualitative aspects of each teaching practice. For example, the mere fact that Mr. Dalton acted in a way that promoted metacognition 13 times in one class period does not necessarily mean that his actions were more effective than a single instance of promoting metacognition by another teacher. In addition, the chart does not indicate the manner in which teachers implemented the practices. For example, Miss Atkinson's way of promoting metacognition was by asking metacognitive questions whereas Mr. Dalton's and Mr. Bailey's way of doing so was primarily by modeling metacognition. Despite the limitations of what it can show, the chart is a helpful tool for summarizing the prevalence of the set of practices among the group of teachers in this study.

Collaboration with Colleagues

Three of the teachers in this study—Miss Atkinson, Mr. Fulbright, and Mr. Dalton—taught at Northridge High School. Their mathematics department relied heavily on collaboration, and all three of the teachers spoke about the advantages of collaborating with their colleagues. One advantage they mentioned was sharing ideas for problems or activities to use in the classroom. All the mathematics teachers at Northridge contributed to the writing and refining of problems and investigations, and all the problems were posted to a shared computer drive to which all teachers had access. As Mr. Fulbright said, "If something works, we spread it around." Collaborating with colleagues is not a teaching practice, but according to the teachers at Northridge, collaboration was a significant factor in their being equipped and encouraged to teach through problem solving.

Miss Atkinson mentioned several benefits of collaboration: (a) sharing ideas and generating new ones, (b) refining problems, for example by removing confusing wording, (c) being challenged and deepening one's own mathematical knowledge, (d) gaining courage to try new things, and (e) decreasing the stress of teaching. She explained,

> If you can collaborate, (a) it cuts down on your stress because you're sharing the load, and (b) you really get a lot more ideas, partly because you get … multiple perspectives, … but also through the collaborative process and through sharing and generating even more ideas on top of that. Sit down and brainstorm, "Well what if we did this?" and then all of a sudden, boom, something completely different happens and it can be really powerful and helpful. … Because you have so many people looking at things differently, you can head off a lot of potential problems that you might see in the classroom, like I

> know that problem [I mentioned earlier] would have confused a lot of my students, worded the way it was.

Miss Atkinson went on to say how much she enjoyed collaborating with her colleagues and how much she had benefited from working with them:

> I love being able to collaborate with people. I don't think I could ever go back to living trapped in my classroom for 6 periods a day and never having the interaction with my colleagues. And I feel challenged by them. I feel like I'm always deepening my own knowledge. … Now, because I know I've got support around me, I just attack [difficult problems] head on and then somehow have the courage to go in and try something different with my students even if I haven't completely figured it out yet.

Mr. Dalton also mentioned the value of working together with colleagues. He noted, "Collaboration is really important for several reasons." First, he mentioned the pragmatic reason that if there are multiple sections of the same course, it is important that teachers coordinate to make sure they are actually teaching the same content and staying on the same schedule. Second, collaboration "helps you become a better teacher, to think about ways of approaching this subject or an interesting problem to do." A third reason for collaborating is to learn by sitting in on each other's classes. He mentioned a teacher who was teaching a multivariable calculus course for the first time and sat in on another teacher's section of the course to see how she did things. Finally, Mr. Dalton said that when teachers collaborate, they share advice about what works and what does not:

> You just learn a lot, talking with your colleagues about what they've done. "I tried this; don't try it." "Don't do it that way; maybe if you did it this way it'll work." Just talking over what happened last week and what you're planning to do next week is, I think, really important.

Miss Atkinson pointed to a particular incident in which teachers met for the sole purpose of talking about how to help students develop as problem solvers. She recalled,

> Several of the teachers during a … department meeting … said, "I'm trying to figure out how to encourage better problem solving among my students and I don't know how to get them to be better at it." And so we … got together and we worked on a problem

> together. Mr. Dalton went out and found some problem that almost made my head explode, but we all worked on it together as a group until we found a solution, and we were all sharing our thoughts and ideas, but at the same time we were monitoring what we were doing, and trying to figure out how we could model that in our classrooms, how we could encourage that sort of behavior in our classrooms. … And so, "What are the types of things that we're doing? What can we say to … our students to try to encourage better collaboration and better problem solving in our classes?"

Note the focus on the metacognitive aspects of problem solving and the desire of these teachers to model problem solving, particularly metacognition, in the classroom. Collaboration with colleagues is not a requirement for teaching through problem solving, but as the Northridge teachers' experience shows, it can certainly help support and facilitate this approach to teaching.

Summary

This chapter contains a description of teachers' beliefs about problem solving and what makes a good problem. For each teacher, problem solving played a central role in his or her teaching. Good problems, they believed, engaged students and challenged them to think mathematically. Assigning meaningful problems was more important to the teachers than assigning large numbers of problems, although each teacher recognized that practice is important for becoming a good problem solver. The bulk of the chapter consists of descriptions of the practices the teachers used to teach through problem solving. These practices are consistent with the advice of mathematics education experts in the past several decades. To help students develop as problem solvers, experts have said, some or all of these practices should be part of a teacher's repertoire: (a) teaching problem-solving strategies, (b) modeling problem solving, (c) limiting teacher input, (d) promoting metacognition, and (e) highlighting multiple solutions. The chapter concludes with a description of the collaboration among the teachers at Northridge High School. Collaborating with colleagues is not a teaching practice, but when teachers work together they are able to support one another in the task of teaching through problem solving by sharing ideas and advice.

CHAPTER 5

CONCLUSION

If problem solving is what mathematics is all about, then mathematics teachers should be in the business of helping students develop their problem-solving abilities. One way to help is to teach mathematics through problem solving. In this approach, problems are a means by which students learn new mathematical concepts and synthesize mathematical knowledge. This chapter contains a summary of the purpose and results of the present study. The summary is followed by a discussion of some connections between Schoenfeld's (1985, 1992) framework for mathematical problem solving and practices for teaching through problem solving. That discussion is followed by a description of limitations of the study and possible next steps for research in teaching through problem solving. The chapter concludes with implications of the study for mathematics educators.

