A Modular Compactification of the Space of Elliptic K3 Surfaces

by

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(Under the Direction of Valery Alexeev)

Abstract

We describe work towards a compact moduli space of elliptic K3 surfaces with marked divisor given by a small multiple of the sum of rational curves in an ample class of sufficiently large degree.

Index words: K3 Moduli, Elliptic surfaces.

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Chapter 1

Introduction

The problem of finding a natural compact moduli space of (some class of) algebraic K3 surfaces has a long history. Immediately from the global Torelli theorem of Piatetski-Shapiro and Shafarevich, which expresses the moduli of K3's with some polarization as a quotient of a type IV symmetric domain, we have a natural compactification, the Baily-Borel compactification. Unfortunately the resulting space is quite singular, and so the search for compactifications with rich geometric meaning continues. Friedman and Scattone [Fri84] [FS86] discuss partial compactifications from a Hodge theoretic perspective, which agree with Mumford's toroidal construction. Scattone [Sca87] raises the question, as yet unanswered, as to whether there exist natural toroidal compactifications of a moduli of algebraic K3's.

Various authors have used GIT to address the same problem. In particular, Shah [Sha80] produced a geometric compactification of degree 2 K3's as a blowup of a quotient of the space of plane sextics. Also of relevance to the current project, Miranda [Mir89] considered a GIT quotient of the space of Weierstrass equations to obtain a compactification of a moduli space of elliptic K3's.

Recently a technology has been developed to produce in a canonical way compactifications of moduli of surfaces of general type, or pairs (X, B) of log general type, introduced by Kollár and Shepherd-Baron [KSB88], and Alexeev [Ale94]. It is tempting to use this method to study the K3 case. This was done by Laza [Laz12], who studied the case of degree 2 pairs (X, B) with B arbitrary. If one wishes to remove the element of choice for B, there are several options for canonically associating an ample divisor to a polarized K3. For example, in the degree 2 case one may take the ramification of the involution. This situation is currently being studied by Alexeev and A. Thompson. Alternatively, it was suggested to study $B = \sum \epsilon B_i$, where B_i are the rational curves in the polarization class. We provisionally call the corresponding moduli spaces $\overline{F_{2d}^{RC}}$ (for *rational curves*), where $2d$ is the degree of the polarization.

In this thesis I describe work towards continuing this program in the case of (jacobian) elliptic K3 surfaces using the latter approach. We call the resulting space $\overline{\mathcal{F}_{\text{ell}}}$. Beyond it's intrinsic interest this lends insight into the (much harder) problem of describing $\overline{F_{2d}^{RC}}$, since $\overline{\mathcal{F}_{\text{ell}}}$ is exactly the intersection of $\overline{F_{2d}^{RC}}$ for all d (though the modular interpretation changes slightly with small d). The main "result" of this work is the conjecture:

Conjecture 1.0.1. The normalization $\overline{\mathcal{F}_{\text{ell}}}^{\nu}$ of $\overline{\mathcal{F}_{\text{ell}}}$ is a toroidal compactification $\overline{\mathcal{F}}_{\text{ell}}^{\mathcal{J}}$ of the period domain $\mathcal{F}_{\text{ell}}$ corresponding to the fan $\mathcal{J}$ described in chapter (12).

The main actual results are paraphrased below:

Theorem 1.0.2. *Let (X, B) be a stable pair parameterized by $\overline{\mathcal{F}_{\text{ell}}}$. Then X is a Weierstrass fibration over a chain of rational curves, and $B = \epsilon \sum_{i=1}^{24} f_i + \delta s$, where f_i are some fibers and s is the section.*

The specific pairs that occur are of course enumerated when the theorem is fully stated. This allows a description of each boundary stratum. In particular:

Theorem 1.0.3. *The boundary strata of* $\overline{\mathcal{F}_{\mathrm{ell}}}^{\nu} \setminus \mathcal{F}_{\mathrm{ell}}$ *are isomorphic to the boundary strata of* $\overline{\mathcal{F}}_{\mathrm{ell}}^{\mathcal{J}}$.

We now briefly discuss the layout of this document. Part I consists of background material, with the aim being to introduce just enough concepts and notation to understand the second part. While the reader may safely skip to the new results, certain crucial calculations are performed as examples when the appropriate concepts are introduced. The reader is referred back to these when they are used. Also, the level of detail in the exposition varies not only depending on the results needed but also on the technicality of the proofs. While toric geometry (3), elliptic surfaces (6), and lattices (5) are elementary, the Minimal Model Program (4), Mixed Hodge Theory (7), and toroidal compactification (8) are intimidating machines, and it would lead too far afield to even hint at the proofs of many key results. In any case the proofs provided more to provide the flavor of the arguments than to completely develop the results. The most assiduous reader is referred to the references.

The second part details new results. First the possible stable pairs are enumerated (9). Next the type III boundary components are parameterized (10). Explicit semistable models are constructed for degenerations and it is shown that all the possible limits described earlier in fact occur (11). Finally the fan $\mathcal{J}$ is described, and the isomorphism of boundary strata is shown (12).

Finally, a word on the technical approach used. Many statements here have a distinctly "old fashioned" feel, and it is likely that a modern approach using stacks and log geometry would simplify much of this work. The reason for this is two fold. First, the old fashioned approach was easier for me to learn. Since a dissertation is partly a historical artifact of the student's learning process this shapes the exposition enormously. Second, some of the main inspirations for this study predate the development of these tools, and so they need to be integrated along with the primary sources. I hope that readers will be able to shape this discussion to their personal tastes.

Part I

Background

Chapter 2

K3 Surfaces

2.1 Introduction and Examples

K3 surfaces were first introduced by Weil in the late 1950's. He chose the name "K3" in honor of Kummer, Kodaira, and Kähler, as well as the Himalayan peak. The definition is:

Definition 2.1.1. A *K3 surface* over k is a complete nonsingular algebraic surface X/k with

1 $K_X = 0$

2 $H^1(X, \mathcal{O}_X) = 0$

A *polarized* K3 surface is a pair (X, L) where X is a K3 surface and $L \in \mathrm{Pic}(X)$ is an ample line bundle. The *degree* of the surface is L^2.

In this work, we will assume $k = \mathbb{C}$ unless otherwise specified. In the category of complex manifolds notice that the above definition still holds. We recall from the classification of surfaces that the only complete algebraic surfaces with trivial canonical class are abelian surfaces and K3 surfaces, so we can replace "2" in the definition with any other property that rules out abelian surfaces, for example:

2' $\pi_1(X)$ trivial.

Notice this also rules out Kodaira surfaces (which have odd first Betti number), so in fact serves as an alternate definition of complex analytic K3 surfaces.

Because of the fact that they naturally form an analytic family, for algebraic purposes it is often easier to deal with polarized K3's.

Example 2.1.2. A double cover $\pi : X \to \mathbb{P}^2$ branched over a sextic with $L = \pi^*\mathcal{O}(1)$ describes a K3 surface of degree 2.

K3 surfaces of small degree generically occur as complete intersections. In particular, smooth quartics in $\mathbb{P}^3$, and smooth $2,3$ or $2,2,2$ complete intersections in $\mathbb{P}^4$ and $\mathbb{P}^5$, respectively are K3's.

The smooth minimal model of the quotient of an abelian surface by the involution $p \mapsto -p$ is a K3. Such surfaces are known as *Kummer surfaces*, and in some sense are dense among all K3 surfaces.

Finally, a smooth elliptic surface with affine equation

$$y^2 = x^3 + Ax + B,\ \ A \in \Gamma(\mathcal{O}(8)), B \in \Gamma(\mathcal{O}(12))$$

defines a K3 surface, an *elliptic K3*, the main subject of this work.

2.2 Basic Properties

We collect various useful facts about K3 surfaces here, mostly following Huybrechts [Huy]. Noting that $h^0(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X)$ by Serre duality and $h^1(X, \mathcal{O}_X) = 0$ by definition the Riemann-Roch theorem reduces to:

$$\chi(L) = \frac{L \cdot L}{2} + 2.$$

From this one sees that any class l with $l^2 \geq 0$ has either l or $-l$ effective. Moreover, the arithmetic genus of an element of $|l|$ is $\frac{l^2}{2} + 1$, by adjunction.

We proceed to describe the cohomology of a K3 surface.

Proposition 2.2.1. *Let X be a K3 surface over $\mathbb{C}$. Any numerically trivial class $l \in \text{Pic}(X)$ is in fact trivial. In particular $\text{Pic}^0(X) = 0$ and the Chern class map $\text{Pic}(X) \to H^2(X, \mathbb{Z})$ is injective.*

Proof. First assume for the sake of contradiction that $l \neq 0 \in \text{Pic}(X)$ is numerically trivial. Let L be any ample class. The fact $l \cdot L = 0$ implies $h^0(l) = 0$. Similarly $h^0(-l) = 0$ so by Serre duality $h^2(l) = 0$ and so $\chi(l) \leq 0$. But then the Riemann Roch formula shows $l^2 < 0$, contradicting the assumption of numerical triviality.

For the statement on the Chern class map, consider the long exact sequence obtained from the exponential sequence:

$$H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O})$$

Now $H^1(X, \mathcal{O}) = 0$ and $H^1(X, \mathcal{O}^*) = \text{Pic}(X)$ both by definition, so the result follows. □

Proposition 2.2.2. *Let X be a K3 surface over $\mathbb{C}$. $H^2(X, \mathbb{Z})$ equipped with the cup product is an even unimodular lattice of signature $(3, 19)$.*[1]

Proof. Observe that $\text{Pic}(X)$ is torsion free, since X has no nontrivial ètale cover. Hence the exponential exact sequence used in the proof of the previous proposition gives $H^2(X, \mathbb{Z})$ torsion free as well. Noether's formula

$$\chi(\mathcal{O}_X) = \frac{c_1^2 + c_2}{12}$$

[1] See 5 for the definition of these terms.

gives $\chi(X) = 24$. The Hodge diamond (see chapter 7 for details and references) is then:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 19 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

so the claim on the rank and signature of $H^2(X,\mathbb{Z})$ follows. Finally, $H^2(X,\mathbb{Z})$ is unimodular by Poincar duality and even by the Riemann Roch formula. □

One denotes by L_{K3} the (unique up to isomorphism) abstract lattice isomorphic to $H^2(X,\mathbb{Z})$.

Chapter 3

Toric Geometry

Here we review the facts of toric geometry that will come in handy later. A standard reference is the book by Cox, Little, and Schenck [CLS11].

Definition 3.0.3. Let $\mathbb{G}_m(k)$ be the multiplicative group variety $\operatorname{Spec} k[x, x^{-1}]$, and write $T^n(k)$ for the n dimensional *torus* $\mathbb{G}_m^n(k) = \operatorname{Spec} k[x_1^{\pm 1} \dots x_n^{\pm 1}]$ (again considered as a group variety). In cases where n is understood or irrelevant we will simply write T. In our situation we will usually work over $\mathbb{C}$, and often choose to write "$\mathbb{C}^*$" and "$\mathbb{C}^{*n}$ for the group varieties $\mathbb{G}_m(\mathbb{C})$ and $T^n(\mathbb{C})$, respectively.

A *toric variety* is a variety with an action of T with a dense orbit and connected stabilizers. One calls the dense orbit the *interior* and its complement the (toric) *boundary.* We assume that toric varieties are normal unless otherwise stated.

One associates two lattices to T, called M and N. (In the context of toric varieties only we use the word lattice to refer to a free abelian group with no extra structure. Compare the definition in 5).

M, or the monomial lattice, represents the monomials in the ring $k[x, x^{-1}]$. Equivalently, these are the possible weights for an action of T on $\mathbb{G}_m$.

Definition 3.0.4. Let S be an additive (commutative) monoid and k a field. The *monoid algebra* $k[S]$ of S over k is the commutative k algebra generated by $\{[s], s \in S\}$ with the condition that $[s_1][s_2] = [s_1 + s_2]$.

So we can now write $T = \mathrm{Spec}(k[M])$.

N, or the lattice of one parameter subgroups[1], is the free abelian group dual to M and represents all homomorphisms $\mathbb{G}_m \to T$. (The pairing is simply by noting such a homomorphism sends monomials to monomials. Alternatively, given any linear action of T on k determined by some weight f we can restrict to an action of any one parameter subgroup $\phi : \mathbb{G}_m \to T$, with weight $\langle \phi, f \rangle$.

3.1 Affine Toric Varieties

If $X = \mathrm{Spec}\, R$ is an affine toric variety, and $x \in X$ is contained in the dense T orbit, write $T' = T/\,\mathrm{Stab}\, x$, and let the corresponding sublattice of M be M', with its dual being the quotient N'. T' has a well defined action on the dense orbit Tx, which extends to X. Hence X is also a toric variety for T'. Now T' has a (noncanonical) dominant embedding into X, so there is an injection $R \to k[M'] \to k[M]$. We can choose generators of $r_i \in R$ that diagonalize the action of T', where T' acts with weights $m_i \in M$ on r_i. Thus R is a direct sum of weight spaces. Indeed, let the cone $\sigma^\vee$ be the cone in $M \otimes \mathbb{R}$ generated by r_i. Then we claim $R = k[\sigma^\vee \cap M]$. Indeed clearly $R = k[\langle m_i \rangle]$, and if $m \in \sigma^\vee \cap M$ then $nm = \sum a_i m_i$, with $n, a_i \in \mathbb{Z}$, so x^m satisfies the monic equation $Y^n = \prod x^{a_i m_i}$ in Y, so by normality $x^m \in R$. Thus:

Proposition 3.1.1. *Affine toric varieties X are in bijection with:*

- *Rational polyhedral cones $\sigma^\vee$ in $M \otimes \mathbb{R}$, where $M \longleftrightarrow \mathrm{Spec}\, k[\sigma^\vee \cap M]$*

[1]The words "M̲onomials" and "oN̲e parameter subgroups" are M̲N̲emonics

- *Rational polyhedral cones σ in $N \otimes \mathbb{R}$, by first taking the dual cone $\sigma^\vee$ We interpret $\sigma \cap M$ as the set of one parameter subgroups $\mathbb{C}^* \to X$ that have a limit at 0, i.e. extend to $\mathbb{C} \to X$.*

Where "rational polyhedral cone *in $L \otimes \mathbb{R}$" simply means a cone generated by a finite number of elements of $L \subset L \otimes \mathbb{R}$.*

In either case we write $TV(C)$ for the toric variety corresponding to the cone C, where will be clear from context which construction is being used.

We will frequently abuse notation by writing $\sigma, \sigma^\vee$ for the monoids $\sigma \cap N, \sigma^\vee \cap M$.

Note that when dealing with toric varieties defined by cones we consider the natural viewpoint to be in the lattice N.

The faces of the cone $\sigma \subset N \otimes Q$ are of special importance. Let $\sigma' \subset \sigma$ be any face. Then the associated toric varieties both contain the same torus and $TV(\sigma) \subset TV(\sigma')$. There is a bijection between faces σ' of σ and T orbits ("strata") in X. This bijection works by associating σ' to the unique closed torus orbit of $TV(\sigma')$, which is isomorphic to the torus $TV(\sigma'^{\perp} \subset M)$. Note this bijection is dimension reversing, i.e. if σ' has codimension n, then the corresponding toric stratum has dimension n.

3.2 Non-Affine and Projective Toric Varieties.

From the proceeding discussion one notes that two rational polyhedral cones $\sigma_1, \sigma_2 \subset N \otimes \mathbb{R}$ that intersect along a face can glue to a toric variety. We thus define:

Definition 3.2.1. Let L be a lattice. A collection Σ of cones in $L \otimes \mathbb{R}$ is called a *fan* if:

- If $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2 \in \Sigma$.

- If σ_1 is a face of $\sigma_2 \in \Sigma$, then $\sigma_1 \in \Sigma$.

Given an arbitrary toric variety $T \hookrightarrow X$ and $x \in X$ the subring R_x of $k[M]$ consisting of functions on the interior that extend to x is torus invariant and since X is normal defined by a cone σ_x. One thinks of σ_x as the closure of the cone generated by all one parameter subgroups limiting to x. It turns out σ_x is rational polyhedral and so defines an affine toric subvariety in X. Hence (modulo details):

Proposition 3.2.2. *A toric variety X is determined by a fan Σ in $N \otimes \mathbb{R}$. Conversely, any such fan determines a toric variety, which we write $TV(\Sigma)$.*

$TV(\Sigma)$ is proper if and only if Σ has support equal to $N \otimes \mathbb{R}$.

In the case of projective toric varieties there is another picture that is more intuitive. Let (X, L) be a pair of a toric variety and a very ample line bundle (a *polarized toric variety*). Then we note that $\mathrm{Pic}(X)$ is discrete (the action of a generic one parameter subgroup of T pushes an arbitrary divisor on X to one supported on the boundary), so L is T invariant. We choose a linearization of L. That is, we construct an action of T on the ring

$$R_X = \bigoplus H^0(X, nL)$$

(graded by n) compatible with the multiplication. Just as in the affine case each graded piece decomposes as a direct sum of weight spaces. In particular $\mathrm{Gr}_1 R_X = \langle x^{m_1}, x^{m_2} \ldots \rangle$ where m_i are the lattice points in a polytope $P_{X,L}$ in M. Notice that the choice of linearization only affects the result by translation of P_X. Conversely given any polytope $P \subset M$ one considers the cone C_P generated by $(P, 1)$ in $M \oplus \mathbb{Z}$. Then the last coordinate gives a natural grading on $k[C_P]$ and $\mathrm{Proj}\, k[C_P]$ defines a polarized toric variety that we write $TV(P)$. Note that if m_i are the corners of the polytope P_X we can cover X by the affine charts $x^{m_i} \neq 0$. Explicitly dehomogenizing the ring R_X shows that each chart corresponds to the normal cone (in N) of P_X at the point m_i. Summarizing:

Proposition 3.2.3. *There is a bijection between polarized toric varieties (X, L) and lattice polytopes.*

The fan Σ_X is the fan of inward facing normal vectors to $P_{(X,L)}$.

In particular n dimensional faces of the polytope are in bijection with n dimensional torus orbits of X.

Observe the figure 3.1 for a diagram showing the polytopes and fans corresponding to $(\mathbb{P}^2, \mathcal{O}(3))$ and $(\mathbb{F}_1, 2s + 4f)$.

Example 3.2.4. We give an example of a toric variety not of finite type with a group action, and show that there is a well defined quotient in a neighborhood of the boundary. (This is one of the standard constructions of the *Tate curve*).

Let $M = \mathbb{Z}^2, T = \operatorname{Spec} k[M]$. Let $\mathbb{Z}^+$ act on M by the matrix $\phi = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Define the cone $\sigma_0 = \langle (0,1), (1,1) \rangle$, and write $\sigma_i = \phi^i \cdot \sigma_0$. Then the collection of all σ_i form the maximal dimensional cones of a fan Σ. The corresponding variety $T = TV(\Sigma)$ is glued from an infinite collection of planes. T can be thought of as the result of blowing up $\mathbb{A}^1 \times \mathbb{P}^1$ at a toric fixed point of $0 \times \mathbb{P}^1$ and then blowing up infinitely often at toric fixed points on exceptional divisors, with the resulting collection of (strict transforms of) exceptional divisors being an infinite chain of -2 curves, $\cup D_i$. Write $T_0 = \cup D_i$. Note that the projection $\pi_1 : \mathbb{A}^1 \times \mathbb{P}^1 \to \mathbb{A}^1$ induces a map $\pi : TV(\Sigma) \to \mathbb{A}^1$. $T_0 = \pi^{-1} 0$.

Clearly the action of $\mathbb{Z}$ on Σ induces an action on T. We cannot take a meaningful quotient of the whole variety, but we can do so in a neighborhood of $\pi^{-1}(0)$. Indeed, consider an n-th order neighborhood of 0 in $\mathbb{A}^1$, $S_n = \operatorname{Spec} k[x]/x^n$. Now $T \times_{\mathbb{A}^1} S$ is supported on $\pi^{-1}(0)$, so ϕ^2 acts freely and a quotient exists. Dividing by the remaining order 2 group we construct $E_n = T \times_{\mathbb{A}^1} S_n/\phi$. It is apparent that if $m > n$ there is an embedding $E_n \to E_m$, so we can produce a formal scheme $\mathcal{E}$ as the limit.

Now let $\mathcal{L}$ be the line bundle $\mathcal{O}_T(\sum \frac{i^2-i}{2} D_i)$. With some work we can show that Ł$|_{E_n}$ is ample. By "Grothendieck's Algebrization Theorem", then, $\mathcal{E}$ is a formal neighborhood of the fiber over the closed point in some actual scheme E over $\operatorname{Spec} k[[x]]$.

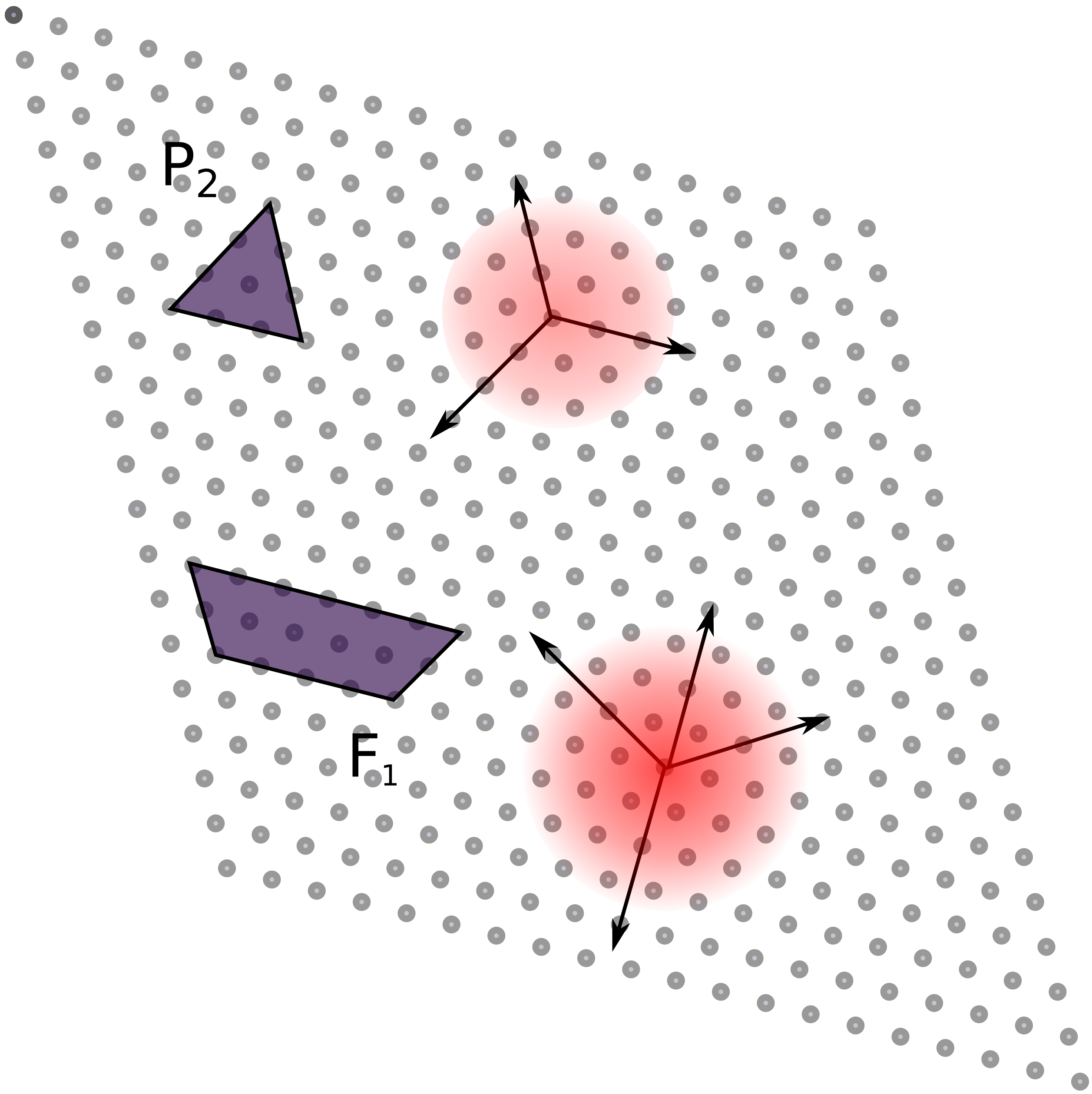

Figure 3.1: Diagram of polytopes and normal fans for some surfaces.

Chapter 4

The Minimal Model Program and Moduli of Stable Pairs

In this section we recall definitions and results from the minimal model program (MMP) and how they apply to the compactification of moduli spaces (the KSBA program). Fundamental references are [KM98] for the MMP, and [KSB88], [Ale94], [Ale96a], [Ale96b] for the KSBA machinery. J. Kollár is producing a book on the subject, [Kol]. See also the introduction to the expository notes [Ale15], which this review loosely follows, and contain many examples useful to the current project.

4.1 Prehistory: Moduli of Pointed Curves

The KSBA program is motivated by, and a generalization of, well known compact moduli spaces for curves with marked points. In particular, recall:

Definition 4.1.1. Fix real numbers $0 < b_i \leq 1, i = 1 \ldots n$. A *weighted stable curve* of genus g with weights b_i is a (reduced and connected but not necessarily irreducible) curve C with arithmetic genus g and n marked points p_i satisfying the two conditions:

Not too bad singularities C is at worst nodal and the points p_i are contained in the smooth locus. p_i may coincide, but in no case should the sum of the weights corresponding to coincident points exceed 1.

Numerical Condition The divisor $K_C + \sum b_i p_i$ is ample on C, where K_C is a generic divisor of the dualizing sheaf. This condition is trivial on components of C with genus > 1, says that any genus one component contains either a marked point or a node, and says that on any rational component the sum of the weights of the points on that component plus the degree of the double locus on that component must be > 2.

We then have the following theorem of Hassett[Has03] generalizing the classical result of Deligne and Mumford[DM69] for weights $b_i = 1$:

Theorem 4.1.2 (Hassett, after Deligne-Mumford). *For any choice of weights b_i and genus g the moduli of weighted stable curves is represented by a smooth proper Deligne-Mumford stack $\overline{\mathcal{M}_{g,b_i}}$. The coarse moduli space is a projective variety.*

Recall in particular how one takes limits: if one has a family of marked (say, for simplicity, smooth) curves $\mathcal{C} \to \Delta^\circ$, where $\Delta^\circ = \Delta \setminus \{0\}$ is a small punctured curve, then one first completes to an arbitrary family $\mathcal{C}' \to \Delta$. The semi-stable reduction theorem asserts one can perform a sequence of blowups and base changes to obtain (keeping the notation C') a family $\mathcal{C}' \to \Delta$ with smooth total space and reduced normal crossing central fiber C_0. By further blowing up and base changing one can assume the strict transform of the markings of $\mathcal{C}$ meet C_0 in distinct smooth points. One now proceeds to contract components of C_0 where the "Numerical Condition" above is not met. An assertion of the theorem is that this is possible and that the resulting curve C_0 is independent of the choices made.

4.2 Singularities in the Minimal Model Program

We wish to generalize to the case of a pair consisting of a surface with a marked divisor. Our first step should be to define an appropriate class of singularities, replacing the first condition in the definition of a weighted stable curve. We will in fact need to discuss both surface and 3-fold singularities (i.e. the singularities in 1 parameter families).

Let (X, B) be a pair of a normal variety and a $\mathbb{R}$ divisor (i.e. $\mathbb{R}$ linear combination of effective Weil divisors). Recall Hironaka's resolution of singularities (or a slight generalization): we can find a birational morphism $f : Y \to X$ such that Y is nonsingular and $\cup f_*^{-1}B_i \cup E_j$ is normal crossing, where $f_*^{-1}B_i$ are the strict transforms of the components of B and E_j are the exceptional divisors of f. We define:

Definition 4.2.1. Assume $K_X + B$ is $\mathbb{R}$-Cartier (i.e. can be written as a linear combination of Cartier divisors, so that $f^*(K_X + B)$ makes sense). Write:

$$K_Y = f^*(K_X + B) + \sum_{D_i} a_i D_i$$

where D_i are distinct irreducible divisors. The numbers a_i are called the *discrepencies.*

If $a_i \geq 0$ one says the pair (X, B) is *canonical.*

If $a_i \geq -1$ one says the pair (X, B) is *log canonical.*

If additionally $a_i > -1$ for D_i not in the strict transform of B one says (X, B) is *log terminal.*

We demonstrate this definition for some pairs on a smooth surface.

Example 4.2.2. Let $X = \mathbb{A}^2$ and $B = \frac{1}{2}C$, where C is the curve $x^2 = y^{n+1}$. One calls the singularity type of C a *type A_n curve singularity.* For simplicity assume n is odd, so $n+1 = 2m$. Blowing up the singular point produces a surface X_1 with an exceptional divisor E_1 and a type A_{n-2} singularity on $f_*^{-1}C \cap E_1$ (for this example, we abuse notation by always letting

f denote the current map to X). Blowing up this singularity to produce X_2 introduces a new exceptional divisor E_2, and now $f_*^{-1}C$ has a type A_{n-4} singularity. Inductively then we have X_m being a log resolution, and $K_{X_m} = \sum_i iE_i = f^*(K_X + B) - \frac{1}{2}f_*^{-1}C$, so the discrepencies along the exceptional curves are 0, and the pair is log terminal (indeed, canonical).

Similarly, let $X = \mathbb{A}^2$ and $B = \frac{1}{2}C$, with C the curve $yx^2 = y^{n-1}$. One calls this curve singularity type D_n. For simplicity assume n even, say $n = 2m + 2$. Then a single blowup produces X_1 with a type A_{n-5} singularity on the exceptional divisor E_1. The remaining blowups to produce a resolution X_m happen as in the previous example. One checks that $K_{X_m} = \sum_i iE_i = f^*(K_X + B) - \frac{1}{2}\sum iE_i - \frac{1}{2}f_*^{-1}C$, so this singularity is also log terminal.

Continuing on the theme with $X = \mathbb{A}^2, B = \frac{1}{2}C$ let C have a triple tacnode, say $x^3 = xy^4$. A single blow up at the singularity produces X_1 with a type D_4 curve singularity on the exceptional divisor, which is resolved by a second blowup to X_2. Now, though $K_{X_2} = E_1 + 2E_2 = f^*(K_X + B) - \frac{1}{2}E_1 - E_2 - \frac{1}{2}f_*^{-1}C$. This is still log canonical, but strictly so, in the sense that adding any additional effective $\mathbb{R}$ divisor passing through the singularity of C will cause the pair to no longer be log canonical.

Finally, let $X = \mathbb{A}_2$ and $B = \frac{1}{2}C + F$, where $C = V(x^2 = y^{n+1})$ and $F = V(y)$. Proceeding as in the previous cases we find that $K_{X_m} = f^*(K_X + B) - \sum E_i - \frac{1}{2}f_*^{-1}C$, so this pair is log canonical. Note the similar computation with C having a type D singularity fails.

The results in the example are actually statements about surface singularities in disguise, due to the following fact:

Proposition 4.2.3. *Let $Y \to X$ be a double cover of X branched over a divisor B. Then Y is log canonical iff $(X, \frac{1}{2}B)$ is.*

One can easily generalize our computation to include the exceptional curve singularities

$$E_6 : x^3 = y^4, E_7 : x^3 = xy^3, E_8 : x^3 = y^5.$$

The corresponding surface singularities one calls *ADE singularities*[1] and denotes by the same symbols A_n, D_n, E_n. We get:

Proposition 4.2.4. *ADE surface singularities are log terminal (in fact they are canonical). A type A_n singularity with a curve passing through it in a generic direction is strictly log canonical, but not for types D_n, E_n. The simple elliptic singularities (double covers branched over a triple tacnode) are strictly log canonical by themselves.*

We now introduce a companion definition to that of log canonical singularities for non-normal varieties. This concept is due to Kollàr and Shepherd-Barron, but the formulation here is from Alexeev.

Definition 4.2.5. A pair (X, B) is *semi log canonical* or *slc* if:

- X is reduced and has at worst nodal singularities in codimension 1 and B has no components in common with the double locus of X.
- X satisfies Serre's condition S_2.
- The normalization $\nu : X^\nu \to X$ has $(X^\nu, D^\nu + \nu^{-1}B)$ log canonical.

Remark 4.2.6. Recall Serre's theorem that normality is equivalent to regularity in codimension 1 and S_2, where S_2 is an algebraic analogue of Hartog's theorem, stating that any regular function defined away from a set of codimension ≥ 2 extends uniquely. In our situation it will always be satisfied, so we view it as a technicality.

Finally, we need a hard result due to Kawakita [Kaw07], confirming a conjecture of Shokurov and Kollár [KA92] relating singularities in a variety to those in a subvariety. We state it for surfaces contained in threefolds, since this is the case we need, and for a long time was the highest dimension known.

[1] Variously *ADE* singularities, *rational double points*, *Kleinian singularities*, etc.. They are exactly the canonical surface singularities, though we don't need this fact.

Theorem 4.2.7 (Inversion of Adjunction). *Let (X, B) be a 3-fold, $S \subset X$ a reduced divisor sharing no components with the support of B. Then $(X, B + S)$ is log canonical on some neighborhood of S if and only if $(S, B|_S)$ is semi log canonical.*

4.3 The Minimal Model Program and Application to Moduli

Here we recall the results we need from the minimal model program and apply them to the moduli of surfaces. Recall that for surfaces of general type the canonical model can be formed by taking any smooth projective birational model X and writing:

$$X_{can} = \text{Proj}(\oplus H^0(O_X(nK_X)))$$

or in the relative setting $F : X \to S$:

$$X_{can} = \text{Proj}_S(\oplus(f_*\mathcal{O}_X(nK_X)))$$

and that X_{can} may be obtained by contracting -1 and -2 curves in fibers of f.

(Here, and for the remainder, we formally write $H^0(\sum d_i D_i) = H^0(\sum \lfloor d_i \rfloor D_i)$.)

One aim of the minimal model program is to obtain a similar procedure for varieties of any dimension. Some results are conjectural for higher dimension, but we only need the 3 dimensional version, which is known.

Recall:

Definition 4.3.1. An $\mathbb{R}$-Cartier divisor $D \subset X$ is *nef* if $D \cdot C \geq 0$ for all curves $C \subset X$.

D is *big* if $\limsup h^0(\mathcal{O}_X(nD))/n^{\dim X} > 0$, i.e. the linear system $|ND|$ gives a map to a variety of the same dimension as X for some (but not necessarily all) large enough N.

The main result of the Minimal Model Program is:

Theorem 4.3.2 (Minimal Model Program)**.** *Let* (X, B) *be a smooth projective variety,* $\dim X = 3$*, and* B *a normal crossing* $\mathbb{R}$ *divisor. Assume* $K_X + B$ *is* $\mathbb{R}$*-Cartier and big. Then*

- *The canonical model* $\mathrm{Proj}(\oplus \mathcal{O}_X(n(K_X + B)))$ *exists.*
- X_{can} *is independent of the model* X *chosen.*
- X_{can} *has at worst log canonical singularities.*
- X_{can} *can be produced algorithmically by a sequence of* "divisorial contractions" *(i.e. contracting divisors onto subvarieties of higher codimension) and flips.*

Similarly, given a map $f : X \to S$ *and assuming* $K_X + B$ *is* $\mathbb{R}$*-Cartier and* f *big (i.e. big when restricted to a generic fiber) the relative canonical model*

$$\mathrm{Proj}_S(\oplus (f_* \mathcal{O}_X(nK_X)))$$

exits, is independent of the choice of birational model, has log canonical singularities, and can be arrived at constructively.

The only part of this statement that is mysterious is the notion of a flip. For brevity we note that for our examples the requisite flips can be effected by simple *flops* on the underlying 3 fold, where the flops of interest can be of two types:

Atiyah Flop Let X_- be a threefold and $C \simeq \mathbb{P}^1 \subset X$ have normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Blowing up C results in an exceptional divisor isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with one of the ruling being the fibers of the blowup. Blow down the "other way" to produce a new surface X_+. This is the Atiyah flop.

Pagoda Flop This is a generalization of the above. Let $C \simeq \mathbb{P}^1 \subset X_-$ have normal bundle $\mathcal{O}(-2) \oplus \mathcal{O}(0)$. Blowing up along C produces an exceptional divisor isomorphic to the Hirzebruch surface $\mathbb{F}_2$, the exceptional section of which is a new curve with normal bundle either $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ or $\mathcal{O}(-2) \oplus \mathcal{O}(0)$. In the first case, perform an Atiyah flop and blow down the (transforms of) the exceptional divisors of the previous blowup. In the second we blow up more until eventually we arrive at a curve with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, at which point we flop and blow down all the previous exceptional divisors.

To apply MMP to moduli, we first define the objects we wish to parameterize:

Definition 4.3.3. A *stable pair* (X, B) is a pair of a surface and an $\mathbb{R}$ divisor B satisfying the conditions:

Not too bad singularities (X, B) has semi log canonical singularities.

Numerical Condition The divisor $K_X + B$ is ample.

The first use of the MMP in moduli is showing how to produce limits of one parameter families. Indeed, let $(\mathcal{X}, \mathcal{B}) \to \Delta^\circ$ be such a family, with $(\mathcal{X}, \mathcal{B})$ log canonical and $K_{X_t} + B_t$ $\mathbb{R}$-Cartier and ample on each fiber (X_t, B_t). One finds a log resolution of $(\mathcal{X}, \mathcal{B})$ and apply the semistable reduction theorem (possibly base changing) to produce a family $(\overline{\mathcal{X}}, \overline{\mathcal{B}})$ with central fiber (X_0, B_0), where X_0 is reduced and normal crossing. Applying the relative MMP produces a unique family over Δ, independent up to base change of the choices made in the construction.

This shows in some sense the properness of the moduli functor. Showing representability is a lot more delicate, so we simply assert it, in a form specialized to K3 surfaces:

Theorem 4.3.4. *(Verbatim from Alexeev [Ale15, Corollary 1.5.5]) For any $d \in 2\mathbb{N}$ there exists a small irrational ϵ such that the moduli space P_d of stable K3 surface pairs $(X, \epsilon H)$*

such that $H^2 = d$ is an open subset of a proper coarse moduli space $\overline{P}_d$ of stable slc pairs $(X, \epsilon H)$. Further:

- *There exists $N \in \mathbb{N}$ such that for all stable pairs parameterized by $\overline{P}_d$ one has $NK_X \sim 0$.*

- *For any family in the closure of P_d in $\overline{P}_d$, one has $K_X \sim 0$ and H is Cartier.*

The case of $d = 2$ has been examined in detail by Laza in [Laz12].

4.4 Application to the Moduli of Varieties of Non-General Type

The above approach can sometimes be specialized to produce "good" moduli spaces for varieties that are not of general type. The idea is to uniformly and uniquely associate a $\mathbb{R}$ divisor B to each variety X such that (X, B) is a stable pair. Some examples are:

del Pezzo Surfaces Letting $B = \sum B_i$, where B_i are the lines on a del Pezzo produces a space studied by Hacking, Keel, and Tevelev [HKT09].

Polarized Abelian Varieties Letting $B = \epsilon\Theta$ allows one to produce a moduli space of abelian varieties, as was shown by Alexeev [Ale02].

K3 Surfaces, degree 2 Degree 2 K3 surfaces are double covers of a rational surface. Letting $B = \epsilon R$, where R is the ramification divisor of this map produces a moduli space currently being studied by Alexeev and Alan Thompson.

K3 Surfaces, any degree For an arbitrary polarized $K3$ surface we can let $B = \sum \epsilon B_i$, where B_i are the rational curves in the polarization class. The present work aims to describe details of this space restricted to the elliptic locus.

Chapter 5

Some Lattice Theory

5.1 Definitions and general theory.

The object of study for this chapter are *lattices*. Basic references are [Ser73] for the structure theorems and [CS99] for more detailed theory of unimodular lattices and reflection groups. The primary reference for Vinberg's algorithm and related results is the Russian [Vin72]. I choose to give references to the English [Vin75].

Definition 5.1.1. A *lattice* is a free abelian group equipped with a ($\mathbb{R}$ valued) quadratic form. There is an associated symmetric bilinear form which we will denote either with angle brackets ("$\langle , \rangle$") or as multiplication if the meaning is clear.

A lattice is *integral* if the associated bilinear form is integral.

If L is a lattice, the *dual lattice*, denoted L^* is defined as:

$$L^* = \{l \in L \otimes \mathbb{R} | \langle l, m \rangle \in \mathbb{Z} \forall m \in L\}$$

(where the form on L is extended by linearity.) Note that a lattice is integral if and only if $L \subset L^*$.

An isomorphism of lattices is an isomorphism of abelian groups compatible with the bilinear form. The set of all automorphism of a lattice L is the *orthogonal group*, denoted $\mathrm{O}\, L$.

A *sublattice* is a subgroup with the bilinear form obtained by restriction. A sublattice $L' \subset L$ is *primitive* if $nl \in L' \implies l \in L'$ for all $n \in \mathbb{Z}, l \in L$.

If L and M are lattices with bilinear forms given by matrices K_L, K_M the direct sum $L \oplus M$ is the group direct sum with form given by the matrix $\left(\begin{smallmatrix} K_L & 0 \\ 0 & K_M \end{smallmatrix}\right)$.

We collect basic terminology below:

Definition 5.1.2. A lattice is *irreducible* if it cannot be expressed as a direct sum of sublattices.

The *discriminant* of a lattice is the determinant of a Gram matrix of associated form. More generally, the *discriminant group* $\mathrm{Disc}\, L$ of a lattice L is L^*/L. The form on $L \otimes \mathbb{Q}$ induces a well defined *discriminant form* on L^*/L, taking values in $\mathbb{Q}/\mathbb{Z}$.

A lattice is *nondegenerate* if the associated form is (equivalently, if the discriminant is nonzero).

The *radical* of a lattice L is the maximal subspace $L' \subset L$ such that $\langle l', l \rangle = 0$ for all $l' \in L', l \in L$. (i.e. the nullspace of the Gram matrix).

A lattice L is *isotropic* if the quadratic form is 0.

A integral lattice is *unimodular* if it has discriminant 1 (so $L = L^*$).

An integral lattice is *even* if the quadratic form takes values in $2\mathbb{Z}$. Otherwise it is *odd*.

Recall that any quadratic form can be diagonalized over $\mathbb{R}$ and that the number of positive and negative terms is independent of the diagonalization chosen ("Sylvester's Law of Inertia"). Hence, we define:

Definition 5.1.3. The *signature* (r, s) of a nondegenerate lattice is the number of positive and negative, respectively, terms in a diagonalization of the quadratic form of L.

A non-degenerate lattice is *positive* (resp. *negative*) *definite* if it has signature $(r, 0)$ (resp. $(0, s)$).

A degenerate lattice L with $\langle L, L\rangle \geq 0$ (resp. ≤ 0) is *positive* (resp. *negative*) *semidefinite*

If L is neither definite nor semidefinite it is *indefinite.*

A definite lattice will also be called *elliptic.* A semidefinite lattice with rank 1 radical is called *parabolic.* A lattice with signature $(1, n)$ is called *hyperbolic.*

Example 5.1.4. The rank 2 lattice U with bilinear form given by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is non-degenerate, even, and unimodular with signature $(1, 1)$.

For any (r, s) the rank $r + s$ lattice $I_{r,s}$ with form given by the block matrix $\left(\begin{smallmatrix} I_r & 0 \\ 0 & -I_s \end{smallmatrix}\right)$ is odd and unimodular with signature (r, s).

The subset of $\mathbb{R}^8$ (with bilinear form given by $-I_8$, i.e. the negative of the standard one) of vectors either in $\mathbb{Z}^8$ or in $(\mathbb{Z} + \frac{1}{2})^8$ with even coordinate sum is a negative definite even unimodular lattice which one calls E_8 [1].

The subset of $\mathbb{R}^{16}$ satisfying the similar conditions is an even unimodular lattice called D_{16}^+.

Example 5.1.5. Complete surfaces with their intersection form are an important source of lattices. For a rational surface X for example, $H^2(X) = \operatorname{Pic} X \simeq I_{1,\rho(X)-1}$. If instead X is a K3 surface $H^2(X) \simeq II_{3,19}$. For this reason we will use the notation $L_{K3} = II_{3,19}$.

In general, the classification of lattices, even unimodular ones, is a hard problem. However we have the following strong result for indefinite lattices[Ser73]:

Theorem 5.1.6. *There is a unique (up to isomorphism) indefinite odd unimodular lattice* $I_{r,s}$ *of signature* (r, s) *for each* (r, s), $rs \neq 0$.

For each $(r, s), rs \neq 0$ *such that* $r - s \equiv 0 \pmod 8$ *there is a unique even unimodular lattice* $II_{r,s}$.

[1] This is the convention most useful for geometry. Other authors may use the symbol E_8 to refer to the positive definite lattice obtained by negating our quadratic form.

In the definite case, the problem is only solved in low rank. In particular, for the even unimodular case[CS99]:

Theorem 5.1.7. *E_8 is the unique even unimodular lattice of signature $(0,8)$.*

$E_8 \oplus E_8$ and D_{16}^+ are the only even unimodular lattices of signature $(0,16)$.

There are exactly 24 even unimodular lattices of signature $(0,24)$.

(The problem apparently becomes very hard after this. There are at least $8 \cdot 10^{16}$ lattices of signature $(0,32)$.)

5.2 Even Root Lattices

For this section let L denote an even lattice.

Definition 5.2.1. Let L be an arbitrary even lattice. A vector $v \in M$ is a *root* if $v^2 = -2$. The set of all roots is denoted Φ_L.

The *root sublattice* $R(L)$ is the sublattice of L spanned by its roots.

If L is equal to its root sublattice, one says it is a *root lattice*.

For a definite lattice there are clearly a finite number of roots. Part of the utility of this concept come from the fact that each pair $\pm v$ of roots induces a reflection $R_v : w \mapsto w + \langle w, v \rangle v$ on L. This is clearly a automorphism of order 2, so we define:

Definition 5.2.2. The *Weyl group* $W(L) \subset \mathrm{O}(L)$ of a lattice L is the subgroup of automorphisms of L generated by root reflections R_v.

The fixed locus of a reflection is a *reflection hyperplane*.

Example 5.2.3. Let L be the sublattice of $I_{0,n+1}$ of vectors with coordinate sum 0. One calls this root lattice A_n. The roots are exactly $\alpha_{i,j} = e_i - e_j$, and the corresponding reflections interchange the i and j coordinates, so $W(A_n) \simeq S_{n+1}$, acting by permuting the coordinates.

Similarly let L be the sublattice of $I_{0,n}$ with even coordinate sum. One calls this root lattice D_n. The roots have the form $\pm e_i \pm e_j$, with the corresponding reflections being either interchanging the i and j coordinates or negating them. $W(D_n)$ is an extension of S_n, in particular an index 2 normal subgroup of the wreath product $S_n \ltimes \mathbb{Z}_2^n$.

Finally consider the unimodular lattice D_{16}^+ defined previously. This is not a root lattice, as its root sublattice is just D_{16} (i.e. the vectors with integral coordinates).

5.2.1 Negative Definite Root Lattices

Let L be a (negative) definite root lattice and $f : L \to \mathbb{R}$ be a generic[2] linear form. Then $L \setminus \{0\} = L_+ \cup L_-$, where $L_+ = \{l \in L | f(l) > 0\}, L_- = \{l \in L | f(l) < 0\}$. Let $\{\alpha_i\}_i$ be the set of minimal roots in L_+, in the sense that $\alpha_i \neq u + v$ for any $u, v \in L_+$. We have:

Proposition 5.2.4. *α_i is a basis of roots of L. Every root $\alpha \in L_+$ can be is a nonnegative integral combinations of the α_i, i.e. $\alpha = \sum a_i \alpha_i,\ a_i \in \mathbb{Z}_{\geq 0}$.*

The proof is just long enough to omit, see [FH91] or your favorite representation theory text. As a corollary, note:

Lemma 5.2.5. *If α is a simple root and $\beta \in L_+$ any positive root, then $R_\alpha(\beta) \in L_-$ if and only if $\alpha = \beta$.*

Proof. Indeed, $\beta - R_\alpha(\beta) \in \langle \alpha \rangle$, so if $\gamma \neq \alpha$ appears with nonzero (positive) coefficient in the expression of β in terms of simple roots it does in $\mathbb{R}_\alpha(\beta)$ as well. □

We refer to α_i as *simple roots* (with respect to f, or the partition L_+, L_-). The utility of this concept is due to the fact that the configuration of simple roots is an invariant of the lattice.

[2] That is, an injective group homomorphism.

Proposition 5.2.6. *The cone $\sigma = \{l \in L | \alpha_i \cdot l \geq 0\}$ forms a fundamental domain for $W(L)$.*

Proof. The reflection hyperplanes perpendicular to roots of L divide $L \otimes \mathbb{R}$ into some finite number of regions. We need to show that the cone σ is one of these regions, that $W(L)$ acts transitively on them, and that $\mathrm{Stab}(\sigma) \subset W(L) = \{1\}$.

No hyperplane $\alpha^\perp$ can pass though the interior of σ. Indeed assume the contrary. Then there are $l \in \sigma$ and $\alpha \in \Phi_L$ with $\alpha \cdot l = 0$ and $\alpha_i \cdot l > 0$ for all simple roots α_i. WLOG we assume $\alpha \in L_+$ so write $\alpha = \sum a_i \alpha_i,\ a_i > 0$. But this implies $\alpha \cdot l > 0$, a contradiction.

Now reflections in simple roots map σ to any adjacent region, which can be further mapped to any region adjacent to it, and so on. Thus the transitivity claim follows from the connectedness of $L \otimes \mathbb{R}$.

Finally, if $w \in \mathrm{Stab}(\sigma)$ we need to show that $w = 1$. Assume not, and w preserves σ, and so preserves L_+, L_-. Let $w = \sigma_1\sigma_2 \ldots \sigma_n$ be a representation of w as a product of reflections through simple roots of minimal length. Then $\sigma_n = R_{\alpha_n}$ takes the simple root α_n from L_+ to L_-. Let $w' = \sigma_i \ldots \sigma_n$ be the subword of w of minimum length that takes α_n to L_+. Now by the lemma 5.2.5 $\sigma_i = R_\alpha$, where $\alpha = \sigma_{i+1} \ldots \sigma_n(\alpha_n)$. Since the conjugate of a reflection is a reflection in the obvious way we can write $w' = \sigma_{i+1} \ldots \sigma_{n-1}\sigma_n\sigma_n$, contradicting the minimality of the original expression for w. □

Corollary 5.2.7. *The Weyl group acts simply transitively on sets of simple roots.*

A fundamental domain of the action of $W(L)$ on $L \otimes \mathbb{R}$ is called a *Weyl chamber*. Note that the set of all Weyl chambers and their faces form a fan, which we call Σ_L. The specific chamber in 5.2.6 is called the *dominant chamber*.

The combinatorics of definite root systems and the associated root systems are studied in representation theory, and the interested reader is referred to, for example, [FH91].

There is also a unique maximal root $\widetilde{\alpha}$ with respect to f. Clearly if $\{\alpha_i^\vee\}_i$ is a dual basis to the simple roots, $\widetilde{\alpha} \cdot \alpha_i^\vee$ is maximal for each i, i.e. $\widetilde{\alpha}$ is the maximal linear combination of roots that is still a root.

We can encode the configuration of simple roots in a diagram, the so-called *Dynkin diagram*[3]. The Dynkin diagram is a graph with one node for each simple root and an edge connecting each pair of non-orthogonal simple roots. If $\alpha_i \alpha_j = 1$ the edge is left undecorated, otherwise it is marked with the product $\alpha_i \alpha_j$. Note that a root lattice is irreducible if and only if the corresponding Dynkin diagram is connected. The Dynkin type diagram whose nodes consist of the simple roots and the minimal root $-\widetilde{\alpha}$ is called an *affine* or *extended* Dynkin diagram. The *rank* of the Dynkin diagram corresponding to a definite root lattice is the rank of that lattice and the rank of an extended diagram is the rank of the corresponding (non-extended) diagram.

Example 5.2.8. Choose a generic linear form on $I_{0,n+1}$ such that $f(e_i) > f(e_{i+1})$. The simple roots of $A_n \subset I_{0,n+1}$ are $\alpha_i = e_i - e_{i+1}$. The Dynkin diagram is as shown 5.1. The maximal root $\tilde{\alpha} = e_1 - e_{n+1}$ is also shown.

Similarly, choose a generic linear form on $I_{0,n}$ such that $f(e_i) > f(e_{i+1}) > 0$. The simple roots of $D_n \subset I_{0,n}$ are $\alpha_i = e_i - e_{i+1}, i = 1 \ldots n-1$ and $\alpha_n = e_{n-1} + e_n$. The Dynkin diagram is as shown, along with the maximal root $\tilde{\alpha} = e_1 + e_2$.

Finally, choose a linear form on $I_{0,8}$ with $f(e_1) >> f(e_i) > f(e_{i+1}) > 0, i > 1$. The simple roots of $E_8 \subset I_{0,8}$ are

$$e_i - e_{i+1}, i = 2 \ldots 7$$

$$e_7 + e_8$$

$$\frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8),$$

[3] Variously named after Coxeter, Dynkin, and Vinberg. We use Dynkin for the crystallographic and affine cases, and Vinberg for the hyperbolic case. Coxeter is responsible for the abstract theory and so we will refer to the diagrams in general as Coxeter diagrams.

with Dynkin diagram as shown. $\tilde{\alpha} = e_1 + e_2$.

Connected subdiagrams of the E_8 diagram clearly give irreducible root lattices, the new ones of which we label E_6 and E_7 (see the definition below). The corresponding Dynkin diagrams and affine Dynkin diagrams are shown.

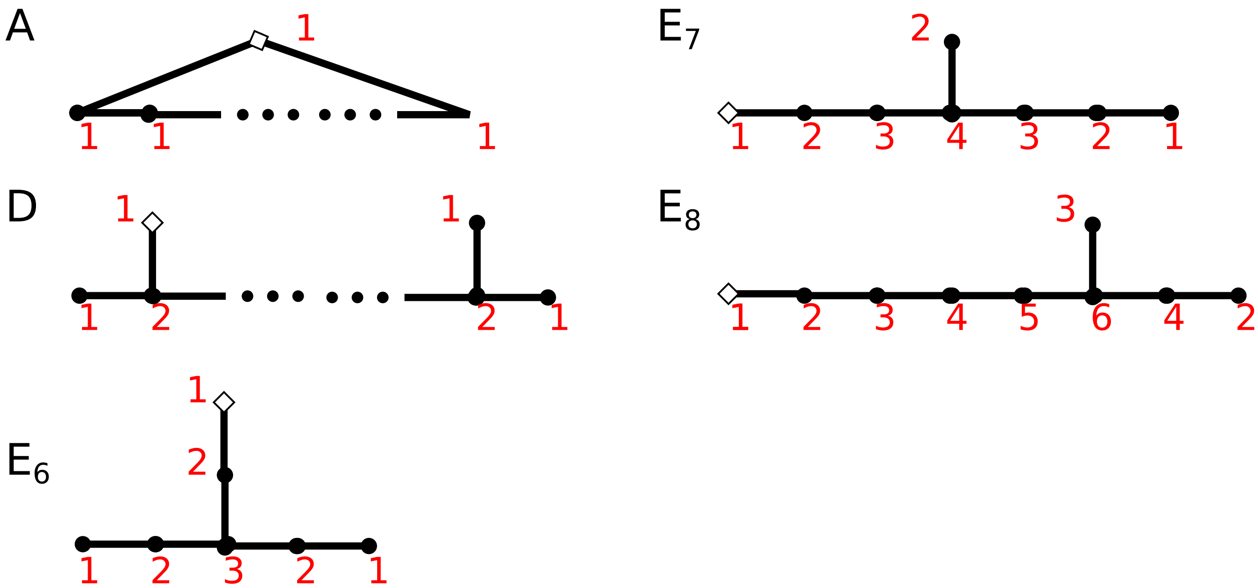

Figure 5.1: Dynkin diagrams for types A, D, and E. The lowest root $-\tilde{\alpha}$ is also shown, as the empty vertex, forming the corresponding extended Dynkin diagrams $\widetilde{A}$, $\widetilde{D}$, $\widetilde{E}$. The red numbers indicate the coefficients in the unique relation among the roots.

The irreducible negative definite even root lattices are easily classified:

Proposition 5.2.9. *The irreducible definite even root lattices are exactly type A_n, D_n, or one of the three exceptional types E_6, E_7, E_8.*

The technology of Dynkin diagrams is quite strong, and was first developed by Dynkin [Dyn52] (Russian). As a useful start one has a complete description of the orthogonal group $\mathrm{O}(L)$ and the fan Σ_L:

Proposition 5.2.10. *The automorphism group of a even definite root lattice L is a semidirect product*

$$\mathrm{O}(L) = D \ltimes W(L)$$

where D is the group of automorphisms of the Dynkin diagram of L.

The facets of a Weyl chamber (and therefore the orbits of cones in Σ_L) are in bijection with subdiagrams of the Dynkin diagram of L.

Subspaces of $L \otimes \mathbb{R}$ spanned by roots are of the form $\sigma^\perp$ for $\sigma \in \Sigma_L$.

Proof. L is spanned by its simple roots, which D permutes, giving the embedding $D \subset \mathrm{O}(L)$. Let σ be the fundamental Weyl chamber. Any $o \in \mathrm{O}(L)$ stabilizing σ must permute the facets of L, so in fact $\mathrm{Stab}(\sigma) = D$. If $\alpha \in L$ is a simple root and $d \in D$ is arbitrary then $dR_\alpha d^{-1} = R_{d\alpha}$, so $W(L) \triangleleft \mathrm{O}(L)$. Finally, since $W(L)$ acts transitively on Weyl chambers any element of $\mathrm{O}(L)$ can be written as wd for some $w \in W(L), d \in D$. This proves the claim on the structure of $\mathrm{O}(L)$.

The claim on the facets of a Weyl chamber simply says that the chamber is a simplicial cone, which it must be since it has $\dim L \otimes \mathbb{R} + 1$ facets (perpendicular to each α_i).

Finally suppose $L' \subset L$ is a primitive sublattice spanned by its root sublattice. Choose f_1 to be a generic linear form vanishing on L' and f_2 to be a generic linear form on L. For sufficiently small ϵ the simple roots corresponding to the form $f = f_1 + \epsilon f_2$ contain a simple root basis for L', which by the previous point span a cone of Σ_L. □

5.2.2 Semidefinite Root Lattices

The parabolic case is similar, and mostly follows from the previous analysis. Let L be a rank $n+1$ irreducible semidefinite even root lattice and $\langle z \rangle$ its radical. Then $\overline{L} = L/\langle z \rangle$ is an even definite root lattice, and sections $\overline{L} \to L$ correspond to points in $\overline{L}^*$ (more precisely, they're a principal $\overline{L}^*$ homogeneous space). Indeed, choose one section $s : \overline{L} \to L$. Then any other section has the form $s'(v) = s(v) + (w \cdot v)z$ for some $w \in \overline{L}^*$. Moreover, each root $\alpha \in L, \alpha = az + s(\alpha')$ determines an affine function $w \mapsto w \cdot (\alpha') + a$ on $\overline{L}^*$ and a corresponding reflection $\overline{R}_\alpha$, so the Weyl group acts on $\overline{L}^*$.

Choose a system of simple roots for $\overline{L}$ and lift them via s to obtain roots $\{\alpha_i\}_1^n \subset L$. We wish to find a fundamental domain, or *alcove*, for the action of $W(L)$ on $\overline{L}^*$, by adding additional facets to the chosen Weyl chamber.

Lemma 5.2.11. *Write $\alpha_0 = z - \widetilde{\alpha}$. Then $\{w | w \cdot \alpha_i \geq 0, i = 0 \dots n\}$ defines an alcove in $\overline{L}^*$.*

Proof. We proceed similarly to the previous case (5.2.6).

Call the purported alcove S.

The reflection hyperplane for α_0 meets the one dimensional faces $\mathbb{R}^+ \alpha_i^\vee$ of the Weyl chamber at $\frac{1}{\alpha_i^\vee \cdot \widetilde{\alpha}} \alpha_i^\vee$, the minimum positive multiple for any reflection hyperplane. Hence S is a connected component of $\overline{L}^* \otimes \mathbb{R} \setminus$ (reflection hyperplanes).

As before $W(L)$ acts transitively on these regions since $\overline{L}^* \otimes \mathbb{R}$ is connected.

Finally note that $S \cap \alpha_0^\perp$ contains no lattice points of $\overline{L}$, and $W(L)$ preserves $\overline{L}$, so any element of $w \in W(L)$ stabilizing S fixes $0 = S \cap \overline{L}$. But now by the definite case (5.2.6) w is trivial, so $W(L)$ acts simply transitively on the complement of reflection hyperplanes. □

We again refer to the walls of any given alcove as *simple roots.*

Theorem 5.2.12. *The irreducible parabolic root lattices are classified by the affine Dynkin diagrams $\widetilde{A}_n, \widetilde{D}_n \widetilde{E}_n$.*

Proof. The preceding discussion shows how to associate an affine Dynkin diagram to a parabolic lattice $\widetilde{L}$, where L is one of A_n, D_n, E_n. For the converse, given an affine Dynkin diagram simply endow the free abelian group $\widetilde{L}$ spanned by the nodes α_i with the quadratic form indicated by the diagram. That is

$$\alpha_i^2 = -2, \alpha_i \alpha_j = 1 \text{ if connected by an edge } 0 \text{ else.}$$

There is a linear combination $z = \sum g_i \alpha_i$ with $z \cdot \alpha_i = 0$ for all i (use the coefficients shown in red in 5.1). $L/\langle z \rangle$ may then be identified with the definite lattice L. □

We have a direct analog to 5.2.10, for which we omit the proof.