Summary of the Purpose and Results of the Present Study

In this study, I examined the beliefs and practices of four high school mathematics teachers who taught through problem solving. They were similar in their basic philosophy of teaching but unique in the ways they carried out the task of helping students develop as problem solvers. There were commonalities in their practices—for example, every teacher promoted metacognition in some way—and there were differences as well—for example, not all the teachers regularly modeled problem solving.

The goal of the study was not to develop a prototype of a teacher who teaches through problem solving. Nor was the goal to prescribe particular practices for all mathematics teachers to use. Rather, the study shows that there are many ways of helping students grow in their problem-solving ability. The primary goal of the study was to describe practices for teaching through problem solving so as to yield a better understanding of what successful teachers do. Mathematics education researchers have offered advice about problem-solving instruction for many years, and I wanted to see how that advice was being put into practice. Knowing what teaching through problem solving looks like can help teachers who desire to implement practices that support this instructional approach.

Teaching through problem solving begins with choosing good problems. Researchers have suggested that good problems are “accessible and engaging to the students, building on what they know and can do” (Schoen, 2003, p. xi). Furthermore, a good problem is “clearly stated, … involves an important real-world context or mathematical context that has the potential to attract and maintain students’ interest, [and] can be solved with a range of methods” (Grouws, 2003, p. 134). As a group, the four teachers in the study chose problems consistent with the advice of mathematics education researchers. The teachers in this study believed problems should be challenging yet manageable and should engage the students’ interests. For example, one teacher explained that he tried to find problems that were “simultaneously engaging and doable.” Some of the teachers in this study expressed the importance of wording problems carefully to help students “discover what you want them to discover.” All four teachers assigned some “real-world” problems and other problems that did not have direct application beyond pure mathematics. Examples of applications include problems about energy use, distances between subway stops, and population growth. One teacher described good problems by saying, “Even if [a problem] permits only a single answer, … there still might be different approaches to get to it.”

Good problems also “foster students’ understanding of important mathematical ideas and techniques” (Marcus & Fey, 2003, p. 55) and “integrate multiple topics” (NCTM, 2000, p. 52). A teacher can use problems to introduce mathematical topics or to help students synthesize their mathematical knowledge. All of the teachers in this study viewed problem solving as a way to accomplish both of these goals. One teacher began each class with the Problem of the Day that he used as an "entry point" for the lesson. For the most part, he used these problems to introduce new mathematical content. Some problems were extensions of the Problem of the Day and required students to use a concept or practice a skill they had learned previously. Another teacher assigned problems as a way of "pulling ideas together." A third teacher distinguished between problems that led students to discover a new concept and problems that required students to apply mathematical content they had already learned to a novel situation. This distinction determined how he worded problems. If his purpose was for students to learn a new concept, then he tried to word the problem in such a way as to provide clear guidance to that end. On the other hand, if his purpose was for students to explore freely and to apply or synthesize previous knowledge in a new context, he worded the problem to give no direction at all regarding what strategy to use.

The group of four teachers implemented teaching practices—in addition to assigning good problems—that were in line with the advice found in the mathematics education literature regarding teaching through problem solving. Some practices were more prevalent than others, but as Figure 11 (see p. 123) shows, the teachers as a group regularly engaged in the following: (a) teaching problem-solving strategies, (b) modeling problem solving, (c) limiting teacher input—for example, having students work in groups, (d) promoting metacognition, and (e) highlighting multiple solutions. In addition, the group of teachers made implicit references to some of Polya's phases of problem solving—specifically understanding the problem and looking back. The following paragraphs describe what the mathematics education literature has to say about each of these practices and include examples from the data from the present study—both classroom observations and teacher interviews—to illustrate how the teachers implemented the practices.

One practice for teaching through problem solving is *teaching problem-solving strategies*, a practice that two of the teachers in this study regularly used. According to the NCTM (2000), “students must become aware of [problem-solving] strategies *as the need for them arises*, and as they are modeled during classroom activities, the teacher should encourage students to take note of them” (p. 54, emphasis added). In line with this advice, the teachers in the present study who taught problem-solving strategies did so in the context of solving problems, rather than teaching strategies in isolation. For example, one teacher used a problem about a climbing rope (see Figure 10) to teach general problem-solving strategies like using unsuccessful attempts to lead to successful ones, and specific problem-solving strategies like keeping “chunks” together when solving an equation. Often the teachers’ presentation of problem-solving strategies occurred while they modeled problem solving, which is the second teaching practice I describe.

Mathematics education experts ever since Polya (1957, 1962, 1965) have advised teachers to *model problem solving* (e.g., Grouws, 2003; Levasseur & Cuoco, 2003). Polya (1957) described how modeling might work and potential benefits for students who observe their teacher modeling problem solving:

> When the teacher solves a problem before the class, he should dramatize his ideas a little and he should put to himself the same questions which he uses when helping the students. Thanks to such guidance, the student will eventually discover the right use of these questions and suggestions, and doing so he will acquire something that is more important than the knowledge of any particular mathematical fact. (p. 5)

Two of the teachers in this study modeled problem solving on a regular basis and one did so on one of the days I observed. That is not to say the other teacher did not model problem solving, but I did not observe her doing so. One teacher modeled problem solving both in his verbal discussions of problems during class and in his written solutions to homework assignments. The most common way of modeling problem solving the teachers used was thinking aloud. When solving problems in front of the class, both teachers verbalized their thought processes, so for them, modeling metacognition was part of modeling problem solving. For example, one teacher explained his thought processes while solving a problem that required integration by parts. He described how he made his decisions about which part of the function to designate as *u* and which to designate as *dv*.