Proposition 5.2.13. *The automorphism group of a parabolic root lattice L is a semidirect product*

$$\mathrm{O}(L) = D \ltimes W(L)$$

where D is the group of automorphisms of the Dynkin diagram of L.

The faces of an alcove are in bijection with subdiagrams of the Dynkin diagram of L.

Definite subspaces of $L \otimes \mathbb{R}$ spanned by roots are of the form $\sigma^\perp$ for $\sigma \in \Sigma_L$.

5.2.3 Hyperbolic Root Lattices and Vinberg's Algorithm

A hyperbolic root lattice L defines two cones of vectors with positive square. (*Time-like* cones, after relativity theory. The set of isotropic vectors is a *light cone.*) Choose one such cone C_+ and define the hyperbolic space $H^n = \{v \in C_+/\mathbb{R}^\times\}$. Then we can view $W(L)$ as acting on H^n, similarly to the case of the positively curved space S^{n-1} in the definite case and the flat space $\overline{L}^*$ in the semidefinite case. Alternatively, we view $W(L)$ as acting on the cone C^+. The reflection hyperplanes divide C^+ into a fan, which we will call the *Vinberg fan.*

Analysis of the type employed previously is complicated by the fact that a fundamental domain of $W(L)$ may not be simplicial. With a bit more work similar results can be obtained, as was noticed by Vinberg [Vin75].

First, choose a "controlling vector" $v_0 \in L$ with $v_0^2 > 0$ such that v_0 lies on at least n separate reflection hyperplanes. Then $\mathrm{Stab}(v_0) \subset W(L)$ is the Weyl group of the definite lattice $v_0^\perp$, so choose a Weyl chamber defined by, say, $\alpha_i \cdot v \geq 0$ for simple roots α_i. We will proceed to add more simple roots corresponding to the facets of the unique fundamental domain containing v_0 and contained in the choice of Weyl chamber. The key is that there is a mechanized way to do this.

Proposition 5.2.14 (Vinberg's Algorithm [Vin75]). *The algorithm is iterative. At the n'th stage of the algorithm we will add all the simple roots α with $\alpha \cdot v_0 = n$. Start with the set of simple roots α with $\alpha \cdot v = 0$ (this is stage $i = 0$). For the n'th stage add all roots α such that $\alpha \cdot v = n$ and $\alpha \cdot \alpha' \geq 0$ for all simple roots α' previously constructed.*

Note that there may be an infinite number of simple roots.

Proposition 5.2.15. *The Vinberg diagrams of the lattices $II_{1,9}$ and $II_{1,17}$ are as shown in the figure. In both cases the lattice is a root lattice spanned by the simple roots. In the $II_{1,9}$ case the simple roots form a basis, whereas in the $II_{1,17}$ case there is the unique relation*

$$3\alpha_1 + 2\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + \cdots - 4\alpha_{14} - 5\alpha_{15} - 6\alpha_{16} - 4\alpha_{17} - 2\alpha_{18} - 3\alpha_{19}$$

, i.e. the linear combination corresponding to the red numbers on the diagram.

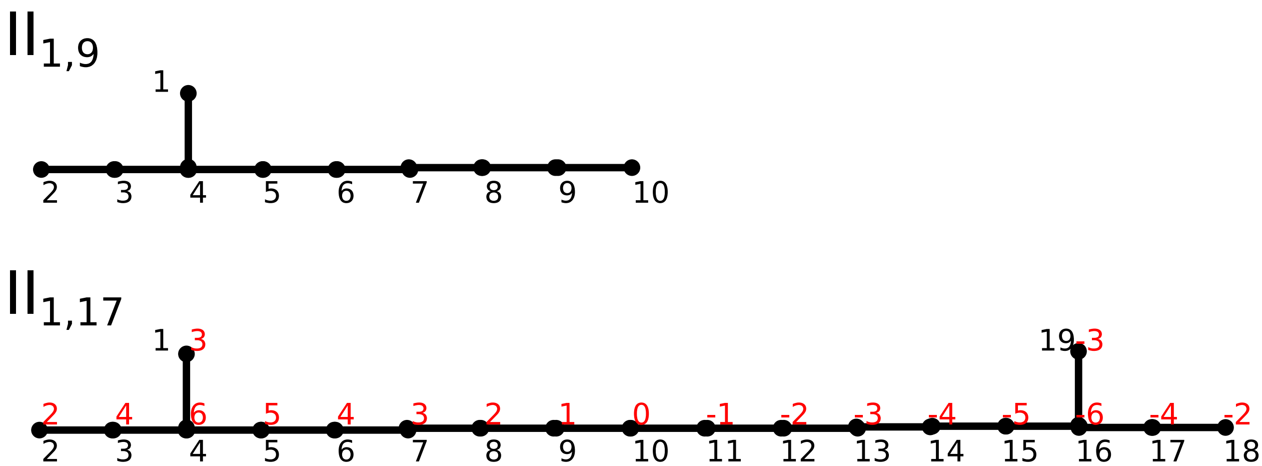

Figure 5.2: Vinberg diagrams for the even unimodular lattices $II_{1,9}$ and $II_{1,17}$. The black numbers label the simple roots α_i The red numbers correspond to the relation among the simple roots for $II_{1,17}$

Proof. $II_{1,9}$ Write $II_{1,9} = E_8 \oplus U$ and choose the controlling vector $v_0 = (0, (1, 1))$. Choosing a simple root basis α_i of E_8 we see that the 9 roots $(\alpha_i, 0), (0, (1, -1))$ define a simple basis for $v_0^\perp$, thus completing stage 0.

For stage 1, we find the unique root $(-\widetilde{\alpha}, (1,0))$. It's easily checked that there are no more simple roots. The Vinberg diagram is as shown. The simple roots form a basis by inspection.

$II_{1,17}$ Similarly, write $II_{1,17} = E_8 \oplus E_8 \oplus U$ and let $v_0 = (0, 0, (1, 1))$. Then $(\alpha_i, 0, 0)$, $(0, \alpha_i, 0)$, and $(0, 0, (1, -1))$ are a simple root basis for $v_0^\perp$. These are the roots labeled $1 \dots 8, 12 \dots 19$, and 10, respectively, in the diagram. This is stage 0.

For stage 1, there are two roots: $(-\widetilde{\alpha}, 0, (1, 0))$ and $(0, -\widetilde{\alpha}, (1, 0))$. These are the roots labeled 9 and 10. There are no more simple roots, and the Vinberg diagram is shown.

The fact that the 19 simple roots span $II_{1,17}$ is by inspection, and so there is exactly one linear relation among them. That is, there is up to scaling one combination of simple roots that pairs as 0 with each. In terms of the diagram this means that the sum of the coefficients on the node adjoining any given node must sum to twice the coefficient at that node, a condition that we immediately check.

□

This is as convenient a point as any to introduce a not entirely standard definition (and a nonstandard one).

Definition 5.2.16. Let T_n represent the A_n sublattice of $II_{1,9}$ spanned by the roots $\alpha_{10}, \alpha_9 \dots \alpha_{10-n}$.

We define $E_{9-n} = T_n^\perp$. Note this agrees with the standard definitions for E_8, E_7, E_6. The definitions $E_5 = D_5, E_4 = A_4, E_3 = A_1 \oplus A_2$ are common but not completely standard.

$E_2 = \langle \alpha_2, \alpha_3 - \alpha_1 \rangle$ and $E_1 = \langle 2\alpha_2 + \alpha_3 - \alpha_1 \rangle$ are not root lattices. They have Gram matrices $\left(\begin{smallmatrix} -2 & 1 \\ 1 & -4 \end{smallmatrix}\right)$ and (-8), respectively.

Observe ([Dyn52]) there is a unique conjugacy class of A_n sublattice in E_8 for $n \neq 7$. These are primitive sublattices. The other embedding of A_7 is non-primitive and has $A_7^\perp = \langle \alpha_2 \rangle$ we call this lattice E_1'.

The Vinberg diagram encodes information about the Vinberg fan, in a manner entirely analogous to the elliptic and parabolic cases though the analysis is more complicated. We introduce an auxiliary definition:

Definition 5.2.17. A Coxeter diagram is *elliptic* if it is a disjoint union of diagrams for irreducible definite root systems. Equivalently the associated Gram matrix is negative definite.

Similarly a Coxeter diagram is *parabolic* if it is the disjoint union of affine diagrams associated to parabolic lattices.

The *rank* of an elliptic or parabolic diagram is the sum of the ranks of its components.

The analog of 5.2.10 and 5.2.13 can now be stated. The reader is referred to Vinberg's text for proof.

Proposition 5.2.18 (Vinberg, [Vin75])**.** *The automorphism group of a hyperbolic root lattice L is a semidirect product*

$$\mathrm{O}(L) = D \ltimes W(L)$$

where D is the group of automorphisms of the Vinberg diagram of L.

Assume that D is finite.

Then $W(L)$ orbits of cones in the interior of the Vinberg fan are in a bijection with elliptic subdiagrams of the Vinberg diagram.

Definite subspaces of $L \otimes \mathbb{R}$ spanned by roots are of the form $\sigma^\perp$ for the cones σ determined by elliptic subdiagrams.

The $W(L)$ orbits of isotropic vectors in L are in bijection with parabolic subdiagrams of rank $\operatorname{rank} L - 2$.

Semidefinite subspaces of $L \otimes \mathbb{R}$ spanned by roots are of the form $\sigma^\perp$ for the cones σ determined by parbolic subdiagrams.

Inspecting the Dynkin diagram for $II_{1,17}$ yields:

Corollary 5.2.19. *The lattice $II_{1,17}$ has two $\mathrm{O}(II_{1,17})$ orbits of isotropic vectors, corresponding to the $\widetilde{E}_8 \oplus \widetilde{E}_8$ and $\widetilde{D}_{16}$ subdiagrams shown 5.3.*

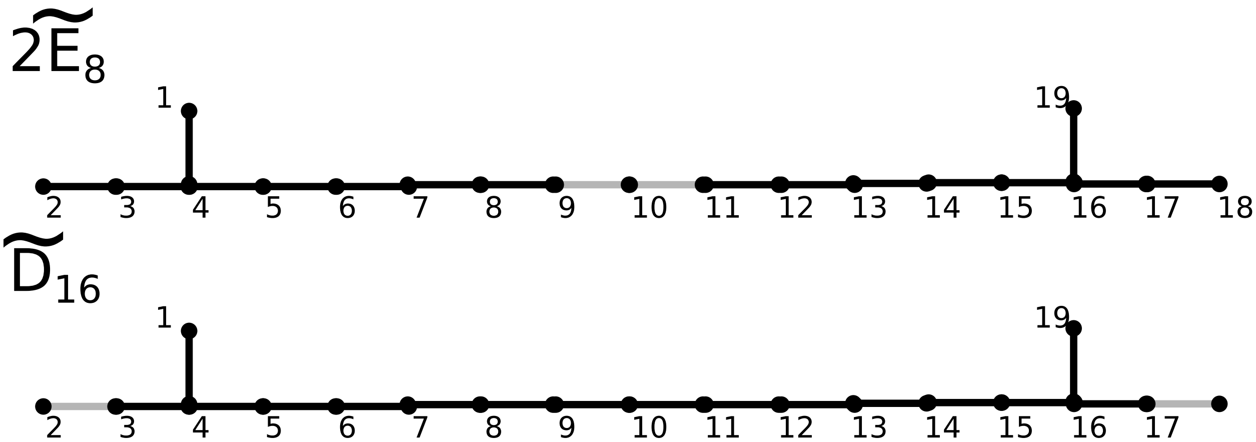

Figure 5.3: Parabolic subdiagrams of the Vinberg diagram for the lattice $II_{1,17}$.

Chapter 6

Elliptic Surfaces

Here we review basic facts about elliptic curves and surfaces that we will need later. Basic references are [Sil86] for curves and [Mir89] for surfaces.

Definition 6.0.20. An *elliptic curve* $(E, 0)$ over a field K is a smooth genus 1 curve over K along with a choice of rational point 0. We assume K has characteristic 0.

We recall some basic facts of elliptic curve theory.

Proposition 6.0.21. *Every elliptic curve* $(E, 0)$ *is isomorphic to a plane curve* $(V(y^2 = x^3 + Ax + B), \infty)$*, where* ∞ *is the unique flex point at infinity. This representation is unique up to a rescaling:*

$$\begin{aligned} V(y^2 = x^3 + Ax + B) = & V(y^2 = x^3 + A'x + B') \\ & \iff A' = t^4 A, B' = t^6 B \textit{ for some } t \in K^* \end{aligned}$$

Such a representation is called a Weierstrass equation *of the curve.*

A given Weierstrass equation $y^2 = x^3 + Ax + b$ *defines a nonsingular curve iff the* discriminant*:*

$$\Delta = 4A^3 + 27B^2$$

vanishes.

Over an algebraically closed field the function $j = \frac{A^3}{\Delta}$ *classifies elliptic curves up to isomorphism.*

Over a non-algebraically closed field, the j *function classifies elliptic curves up to quadratic, cubic and biquadratic twists (see below).*

The automorphism group of a curve is $\mathbb{Z}/2$ *if* $j \neq 0, 1$, $\mathbb{Z}/4$ *if* $j = 1$, *and* $\mathbb{Z}/6$ *if* $j = 0$.

Note the choice of normalization of the j function used here is the same as Miranda [Mir89] and omits the factor of 12^3 commonly used by number theorists.

Definition 6.0.22. If $C = V(y^2 = x^3 + Ax + B)$ is an elliptic curve, a *quadratic twist* of C is any curve with Weierstrass equation $y^2 = x^3 + d^2Ax + d^3B$ for some $d \in K^*$.

Similarly, if C has j invariant 1, then $C = V(y^2 = X^3 + Ax)$ and we define a *biquadratic twist* to be any curve $C = V(y^2 = X^3 + dAx), d \in K^*$.

If C has j invariant 0, then $C = V(y^2 = X^3 + B)$ and we define a *cubic twist* to be any curve $C = V(y^2 = X^3 + dB), d \in K^*$.

Remark 6.0.23. The correct way to look at twisting and the fact that the j function is only a complete invariant up to twists is to assert that the moduli space of elliptic curves is in fact represented by a Deligne-Mumford stack with the j line being only the coarse moduli space. The automorphism group of the generic point is $\mathbb{Z}/2$ and the automorphism group of the points over 0 and 1 are $\mathbb{Z}/6$ and $\mathbb{Z}/4$, respectively.

The theory of elliptic surfaces is parallel.

Definition 6.0.24. An *elliptic surface* is a surface X, along with a map $\pi : X \to C$ to a curve and a section[1] $s : C \to X$ such that the general fiber of π is a smooth genus 1 curve and $\pi \circ s = \text{id}$.

[1]Not all authors require a section.

X is *minimal* if it is relatively minimal over C.

Again we assume characteristic 0.

In other words, for our purposes an elliptic surface is simply a projective model of an elliptic curve over the generic point of the base C.

We explicitly describe the action of twists on surfaces with constant j invariant (for ease of calculation).

Example 6.0.25. Consider a trivial elliptic fibration $X = E \times \mathbb{P}^1$ with arbitrary j invariant, and let $C \to \mathbb{P}^1$ be a double cover with hyperelliptic involution τ. Say, locally, $C = V(y^2 - d)$ with d square free. Then $X \times_{\mathbb{P}^1} C$ has an involution (i, τ), where i is the involution on E. The quotient of $X \times_{\mathbb{P}^1} C$ by this involution is an new (singular) elliptic surface with the same j function, where the elliptic curve over the generic point is the quadratic twist by d. The singular fiber introduced over a zero of d has multiplicity 2 and 4 singular points of type A_1, corresponding to the fixed points of the involution. Blowing up at these points produces the minimal smooth model, where the new singular fibers are type $\widetilde{D}_4$ configurations of smooth rational curves. (In general a *type* $\widetilde{\Phi}$ *configuration* is a collection of lines with dual graph isomorphic to the corresponding affine Dynkin diagram and multiplicities given by the coefficients of the relation on the roots. See figure 6.1.)

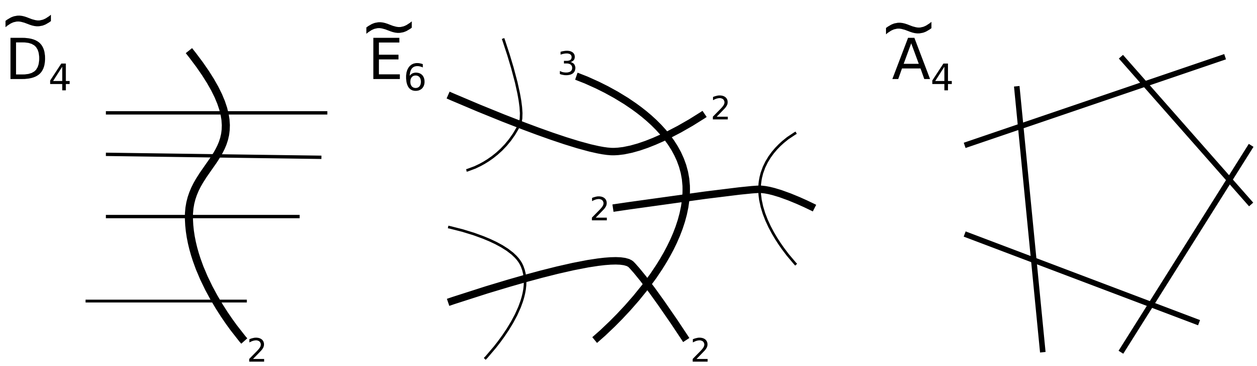

Figure 6.1: Fiber configurations of types $\widetilde{D}_4, \widetilde{E}_6, \widetilde{A}_4$. The numbers next to thick curves indicate the multiplicity of that curve. Compare figure 5.1.

Similarly, if $j(E) = 0$ (so E has an order 6 automorphism i_6) we take a cyclic $\mathbb{Z}/6$ cover $C \to \mathbb{P}^1$, say $C = V(y^6 - d)$. Then quotienting $X \times_{\mathbb{P}^1} C$ by the action (i_6, τ_6) gives a singular surface corresponding to the cubic twist by d. We can resolve by blowing up. The form of a fiber over a point p varies depending on $v_p(d)$:

$v_p(d) = 0$ Smooth fiber.

$v_p(d) = 1$ Cuspidal curve.

$v_p(d) = 2$ Three rational curves meeting at a point.

$v_p(d) = 3$ $\widetilde{D}_4$ configuration.

$v_p(d) = 4$ $\widetilde{E}_6$ configuration.

$v_p(d) = 5$ $\widetilde{E}_8$ configuration.

For example, in the case $v_p(d) = 5$ we have the fiber over p occurring with multiplicity 6 and having 3 quotient singularities of types $\frac{(1,-1)}{6}$, $\frac{(1,-1)}{3}$ and $\frac{(1,-1)}{2}$ (that is, types A_5, A_2, A_1). Blowing up gives the claimed fiber.

In the remaining case $j(E) = 1$ E has an order 4 automorphism i_4, and taking cyclic $\mathbb{Z}/4$ covers of $C \to \mathbb{P}^1$ and quotienting $X \times_{\mathbb{P}^1} C$ by the action (i_4, τ_4) gives singular fibers containing singular points of the surface with form depending on $v_p(d)$:

$v_p(d) = 0$ Smooth fiber.

$v_p(d) = 1$ Two rational curves meeting at a tacnode

$v_p(d) = 2$ $\widetilde{D}_4$ configuration.

$v_p(d) = 3$ $\widetilde{E}_7$ configuration.

Finally, we note that given a smooth elliptic surface $X \to \mathbb{P}^1$ where the j function has a simple pole over (say) 0 and the fiber over 0 is an irreducible nodal curve , base changing by $\mathbb{P}^1 \to \mathbb{P}^1 : x \mapsto x^n$ produces a type A_n surface singularity. After resolving the fiber over 0 is a type $\widetilde{A}_n$ configuration. If we perform a further quadratic twist over 0 (by base changing $x \mapsto x^2$ and then dividing by the composition of $x \mapsto -x$ and the hyperelliptic involution) we arrive at a surface with singularities that resolve to a type $\widetilde{D}_{n+4}$ configuration.

Singular Fibers of Elliptic Surfaces The possible singular fibers of the relatively minimal model of an elliptic surface were classified by Kodaira [Kod63]. Over $\mathbb{C}$ the theory is essentially topological, the isomorphism class of the fiber being determined by the monodromy around that fiber. Indeed, the above example considers all the cases, which is easily seen by pulling back to the universal elliptic curve. We record this below:

Proposition 6.0.26 (Kodaira [Kod63])**.** *The singular fibers of the relatively minimal model of an elliptic surface are given in the table, where "Name" is the Kodaira label and e is the contribution to the Euler characteristic.*

Name	*Configuration in minimal model*	j	e
I_0	*Elliptic Curve*	$\neq \infty$	*0*
I_0^*	$\widetilde{D}_4$	$\neq \infty$	*6*
I_n	$\widetilde{A}_{n-1}$	∞	n
I_n^*	$\widetilde{D}_{n+4}$	∞	$n+6$
II	*Cuspidal Curve*	*0*	2
IV	*Three rational curves meeting at a point.*	*0*	*4*
IV^*	$\widetilde{E}_6$	*0*	*8*
II^*	$\widetilde{E}_8$	*0*	*10*
III	*Two rational curves meeting at a tacnode.*	*1*	*3*
III^*	$\widetilde{E}_7$	*1*	*9*

Proof. Since the statement is local on the base, all the claims except the Euler characteristic follow from the example. By a quadratic twist of a trivial surface $E \times \mathbb{P}^1$ we obtain a rational elliptic surface with 2 type I_0^* fibers. Similarly, a quadratic twist of any surface with an I_n singularity can be performed to obtain a surface with the I_n singularity replaced by an I_n^* singularity and an additional I_0^* fiber. Since the fundamental line bundle's degree (6.1.1) is increased by one, and χ increased by 12, we see the local Euler characteristic increased by 6. The same argument relates the Euler characteristics of fibers of types II, III, IV with those of types II^*, III^*, IV^*. But cubic and biquadratic twists of trivial elliptic surfaces can produce rational elliptic surfaces with fiber types $6II, 4III, 3IV$. Dividing the total χ of 12 by the number of fibers gives the result. □

6.1 Weierstrass Fibrations

We proceed to globalize the concept of the Weierstrass model of an elliptic curve. The appropriate definition is:

Definition 6.1.1 (Miranda, [Mir89]). A *Weierstrass fibration* Is a surface X with a flat proper map $\pi : X \to C$ to a curve C, where the general fiber is smooth, every geometric fiber has arithmetic genus 1, and there is the additional data of a section $s : C \to X$ meeting every fiber at a smooth point.

An important invariant of a Weierstrass fibration is the *fundamental line bundle* $\mathbb{L} = (N_{s/X})^{-1}$ (equivalently $(R^1\pi_*\mathcal{O}_X)^{-1}$).

Observing the classification of singular fibers we see that contracting rational fiber components of a (smooth) elliptic surface X' disjoint from the section produces a Weierstrass fibration. The result is clearly a birational invariant of X' (although there are other Weierstrass fibrations birational to it).

We will accept the following fact ([Mir89][II.2])

Proposition 6.1.2. *Let X be an elliptic surface. X is birational to a double cover of $\mathbb{F}_{2\deg\mathbb{L}} = \mathbb{P}(\mathcal{O} \oplus \mathbb{L}^2)$ branched over the exceptional section and a trisection T.*

Assuming this, one can complete the construction of the Weierstrass equation:

Lemma 6.1.3. *There are coordinates on $\mathbb{F}_{2\deg\mathbb{L}}$ such that the trisection $T = V(X^3 + AX^2 + B)$ for some $A \in H^0(\mathbb{L}^4)$, $B \in H^0(\mathbb{L}^6)$.*

Corollary 6.1.4. *X can be written as a divisor in the $\mathbb{P}^2$ bundle $\mathbb{P}(\mathcal{O} \oplus \mathbb{L}^{-2} \oplus \mathbb{L}^{-3})$ with equation $Y^2Z = X^3 + AXZ^2 + BZ^3$ (where the section is then $X = Z = 0$).*

Definition 6.1.5. *Weierstrass data* for a Weierstrass fibration consist of a line bundle $\mathbb{L}$, and a pair of sections $A \in |\mathbb{L}^4|, B \in |\mathbb{L}^6|$. We define the *discriminant* $\Delta = 4A^3 + 27B^2 \in \mathbb{L}^{12}$.

We will frequently abuse notation and write the divisors of A, B, Δ as A, B, Δ, respectively.

We now describe the singular fibers of a Weierstrass fibration in terms of the trisection. The following two propositions are the content of Miranda's "a, b, δ" table ([Mir89][IV.3]), broken up for easier reading.

Proposition 6.1.6. *The singular fibers of a Weierstrass fibration are as follows, where the "Kodaira fiber" column indicates which fiber type, if any, of a minimal (smooth) elliptic surface yields the corresponding fiber in the Weierstrass fibration when the components not meeting the section are contracted:*

Singularity	*Kodaira Fiber*	*Configuration of T*
Smooth	I_0	T *meets* f *in distinct points*
A_{n-1}	I_n	*A type A double point,* $x^2 = y^{n+1}$
D_4	I_0^*	*Ordinary triple point.*
D_{n+4}	I_n^*	*A double point with local equation* $yx^2 = y^{n-1}$
Smooth	II	T *is flexed to* f
A_2	IV	T *meets* f *three times at a cusp.*
E_6	IV^*	*A triple point with local equation* $x^3 = y^4$
E_8	II^*	*A triple point with local equation* $x^3 = y^5$
A_1	III	T *meets* f *three times at a node.*
E_7	III^*	*A triple point with local equation* $x^3 = xy^3$
Elliptic or worse	*None*	*A triple tacnode.*

The fiber is a nodal curve in type I_n *and a cuspidal curve in all other cases.*

Further, the singularity type can be read directly off of the Weierstrass data:

Proposition 6.1.7. *Let* $\pi : X \to C$ *be a Weierstrass fibration with a chosen fiber* f *and* $p = \pi f \in C$*. The Kodaira type of* X *on* f*, as well as the* j *invariant and degree of the discriminant can be determined by the valuations* $v_p(A), v_p(B)$ *of the sections* A, B *as follows.*

Fiber type	$v_p(A)$	$v_p(B)$	j	$v_p(\Delta)$
I_0	*0*	*0*	$\neq 0,1$	*1*
	≥ 0	*0*	*0*	*1*
	0	≥ 0	*1*	*1*
I_n (A_{n-1})	*0*	*0*	∞	n
I_0^* (D_4)	*2*	*3*	$\neq 0,1$	*4*
	≥ 2	*3*	*0*	*4*
	2	≥ 3	*1*	*6*
I_n^* (D_{n+4})	*2*	*3*	∞	$6+n$
II	≥ 1	*1*	*0*	*2*
IV (A_2)	≥ 2	*2*	*0*	*8*
IV^* (E_6)	≥ 3	*4*	*0*	*8*
II^* (E_8)	≥ 4	*5*	*0*	*10*
III (A_1)	*1*	≥ 2	*1*	*3*
$III*$ (E_7)	*3*	≥ 5	*1*	*9*
Elliptic	≥ 4	≥ 6	*	≥ 12

Where the j invariant at an elliptic singularity may be arbitrary.

The results of (6.1.7) simply the expression of the twists and base changes in the example (6.0.25) in terms of Weierstrass equations. The non-notational part of 6.1.6 then follows immediately.

Observing the tables and recalling the discussion of surface singularities (4.2.4) we have the following corollary:

Corollary 6.1.8. *Let $X \to C$ be a Weierstrass fibration and consider the pair $(X, B), B = \epsilon(s + \sum f_i) + \sum F_j)$ for small $\epsilon > 0$, where F_j are distinct fibers and f_i are the singular fibers not in $\{F_j\}_j$. Then (X, B) is log canonical if and only if X has at only rational double point singularities and the fibers F_j contain at at worst type A_n singularities.*

In terms of the Weierstrass data, (X, B) is log canonical if and only if the divisors A and B are disjoint from the points on C corresponding to F_j and A and B do not simultaneously vanish to order 4 and 6, respectively.

6.2 The Mordell-Weil Lattice

Given an elliptic surface $\pi : X \to C$ (with marked section s) one can put a group structure on the set of sections $\sigma : C \to X, \pi \circ \sigma = \text{id}$. This is of course the group of rational points of the generic fiber of π and is called the *Mordell-Weil group*, or $\text{MW}(X)$. The subgroup $\text{MW}(X)^\circ$ of $\text{MW}(X)$ consisting of sections passing through the identity components of each singular fiber is also important, and named the *narrow Mordell-Weil group*. Following Shioda ([Shi90]), we will define a canonical bilinear form on $\text{MW}(X)$, allowing us to view $\text{MW}(X)/\text{MW}(X)_{tors}$ as a lattice[2].

Indeed, $NS(X)$ is already an (indefinite) lattice. Define the *trivial* sublattice $T \subset NS(X)$ as the sublattice spanned by the section and all fiber components. The orthogonal projection $\phi : NS(X) \to T^\perp \otimes \mathbb{Q}$ induces a well defined map $\text{MW}(X)/\text{MW}(X)_{tors} \to T^\perp \otimes \mathbb{Q}$. If $p_1, p_2 \in \text{MW}(X)$ then one defines $\langle p_1, p_2 \rangle = \langle \phi(\overline{p}_1), \phi(\overline{p}_2) \rangle$.

We will mostly need to know about the Mordell-Weil lattice in the case of rational elliptic surfaces, in which case the following description is available:

Proposition 6.2.1 (Shioda 10.3)**.** *Let X be a rational elliptic surface, $T \subset NS(X)$ be the trivial lattice as defined above. Write $L = T^\perp$. Then*

- $\text{MW}(X)^\circ = L$ *as lattices.*
- $\text{MW}(X)/\text{MW}(X)_{tors} = L^*$ *as lattices.*
- *Let T' be the primitive closure of T. Then* $\text{MW}(X)_{tors} = (T'/T) \cap \langle s, f \rangle^\perp$.

[2] This pairing is essentially the canonical height on the generic fiber, constructed in a much easier manner due to the special nature of our situation.

Finally note that Oguiso and Shioda explicitly calculated all the Mordell-Weil lattices that occur, and give them as a table in [OS91].

6.3 Elliptic K3 Surfaces

In this brief section we collect some important facts and definitions about elliptic K3 surfaces.

Definition 6.3.1. An *elliptic K3 surface* is a Weierstrass fibration $\pi : X \to \mathbb{P}^1$ with at worst ADE singularities such that the minimal model is a K3 surface.

Notice that set theoretically elliptic K3's are in bijection with smooth K3's equipped with the extra structure of being an elliptic surface. The reason for the definition given is to avoid having to deal with a nonseparated moduli problem.

Specializing the above discussion to elliptic K3 surfaces, we see that the fundamental line bundle is $\mathcal{O}(2)$ and that there are 24 singular fibers, counted with multiplicity. The possible types of singular fibers have been classified by Shimada [Shi00].

Examples 6.3.2. Starting with any rational elliptic surface $Y \to \mathbb{P}^1$ a 2:1 base change (and contracting -2 curves away from the section) produces an elliptic K3.

An elliptic K3 can also be obtained from a rational elliptic surface by a general quadratic twist.

Several modular surfaces are elliptic K3's, such as those corresponding to the groups $\Gamma(4)$, $\Gamma_1(7)$, and $\Gamma_0(12)$. The full list of 9 possibilities is due to Sebbar [Seb01].

Finally we note that there is a coarse moduli space of elliptic K3 surfaces, which we call $\mathcal{F}_{\text{ell}}$.

Theorem 6.3.3 ([CD07])**.** *The locally symmetric space*

$$\mathcal{F}_{\text{ell}} = \mathrm{O}(II_{2,18})\,\mathrm{O}(2,18)/\,\mathrm{SO}(2) \times \mathrm{SO}(18)$$

is a coarse moduli space for elliptic K3 surfaces.

This is discussed in more (but still incomplete) detail in the next chapter (7.3). For a complete discussion see Clingher and Doran ([CD07]).

Chapter 7

Hodge theory of Kulikov degenerations

In this chapter we briefly recall some classical analytic results on degenerations of K3 surfaces. We start by discussing the Kulikov-Persson-Pinkham theorem on the structure of nice models of degenerations ("Kulikov degenerations"). We then mention Friedman's criterion classifying exactly which surfaces may appear as the central fiber in a Kulikov degeneration. Finally we discuss the Hodge theory of K3 surfaces and Kulikov degenerations, starting with the general setup before discussing smooth surfaces and the global Torelli theorem and finishing with a description of Hodge theoretic aspects of Kulikov degenerations. The results of this chapter are quite technical and for the most part we don't even attempt to sketch proofs.

The primary source for the theory of Kulikov degenerations is of course Kulikov's paper [Kul77], but the recommended starting point is Persson and Pinkham's proof of the key result [PP81]. The detailed study of degeneration of K3's was undertaken by Friedman [Fri84] and Friedman-Scattone [FS86]. Scattone later studies the moduli theory [Sca87]. A useful general reference is [Per77]. For a moderately quick general introduction to Hodge theory in our context try [Loo11].

7.1 Kulikov Degenerations

A central theme has been given a degeneration $\mathcal{X} \to \Delta^\circ$ how to create a good model $\overline{\mathcal{X}} \to \Delta$. The aim of the KSBA approach outlined previously (4) is to produce models that are essentially unique, though perhaps somewhat singular. There is a complementary body of classical work studying smooth models. We define the central object:

Definition 7.1.1. Let $\mathcal{X} \to \Delta^\circ$ be a degeneration of K3 surfaces. A *Kulikov model* or *Kulikov degeneration* is a degeneration $\overline{\mathcal{X}} \to \Delta$ satisfying

1. $\overline{\mathcal{X}}$ is semistable, i.e. smooth with reduced normal crossing central fiber.

2. $K_{\overline{\mathcal{X}}} \simeq \mathcal{O}_{\overline{\mathcal{X}}}$.

The main result then is an enhanced semistable reduction theorem:

Theorem 7.1.2 (Kulikov [Kul77] Persson-Pinkham [PP81])**.** *After base change every degeneration $\mathcal{X} \to \Delta^\circ$ has a Kulikov model $\pi : \overline{\mathcal{X}} \to \Delta$. Moreover the central fiber $X_0 = \pi^{-1}0$ is of one of three types:*

- *(Type I) Smooth K3 surface.*

- *(Type II) A chain $X_0 = Y_1 \cup Y_2 \cdots \cup Y_n$ were $Y_i \cap Y_j = D$, some fixed genus 1 curve if $j = i \pm 1$ (otherwise empty), Y_1, Y_n are rational, and $Y_i, i \neq 1, n$ are elliptic ruled.*

- *(Type III) A union of rational surfaces, satisfying the triple point formula, such that the dual graph is a triangulation of the sphere and each the double locus on each component is an anticanonical cycle.*

We give a name to the surfaces satisfying the numerical conditions on X_0 in the theorem:

Definition 7.1.3. A surface of any one of types above satisfying the triple point formula will be called a *combinatorial K3 surface*[1] of the appropriate type. We call a type II surface *short* if it has only 2 components[2].

Kulikov models are far from unique, but are convenient and can provide a starting point to produce unique models (for example by running the MMP). The interesting question arises as to which combinatorial K3 surfaces may actually arise as the central fiber of a Kulikov degeneration. This was answered by Friedman [Fri83].

Definition 7.1.4. Let X be a combinatorial K3 surface. X is said to be *d-semistable* if

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = \mathcal{O}_{X_{sing}}.$$

Write $X = \bigcup V_i$ as a union of irreducible components. For each V_i define a divisor ξ_i by its pullback on the normalization $\sqcup V_i$ by $\nu^*\xi_i = \sum_j (V_i \cdot V_j)|_{V_i} - (V_i \cdot V_j)|_{V_j}$. Then X is d-semistable if each ξ is a Cartier divisor on X.

The main theorem is then:

Theorem 7.1.5 (Friedman)**.** *A combinatorial (analytic) K3 surface X is smoothable if an only if it is d-semistable. If it is smoothable then the smoothing component of the deformation space is smooth and 20 dimensional.*

A polarized combinatorial K3 surface (X, H) is smoothable if and only if it is d-semistable, and if so the smoothing component of the deformation space is smooth and 19 dimensional.

The analytic statement is from [Fri83] and the polarized one from [FS86].

[1] Note that, unlike some authors, we do not include an analytic condition, e.g. d-semistability.

[2] Friedman uses the word "stable", which already has several meanings here.

Friedman [Fri84][Theorem 2.3] shows that if $\overline{\mathcal{X}}$ is a type II degeneration then after birational modification there is an equivalent family with X_0 a short d-semistable type II combinatorial K3 surface. Henceforth we restrict ourself to short type II surfaces.

7.2 Definitions and Common Results

The basic definitions are:

Definition 7.2.1. A (pure) *Hodge structure* of weight n on a free abelian group L is a decreasing filtration

$$L \otimes \mathbb{C} = F^0 \supset F^1 \cdots \supset F^n$$

such that if $p+q = n+1$ then $F_p \cap \overline{F}^q = 0$ (one says that $F, \overline{F'}$ are *opposite*). Equivalently, there is a decomposition

$$L = \oplus_{p+q=n} H^{p,q}$$

such that $H^{p,q} = \overline{H}^{q,p}$.

A *Mixed Hodge structure* on L is a pair of a decreasing *Hodge filtration* $F^\bullet$ and a rationally defined increasing *weight* filtration W with the property that the Hodge filtration induces a Hodge structure of weight i on the weight graded pieces $\operatorname{Gr}_i^W L$.

Proposition 7.2.2 ([Wel08])**.** *Every smooth compact Kähler (in particular, projective) variety X has a pure Hodge structure of weight i on each $H^i(X, \mathbb{Z})$, with*

$$H^{p,q} = H^q(X, \Omega^p)$$

Note that this is the E_1 page for the spectral sequence associated to the complex

$$\mathcal{O}_x \to \Omega^1 \to \Omega^2 \dots$$

with the filtration

$$F^p = \dots 0 \to \Omega^p \to \Omega^{p+1} \dots .$$

The content of the theorem is mostly in the degeneration of this sequence at the E_1 page.

It is conventional to display this information diagrammatically, in the so-called *Hodge diamond*, where the i'th row from the bottom represents the dimensions of the graded pieces of the filtration on H^i, the so-called *Hodge numbers.* In our case:

Proposition 7.2.3. *The Hodge numbers of a K3 surface are given by the diamond:*

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 19 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Proof. The only Hodge number that is not obvious from the definition of a K3 surface is $h^{1,1}$. But this follows from knowing $\chi(X) = 24$ (directly in the elliptic case, or by Noether's formula in the general case). □

In the case of a degeneration $\mathcal{X} \to \Delta^\circ$ that can be completed to a family of smooth K3 surfaces the variation of pure Hodge structures is all we need to understand. In the case where the limit is singular, however, there is more to do. In particular, we will associate 2 distinct mixed Hodge structures to $\overline{\mathcal{X}} \to \Delta$: one depending only on the central fiber and one depending only on the general fiber $\mathcal{X} \to \Delta^\circ$ (and on a choice of tangent direction to the point $0 \in \Delta$, a slight technicality.)

The holomorphic universal cover of Δ° is the upper half plane H^+. Write X_∞ to be the pullback of $\mathcal{X}$ to H^+ (since the base is contactable, this is deformation retracts to any smooth fiber). The deck transformations on H^+ induce the monodromy action on $T : H^2(\mathcal{X}) \to H^2(\mathcal{X})$, where $\mathcal{X} \simeq L_{K3}$.

We recall general results on the monodromy T.

Proposition 7.2.4. • *T is quasi-unipotent in general, and unipotent for semistable degenerations. Hence the logarithm $N = 1 - T + \frac{(1-T)^2}{2} \dots$ is well defined.*

- *$N^2 = 0$ for type II degenerations and $N^3 = 0$ for type III.*

- *T is orthogonal with respect to the intersection form on H^2, so N is antisymmetric with respect to the same form.*

An additional algebraic result that is useful is:

Theorem 7.2.5 (Jacobson-Morozov)**.** *Any nilpotent element e in a semisimple Lie algebra can be extended to a triple $\{e, f, h\}$ defining an $\mathfrak{sl}_2$ subalgebra, where e, f, h correspond to the standard matrices*

$$e = \left\{ \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right\} \quad f = \left\{ \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \right\} \quad h = \left\{ \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right\}$$

As a corollary (this pathway is somewhat standard, we loosely follow [Loo11]) we get:

Lemma 7.2.6 ("Jacobson-Morozov lemma")**.** *If N is a nilpotent endomorphism of a finite dimensional vector space, there is a unique filtration $W_\bullet$ such that:*

- *$N(W_i) \subset W_{i-2}$*

- *N_i induces an isomorphism of graded pieces $\mathrm{Gr}_i W \simeq \mathrm{Gr}_{-i} W$*

We will call this filtration the Jacobson-Morozov *filtration $W_\bullet^{JM}$.*

Note that the $\mathfrak{sl}_2$ subalgebra in 7.2.5 is well defined only up to the action of the action of some group, but the filtration is still well defined.

In our case we shift the Jacobson-Morozov filtration on $H^2(X_\infty)$ by 2, i.e.

$$W_k = W_{k-2}^{JM}.$$

The orthogonality statement in 7.2.4 now implies that $W_\bullet$ is self dual in the sense:

$$W_i{}^\perp = W_{n-i-1}$$

where n is the index of nilpotency of N.

Let $\mathcal{D}$ be some space parameterizing pure Hodge structures of appropriate type (we describe this explicitly for K3 surfaces in 7.3). The map $G : H^+ \to \mathcal{D}$ given by $G(\tau) = \exp(\tau N)F(\tau)$ is invariant under translation by $\mathbb{Z}$, so descends to a map $\overline{G} : \Delta^\circ \to \mathcal{D}$. (This construction is well defined up to a factor of $\exp(\alpha N)$ for some α). The big result is:

Theorem 7.2.7 ([Sch73])**.** *The map $\overline{G}$ is holomorphic, and $\lim_{z\to 0} \overline{G}(z) \in \overline{\mathcal{D}}$ corresponds to a filtration $F^\bullet$ such that the pair W, F is a mixed Hodge structure.*

Definition 7.2.8. The mixed hodge structure above is called the *limit mixed Hodge structure* of the degeneration. We will denote the limit mixed Hodge structure for a degeneration $\mathcal{X}$ by $LH^\bullet(\mathcal{X})$.

There is also a mixed hodge structure associated to the variety X_0. In the case of a semistable degeneration we can access this by observing that the maps in the Mayer-Vietoris spectral sequence are maps of Hodge structures.

We now specialize to the cases of type I, II and III degenerations, noting that the case of type I is of special importance since it provides a description of the (coarse) moduli space of polarized K3 surfaces.

7.3 The Global Torelli Theorem and Type I Degenerations

A weight 2 Hodge structure on a rank 22 lattice is said to be of *K3 type* if $\dim H^{2,0} = \dim H^{0,2} = 1$. We can parameterize the possible pure Hodge structures on L_{K3} of $K3$ type in a straightforward way. Indeed, given such a structure we have a distinguished one dimensional subspace $H^{2,0} = \langle \omega \rangle \in H^2(X, \mathbb{C})$. Noting that the intersection form on L_{K3} is simply the restriction of the cup product we have $\omega^2 = 0$ and $\omega \cdot \overline{\omega} > 0$, so a first choice for the space of all Hodge structures of K3 type would be

$$\mathcal{D} = \{\omega \in \mathbb{P}(L_{K3} \otimes \mathbb{C})) | \omega \cdot \omega = 0, \omega \cdot \overline{\omega} > 0\}$$

There is the small detail that $\mathcal{D}$ has 2 components, exchanged under complex conjugation. We resolve this by choosing one arbitrarily, which we call $\mathcal{D}_-$. This is a Hermitian symmetric domain of type IV. The remaining problem is that different period points may correspond to the same Hodge structure, being related by an automorphism of L_{K3}. So we define the period domain:

$$\mathcal{F}_{K3} = \mathcal{D}_- / \Gamma_+$$

where Γ_+ is the subgroup of the orthogonal group $\mathrm{O}(L_{K3})$ that fixes the component $\mathcal{D}_-$.

The period domain $\mathcal{F}_{K3}$ provides an adequate parameter space for complex analytic K3 surfaces. To consider algebraic K3's, one must first fix a class $h \in L_{K3}$, which we require to be ample. We can assume that h is primitive in L_{K3}, and write $h^2 = 2d$. It follows from James's theorem[3] [Jam68] that there is in fact a unique conjugacy class of such vector. So

[3] Any nonsingular lattice of rank less than the index of an even unimodular lattice has a unique conjugacy class of primitive embedding.

we write:

$$L_{2d} = h^{\perp} \in L_{K3}$$

$$\mathcal{D}_{2d} = \text{one component of } \{\omega \in \mathbb{P}(L_{2d} \otimes \mathbb{C}) | \omega \cdot \omega = 0, \omega \cdot \overline{\omega} > 0\}$$

$$\mathcal{F}_{2d} = \mathcal{D}_{2d}/\Gamma_{2d}$$

Where Γ_{2d} is the subgroup of $\mathrm{O}(L_{2d})$ obtained by restricting the elements of $\mathrm{O}(L_{K3})$ fixing h and stabilizing the component $\mathcal{D}_{2d}$. It is in this situation that we can establish a strong Torelli theorem, originally due to Piatetskii-Shapiro and Shafarevic [PSS71]. Friedman's argument [Fri84] is more fitting here, though.

Theorem 7.3.1 (Global Torelli)**.** *The period domain $\mathcal{F}_{2d}$ is a coarse moduli space for K3 surfaces with primitive polarization of degree $2d$.*

In our situation instead of a single polarizing vector we have the additional data of an elliptic fibration, that is a pair s, f of algebraic classes, one for the chosen section and one for a fiber. s, f span a unimodular sublattice isomorphic to H of L_{K3}, so $L_{K3} = \langle s, f\rangle \oplus II_{2,18}$ (by the structure theorem for indefinite unimodular lattices) and thus the sublattice is unique up to conjugation. We repeat the previous construction to obtain a moduli space $\mathcal{F}_{\text{ell}}$ as a subspace of F_{2d} for all d. Explicitly:

Definition 7.3.2. Write

$$\mathcal{D} = \text{one component of } \{\omega \in \mathbb{P}(II_{2,18} \otimes \mathbb{C}) | \omega \cdot \omega = 0, \omega \cdot \overline{\omega} > 0\}$$

and let

$$\Gamma \subset \mathrm{O}(II_{2,18}).$$

be the index 2 subgroup stabilizing the component Γ. The period domain for elliptic K3 surfaces is then defined as

$$\mathcal{F}_{\text{ell}} = \mathcal{D}/\Gamma$$

.

The appropriate Torelli theorem is an application of Dolgachev's ([Dol96]) theory of (psuedo-ample) lattice polarized K3 surfaces, and is carefully explained by Clingher and Doran [CD07].

Theorem 7.3.3 (Torelli Theorem for Elliptic K3 Surfaces ([CD07]).)**.** $\mathcal{F}_{\text{ell}}$ *is a coarse moduli space for elliptic K3 surfaces.*

The Torelli theorem provides a complete description of the Hodge theory of degenerations with X_0 smooth. We now expand to study degenerations with X_0 singular.

7.4 Type II

The statements in this section mostly follow [Fri84].

Lemma 7.4.1. *([Fri84] 3.4) Let X_0 be a short type II surface with the components joined along a genus 1 curve D. Then*

1. $\dim H^2(X_0) = 21$
2. $W_2H^2(X_0)/W_1H^2(X_0)$ *has dimension 19 and is pure of type (1,1).*
3. $W_1H^2(X_0) \simeq H^1(D)$ *as Hodge structures.*

Proof. 1 follows from the Mayer-Vietoris sequence, which has E_1 page

$$\begin{array}{ccccc} H^0(D) & H^1(D) & H^2(D) & 0 & 0 \\ \bigoplus H^0(V_i) & 0 & \bigoplus H^2(V_i) & 0 & \bigoplus H^4(V_i) \end{array}$$

Since we have (by the triple point formula) $K_{V_1}|_D + K_{V_2}|_D = -K_{V_1}^2 - K_{V_2}^2 = 0$ and (since they are rational) $\dim H^2(V_i) = 10 - K_{V_i}^2$. Furthermore, the map $\bigoplus H^2(V_i) \to H^2(D)$ is surjective. Indeed this is merely the statement that some curve on at least one of the V_i meets D at a point. Putting this together we get:

$$\dim H^2(X_0) = \dim H^2(V_1) + \dim H^2(V_2) + \dim H^1(D) - \dim H^2(D) = 21$$

The statement on the graded pieces (2) follows immediately by recalling the weight filtration is given by the columns of the sequence and observing that the Hodge structures on $H^2(V_i)$ are pure of type $(1,1)$.

The final claim is obvious, since there are no nonzero differentials that can affect the term $H^1(D)$. □

We also need to understand the limit Hodge structure of a degeneration. Define $\mathcal{E} = D|_{V_1} - D|_{V_2}$. $\mathcal{E}$ is clearly a isotropic vector in $H^2(\widetilde{X}_0)$. We have

Lemma 7.4.2. *(Verbatim from Friedman [Fri84])*

1. *The Clemens-Schmid exact sequence*

$$H_4(X_0) \to H^2(X_0) \to LH^2(\mathcal{X}) \to LH^2(\mathcal{X})$$

   is exact over $\mathbb{Z}$*.*

2. $W_1 LH^2(\mathcal{X}) \simeq W_1 LH^2(\mathcal{X})$

3. $\mathrm{Gr}_2^W LH^2(\mathcal{X}) \simeq \mathcal{E}^\perp/\mathbb{Z}\mathcal{E}$ *as a sublattice of* $H^2(\widetilde{X}_0)$*.*

4. *The signature of the intersection pairing on* $\mathrm{Gr}_2^W LH^2(\mathcal{X})$ *is (1,17).*

This information becomes especially useful considering the following theorem of Carlson, as quoted in Friedman [Fri84].

Theorem 7.4.3. • *The mixed Hodge structure on $H^2(X_0)$ determines a homomorphism $\psi : \mathrm{Gr}_2^W H^2(X_0) \to J(D)$.*

- *This homomorphism is given geometrically by $\psi(l) = (l|_{V_1} \cdot D) \otimes (l|_{V_2} \cdot D)$.*

- *A class $l \in H^2(X_0)$ is Cartier only if $\psi(l) = 0$.*

For a degeneration of polarized K3 surfaces, write the polarization class as h and write $L = h^{\perp} \subset \mathrm{Gr}_2^W H^2(X_0)$. Clearly L has signature $(0, 17)$. Since h is Cartier we can factor ψ through a map $L \to J(D)$, and so it is of interest to study the structure of L in detail.

The following examples roughly follow Friedman [Fri84, Section 5]. His exposition is more general in most regards, but we choose to compute the entire lattice L, rather than just its root sublattice.

Example 7.4.4. We first calculate the lattice $L = h^{\perp}$ in the case of a family of 2 polarized K3's given as double covers of $\mathbb{P}^2$ over a sextic, where the sextic degenerates towards twice a cubic. There may be a type A_n singularity in the total space along the double locus of the central fiber, but assume for simplicity $n = 0$, i.e. the threefold is smooth in codimension 2[4]. The central fiber is now 2 planes meeting on a cubic D, with (generically) 18 singular points in the 3-fold[5] . We resolve these in such a way that the effect on the central fiber is to blow up one of the planes 18 times at points on D. Call the component of X_0 isomorphic to $\mathbb{P}^2$ Y_1 and the blown up component Y_2. We write $H^2(Y_1) = \langle l_1 \rangle$ and $H^2(Y_2) = \langle l_2, e_1 \dots e_{18} \rangle$.

We now have:

$$\mathcal{E} = 3l_1 - (3l_2 - \sum e_i)$$

$$h = l_1 + l_2$$

The classes $\alpha_i = e_i - e_{i+1}$ are visibly in $\langle \mathcal{E}, h \rangle^{\perp}$ and independent modulo $\mathcal{E}$. These form a full rank A_{17} sublattice of L. But L is in fact a strict overlattice of this root lattice,

[4] If not the central fiber of a Kulikov model will not be short.

[5] If one writes the family as $z^2 = f_3^2 + tf_6$ these are the points over $V(f_3, f_6)$.

since by adding a multiple of $\mathcal{E}$ we can write a class in $\langle \mathcal{E}, h\rangle^\perp$ without using l_1, l_2 only when the coefficient of l_1 (=negative the coefficient of l_2) is divisible by 3. Thus the class $l_1 - l_2 - 6e_1 \in \mathcal{E}^\perp/\mathcal{E}$ generates L/A_{17}, and $[A_{17} : L] = 3$.

Example 7.4.5. Now consider a family of 2 polarized $K3$ surfaces degenerating towards a double cover of $\mathbb{P}^2$ over twice a conic plus some other conic (in our situation , with all surfaces elliptic, we call this the $\widetilde{D}_{16}$ case, for reasons that will soon be clear). As before this representation is not semistable (it has a type A_n singularity. We assume that $n = 1$). A single blowup will produce a semistable family, with central fiber X_0. D, the double curve of X_0 is isomorphic to the conductor of the normalization of the original central fiber. Write $X_0 = Y_1 \cup Y_2$, where $Y_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is the strict preimage of the original central fiber and $Y_2 = \mathrm{Bl}_{16}\mathbb{F}_2$ is a ruled surface blown up at 16 points on the bisection D.

We now calculate $\mathrm{Gr}_2^W H^2(X_0)$ and the lattice L in this case. We write $H^2(Y_1) = \langle s_1, f_1\rangle$ and $H^2(Y_2) = \langle s_2, f_2, e_1 \ldots e_{16}\rangle$, where s_i, f_i are sections and fibers and $s_2^2 = -2$.

We have:

$$\mathcal{E} = 2s_1 + 2f_1 - (2s_2 + 4f_2 - \sum e_i)$$

$$h = s_1 + f_1 + 2f_2$$

We observe that $\alpha_0 = s_1 - f_1$ represents a root, and moreover that any class in $h^\perp$ has even intersection with α_0. Thus $\langle\alpha_0\rangle \simeq A_1$ is actually a direct summand of L. $\alpha_0^\perp$ is unimodular since $-\alpha_0^2 = h^2$ and in fact even since being in $\mathcal{E}^\perp$ implies that it contains an even number of exceptional divisors, counting multiplicity. It is obvious by symmetry that $\alpha_0^\perp \simeq D_{16}^+$, but we explicitly can write:

$$R = \left\{ \sum a_i(-\frac{1}{2}f_2 + e_i) \middle| \sum a_i \equiv 2(\mod 2) \right\}.$$

This is a D_{16} root lattice, with roots

$$\alpha_1 = -f_2 + e_1 + e_2$$

$$\alpha_i = e_{i-1} - e_i$$

and represents every class in $\langle \mathcal{E}, h, \alpha_0 \rangle^\perp$ that can be written without using s_1, f_1 (i.e. those that are written using even coefficients of s_1). Additionally, though, we have the class of:

$$\alpha_{17} = -s_1 - f_1 + s_2 + e_{15} + e_{16}$$

We check that these give a root basis for D_{16}^+

In the case of elliptic surfaces this calculation does not work exactly as before. Instead we start with a family of double covers of $\mathbb{F}^4$, and the natural model is $X_0 = Y_0 \cup Y_1$ with $Y_0 \simeq \mathbb{F}_2$, and $Y_2 \simeq \mathrm{Bl}_{16}\mathbb{F}_2$. The calculation is entirely similar. The two models can be related by flopping one exceptional curve from Y_1 in the first model into Y_0, so now we have Y_1 as the blowup of $\mathbb{P}_2$ at two points. If the points become infinitely close one of the roots (α_0) of L represents an (effective) Cartier divisor, the strict preimage of the first blowup, and flopping the -1 curve in Y_0 not meeting this back into Y_1 results in the model specialized to the elliptic case. In this case since α_0 is Cartier, the map ψ factors through $L/\alpha_0 = D_{16}^+$ and the simple roots of D_{16}^+ are

$$\alpha_1 = -f_2 + e_1 + e_2, \quad \alpha_i = e_{i-1} - e_i$$

with the affine root

$$\alpha_{17} = s_1 + 2f_1 + s_2 + f_2 + e_{15} + e_{16}.$$

7.5 Type III

Similar statements hold for the type III case. We first describe the mixed hodge structure on the central fiber. The content of this proposition is in Friedman-Scattone [FS86], though the style of exposition more closely follows Laza [Laz08].

Let X_0 be the type III combinatorial K3 with n components and dual complex Γ (a triangulation of S^2). Write $L = \mathrm{Gr}_2^W H^2(X_0)$.

Proposition 7.5.1. *With the setup above*

- *L has rank $18+n$ and is of type $(1,1)$.*
- *The mixed hodge structure on $H^2(X_0)$ is an extension of Hodge structures*

$$0 \to W_0 H^2(X_0) \to H^2(X_0) \to L \to 0 \tag{*}$$

- $W_0 H^2(X_0) = H^2(\Gamma)$
- *The possible extensions * are parameterized by maps $\phi : L \to \mathbb{C}^*$ where $\ker \phi$ are exactly the Cartier divisor classes in L.*

Proof. The statement on the structure of H^2 follows from the Mayer-Vietoris spectral sequence. Let V_i be the components of X_0, D_{ij} the double intersections and T_{ijk} the triple points. Then the E_0 page is given by:

$$\begin{array}{ccccc}
\bigoplus \Omega_0(T_{ijk}) & 0 & 0 & 0 & 0 \\
\bigoplus \Omega_0(D_{ij}) & \bigoplus \Omega_1(D_{ij}) & \bigoplus \Omega_2(D_{ij}) & 0 & 0 \\
\bigoplus \Omega_0(V_i) & \bigoplus \Omega_1(V_i) & \bigoplus \Omega_2(V_i) & \bigoplus \Omega_3(V_i) & \bigoplus \Omega_4(V_i)
\end{array}$$

So the E_1 page is:

$$\begin{array}{ccccc} \bigoplus H^0(T_{ijk}) & 0 & 0 & 0 & 0 \\ \bigoplus H^0(D_{ij}) & 0 & \bigoplus H^2(D_{ij}) & 0 & 0 \\ \bigoplus H^0(V_i) & 0 & \bigoplus H^2(V_i) & 0 & \bigoplus H^4(V_i) \end{array}$$

Since the V_i are rational surfaces, $H^2(V_i, \mathbb{Z}) = \mathrm{Pic}(V_i)$. The E_2 page is then:

$$\begin{array}{ccccc} \mathbb{C} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \mathbb{C} & 0 & L \otimes \mathbb{C} & 0 & \mathbb{C}^n \end{array}$$

Where L are the divisor classes l such that $\deg(l|D_{ij} \subset V_i) = \deg(l|D_{ij} \subset V_j)$. The rank of a rational surface is $\rho = 10 - K^2$. If the surface has an anticanonical cycle $\{D_i\}$ of length m we have $\rho = 10 - 2m - \sum D_i^2$. Now we sum over all components, using Euler's formula and the triple point formula. Each of the $3n - 6$ edges appears twice in the sum, and by the triple point formula contributes 2. Each of the $2n - 4$ vertices appears 3 times, contributing -2 each time, so $\dim \bigoplus H^2(V_i) = 10n + 2(3n - 6) - 3 \cdot 2(2n - 4) = 4n + 12$. The map $\bigoplus H^2(V_i) \to \bigoplus H^2(D_{ij})$ is surjective, so the kernel has dimension $18 + n$, as claimed.

The claim that the extensions are parameterized by maps $\phi : L \to \mathbb{C}^*$ is a direct application of Carlson theory [Car80], however the definition there is hard to use. We can directly describe a map with the property that $l \in L$ is a Cartier divisor if and only if $\phi(L) = 1$, which uniquely characterizes Carlson's map. First choose a Cartier divisor l' on some neighborhood of $\bigcup D_{ij}$ such that $\deg(l'|_{D_{ij}}) = \deg(l|_{D_{ij}})$ (this makes sense even if l is not Cartier). If l was Cartier then for any oriented cycle of rational curves C and map $c : C \to \bigcup D_{ij}$ $c^*(l) \in \mathrm{Pic}^0 C = \mathbb{C}^*$ would be well defined. In particular there would be a map $\gamma : H_1(\Gamma) \to \mathbb{C}^*$. $H_1(\Gamma)$ is generated by the oriented boundaries $\partial V_i \subset V_i$, with the one relation $\sum_i \partial V_i = 0$. Hence the obstruction to l being Cartier is $\prod_i l|_{\partial V_i} \otimes l'|_{\partial V_i} = 1$. □

We restate the d-semistability condition in terms of the extension map $\phi : L \to C^*$.

Definition 7.5.2. Let X be a combinatorial K3 surface. For each component V_i define $\xi_i \in L$ as

$$\xi_i = \sum D_{ij} - \sum D_{ji}$$

Note the obvious relation $\sum \xi_i = 0$. Indeed this is the only such relation For a general proof see Laza [Laz08].