The mathematics education literature indicates that *limiting teacher input* is a key part of teaching through problem solving (e.g., Hiebert & Wearne, 2003; NCTM, 2000), and the group of teachers in the present study confirmed that in their teaching practices. Their practices included having students work in groups, which researchers have long claimed is a key element in problem-solving instruction (Kilpatrick, 1985; Posamentier & Jaye, 2006; Schoenfeld, 1985). The teachers also were careful to refrain from telling students too much—in both written and verbal instructions—and allowed students to struggle, which are important practices for teaching through problem solving (Hiebert & Wearne, 2003; NCTM, 2000). Hiebert and Wearne claimed, "The key to allowing mathematics to be problematic for students is for the teacher to refrain from stepping in and doing too much of the mathematical work too quickly" (p. 7). The teachers in this study also encouraged their students to rely on each other and on their own mathematical reasoning, rather than on the teacher, to confirm correct answers, which is a practice advocated by some mathematics education experts (e.g., Schoenfeld, 1994). The teachers frequently responded to students' questions by asking questions in return.

The teachers in the present study *promoted metacognition* in two ways: (a) by modeling metacognition, and (b) by asking metacognitive questions. Grouws (1985) described teachers who model problem solving and noted that modeling includes thinking aloud—that is, modeling metacognition:

> Some teachers regularly and proficiently model the problem-solving process and … this is a positive influence on students. Such teachers pause at times to think, think aloud while considering subsequent steps, … check the reasonableness of answers, and so on. (p. 302)

In the classrooms I observed, modeling metacognition occurred as teachers modeled problem solving, as I described above. The mathematics education literature is full of examples of metacognitive questions, including the following: "Are we making progress or should we reconsider what we are doing?" (NCTM, 2000, p. 55), "How did you decide on a solution method to try?" (Grouws, 2003, p. 137), "Have I made any careless mistakes?" (Posamentier & Jaye, 2006, p. 80), "Is the answer reasonable?" (Grouws, 2003, p. 139), and "What could I have done better?" (Polya, 1962, p. xii). Asking metacognitive questions was a common practice with the group of teachers in the study. Examples of metacognitive questions the teachers asked were very similar to those above and included the following: "Are the decisions I am making actually helpful?" "Is what you're doing getting you anywhere?" "Why did you decide to use that approach?" "Did I do that correctly?" "Are the answers that you're coming up with … making any sense?" and "Do you think there are things you [could] have done to have solved the problem more quickly or efficiently?" These examples show that the data in this study regarding promoting metacognition by asking metacognitive questions support what mathematics education experts have recommended in the literature.

Some mathematics education experts have recommended that teachers *highlight multiple solutions* for any given problem (e.g., Grouws, 2003; Hiebert et al., 1996; Hiebert & Wearne, 2003; NCTM, 2000). For example, the NCTM (2000) has advocated encouraging students to "propose, critique, and value alternative approaches to solving problems" (p. 261). As a group, the teachers in the present study encouraged students to at least recognize and appreciate that a problem could be solved using a variety of approaches. Asking students to solve a problem in more than one way was not a common practice among the teachers in this study, but I did observe one teacher challenge a group of students to think about a problem geometrically after they had solved it algebraically. Only one of the teachers regularly had students share their solutions, although all of the teachers had students share solutions on occasion. All of the teachers noted that some solutions are better than others—for example, some solutions are more efficient, more elegant, or give more insight into the problem than others.

None of the teachers in the study mentioned Polya by name, or phases of problem solving, but the teachers encouraged students to *understand the problem* and to *look back* on a problem. The teachers encouraged their students to look back by reflecting on the problem while solving it—for example, by pausing to see if the chosen method was getting them closer to a solution—and by reflecting on solution after solving the problem—for example, by checking for errors.

There are practices for teaching through problem solving that mathematics education researchers have recommended, but that the teachers in the present study did not regularly implement. One such practice that many Japanese teachers implement is *making connections between solutions*—that is, examining various ways of approaching a problem and comparing the mathematical ideas in those approaches (Smith, 2004; Takahashi, 2008). The teachers in this study highlighted multiple solutions—for example by having students share different ways of solving a problem—and compared solutions based on elegance or efficiency, but did not regularly make connections between solution methods.

Some mathematics education researchers (e.g., Silver, Kilpatrick, & Schlesinger, 1990; Silver, Mamona-Downs, Leung, & Kenney, 1996) have suggested that one way to help students develop as problem solvers is to have the students *pose problems*. Silver et al. (1990) stated,

> Students need practice in formulating mathematical problems for themselves. … Problem posing is almost always overlooked in discussions of the importance of problem solving in the curriculum. … Nonetheless, it ought to be given the same emphasis in instruction that problem solving is beginning to receive. (p. 10–11)

Having students pose problems was not a regular practice among the teachers in the present study. On one occasion a teacher presented a scenario about population growth and asked his students, "What would be a good question to ask? Let's ask an interesting question." But as a group, the four teachers did not incorporate problem posing as a teaching practice on a regular basis.

As I observed and interviewed the teachers, I discovered some aspects of teaching through problem solving that I did not expect. One such discovery was that some of the teachers took *affective considerations* into account when choosing problems. That it, they considered how students would respond emotionally to particular problems. One teacher, when given the choice between two problems that incorporated the same mathematical concepts and addressed similar environmental issues, opted for the problem that would be more heartening and empowering for her students. Another teacher considered the emotional responses of the students in his class before assigning a problem involving potentially uncomfortable data about height and weight.

Another surprising aspect of teaching through problem solving was the importance of *collaboration*. A common theme among the three teachers in the study who taught at the same high school was the importance of collaborating with colleagues. All the mathematics teachers at the school contributed problems and activities to a shared pool for everyone in the department to use. The three teachers in the study talked about the benefits of working together with colleagues to write and refine problems, plan lessons, and even solve problems together as a way of sharing ideas about how to help students become better problem solvers. Collaboration is not a teaching practice, but working with fellow teachers played a key role in the three teachers' ability to teach through problem solving because of the support they could give one another.

The teacher at the other school noted that he did not often collaborate with his colleagues in a way that supported his instructional approach. Compared to the mathematics department where the other three teachers taught, the single teacher's mathematics department was not as unified in their philosophy of teaching mathematics. Where the three teachers taught, there was a department-wide understanding—even a policy—that mathematics should be taught using problems, activities, and investigations. That was not the case at the single teacher's school. Another reason he did not experience the same level of collaboration as the three teachers at the other school was the physical distance between teachers at his school. Whereas the three teachers who taught at the same school were housed in the same building, the mathematics classrooms at the single teacher's school were spread across the campus so that each teacher was fairly isolated geographically. This isolation, along with the lack of motivation that might have existed had there been philosophical agreement, resulted in little if any collaboration to support the teacher's implementation of teaching through problem solving.

Connecting Schoenfeld's Framework to Practices for Teaching Through Problem Solving

In Schoenfeld's (1985, 1992) framework, successful problem solving depends on the problem solver's resources, strategies, control, and beliefs. Two of the assumptions in this study were that successful problem solving depends on those four factors and that teachers who teach through problem solving seek to help their students develop as problem solvers. If teachers are to help students develop as problem solvers, they must consider how to encourage students in matters of resources, strategies, control, and beliefs. I examined the practices teachers in this study used to teach through problem solving, and I connected those practices with the elements in Schoenfeld's framework. There is not a one-to-one correspondence between the teaching practices and the elements of the framework. Rather, each practice addresses more than one element, and each element corresponds to more than one practice.

Resources

The first element of Schoenfeld's (1985, 1992) framework is mathematical resources. Teaching through problem solving depends on teachers assigning good problems. When students have the opportunity to work on meaningful and challenging problems, their problem-solving ability and mathematical knowledge are likely to grow in breadth as well as depth. The teachers in this study assigned problems that covered a wide range of mathematical content. More importantly, teaching through problem solving, with its goal of using problems as a means for learning mathematical content and synthesizing mathematical knowledge, is a powerful way to help students deepen—not just broaden—their knowledge and understanding of mathematics.

Teaching problem-solving strategies can also be a way to strengthen students' resources by encouraging them to examine their knowledge and put it to use. For example, when one teacher suggested that students exploit the symmetry of a polar curve in order to integrate it more efficiently, he called the students' attention to knowledge they already had and showed how to use it to solve a problem.

Modeling problem solving can be a powerful teaching practice that helps students grow as problem solvers. Teachers highlight relevant resources when they demonstrate mathematical concepts and skills. For example, when one teacher wrote up his solutions after every homework assignment, his students could see what resources were required for a particular problem.

Promoting metacognition can have a positive effect on students' resources. Teachers promote metacognition when they encourage their students to be aware of their own knowledge. Asking metacognitive questions challenges students to think about what they know and how they know it. By being reflective and examining what they know and how they are thinking, students can strengthen the parts of their knowledge base they are focusing on. For example, a teacher's question, "How do you picture in your head what the inverse is gonna look like?" may help students solidify their understanding of inverse functions.

Highlighting multiple solutions is a way that a teacher can encourage students to use the full range of their available resources—as well as strategies—to solve a problem. And when students are exposed to various ways to solve a problem, they may see a variety of mathematical concepts at work.

Strategies

There are several ways teachers can help their students increase their knowledge of, and proficiency with, problem-solving strategies. For example, teachers can assign problems that highlight particular problem-solving strategies, as was the case for one teacher, who deliberately chose problems at the beginning of his differential equations course to highlight both general and specific problem-solving strategies that the students would need throughout the course.

It seems obvious that *teaching problem-solving strategies* is likely to result in the growth of students' repertoires of strategies. This growth occurs when teachers present problem-solving strategies in the context of solving a problem as opposed to teaching heuristic strategies in isolation.

When teachers are *modeling problem solving* they are inevitably demonstrating strategy use. For example, in the process of solving a differential equations problem in front of his class, one teacher demonstrated the strategy of keeping a certain “chunk”—that is, a mathematical expression—together in order to make the algebra in the problem more manageable.

Limiting teacher input, particularly refraining from giving students too many instructions about how to solve a problem, encourages students to develop their own problem-solving strategies. A good example of a problem with very few instructions is the Subway Problem (see Figure 5), which gave no guidance, so students had to figure out a strategy on their own.

Promoting metacognition involves encouraging students to monitor their use of problem-solving strategies. Teachers may encourage monitoring by asking questions like, “Why did you choose that strategy?” or “Will that strategy lead you closer to a solution?” On one occasion a teacher said to her students, “Think about some strategies you might employ to get a graph of this. Think about how this is different from the functions we’ve done before. Think about characteristics this type of graph might have.” By making these comments, she emphasized the importance of monitoring strategy use.

Control

Monitoring, or control, has to do with the metacognitive behavior that is necessary for successful problem solving. Teachers can encourage students to be good monitors while problem solving by asking metacognitive questions such as, “Is what you’re doing getting you anywhere?” They can also encourage students to be thoughtful and reflective both during the process of solving a problem and after they have found a solution.

Modeling problem solving often involves modeling metacognition. Modeling metacognition requires that teachers verbalize their thought processes while solving a problem in front of the class, as two of the teachers in this study did. Modeling metacognition also includes explaining thought processes in written form, as the teacher did who wrote out his solutions to homework problems.

The most obvious teaching practice associated with control is *promoting metacognition*. By drawing attention to students' thinking, as well as to their own, teachers communicate that metacognition is important. The teachers in this study asked students many questions to help students develop metacognitive behavior—questions such as, "What do you know?" "Why did you choose that strategy?" and "Are you checking your work?"