Recall X is d-semistable if and only if each of the classes ξ_i is Cartier.

Writing $K = \langle \xi_i \rangle_i$ and $\overline{L} = L/K$ the theorem shows that the mixed Hodge structure of a smoothable surface is defined by a map $\overline{\phi} : \overline{L} \to \mathbb{C}^*$. Indeed, we will show that the map $\overline{\phi}$ effectively determines the limit mixed Hodge structure of a smoothing. (Closely following Friedman and Scattone)

First though, we breifly move on to discuss the relation between a combinatorial K3 surface and its normalization.

Definition 7.5.3. Let V_i be the components and Γ the dual complex of a combinatorial K3 surface. Then a *gluing* of V_i (with Γ implicit) is a specific combinatorial K3 with those components and dual complex.

Similarly, if X is a gluing of V_i and Γ and D some collection of double curves then a *regluing* along D is a gluing of V_i, Γ isomorphic to X away from D.

Friedman ([Fri83]) discusses when a collection of surfaces V_i and dual complex may be glued to form a d-semistable K3. His statement [Fri83][5.14] is unclear (to me) in that it's not clear how to handle double curves with self intersection 0 on one component. We give the version there as well as one adapted to our needs.

First, notice that a choice of orientation on the dual complex of a surface induces isomorphisms $\operatorname{Pic} \partial V_i \simeq \mathbb{C}^*$ for each component and $\operatorname{Hom}(D_{ij}, D_{ij}) \simeq \mathbb{C}^*$ for each double curve. We assume these.

Lemma 7.5.4. *Let D_{ij} be a double curve in a combinatorial K3 surface X' with $D_{ij}^2 = a, D_{ji}^2 = -2-a$. Let X be the combinatorial K3 surface obtained by replacing the isomorphism $D_{ij} \simeq D_{ji}$ with $D_{ij} = \alpha D_{ji}$ with $\alpha \in \mathrm{Hom}(D_{ji}, D_{ji}) = \mathbb{C}^*$. Then $\phi_X(\xi_j) = \alpha^{2+a}\phi_{X'}(\xi_j)$ and $\phi_X(\xi_i) = \alpha^{-a}\phi_{X'}(\xi_i)$.*

Proof. By symmetry we must only prove the statement for ξ_j. As in the proof of 7.5.1, choose a Cartier divisor l' in a neighborhood of the double locus of X' such that $l' \otimes \xi_j|_{V_k}$ is numerically trivial for all components V_k. Then observing the proof of 7.5.1 one sees that $\phi_{X'}(\xi_j) = \prod_k (l' \otimes \xi_j)|_{V_k}$. If we replace X' by X we replace l' by some l, which can be taken to agree with l everywhere except on D_{ij}, where $l \otimes l'^{-1} = \alpha^{\deg \xi_j|_{D_{ij}}} = \alpha^{2+a}$. Then $l \otimes \xi_j|_{\partial V_k} = l' \otimes \xi_j|_{\partial V_k}$ for $k \neq i$ and $l \otimes \xi_j|_{\partial V_k} = \alpha^{2+a}(l' \otimes \xi_j|_{\partial V_k})$. $\square$

Lemma 7.5.5. *Let X be a K3 surface with no double curve having square 0 in either component containing it. Let T be a spanning tree in the dual graph. Then there is a d-semistable regluing of X along T.*

Proof. For each edge D_{ij} in T the possible isomorphisms $D_{ij} \simeq D_{ji}$ are acted on by $\mathbb{C}^*$. The weights of this action on $\phi|_{\langle \xi_i \rangle}$ are as given above 7.5.4. These weights are linearly independent, so there is some gluing such that $\phi(\xi_i) = 1$ for all except at most one ξ, say ξ_j. But $\phi(\xi_j) = (\prod_{i \neq j} \xi_i)^{-1}$, so the gluing is in fact d-semistable. $\square$

We can do a little better. (This lemma is unnatural, it's simply the statement that will be used later.)

Lemma 7.5.6. *Let X be a K3 surface and U_i be some components such that $\mathcal{O}_{U_i}(\partial U_i)|_{\partial U_i} = \mathcal{O}_{U_i}$.*

Let Γ_{good} be the subgraph of the dual graph with edges corresponding to the boundary of the U_i and the double curves with nonzero square in both components containing them.

Call a component N_j negligible *if it has a boundary component with square 0, and let N be a set of double curves containing one with square 0 in each negligible component.*

Then for each tree T of Γ_{good} containing all non-negligible components there is a d-semistable regluing of X along $T \cup N$.

Proof. Using 7.5.4 first note that by regluing along N the ξ_j corresponding to negligible components can be made Cartier without affecting the values of ϕ on the other ξ_i.

Similarly to before the regluings along edges in T give an action of $(\mathbb{C}^*)^T$ on $\phi|_{\langle \xi_i \rangle}$. The weights are given in 7.5.4. These weights are not linearly independent if there is more than one surface U_i, but $(\mathbb{C}^*)^T$ is still easily seen to act transitively on $\mathrm{Hom}(\langle \xi_j \rangle_{V_j \notin \{U_i \cup N_j\}}, \mathbb{C}^*)$; in particular, one may reglue X along T to make $\xi_j, V_j \notin \{U_i\}$ Cartier. But the ξ_i are in the span of $V_j \notin \{U_i\}$ and ∂U_i, and ∂U_i are assumed to be Cartier, so all the ξ_i are too. □

Finally, this is a good place to state a modified form of a proposition of Friedman and Scattone (we omit some of their conclusion, but extend to a lattice polarization. Their proof goes through):

Proposition 7.5.7 (Friedman [FS86][5.5]). *Let X_0 be a type III d-semistable surface with Cartier divisor classes d_1, d_2. Then there is a smoothing $\mathcal{X}/\Delta$ with divisor classes $\bar{d}_1.\bar{d}_2$ specializing to d_1, d_2.*

We move on to discussing the limiting mixed Hodge structure. For convenience we write $W_n = W_n LH^2(X_0), F^n = F^n LH^2(X_0)$, etc..

For a K3 surface, the structure of the monodromy transformation N and the corresponding weight filtration can be made very explicit:

Proposition 7.5.8.

- *For type III degeneration the filtration W takes the form*

$$W_0 \subset W_2 \subset W_4$$

where $\dim W_0 = 1 = \dim W_4/W_2$*, and* W_0 *is isotropic by self-duality.*

- *([FS86] Lemma 1.1) Let γ be a generator of W_0, and choose γ' with $\gamma \cdot \gamma' = 1$. Put $\delta = N\gamma'$. δ is well defined modulo W_0 and we have*

$$Nx = (x \cdot \gamma)\delta - (x \cdot \delta)\gamma$$

Proof. Write $H_m^{p,q} = W_m \cap F^p \cap \bar{F}^q(\mod W_{m-1})$, where $m = p + q$. (These are simply the bigraded pieces of the mixed Hodge structure on H^2). Since $W_3 = \ker N^2$ we have $W_4/W_3 \neq 0$. But $\mathrm{Gr}_4^W = H_4^{2,2}$, and this is a quotient of a one dimensional space so $\dim W_0 = 1 = \dim \mathrm{Gr}_4^W$. Further, we have $\mathrm{Gr}_3^W = H_3^{1,2} \cup H_3^{2,1}$. But $\dim F^2 = \sum \dim H_{2+i}^{2,i} = 1$, and $H_3^{1,2} = \overline{H_3^{2,1}}$, so $\dim \mathrm{Gr}_3^W = \dim \mathrm{Gr}_1^W = 0$.

The choice of γ' was only well defined up to W_2, but then the resulting δ is well defined up to $NW_2 = W_0$.

For the claim on N, observe that if $y \in W_2 = \gamma^\perp$ we have $Ny = a\gamma$, where $a = Ny \cdot \gamma' = -y \cdot \delta$. Since $(\gamma' \cdot \gamma)\delta - (\gamma' \cdot \delta)\gamma = 1 \cdot \delta - 0 \cdot \gamma$ the claimed result holds all of $H^2 = W_2 + \langle \gamma' \rangle$. □

Recall the Wang sequence:

$$0 = H^1(X_\infty) \to H^2(\mathcal{X}) \to H^2(X_\infty) \xrightarrow{1-T} H^2(X_\infty)$$

(Note that topologically $\mathcal{X}$ retracts to a K3 fibration over S^1). The Wang sequence is an exact sequence of mixed Hodge structures, with $H^2(X_\infty) = LH^2(X_\infty)$ being given the limit Hodge structure. Our immediate goal is then to describe the map $H_2(\mathcal{X}) \to LH^2(X_\infty)$.

We start with the exact sequence for the pair $(\overline{\mathcal{X}}, \mathcal{X})$:

$$H^i(\overline{\mathcal{X}}, \mathcal{X}) \to H^i(\overline{\mathcal{X}}) \to H^i(\mathcal{X}) \to H^{i+1}(\overline{\mathcal{X}}, \mathcal{X}) \to H^{i+1}(\overline{\mathcal{X}}) \to H^{i+1}(\mathcal{X}) \to$$

and observe that since X_0 is a retract of $\overline{\mathcal{X}}$, $H^i(\overline{\mathcal{X}}, \mathcal{X}) = H^i(X_0)$ and $H^i(\overline{\mathcal{X}}) = H^i(X_0)$. So our sequence is now:

$$H^i(X_0) \to H^i(X_0) \to H^i(\mathcal{X}) \to H^{i+1}(X_0) \to H^{i+1}(X_0) \to H^{i+1}(\mathcal{X}) \to$$

But Alexander duality gives $H^i(X_0) \simeq H_{6-i}(\overline{\mathcal{X}}, \mathcal{X})$, which is again isomorphic to $H_{6-i}(X_0)$. So finally we have:

$$H_{6-i}(X_0) \to H^i(X_0) \to H^i(\mathcal{X}) \to H_{6-i-1}(X_0) \to H^{i+1}(X_0) \to H^{i+1}(\mathcal{X}) \to$$

Observe that the image of the map $H_4(X_0) \to H^2(X_0)$ is $\langle \xi_i \rangle$ and $H^3(X_0) = 0$ so we can replace $H^2(\mathcal{X})$ in the Wang sequence to get:

$$0 \to \sum \xi_i \to \langle \xi_i \rangle \to H^2(X_0) \xrightarrow{\alpha} H^2(X_\infty) \xrightarrow{N} H^2(X_\infty) \tag{7.1}$$

Note that $(1-T)$ has been replaced with N. This is justified since $\ker(1-T) = \ker N$.

But $\ker N = \delta^\perp \cap W_2$, so we have shown the following claim:

Proposition 7.5.9. *Consider $\overline{L} \subset \mathrm{Gr}_2^W H^2(X_0)$. The sequence above gives an isomorphism $\overline{L} \to \delta^\perp (\mod W_0)$.*

Finally,

Proposition 7.5.10. *([FS86, 4.16]) The limit mixed Hodge structure LH^2 is determined up to a nilpotent orbit by the mixed Hodge structure on $H^2(X_0)$ and the collapsing map $\alpha : H^2(X_0) \to LH^2$.*

Proof. The weight filtration of LH^2 is determined by a choice of W_0. But $II_{2,18}$ has a unique orbit of primitive isotropic vector so one may fix a weight filtration at the outset. The Hodge filtration is determined by a choice of one dimensional subspace $F^2 = \mathbb{C}v \subset W_2 \otimes \mathbb{C}$. The map α in equation 7.1 is a map of mixed Hodge structures, so $\alpha(F^1 H^2(X_0))$ is a subspace of F^1.

In particular $v \in \alpha(F^1H^2(X_0))^\perp$. But $\operatorname{codim} \alpha(H^2(X_0) = 2$ and $\operatorname{codim} F^1(H^2(X_0)) = 1$ and $\alpha(F^1H^2(X_0)) \not\subset F^1$ so there is a 3 dimensional space A of candidates for v. The quadratic form on A is nondegenerate so the possible choices for v with $v^2 = 0$ form a conic in $\mathbb{P}(A)$ (There is the additional linear condition $v \notin W_2$). The possible choices are seen to form a conic in $\mathbb{A}_2$. The group $\exp N$ acts nontrivially therefore transitively. □

Remark 7.5.11. The data of the map α should be unnecessary so long as the central fiber is known. Although there are a priori several ways to embed $\bar{L}$ into $II_{1,17}$, the topology of a smoothing is determined by the central fiber ([PP81][2]).

Chapter 8

Compactifications of $\mathcal{D}/\Gamma$

In this chapter we describe the Baily-Borel and toroidal compactifications of $\mathcal{F}_{\mathrm{ell}} = \mathcal{D}/\Gamma$. While both constructions exist in great generality (quotients of Hermitian symmetric domains by arithmetic groups) we will specialize to the case of interest and only mention other cases in passing.

The Baily-Borel compactification $\overline{\mathcal{F}}_{\mathrm{ell}}^{\mathrm{BB}}$ is canonical and in some sense minimal. Unfortunately this very minimality causes $\overline{\mathcal{F}}_{\mathrm{ell}}^{\mathrm{BB}}$ to be quite singular, and to my knowledge it does not carry a good modular interpretation. On the other hand the toroidal compactifications $\overline{\mathcal{F}}_{\mathrm{ell}}^{\Sigma}$ require the extra data of a fan Σ, but have very mild singularities. It is reasonable to believe that for some Σ a strong modular interpretation exists.

The fundamental reference for this material is [AMRT10]. In the introduction to [Loo03] (where Looijenga defines certain compactifications intermediate between the Baily-Borel and toroidal) there is a very readable introduction, and Kondo ([Kon93]) gives a very concrete description of parts of the theory. We draw inspiration from both sources.

These techniques are, in their easiest formulation, essentially analytic. Thus in this chapter we will always work in the complex analytic category unless otherwise noted.

As will quickly become apparent any toroidal compactification $\overline{\mathcal{F}}_{\text{ell}}^{\Sigma}$ dominates $\overline{\mathcal{F}}_{\text{ell}}^{\text{BB}}$, and so we begin by discussing the Baily-Borel compactification.

8.1 The Baily-Borel Construction

The general strategy of the Baily Borel construction is to enlarge the domain $\mathcal{D}$ by adjoining some collection of "rational boundary components", which are also Hermitian symmetric domains. The resulting space $\mathcal{D}^*$ is given a topology (the so-called Satake topology, which restricts to the analytic topology on the interior and all boundary components) and a sheaf of "analytic" functions, these simply being those continuous functions that restrict to holomorphic functions on each component. If this was done correctly then the quotient $\overline{\mathcal{F}}_{\text{ell}}^{\text{BB}} = \mathcal{D}^*/\Gamma$ is a complex analytic variety. Moreover, there is a natural Γ equivariant line bundle $\mathbb{L}^{BB}$ on $\mathcal{D}^*$ such that $\overline{\mathcal{F}}_{\text{ell}}^{\text{BB}} = \operatorname{Proj} H^0(\mathbb{L}^n)^\Gamma$.

One starts by embedding $\mathcal{D}$ into its compact dual $\hat{\mathcal{D}}$ by the Borel embedding. In our case this is already done

$$\mathcal{D} = \text{one component of } \{w \in \mathbb{P}(II_{2,18} \otimes \mathbb{C}) | w^2 = 0, w \cdot \bar{w} > 0\} \subset \{w | w^2 = 0\} = \hat{\mathcal{D}}$$

We remark for further reference that $\hat{\mathcal{D}}$ is the Grassmanian of positive definite oriented planes in $\mathbb{R}^{20}$. Denote the closure of $\mathcal{D}$ in $\hat{\mathcal{D}}$ as $\overline{\mathcal{D}}$. Note that $\overline{\mathcal{D}} \setminus \mathcal{D}$ is exactly the set of complex lines whose real and imaginary parts span a isotropic subspace of $II_{2,18} \otimes \mathbb{R}$. The *boundary components* of $\overline{\mathcal{D}}$ are simply the locally closed analytic subsets of $\overline{\mathcal{D}} \setminus \mathcal{D}$, and the *rational boundary components* are the boundary components defined over $\mathbb{Q}$. In general the boundary components correspond to the sets stabilized by parabolic subgroups of $\Gamma(\mathbb{R})$, and the rational boundary components to those stabilized by parabolic subgroups of Γ. In our case then we see that rational boundary components are contained in the $\mathbb{C}$ spans of rational isotropic subspaces of $II_{2,18} \otimes \mathbb{R}$. An isotropic vector corresponds to a point and a isotropic plane corresponds to a copy of the upper half plane. Write $\mathcal{D}^*$ as the union of $\mathcal{D}$ and the rational boundary components.

Let $\Gamma' \subset \Gamma$ be a neat[1] finite index subgroup (such things always exist). $\mathcal{D}^*/\Gamma'$ is then a quasi-projective variety, the homogeneous coordinate ring of which is generated by the so-called "automorphic" forms for Γ, which are the sections of a line bundle $\mathbb{L}^{BB}$. $\mathbb{L}^{BB}$ is defined to be the quotient of the tautological line on D^* (equipped with the appropriate topology near the cusps) by the action of Γ'.

The images of the rational boundary components in $\overline{\mathcal{F}}_{\mathrm{ell}}^{\mathrm{BB}}$ are called *cusps*. They are in bijection with Γ orbits of isotropic subspaces in $II_{2,18}$. Specifically:

Proposition 8.1.1. $\overline{\mathcal{F}}_{\mathrm{ell}}^{\mathrm{BB}}$ *has 3 cusps:*

- *A one dimensional cusp corresponding to the orbit of rational isotropic planes L with $L^\perp/L \simeq E_8 \oplus E_8$*
- *A one dimensional cusp corresponding to the orbit of rational isotropic planes L with $L^\perp/L \simeq D_{16}^+$*
- *A zero dimensional cusp corresponding to the unique orbit of a rational isotropic vector.*

Equivalently, Γ has three conjugacy classes of parabolic subgroups, corresponding to the stabilizers of the above subspaces.

Proof. The claims on the one dimensional cusps follow from Vinberg theory (5.2.19).

If $v^2 = 0$ then there exists u with $u \cdot v = 1$. $\langle v, u\rangle \simeq H$ is unimodular so $II_{2,18} = \langle v, u\rangle \oplus II_{1,17}$. Thus any isotropic vector is conjugate to a standard one, proving the last claim. □

[1]A subgroup is *neat* if the eigenvalues of the elements generate a torsion free subgroup of $\mathbb{C}^*$.

We will refer to the one dimensional cusps/boundary components as “type II” and the zero dimensional ones as “type III”, respectively.

8.2 The Toroidal Construction

The idea of the toroidal construction is to model a analytic neighborhood of each cusp as a quotient of a subset of a torus. One can thus attempt to build a compactification modeled by a (usually not of finite type) toric variety near each cusp. The canonical source for this material is [AMRT10].

Let $N(F)$ be the parabolic subgroup stabilizing a rational boundary component F, let $W(F)$ be the unipotent radical of $N(F)$ and let $U(F) = Z(W(F))$ be its center, which can be identified with a (real) vector space. We choose coordinates (z, w, τ) embedding $\mathcal{D} \hookrightarrow (U(F) \otimes \mathbb{C}) \times \mathbb{C}^m \times F$. $U(F)$ acts by real translation in the z coordinate and the fibers of the projection to (w, τ) are translations of $U(F) \times i\sigma_F$ where $\sigma_F \subset U(F)$ is some self-dual cone. Such a parameterization is called a *Siegel domain*[2]. In the special case where $m = 0$ $U(F)$ acts by translation and it is called a *tube domain*.

The quotient $\mathcal{D}/U(F)$ is a bounded open subset of the toric variety $TV(U(F) \cap \Gamma)$. The one parameter subgroups in $TV(U(F) \cap \Gamma)$ which limit to the cusp in $\overline{\mathcal{F}}_{\mathrm{ell}}^{\mathrm{BB}}$ are given by the cone $\sigma_F \in U(F)$. By symmetry $N(F)$ acts on σ_F and so a toric compactification of $\mathcal{D}/U(F)$ near the cusp is given by a complete fan Σ_F subdividing σ_F. We need to impose some obvious conditions to make the corresponding partial compactification of $\mathcal{F}_{\mathrm{ell}}$ exist:

- $N(F) \cap \Gamma$ acts on Σ_F.
- The stabilizer $\mathrm{Stab}(\sigma \in \Sigma_F)$ of each cone $\sigma \in \Sigma_F$ is finite.
- There are a finite number of $N(F)$ orbits in Σ_F.

[2]The precise definition is more specific, but unimportant to our case.

- The cones in Σ_F are rational polyhedral cones.

Fans of this type are called *admissible*. In general there is a compatibility condition among the fans, where Σ_F is determined by $\Sigma_{F'}$ for any $F' \subset \overline{F}$, but this does not come into play in our case, as there will turn out to be only one choice of fan for any one dimensional cusp. In this case $\overline{\mathcal{F}}_{\text{ell}}^{\Sigma}$ is determined by the fan associated to the zero dimensional cusp, which we simply call Σ.

The local compactifications then glue to a global compactification ${\mathcal{F}_{\text{ell}}}^{\Sigma}$ which is a complete projective variety with at worst toric quotient singularities.

8.3 Explicit description of $\overline{\mathcal{F}}_{\text{ell}}^{\Sigma}$

This section describes the construction of the Siegel domains near one dimensional cusps of $\overline{\mathcal{F}}_{\text{ell}}^{\text{BB}}$ more explicitly. The construction around one dimensional cusps is independent of Σ, and the corresponding boundary strata of $\overline{\mathcal{F}}_{\text{ell}}^{\Sigma}$ are described. In the type III case a description is given in terms of Σ. This material is largely adapted from Kondo [Kon93].

8.3.1 Type II Cusps

Let F be a type II boundary component associated with a conjugacy class of isotropic plane in $L \in II_{2,18}$. $L_K = F^{\perp}/F$ is a rank 16 even unimodular negative definite lattice with quadratic form given by some matrix K. Since L_K is unimodular it is a direct summand of $II_{2,18}$. Concretely one may write $II_{2,18} = L_K \oplus H^2$. WLOG assume that the first coordinates of the hyperbolic summands generate F. Explicity, then, we can choose a basis such that the quadratic form of $II_{2,18}$ is given by the matrix $\left(\begin{smallmatrix} 0 & 0 & I \\ 0 & K & 0 \\ I & 0 & 0 \end{smallmatrix}\right)$. Note that with respect to this basis a choice of component of $V(w^2 = 0)$ amounts to choosing an orientation of the real and imaginary parts of the last 2 coordinates.

Using this coordinate system we write down the matrix forms for the parabolic subgroup $N(F)$, its unipotent radical $W(F)$, and the center of the unipotent radical $U(F) = Z(W(F))$.

$$N(F) = \left\{ \begin{pmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{pmatrix} \right\}$$

such that

$$\begin{aligned} &U^t Z = I \\ &X^t K X = K \\ &X^t K Y + V^t Z = 0 \\ &Z^t W + W^t Z + Y^t K Y = 0 \\ &\det U > 0 \end{aligned}$$

since any isomorphism stabilizing F must stabilize the flag $F \subset F^\perp \subset II_{1,17}$. The last condition restricts to the subgroup preserving $\mathcal{D}_-$. The remaining conditions are the definition of orthogonality. The block diagonal Levi subgroup is simply $\mathrm{O}(L_K) \times \mathrm{SL}(2, \mathbb{Z})$. Then:

$$W(F) = \left\{ \begin{pmatrix} I & V & W \\ 0 & I & Y \\ 0 & 0 & I \end{pmatrix} \middle| KY + V^t = W + W^t + Y^t K Y = 0 \right\}$$

and:

$$U(F) = \left\{ \begin{pmatrix} I & 0 & W \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \middle| W + W^t = 0 \right\}$$

We choose affine coordinates (t_1, w, t_{19}) for $\mathcal{D}$ (with $w \in L_K \otimes \mathbb{C}$) by homogenizing with respect to the coordinate (t_{20}) and noting that t_2 is uniquely determined. Write $z = t_1, \tau = t_{19}$, and notice $\tau \in H^+$. These coordinates express $\mathcal{D}$ as a Siegel domain. The open condition can be written $2\Im z \Im \tau + \Im w^t K \Im w > 0$. The cone $\sigma_F \in \mathbb{R}$ is then simply R^+, and cannot be further subdivided. We identify $U(F)$ with $\mathbb{R}$ using $a \mapsto \begin{smallmatrix} 1 & 0 & W_a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}$, where $W_a = \begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}$ then we see that $a \cdot (z, w, \tau) = (z + a, w, \tau)$.

We proceed to discuss the quotient $\mathcal{D}/N(F)$ in a neighborhood of F. First consider $\mathcal{D}/(U(F) \cap \Gamma)$. This is a trivial $\mathbb{C}^*$ bundle over $\mathcal{D}/U(F)$:

$$\mathcal{D}/(U(F) \cap \Gamma) = \Delta^* \times L_K \otimes \mathbb{C} \times H^+$$

We fill in the puncture (i.e. produce the partial compactification corresponding to the to cone σ_F) to get:

$$(D/(U(F))_{\sigma_F} \cap \Gamma) = \Delta^* \times L_K \otimes \mathbb{C} \times H^+$$

Consider now the action of $W(F) \cap \Gamma$. An element of $W(F)/U(F)$ is entirely determined by Y (observe the matrix above) and acts on $(D/(U(F))_{\sigma_F}$ by translation in the second coordinate:$w \mapsto w + Y[\begin{smallmatrix} \tau \\ 1 \end{smallmatrix}]$. The quotient is then a trivial Δ fibration over $E \otimes_{\mathbb{Z}} L_K \times H^+$ where E is the elliptic curve $\mathbb{C}/\langle 1, \tau \rangle$. One thinks of this $\Delta \times \mathcal{E} \otimes_{\mathbb{Z}} L_K$ where $\mathcal{E} \to H^+$ is the usual elliptic curve with universal level structure.

Finally we discuss the action of the block diagonal Levi subgroup. If we write $Z = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in \mathrm{SL}(2, \mathbb{Z})$ the action of (X, Z) is given by

$$\begin{aligned} (\overline{z}, \overline{w}, \tau) \mapsto & (\overline{\frac{dz + c(z\tau + w^T K w)}{c\tau + d}}, X\overline{\frac{w}{c\tau + d}}, (a\tau + b)/(c\tau + d)) \\ &= (\overline{z + \frac{cw^T K w}{c\tau + d}}, X\overline{\frac{w}{c\tau + d}}, (a\tau + b)/(c\tau + d)) \end{aligned}$$

.

That is to say (Z, X) acts on $\mathcal{E} \otimes_{\mathbb{Z}} L_K$ as $Z \otimes X$. (We do not calculate the effect on the first coordinate.)

To summarize:

Theorem 8.3.1. *Let C be a 1 dimensional cusp of $\overline{\mathcal{F}}_{\mathrm{ell}}^{\mathrm{BB}}$ and $\pi : \overline{\mathcal{F}}_{\mathrm{ell}}^{\Sigma} \to \overline{\mathcal{F}}_{\mathrm{ell}}^{\mathrm{BB}}$ be any toroidal compactification.*

- *C is isomorphic to the j line $H^+/\operatorname{PSL}(2, \mathbb{Z})$*
- *$\pi^{-1}(C) = (\mathcal{E} \otimes_{\mathbb{Z}} L)/\mathrm{O}(L) \times \operatorname{PSL}(2, \mathbb{Z})$*

Where $L = E_8 \oplus E_8$ or $L = D_{16}^+$ is the lattice associated with the cusp C.

8.3.2 Type III Cusps

The situation over type III points is in some ways easier, insofar as we work with a tube domain (type I) rather than a type III Siegel domain, and harder, insofar as it involves the choice of fan in a nontrivial way. We first give explicit coordinates exhibiting $\mathcal{D}$ as a tube domain, again closely following Kondo (in principle, some of the expressions will look slightly different!). Write $II_{2,18} = II_{1,17} \oplus H$, where for similarity to the previous case we consider $II_{1,17}$ to have quadratic form induced by the matrix K, and we will let e be our chosen isotropic vector. Using these coordinates, we can write $\mathcal{D}$ as one component of $\{(w, z_{19}, z_{20}) | w^t K w + 2 z_{19} z_{20} = 0, \Im w^t K \Im w + \Im z_{19} \Im z_{20} > 0\}$ Note that $z_{20} \neq 0$[3], so we homogenize with respect to z_{20}. Note that z_{19} is now determined by $z_{19} = -w_t K w/2$. The 2 components differ according to the sign of $\Im z_{19}$. Choose $\Im z_{19} > 0$. We can write $\mathcal{D} = \{w \in L_K \otimes \mathbb{C} | \Im w^t K \Im w > 0\} = \mathbb{R}^{18} + iC$, where C is the forward light cone of $L_K \otimes \mathbb{R}$.

[3]Indeed, if not then the conditions reduce to $\Re w^t K \Re w - \Im w^t K \Im w = 2 \Re w^t K \Im w = 0$ and $\Im w > 0$, which is clearly impossible by the signature of K

Now, we have:

$$N(F) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\}$$

with:

$$\begin{aligned} A^t K A &= K \\ b_2^t K A + d_{22} c_1 &= 0 \\ b_1 = c_2 &= 0 \\ d_{21} &= 0 \\ b_2^t K b^2 + 2 d_{22} d_{12} &= 0 \\ d_{11} d_{22} &= 1 \end{aligned}$$

where d_{ij} are matrix entries of D, and b_i, c_i^t are the columns of B, C^t, respectively.

$$W(F) = U(F) = \left\{ \begin{pmatrix} I & 0 & b_2 \\ c_1 & 1 & d_{12} \\ 0 & 0 & 1 \end{pmatrix} \middle| K b_2 + c_1 = 0, b_1^t K b_1 + 2 d_{12} = 0 \right\}$$

Observe that the action of $N(F)$ on $\mathcal{D} \subset L_K \otimes \mathbb{C}$ is given by

$$\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \cdot w = \frac{1}{d_{22}} (A w + b_1).$$

$\mathcal{D}/(U(F) \cap \Gamma)$ is a bounded neighborhood of the origin in $(\mathbb{C}^*)^{18} \simeq L_K \otimes_{\mathbb{Z}} \mathbb{C}^*$, and we partially compactify this latter torus using the fan Σ, prior to dividing by $(N(F) \cap \Gamma)/(U(F) \cap \Gamma)$. That is, we construct a toric variety $(\mathbb{C}^*)^{18} \hookrightarrow T_\sigma$ for each cone σ in the fan, and glue them as indicated.