Beliefs

Much of what teachers do and say affects students' beliefs about mathematics, problem solving, and themselves as problem solvers. When teachers assign meaningful and engaging problems, for example, students may come to see mathematics as a worthwhile endeavor. Students may develop a belief—or confirm their view—that mathematics is relevant and interesting. Furthermore, they may develop positive beliefs about themselves as problem solvers when they have success in solving meaningful problems. When they are engaged in a meaningful task, their work has purpose.

Teaching problem-solving strategies can have an impact on students' beliefs about mathematics and about themselves as problem solvers. That there are common strategies for solving a range of problems illustrates that mathematics is sensible and connected rather than a disjoint set of ideas and problems. Gaining proficiency in problem-solving strategies can also help students grow more confident in their ability to solve problems.

One component of *modeling problem solving* is demonstrating perseverance and patience. By demonstrating perseverance and patience, teachers model their beliefs about problem solving and can positively influence students' beliefs. One of these beliefs is that mathematics is worth doing even if it involves a struggle. Two of the teachers in this study led class discussions of problems in which they hit one or more roadblocks or errors. They persevered and eventually reached a solution. Another belief teachers can model is that many worthwhile problems cannot be solved in just a few minutes. Schoenfeld (1988) identified the belief that "students who understand the subject matter can solve assigned mathematics problems in 5 minutes or less" (p. 7) as being detrimental to problem solving. Teachers' efforts to model the patience required to solve problems can help to change this belief.

Limiting teacher input can positively influence students' beliefs about mathematics and problem solving. First, working in groups lets students see that mathematics is not just an individual endeavor. Frequently, a group makes more progress in solving a problem than a single problem solver could make. Doing mathematics can also be more enjoyable when working as a group. All of the teachers in this study indicated that students seem to enjoy working in groups. Second, when teachers allow students to struggle, perhaps by not giving them much guidance, they encourage students to believe that perseverance is worth the effort. Solving problems is a worthwhile pursuit even though it is often fraught with roadblocks and difficulties. Third, teachers can help students believe that they are capable of solving problems. One of the beliefs Schoenfeld (1988) listed as having the potential to negatively affect students' abilities to solve problems is, "Only geniuses are capable of discovering, creating, or really understanding mathematics. Corollary: Mathematics is studied passively, with students accepting what is passed down 'from above' without the expectation that they can make sense of it for themselves"

(p. 7). Teachers can help students change this belief by encouraging them to rely on their own mathematical reasoning rather than looking solely to the teacher for the correct answers. When a teacher turns the work of problem solving over to the students, they gain confidence in their own mathematical reasoning and come to believe mathematics is a sensible discipline rather than an arbitrary set of rules set forth by an outside authority.

Finally, by *highlighting multiple solutions*, teachers can help students gain courage to try a unique way of tackling a problem, knowing that their task is not to search for the one right way to solve a problem. Knowing that there is more than one way to solve a problem can also give students confidence in their ability as problem solvers, realizing that just because they did a problem differently from someone else does not mean they did it incorrectly.

Limitations of the Study and Next Steps

This study was a *descriptive case study*. As is true with all case studies, the goal was not to make generalizations about large populations. Not everyone who teaches mathematics through problem solving implements this instructional approach the way the four teachers in this study did. I used a small sample of teachers, so not every practice for teaching through problem solving was represented in their instruction. Furthermore, I observed each teacher for only 2 or 3 weeks, so there may have been practices they employed that I did not observe. More observations over a longer period of time might have revealed a more complete picture of teaching through problem solving. This limitation suggests a possible next step for research in problem-solving instruction: to conduct a longer-term study of teachers who teach through problem solving.

Another limitation of the study is the uniqueness of the teachers and the settings in which they taught. These were four exceptional teachers who taught in schools where they had the freedom to focus on problem solving. Their students were, for the most, well behaved and cooperative. Many teachers do not have these advantages. Further research, then, might include interviewing and observing teachers in a wider variety of settings so that a researcher could create a more thorough description of practices for teaching through problem solving. For example, I would like to know about teachers who teach through problem solving in a typical public school. Do they find it difficult to teach through problem solving and also meet the demands that accompany state standards and standardized testing?

I did not assess the impact of teaching through problem solving. Because I began with the assumption that teaching through problem solving is a valuable instructional approach, I did not consider the effects on students of teaching through problem solving. I was not testing the effectiveness of teaching through problem solving, so I did not seek to determine whether the approach actually helped students develop their problem-solving ability. I certainly observed positive outcomes as I watched students working together on problems, but I have no quantifiable data to show that teaching through problem solving is a superior instructional approach. Further research in problem-solving instruction might include examination of outcomes of teaching through problem solving. One way to assess the impact on students of teaching through problem solving is to examine the problem-solving ability, as well as attitudes and beliefs, of students before and after they complete a course that was taught using this instructional approach. One of the schools in the present study had students in Grades 11 and 12 only. Most students entering as 11th graders had little or no experience in classes that focused on problems and investigative tasks. I would like to follow a group of students through their 2 years at the school to see the change in their ability to solve problems they had not been taught how to solve. In addition, I would like to see how their attitudes about problems and problem solving changed over the course of the 2 years.

Implications

The NCTM (1980, 1989, 2000) has strongly recommended that mathematics teachers emphasize problem solving in their classrooms. If teachers are to follow this recommendation, they must find ways to use problems and problem solving as a means to help students gain deeper mathematical understanding, which in turn can help students become better problem solvers. One way to emphasize problem solving is to teach mathematics through problem solving, but in order to do so, teachers must know what this instructional approach looks like in practice.