Part II

A Modular Compactification

Chapter 9

KSBA Stable Limits

Now that the background material has been introduced, it is reasonable to take some time to remember the goals of this work. Recall (4.3) that there is a compact moduli space of stable pairs $(X, \epsilon B)$, where (X, L) is a K3 surface with an ample line bundle L of degree $2d$ and $B \in |L|$. By choosing $B = \sum C_i$, where C_i are all the rational curves in $|L|$, we can embed F_{2d}, the moduli space of $2d$ polarized K3's, into the space of pairs, and by taking the closure in the space of stable pairs produce a compact moduli space of polarized K3 surfaces, which we call $\overline{F_{2d}^{RC}}$ (for the *rational curves* composing the divisor).

The description of $\overline{F_{2d}^{RC}}$ itself is a problem we won't address here. However one notes that $\mathcal{F}_{\text{ell}}$ embeds as a divisor in each F_{2d} by letting $L = s + (d+1)f$. Thus as a first step one may describe the closure of $\mathcal{F}_{\text{ell}} \subset F_{2d}$ in $\overline{F_{2d}^{RC}}$.

By the work of Bryant and Lueng [BL00] we know that the rational curves in $|L|$ are exactly the curves $s + \sum_1^{d+1} f_i$, where f_i are singular fibers. Hence the pairs we are considering can be written more explicitly as $(X, \epsilon B), B = N_{2d}(s + \frac{d+1}{24} \sum_{i=1}^{24} f_i)$, where N_{2d} is the total number of such curves. For large d we can then change notation and consider pairs (X, B) with

$$B = \epsilon \sum_{i=1}^{24} f_i + \delta s, \quad \epsilon >> \delta.$$

This is justified by the fact that as long as d is reasonably large the corresponding moduli spaces are identical, a corollary to the main theorem which we record below:

Corollary 9.0.2. *Let* $\overline{\mathcal{X}} \to \Delta$ *be a (1 parameter) family with generic point a smooth K3. For* $d_1, d_2 >> 0$ *write the corresponding (uniquely determined) divisors* B_1, B_2*. Then* $(\overline{\mathcal{X}}, B_1)$ *is stable if and only if* $(\overline{\mathcal{X}}, B_2)$ *is.*

Hence we define our main object of study:

Definition 9.0.3. The compact moduli space of elliptic K3 surfaces obtained by embedding $\mathcal{F}_{\text{ell}} \to F_{2d}$ by $L = s + (d+1)f$ is independent of d for large d, as is the corresponding universal family. We call this space $\overline{\mathcal{F}_{\text{ell}}}$.

Goal 9.0.4. Explicitly describe $\overline{\mathcal{F}}_{\text{ell}}$ and the corresponding universal family.

As a first step in this chapter we describe the possible stable limits parameterized by $\overline{\mathcal{F}_{\text{ell}}}$. Specifically, given a one parameter family $\mathcal{X}/\Delta^\circ$ with generic fiber an elliptic K3 surface we will first explicitly show how to complete $\mathcal{X}$ (perhaps after base change) to a stable family. By the general theory the resulting family $\overline{\mathcal{X}}$ is unique up to base change. We then elaborate on this description. We discuss the singularities in the stable model of a degeneration, in the process deriving a formula for the number of triple points in the central fiber of a Kulikov model. Finally we elaborate on the types of components that may occur in a stable limit.

9.1 Description of Limit Pairs

The construction is straightforward and relies on reducing the 2-dimensional problem to a manipulation of data supported on curves.

Recall from the chapter on elliptic surfaces 6.1.5 that an elliptic surface over a curve C is determined by the Weierstrass data $L \in \text{Pic}(C), A \in H^0(4L), B \in H^0(6L)$. We will additionally keep track of the discriminant Δ. As usual, we abuse notation where convenient by using the same symbols to refer to the corresponding divisors.

The next task would be to describe the stability condition 4.3.3 in terms of the Weierstrass data. Fortunately, we already did the necessary work to translate the condition on singularities (6.1.8), and the numerical condition is immediate. The result is:

Lemma 9.1.1. *Let $\pi : X \to \mathbb{P}^1$ be a normal Weierstrass fibration with corresponding Weierstrass data L, A, B. Assume X is of K3 or rational type, i.e. $L = \mathcal{O}(2)$ or $L = \mathcal{O}(1)$. Denote by f_i the singular fibers and choose some other special fibers $F = \sum_i F_i$.*

The pair $(X, F + \delta s + \epsilon \sum_{f_i \notin F} f_i)$ is stable if and only if

- *The divisor $\pi_* F$ is reduced and each point in $\pi_* F$ is contained in at most one of A, B.*
- *Either $\deg A|p < 4$ or $\deg B|p < 6$ at every point $p \in \mathbb{P}^1$*
- *If $L = \mathcal{O}(1)$ then there is at least one special fiber (F is nonempty).*

Similarly if $\pi : X \to \mathbb{P}^1$ is a non-normal Weierstrass fibration with $L = \mathcal{O}(n)$ the pair $(X, B) = (X, F + \delta s + \epsilon \sum_{f_i \notin F} f_i)$ (for some fibers f_i) is stable if and only if the sums of the coefficients of the fibers in the divisor exceeds $i - 2$.

Proof. The first two items give the condition to have log canonical singularities. This statement is simply 6.1.8. Note that if X has log canonical singularities then it must have at least 2 singular fibers, since otherwise A and B , if nonzero, could only vanish at the image of the singular fiber.

The last condition is the numerical condition. Recall $K_X = 0$ if $L = \mathcal{O}(2)$, and $K_X = -f$ if $L = \mathcal{O}(1)$. In either case the last condition guarantees that the class $K_X + F + \delta s + \epsilon \sum_{f_i \notin F} f_i)$ is ample.

In the non-normal case the condition on singularities is simply that no divisor appears in B with coefficient > 1. The normalization of X is the ruled surface $\mathbb{F}_i$ and the conductor

D is a bisection. Since any fiber f has $f \cdot B = 2 + \delta$ and $f \cdot K_X = 0$ we only need to check that $s \cdot B + K_X > 0$. But $s \cdot K_x = i - 2$, $\epsilon >> \delta$, and $s \cdot D = 0$ so the result follows. □

We will now degenerate the Weierstrass data to produce stable limits.

Remark 9.1.2. Morally, we're computing a "stable" map of Deligne-Mumford stacks to the moduli stack of elliptic curves. This is probably a slight generalization of Abramovich and Vistoli's notion of twisted stable maps[AV00], since we allow small weights.

We first informally describe the construction of limits, with a formal statement and proof after. Given a family $\mathcal{X}/\Delta^\circ$ we have an associated family of Weierstrass data and of j maps (the former notion we will avoid giving a formal definition of, for now). Recall that there is a complete moduli space $\bar{M}_{0,24\epsilon}(\mathbb{P}^1, 24)$ of stable pointed maps $(j : \mathbb{C} \to \mathbb{P}_1, p_i)$, where C is normal crossing, $p_a(C) = 0$ and $\deg j = 24$, with $p_i \in C^\circ$ being 24 ordered marked points taken with weight ϵ. Now a general elliptic K3 determines, by its j map, a point in $M_{0,24}$ satisfying:

1. For each component of $j^{-1}(0)$ (resp. $j^{-1}(1)$) there is a deleted neighborhood basis (in the analytic topology) where j has local degree 3 (resp. 2).

2. $p_i \in j^{-1}(\infty)$ for all i

These properties are clearly maintained in the closure of the image of $\mathcal{F}_{\text{ell}}{}^\circ$ which will be called the *Kontsevich compactification* M^K [1]. The closures of the divisors A, B, Δ on $\mathcal{C}$ give divisors on the limit curve C_0. We proceed to modify the map $C_0 \to \mathbb{P}^1_j$ to produce Weierstrass data for the limit stable pair.

Definition 9.1.3. If C is a curve with at most nodal singularities and $p_a(C) = 0$, a *branch* B of C is a union of components such that both B and B^c are connected.

[1] The definition of Looijenga-Heckmann differs in that they do not consider the discriminant to have small weight.

To produce the stable limit, contract all branches B of C_0 with total degree $\deg B < 12$ to get a map $j : \widetilde{C}_0 \to \mathbb{P}^1$ and let $\widetilde{A}, \widetilde{B}, \widetilde{\Delta}$ be the images of the corresponding divisors in C_0. The result is Weierstrass data giving a stable model of the degeneration.

Formally:

Theorem 9.1.4. *Stable limit pairs (X, B) are of one of the following forms.*

- *X is an elliptic K3 (with ADE singularities), $B = \epsilon \sum f_i + \delta s$.*
- *X is the double cover of $\mathbb{F}_4$, branched over the divisor $s + s_1 + 2s_2$, where the s_i intersect transversely. There are four cuspidal fibers c_i, and $B = \epsilon(2\sum c_i + \sum_{j=1}^{16} f_j) + \delta s$, where f_j are some additional fibers.*
- *X is a double cover of of a ruled chain $F \simeq \mathbb{F}_2 \cup \mathbb{F}_0 \ldots \mathbb{F}_0 \cup \mathbb{F}_2$ branched over $s + T$, where $T|_{\mathbb{F}_2} \simeq 3s + 6f$ is reduced and $T|_{\mathbb{F}_0} = s_1 + 2s_2, s_i \simeq s$. In this case $B = \epsilon(\sum f_i + \sum g_j) + \delta s$, where the f_i are the singular fibers of the components at the end of the chain, and the g_i are fibers in the middle components, with the requirement that each component contains at least one of the g_i.*
- *Similar to the above, but $T|_{\mathbb{F}_2} = s_1 + 2s_2$ is non-reduced at one or both ends of the chain and s_1, s_2 intersect transversely. The marked fibers are then twice each cuspidal fiber and some additional fibers, for a total of 24 in the entire surface.*

Recall that we do not yet make any claim as to whether an arbitrary pair of one of these forms (other than actual K3 surfaces) is in fact smoothable.

Proof. The figure 9.1 gives a schematic representation of the construction.

Start with Weierstrass data corresponding to a family of elliptic K3 surfaces with ADE singularities on $\mathcal{Y} = \mathbb{P}^1_{\Delta^o}$, so $A \in H^0(O(8)), B \in H^0(O(12))$. By base changing (marked "base change" in the figure) we can assume $A = A_1 + A_2 \cdots + A_8, B = B_1 + B_2 \cdots + B_{12}$, $\Delta = \Delta_1 + \cdots + \Delta_{24}$. We base change again and blow up $\mathbb{P}^1_\Delta$ in the central fiber Y_0 such that:

- The new central fiber Y_0' is reduced.

- The closures $\overline{A}, \overline{B}, \overline{\Delta}$ of the divisors A, B, Δ are in $Y_0'^{\circ}$, the smooth locus of Y_0'.

- For any pair D_1, D_2 where $D_i \in A_j, B_j, \Delta_j$ $\overline{D}_1 \cdot Y_0 = \overline{D}_2 \cdot Y_0$ only if $D_1 = D_2$.

Note that if the original family was semistable, i.e. the divisors A and B were disjoint, this would resolve the indeterminacy of the j map.

The dual graph of the central fiber Y_0' is now a tree, where the leaves are exceptional curves of the first type. Moreover for any $p \in Y_0'$ we have either $v_p(A) < 4$ or $v_p(B) < 6$, since this condition holds generically. We now proceed to iteratively contract any leaf C where $\bar{A}C < 4$ or $\bar{B}C < 6$. This is marked "contract" in the figure. Call this new smooth family $\bar{\mathcal{Y}}$.

By construction, $\bar{Y}_0$ is a tree where each leaf C has $AC \geq 4$ and $BC \geq 6$, so it in fact takes the form of a chain of smooth rational curves. If Y_0 has more than one component, A and B have degrees 4 and 6, respectively, on both end components, and 0 otherwise. Define L to be the unique line bundle restricting to $\mathcal{O}(2)$ on the generic fiber that has degree 1 on the end components, and 0 otherwise. If there is only one component, A and B have degrees 8 and 12, respectively, and we define $L = \mathrm{O}_{\mathcal{Y}}(2)$. In either case, L, $\bar{A}$, and $\bar{B}$ give Weierstrass data on $\mathcal{Y}$, and so an elliptic surface $\pi : \mathcal{X} \to \bar{\mathcal{Y}}$. Moreover $\bar{\Delta}$ is contained in the discriminant locus of π. We write $B = \epsilon\pi^*\Delta + \delta s$. Note that $(A + B + \Delta)|_{Y_0}$ still has smooth support, since we only contracted leaves.

We note that for any point $p \in Y_0$ either p has multiplicity less than 4 in $A|_{Y_0}$ or has multiplicity less than 6 in $B|_{Y_0}$, since if p failed both conditions it would have been produced by contracting a leaf C that already had $AC \geq 4$ and $BC \geq 6$. Noting that the divisors A and B contain no components of Y_0, we see that the surface X_0 has at worst log terminal singularities on each component (see lemma 9.1.1).

It remains to show that the pair (X_0, B) is slc along the double locus in each component. Again, we note by construction that the double locus does not contain any of the marked fibers f_i, and that it has at worst A_n singularities (since the corresponding points on Y_0 are disjoint from $A+B$). Moreover these A_n singularities can be resolved by blowing up in such a way as pullback of the double locus on that component is a cycle of -2 curves, hence the pair is log canonical.

Finally, we apply MMP to produce a stable model from the slc model we now have. The criterion for stability of a component (see lemma 9.1.1) is that it contains at least one marked fiber. This is automatic for leaves, but may not be the case for interior components, in which case MMP corresponds to a series of divisorial contractions (i.e. blowdowns).

The remaining claim of the theorem is that for the non-normal components of X_0, each cuspidal fiber occurs at least twice in the list of marked fibers. This follows from direct calculation. Indeed, working in a formal neighborhood in $\mathcal{X}$ of a cuspidal fiber of X_0 we have the trisection of $\mathcal{Y}$ being given by a polynomial $x^3 - sx^2 + t^n(a_3x^3 + a_2x^2 + a_1x^1 + a_0) \in \mathbb{C}[[s,t]][x]$, where t is the parameter of the degeneration and s is a parameter on the base curve. We can change variables so that one of $a_0, a_1, a_2 \neq 0$. Recalling that the discriminant of a general cubic is given by

$$\Delta(ax^3 + bx^2 + cx + d) = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd$$

we see that on the generic fiber $t \neq 0$ Δ has degree at least 2 in s. $\square$

9.2 Singularities of Stable Models

From the construction we can now rapidly describe the singularities of a stable model. Indeed, let $\mathcal{Y}$ be the base curve of the fibration. The central fiber is produced by contracting some set of -2 curves in a chain of smooth rational curves on a smooth model, so $\mathcal{Y}$ has type A singularities. Write $Y_0 = f_0 \cup f_1 \cup \cdots \cup f_n$, where intersections are given by:

$$\begin{aligned} f_{i-1}f_i &= \frac{1}{c_i} \\ f_i^2 &= -\frac{1}{c_i} - \frac{1}{c_i+1} \\ f_if_j &= 0 && \text{if } |i-j|>1 \end{aligned}$$

Define $d_i = \widetilde{\Delta} f_i$, where $\widetilde{\Delta}$ is the closure of the discriminant on the generic fiber. We wish to find a_i such that $\Delta = \widetilde{\Delta} + \sum_{i=1}^{n-1} a_i f_i$. Since $\Delta \in H^0(L^{12})$:

$$\Delta f_i = \begin{cases} 12 \text{ if } i = 0, n \\ 0 \text{ otherwise} \end{cases}$$

which gives the relations:

$$d_i = \frac{a_i - a_{i-1}}{c_i} - \frac{a_{i+1} - a_i}{c_{i+1}} \text{ if } 0 < i < n$$

$$12 - d_0 = \frac{a_1 - a_0}{c_1} \text{ if } i = 0$$

$$12 - d_n = \frac{a_{n-1} - a_n}{c_n} \text{ if } i = 0$$

We see then that the values of c_i, d_i give affine conditions uniquely determining a_i, and that everything is determined by a_0 and the c_i. One can think of this concretely by resolving the singularities of $\mathcal{Y}$ to produce a semistable family with central fiber $Y_0' = f_0' \cup \cdots \cup f_N'$, and observing that (for this family) the d_n are determined by the change of slope of the function $(1 \ldots N) \to \mathbb{N} | i \mapsto a_i$ at the point n, with the initial and final slopes being $12 - d_0$ and $d_N - 12$, respectively. The c_i are the lengths of sections of constant slope.

The singularities in codimension 2 of the stable model are then given by type A_{c_i-1} over the double fibers and type A_{a_i-1} along the double locus of each non-normal component.

We now can build on this to give a description of the number of triple points associated to an arbitrary KSBA stable degeneration based on the singularities of the threefold.

Lemma 9.2.1. *Let $\mathcal{X}$ is a degeneration with singularities of type A_{c_i-1} along the fibers in the double locus of the central fiber, and of type A_{a_i-1} along the double locus of each non-normal component of X_0, the number of triple points of the degeneration is given by $\sum c_i(a_{i-1}+a_i)$, where we formally consider a normal component of X_0 to have $a_i = 0$.*

Proof. The degeneration is locally toric, where at a triple point of X_0 the degeneration can be given using Mumford's construction where the slopes of the piecewise affine function the components are $(0,0), (c_i, 0), (0, a_{i-1}), (c_i, a_i)$. A toric resolution of this degeneration then corresponds to a triangulation of the lattice polygon defined by these slopes, with each triangle corresponding to a triple point. The number of triangles obtained is twice the area, so the result follows. □

For further use we introduce coordinates to write the number of triple points as a quadratic form in the algebraically defined combinatorics of the degeneration.

Lemma 9.2.2. *Let $\mathcal{X}$ be a maximal degeneration with X_0 having type $E_0 \oplus A_0^{18} \oplus E_0$. Let the singularities along the fibers in the double locus be of types A_{a_i-1}, where $i = -9 \ldots 9$ (this numbering makes the statement simpler). Then we have $\sum ia_i = 0$ and the number of triple points is given by the quadratic form $\langle \sum a_i e_i, \sum a_i e_i \rangle$ induced by the bilinear form with $\langle e_i, e_j \rangle = -\min(i,j)$, restricted to the sublattice $0 = \sum ia_i$.*

Proof. Still using notation from the previous section (excepting the different index convention) we have $d_{-10} = d_9 = 3$, since a rational elliptic surface with an A_8 singularity has 3 other singular fibers, and $c_{-10} = c_9 = 0$, since type E surfaces have surjective j map. We see that $c_j = \sum_{i \le j} -ia_i$ (hence the condition $0 = \sum ia_i$). The number of triple points is $\sum (c_{i-1}+c_i)a_i$, and upon expanding we observe that the monomial $a_i a_j$ occurs with coefficient -2 $\min(i,j)$ if $i \neq j$, and $-i = -j$ otherwise. □

The numerically admissible choices for a_i form a cone, which we can write explicitly:

Proposition 9.2.3. *The cone of a_i with $\sum ia_i = 0$ and $a_i, c_i \ge 0$ for the corresponding c_i is 19 sided, with walls*[2]

$$\beta_1 = 8e_{-9} - 9e_{-8}, \alpha_i = e_{i-1} - 2e_i + e_{i+1}, \beta_2 = -9e_8 + 8e_9$$

The intersection form of these walls is given by an A_{19} type diagram, but with $\beta_i^2 = -72$ and $\beta_1\alpha_{-8} = \beta_2\alpha_8 = 9$

Proof. Direct calculation. Note that the condition $c_i \ge 0$ is already implied by $a_i \ge 0$, and that $\{(a_i)_i | \sum ia_i = 0\}^\perp = \langle e_9 \rangle$, so simply write down the primitive vectors $x\hat{e}_i + ye_9 \in e_9^\perp$ pairing positively with some interior element of the cone. □

We state similar results for the other types of maximal degeneration:

Proposition 9.2.4. *As above, let $\mathcal{X}$ have X_0 of type $D_0 \oplus A_0^{16} \oplus D_0$. We have $c_{-9} - \sum ia_i = c_8$ and the additional parity conditions $2 \mid c_{-9}, 2 \mid c_8$. The number of triple points is given by the quadratic form $\langle c_{-9}f_{-9} + \sum a_ie_i + c_8f_8, c_{-9}f_{-9} + \sum a_ie_i + c_8f_8 \rangle$ induced by the bilinear form:*

$$\langle e_i, e_j \rangle = -\min(i,j)$$

$$\langle f_1, e_j \rangle = 1$$

$$\langle f_1, f_1 \rangle = 0$$

[2] Where w being a "wall" means that $w \cdot x \ge 0$ defines a facet of the cone.

(f_2 is isotropic).

The cone of (a_i) satisfying the above conditions and with $a_i \geq 0$ is 19 sided with walls:

$$\gamma_1 = -8f_1 + e_{-8}, \delta_1 = 2f_1 - 2e_{-8} + 2e_{-7}, \alpha_i = e_{i-1} - 2e_i + e_{i+1},$$
$$\delta_2 = 2e_7 - 2e_8 - 2f_2, \gamma_2 = e_8 + 8f_2$$

The intersection form is again given by an A_{19} diagram, with

$$\gamma_i^2 = -8, \delta_i^2 = -4, \gamma_i\delta_i = 2, \delta_1\alpha_{-7} = \delta_2\alpha_7 = 2.$$

Proof. We now have $d_{-9} = d_8 = 4$, so $c_j = d_{-9} + \sum_{i \leq j} -ia_i$. The relation on the variables is $c_8 = c_{-9} + \sum -ia_i$, and reading the coefficients of $\sum(c_{i-1} + c_i)a_i$ gives the claimed quadratic form. The claims on the cone again follow by writing down the dual vectors for each of e_i and f_i, and observing the intersection form. □

Similarly for the remaining case (proved identically):

Proposition 9.2.5. *The cone for type $E_0 \oplus A_0^{17} \oplus D_0$ has walls β_1, $\alpha_{-8} \ldots \alpha_7$, δ_2, γ_2, with notation as above.*

These cones are clear candidates for the monodromy cones in the fan for a toroidal description of $\overline{\mathcal{F}}_{\text{ell}}$. We provisionally call them $\mathcal{M}$ *cones.*

9.3 Components of Stable Pairs

More specifically, the possible components of a degenerate K3 are then

- (In the middle of a chain) A non-normal surface whose normalization is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, with double locus two $(0,1)$ curves, and the ruling being given by the $(1,0)$ curves. We call these A_n components.

- (At the end of a chain) A non-normal surface whose normalization is isomorphic to $\mathbb{F}_1$ with the double locus being a bisection. In this case the polarizing divisor contains each fiber tangent to the double locus twice, as well as n "extra" fibers. We call these D_n components.

- (At the end of a chain) A Weierstrass fibration corresponding to a rational elliptic surface. If there are $n+3$ singular fibers away from the attaching fibers we call this a type E_n component. There are two distinct families (see below) which we call E_1 and E_1'

- (At the end of a chain with 2 components) A Weierstrass fibration corresponding to a rational elliptic surface with nonsingular attaching fiber. We call this an $\widetilde{E}_8$ component.

- The surface is irreducible, with exactly one non-normal component. We call this type $\widetilde{D}_{16}$.

As notation, if a surface is a chain of surfaces $X = X_1 \cup X_2 \ldots$, where each component X_i is named after a lattice L_i, we will associate the surface to the lattice $L_1 \oplus L_2 \ldots$, where we will routinely abuse notation such that the order of the sum is meaningful, corresponding to the order of the in the chain.

Definition 9.3.1. We refer to any chain of components as above with 24 marked fibers as a *stable elliptic K3*.

For convenience, we explicitly describe the possible components of a type III degeneration as double covers of ruled surfaces.

A_n A double cover of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, with trisection being $T = s + 2s'$, for s, s' sections.

D_n A double cover of $\mathbb{F}_2$, with trisection $T = s + 2s'$.

E_n A double cover of $\mathbb{F}_2$, with trisection having intersecting the marked fiber twice at a point, where there is a A_{8-n} singularity oriented transversely to the marked fiber.

By blowing up at the A_n singularity of an E_n surface, and blowing down the resulting -1 curve one can arrive at an equivalent model that is easier to deal with. We record the resulting constructions below.

E_8 A double cover of $\mathbb{F}_2$ with trisection simply tangent to the marked fiber.

E_7, E_6 Start with $\mathbb{P}^2$ and a quartic f_4. Blow up at a smooth point on the quartic that is not a flex, and blow up again at the intersection of the exceptional divisor and strict pre-image of f_4. Blowing down the new exceptional curve results in a ruled surface of type $\mathbb{F}_2$ with trisection the strict transform of f_4 and marked fiber the strict transform of the exceptional divisor of the second blowup. This is Weierstrass data for a type E_7 component except when the initial point lies on a bitangent, in which case it is an E_6 component.

E_5, E_4 Similarly, start with $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and choose a $3,2$ curve $f_{3,2}$. Blowing up at a point of tangency to a $(1,0)$ curve[3] further blow up twice along the intersections of the strict pre-image of $f_{3,2}$ and the exceptional divisors, and then blow down twice to arrive at $\mathbb{F}_2$ with a trisection. The result is Weierstrass data for a component of type E_5, unless the $(0,1)$ curve through the initial point was tangent to $f_{3,2}$, in which case we obtain an E_4 surface.

E_3, E_2, E_1, E_1', E_0 Similarly, start with $\mathbb{P}^2$ and an irreducible (not necessarily nonsingular) cubic f_3 with one flex point p_0 distinguished. Blow up at a point p_1 not on the line tangent to p_0 and not a double point of f_3 and write the pullback of f_3 as f_3'. Further blow up three times along the strict preimage of f_3 at the chosen flex point, and blow

[3]There are generically 8 choices for the location of the first blowup. Indeed, by adjunction $p_a(f_{3,2}) = 2$, and so the 3:1 projection to a $(0,1)$ curve is ramified at 8 points.

down 3 times. The result is an $\mathbb{F}_2$ with trisection given by the strict preimage of f_3'. The Weierstrass data for all type $E_n, n \leq 3$ surfaces arises this way, as follows:

E_3 The line $\overline{p_0 p_1}$ is transverse to f_3.

E_2 $\overline{p_0 p_1}$ is tangent to f_3 at a smooth point and $p_1 \notin f_3$.

E_1 $\overline{p_0 p_1}$ is tangent to f_3 at p_1.

E_1' $\overline{p_0 p_1}$ meets f_3 at a node and $p_1 \notin f_3$.

E_0 $\overline{p_0 p_1}$ passes through the cusp of f_3.

One can now easily write down a parameter space for type E_n components (including type E_1'). Be warned that this result is in some sense weaker (in that it doesn't reveal the orbifold structure) than we will prove later (10.1.1). It is complementary, though, in that it provides explicit equations for the surfaces as a double cover of a ruled surface.

Theorem 9.3.2. *The coarse moduli spaces for type E_n components are as follows:*

E_8 $\mathbb{A}^8$

E_7 $\mathbb{A}^7/\mathbb{Z}_2$ *where* $\mathbb{Z}_2$ *acts multiplicatively with weights* $(0,0,0,0,1,1,1)$.

E_6 $\mathbb{A}^6/\mathbb{Z}_3$ *where* $\mathbb{Z}_3$ *acts multiplicatively with weights* $(0,0,1,1,2,2)$.

E_5 $\mathbb{A}^5/\mathbb{Z}_4$ *where* $\mathbb{Z}_4$ *acts multiplicatively with weights* $(0,1,2,2,3)$.

E_4 $\mathbb{A}^4/\mathbb{Z}_5$ *where* $\mathbb{Z}_5$ *acts multiplicatively with weights* $(1,2,3,4)$.

E_3 $\mathbb{A}^3/\mathbb{Z}_6$ *where* $\mathbb{Z}_6$ *acts multiplicatively with weights* $(2,3,4)$.

E_2 $\mathbb{A}^1 \times \mathbb{G}_m$

E_1 $\mathbb{G}_m$

E_1' $\mathbb{A}^1$

E_0 *Rigid.*

Before giving the proof, recall a fact from childhood will be used repeatedly.

Lemma 9.3.3. *If* $r^{n-i}s^i + a_{n-i-1}r^{n-i-1}s^i + \cdots + a_n s^n$ *is a polynomial on* $\mathbb{P}^1_{r,s}$ *then there is a change of variables* $r \mapsto r'$ *expressing the form as* $r^{n-i}s^i + a'_{n-i-2}r^{n-i-2} + \dots a'_n s^n$*. This is unique up to scaling the coordinate.*

Proof. Indeed, take $r \mapsto r - \frac{a_{n-i-1}}{n-i}$. □

The starting data for each of the constructions is a pair isomorphic to a toric variety and ample divisor. The proof of the theorem proceeds by rigidifying the models by choosing a toric structure and fixing the values of some coefficients of monomials in the equation determining the divisor, thus putting the equation into "standard form". The possible rigid pairs are then an affine space. In general there may be several ways to put a given pair into the standard form, related by the action of a finite group.

Proof. Write the divisor as $V(\sum a_{i,j}x^{i,j})$. The coeffecients $a_{i,j}$ of the rigid models for $E_8 \dots E_4$ are shown in the figure 9.2, with "*" denoting those $a_{i,j}$ that can vary arbitrarily. We explain the process in the first two cases, the remaining cases being similar.

In the E_8 case one starts with the Weierstrass form, which amounts to fixing a section of $\mathbb{F}_2^\circ$ and sets $a_{\bullet,1} = 0$. By choosing the special fiber to be in the toric boundary we get

$$4a_{0,2}^3 + 27a_{0,0}a_{0,3}^2 = 0$$

with $a, b \neq 0$. Applying the childhood lemma gives a unique choice for the other boundary fiber making $a_{1,2} = 0$. Note that $a_{1,3} \neq 0$, since the divisor is assumed to be nonsingular along the special fiber. We have now fixed the structure of a toric variety on $\mathbb{F}_2$. In particular the only remaining freedom is the action of T^2. Since the lattice points $(1,3), (0,3), (0,2)$ span the M lattice there is a unique $t \in T^2$ that makes $a_{1,3} = 1, a_{0,3} = 2, a_{0,2} = 3$. But then by 9.3 $a_{0,0} = 1$ as well. The coefficients corresponding to lattice points marked "*" can be chosen arbitrarily, giving A_8.

In the E_7 case one wishes to give $\mathbb{P}^2$ the structure of a toric variety such that the flag consisting of the given point p on the quartic f_4 and the line tangent to f_4 at that point is in the toric boundary. This amounts to saying $a_{0,0} = a_{1,0} = 0$. Since p was not a flex we have $a_{3,0} \neq 0$, so by the childhood lemma we can choose the other boundary line through p in such a way as to make $a_{1,1} = 0$. Finally since p was a smooth point of f_4 we must have $a_{0,1} \neq 0$, so by the childhood lemma we can choose the remaining boundary divisor such that $a_{0,2} = 0$. Since p was assumed not to lie on a bitangent one has $a_{4,0} \neq 0$. The lattice points $(0,1),(2,0),(4,0)$ span an index 2 sublattice $M' \subset M$, and so there there are 2 elements of the torus orbit of f_4 with $a_{0,1} = a_{2,0} = a_{4,0} = 1$, related by $M/M' = \mathbb{Z}_2$ acting with weight i on $a_{i,j}$.