This study shows that teaching through problem solving is possible. Real teachers in real classrooms with real students have implemented many of the teaching practices that mathematics education researchers have suggested. The four teachers in the study were unique—they each had a reputation for being successful—but their practices are not unattainable. Just as investigating expert problem solvers can give educators and researchers insights into what successful *problem solving* looks like, so investigating good teachers can give educators and researchers insights into what successful *teaching through problem solving* looks like. This study has provided such insights. Therefore the study can contribute to the body of mathematics education literature, which lacks research on teaching through problem solving.

The description of practices for teaching through problem solving that I describe in this dissertation can be helpful for any teacher interested in implementing such an instructional approach. Teaching through problem solving looked different for each of the teachers in the study, which means that a teacher interested in this instructional approach need not seek to copy a particular teacher in order to focus on problem solving in his or her instruction. Rather, a teacher may consider the wide range of practices examined in this study as providing an overview of the possibilities he or she might incorporate into his or her teaching.

If, for example, teachers would like to change the way they select problems to assign, they might consider some of the criteria teachers in this study used. They might consider ways to use problems as a means to empower students to address real-world scenarios such as energy use. They might choose to follow the advice of one teacher to be very careful about the wording of a problem, keeping in mind the problem's purpose. Teachers might find it suitable to use problems to highlight particular problem-solving strategies, or to use a problem as an entry point for a lesson.

Teachers interested in teaching through problem solving might challenge themselves to try one or more of the practices the teachers in this study used. Implementing these practices may require small steps such as thinking aloud while solving a problem in front of the class or refraining from intervening when students hit a roadblock while solving a problem. A teacher may try to ask some metacognitive questions or ask students to look back on a problem after they have solved it. Teaching through problem solving is difficult, but implementing one or two practices can be quite manageable for a teacher who sees the value in it.

There may be teachers who are resistant to a problem-solving approach to instruction but whose administration or school board require them to use a problem-based curriculum. In this case, the present study may be helpful as a description of teachers who have successfully taught through problem solving. Having an idea about what teaching through problem solving looks like may give teachers courage to try new things.

For teachers who have never heard of teaching through problem solving, the present study may pique their interest. It may never have occurred to some teachers that there is a way to teach mathematics other than the way they were taught. Ideally, the teachers in this study will inspire other teachers to broaden their horizons by trying new ideas. At the very least, this study may cause teachers to think differently about mathematics education and the possibilities for what can happen in the classroom. Thinking differently may lead to teaching differently.

REFERENCES

Allevato, N. S. G., & Onuchic, L. R. (2008). *Teaching mathematics in the classroom through problem solving*. Paper presented at the 11th International Congress on Mathematical Education, Monterrey, Mexico.

Arcavi, A., Kessel, C., Meira, L., & Smith, J. (1998). Teaching mathematical problem solving: A microanalysis of an emergent classroom community. In A. H. Schoenfeld, E. Dubinsky, & J. Kaput (Eds.), *Research in collegiate mathematics education 3* (pp. 1–70). Providence, RI: American Mathematical Society.

Barrett, G. B., & Compton, H. L. (2003). Snapshots of a precalculus course focused on problem solving. In H. L. Schoen (Ed.), *Teaching mathematics through problem solving: Grades 6–12* (pp. 85–91). Reston, VA: National Council of Teachers of Mathematics.

Bellman, A. (2003). Who is in charge here? In H. L. Schoen (Ed.), *Teaching mathematics through problem solving: Grades 6–12* (pp. 119–126). Reston, VA: National Council of Teachers of Mathematics.

Bennett, J. M. (2007). *Holt Mathematics Course 1*. Orlando, FL: Harcourt.

Berg, B. L. (2007). *Qualitative research methods for the social sciences* (6th ed.). Boston, MA: Pearson.

Bloom, B. S. (Ed.). (1956). *Taxonomy of educational objectives, handbook 1: Cognitive domain*. New York, NY: McKay.

Boaler, J. (2008). *What's math got to do with it? Helping children learn to love their least favorite subject—and why it's important for America*. New York, NY: Viking.

Branca, N. A. (1980). Problem solving as a goal, process, and basic skill. In S. Krulik & R. E. Reys (Eds.). *Problem solving in school mathematics* (1980 Yearbook of the National Council of Teachers of Mathematics, pp. 3–8). Reston, VA: National Council of Teachers of Mathematics.

Campione, J. C., Brown, A. L., & Connell, M. L. (1988). Metacognition: On the importance of understanding what you are doing. In R. Charles & E. A. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 93–114). Reston, VA: National Council of Teachers of Mathematics.

Charles, R. I., & Lester, F. K. (1984). An evaluation of a process-oriented mathematical problem-solving instructional program in Grades 5 and 7. *Journal for Research in Mathematics Education*, *15*, 15–34.

Cohen, D. K. (1990). A revolution in one classroom: The case of Mrs. Oublier. *Education Evaluation and Policy Analysis*, *12*(3), 311–329.

Garofalo, J., & Lester, F. K. (1985). Metacognition, cognitive monitoring, and mathematical performance. *Journal for Research in Mathematics Education*, *16*, 163–176.

Grouws, D. A. (1985). The teacher and classroom instruction: Neglected themes in problem-solving research. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 295–307). Hillsdale, NJ: Erlbaum.

Grouws, D. A. (2003). The teacher's role in teaching mathematics through problem solving. In H. L. Schoen (Ed.), *Teaching mathematics through problem solving: Grades 6–12* (pp. 129–141). Reston, VA: National Council of Teachers of Mathematics.

Guernon, V. E. (1989). *The effects of teaching heuristics within the context of a prescriptive metacognitive control system on the problem-solving performance of eighth-grade general mathematics students*. ProQuest Dissertations and Theses. Retrieved from http://search.proquest.com/docview/303701788?accountid=10285

Harwell, M. R., Post, T. R., Maeda, Y., Davis, J. D., Cutler, A. L., Anderson, E., et al. (2007). Standards-based mathematics curricula and secondary students' performance on standardized achievement tests. *Journal for Research in Mathematics Education*, *38*, 71–101.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, et al. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher, 25*(4), 12–21.