We now turn to E_3, E_2, E_1, E_1'.

For E_3, we have the starting data of an irreducible plane cubic f_3, a flex p_0 and another point p_1 not on the line through the flex such that $\overline{p_0p_1}$ is transverse to f_3. By starting with a Weierstrass equation and translating $x \mapsto x+\alpha$ we can assume that $f_3 = V(y^2 - x^3 - a_1x^2 - a_2x - a_3), a_3 \neq 0$ with p_0 the point at infinity and $p_1 = (0, y_1)$. By scaling $x \mapsto t^2x, y \mapsto t^3y$ we can assume $a_3 = 1$. There were 6 possible choices for scaling related by the group $\mathbb{Z}_6$ acting with weights $(2,3,4)$ on the coordinates (a_1, y_1, a_2).

Similarly for E_2 we can choose coordinates where $f_3 = V(y^2 - x^3 - a_1x^2 - x)$ and $p_1 = (0, y_1), y_1 \neq 0$. The choices are related by $\mathbb{Z}_4$ acting with weights $(1,2)$ on (y_1, a_1).

For E_1' write $f_3 = V(y^2 - x^3 - x^2),\ p_1 = (0, y_1),\ y_1 \neq 0$. The choices made are related by $\mathbb{Z}_2$ acting with weight 1 on y_1.

For E_1 write $f_3 = V(y^2 - x^3 - a_1x^2 - x),\ p_1 = (0,0)$. The choices made are related by $\mathbb{Z}_4$ acting with weight 2 on y_1.

E_0 is clearly rigid. □

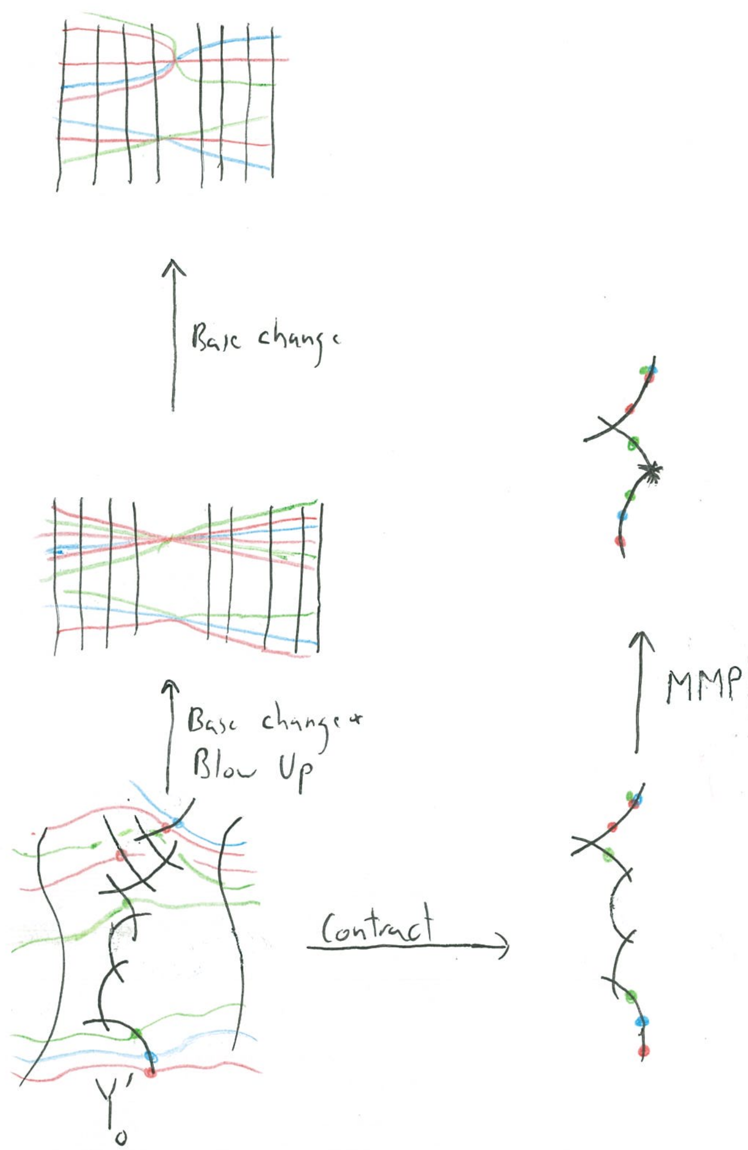

Figure 9.1: Diagram of the stable reduction process for Weierstrass data corresponding to an elliptic K3 surface. The red, blue, and green curves represent the divisors A, B and Δ, respectively. Not all components of these divisors are shown.

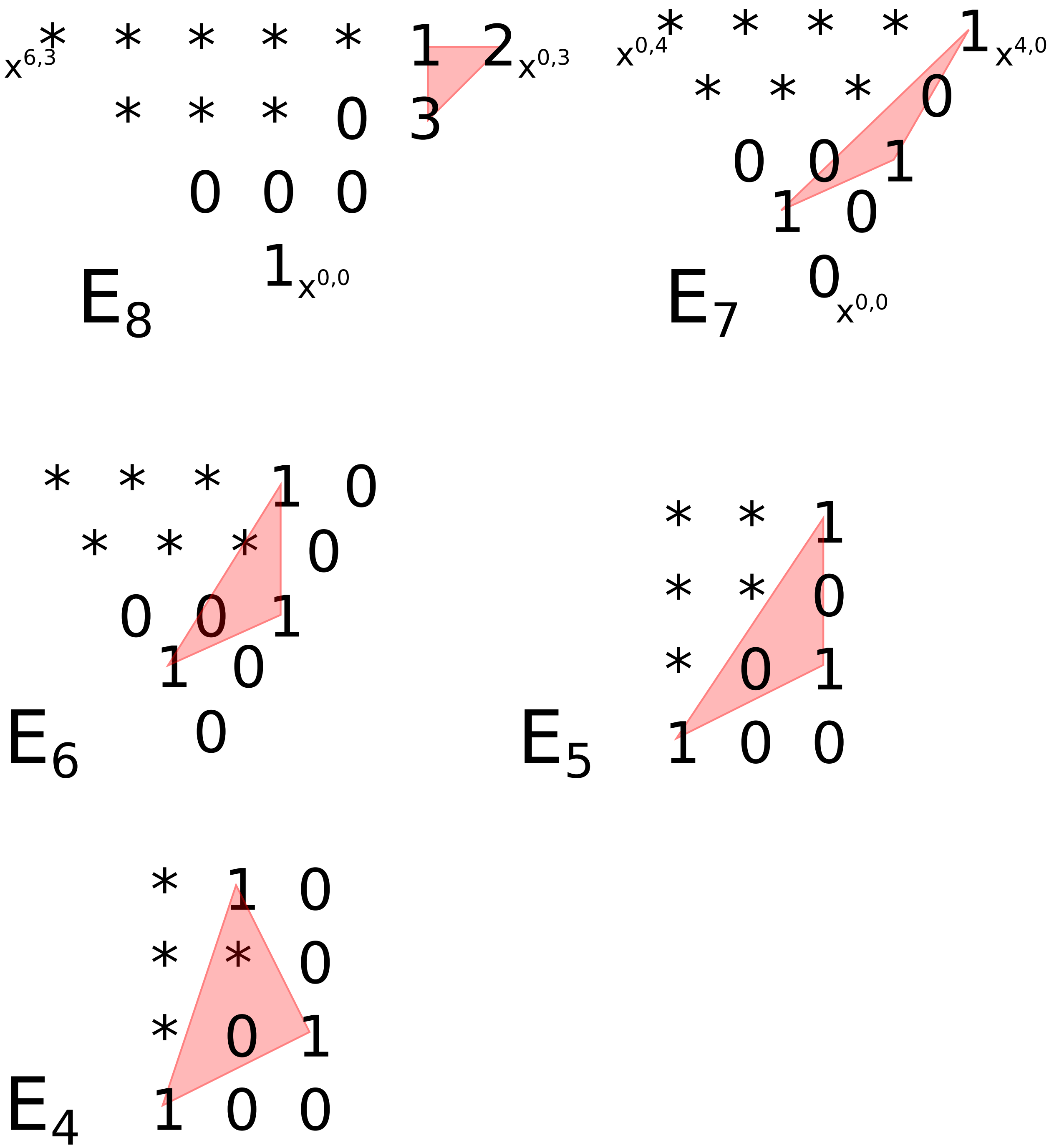

Figure 9.2: Diagrammatic representation of the permissible coefficients in the equation of a divisor in standard form corresponding to the constructions for $E_n, 4 \leq n \leq 8$ given in the text. The red triangles show nonzero constant coefficients for the standard form. The area of these determines the number of distinct ways to put the corresponding pair into standard form.

Chapter 10

Torelli type theorems for components of degenerations and applications.

We now move on to the problem of describing the boundary $\overline{\mathcal{F}_{\text{ell}}} \setminus \mathcal{F}_{\text{ell}}$.

10.1 Torelli Theorems

Recall (9.1.4) that the stable models of such limits are chains of surfaces where the end components are either rational elliptic (type E_n) or a non-normal surface with 2 cuspidal fibers (type D_n) and the middle components are non-normal surfaces obtained by identifying 2 sections of $\mathbb{P}^1 \times \mathbb{P}^1$. All component surfaces come with a marked choice of section and some marked fibers (see chapter 9 for details). There are no moduli in the gluing of such a surface since the surfaces are glued along nodal curves with an additional marked point (from the choice of section). As such we can attempt to parameterize the possible limits by separately describing the moduli of each possible component. Here "possible component" means any surface of the types listed in 9.3 and "possible limit" means any chain of possible components. In particular we postpone showing that all "possible" limits actually occur to a later chapter.

As was already hinted at in the notation we first assign a lattice L_V and group Γ_V to each component V of a type II or III degeneration. For components of type III degenerations we will continue to use the same notation for the lattice and the corresponding surface. Recall the definitions of the lattice E_n given in (5.2.16). The assignments are given by the table:

Surface Type	L_V	Γ_V
$\widetilde{D}_{16}$	D_{16}	$\operatorname{Aut} D_{16}$
$\widetilde{E}_8$	E_8	$W(E_8)$
E_n	E_n	$W(E_n)$
D_n	D_n	$\operatorname{Aut} D_n$
A_n	A_n	$W(D_n)$

Observe that the lattice E_n is the narrow Mordell-Weil lattice associated to a generic surface of that type (6.2.1).

For type II surfaces there is also an associated elliptic curve J_V. For $\widetilde{E}_8$ J_V is the Jacobian of the attaching fiber. In the $\widetilde{D}_{16}$ case it is the Jacobian of the conductor of the normalization. For the other types we write $J_V = \mathbb{G}_m$. (For E_n components this is naturally associated to the Jacobian of the attaching fiber, but this association is less obvious for types A and D.)

The reader may ask why the type II components are not associated with the appropriate semidefinite lattices. This is probably an artifact of the piecemeal approach taken here. A more uniform treatment may be able to resolve the issue. The slight irregularity in the choice of Γ_V is essential, though: it is responsible for much of the non-normality of $\overline{\mathcal{F}}_{\text{ell}}$.

The Torelli theorem is as expected:

Theorem 10.1.1. *The coarse moduli space for components of type $\mathcal{V}$ is given by* $\operatorname{Hom}(L_{\mathcal{V}}, J_{\mathcal{V}})/\Gamma_{\mathcal{V}}$.

We call the map $L_V \to J_V$ associated to a surface V a *period point.*

Proof. We prove the result for non-normal components in a case by case manner, leaving the harder case of describing rational elliptic surfaces as a lemma.

A_n We describe the moduli of $n+1$ points p_i on a line with 0 and ∞ marked. Starting with the points p_i we need to produce a map $A_n \to \mathbb{C}^*$, where $A_n = \{(e_1 \dots e_{n+1}) \in \mathbb{Z}^{n+1} | \sum e_i = 0\}$. Indeed, we simply choose an isomorphism $\mathbb{P}^1 \setminus \{0, \infty\} \simeq \mathbb{C}^*$, and map $e_i \to p_i$. While this map depends on the choice of isomorphism (i.e. which point we label 1), the map from $A_n \subset \mathbb{Z}^{n+1}$ does not. The converse construction is clear. Note that $W(A_n)$ acts by permuting the p_i.

D_n Here we are given n points $p_i \neq q_1$ on a line with a marked origin q_1 and two other marked points q_2, q_3. The double cover of $\mathbb{P}^1$ branched over $2q_1 + q_2 + q_3$ is a nodal curve C. We choose an isomorphism $C^\circ \simeq \mathbb{C}^*$ where q_2, q_3 correspond to the 2-torsion points, and lifts $\widetilde{p_i}$ of the p_i. If we write $D_n = \{(e_1 \dots e_n) \in \mathbb{Z}^n | \sum e_i \equiv 0 (\mod 2)\}$ then the choice of automorphism and lifts gives we have a well defined map $D_n \to \mathbb{C}^* | e_i \mapsto \widetilde{p_i}$. The group $\operatorname{Aut} D_n \simeq S_n \ltimes (\pm 1)^n$ exactly accounts for the choices of labels and lifts. Conversely any map $D_n \to \mathbb{C}^*$ determines a map $\mathbb{Z}^n \to \mathbb{C}^*$ (i.e. the $\widetilde{p_i}$) up to ± 1, so determines the p_i.

$\widetilde{D}_{16}$ Similarly, given a map $D_{16} \to E_j$, where E_j is an elliptic curve we choose a map $\mathbb{Z}^{16} \to E_j | e_i \mapsto \widetilde{p_i}$ extending it. This is well defined up to translation by a 2 torsion point. But then the images p_i of $\widetilde{p_i}$ in $(\mathbb{P}^1, 4\text{pts.})$ are well defined up to automorphism, since translation by 2 torsion induces automorphisms of the pair $(\mathbb{P}^1, 4\text{pts.})$. Conversely, given p_i the lifts $\widetilde{p_i}$ are determined up to ± 1, so the choices involved in the construction are related by $\operatorname{Aut} D_{16}$.

□

The remaining task is to derive the description of the moduli of rational elliptic surfaces with a marked fiber of type I_n. The analogous result is attributed to Looijenga for the moduli of del Pezzo pairs, though hard to find stated in the appropriate form. We give an independent proof, the strategy of which is to express the relatively minimal model of the pair X, D as a blowup of $\mathbb{P}^2$ at 9 points on a cubic C (or in the case of E_1', 4 points on the boundary of an appropriate toric surface), and then demonstrate that the choices involved in the construction are parameterized by the appropriate Weyl group. To start, some definitions:

Definition 10.1.2. Write $I_{1,n} = \langle l, e_0 \dots e_{n-1} \rangle$ with $l^2 = 1, e_i^2 = -1$. The *simple roots* of $(-3l + \sum e_i)^\perp$ with respect to this basis are $\alpha_{i+1} = e_i - e_{i+1}$ and $\alpha_1 = l - e_0 - e_1 - e_2$.

A *marking* of a rational surface X is an isomorphism $I_{1,n} \simeq \operatorname{Pic}(X)$ such that $K_X = -3l + \sum e_i$.

A *geometric marking* of X is a marking induced by a representation of X as an n-fold iterated blowup of $\mathbb{P}^2$, where e_i is the pullback of the $(i-1)$st exceptional divisor and l is the pullback of a line. The corresponding basis of $\operatorname{Pic} X$ is called a *geometric basis*.

The following lemma is Dolgachev's [Dol12, 8.2.35].

Lemma 10.1.3. *Let X be a surface obtained by blowing up $\mathbb{P}^2$ at n points in almost general position, with $n \leq 8$. Let $\phi : I_{1,n} \to \operatorname{Pic}(X)$ be any isomorphism with $\phi(-3l + \sum e_i = K_x)$. Then there is a unique sequence of -2 curves c_i such that the composition of reflections $\sigma = \prod \sigma_{c_i}$ has $\sigma \circ \phi \circ \sigma^{-1}$ a geometric marking.*

The same conclusion holds for $n = 9$, under the condition that e_0 is the class of a -1 curve.

Proof. We use induction on n for $n \leq 8$, the result being clearly true for $n = 1$. Let c_j be the classes of -2 curves. Note that the group generated by reflections in c_j acts transitively on the corresponding chambers[1]. Hence there exists some element of this reflection group that sends e_0 to a curve e_0' with $e_0' \cdot c_j \geq 0$ for all j. e_0' is easily seen to be a -1 curve ([Dol12, 8.2.22]), so can be blown down, giving $p : X \to X'$. Choosing a basis l, e_i' for $\operatorname{Pic}(X')$ with $p^* e_i' = e_i$ we apply the induction hypothesis to get a unique composition of reflections $\prod \sigma_{c_i'}$

[1] That is, connected components of $\operatorname{Pic}(X) \otimes \mathbb{R} \setminus \cup c_j^\perp$.

giving a geometric marking of X'. But the curves c_i' pull back to -2 curves, so the result follows.

In the case $n = 9$, the assumption that e_0 is a -1 curve allows one to reduce to the $n = 8$ case as above. $\square$

Definition 10.1.4. Let X be a rational elliptic surface with marked fiber D of type I_n. Let $T \subset \mathrm{Pic}(\widetilde{X})$ be the (type A_n) lattice spanned by the nonidentity components of $\widetilde{D}$. A marking $\langle l, e_0, \dots e_8 \rangle$ is said to be *adapted* if all the effective roots of T are simple roots relative to this basis, and $\alpha_8 \in T$. Write $E = T^\perp$ and observe that the choice of marking induces an isomorphism of E with a standard type E lattice.

Lemma 10.1.5. *Let (X, D) be the relative minimal model of a type E_n (not including E_1') component. Then* $\mathrm{Pic}\, X$ *has an adapted geometric basis.*

Proof. By Dynkin's results the root lattice T embeds primitively in $\langle s, f \rangle^\perp \simeq E_8$ and this embedding is unique up to $\mathrm{O}\, E_8$. Recalling the definition of E_n (5.2.16) we then have $T^\perp \simeq E_n$. The root sublattice extends to a basis of simple roots α_i for $K^\perp \in \mathrm{Pic}\, X$, adding the root α_{n+1}. Choose $e_0 \in \mathrm{Pic}\, X$ to be the unique class with $e_0 \cdot K_X = 1, e_0 \cdot \alpha_2 = 1, e_0 \cdot \alpha_2 = -1$ and $e_0 \cdot \alpha_i = 0$ for $i > 2$. Then there is an adapted basis $\langle l, e_i \rangle$ where the α_i are the simple roots.

By 10.1.3 this basis becomes geometric after some number of reflections in -2 curves. Note that the root α_{n+1} can never be effective, though: the only -2 curves intersecting the fiber D are components of D. Thus reflections in -2 curves don't change the adaptedness of the basis. $\square$

The choice of adapted basis determines a choice of orientation on D. We now show that the different adapted bases giving the same orientation are related by the Weyl group $W(E_n)$

Lemma 10.1.6. *With notation as above* $\mathrm{Fix}(T) \subset \mathrm{O}(K_X^{\perp}) = W(E_n)$.

Proof. Since $E_n \perp T$ by definition $W(E_n)$ fixes T. Conversely the embedding of the root sublattice $E_n \oplus T \hookrightarrow K_X^{\perp}/K_X = E_8$ determines an isomorphism $\mathrm{Disc}\, E_n \simeq \mathrm{Disc}\, T$. The diagram automorphism of E_n acts non-trivially on $\mathrm{Disc}\, E_n$ so cannot extend to an automorphism of E_8. □

We may now introduce the period map.

Lemma 10.1.7. *The restriction* $\mathrm{Pic}\, X \to \mathrm{Pic}\, D$ *gives a well defined element* $\phi_X \in \mathrm{Hom}(E_n, \mathbb{C}^*)/W(E_n)$.

Proof. The choice of an orientation on D gives an isomorphism $\mathrm{Pic}\, D \simeq \mathbb{C}^*$. By 10.1.6 the various choices of adapted basis with a given orientation are related by $W(E_n)$, so ϕ is well defined after fixing an orientation. But any adapted basis with one orientation on D is pulled back via the hyperelliptic involution from one with the opposite orientation on D and the same period ϕ_X, so we're done. □

Proposition 10.1.8. *The period map is a bijection.*

Proof. Given $\phi : E_n \to \mathbb{C}^*$ we must build a model. Start with a nodal cubic $C \subset \mathbb{P}^2$ and write l for the class of a line. Choose one branch of C through the node and blow up $8-n$ times along the strict preimage of this branch of C. The result is a rational surface with an anticanonical cycle of $9-n$ lines. Call the pullback of the class of the $i'th$ exceptional divisor e_{10-i}, so the new components of the anticanonical cycle are $e_i - e_{i+1}$. We now want to find n points on the strict preimage $\tilde{C}$ to blow up. Using the basis and simple roots for E_n given above note that $c = 2\alpha_2 + \alpha_3 - \alpha_1 = l - 3e_0 \in E_n$. Choose $p_0 \in \tilde{C}$ with $\phi c = l \cdot \tilde{C} - 3p_0$. Up to isomorphism there is only one possibility. Now for each root $\alpha_i = e_i - e_{i+1} \in E_n$ choose p_i with $\phi(\alpha_i) = p_i - p_{i+1}$ Blow up at p_i to form a degree 1 (weak) del Pezzo surface $\tilde{X}$, call the classes of the exceptional divisors corresponding to p_i e_i. $|K_{\tilde{X}}|$ has a unique base point which we blow up to form X_ϕ. Call the last exceptional divisor e_{n+1}. It is clear by construction that $\phi_X = \phi$.

For injectivity one needs to show that X_ϕ is unchanged after conjugating ϕ by $W(E_n)$. We only need to show this for reflections in simple roots. Observing the construction we see that reflection in a root $e_i - e_j$ simply interchanges the points $p_i - p_j$. Reflection in a root $l - e_i - e_j - e_k$ takes the point p_i to the third point p_i' on the line $\overline{p_j p_k}$. But the blowup of p_i, p_j, p_k is isomorphic to the blowup of p_i', p_j', p_k'. □

Finally we need to deal with E_1'.

Proposition 10.1.9. *A surface of type E_1' is determined by $\phi : \mathrm{MW}^\circ(X) \to \mathbb{C}^*$, modulo inversion.*

Proof. Observe that $MW^0(X)$ is cyclic, and generated by any section not intersecting the identity. Our model will be built from $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up, as outlined in the accompanying diagram10.1.

1. Blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at the toric fixed points.

2. Blow up p_1, p_2 on the curves e_2, e_3 meeting $\tilde{s}_1$, the strict preimage of a section. Call the exceptional divisors e_5, e_6. Up to the torus action there is no choice involved here.

3. Observe there is a unique curve c meeting the boundary once on e_1 (and meeting e_6). Call $q = c \cdot e_1$. Choose p_3 to be the unique point on e_1 with $\phi\alpha_1 = p_3 - q$ and blow up at p_3, calling the exceptional divisor e_7.

4. The resulting surface is a degree 1 del Pezzo, so blow up the base point of $|-K|$ to form X.

Letting e_7 be the identity section we see this is a E_1' surface, since it contains 2 torsion in the Mordell-Weil group. Indeed, the section $2(e_6 - e_7)$ projects to a principal divisor in $T^\perp$ as shown in the figure (see 6.2). □

Remark 10.1.10. The explicit description of these quotients is fairly well known. In particular if L is a root lattice then the ring of invariants $k[L]^{W(L)}$ is isomorphic to the monoid ring $k[\Lambda^+ \cap L]$, where Λ^+ is the dominant Weyl chamber. Since the monoid of domininant weights is freely generated the ring of invariants is easily calculated and the affine variety is the quotient of $\mathbb{A}^{\operatorname{rank} L}$ by an action of $\operatorname{Disc} L$. Compare to the result 9.3.2.

Remark 10.1.11. The reader may wonder why we refer to this theorem as a Torelli theorem. It is an honest Torelli theorem in the case of E_n and $\widetilde{E}_8$, in which case it parameterizes the surfaces by the mixed Hodge structure of their interior (i.e. the restriction of $\operatorname{Pic} V$ to $\operatorname{Pic} \partial V$). In the other cases it can be seen either as part of a (as yet unproved) Torelli theorem for a neighborhood of V in $\overline{\mathcal{F}}_{\text{ell}}$ or as reflecting the period map for some component in Kulikov models that get contracted in the stabilization process (a fact we also don't show).

10.2 Automorphisms of stable pairs

Since toroidal compactifications of $\mathcal{F}_{\text{ell}}$ are constructed in an essentially Hodge theoretic manner, knowing the automorphisms of stable pairs accounts for stacky behavior of the moduli space. Some of this is already apparent in the representation of $\overline{\mathcal{F}}_{\text{ell}}$ as an orbifold, however highly degenerate surfaces may have some extra automorphisms.

Much of the complication is caused by the existence of automorphisms that act nontrivially on the base.

Lemma 10.2.1. *Let (V, B) be a component of a stable surface of type E_n (including E_1'). The group $H(V) \subset \operatorname{Aut}(V, B)/\mathbb{Z}_2$ of automorphisms modulo the hyperelliptic involution that act trivially on* $\operatorname{Pic} V$ *is*

- $\mathbb{Z}/3$ *if V has type E_0*
- $\mathbb{Z}/2$ *if V has type E_1*

- *Trivial otherwise.*

Proof. Let $\widetilde{V}$ be the smooth minimal model. The map $\widetilde{V} \to V$ contracts ADE configurations of -1 curves. If the configuration spans a primitive sublattice then each reduced curve in the fiber is intersected by a section. This is not true in the case of a configuration of -2 curves not spanning a primitive sublattice. These are exactly the cases of an E_8 or D_8 configuration or one type of E_7 configuration. By Dynkin's results (**??**) one sees that in all cases there are no nontrivial automorphisms of the dual graph of the configuration of -2 curves fixing the components intersecting sections. Therefore we see an automorphism acts trivially on $\operatorname{Pic} V$ if and only if it acts trivially on $\operatorname{Pic} \widetilde{V}$. Since a type E surface has at least 3 singular fibers we can choose an isomorphism of the base curve with $\mathbb{P}^1$ such that the automorphisms in $H(V)$ are induced by multiplication by roots of unity on the base curve. The quotient of a type E_n surface by an automorphism of order m is a rational elliptic surface where the image of the I_{9-n} fiber is a $I_{(9-n)/m}$ fiber. Since the automorphism fixes $H^2(\widetilde{V})$, the pullback map on the Mordell-Weil group must be an isomorphism. But this can certainly never happen when $MW(V)$ maps surjectively to the component group of the I_{9-n} fiber, since a section generating the component group could only be the pullback of a section passing through a singular point of the $I_{(9-n)/m}$ fiber, contradicting the fact that all sections pass through smooth points of their fibers. Either by observing the construction of type E_n surfaces in the proof of 10.1.1 or by glancing at the table of Mordell-Weil groups in Oguiso-Shioda [OS91] we see the only possibilities are E_0 and E_1. The unique surface of type E_0 is a triple cover of the surface with fiber type $IV^*I_1I_3$. If V is of type E_1 then there is a double cover $\pi : V \to V'$ where V' is a surface of type $I_0^*I_1I_1I_4$. In both cases one can verify that the pullback is an isomorphism of Mordell-Weil groups. Indeed, checking Oguiso-Shioda [OS91] we see that the torsion parts of the Mordell-Weil group are the same, and that the lattice structure on the non-torsion part of type $I_0^*I_1I_1I_4$ has one generator of square $\frac{1}{8}$, whereas that of a type E_1 surface (Kodaira type $I_9I_1I_1I_1$) has a single generator of square $\frac{1}{2}$. Since the double cover multiplies the intersection form by 2, $\pi^* : \mathrm{MW}(V') \to \mathrm{MW}(V)$ must be an isomorphism. There is a one dimensional family of type $I_0^* I_1 I_1 I_4$ surfaces, so all E_1 surfaces can be obtained from them. □

We formally set $H(V) = 0$ for V of types A or D.

The automorphisms of components are then a semidirect product with the group that changes the marked period map:

Lemma 10.2.2. *Let (V, B) be a component of a stable surface of type III with period map $\phi : L_V \to \mathbb{C}^*$. Then*

$$\mathrm{Aut}(V, B) = (\mathrm{Stab}_{\Gamma_V}(\phi)/W(\ker \phi)) \ltimes H(V)$$

with $H(V)$ as above.

Proof. In the type E case observe that $\ker(\Gamma_V \to \mathrm{O}(\mathrm{MW}(V))) = W(\ker \phi)$, so the result follows from the previous lemma.

In the type A and D cases, any automorphism must permute the marked fibers. Using the standard embeddings $A_{n-1}, D_n \hookrightarrow I_n$ the roots are all of the form $e_i - e_j$ in the type A case and $\pm e_i \pm e_j$ in the second. Observing the constructions in the proof of 10.1.1 we see that $\phi(\alpha) = 1$ for a root α if and only if two marked fibers coincide. Conversely, if several marked fibers coincide then $\ker \phi$ contains a type A sublattice whose Weyl group permutes them. Thus $\ker(\mathrm{Stab}(\phi) \to \mathrm{Aut}(V, B)) = W(\ker \phi)$. □

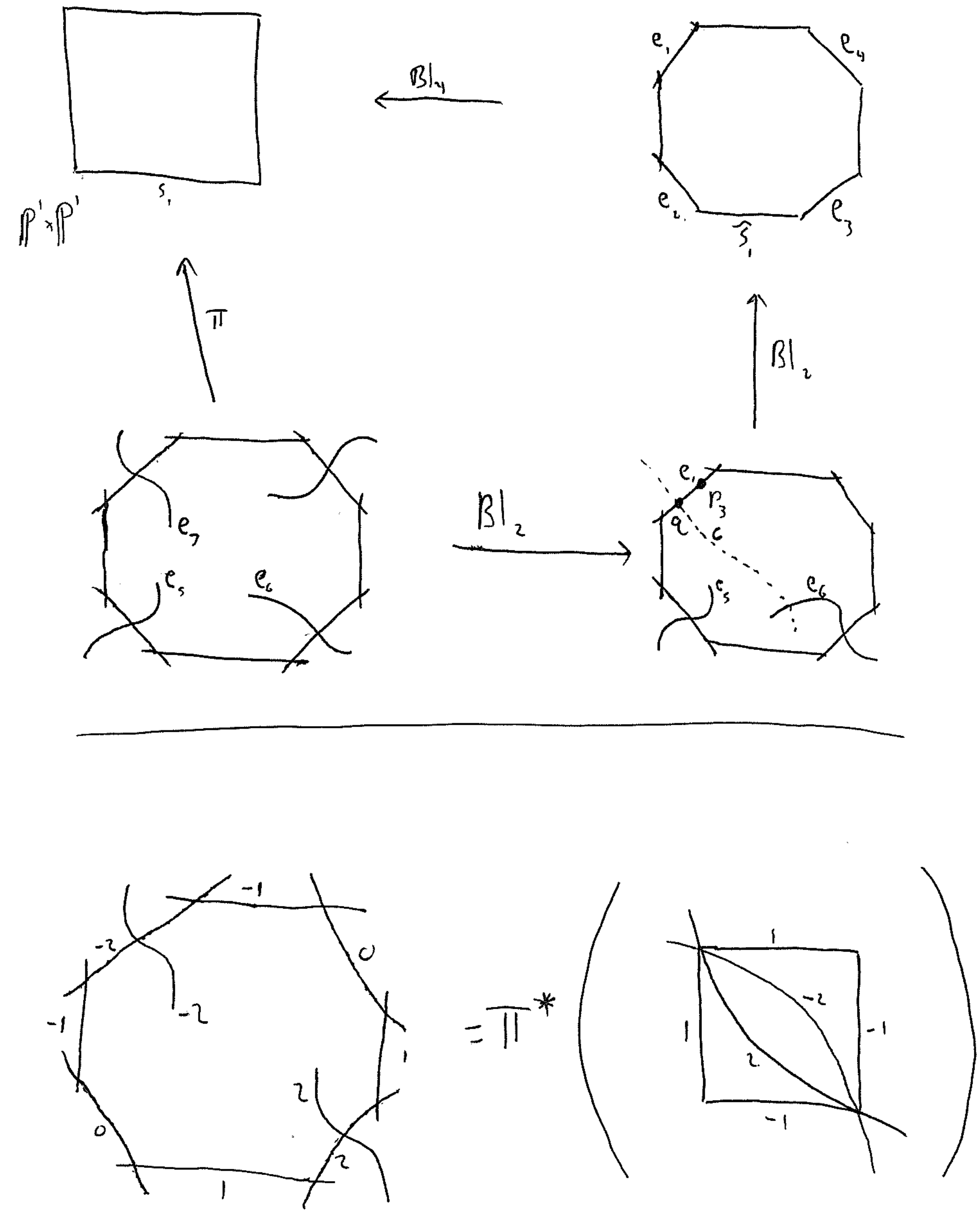

Figure 10.1: Diagram showing the construction of surfaces of type E_1'.