Hiebert, J., & Wearne, D. (2003). Developing understanding through problem solving. In H. L. Schoen (Ed.), *Teaching mathematics through problem solving: Grades 6–12* (pp. 3–13). Reston, VA: National Council of Teachers of Mathematics.

Huntley, M. A., Rasmussen, C. L., Villarubi, R. S., Sangtong, J., & Fey, J. T. (2000). Effects of Standards-based mathematics education: A study of the Core-Plus Mathematics Project algebra and functions strand. *Journal for Research in Mathematics Education*, *31*, 328–361.

Johnson, D. R. (1990). *Every minute counts: Making your math class work*. White Plains, NY: Seymour.

Kahan, J. A., & Wyberg, T. R. (2003). Mathematics as sense making. In H. L. Schoen (Ed.), *Teaching mathematics through problem solving: Grades 6–12* (pp. 15–25). Reston, VA: National Council of Teachers of Mathematics.

Kilpatrick, J. (1978). Variables and methodologies in research on problem solving. In L. L. Hatfield (Ed.), *Mathematical problem solving* (pp. 7–20). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Kilpatrick, J. (1985). A retrospective account of the past 25 years of research on teaching mathematical problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 1–15). Hillsdale, NJ: Erlbaum.

Lambdin, D. V. (2003). Benefits of teaching through problem solving. In F. K. Lester (Ed.), *Teaching mathematical through problem solving: Prekindergarten–Grade 6* (pp. 3–13). Reston, VA: National Council of Teachers of Mathematics.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Freil, S. N., & Phillips, E. D. (1991–1997). *Connected mathematics*. White Plains, NY: Seymour.

Lappan, G., Fey, J. T., Fitzgerald, W., Friel, S. N., & Phillips, E. D. (2002). *Connected Mathematics*. Glenview, IL: Prentice Hall.

Lappan, G., Fey, J. T., Fitzgerald, W., Friel, S. N., & Phillips, E. D. (2006). *Connected Mathematics 2*. Boston, MA: Prentice Hall.

Larson, R. E., Kanold, T. D., & Stiff, L. (1993). *Algebra 2*. Lexington, MA: Heath.

Lester, F. K., Jr. (1985). Methodological considerations in research on mathematical problem-solving instruction. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 41–69). Hillsdale, NJ: Erlbaum.

Lester, F. K., Jr. (1994). Musings about mathematical problem-solving research: 1970–1994. *Journal for Research in Mathematics Education*, *25*, 660–675.

Lester, F. K., Jr., & Charles, R. (Eds.). (2003). *Teaching mathematics through problem solving: Prekindergarten–Grade 6*. Reston, VA: National Council of Teachers of Mathematics.

Levasseur, K., & Cuoco, A. (2003) Mathematical habits of mind. In H. L. Schoen (Ed.), *Teaching mathematics through problem solving: Grades 6–12* (pp. 27–37). Reston, VA: National Council of Teachers of Mathematics.

Marcus, R., & Fey, J. T. (2003). Selecting quality tasks for problem-based teaching. In H. L. Schoen (Ed.), *Teaching mathematics through problem solving: Grades 6–12* (pp. 55–67). Reston, VA: National Council of Teachers of Mathematics.

Maxwell, J. A. (2005). *Qualitative research design: An interactive approach* (2nd ed.). Thousand Oaks, CA: Sage.

Mayer, R. E. (1985). Implications of cognitive psychology for instruction in mathematical problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 123–138). Hillsdale, NJ: Erlbaum.

McCaffrey, D. F., Hamilton, L. S., Stecher, B. M., Klein, S. P., Bugliari, D., & Robyn, A. (2001). Interactions among instructional practices, curriculum, and student achievement: The case of standards-based high school mathematics. *Journal for Research in Mathematics Education*, *32*, 493–517.

Montague, M. (2007). Self-regulation and mathematics instruction. *Learning Disabilities Research and Practice*, *22*, 75–83.

National Council of Teachers of Mathematics. (1980). *An agenda for action: Recommendations for school mathematics of the 1980s*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Research Council. (1989). *Everybody counts: A report to the nation on the future of mathematics education*. Washington, DC: National Academy Press.

Polya, G. (1957). *How to solve it* (2nd ed.). New York, NY: Doubleday.

Polya, G. (1962). *Mathematical discovery* (Vol. 1). New York, NY: Wiley.

Polya, G. (1965). *Mathematical discovery* (Vol. 2). New York, NY: Wiley.

Posamentier, A. S., & Jaye, D. (2006). *What successful math teachers do, Grades 6–12: Seventy-nine research-based strategies for the standards-based classroom*. Thousand Oaks, CA: Corwin.

Riordan, J. E., & Noyce, P. E. (2001). The impact of two standards-based mathematics curricula on student achievement in Massachusetts. *Journal for Research in Mathematics Education*, *32*, 368–398.

Santos-Trigo, M. (1998). Instructional qualities of a successful mathematical problem-solving class. *International Journal of Mathematical Education in Science and Technology*, *29*(5), 631–646.

Schoen, H. L. (Ed.). (2003). *Teaching mathematics through problem solving: Grades 6–12*. Reston, VA: National Council of Teachers of Mathematics.

Schoen, H. L., Cebulla, K. J., Finn, K. F., & Fi, C. (2003). Teacher variables that relate to student achievement when using a standards-based curriculum. *Journal for Research in Mathematics Education*, *34*, 228–259.

Schoenfeld, A. H. (1980). Heuristics in the classroom. In S. Krulik & R. E. Reys (Eds.), *Problem solving in school mathematics* (1980 Yearbook of the National Council of Teachers of Mathematics, pp. 9–22). Reston, VA: National Council of Teachers of Mathematics.