Chapter 11

Construction of type III Degenerations

In this chapter we describe the structure of a Kulikov model for any type III degeneration, as well as how to produce a d-semistable model from a given stable model and "monodromy" (corresponding to the threefold singularities along the double curves in the central fiber, i.e. a point in the appropriate $\mathcal{M}$ cone 9.2).

Given a degeneration $\mathcal{X} \to \Delta$ to a stable pair (X_0, B) we first produce a standard semistable model. Recall that the base curve for X_0 is a chain of rational curves. Arbitrarily label the ends "right" and "left". Using the notation of section 9.2, $\mathcal{X}$ has type A_{a_i-1} singularities above the nodes of the base curve C, and type A_{d_i-1} singularities along the double curves in each component. We can resolve the singularities by first blowing up the base surface (in any order) to resolve its singularities (this gives a new base curve $\tilde{C}$), and then repeatedly move from right to left down the chain, blowing up the singularity along each component of the base. After some small resolutions this will be a Kulikov model with central fiber X_0'. Notice that the fibration on X_0 extends to a map from $X_0' \to \tilde{C}$.

Notation 11.0.3. As a matter of notation, we will introduce some terms for the anatomy of such a model.

We call the strict preimages of the components of X_0 (i.e. those components intersecting the section) $V_{i,0}$, with i going from right to left. The "row" of components $V_{\bullet,0}$ adjoins 2 other rows. We choose one and label it $V_{\bullet,1}$, and continue to thus label the components in succesive rows in a similar manner.

The preimage of each non-end component of $\tilde{C}_i \subset \tilde{C}$ will be called a "ring" R_i.

The irreducible curves in X_0' come in two types. The ones contained in a fiber we call *vertical*. All others are *horizontal*.

The components and horizontal curves farthest away from the originals we refer to as the "spine".

We may choose the small resolutions in the construction so as the exceptional divisors all end up in $V_{i,j}$ rather than $V_{i,j+1}$. The result of this process for a surface of type $E_5A_2A_3E_5$ is shown in the diagram below 11.1.

We describe the resolution process in more detail. One goal of this discussion is to demonstrate the following lemma:

Lemma 11.0.4. *The dual complex $\Gamma_{X_0'}$ and normalization $X_0'^{\nu}$ of the limit fiber X_0' produced by this process depend only on:*

- *The pair (X_0, B)*
- *The surface singularities along the double curves of X_0, given by the numbers a_i, c_i.*
- *For each type D component in X_0 with period $\phi : D_n \to \mathbb{C}^*(\mod \operatorname{Aut}(D_n))$ a choice of lift $\tilde{\phi} : D_n \to \mathbb{C}^*(\mod W(D_n))$.*

Moreover the construction of $\Gamma_{X_0'}$ and $X_0'^{\nu}$ given these data makes sense for any choice of (X_0, B).

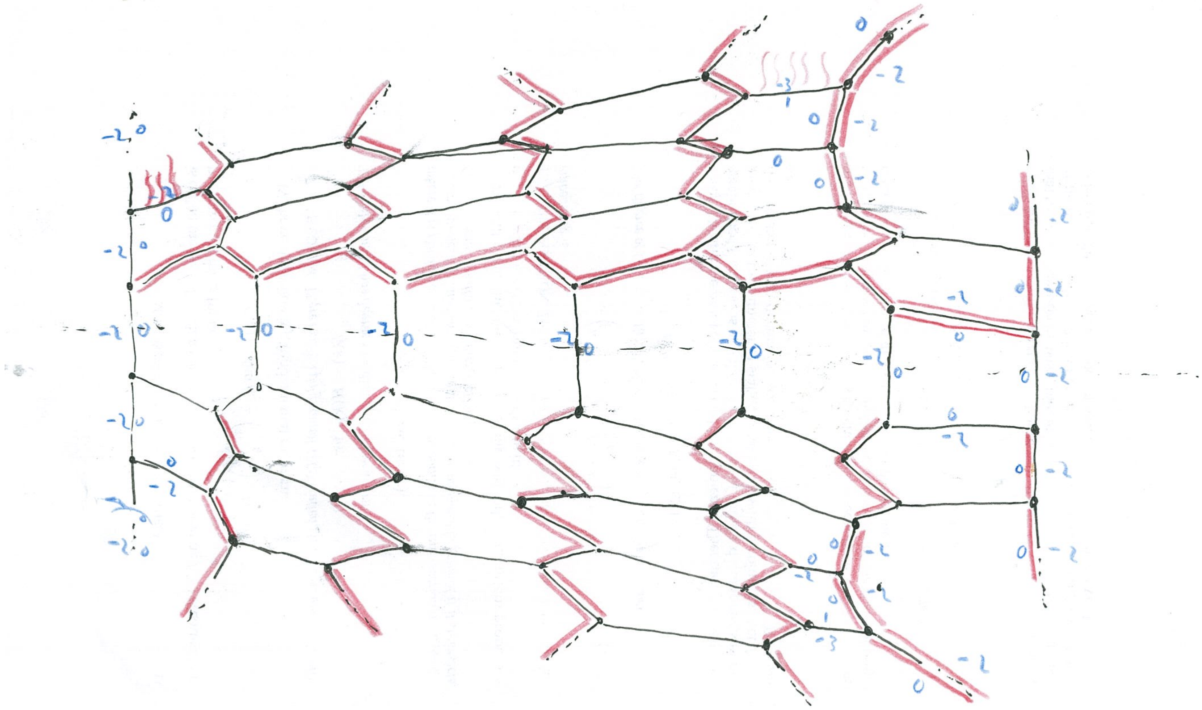

Figure 11.1: Diagram of the central fiber of a Kulikov degeneration with KSBA stable model $E_5A_2A_3E_5$. The blue numbers show the self intersection of the labelled curve on the appropriate component.

Notice that the specific surface X_0' is still highly dependent on our choice of procedure for performing the resolution. The point of the lemma is that very little information is needed about the 3-fold to construct X_0. In particular we define:

Definition 11.0.5. Starting with X_0, a_i, c_i any one of the (at most 4 choices of) surfaces X_0' produced as shown below defines a *formal resolution* $(\Gamma_{X_0'}, X_0'^{\nu})$.

The lemma says that formal resolutions exist.

The first step is to resolve the singularities in the base curve. The effect on X_0 is to add in $c_i - 1$ non-normal components with normalization isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ between the i and $(i+1)$ components of X_0. We formally call these type A_{-1} components.

We will reuse the notation a_i to refer to the order of vanishing of Δ along the corresponding component of $\tilde{C}$, labeled so that the index 1 refers to the leftmost component. In order for this process to work it is required that the multiplicity of Δ on the image of any type D_n component be even.

We will first look at the double cover model of the pre-image of the interior $\tilde{C}_i^\circ$ of each component of the base curve $\tilde{C}$. This is illustrated for types A and D in the figure 11.2. For type E components, $a_i = 0$, so there is nothing to do. For type A_n components one needs to blow up the double curve in T $\lfloor \frac{a_i}{2} \rfloor$ times. If a_i was even the new fiber's top component is $\tilde{C}_i^\circ \times \mathbb{P}^1$ with a reduced bisection B disjoint from the double curve and tangent to each of the marked fibers from the original surface. This is unique up to isomorphism. If a_i was odd the bisection remains non-reduced. The trisection $\mathcal{T}$ in the threefold is simply tangent to the central fiber and has singularities with local equation $st - y^2 = 0$ at the points meeting the marked fibers.

The type D_n case has slightly more subtlety. Again one blows up the double curve in T $\frac{a_i}{2}$ times. $\mathcal{T}$ now meets the middle surfaces in a pair of fibers and the top surface in a bisection B. Again the map $B \to \tilde{C}_i^\circ$ is branched exactly over the marked points. Given the original component there are 2 distinct choices for the top component. In the case of type $D_{n>0}$ these form a connected family, however in the case of type D_0 the divisor B is always reducible and there are 2 distinct choices , one where both components of B intersect the boundary and one where one does twice.

The task now is to keep track of the blowing up procedure to include the vertical boundary of the components and describe the gluing. Since the threefold is smooth, each blowup originally introduces a ruled surface. The triple point formula tells which ruled surface we get: if the curve being blown up has self intersection i in the central fiber, the double curve has square $-i-2$ on the exceptional divisor if it is not over the end of a chain and $-i-1$ if it was. A component $V_{i,j}$ with $j < 2a_i$ will be further blown up at most once each on the left and right boundary fibers. A component with $j \geq 2a_i$ may be blown up further, but in any case the blowing up always occurs on the trisection T, so is determined entirely by the geometry of X_0.

We then have:

Theorem 11.0.6. *Every stable elliptic K3 is smoothable.*

Proof. Let X_0' be a formal resolution of X_0. Then we first note that the strict preimages of the double curves in X_0 can be glued as they were in X_0, so the section still exists as an effective Cartier divisor. The main task is then to show there is a d-semistable regluing of X_0' along curves not intersecting the section. It is perhaps quicker for the reader to check the hypothesis for 7.5.6 for herself than it is to describe an argument. Such a reader may skip the next paragraph.

Notice that every component other than type E is ruled surface. If a component were to have more than one vertical boundary divisor on either side 2 divisors on that side would be -1 curves, and at least one would connect the component to one closer to the row of components containing the section. A component with only one curve in one of its boundary fibers would be negligible. The only remaining task is to show that the non-negligible, non type E components containing the section can all be connected. But this is clear, since the boundary of all except the rightmost such component contains -1 curves in different fibers (the rightmost is negligible).

We apply 7.5.6 to get a d-semistable gluing. By 7.5.7 there is a smoothing where the classes of the fiber and the section remain Cartier, so by cohomology and base change the corresponding divisors extend. Moreover, examining the description for the resolution over components of $\tilde{C}$ given in discussion above note that the marked fibers are exactly the points where the discriminant vanishes more than a_i times. Hence the smoothing is in fact a smoothing of stable elliptic K3's. $\square$

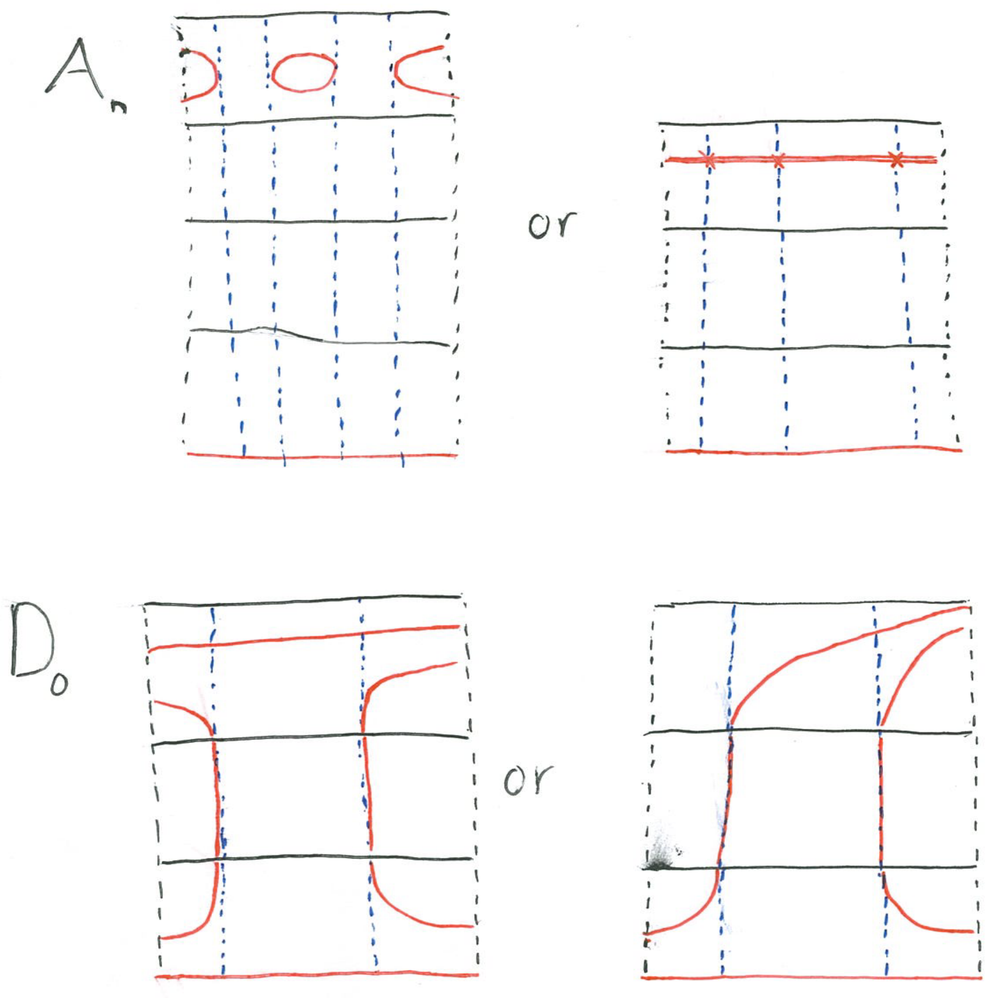

Figure 11.2: Diagram showing the resolution of the double cover representation of a stable elliptic K3 over the interior of a type A or type D component. The red curve represents the branch divisor. In the second case for type A the 3-fold has double point singularities at the points shown in the non-reduced component of the branch curve.

Chapter 12

Description of the Conjectural Fan

In this chapter we briefly review the properties of the Vinberg (reflection) fan $\mathcal{V}$ on a positive light cone of $II_{1,17}$, before introducing a certain subdivision of $\mathcal{V}$ which we shall call $\mathcal{J}$. We conjecture that $(\overline{\mathcal{F}}_{\text{ell}})^{\nu} = \overline{\mathcal{F}}_{\text{ell}}^{\mathcal{J}}$. In support of this conjecture we describe a combinatorial bijection between the cones of $\mathcal{J}$ and the strata of $\overline{\mathcal{F}}_{\text{ell}} \setminus \mathcal{F}_{\text{ell}}$.

We recall that there are 2 parabolic subdiagrams ($\widetilde{E}_8 \oplus \widetilde{E}_8$ and $\widetilde{D}_{16}$) and that elliptic subdiagrams can be obtained by deleting any subset of nodes with the requirement that at least one node on the left and right of the center is deleted. Subdiagrams of the Vinberg diagram correspond to both cones in the Vinberg fan and negative definite sublattice containing a full rank root sublattice. They are named by the type of the root lattice, abusing notation so as the order of the sum matters. (Note this system still isn't perfect.)

Example 12.0.7. As an example we explicitly calculate primitive generators of the Vinberg rays of types $E_a \oplus A_{17-2a} \oplus E_a$ and use computer calculations to determine the squares of rays of types $D_{8+a} \oplus E_{9-a}$ and $E_{9-a} \oplus A_{a+b-1} \oplus E_{9-b}$. Observing the simple roots α that we require $(\alpha, c_a) = 0$ we see that in each E_8 summand c_a projects as $(s_1 \dots s_8)$ with

- $\sum s_i = 0$

- $s_i = s_{i+1}$ for $i < 7, i \neq a$

(resp. for t_i). Moreover, in this situation we see that the coordinates in H are exactly $(s_7 - s_8, s_7 - s_8)$ (this forces the projections to the two E_8 summands to agree). Clearly the primitive member of this ray has coordinates $((9-a, 9-a), (0, \ldots, 0, 1, \ldots, 1, 1-a), (0, \ldots, 0, 1, \ldots, 1, 1-a)$, where there are $a-1$ zeros in each E_8 block. $c_a^2 = 18 - 2a$.

For the $D_{8+a} \oplus E_{9-a}$ case, we machine computation to show that $c_a^2 = 4a$ for a odd and a for a even. (In this and the following example, the need for machine computation is obviated by the primitivity computation below.

For $E_{9-a} \oplus A_{a+b-1} \oplus E_{9-b}$ machine computation shows $c_{a,b}^2 = \mathrm{lcm}(a, b, a+b)$.

Primitivity of root lattices corresponding to Vinberg cones.

In the previous example one notices that many rays corresponded to non-primitive root lattices. We analyse this phenomenon in slightly more generality:

Proposition 12.0.8. *Let $\sigma \subset N$ be a rational polyhedral cone with a single relation $\sum a_i w_i = 0$ among the walls $w_i \cdot \_ \geq 0$. Let F be a face of σ with $L = \langle w_i \rangle_{i \notin I} \subset M$. Let $\overline{L} = F^\perp$ be the primitive closure. Then $\overline{L}/L$ is cyclic of order* $\gcd\{a_i\}_{i \in I}$.

Proof. Let $v \in \overline{L} \setminus L$ with $nv \in L$. We may assume $v = \sum_{iinI} b_i w_i$. Then there is a relation $nv = \sum_{i \notin I} c_i w_i$ and so $nb_i = ma_i$ for some m. Hence $\overline{L}/L$ is generated by $\frac{1}{\gcd\{a_i\}_{i \notin I}} \sum_{i \in I} a_i w_i$. □

The subdivision $\mathcal{J}$

We now divide the maximal dimensional cone in $\mathcal{V}$ by 4 additional hyperplanes, and describe certain details of the resulting structure. Indeed, let $c_1 = 2e_2 + e_3 - e_1, c_2 = 2e_{18} + e_{17} - e_{19}$ and $d_1 = e_3 - e_1, d_2 = e_{17} - e_{19}$ (using the numbering shown in figure **??**), and divide the Vinberg cell by the hyperplanes defined by c_i, d_i. The rays of this cone correspond to rank

17 negative definite sublattices L spanned by some roots and possibly some of the c_i, d_i. We will consistently refer to such lattices as the maximal root sublattice adjoining at most one extra root at each end (noting that $e_2 \in \langle c_1, d_1 \rangle$).

The only way for c_1 to be in this lattice and not in the root sublattice $R(L)$ is when the root sublattice does not contain at least two of e_1, e_2, e_3. (Similarly for c_2). Similarly d_1 is in $L \setminus R(L)$ only if e_1, e_3 are not in L

We now describe the maximal dimensional cones in the fan.

Proposition 12.0.9. *Up to* $\mathrm{Aut}(II_{1,17})$ *there are 6 orbits of maximal dimensional cone in* $\mathcal{J}$*, as follows:*

1. $V \cap c_1^- \cap c_2^-$, *where* V *is the Vinberg cell and* x^- *represents the negative half space relative to* x*. The 19 walls of this cone are perpendicular to the roots* $\alpha_2 \dots \alpha_{18}$ *and the extra vectors* c_i*.*
2. $V \cap c_1^+ \cap c_2^+ \cap d_1^- \cap d_2^-$ *with 19 walls perpendicular to* $\alpha_3, \alpha_4 \dots \alpha_{17}$ *and* d_i, c_i*.*
3. $V \cap c_1^+ \cap c_2^+ \cap d_1^+ \cap d_2^+$ *with 19 walls perpendicular to* $\alpha_1, \alpha_2, \alpha_4 \dots \alpha_{16}, \alpha_{18}, \alpha_{19}$ *and* d_i*.*
4. $V \cap c_1^- \cap c_2^+ \cap d_2^-$ *with 19 walls perpendicular to* $\alpha_2 \dots \alpha_{17}$ *and* c_i, d_2*.*
5. $V \cap c_1^- \cap c_2^+ \cap d_2^+$ *with 19 walls perpendicular to* $\alpha_2 \dots \alpha_{16}, \alpha_{18}, \alpha_{19}$ *and* c_1, d_2*.*
6. $V \cap c_1^+ \cap c_2^+ \cap d_1^- \cap d_2^+$ *with 19 walls perpendicular to* $\alpha_3 \dots \alpha_{16}, \alpha_{18}, \alpha_{19}$ *and* c_1, d_i*.*

Proof. Machine computation using Porta. □

Corollary 12.0.10. *A maximal cone of* $\mathcal{J}$ *in* $c_i^+ \cap d_i^-$ *reflects through the wall* $d_i^\perp$ *to another maximal cone.*

Proof. Indeed, observing the list we only need to note that $\alpha_1 d_1 = -\alpha_3 d_1$, $c_1 d_1 = -2\alpha_2 d_1$, $\alpha_1 - \alpha_3 = d_1$, and $c_1 - 2\alpha_2 = d_1$, since all the other walls are perpendicular to d_1. The statement with d_2 is symmetric. □

Proposition 12.0.11. *The maximal cones in $\mathcal{J}$ are rationally isomorphic. They are 18 dimensional cones given by inequalities $e_i \cdot x \geq 0$ with the one relation $\sum_{i=-9}^{9} ie_i = 0$. The faces of this cone are perpendicular to subsets of the e_i containing neither $\{e_i\}_{i=1}^{9}$ nor $\{e_i\}_{i=11}^{19}$, or perpendicular to both subsets.*

Proof. We write down the relations among the walls of the cones above, and observe that they are all rationally isomorphic to the given cone.

1. $3(-c_1) + 8\alpha_2 + 7\alpha_3 \cdots - 7\alpha_{17} - 8\alpha_{18} - 3(-c_2) = 0$
2. $c_1 + 4(-d_1) + 7\alpha_3 \cdots - 7\alpha_{17} - 4(-d_2) - c_2 = 0$
3. $2\alpha_2 + 4d_1 + 7\alpha_1 + 6\alpha_4 + 5\alpha_5 \cdots - 5\alpha_{15} - 6\alpha_{16} - 7\alpha_{19} - 4d_2 - 2\alpha_{18} = 0$
4. $3(-c_1) + 8\alpha_2 + 7\alpha_3 \cdots - 7\alpha_{17} - 4(-d_2) - c_2 = 0$
5. $3(-c_1) + 8\alpha_2 + 7\alpha_3 \cdots - 5\alpha_{15} - 6\alpha_{16} - 7\alpha_{19} - 4d_2 - 2\alpha_{18} = 0$
6. $c_1 + 4(-d_1) + 7\alpha_3 \cdots - 5\alpha_{15} - 6\alpha_{16} - 7\alpha_{19} - 4d_2 - 2\alpha_{18} = 0$

The remaining claim regards the face structure of the cone C. It suffices to show that $C \cap \bigcap_{i \neq j,k} e_i = 0$ is nonempty if and only if $ij \leq 0$, and that the rays defined when $ij = 0$ are identical. Indeed, if $x \in \bigcap_{i \neq j,k} e_i^{\perp}$, $x \cdot (je_j + ke_k) = 0$ so $x \cdot e_j$ and $x \cdot e_k$ have the same sign iff $jk < 0$. If $jk = 0$ then x lies in the one dimensional subspace $\{e_i\}_{i \neq 0}^{\perp}$. □

We want to show that the fan $\mathcal{J}$ somehow corresponds to the boundary of $\overline{\mathcal{F}}_{\mathrm{ell}}{}^{\nu}$. Unfortunately we don't have a description of how the boundary strata intersect. We can however give a reasonable guess:

Assumption 12.0.12. The components of a non-minimal type III stable surface can be smoothed independently. The results of a smoothing are as follow:

E_n $A_{m-1} \oplus E_{n-m} \rightsquigarrow E_n$

D_n Smoothing the attaching fiber: $A_{m-1} \oplus D_{n-m} \rightsquigarrow D_n$

$D_n, n > 0$ Smoothing the double locus: $D_n \rightsquigarrow E_{n+1}$

D_0 Smoothing the double locus: $D_0 \rightsquigarrow E_1$ for some monodromies.

D_0 Smoothing the double locus: $D_n \rightsquigarrow E_1'$ for other monodromies.

A_n $A_{m-1} \oplus A_{n-m} \rightsquigarrow A_n$

The manner in which the D_0 components smooth is what distinguishes the various cones with the same maximal degeneration.

The problem is now a combinatorial matter.

Proposition 12.0.13. *The combinatorial types of degeneration are in bijection with orbits of cones in $\mathcal{J}$ modulo reflections in d_i.*

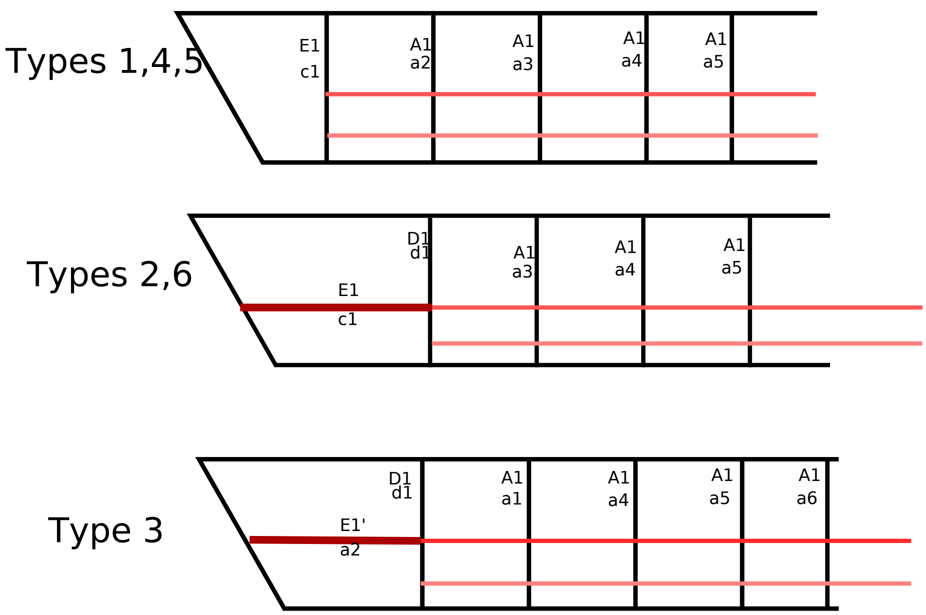

Figure 12.1: Left hand sides of maximally degenerate surfaces, showing correspondence between walls of the corresponding cone and smoothable curves, and the possible simple smoothings.

Proof. We simply show how to identify maximal dimensional orbits with maximal degenerations, and then show how the faces correspond to smoothings. The result then follows from showing the identification is the same on the intersection of any two maximal dimensional cones in different orbits.

The left hand sides of Weierstrass diagrams are shown for the degenerations above.

For the smoothing, observe the diagrams for the types of cones, where curves to be smoothed are labelled with both the wall of the maximal cone and the type of component obtained by smoothing that curve. Recalling that the borders between cones of different types are perpendicular to d_i, c_i, we need to check that smoothing the corresponding curves in different starting diagrams gives an identically marked diagram. Explicitly:

- Smoothing the curve corresponding to c_1 in the type 1 diagram results in a E_1 component and curves marked $\alpha_2, \alpha_3 \ldots$
- Smoothing the curve corresponding to c_1 in a type 2 or 6 component results in a surface of type $E_1, A_0 \ldots$ with curves marked $d_1, \alpha_3, \alpha_4 \ldots$
- Smoothing the curve corresponding to d_1 in a type 2 or 6 component results in a surface of type $D_1, A_0 \ldots$ with marked curves $c_1, \alpha_3, \alpha_4 \ldots$
- Smoothing the curve marked d_1 on a type 3 component results in a surface of type $D_1, A_0 \ldots$ with curves corresponding to $\alpha_2, \alpha_1, \alpha_4 \ldots$

The first and second cases agree because $\langle \alpha_2, c_1 \rangle^{\perp} = \langle d_1, c_1 \rangle^{\perp}$. The third and fourth agree because $\langle d_1, c_1 \rangle^{\perp} = \langle d_1, \alpha_2 \rangle^{\perp}$ □

Relation to $\mathcal{M}$ cones Finally we demonstrate a rational (isometric) isomorphism between the maximal cones above and the $\mathcal{M}$ cones introduced previously in section 9.2 consistent with both the construction of the previous chapter and the interpretation of the strata in this one.

Type 1 $\beta_i \mapsto 3c_i,\ \alpha_i \mapsto \alpha_i$

Type 2 $\gamma_i \mapsto c_i,\ \delta_i \mapsto -d_i,\ \alpha_i \mapsto \alpha_i$

Type 3 $\gamma_1 \mapsto 2\alpha_2,\ \gamma_2 \mapsto 2\alpha_{18},\ \delta_i \mapsto d_i,\ \alpha_i \mapsto \alpha_i$

Type 4 $\beta_1 \mapsto -3c_1,\ \gamma_2 \mapsto c_2,\ \delta_2 \mapsto -d_2,\ \alpha_i \mapsto \alpha_i$

Type 5 $\beta_1 \mapsto -3c_1,\ \gamma_2 \mapsto 2\alpha_1 8,\ \delta_2 \mapsto d_2,\ \alpha_i \mapsto \alpha_i$

Type 6 $\gamma_1 \mapsto c_1,\ \gamma_2 \mapsto 2\alpha_{18},\ \delta_1 \mapsto -d_1,\ \delta_2 \mapsto d_2,\ \alpha_i \mapsto \alpha_i$

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