Schoenfeld, A. H. (1983). Beyond the purely cognitive: Belief systems, social cognitions, and metacognitions as driving forces in intellectual performance. *Cognitive Science*, *7*, 329–363.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A. H. (1987a). What's all the fuss about metacognition? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189–215). Hillsdale, NJ: Erlbaum.

Schoenfeld, A. H. (1987b). Polya, problem solving, and education. *Mathematics Magazine*, *60*, 283–291.

Schoenfeld, A. H. (1988). When good teaching leads to bad results: The disasters of "well-taught" mathematics courses. *Educational Psychologist*, *23*, 145–166.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). New York, NY: Macmillan.

Schoenfeld, A. H. (1994). Reflections on doing and teaching mathematics. In A. H. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 53–70). Hillsdale, NJ: Erlbaum.

Schoenfeld, A. H. (2002) Making mathematics work for all children: Issues of standards, testing, and equity. *Educational Researcher, 31*, 3–25.

Schroeder, T. L., & Lester, F. K. (1989). Developing understanding in mathematics via problem solving. In P. R. Trafton & A. P. Shulte (Eds.), *New directions for elementary school mathematics* (1989 Yearbook of the National Council of Teachers of Mathematics, pp. 31–42). Reston, VA: National Council of Teachers of Mathematics.

Silver, E. A. (1982). Knowledge organization and mathematical problem solving. In F. K. Lester & J. Garofalo (Eds.), *Mathematical problem solving: Issues in research* (pp. 15–25). Philadelphia, PA: Franklin.

Silver, E. A., Kilpatrick, J., & Schlesinger, B. (1990). *Thinking through mathematics: Fostering inquiry and communication in mathematics classrooms*. New York, NY: College Entrance Examination Board.

Silver, E. A., Mamona-Downs, J., Leung, S. S., & Kenney, P. A. (1996). Posing mathematical problems: An exploratory study. *Journal for Research in Mathematics Education*, *27*, 293–309.

Smith, M. (2004). Beyond presenting good problems: How a Japanese teacher implements a mathematics task. In R. N. Rubenstein (Ed.), *Perspectives on the teaching of mathematics* (2004 Yearbook of the National Council of Teachers of Mathematics, pp. 96–106). Reston, VA: National Council of Teachers of Mathematics.

Smith, S. A., Charles, R. I., Keedy, M. L., Bittinger, M. L., & Orfan, L. J. (1988). *Algebra*. Menlo Park, CA: Addison-Wesley.

Stanic, G. M., & Kilpatrick, J. (1988). Historical perspectives on problem solving in the mathematics curriculum. In R. I. Charles & E. A. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 1–22). Reston, VA: National Council of Teachers of Mathematics.

Stein, M. K., Boaler, J., & Silver, E. A. (2003). Teaching mathematics through problem solving: Research perspectives. In H. L. Schoen (Ed.), *Teaching mathematics through problem solving: Grades 6–12* (pp. 245–256). Reston, VA: National Council of Teachers of Mathematics.

Takahashi, A. (2008). *Beyond show and tell*: Neriage *for teaching through problem-solving—ideas from Japanese problem-solving approaches for teaching mathematics*. Paper presented at the 11th International Congress on Mathematical Education, Monterrey, Mexico.

Thompson, D. R., & Senk, S. L. (2001). The effects of curriculum on achievement in second-year algebra: The example of the University of Chicago School Mathematics Project. *Journal for Research in Mathematics Education*, *32*, 58–84.

Wilson, J. W., Fernandez, M. L., & Hadaway, N. (1993). Mathematical problem solving. In P. S. Wilson (Ed.), *Research ideas for the classroom: High school mathematics* (pp. 57–78). New York, NY: Macmillan.

Wood, T., & Sellers, P. (1997). Deepening the analysis: Longitudinal assessment of a problem-centered mathematics program. *Journal for Research in Mathematics Education*, *28*, 163–186.

APPENDIX: Interview Questions

First interview

1. What comes to mind when you hear the term *mathematical problem solving*?
2. How do you incorporate problem solving [or *problem-based teaching*] into your math classes?
3. Tell me about when and how you came to see problem solving as a necessary focus in your classroom.
4. If you have not always taught with this focus, how did your teaching change when you started focusing on problem solving?
5. Tell me about a lesson recently that went particularly well with regard to problem solving.
6. Could you give me an example of a good math problem you've used in class recently? What made it a good problem? What do you look for in a good problem?
7. Is there a problem you have used in the past that you know longer use? If so, why do you no longer use it?
8. How do you want your students to view problem solving?
9. What are the main things you do on a regular basis to emphasize problem solving?
10. How long do you let students go down an unproductive solution path before stepping in to give guidance?

Second interview

1. Are problems/investigations/projects that you assign used as a means to help students learn new material, or do problems/etc. always come *after* students have learned the content required to solve/complete them?

2. I asked last time if there were *problems* that you've used in the past that you no longer use. Are there any *teaching strategies* or *techniques* you used in the past that you no longer use?
3. In regard to teaching through problem solving, are there ideas you've considered incorporating in class that you'd like to try in the future?
4. Do you ever have students share their solutions with others in the class? If so, in small groups? Whole class discussion?
5. Do you ever ask students to solve a problem in more than one way?
6. Could you talk about the importance of collaborating with colleagues? Do you ever discuss teaching strategies or philosophy with colleagues?

Third (email) interview

1. For students who come into your class having experienced only traditional (e.g., lecture-based) mathematics teaching and learning, are there things you do at the beginning of your course to help them acclimate to a problem-based approach?
2. Do you think your teaching changes as the course progresses, either in terms of helping students acclimate (as in question #1) or otherwise? If so, how? Does the nature of the activities/problems/investigations change over the course of the trimester/year?